Leibniz as Reader and Second Inventor: The Cases of Barrow and Mengoli

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1 Introduction

During his stay in Paris in the years 1672–1676 Leibniz acquired a wealth of knowledge in mathematics and discovered significant results within a short time. But in respect of some of his findings he had to recognize that he was not the first mathematician to treat them successfully. The best known example is, of course, the calculus, where it was Isaac Newton who anticipated him. But there are other mathematicians who likewise anticipated Leibniz and whose writings were much more easily available to him. During his initial steps towards the calculus in 1673, for example, neither the use of the infinitesimal characteristic triangle, nor the transmutation of curves, nor even recognition of the relationship between the calculation of tangents and areas were completely new insights. Several mathematicians had acquired knowledge of such methods and had worked with them. Indeed, all the examples mentioned had already been published by Isaac Barrow. But even with Leibniz's first mathematical success in Paris, when he solved the problem of the summation of the reciprocal triangular numbers that Christiaan Huygens had set him in 1672, both the specific result and the general method of solution had already been discovered by Pietro Mengoli. Moreover, the general method had also been found by François Regnauld, as Leibniz learned during his visit to London early in 1673. Another example is provided by the arctan series for the circle, which Leibniz formulated in 1673. Unbeknown to him, this series had already been discovered by James Gregory. It was not until April 1675 that he found out about this prior discovery – in a letter from Henry Oldenburg which also contained a sine series of

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Newton[1]. And last but not least, the rules for the quadrature of the higher parabolas and hyperbolas with arbitrary real exponents had been published earlier by John Wallis.

In order to provide data for a comparative study of Leibniz's treatment of predecessors in these topics, it seems necessary first to investigate the extent of Leibniz's knowledge of their results and his use of the sources available to him. This paper aims to contribute to the issue in exploring the cases of two mathematicians who anticipated results found by Leibniz, the more prominent Isaac Barrow (Part I)[2] and the lesser-known Pietro Mengoli (Part II)[3]. Since E. W. v. Tschirnhaus in a letter to Leibniz argued that the calculus provided nothing essentially new in comparison to the methods in Barrows *Lectiones geometricae*[4], the suspicion was raised from time to time that Leibniz had gained benefit in a decisive way from reading this book in finding and developing the differential and integral calculus[5]. After publication of the relevant portion of Leibniz's manuscripts concerning the prehistory and early history of the calculus in the Academy Edition this question can be investigated on a secured basis of original texts[6]. In the case of Pietro Mengoli on the other hand, an investigation of Leibniz's studies on series and on the arithmetic circle quadrature seems especially promising, because Leibniz wanted to publish in this work most of the results mentioned before.[7]

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[1] Leibniz (1923), A III 1 No. 49_2; OC (=Oldenburg 1965) XI No. 2642.

[2] Part I is based on Probst (2011).

[3] Part II is based on a talk "Die Rezeption der Reihenlehre von Pietro Mengoli durch Leibniz in der Zeit seines Parisaufenthalts (1672–1676)" presented at the meeting of the Fachsektion Geschichte der Mathematik der DMV Lambrecht (Pfalz) in 2007 (print forthcoming); an English version entitled "The Reception of Pietro Mengoli's Work on Series by Leibniz (1672–1676)" was presented at the Joint International Meeting UMI-DMV in Perugia (18–22 June 2007).

[4] See Tschirnhaus to Leibniz [April/May 1679] (A III 2 No. 301, 708–712). Barrow (1670), title print in Barrow (1672), title prints with additions Barrow (1674), Archimedes (1675); see Mahnke (1926, pp. 20–22).

[5] The thesis of a dependence of Leibniz's calculus from Barrow was again put forward by J. M. Child in Barrow (1916) and extensively developed in Leibniz (1920). Mahnke (1926), Hofmann (1974, pp. 74–78), and Mahoney (1990, pp. 236–249), denied such a dependence, Feingold (1993, pp. 324–331), repeated Child's claims; Feingold added an investigation of the correspondence and the discussions during the lifetime of Leibniz and of parts of the later research. For a critique see Wahl (2011). The question has been raised again by Blank (2009, pp. 608–609). Recent publications by Nauenberg (2014) and Brown (2012, pp. 58–60), side with Child and Feingold.—A balanced evaluation of the methods and results of Barrow and Leibniz is presented in Breger (2004).

