Comparability of Infinities and Infinite Multitude in Galileo and Leibniz

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1 Introduction

Galileo's discussion of the infinite in *Discourses and Mathematical Demonstrations Concerning Two New Sciences* (1638) hardly wants for recognition. But its importance for Leibniz's philosophy has not always been appreciated. Nor, I think, has Galileo's own view of the infinite in *Two New Sciences* yet been properly understood. A close study of Galileo's paradox of the natural numbers and his answer to it can throw new light on Galileo's own position and, with its elements in view, the influence of Galileo on Leibniz comes into high relief. A number of new points of interpretation of Galileo will be on offer in what follows, some likely to be controversial. Contrary to the customary account, for instance, I hold that Galileo allows for judgments of equality among infinite classes; indeed they are readily found in his mathematical and philosophical work. As I see it, his celebrated denial that the terms 'greater', 'less' and 'equal', apply in the infinite is in fact limited to unbounded magnitudes, but consistent with judgments of cardinal equality among infinite multitudes that are bounded in magnitude and thus, as magnitudes, finite. Galileo's denial of comparability nonetheless poses a threat to two important mathematical principles, Euclid's Axiom and the Bijection Principle of Cardinal Equality, and I consider two sorts of strategies for reconciling those principles with Galileo's position. One strategy, suggested by Eberhard Knobloch,[1] appeals to Galileo's use of the distinction between *quanti* and *non quanti*. The other is due to Leibniz and involves a distinction between totalities and pluralities. I argue that the first strategy cannot save Galileo's account from having to relinquish at least one of the two mathematical principles. Leibniz's strategy offers a more promising way to escape from the paradox while leaving both principles intact, although it imposes a peculiar metaphysical cost of its own. Spelling out the details of Leibniz's solution further reveals

[1] Knobloch (1999; 2011).

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how intimately related his account of the term ‘infinite’ is to Galileo’s discussion and draws out key contrasts between their respective views of comparability and their definitions of ‘infinite’.

1.1 Galileo’s Paradox of the Natural Numbers

Early in the discussion of the dialogue’s First Day, Galileo offers a striking proof for the claim that “one infinity cannot be said to be greater or less than or equal to another” (EN 8:78/D40).[2] The context is one in which Galileo, via his spokesman Salviati, is looking to defend the coherence of the idea that a finite quantity such as a line or a solid might contain an infinity of indivisible points. Simplicio has detailed an objection: it seems a longer line would then contain an infinity of points greater than the infinity contained in a shorter line, implying an infinity greater than the infinite, “a concept not to be understood in any sense” (EN 8:77/D 39). Galileo’s proof would cut off the objection by disallowing any comparison of size among ‘infinites.’ It is the proof itself, though, not the picture of matter being defended, that is our present concern.

Galileo takes the natural numbers as his example of an infinite and argues as follows. Since the natural numbers include both the square numbers and non-square numbers, there are more [*esser più che*] naturals than squares. Yet there are just as many squares as there are roots, since every root has its own square and every square its own root; and there are just as many naturals as roots, since every natural is a root and every root is a natural. So it follows that there are just as many [*siano quanti*] squares as naturals. We thus appear to have contradictory results: the natural numbers are both greater than and equal to the square numbers, which is absurd. (Cf. EN 8:78–79.)

Galileo’s paradox of the natural numbers, then, appears to derive a contradiction from the idea that one infinite can be said to be greater or less than or equal to another. As is readily noted, the proof trades on two different standards for comparison. By the standard of ‘proper inclusion’, there are more naturals than squares since the natural numbers properly include the squares, i.e. the naturals include non-square

[2] Primary texts are abbreviated as follows. For Galileo: EN = *Opere*, Edizione Nazionale, ed. Antonio Favaro (Florence 1898). For Leibniz: A = Berlin Academy Edition, *Sämtliche Schriften un Briefe. Philosophische Schriften*. Series VI. Vols. 1–4. (Berlin: Akademie-Verlag, 1923–99); GP = Gerhardt, *Die Philosophischen Schriften*, Vols. 1–7. Ed. C.I. Gerhardt (Berlin: Weidmannsche, Buchhandlung 1875–1890); GM = *Mathematische Schriften von Gottfried Wilhelm Leibniz*, Vols. 1–7. Ed. C.I. Gerhardt (Berlin: A. Asher; Halle: H.W. Schmidt 1849–1863). References to EN, GP and GM are to volume and page numbers; those to A are to series, volume and page. Translations of Galileo generally follow those of Stillman Drake (abbreviated ‘D’), *Galileo Galilei: Two New Sciences, Including Centers of Gravity and force of Percussion*, 2nd Ed, (Toronto: Wall and Emerson, Inc. 1974), and those of Leibniz generally follow Richard Arthur (abbreviated ‘Ar’), *G.W. Leibniz: The Labyrinth of the Continuum: Writings on the Continuum Problem, 1672–1686* (New Haven: Yale University Press 2001). I have sometimes modified translations without comment.

numbers as well as all the square numbers. Or as we might say, the squares form a proper subclass of the natural numbers.[3] By the standard of 'one-one maps', however, there are just as many squares as naturals, since the two classes can be mapped one-one into each other, implying a 'one-one correspondence' (or *bijection*) between them. In the case of finite classes the standards are always in agreement: no finite class can be mapped one-one into one of its own proper subclasses, and finite classes are always greater than their proper subclasses. Only in the infinite can the standards conflict .

Galileo in effect treats the two standards of comparison as equally sound and suggests that we are mistaken to extend either one from the finite case to the infinite case. He recommends abandoning comparisons altogether in the infinite. History has instead taken sides in order to resolve the paradox, and it has favored the standard of one-one maps over that of proper inclusion. Classes X and Y are equal in size just in case there is a one-one correspondence between them, the proper inclusion of one in the other notwithstanding. Developments in transfinite set theory due to Cantor would establish this as a consistent approach to the paradox,[4] and subsequent orthodoxy was to hold, in Russell's words, "it is actually the case that the number of square (finite) numbers is the same as the number of (finite) numbers."[5]

That is all familiar enough. Writers on the topic tend to be orthodox Russellians on this point today. Galileo's own analysis of the paradox is hardly refuted by the preference of history,[6] however, and what he has to say is quite interesting. Here in the words of Salviati:

> I don't see how any other decision can be reached than to say that all the numbers [*tutti i numeri*] are infinitely many [*infiniti*]; all the squares infinitely many; all their roots infinitely many; that the multitude [*moltitudine*] of squares is not less than that of all numbers, nor is the latter greater than the former. And in the final conclusion, the attributes of equal, greater and less have no place in infinite, but only in bounded quantity [*quantità terminate*]. (EN 8:79/D 41)

The denial that such comparisons are possible in the infinite is Galileo's signature conclusion here. But his words convey a few more ideas worth drawing out. The

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[3] A quick note on terminology. In what follows I sometimes use the terms 'class', 'subclass', etc., for convenience, but without meaning to imply that the many elements of a class thereby form a *set* or *totality* or other 'single object.' (Mostly here it will cause no harm to read 'class' and 'set' as equivalent, but sometimes it will lead astray, so beware.) Occurrences of 'class', etc., can, with appropriate shifts in syntax, always be replaced by suitable plural expressions—e.g., 'the natural numbers' instead of 'the class of natural numbers'—or by terms such as 'multitude' or 'plurality' that cancel the implication of one thing formed from many. In contexts in which greater precision is required to convey the intended meaning, and avoid unwanted implications, I shall use unambiguous terms.

[4] Cantor writes, "There is no contradiction when, as often happens with infinite aggregates, two aggregates of which one is a part of the other have the same cardinal number" (Cantor 1915, p. 75; noted in Parker (2009)).

[5] Russell (1913, p. 198).

[6] For a very illuminating discussion of the history of the idea of measuring the size of the natural number collections, as it has evolved up to Cantor, plus some contemporary alternatives to Cantor, see Mancosu (2009).

multitude of (natural) numbers is infinite, as are those of the squares and their roots. This is justified, presumably, by the one-one correspondences between the squares and the roots and the roots and the naturals. Thus it seems that even if a one-one correspondence isn't sufficient for claiming that two classes are equal, it is sufficient for claiming that a class is infinite if there is a one-one map from some infinite class into it—or, at least, a class is infinite if there is a one-one map from the natural numbers into it.

Galileo does not define 'infinite' explicitly in *Two New Sciences*. Still, his suggestion that a class is infinite if the natural numbers can be mapped into it can itself serve well as an intuitive definition. And his discovery that the class of natural numbers can be mapped one-one into a proper subclass of itself suggests a structural property of classes that would later be elevated by Dedekind into a definition of 'infinite': infinite classes are exactly those that can be mapped into one of their own proper subclasses. In Dedekind's words, 'A system S is said to be infinite when it is similar to a proper part of itself.'[7] In the terms of modern-day mathematics, Galileo discovered that the natural numbers are 'Dedekind infinite' .

1.2 Parts, Wholes and Euclid's Axiom

Unlike Dedekind, Galileo does not use the word 'part' or 'proper part' in an explicitly formal statement of his mathematical principles. The language of parts is not absent from his discussion, however, for he does say, in passing, that the squares form a part of the natural numbers—a 'tenth part' of the first hundred numbers, a 'hundredth part' of the first ten thousand, etc.—and that the non-squares form a 'greater part' [*maggior parte*] than the squares (EN 8: 79). But it is at best equivocal evidence that he means to use the language of parts and wholes for his technical mathematical vocabulary. And Galileo does not say that the natural numbers form a 'whole' or 'totality' or even a 'system'. His phrase *tutti i numeri* might suggest this, since *tutti* can have the force of 'whole' in some uses,[8] but taken straightforwardly what Galileo says is simply 'all the numbers' and likewise 'all the squares' and 'all the roots'. Moreover, his use of the term *moltitudine*, or 'multitude', in claiming that the multitude of squares, that of natural numbers and that of roots are all infinite, seems gauged to avoid the supposition that there is a single totality, a single mathematical object, made up of all the squares or all the naturals or all the roots.

The language of parts and wholes is nonetheless a natural one in which to frame the discussion, and it also offers an idea that likely lies behind Galileo's appeal to the standard of proper inclusion in his initial claim that the natural numbers as a

[7] In the usual formula: S is infinite if and only if there is a one-one map Φ from S into S with some element of S not in the range of Φ. *Was sind und was sollen die Zahlen?*, Sect. 64; cf. Sect 66.

[8] Crew and de Salvio's 1914 translation renders *tutti i numeri* as "the totality of all numbers" (cf. p. 31). In the same lines it also inserts 'number' in "the number of squares is infinite" and "the number of roots is infinite", where the corresponding term does not occur in the original. Drake's translation steers clear of those interpolations.

class are greater than the squares. That idea is the following principle: The whole is greater than the part. It is sometimes called 'Euclid's Axiom' for its occurrence as Common Notion V at the start of Book One of the *Elements*. If the square numbers form a part of the natural numbers, then by Euclid's Axiom the whole of the natural numbers must be greater than the part formed by the square numbers alone. When Dedekind says that in an infinite system the proper part is equal ('similar') to the whole, his position implies the falsity of Euclid's Axiom.

It is not hard to think Galileo's position has the same consequence, if somewhat more subtly. When he recommends that we drop the terms 'greater', 'equal' and 'less' from use in application to infinites, he is in effect abandoning Euclid's Axiom, at least in the case of the infinite. For the result would be to deny that the whole is greater than the part in this case, even if the equality of part and whole is not asserted. As we shall see, Leibniz interprets Galileo in just this way. At the moment it is enough to observe the potential implication for Euclid's Axiom, and to note that Galileo himself may not quite be committed to it, since it is not evident to what extent he embraces the language of parts and wholes for classes of numbers.

The minimal reading of Galileo's own position is just that the extension of Euclid's Axiom from the finite to the infinite is incorrect. He is explicit in warning against taking such extensions for granted. "These are some of those difficulties that arise", he writes, "that derive from reasoning about infinites with our finite minds and giving to them those attributes that we give to the finite and the bounded" (EN 8: 77–78/D39-40). It will then remain to say *why* it is incorrect to apply terms of comparison, or principles like Euclid's Axiom, in the infinite case. The paradox only points up an inconsistency, perhaps showing *that* the extension is invalid; it does not explain the underlying problem.

It is open to Galileo to deny the applicability of Euclid's Axiom in the infinite case without thereby rejecting the axiom itself, if a condition of its terms can be seen not to hold in the infinite. The involvement of the concepts of part and whole in the axiom indicates one possible avenue for doing this: if we should say that infinite multitudes cannot form wholes, then Euclid's Axiom will be seen not to apply in the infinite case without thereby being overturned by a counterexample. If there are no infinite wholes, then there is no whole that fails to be greater than its parts. Another possible avenue might be to identify some less explicitly stated condition of Euclid's Axiom, say, some requirement that the terms it compares be in some way 'measurable' or 'quantifiable', and then see if it can be denied that 'infinites' meet this condition. If it can be held that infinites are not measurable or quantifiable in the relevant way, then denying that Euclid's Axiom applies to them might fall short of rejecting the axiom itself. Again, its inapplicability would not be due to the existence of counterexamples.

I belabor those points because each of those two avenues has been suggested as a possible route of escape from the prospect of having to deny Euclid's Axiom. The first—denying that infinite multitudes form a whole—is proposed by Leibniz, who, commenting on Galileo's paradox of the natural numbers in his notes on *Two New Sciences*, writes:

> Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one [*Unum*] and not a whole [*totum*]. (A VI, 3, 158/Ar 9)

As before, it is not quite true that Galileo says the whole is not greater than the part in the infinite, since his mention of parts is not clearly committal and he does not write of the number classes directly in terms of wholes. (Perhaps Leibniz, like some of Galileo's later readers, sees the language of wholes [*totum*] in Galileo's phrase *tutti i numeri*. This would not be surprising, since part-whole terminology was a common feature of mathematical language in the seventeenth century.) We shall consider Leibniz's own position in due course.

The second avenue of escape—that of denying that infinites are suitably measurable or quantifiable—is suggested as Galileo's own view by Eberhard Knobloch, who calls attention to Galileo's careful distinction between those things which are true quantities or 'quantified' (*quanti*) and those which are 'non-quantified' (*non-quanti*) in the treatment of infinites and indivisibles in *Two New Sciences*. Knobloch writes,

> An 'infinite quantity' ('quantità infinita') would according to Galileo's conception actually be a 'contradiction in terms', because an infinite lacks precisely those properties which characterize a quantity. […]
> Correspondingly, the Euclidean axiom 'The whole is greater than the part' is not invalidated in the sense that the logical opposite is valid in the domain of infinite sets, that is, that an infinite set is smaller than or equal to one of its parts. Rather it is invalidated in the sense that it cannot be applied there, simply because there are not quantities which could be compared.[9]

Knobloch's analysis of Galileo's position is an important one, and we shall turn shortly to consider the content of the distinction between *quanti* and *non-quanti*. For now it is enough to note that it stands as an alternative to Leibniz's proposed route of escape. Each one in its own way allows us to see how Galileo's suggestion that infinites cannot be compared need not automatically imply the rejection of Euclid's Axiom. And each would give us a way to explain why the axiom is not rendered invalid: either there are no infinite wholes or there are no infinite quantities, and hence there are no counterexamples.

1.3 The Same Question Revisited: The Bijection Principle

Just as Galileo's denial of comparison among infinites poses at least a *prima facie* threat to Euclid's Axiom, so too it poses a threat to the idea that one-one correspondence between classes is a valid standard of equality—and, more generally, a threat to the validity of using one-one maps to determine comparisons of size among sets or classes. To the modern eye this may seem the more troubling element of Galileo's position, since in the wake of Cantor, Euclid's Axiom has been set aside while the standard of one-one maps has come into its own as a vital piece of mathematical theory and practice.

[9] Knobloch (1999), p. 94.

A few words of clarification are in order about the intended principle of equality based on one-one maps and what we shall call it.[10] In contrast to some writers,[11] Galileo is addressing the idea of comparison between multitudes without any obvious presupposition of number.[12] His expressions for equality in the relevant passages are simply *tanti quanti* and *altrettanti* with the sense of 'precisely as many as'. Thus for now in representing his view we can step back from the idea of assigning a number to a multitude, and ask more minimally whether one-one correspondence, or bijection, implies equality of relative size, though of course the relevant notion of size is the cardinal one of 'many-ness' rather than, say, the metrical one of 'much-ness' or measure. Likewise for the related definitions of 'greater' and 'less': X is greater than Y if and only if Y can be mapped one-one into X but X cannot be mapped one-one into Y, and *vice versa* for 'less', but neither need be taken to imply a claim about number or absolute size. We shall adopt the precise if anachronistic label 'the Bijection Principle of Cardinal Equality'—or just 'the Bijection Principle'—for the principle that says X and Y are equal if and only if there is a one-one correspondence (bijection) between their elements. (Better still is the plural form: there are just as many Xs as Ys if and only if there is a one-correspondence between the Xs and the Ys.)

The issue before us is what to make of Galileo's abandonment of the Bijection Principle in the infinite. There are both conceptual and historical questions to consider. Take first the purely analytical question of whether this means that the Bijection Principle is simply invalid on Galileo's terms. Such a result did not follow in the case of Euclid's Axiom; there are ways of leaving the axiom intact while withholding it from the infinite. Yet unlike Euclid's Axiom, the Bijection Principle is not phrased in terms of parts and wholes. So if we follow Galileo in denying the comparability of infinites, there is no taking Leibniz's escape from the conclusion that infinites are counterexamples to the Bijection Principle by denying that infinites are wholes. If we take the Bijection Principle to apply only to *sets*, a clear version of Leibniz's tactic remains available. If an infinite multitude does not form a set—if, in

[10] My discussion is indebted to Parker (2009), who, defensibly, calls our two principles 'Euclid's Principle' and 'Hume's Principle'. If Galileo had not rejected the one-one maps standard in the infinite case, we should call it 'Galileo's Principle'. For reasons to think Archimedes made use of this principle in application to infinite classes, see Netz et al. (2001–2002).

[11] Notably those involved in discussion of a similar principle of equality sometimes called 'Hume's Principle': the number of Fs equals the number of Gs iff there is a one-one correspondence between the Fs and the Gs. The principle is so-called for Frege's reference, in Sect. 73 of the *Foundations of Mathematics*, to Hume's remark, in *Treatise* I.iii.1, "When two numbers are so combin'd as that one has always an unite answering to every unite of the other, we pronounce them equal." Yet both of those authors have their sights on slightly more restricted conditions than the ones Galileo considers. Frege takes one-one correspondence between classes to imply the existence of a *number* that measures them; Hume \t "*See* Principle" is expressly considering a standard of equality for numbers, where the numbers themselves are conceived as made up of units. It is in this vein that one-one correspondence is sometimes said to be a criterion of 'equinumerosity': equality of number.

[12] Or he appears to be doing so. Below I shall suggest his account of comparisons of infinite number classes turns out to involve an infinite number after all; that is, if, *per impossibile*, there were such a comparison, it would have to involve an infinite number.

Russell's phrase, it is a *proper class*—then perhaps it does not fall within the scope of the Bijection Principle, and so denying that it is comparable to other infinites via one-one maps does not thereby make it a counterexample to the principle. (Here our use of the neutral term 'class' rather than 'set' in referring to multitudes of things matters.[13]) Likewise, a version of Knobloch's strategy is available if we can read the Bijection Principle as tacitly requiring its terms of comparison to be *quanti* and then hold that infinite multitudes are *non-quanti*. The two strategies might be regarded as nearly equivalent, since a natural thought is that something is mathematically quantifiable or fit for mathematical measurement only if it can be understood as a 'single object,' such as a set. If infinites are not unities, but only uncollected multitudes like proper classes, they might on that ground be regarded as *non-quanti* and hence not candidates to be counterexamples to the Bijection Principle.

The question of whether Euclid's Axiom or the Bijection Principle face counterexamples matters because each has, in its time, been thought to capture or reflect something deep about the idea of a mathematical quantity. Euclid's Axiom was regarded as nearly constitutive of the very idea of quantity.[14] The Bijection Principle has something of the same position with respect to the idea of cardinality today. If either one were shown to be incorrect in clear cases, there would be reason to doubt whether the related understanding of quantity or cardinality were truly secure. Even damming up counterexamples on the far side of the distinction between the finite and the infinite is not automatically a satisfactory solution if we would otherwise take ourselves to see clearly that infinite classes meet the conditions of quantity or cardinality. If the properties we appeal to in the finite in order to justify our mathematical reasoning are patently also exemplified in the infinite but then lead into contradiction, we should doubt whether our original appeal was sound. That is, we should doubt whether it was sound unless we can explain why the extension of those properties to the infinite case is invalid.

Denying that infinites are wholes or that they are truly quantities can be a step in the direction of an explanation. "Our mathematical justifications in the finite case", we could say, "presupposed that the objects of study are wholes or quantities. The infinite case provided an initial appearance of this, but it was only an illusion. There simply are no 'infinite wholes' or 'infinite quantities' to which they may be applied. So there are no counterexamples to our principles." Of course, this only works if we have some ground for saying that infinites are not wholes or quantities independently of the paradox; otherwise, we are just left "wielding the big stick"—i.e., pointing to the contradiction—rather than offering an explanation.[15] I suspect the proposed escape routes, whether Leibniz's or the one Knobloch sees in Galileo,

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[13] See footnote 3 above.

[14] Bolzano, for example, explicitly defended the primacy of Euclid's Axiom against the Bijection Principle, writing that even two sets that stand in a one-one correspondence "can still stand in a relation of inequality in the sense that the one is found to be a whole, and the other a part of that whole" (Bolzano 1950, p. 98). For discussion of Bolzano, see Parker (2009) and Mancosu (2009). Even Russell acknowledged that "the possibility that the whole and part may have the same number of terms is, it must be confessed, shocking to common sense" (1903, p. 358).