[6] See especially the volumes A VII 4 (1670–1673) and A VII 5 (1674–1676) concerning infinitesimal mathematics.

[7] The studies on series (1672–1676) are printed in A VII 3. The main manuscript text on the arithmetical circle quadrature has been published for the first time completely in Leibniz (1993). Together with the remaining relevant manuscripts from 1673 to 1676, *De quadratura arithmetica* has been published in 2012 in the Academy edition in vol. A VII 6 No. 51, 520–676.

2 Part I: The Reception of Isaac Barrow's *Lectiones Geometricae* (1670) by Leibniz in Paris (1672–1676)

2.1 References to Barrow and Marginal Notes in Leibniz's Copy of the Lectiones Geometricae

Isaac Barrow was one of the first rank of contemporary mathematicians, whose name was known to Leibniz already in his early years in Germany: In *De arte combinatoria* (1666) and in the *Nova methodus discendae docendaeque jurisprudentiae* (1667), he referred to the mathematical symbols that Barrow had used in his edition of Euclid's *Elements* of 1655[8]. In August 1670 Henry Oldenburg informed him of the publication of Barrow's *Lectiones* Opticae (1669) and L*ectiones geometricae* (1670) (A II 1 (2006) No. 27, 99; OC VII No. 1506, 111). Leibniz in a letter to Martin Fogel in January 1671 mentioned only the *Lectiones Opticae* (A II 1 (2006) No. 38, 126–127). Two years later, during his stay in London (January-February 1673) Leibniz acquired the edition of 1672, in which the two works were sold together with a common titlepage (Hanover, Gottfried Wilhelm Leibniz Bibliothek, Leibn. Marg 0)[9]. In his notes on this journey, *Observata in itinere Anglicano*, Leibniz wrote that he had heard that Barrow tackled an optical phenomenon that he had not been able to explain (A VIII 1 No. 1, 6). In April 1673 in a letter to Oldenburg he referred to this statement in Barrow's *Lectiones Opticae* and told him that Huygens and Mariotte declared that they were able to solve the problem concerned (A III 1 No. 17, 87; OC IX No. 2208, 595–596). Another note in the *Observata* could possibly relate to the *Lectiones geometricae* of Barrow, as has been suggested already by Gerhardt[10]; Leibniz wrote: "Tangents to all curves. Development of geometrical figures by the motion of a point in a moving line."[11] Since Leibniz was familiar with the ancient idea of the generation of a line by a flowing point and already in 1671 wanted to construct all possible lines by the composition of rectilinear motions, his note suggests that he was confronted with this issue again in London[12]. The second sentence goes well with a passage on page 27 of the *Lectiones geometricae*, underlined in Leibniz's personal copy: "For every line that lies in a plane can be generated

[8] Euclid (1655), „Notarum explicatio", facing page 1; see A VI 1 No. 8, 173; A VI 1 No. 10, 346.

[9] The copy is available online at: http://digitale-sammlungen.gwlb.de/goobit3/ppnresolver/?PPN=688854583. (All pictures of figures in Barrow's *Lectiones geometricae* in this paper are taken from this copy by courtesy of the Gottfried Wilhelm Leibniz Bibliothek Hanover.) The marginal notes to the *Lectiones opticae* are printed in A VIII 1 No. 26, 206–209; the marginal notes to the *Lectiones geometricae* are to be found in A VII 5 No. 43, 301–309; concerning the dating of these notes see 301.

[10] Gerhardt (1891, pp. 157–158); Leibniz (1920, p. 160).

[11] Leibniz (1920, p. 185); "Tangentes omnium figurarum. Figurarum geometricarum explicatio per motum puncti in moto lati." (A VIII 1 No. 1, 5).