[15] On "wielding the big stick", see Michael Dummett (1994).

may turn out to be cases of the big stick and not truly explanations. But in any case it should be clear that there is something at stake here (if different stakes for audiences from different eras) in asking whether Euclid's Axiom or the Bijection Principle is in jeopardy of admitting counterexamples even in the infinite case.

1.4 Infinite Multitude and Non Quanti in Galileo

Back to Galileo and a few historical questions. Is Galileo's concept of *non quanti* meant to cover the case of the infinite multitude? I think the answer is no, or at least I believe that infinite multitudes do not automatically qualify as *non quanti* for Galileo, though in special cases they may do so. To cast enough light on Galileo's view here, a fairly close look at the texts will be required, though technical matters can be kept to a minimum.

The distinction between *quanti* and *non-quanti* in *Two New Sciences* occurs in connection with indivisibles, in particular with the idea that quantities such as lines or circles or solid bodies might contain or be resolved into infinitely many indivisible parts. Two sections of the dialogue are most explicit in discussing the idea of *non quanti*. In both, Galileo, in the voice of Salviati, appeals to the hypothesis of the composition of matter from indivisibles and the presence within it of indivisible vacua or void spaces to make sense of the possibility of the expansion or contraction in size of a finite quantity. These expansions and contractions ('rarefaction' and 'condensation') are themselves introduced to resolve the ancient paradox of the wheel, concerning the motion of concentric circles rolling along a line. The puzzle is that it appears that the smaller interior circle and the larger outer one will traverse equal distances in the course of a single revolution despite the difference in their circumferences. (See Fig. 1.)

Galileo approaches the problem by developing an analysis of the motion of concentric polygons and then extending it to that of the circles, taking the circles as

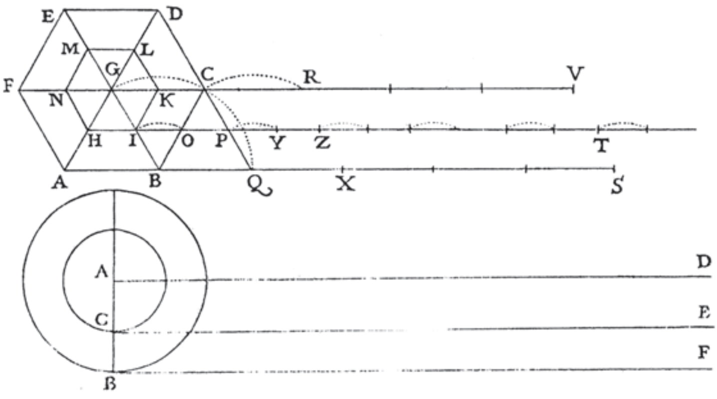

Fig. 1 Galileo's analysis of 'Aristotle's Wheel.' (EN 8: 68)

infinilateral polygons. The details of the analysis are, as Galileo's Sagredo notes, 'intricate,' and need not detain us.[16] The key point is that Galileo finds that the motion of the larger polygon passes over a line approximately equal in length to that traversed by the smaller polygon, but with the interposition of 'skipped over' void spaces into the line traversed by the smaller. (The presence of the void spaces compensates for the difference in the lengths directly marked out on the line through contact by the sides the two polygons, whose perimeters are, of course, unequal.) As the number of sides is increased, the sides and void spaces become smaller and the lengths of the lines measured out by the motions come closer to equality. Advancing now to the limit case of the motion of two concentric circles, the distinction between *quanti* and *non quanti* appears when Salviati says:

> And just so, I shall say, in the circles (which are polygons of infinitely many sides), the line passed over by the infinitely many sides of the larger circle, arranged continuously <in a straight line>, is equal in length to the line passed over by the infinitely many sides of the smaller, but in the latter case with the interposition of just as many voids [*d'altrettanti vacui*] between them. And just as the sides are not quantified [*lati non son quanti*], but are infinitely many [*ma bene infiniti*], so too the interposed voids are not quantified [*vacui non son quanti*], but are infinitely many; that is, for the former <line touched by the larger circle there are> infinitely many points all filled, and for the latter <line touched by the smaller circle>, infinitely many points, part of them filled points and part of them voids. (N 8:71/D 33)

As Knobloch observes,[17] Galileo shifts from having a little earlier spoken of the lines as 'measured' (*misurata*) by the finitely many sides of the finite polygons to saying only that they are 'passed over' (*passata*) by the infinitely many sides of the circles. The sides are no longer strictly fit to measure the lines they touch: they are *lati non quanti*.

There is a delicate question here in interpreting Galileo's remarks about the *lati* and *vacui non quanti*. What is quite clear is that the sides and voids themselves are *non quanti* in the sense that each individual sidelet or void space has no measure; each is an indivisible point that cannot mark a unit of measure of a line or a body. So being *non quanti* is an intrinsic characterization of an indivisible point, whether filled or unfilled. Less clear is whether the characterization of the sides and voids as *non quanti* is also supposed to apply to their being *infinitely many*. Is an infinite multitude, simply by virtue of its infinitude, 'not quantified'? When Galileo says the sides and voids *non son quanti*, *ma bene infiniti* ("are not quantified, but infinitely many"), this could be taken to contrast being *quanti* with being infinitely many, so that being infinite in multitude is itself a further case of being non-quantified. Or it may instead be taken to clarify the fact that although the sides and voids are *non quanti*—each of them not itself a measurable unit—there are nonetheless infinitely many of them, without thereby implying that *non quanti* applies also the idea of infinite multitude. Which is it?

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[16] It should be noted that Galileo's analysis is mistaken—at most one of the two circles *rolls* along the tangent, the other merely revolves continuously along it with the illusion of rolling—but our interest concerns the elements of his analysis, not the quality of his solution. For detailed discussion see Drabkin (1950), Costabel (1964), and Knobloch (1999, 2011). See also Mancosu (1996, pp. 121–122).

[17] Knobloch (1999, p. 92).

Unsurprisingly, there are two concepts of quantity to be considered in asking after the meaning of *non quanti* for Galileo, the *metrical* and the *cardinal*. Calling indivisibles *non quanti* is an intrinsic description of points, denying them metrical properties. If *non quanti* is also supposed to characterize infinite multitudes as such, it is then a description denying infinite multitudes cardinal properties. Certainly it is a description denying such multitudes a definite cardinality or number, which would already have been a commonplace view of the time. But perhaps the phrase *non quanti* is further laying the groundwork for denying infinite multitudes very general cardinal properties of comparability: qualifying as 'greater', 'less' or 'equal' in the sense of being more, fewer or equally many. It is this cardinal sense of being *non quanti* that is crucial to asking whether Galileo's reply to the paradox of the natural numbers requires him to deny Euclid's Axiom and the Bijection Principle.

I suspect that Galileo is fairly consistently thinking of *non quanti* through a metrical lens rather than a cardinal one in the passage just reviewed above. After all, he quite explicitly says the line passed over by the smaller circle contains 'just as many' voids (*altrettanti vacui*) as the infinitely many sides (*infiniti lati*) of the smaller circle, even while going on to say that the voids and sides are *non quanti*. A cardinal conception of *non quanti* that disallows comparison of infinite multitudes should have ruled that out. Moreover, the basis for the judgment of equality in that very example is paradigmatically cardinal. In the case of finite polygons, the equal number of sides and voids is established by the one-one correspondence of the sides of the revolving polygons to the parts of the line successively touched or skipped over. In the infinite case of the circles, there will likewise be a succession of touches and skips to establish a one-one correspondence. It is this one-one correspondence which underwrites Galileo's claim that there are just as many voids in the line as sides on the circle, understood as an infinite polygon. So the fact that the sides and voids are *non quanti* does not yet preclude the claim of cardinal comparability. Perhaps when Galileo later denies the possibility of comparison among infinite multitudes, in examining the paradox of the natural numbers, it is not because he thinks infinite multitudes automatically qualify as *non quanti*.

Cardinal notions are at work in his thought as he discusses *non quanti*, and this is evident in the lines that follow immediately on those of the prior passage. Galileo elaborates the idea of composing a finite quantity from an infinity of *non quanti* in order to show how the doctrine of indivisible voids can be deployed to make sense of the possibility of the expansion (and presumably contraction) of lines or solids into spaces of different sizes. Salviati continues with his explanation:

> Here I want you to note how, if a line is resolved and divided into parts that are quantified [*in parti quante*], and consequently numbered [*numerate*], we cannot then arrange these into a greater extension than that which they occupied when they were continuous and joined, without the interposition of just as many [*altrettanti*] void<finite>spaces. But imagining the line resolved into unquantifiable parts [*parti non quante*]—that is, into infinitely many indivisibles—we can conceive it immensely [*immenso*] expanded without the interposition of any quantified void spaces, though not without infinitely many indivisible voids.
>
> What is thus said of simple lines is to be understood also of surfaces and solid bodies, considering those as composed of infinitely many unquantifiable atoms [*infiniti atomi non quanti*]; for when we wish to divide them into quantifiable parts [*parti quante*], doubtless we cannot arrange those in a larger space than that originally occupied by the solid unless

> quantified voids [*quanti vacui*] are interposed—void, I mean, at least of the material of the solid. But if we take the highest and ultimate resolution <of surfaces and solid bodies> into the prime components unquantifiable and infinitely many [*componenti non quanti ed infiniti*], then we can conceive such components as being expanded into immense space [*in spazio immenso*] without the interposition of any quantified void spaces, but only of infinitely many unquantifiable voids [*vacui infiniti non quanti*]. In this way there would be no contradiction in expanding, for instance, a little globe of gold into a very great space without introducing quantifiable void spaces [*spazii vacui quanti*]—provided, however, that gold is assumed to be composed of infinitely many indivisibles. (EN 8: 71–72/D 33–34)

There is much to say about this passage, but we shall focus on just a few points. With the conception of lines and solids as composed of infinitely many *non quanti* indivisibles in mind—a familiar precursor to contemporary point-set analysis of the continuum—Galileo is observing, correctly, how the metrical properties of collections are not directly determined by those of their elements if the elements are allowed to be both infinitely many and to have, individually, no positive measure. The same infinite collection of *non quanti* points might constitute a line of any finite length, or a globe of any size, depending on how the points are arranged. Or more carefully: any assignment of measure might be consistent with a collection of infinitely many *non quanti* points; there is no contradiction in assigning different sizes to such collections.[18] Galileo's appeal to the presence of *non quanti* voids in the lines or solids to explain the differences in measure for the different arrangements seems questionable. It's not clear why *non quanti* voids should expand things any more than *non quanti* atoms would on their own; the appeal to voids seems to serve as a placeholder for whatever it is that makes the difference in the 'arrangement' of the *non quanti* atoms to yield different measures.[19]

The emphasis in most of the passage is on a metrical concept of *non quanti*: the individual points have no measure, and this allows a consistent assignment of any measure at all to lines or bodies composed of them. Cardinality is something of a background condition: there must be infinitely many such *non quanti* points if they are to constitute lines or solids of finite measure in the first place. A finite number of such points cannot suffice. Galileo expressly argues for this claim in response to a different objection to the idea of composing lines from indivisibles. If a line could consist of only finitely many points, it could consist of an odd number of them; but in that case what we might call 'the bisection principle', that a continuous line can always be divided into two equal parts, would require that the middle indivisible be cut, contrary to hypothesis.[20] Galileo replies:

[18] This runs parallel to the classical contemporary point-set analysis, which allows unions of infinitely many zero-dimensional points (or singletons) to have any positive measure, though with the proviso, on the contemporary account, that the cardinality of the union be uncountable; countably infinite unions of points would still have measure zero. See Skyrms (1983).