[12] The flowing point is already mentioned by Aristotle, *De anima*, 409a 4–5; for Leibniz's discussion of the generation of lines by the composition of rectilinear motions see the *Theoria motus abstracti* (A VI 1 No. 41, 270–271).

by the motion of a straight line parallel to itself, and the motion of a point along it; every surface by the motion of a plane parallel to itself and the motion of a line in it (that is, any line on a curved surface can be generated by rectilinear motions); in the same way solids, which are generated by surfaces, can be made to depend on rectilinear motions."[13] The first sentence of the note in the *Observata* could also refer to Barrow's book where a large part deals with the construction of the tangents of different curves. However, other interpretations are possible: On 8 February 1673, Leibniz took part in a meeting of the Royal Society, during which a letter from René François de Sluse containing an exposition of his method of tangents was read[14]. The letter was published in the current issue of the *Philosophical Transactions*[15], and Leibniz conveyed a copy of the printed version to Huygens in Paris (see A III 1 No. 6, 31–32) and made a personal copy of most of the article. Paraphrasing Sluse's introductory remarks he gave the excerpt the title: "Method to draw tangents to all kinds of curves, without laborious calculation, which can be taught to a boy ignorant of geometry"[16]. The similarities between "tangents to all curves" and "tangents to all kind of curves" are striking. However, motions are not used in Sluse's method of tangents. Another possibility could be a reference to Wallis (1672); this article had been printed a year earlier in the *Philosophical Transactions*. The motion of a point ("motus puncti") is used by Wallis, especially on pages 4014–4016. Perhaps in connection to the reading of Sluse's letter there had been talks where the article by Wallis was mentioned[17].

The rest of the underlined passages in the first part of Leibniz's copy of the *Lectiones geometricae*, which probably originated in the early stages of reading, relates twice (pages 13 and 17) to the concept of motion in geometry, in the third (page 21) Barrow justifies using the terminology of indivisibles (see A VII 5 No. 43, 302). Whether the marginal notes on pages 131–133 and page 136 concerning the classification of curves using their equations already came about at this first reading or only later in Hannover, probably cannot be established. Leibniz uses the equality sign "=" both before mid-1674 as well as from 1677 on. There seems to be no direct evidence for a further reading of the *Lectiones Geometricae* before the autumn of 1675. Only Leibniz's expression of regret in his reply to Oldenburg, dated 12 June 1675, on having heard the news that Barrow had retired from active mathematical

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[13] Barrow (1916, p. 49); "Omnis, inquam, in uno plano constituta linea procreari potest e motu parallelo rectae lineae, et puncti in ea; omnis superficies e motu parallelo plani, et lineae in eo (lineae scilicet alicujus e rectis modo jam insinuato motibus progenitae) consequenter et linea quaevis etiam in curva superficie designata rectis motibus effici potest" (Barrow 1672, 27; see A VII 5 No. 43, 302). For a comprehensive analysis of Barrow's treatment of curves and motion see Mahoney (1990, 203–213).

[14] Neither Wallis nor Barrow or Newton were present at this meeting, and Leibniz did not meet them during his stay in England or later.

[15] Sluse (1673).

[16] "Methodus ducendi tangentes ad omnis generis curvas, sine calculi laboris, quam etiam puer ἀγεωμέτρητος doceri possit" (A VII 4 No. 6, 70–71).

[17] See the note of the editors to A VII 4 No. 17, 360, which suggests that Leibniz has read this article in spring 1673. It is sure that Leibniz knew the paper in August 1673 (A VII 4 No. 40, 661).

research because of other commitments could be an indication that by now he was familiar with the contents of the book: "I regret that Barrow has done with geometry, for I was still in expectation of many distinguished things from him"[18].

Leibniz probably received new grounds to consider the *Lectiones Geometricae*, when he had several meetings with Tschirnhaus in October 1675. His compatriot, who had recently arrived from England[19], owned a copy of the edition of *Lectiones geometricae* with additions printed on pages 149–151[20]. Leibniz noted at the end of his copy of the 1672 edition, in which these additions are missing, that he had seen these "addenda"[21]. This note as well as the marginal notes on page 85 and the note on the related figures No. 122 and 125 in Leibniz's personal copy may have originated in the context of these meetings. The single marginal note to the text of page 85 says, "I know for some time" ("Novi dudum")[22], and on the pages with figures No. 122 and 125, corresponding to the text on pages 85–89, Leibniz expressed some of the results of Barrow with his new integral symbol. The use of the integral symbol shows that these notes were not written earlier than the end of October 1675 (Fig. 1).[23]

2.2 *Readings Without References to Barrow*

Leibniz read at least selectively Barrow's *Lectiones geometricae* during the following months. This is documented by his marginal notes and additions to figure No. 119 concerning the quadratrix curve (Fig. 2).