[19] Perhaps an expansion by mere rearrangement of *non quanti* atoms would seem to violate conservation principles, whereas the interposition of *non quanti* voids would not, if void is not a conserved quantity, so to speak.

[20] For provocative discussion of the bisection principle and its possible denial, see Benardete (1964, pp. 240 ff).

> In this, and other objections of this kind, satisfaction is given to its partisans by telling them that not only two indivisibles, but ten, or a hundred, or a thousand do not compose a divisible and quantifiable magnitude [*grandezza divisible e quanti*]; yet infinitely many of them may do so. (EN 8:77/D 39)

Nearly all the uses of the idea of the distinction between *quanti* and *non quanti* in those passages is devoted to the metrical concept. Cardinal properties are involved only in a rather indirect way, when the *non quanti* components are allowed to be infinitely many in order to free up the metrical properties of the finite quantities composed of them. There is no indication of Galileo holding that infinite multitudes may not be judged equal in cardinal terms because they are *non quanti*. The indivisible parts, components, atoms and voids are *non quanti*; the status of infinite multitudes as such remains out of the spotlight.

There is a single phrase at the start of the long passage above, from EN 8:71, that would seem to imply that infinite multitudes cannot be *quanti*, when Galileo describes a line as "divided into parts that are quantified and consequently numbered [*consequenza numerate*]." Taken at face value, this says that being *quanti* entails being numbered, which could well mean having a finite cardinality, rather than, say, just being 'reckoned' into measurable units. If being *quanti* directly implies a finite cardinality in this way, then infinite multitudes will trivially be *non quanti*, and counting an infinite multitude as a 'quantity' would be a 'contradiction in terms', as Knobloch puts it. It is unclear how much weight to assign to this line and to this potential reading of it, but interpreters following Knobloch's lead will want to fasten onto it as evidence that infinite multitudes, just in virtue of cardinality, are automatically *non quanti* for Galileo.

The distinction between *quanti* and *non quanti* comes back to the fore most clearly a little later when Galileo returns to the puzzle of the concentric circles and his solution to it based on the analysis of rotating polygons. Again we can sidestep the details of the analysis and focus on what Galileo says about the limit case of circles:

> If we were to apply similar reasoning to the case of circles, we should have to say that where the sides of any polygon are contained within some number, the sides of any circle are infinitely many: the former are quantified [*quanti*] and divisible, the latter unquantifiable [*non quanti*] and indivisible. (EN 8: 95/D 56)

Although it is clear that the sides of the polygon are finite, *quanti* and divisible, whereas those of the circle are infinite, *non quanti* and indivisible, it is not obvious that *quanti* and *non quanti* refer to more than just the metrical properties of the sides, as one of three distinct categories of properties, roughly: cardinality, measure, and divisibility. And that is, in fact, how I am inclined to read the passage.

For the final reason that leads me to interpret *non quanti* in Galileo as only an intrinsic metrical characterization of indivisibles, and not applying to infinite multitudes just in virtue of their being cardinally infinite, consider a key element of the demonstration of Theorem 1, Proposition 1, in *Two New Sciences*. This proposition is the mean-value theorem for free fall, i.e., the law of falling bodies which says that the time required for an object traveling with uniformly accelerated motion across a given distance is the same as that which would be required for an object traveling

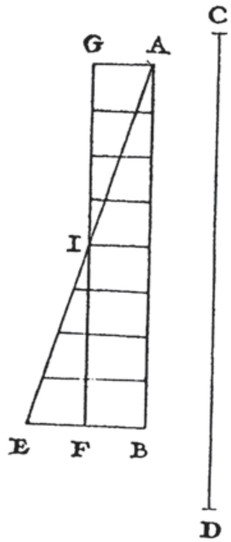

Fig. 2 Diagram for proof of the mean velocity theorem for falling bodies. (EN 8:208)

with uniform motion of half the maximum and final degree of speed of the first. Galileo's proof employs a version of the method of indivisibles, taking aggregates of 'all parallels'—thus *infinite* aggregates of parallels—cutting across the triangles and quadrilaterals that contain them. (See Fig. 2.)

Points on the line AB are taken to represent instants of time, and the parallels drawn from those points across to AIE or GF in the figures in Galileo's diagram represent degrees of speed, either increasing or 'equable.' The parallels do the work of establishing a one-one correspondence between the instants of time and the degrees of increasing speed, as well as a one-one correspondence between the instants and the degrees of equable speed. On the strength of those correspondences, Galileo is able to conclude that there are "just as many degrees of speed not increased but equable", and "there are just as many momenta of speed consumed in the accelerated motion as in the equable motion" (EN 8:208–209/D 165–166). 'Just as many' here is the Latin *totidem*. It is, again, a paradigmatically cardinal treatment of comparison of the aggregates of parallels, despite their being infinitely many. Knobloch is quite right to observe that Galileo does not treat the parallels as having a *sum*; Galileo's 'aggregate' does not indicate that the indivisible points or parallels can be added together.[21] Despite this, the aggregates of points and parallels have cardinal properties: one-one correspondence implies 'just as many,' despite the fact that the points and parallels are infinite in multitude.

In my view, Galileo's use of one-one correspondences between infinite multitudes to justify cardinal claims of equality—the non-numerical but precise claim that there are just as many Xs as Ys—is the strongest evidence that Galileo does not regard infinite multitudes *per se* as falling under the rubric of *non quanti*, which

[21] Knobloch (1999, p. 93; 2011).

seems to be primarily a metrical concept rather than a cardinal one. There is a risk in this argument, of course, since Galileo's claims of cardinal equality between infinite multitudes also appear to run directly contrary to the very philosophical pronouncement that is driving our present inquiry, namely, that "one infinity cannot be said to be greater or less than or equal to another" (NE 8: 78/D40).[22] It may thus 'prove too much' to appeal to Galileo's mathematical practice, when evaluating his considered philosophical position. The practice in some places appears to be patently at odds with the philosophy, even apart from the question of *non quanti*; perhaps it is incautious to draw a philosophical conclusion from it.

In fact, however, the cases are not truly parallel. The clash between the proof of *Thm*. 1, *Prop*. 1, described on the Third Day and Salviati's denial of comparability among infinites during the First Day's discussion of paradoxes at least takes place across relatively distant points within *Two New Sciences*. And of course the 'mathematical demonstrations' are attributed to 'the Author' of the Latin treatise *On Local Motion*, i.e. Galileo, and not necessarily endorsed in every respect by the more philosophically drawn Salviati. As we saw above, however, Galileo's (Salviati's) assertions of cardinal equality among infinite multitudes of indivisibles—between the sides and the voids—occur even in the same breath as his careful efforts to distinguish *quanti* from *non quanti*. If infinite multitudes are supposed to be cardinally incomparable on grounds of being *non quanti*, Galileo's analyses of the rolling circles and polygons in those texts are then grossly mistaken on their own terms. This strikes me as a needlessly damaging interpretation. A more natural and less destructive reading is to take *non quanti* as a metrical concept that is not meant to cover just any case of infinite multitude, and further to take the denial by Salviati, later in the First Day, of comparability among infinites to be based on a somewhat different mix of considerations, which we shall consider shortly below.

To sum up, Galileo's distinction between *quanti* and *non quanti* appears to be metrical rather than cardinal in character. While it marks out a crucial difference between the intrinsic properties of divisible parts and those of indivisible ones—only the former are *quanti* and suitable for 'measure'—it does not by itself rule out judgments of cardinal equality among infinite multitudes.

Before pressing ahead with this result, it is worth noting that we should not be too quick to pull apart the metrical and cardinal concepts of quantity in Galileo, as if they were wholly severable in his thought.[23] The idea of cardinality as something determined by one-one maps—by *functions* on *sets*—would not be properly

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[22] Another possibility is that Galileo's denial that 'greater', 'less' and 'equal' apply in the infinite is carefully consistent with his judgments that there as just as many Xs as Ys in some cases of infinite multitudes: perhaps the Xs and the Ys can be *just as many* without falling under the term 'equal'. If so, however, it would seem to be only a matter of a word, as no richer notion of cardinal equality seems available for which 'just as many' is not sufficient. A more substantial possibility here would be that for Galileo, 'greater', 'less' and 'equal' are essentially metrical notions, and their cardinal counterparts 'more', 'fewer' and 'equally many' cannot be applied on the basis of one-one maps without corresponding geometrical judgments in place as well. I am more sympathetic to this idea but cannot pursue it here; a few related points are discussed below.

[23] I am indebted here to Katherine Dunlop.

distilled until well into the eighteenth century or even later. Galileo's handling of one-one relations between elements of multitudes, especially infinite multitudes, is typically mediated by geometrical relations between the mathematical objects that contain those elements. In the case of Aristotle's wheel, for instance, the one-one correspondences between sides of the rotating polygons (including the circles) and parts and voids in the lines are established under the aegis of geometrical relations between sides of the polygons and the parts of the lines they touch. For instance, the judgment that the line passed over by the smaller circle contains as many voids as there are sides on the circle is based on the following consideration. For every length of the line BF passed over by a side of the larger circle, an equal segment of the equal line CE must be passed over by a side of the smaller circle. (See, again, Fig. 1.) Since each side of the smaller circle is shorter than the segment of the line CE that it passes over, the side touches only some part of that segment; thus there must a be void interval remaining in that segment which the circle 'skips over' in its passage. (Why must each side of the smaller circle be shorter than those of the larger circle? Because their ratio is supposed to be preserved when we shift from the case of finite polygons to the case of infinilateral circles.) The one-one correspondences among the elements of the figures—sides, parts, voids—are thus fixed within a pattern of systematic geometrical relations between the figures themselves.

Nowadays measurements of cardinality require that the two objects compared are *sets*, and this remains a natural counterpart (and perhaps remnant) of the earlier thought that the two measured objects are wholes. The measurements themselves are effected by functions, which need not require any rule or procedure by which to relate the elements of the two sets. In the early modern context, however, the idea of a completely *arbitrary* relation between the elements—a purely 'combinatorial' concept of function—would have been rather alien. Thus measurement of cardinality still needed to be supplemented by other considerations, ones in which metrical concepts often had important roles to play. When we find, as we shall below, Galileo limiting judgments of cardinal equality among infinite multitudes to cases with specific metrical constraints—namely, that the objects be *bounded* quantities—there should be no surprise. This arises organically from the way in which geometrical concepts remain coeval with more purely arithmetical ones in his thought.