The marginal notes (two equations) were written first and are partly overwritten by the additions to the figure which are made in a different ink. The notes are probably related to a manuscript from June 1676, *De Quadratrice* (A VII 5 No. 86), which is based on the investigation of the quadratrix by Barrow: Leibniz sketches a similar figure in his manuscript: several points are designated with the same letters and he adopts two equations directly from the text of Barrow. His additions to figure No. 119 in the *Lectiones geometricae* are, however, related to a manuscript written

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[18] OC XI No. 2672, 333; "Barrovium geometrica missa fecisse doleo; nam multa ab eo praeclara adhuc exspectabam." (A III 1 No. 55, 256.)

[19] See Mayer (2006) .

[20] See J. Collins to J. Gregory, 19/29 October 1675, printed in: Turnbull (1939, p. 342). Perhaps Tschirnhaus owned the edition of the *Lectiones geometricae* that had been added to Barrow's edition of Archimedes (1675). There are notes from a talk between Leibniz und Tschirnhaus in February 1676 that refer to this edition (A VII 1 No. 23, 180–181).

[21] The additions contain solutions to three problems concerning the arc length of curves, a generalization of the quadrature of the cycloid and several propositions on maxima and minima based on tangent properties.

[22] Barrow's theorem XI, I on the area under the curve of the subnormals corresponds to $\int yy'dx = \int ydy = \frac{y^2}{2}$. Leibniz proved an equivalent proposition in 1673 (A VII 4 No. 27, prop. 6, 467–468).

[23] The notes on the right side of Fig. 123 refer to Fig. 125.