1.5 Euclid's Axiom Revisited: Infinity, Magnitude and Infinite Number in Galileo

If our conclusions in the last section are right, then when Galileo denies comparability among infinite classes of numbers, no simple appeal to the distinction between *quanti* and *non quanti* will offer a route of escape from overturning the Bijection Principle. Infinite multitudes are not automatically *non quanti*, at least with respect to cardinal comparisons, and therefore they are not out of play as potential counterexamples to the Bijection Principle. Likewise, Knobloch's original proposal that Euclid's Axiom is not invalidated on Galileo's account because infinites are *non* *quanti* seems not to be fully sustained either. If the various infinite number classes are *non quanti* and thereby outside the scope of Euclid's Axiom, that remains to be shown; it does not follow merely from their being infinite in multitude.

Yet Knobloch's defense of Galileo in the case of Euclid's Axiom strikes very close to the truth. The most natural concept of size for the idea that the whole is greater than the part is a metrical one rather than a cardinal one. So an extension of Galileo's distinction between *quanti* and *non quanti* parts of lines or solids to multitudes of those parts could comfortably rule that only multitudes with a finite total measure can count as *quanti* and thus qualify for comparison. For whereas the concept of infinite multitude might admit of precise mathematical handling in terms of maps, classes, etc., the concept of infinite measure—infinite *magnitude*, as we might say—is less amenable to mathematical analysis and would arouse more skepticism. And Galileo himself seems to confine mathematical analysis to objects that are in some way limited in magnitude. As Knobloch notes, Galileo's term for this is *terminata* or 'bounded.'[24] When considering a line or a circle or a solid as composed of infinitely many indivisibles, it qualifies as a quantity fit for mathematical treatment only if it is itself a bounded magnitude. Galileo remarks on the "infinite difference and even repugnance and contrariety of nature in a bounded quantity in passing over to the infinite" (EN 8:83/D 46):

> Consider, then, what a difference there is<in moving from>a finite to an infinite circle. The latter changes its being so completely as to lose its existence and its possibility of being<a circle>. For we understand well that there cannot be an infinite circle, from which it follows as a consequence that still less can a sphere be infinite; nor can any other solid or surface having shape be infinite. What shall we say about this metamorphosis in passing from finite to infinite? (EN 8:85/D 47)

In fact Galileo does not say exactly what his answer to this question is, beyond the idea that the nature of the objects in question changes or is lost entirely in passing to the infinite, and, in his earlier words of admonition, "These are among the marvels that surpass the bounds of our imagination and that must warn us how gravely one errs in trying to reason about infinites by using the same attributes that we apply to finites; for the natures of these have no necessary relation between them" (EN 8:83/D 46). But the warning is clearly about passing from the intelligible to the unintelligible, and as Knobloch keenly observes, Galileo's language and argumentation clearly evoke that of Nicholas of Cusa, who originated the distinction between (in the Latin) *quanta* and *non quanta* and imbued the whole topic with almost mystical significance.[25] If Galileo does not quite say that the infinite circle and sphere are *non quanti*, the allusions to Nicholas may in effect do this for him.

With that in mind, a second look at the lesson Galileo draws from the paradox of the natural numbers readily finds the same concern to limit comparisons to bounded quantities: "And in the final conclusion, the attributes of equal, greater and less have no place in infinite, but only in bounded quantity [*quantità terminate*]" (EN 8:79/D

[24] Knobloch (1999, p. 92). See also Knobloch (2011).

[25] See Nicholas ([*1440*] 1985), *De Docta Ignorantia*, Book 1, Chaps. 11–23; see especially Chap. 14 for use of the distinction between *quanta* and *non quanta*.

41). This is revealing, for it shows that Galileo sees the crux of the problem to be the unboundedness of the multitude of natural numbers. His contrast between 'infinite' and 'bounded' strongly suggests that he is thinking of magnitude rather than multitude. 'Bounded' of course has cardinal as well as metrical senses, but it would be redundant here to point out that the infinite multitude of natural numbers is cardinally unbounded. By contrast, pointing out that the natural numbers taken together are infinite and unbounded in magnitude distinguishes them in a special way, for an infinite multitude is not always unbounded in magnitude. Even on Galileo's view, infinitely many points may compose a bounded quantity such as a finite globe. Because the individual points themselves are *non quanti*, taking infinitely many of them in aggregate need not total up to an unbounded quantity; they can consistently be taken to compose a bounded quantity of finite magnitude such as a little globe of gold. The same could not be true of *quanti* parts, unless they happen to form a convergent infinite series (e.g., 1/2 + 1/4 + 1/8 + 1/16, &c., whose sum is equal to 1). Apart from this exceptional case (which was much disputed at the time and is not mentioned by Galileo in connection with the topic of *quanti* and *non quanti*), any infinite collection of finite quantities will surpass any finite size and thus be unbounded.

When Galileo holds the multitude of natural numbers to be infinite and unbounded, it seems he is regarding each natural number as if it were a 'quantifiable' part of all the naturals. This is not so strange an idea, and it shows that the concept of number at work includes metrical as well as cardinal elements. Taking the natural numbers to be multiples of the number one as the basic unit of 'arithmetical measure', each natural itself is a finite quantity. If all the naturals are taken together there is no way for the total to be bounded, i.e. less than or equal to some finite magnitude, i.e. the magnitude of some finite number. Their aggregate would instead appear to be an infinite magnitude, as if a colossal infinite number made up of all the finite numbers or infinitely many copies of the unit number one.

If this is how the infinity of natural numbers is understood by Galileo, then there is a case to be made that it too qualifies as 'non-quantifiable', like the infinite globe or infinite circle, not in virtue of its smallness but in virtue of its greatness. But it is metrical greatness—greatness of magnitude—rather than cardinal greatness that yields the verdict of *non quanti*. For there is no cardinal difference between the infinity of natural numbers and the infinity of indivisibles in a little globe of gold, or the infinity of sides of a circle. There is textual evidence that Galileo thinks of the idea that the natural numbers are susceptible to comparative measurement as representing the natural numbers as components in a single colossal number with infinitely many finite parts. And it comes directly in his discussion of the paradox of the natural numbers. Having established the one-one correspondences among squares, roots and naturals, Galileo notes:

> That being the case, it must be said that square numbers are as numerous as all numbers, because they are as many as their roots, and all numbers are roots. Yet at the outset we said that all the numbers were many more than the squares, the greater part [*maggior parte*] being non-squares. Indeed, the multitude of squares diminishes in ever-greater ratio as one moves on to greater numbers, for up to one hundred there are ten squares, which is to say one-tenth part [*parte*] are squares; in ten thousand, only one-hundredth part [*parte*] are squares; in one million, only one-thousandth. (EN 8:78–79/D 40–41)

The farther the series of numbers is finitely extended, the smaller the share of squares becomes, so it seems especially peculiar how in the infinite case a transition occurs to make the squares and naturals equal. Now notice exactly how Galileo phrases his point in the line that follows:

> Yet in the infinite number [*numero infinito*], if one can conceive that, it must be said that there are as many squares as all numbers together. (Ibid.)

The comparison between the squares and the naturals that shows them to be equal takes place ‘in the infinite number’, of which, it seems, the squares form a share or part. So it appears that comparison of infinite multitudes of numbers, at least, presupposes a single number after all—not in order to stand as the absolute cardinality of the multitudes being compared, but to be a common object of which each multitude is some proportional part and can be compared in ratio to the other. The common object in this case is a number because it is composed of numbers, the infinite aggregate of all numbers. Perhaps if comparisons were being made among infinite multitudes of other types, the need for a common object would not automatically entail the existence of an infinite number but of something else, say, the aggregate of all stars in the heavens, or, to use a case straight from Galileo’s texts, the aggregate of all parallels in a given figure or points in a line or sides on a circle. Galileo is not explicit about the details. But it is clear enough in the case of the natural numbers that comparisons among its infinite subclasses are conceived by Galileo to involve an infinite number as the common object containing the compared classes as parts.

The same strand of thought is carried through when Galileo returns to the paradox later in the dialogue. It appears this time when Salviati introduces the revisionary suggestion that infinity be regarded, if anything, not as a number standing at the far end of the natural numbers but instead as the number one:

> In our discussion a little while ago, we concluded that in the infinite number [numero infinito], there must be as many squares or cubes as all the numbers because both<squares and cubes>are as many as [tante…quanti] their roots, and all numbers are roots. Next we saw that the larger the numbers taken, the scarcer became the squares to be found among them, and still rarer, the cubes. Hence it is manifest that to the extent that we go to greater numbers, by that much and more do we depart from the infinite number. From this it follows that turning back (since our direction took us always farther from our desired goal), if any number may be called infinite, it is unity. And truly, in unity are those conditions and necessary requisites of the infinite number. I refer to those<conditions>of containing in itself as many squares as cubes, and as many as all the numbers<contained>. (EN 8: 82–83/D 45)

Leave aside the new suggestion that the infinite number is unity or one. What matters for us is Galileo’s consistent appeal to the idea of an infinite number in his handling of comparisons between the squares, cubes, roots and naturals. For these classes of numbers to be compared to one another, it seems as if they must be able to stand in ratio to one another and to form parts or shares of some single common whole, the infinite number formed out of all the natural numbers taken together (at least prior to Salviati’s revisionary identification of the infinite number with unity). Such a number would indeed be infinite in magnitude, unbounded, and comfortably regarded as ‘non-quantifiable’ on account of its greatness.

It is instructive to note that in the instances in which Galileo relies on comparisons of equality among infinite multitudes in his avowed mathematical arguments, those multitudes are confined to bounded spaces of finite magnitude. His proof of Thm. 1, Prop. 1 in *On Local Motion*, for example, trades on the cardinal equality of degrees of speed and instants in time; but these 'infinites' are represented as aggregates of points or parallels within bounded geometrical figures. Unbounded magnitude is in effect ruled out from the start, and with that safe harbor established it seems Galileo is prepared to make use of cardinal comparison, or at least cardinal equality, among infinites. 'Safe harbor' perhaps suggests too much, for we have not seen whether it is a *sufficient* condition for infinite collections to be cardinally comparable that they be bounded in magnitude. The present suggestion is only that Galileo seems to treat it as a *necessary* condition, one which the aggregate of parallels in a bounded figure satisfies but the collection of all natural numbers does not. What else, precisely, is required for a sufficient condition for cardinal comparability is not immediately clear; though as suggested before, perhaps some geometrical relation or finitary rule or procedure would be expected in order to establish the one-one mapping among the elements.

The result of all this would appear to be that, given Galileo's reliance on the idea of an infinite number as a common whole in framing comparisons involving all the natural numbers taken together, his distinction between *quanti* and *non quanti* can be extended to count the infinity of natural numbers as 'non-quantified'—not because they are infinite in multitude but because the infinite number they compose would be unbounded in magnitude and thus unfit for comparison. This then also preempts the natural numbers, and any similar infinite class of numbers, from constituting a potential counterexample to Euclid's Axiom. Such infinites, by virtue of their magnitude, are *non quanti* and unfit for comparison, and thus do not fall within the scope of Euclid's Axiom in the first place. So with respect to the specific paradox of the natural numbers, Knobloch's diagnosis appears to be correct.

Yet not all infinite multitudes will likewise amount to infinite magnitudes when taken all together, and those which do not, such as infinite multitudes of indivisibles in bounded lines or solids, may still be open for comparison in cardinal terms by means of one-one maps. There are as many sides in the circle as indivisible parts in the line it traverses. The difficulty now reappears just as it emerged in Simplicio's original objection. If finite and bounded quantities consist of infinitely many indivisible points—their *non quanti* parts—examples can easily be found in which the whole is equal to the part, by use of one-one correspondences and the Bijection Principle. Parallels can be dropped from all the points on the diagonal of a square to all those on one of its sides, and yet by rotating that side back up to the diagonal, the points on the side can be put into one-one correspondence with those of only a part of the diagonal. Thus all the points on the diagonal can be put into one-one correspondence with the subclass of those points composing the part of it congruent with the rotated side.[26] Again Euclid's Axiom and the Bijection Principle conflict: the diagonal is both greater than and equal to its part.

[26] This so-called Diagonal Paradox was well-known by the seventeenth century; see Leibniz's use of it at A VI, 3, 199. As noted by Lison (2006/2007, p. 199fn3), the example goes back at least to

If Galileo's denial of comparability is limited to cases in which infinite multitudes constitute aggregates of infinite magnitude, there is no escape from contradiction in the finite bounded case. But if his denial is taken to apply across the board even to those infinite multitudes that compose only bounded magnitudes, then it seems either Euclid's Axiom or the Bijection Principle (or both) must be a casualty of his analysis after all.

Similarly, a limited denial of comparability of infinites would allow Galileo's use of the Bijection Principle in his mathematical practice to remain in harmony with his philosophical pronouncements, since he appears to confine that use to infinite multitudes housed within bounded quantities. But an across-the-board denial of comparability among infinites will yield a clash with his mathematical practice of applying the Bijection Principle in reasoning about infinite classes of indivisibles in the bounded cases.

The lesson of our inquiry is that the distinction between *quanti* and *non quanti* will not allow Galileo a fully reconciled position that avoids contravening at least some major mathematical principle of comparison. If all infinite multitudes are *non quanti*, then Galileo's ready use of cardinal comparison among infinite multitudes in bounded cases is, as we have seen, grossly at odds with his philosophy on this point, even in the very texts in which he articulates his notion of *non quanti*. On the other hand, if infinite multitudes are *non quanti* only when they constitute unbounded magnitudes—as I have been suggesting—then paradox reemerges in cases in which infinite multitudes compose only finite and bounded magnitudes, and Galileo has no choice but to abandon either Euclid's Axiom or the Bijection Principle.

Unless, that is, there is altogether another way for him to solve paradox.

1.6 Leibniz's Alternative

In notes written in 1672 on *Two New Sciences*, Leibniz reviews Galileo's paradox of the natural numbers:

> He [Galileo] thinks that one infinity is… not greater than another infinity… And the demonstration is worth noting: Among the numbers there are infinite roots, infinite squares, infinite cubes. Moreover, there are as many roots as numbers. And there are as many squares as roots. Therefore there are as many squares as numbers, that is to say, there are as many square numbers as there are numbers in general [*in universum*]. Which is impossible. (A VI, 3, 158)

Leibniz sees that the equality of the squares with the natural numbers will violate Euclid's Axiom and that one option is simply to reject the axiom in the case of the

Ockham. Note also that similar examples can be constructed even with multitudes containing only *quanti* parts, provided their total measure remains finitely bounded. Considering a line segment as composed of a sequence of geometrically decreasing non-overlapping subsegments, one can easily construct a one-one correspondence between the subsegments of the side of a square and the subsegments of the diagonal, even though the side can be shown by rotation to be equal to a part of the diagonal.

infinite, which he takes to be Galileo's solution. Yet, as we observed once before, he regards this as unacceptable, an admission that the axiom itself has counterexamples, and he appears to see another way out of the paradox:

> Hence it follows either that in the infinite the whole is not greater than the part, which is the opinion of Galileo and Gregory of St. Vincent, and which I cannot accept; or that infinity itself is nothing, i.e. that it is not one [*Unum*] and not a whole [*totum*]. (Ibid.)

What is Leibniz's alternative to (what he takes to be) Galileo's answer? First let us understand the elements of his reply.[27] When Leibniz denies that infinity is one or a whole he is not saying that there is no such thing as infinity, but rather he is denying that an infinity of things forms a unity or single whole. It is not the number one (contrary to Salviati's revisionary suggestion), nor is it to be understood as a single set with infinitely many elements. Leibniz's position here is subtle. There are actually infinitely many natural numbers, on his view, but they do not form a totality. There is only a plurality or multitude of them, but no one thing, no single object, to which they all belong as constituents. There is no set of all natural numbers, so to speak, but only a proper class. When Leibniz says "infinity itself is nothing" he is not denying that there are infinitely many numbers; he means there is nothing over and above the natural numbers themselves. To call them 'infinite' is not to posit a special entity—a super-number or super-whole embracing all the naturals and larger than every finite number. It is, rather, to describe the multitude of natural numbers in a particular way. Exactly what he means by 'infinite' we shall discuss shortly below.

It is fairly straightforward to see how this can solve the paradox. By denying that an infinite multitude can form a whole, Euclid's Axiom is taken out of play, and along with it the consequence that there must be more naturals than squares. The infinite multitude of natural numbers is not a whole of which the square numbers form a part, so the infinity of natural numbers does not have to be said to be greater than the infinity of the squares. This removes the contradiction that the naturals are both greater than and equal to the squares. Note that it also does so without contravening Euclid's Axiom. If there are no infinite wholes, then there is no whole that can fail to be greater than the part, and so no counterexample to the axiom.

It is less clear whether infinite multitudes can or cannot be compared to one another at all on Leibniz's view. Unlike the earlier strategy of looking to classify infinites as *non quanti* and thereby preclude them from comparison altogether, denying that infinites form wholes only takes them out of the specific jurisdiction of Euclid's Axiom. It remains open, at least, to compare them using one-one maps and the Bijection Principle. What comparisons, if any, does Leibniz allow under this principle? We will not resolve the issue completely in the present essay, but a few points may be instructive nonetheless.

I think the matter turns on what is involved in the idea of there being 'as many' elements in one class as in another. As before, one thought is that in order for there to be as many Fs as Gs, the must be a *number* that records how many Fs there are,

[27] For related discussion, see Levey (1998) and Arthur's introduction to Leibniz (2001).

and likewise a number to record the cardinality of the Gs. If those two numbers are equal, then there are as many Fs as Gs, and not otherwise. For reasons we shall not pursue, however, Leibniz holds that there is no such thing as an infinite number, no number that could say of an infinite class just how many elements there are in it.[28] To call a class infinite is not to assign it a number, on his view. If infinite classes are without number, and if there are as many Fs as Gs only if they are the same in number, then it cannot be said that there are as many Fs as Gs when they are infinite. There may well be one-one correspondences between the Fs and the Gs, but this will not automatically yield the conclusion that they are equal in any sense beyond the idea that both classes are infinite.

Leibniz does not, to my knowledge, explicitly say that equality of class size—in the sense of 'as many Fs as Gs'—requires equality of number. But it is clear that he thinks the question of equality of number is relevant to the analysis of Galileo's paradox. For in a subsequent discussion of the paradox, in the dialogue *Pacidius to Philalethes* (1676), he systematically frames the issue not in terms of there being *as many* squares as numbers, as he had in the earlier notes on *Two New Sciences*, but in terms of 'the number of all squares' and 'the number of all numbers' (A VI, 3, 550). The interlocutors consider Galileo's response to the paradox and reject it:

> CHARINUS: Please allow me to hear first from Gallutius what Galileo said.
> GALLUTIUS: He said: the appellations 'greater', 'equal' and 'less' have no place in the infinite.
> CHARINUS: It is difficult to agree with this. For who would deny that the number of square numbers is contained in the number of all numbers, when squares are found among all numbers? But to be contained in something is certainly to be a part of it, and I believe it to be no less true in the infinite than in the finite that the part is less than the whole.
> (A VI, 3, 551/Ar 178–179)

The commitment to Euclid's Axiom is clear, as is the deployment of the language of parts and wholes in describing the relation between the squares and the natural numbers. In fact the view here seems to integrate 'number', 'whole', 'part' and 'containment' quite fully. Not only are the square numbers among all the numbers, but also the *number* of squares is a *part* of the number of all numbers; the latter number is the whole that contains the part. Leibniz's strategy for solving the paradox by denying that the natural numbers forms a whole is thus expressed in subsequent lines by Charinus through the claim, "there is no number of all numbers at all, and that such a notion implies a contradiction" (ibid.). The denial that there is such a number is the denial that the natural numbers form a whole. Leibniz's principal spokesman, Pacidius, then praises Charinus's answer as "very clear, and if I am any judge, true",

[28] Leibniz has a few different lines of argument to offer against infinite number, including, notably for us, a deployment of Galileo's paradox that expressly says infinite numbers are "impossible" because "it is impossible that this axiom"—Euclid's Axiom—"fails" (A III, 1, 11). (This passage comes from some remarks by Leibniz on Galileo's paradox, written in 1673.) Here an infinite number is presumably understood as a whole constituted of infinitely many units or the combination of all natural numbers, themselves taken as wholes composed of units. For some related discussion, see Levey (1998).

and adds, “for it is necessary that what has contradictory consequences is by all means impossible” (ibid.)

Again, Galileo does not quite say directly that the natural numbers form a whole; Leibniz is reading this into Galileo’s position. But as we saw, Galileo’s account of what it would be for the natural numbers to be comparable to the squares or cubes, etc., seems to involve the existence of an infinite number, made up of all the numbers, as the common object of which the squares form a part. So while Galileo does not say that the (cardinal) equality of two infinite multitudes must be equality in number, in the sense that there is some single number that says precisely how many elements there are in each multitude, nonetheless for the case of comparison of infinite number classes his analysis does seem to require the existence of an infinite number in which those classes are contained. It is not evident that Leibniz has discerned all this in Galileo—he does not describe Galileo’s account in enough detail to tell—but when he attributes to Galileo the idea that the natural numbers form a whole, it seems he has hit upon the truth.

Leibniz’s solution to the paradox does not require him to reject Euclid’s Law. And it seems also to leave the Bijection Principle intact, if only because the principle is silent about the distinction between whole and multitudes, and silent also about the concept of number, and so is not called into question by Leibniz’s denial of infinite wholes and infinite numbers. It remains to ask whether Leibniz himself will endorse the Bijection Principle or any related form of comparability of infinite classes using one-one maps; we turn to that in the next section. Before doing so, however, it should be noted that Leibniz’s solution does not completely disarm Galileo’s paradox .