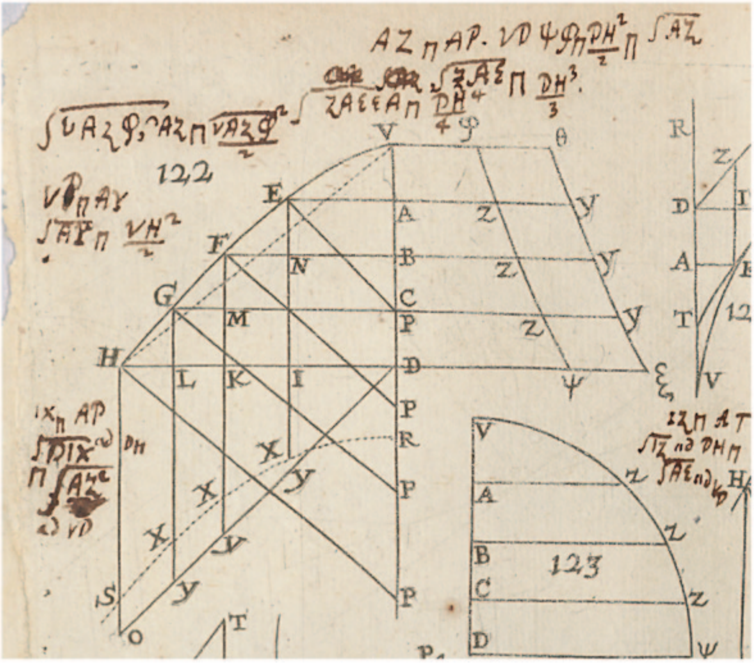


Fig. 1 Leibniz's marginal notes to Barrow's Fig. 122

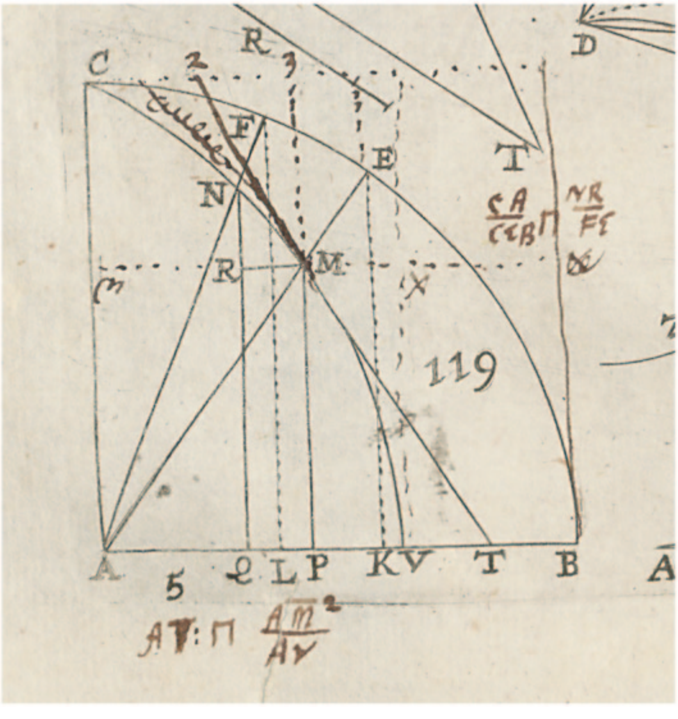


Fig. 2 Leibniz's notes and additions to Barrow's Fig. 119

in Hannover, *De Quadratura quadratricis* (LH 35 XIII 1 fol. 236–239), dating from 6/[16] July 1677.

Another example from spring of 1676 is the *Praefatio opusculi de Quadratura Circuli Arithmetica* (A VII 6 No. 19, 169–177; GM V, 93–98): This is the only manuscript of the Paris period known so far where Leibniz mentions the circle approximation published by Adriaan Metius (A VII 6 No. 19, 173; GM V, 95)[24]. In a small note in this manuscript, not included in Gerhardt's edition, Leibniz wrote down two inequalities. These two inequalities are the results of the approximation method by which Barrow derived the result of Metius in the *Lectiones geometricae*. Barrow expressed the results (for a circle with a diameter of 113 units) in the following words: "the whole circumference, calculated by this formula, will prove to be greater than 355 less a fraction of unity", and "the whole circumference is less than 355 plus a fraction"[25], Leibniz used symbolic formulas: "$c \ulcorner 355-\frac{1}{b}$" and "$c \urcorner 355+\frac{1}{b}$"[26].

These two instances where it is sure that Leibniz used Barrow's book without reference are from the year 1676 and therefore could not have any influence on Leibniz's invention of the calculus which took place earlier. In addition, their thematic relevance for the calculus is a minor one. But the fact that Leibniz does not refer to his source in both cases, suggests further investigation into his manuscripts of 1673–1675. Perhaps there can be more adoptions from Barrow than these two. We know that in the use of the infinitesimal characteristic triangle, the transmutation of curves, and the insight into the relationship between the determination of tangents and of areas Barrow had preceded Leibniz[27]. If there is any adoption of Barrow's methods and results concerning the invention of the calculus it should be possible to discover it in manuscripts dealing with these topics.

2.3 Transmutation Method and Characteristic Triangle

With regard to the transmutation method and the characteristic triangle of Leibniz, whose dependence on Barrow J.M. Child had claimed[28], D. Mahnke has defended the independence of the development of Leibniz[29]. His argument on the basis of then unpublished manuscripts can now be confirmed by means of those texts recently published in A VII 4[30].

[24] The circle approximation $\pi \approx \frac{355}{113}$ found by Adriaan Anthonisz (ca. 1543–1620) was published by his son Adriaan Metius in Metius (1625, 178–179), and in Metius (1633, 102–103).

[25] Barrow (1916, 150 and 151); "tota circumferentia major quam 355, minus fractione unitatis" and "fore Totam circumferentiam minorem quam 355, plus fractione" (Barrow 1672, 103).

[26] A VII 6 No. 19, 172; the symbols $\ulcorner$ and $\urcorner$ are equivalent to the modern symbols $>$ and $<$ for "greater than" and "smaller than".

[27] For Barrow's transformation methods see Mahoney (1990, 223–235).

[28] See Leibniz (1920), especially 15–16, 172–179.

[29] Mahnke (1926, 8–43).

[30] See xxii-xxiii in the introduction to A VII 4.

The transmutation method of Leibniz is a special method of integral transformation and proceeds in two steps: first, the decomposition of a curve segment into infinitesimal triangles starting from a common endpoint and infinitesimal baselines, the elements of arc, which together form the arc of the curve segment. Second, the use of the similarity of the infinitesimal triangle consisting of the elements of abscissa, ordinate, and arc (or tangent) that Leibniz calls the characteristic triangle of the curve and certain finite triangles (e.g., the triangle formed by the ordinate, tangent and subtangent of the curve), which allows the establishing of proportional equations between finite and infinitesimal sides of the triangles considered. Based on his investigation of surfaces of revolution and the associated determination of arc lengths Leibniz pursues in the spring of 1673 the idea to divide the area under a curve into triangles with a common vertex in the center of gravity of the arc and infinitesimal bases on the arc of the curve (A VII 4 No. 5, 63–64). After that he tries several approaches to implement this idea using the example of the parabola and the circle (A VII 4 No. 5, 64–69; A VII 3 No. 17, 202–227; A VII 4 No. 10_1, 140; A VII 4 No. 10_2, 156–158; A VII 4 No. 12_1, 174–176). Later Leibniz learns about the results of the rectification method of H. van Heuraet (A VII 3 No. 16)[31] and immediately tries to form infinite series of numbers whose sum would, for example, give a result for the rectification of an arc of a parabola (A VII 3 No. 17).

Leibniz uses different starting points for the decomposition of the area, from centers of gravity he moves to any point on the axis of the curve and finally uses the apex of the curve, drawing chords from the apex to the points on the arc of the curve as Barrow had done before him (*Lectiones geometricae* XI, § XXIV, 92). This means that only after a series of general considerations and several investigations did Leibniz arrive at the point where Barrow started his transmutation. Since the area of an infinitesimal triangle with an element of arc or tangent as a baseline and the vertex at the apex of the curve is determined, when the altitude of this triangle (i.e. the perpendicular from the apex to the tangent) is determined, Leibniz gains from this the following segment theorem: The area of the curve segment is equal to the sum of the areas of these triangles and therefore equal to half the sum of the products from the baselines and altitudes of these triangles. It happened a few times that Leibniz forgot to halve the sum of the infinitesimal rectangles (e.g., A VII 4 No. 16, 271). Using perpendiculars from the axis to the tangent Leibniz forms different right triangles between axis, tangent and perpendiculars to the axis or to the tangent (e.g., A VII 4 No. 5, 64; A VII 4 No. 10_2, 156; A VII 3 No. 17, 202–203, 210, 220, 222; see Fig. 3). By contrast, Barrow formed his right-angled triangles with perpendiculars to the chords (Fig. 4):

[31] Leibniz marks neighbouring points on the abscissa in the related drawing and calls the distances between these points arbitrarily small ("quantumvis parvae"), but in the following he only deals with the ordinates (A VII 3 No. 16, 200). He does not seem to have noticed the characteristic triangle used by van Heuraet. Perhaps Leibniz at this time did not yet know the original publication Heuraet (1659), but used the presentation of the result in Huygens (1673, 69–73). The related drawings in this book (p. 70–71) do not contain characteristic triangles, and Leibniz added them on page 70 in his own copy of the book he had received from Huygens (A VII 4 No. 2, 32).