Just as the appeal to the distinction between *quanti* and *non quanti* left a version of the paradox untouched when reconfigured from an unbounded infinite case to a bounded case—for instance, the Diagonal Paradox—so too there remains a problem for Leibniz’s strategy. For in order to prevent Euclid’s Axiom and the Bijection Principle from coming into conflict by appeal to the distinction between wholes and pluralities, he will have to deny that *any* infinite multitude forms a whole. This goes for bounded and unbounded infinite multitudes alike. In mathematical examples of the unbounded case, this means that the classes of natural numbers, of squares, etc., are not wholes. In a natural-world example, this means that the infinite universe itself does not form a whole—a conclusion Leibniz expressly draws on the basis of his analysis of Galileo’s paradox.[29] But both mathematics and the natural world also provide bounded cases of infinite multitudes: the Diagonal Paradox for finite, bounded geometrical figures; and any given finite body in nature, since according to Leibniz every body is actually infinitely divided into parts.

[29] Leibniz writes: “*God is not the soul of the world* can be demonstrated; for the world is either finite or infinite. If the world is finite, certainly God, who is infinite, cannot be said to be the soul of the World. If the world is supposed to be infinite, it is not one Being or one body *per se* (just as it has elsewhere been demonstrated that infinite in number and in magnitude is neither one nor a whole, but infinite in perfection is one and a whole). Thus no soul of this sort can be understood. An infinite world, of course, is no more one [Being] and a whole than an infinite number, which Galileo has demonstrated to be neither one nor a whole” (Leibniz 1948, p. 558).

In the case of geometrical figures, it is easy enough for Leibniz to accept the verdict that they are not truly wholes. After all, they are, on his view, only *entia rationis* and not real beings. The same conclusion is harder to accept in the case of bodies, for it will quickly preclude any body from truly being a single whole, calling into question its reality and indeed the reality of the entire corporeal world. Certain interpretations of Leibniz might welcome this result, while others would find it an unhappy fit with his views at least for important parts of his career.[30] Leaving aside the implication for 'idealist' and 'realist' interpretations of Leibniz, however, it should be noted that Leibniz shows no sign of intending his analysis of Galileo's paradox to issue in a denial of the reality of finite corporeal beings. Since Galileo, like Leibniz, regards bodies as infinitely divided into parts, if he were to take Leibniz's path of escape from overturning Euclid's Axiom and the Bijection Principle, he too would face this discomforting consequence for the natural world. Thus although Leibniz's distinction between wholes and multitudes might provide a way to resolve the contradiction in mathematics alone, the paradox continues to hold some hostages in metaphysics.

1.7 Comparability and the Definition of Infinite in Leibniz and Galileo

There are a few last pieces of Leibniz's view that still need to be articulated, concerning comparability, number and infinity. Leibniz rejects both infinite totalities and infinite numbers. Does he also reject the idea that infinite classes can be comparable? The squares and the naturals cannot literally be equal *in number*, and they cannot be equal parts of a single common object, the infinite number. But might they nonetheless be equal in the general sense of being 'just as many', in virtue of there being a one-one correspondence between them? That is, does Leibniz endorse the Bijection Principle even for infinite classes?

In at least one clear statement, first noted by Russell, Leibniz denies the claim of equality for infinite classes despite the existence of a one-one correspondence between them:

> There is an actual infinite in the mode of a distributive whole, not of a collective whole. Thus something can be enunciated concerning all numbers, but not collectively. So it can be said that to every even<number>corresponds its odd<number>, and vice versa; but it cannot be accurately said that the multitudes of odds and evens are equal. (GP II, 315)

The link between speaking of 'all numbers' and taking them as a 'collective whole' is explicit, and it seems that equality of the odds and the evens has to be denied precisely because such infinite classes do not form collective wholes (presumably 'distributive whole' just means multitude or plurality here). Leibniz's solution to Galileo's paradox protects Euclid's Axiom but in turn leads him to reject the Bijection Principle for the infinite case—*if*, that is, this passage reflects his considered

[30] For an overview of this dispute in the interpretation of Leibniz, see Levey (2011).

position. It should be noted that the passage occurs on a separate slip of paper enclosed in Leibniz's copy of his letter of 1 September 1706 to Bartholomew Des Bosses, and Leibniz has crossed it out.[31] It is hard, therefore, to know how much weight to give it.

Even if it were Leibniz's view that the odds and evens are not equal, it cannot be the entirety of Leibniz's position to deny comparisons involving infinite classes. For he has to preserve at least one crucial claim of comparability for infinite classes, namely, that there are more elements in an infinite class than in any finite one. This is essential to Leibniz's definition of infinite:

> When it is said that there are infinitely many terms, it is not being said that there is some specific number of them, but that there are more than any specific number. (GM III, 566)

This is no fleeting aspect of Leibniz's philosophy of mathematics but his deeply held definition of 'infinite' with ramifications across his thought, and thus he is quite committed to the coherence of this comparison between infinite and finite classes. Interestingly, Galileo, in the character of Sagredo, denies that infinite quantities can be compared even with finite ones, apparently in an echo of Aristotle's prohibition, in *De Caelo* 274a10 and 274b12, against ratios between finite and infinite; cf. EN 8: 79–80/D 42.[32] (Again, though, this may be limited to comparisons of magnitudes and not to multitudes.)

Taken just at face value, and if we include the crossed-out remark on the slip in the letter to Des Bosses, Leibniz may seem to have landed in almost the inverse of Galileo's position: infinite multitudes cannot be said to be equal, but the infinite can be compared to the finite and judged to be (cardinally) greater. The particular example Leibniz offers in denying equality among infinites is one about which Galileo would agree. For the evens and the odds would each constitute an unbounded infinite magnitude if taken all together and thus fail to be *quanti* for Galileo, thereby preempting a judgment of equality.

Unlike Galileo, whose denial of comparability among infinites turns out to be linked to the concept of magnitude—infinite magnitudes are unbounded and thus *non quanti* and incomparable—Leibniz's denial of comparability (or at least equality) among infinites relies only on the distinction between 'collected' wholes or totalities and multitudes, without any obvious reference to the concept of magnitude. So whereas Galileo can allow infinite multitudes of finite and bounded magnitude to be among the *quanti* and hence comparable, it is not clear that Leibniz can make the parallel allowance that those infinite multitudes of finite and bounded magnitude can count as wholes. Given similar opportunities to treat bounded mathematical magnitudes with infinitely many elements as wholes, he appears to take steps not to do so.[33]

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[31] See Leibniz (2007), p. 409.

[32] Drake suggests that Galileo neither fully accepted nor fully rejected Aristotle's principle (1974, 42fn26).

[33] See Levey (1998, 1999, 2003). For defense of the view that Leibniz's strictures against infinite wholes only preclude wholes of infinite magnitude, and thus can allow wholes of finite magnitude that include infinitely elements, see Brown (2005).

Either Leibniz has an unstated subtlety at work here in his restrictions on comparability—one that allows the claim of 'more than' between infinite and finite classes but rules out the claim of equality among the evens and odds—or he is inconsistent in his statements across texts. I am inclined to think the denial of cardinal equality in the crossed-out passage is an outlier and not Leibniz's considered view. As noted earlier, however, this matter will not be entirely resolved here. In any case, for now it is enough to note that insofar as Leibniz requires comparability of an infinite class with a finite one for his definition of 'infinite', infinite multitude is not *eo ipso* a barrier to it.

On the definition of 'infinite' there is a further illuminating, and perhaps ironic, connection between Galileo and Leibniz. In *Two New Sciences*, Galileo has the interlocutors consider the question of how many *quanti* parts a bounded continuum such as a finite line segment can be divided into. Finitely many or infinitely many? Salviati's answer is that there are neither finitely many nor infinitely many, but something intermediate between the two:

> *Salviati*: To the question which asks whether the quantified parts in the bounded continuum are finite or infinitely many [*infinite*], I shall reply exactly the opposite of what Simplicio has replied; that is, 'neither finite nor infinite.'
> *Simplicio*: I could never have said that, not believing that any middle ground is to be found between the finite and the infinite, as if the dichotomy or distinction that makes a thing finite or else infinite were somehow wanting and defective.
> *Salv.*: It seems to me to be so. Speaking of discrete quantity it appears to me that there is a third, or middle, term; it is that of answering to every [*ogni*] designated number. Thus in the present case, if asked whether the quantified parts in the continuum are finite or infinitely many [*infinite*], the most suitable reply is 'neither finite nor infinitely many, but so many as [*ma tante che*] to correspond to every specified number.' To do that, it is necessary that these be not included within a limited number, because then they would not answer to a greater<number>; yet it is not necessary that they be infinitely many, since no specified number is infinite [*infinito*]. And thus at the choice of the questioner we may cut a given line into a hundred quantified parts, into a thousand, and into a hundred thousand, according to whatever number he likes, but not into infinitely many<quantified parts>.
> (EN 8: 81/D 43–44)

Galileo's 'middle term' between finite and infinite is almost exactly Leibniz's official definition of infinite, differing only in whether the parts in the multitude are so many as to *correspond to* any specified number or to be *more than* any specified number. This quickly comes to the same thing, since, as Galileo points out, corresponding to any specified number requires not being included in any limited number; so for any given number *n*, the parts in the multitude must exceed *n* anyway. In his 1672 notes on this passage, Leibniz just describes this as saying that the parts in the continuum are 'indefinite.' No doubt this captures some of Galileo's intention in placing emphasis on the 'choice of the questioner.' But the relation between the variables here—between the number chosen and the comparison of the multitude with that number—is no mere matter of indefiniteness, and it later becomes the key to Leibniz's considered definition.

As is now recognized among commentators, Leibniz advances a *syncategorematic* analysis of the term 'infinite' rather than a *categorematic* one.[34] To say that a class is infinite is not to assign it a single infinite number but to describe it in terms of logical relations among variables referring to finite numbers: the Fs are infinite iff for any n, there are more than n Fs. By contrast, a categorematic analysis of 'infinite' would make reference to an infinite number: the Fs are infinite iff there is a number k of Fs such that k is greater than any finite number n. (The order of quantifiers is crucial of course.) This is not wholly an innovation by Leibniz. The distinction between categorematic and syncategorematic terms goes back at least to Priscian, and the use of this distinction in the analysis of the term 'infinite' predates Galileo and Leibniz by a few centuries.[35] What is notable here for us is that Galileo clearly reveals his own interpretation of 'infinite' to be a categorematic one. When he writes of the multitude of *quanti* parts in a bounded continuum that they are so many as to correspond to any specified number 'yet it is not necessary that they be infinite, since no specified number is infinite,' he is expressly holding that for the multitude to be infinite it must correspond to an infinite number—the unmistakable signature of the categorematic account. What Galileo takes to define an intermediate status between infinite and finite is exactly what Leibniz later appropriates for his own syncategorematic analysis of 'infinite'.

A last point is important to bring out in Leibniz's account. Taking care with his statement of what is meant by saying there are infinitely many terms, very precise definitions of 'finite' and 'infinite' can be formulated. A class X is infinite if and only if for any number n, there are more than n elements of X. And by negation, then, a class X is finite if and only if for some number n, there are *not* more than n elements of X. What does 'more than n elements of X' mean? Here we draw upon Leibniz's subscription to the idea that a number is an aggregate of unities, for instance, that $6=1+1+1+1+1+1$ (cf. A VI, 3, 518). Leibniz is explicit about this when he defines 'integer number':

> An integer number is a whole [totum] collected from unities. (LH XXXV, 1, 9, f. 7r-v)[36]

Each 'integer number' or natural number is then itself a class, indeed a whole, and it is open to compare other classes to it. It is also clear that Leibniz consciously intends for his definition of an infinite multitude to be understood in terms of comparisons with numbers taken as wholes composed of unities; for example, he asserts

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[34] See Ishiguro (1990), Knobloch (1994, 2002), Bassler (1998), Arthur (2008), (2009) and (2013), and Levey (2008). In the present paper it's worth noting that the same crossed-out passage that contains the denial of cardinal equality between the evens and odds, Leibniz begins: "*There is a syncategorematic infinite* or passive power having parts, namely, the possibility of further progress by dividing, multiplying, subtracting, or adding. And *there is a hypercategorematic infinite*, or potestative infinite, an active power having, as it were, parts eminently but not formally or actually. This infinite is God himself. But *there is not a categorematic infinite* or one actually having infinite parts formally" (GP II, 314–315). Translated by Look and Rutherford in Leibniz (2007, p. 53).

[35] William Heytesbury may have been the first to defend a syncategorematic analysis of 'infinite'; see *sophisma* xviii of his *Sophismata*, in Pironet (1994).

[36] Quoted in Grosholz and Yakira (1998, p. 99).

the existence of infinitely many bodies in this way: ‘Bodies are actually infinite, that is, there exist more bodies than there are unities in any given number’ (A VI, 4, 1393/Ar 235).

It then remains only to define ‘more than *n* elements of X’. No appeal to Euclid’s Axiom that the whole is greater than the part is available here to license the claim that an infinite multitude contains more elements than unities in a natural number, since Leibniz rules out infinite wholes. The most natural route at this point is not via Euclid’s Axiom at all but by appeal to the standard of one-one maps in combination with the idea of a natural number as a totality of elements: there are more than *n* elements of X just in case there is no one-one map from X into the natural number *n*. This now allows us to state Leibniz’s definitions of ‘infinite’ in canonical terms: X is infinite iff there is no one-one map from X into any natural number.

Assiduous readers will note that Leibniz’s definition of ‘infinite’ is not the same as the definition suggested in our earlier discussion of Galileo. Whereas Galileo’s definition considers maps of all the natural numbers into classes—a class is infinite just in case there is such a map, and finite otherwise—Leibniz’s definition considers maps from classes into individual natural numbers. Dedekind’s definition takes yet another angle, considering maps from classes into themselves: a class is infinite just in case it can be mapped into a proper part of itself. All these definitions are at least conceptually distinct. Intriguingly, Galileo’s and Dedekind’s definitions turn out to be equivalent, while Leibniz’s definition turns out to be different, and weaker, if only barely so. And it is Leibniz’s definition that is now the standard definition of infinite in mathematics, while Galileo’s and Dedekind’s has come to be regarded as a special case called ‘Dedekind infinite’. But that is a story for another day.[37]

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References

Arthur, Richard. 2008. Leery bedfellows: Leibniz and Newton on the status of infinitesimals. In *Infinitesimals differences: Controversies between Leibniz and his contemporaries*, ed. U. Goldenbaum and D. Jesseph, 7–30. Berlin: Walter de Gruyter.

Arthur, Richard. 2009. Actuals infinitesimals in Leibniz’s early thought. In *The philosophy of the young Leibniz (Studia Leibnitiana*Sonderheft 35), ed. M. Kulstad, M. Laerk, and D. Snyder, 11–28. Stuttgart: Franz Steiner Verlag),

Arthur, Richard. 2013. Leibniz’s syncategorematic infinitesimals. *Archive for History of Exact Sciences* 67:553–593.

[37] See my ‘Leibniz’s Analysis of Galileo’s Paradox,’ ms.

Bassler, O. Bradley. 1998. Leibniz on the indefinite as infinite. *The Review of Metaphysics* 15: 849–874.
Benardete, José. 1964. *IInfinity: An essay in metaphysics*. Oxford: Clarendon Press.
Bolzano, Bernard. 1851/1950. *Paradoxes of the infinite* (translated by F. Prihonsky). London: Routledge.
Brown, Gregory. 2005. Leibniz's mathematical argument against a Soul of the World. *British Journal for the History of Philosophy* 13:449–488.
Cantor, Georg. 1915. *Contributions to the founding of the theory of transfinite numbers* (Translated and edited by Philip Jourdain). New York: Dover.
Costabel, Pierre. 1964. La roue d'Aristote et les critiques françaises à l'argument de Galilée. *Revue d'Histoire des Sciences* 17:385–396.
Drabkin, I. E. 1950. Aristotle's wheel: Notes on the history of a paradox. *Osiris* 9:126–198.
Dummett, Michael. 1994. What is mathematics about? In *Mathematics and mind*, ed. Alexander George, 11–26. Oxford: Oxford University Press.
Galilei, Galileo. 1898. *Opere,* Vols. 1–8. ed. Antonio Favaro. Florence: Edizione Nazionale.
Galilei, Galileo. 1974. *Two new sciences, including centers of gravity and force of percussion,* 2nd edn. (Translated and edited with commentary by Stillman Drake). Toronto: Wall and Emerson, Inc.
Grosholz, Emily, and Elhanan Yakira. 1998. *Leibniz's science of the rational*. *Studia leibnitiana*—Sonderhefte 26. Stuttgart: Franz Steiner Verlag.
Ishiguro, Hidé. 1990. *Leibniz's philosophy of logic and language,* 2nd edn. Cambridge: Cambridge University Press.
Knobloch, Eberhard. 1994. The infinite in Leibniz's mathematics: The historiographical method of comprehension in context. In *Trends in the Historiography of Science,* ed. Kostas Gavroglu et al., 265–278. Dordrecht: Kluwer.
Knobloch, Eberhard. 1999. Galileo and Leibniz: Different approaches to infinity. *Archive for History of Exact Sciences* 54:87–99.
Knobloch, Eberhard. 2002. Leibniz's rigorous foundation of infinitesimal geometry by means of riemannian sums. *Synthese* 133 (1–2): 59–73.
Knobloch, Eberhard. 2011. Galileo and German thinkers: Leibniz. In *Galileo e la scuola galileiana nelle Università del Seicento*, ed. L. Pepe, 127–139. Bologna: Cooperativa Libraria Universitaria Bologna.
Leibniz, G.W. 1849–1863. *Mathematische Schriften von Gottfried Wilhelm Leibniz*. Vols. 1–7 (Ed. C.I. Gerhardt). Berlin: A. Asher; Halle: H.W. Schmidt.
Leibniz, G.W. 1875–1890. *Die Philosophischen Schriften*. Vols. 1–7 (Ed. C.I. Gerhardt). Berlin: Weidmannsche Buchhandlung.
Leibniz, G.W. 1923–1999. *Samtliche Schriften un Briefe. Philosophische Schriften*. Series VI. Vols. 1–4. Berlin: Akademie-Verlag.
Leibniz, G.W. 1948. *Texts inédits d'apres les manuscits de la bibliothéque provinciale de Hanover*. Vols. 1–2. Edited by Gaston Grua. Paris: P.U.F.
Leibniz, G.W. 2001. *Labyrinth of the continuum: writings on the continuum problem, 1672–1686*. Translated and edited with commentary by Richard T. W. Arthur. New Haven: Yale University Press.
Leibniz, G.W. 2007. *The Leibniz-Des Bosses correspondence*. Translated and edited with commentary by Brandon Look and Donald Rutherford. New Haven: Yale University Press.
Levey, Samuel. 1998. Leibniz on mathematics and the actually infinite division of matter. *The Philosophical Review* 107 (1): 49–96.
Levey, Samuel. 1999. Leibniz's constructivism and infinitely folded matter. In *New essays on the rationalists,* ed. Rocco J. Gennaro and Charles Huenemann, 134–162. New York: Oxford University Press.
Levey, Samuel. 2003. The interval of motion in Leibniz's *Pacidius Philalethi*. *Noûs* 37:317–416.
Levey, Samuel. 2008. Archimedes, infinitesimals and the law of continuity: On leibniz's fictionalism. In *Infinitesimals differences: Controversies between Leibniz and his contemporaries*, ed. U. Goldenbaum and D. Jesseph, 107–134. Berlin: Walter de Gruyter.

Levey, Samuel. 2011. On two theories of substance in Leibniz: Critical notice of Daniel Garber, *Leibniz: Body, Substance, Monad. The Philosophical Review* 120:285–319.

Lison, Elad. 2006/2007. The philosophical assumptions underlying Leibniz's use of the diagonal paradox in 1672. *Studia Leibnitiana* 38/39:197–208.

Mancosu, Paulo. 1996. *Philosophy of mathematics and mathematical practice in the seventeenth century*. New York: Oxford University Press.

Mancosu, Paulo. 2009. Measuring the size of infinite collections of natural numbers: Was cantor's theory of infinite numbers inevitable? *Review of Symbolic Logic* 2:612–646.

Netz, R., K. Saito, and N. Techenetska. 2001–2002. A new reading of method proposition 14 from the Archimedes Palimpsest (Parts 1 and 2). *Sciamus* 2–3:9–29 and 109–125.

Nicholas of Cusa. 1440/1985. *On learned ignorance.* In *Nicholas of Cusa on Learned Ignorance: A Translation and Appraisal of De Docta Ignorantia,* ed. Jasper Hopkins, 2nd ed. Minneapolis: Arthur J. Banning Press.

Parker, Matthew W. 2009. Philosophical method and galileo's Paradox of infinity. In *New perspectives on mathematical practices,* ed. Bart Van Kerkhove, 76–113. Singapore: World Scientific.

Pironet, Fabienne, ed. 1994. *Guillaume Heytesbury, Sophismata asinina. Une introduction aux disputes médiévales. Présentation, édition critique et analyse*. , ed. Bart Van Kerkhove, 76–113. Collection Sic et Non; Paris: Vrin.

Russell, Bertrand. 1903. *The principles of mathematics*. Cambridge: Cambridge University Press.

Russell, Bertrand. 1914. *Our knowledge of the external world*. Chicago: Open Court Publishing Co.

Skyrms, Brian. 1983. Zeno's Paradox of Measure. In *Physics, philosophy and psychoanalysis,* ed. R. S. Cohen and L. Lauden, 223–254. Dordrecht: D. Reidel Publishing Company.