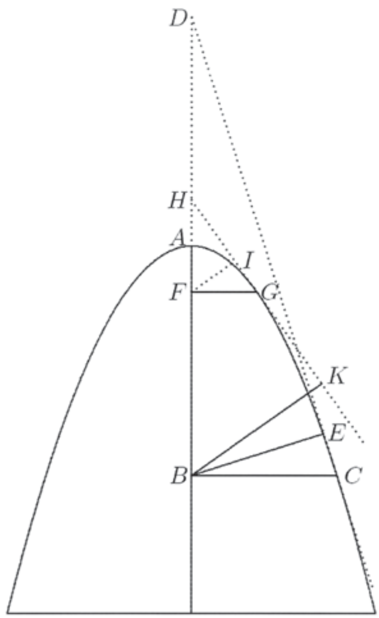


Fig. 3 A VII 3 No. 17_3, 220

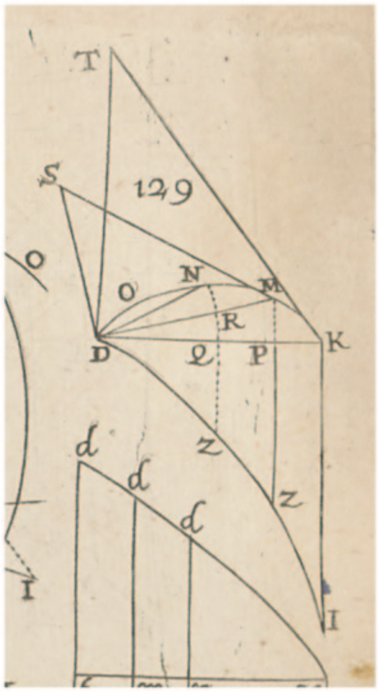
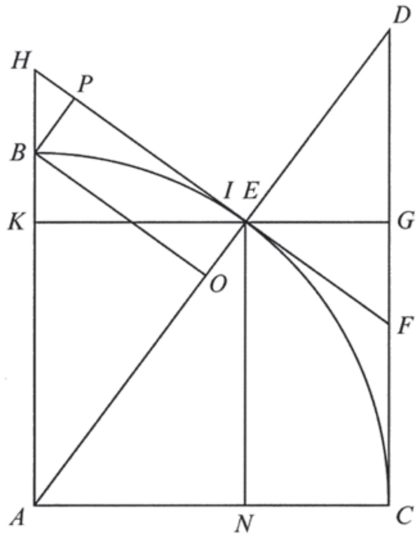


Fig. 4 Barrow's Fig. 129

Fig. 5 A VII 4 No. 22, 392



The right-angled triangles are already similar triangles to the infinitesimal triangle Leibniz later (starting with A VII 4 No. 24[32]) calls the "characteristic" triangle of the abscissa, ordinate and tangent differences, and which is the basis for his transmutation method. Leibniz first carries out area transformations using finite similar triangles by setting equivalent various products of pairs of these sides (A VII 4 No. 21 and 22). After establishing a proportional equation for the circle, which is essentially the same as $\tan = \frac{\sin}{\cos}$, Leibniz states:

"$\frac{AE}{EN} = \frac{HE}{NA}$ [Fig. 5]. Therefore, $AE \cdot NA = EN \cdot HE$. From this proposition follows the quadrature of the sine curve and nearly everything in Pascal's treatises of the sines and arcs of the circle and the cycloid"[33].

The use of products of geometrical quantities corresponds to the transformation methods, which he found earlier in the writings of Pascal and Fabri. Subsequently, in the summer of 1673, Leibniz carried out two systematic studies on the trigonometric quantities in the circle, using finite (A VII 4 No. 26) and infinitesimal (A VII 4 No. 27) right triangles, establishing more than 150 equations for area transformations. While he already succeeded in this transmutation with the tangent of the half-angle (A VII 4 No. 27, 489–494) a result that shortly afterwards would lead him to

[32] There is already a characteristic triangle in A VII 4 No. 10_2, 156, but this was probably added only later in connection with the additional remarks 158 l. 4–7. The psychological importance of the discovery of the characteristic triangle for Leibniz is indicated by the fact that in this example, as in many others documented after August 1673, he marked the vertices of the characteristic triangles with his initials *G, W, L*.

[33] "$\frac{AE}{EN} = \frac{HE}{NA}$ Ergo $AE \cap NA = EN \cap HE$. Ex hac propositione pendet quadratura figurae sinuum, et pleraque omnia in Pascalii tract. de sinubus arcubusque circuli, deque cycloeide." (A VII 4 No. 22, 396–397; drawing 392.).

the circle series (A VII 4 No. 42), he formulated transmutation theorems for general curves also with other quantities (A VII 4 No. 27, 495; A VII 4 No. 39_1, 617 and 621). Leibniz uses the term "characteristic triangle" for the first time not in the case of the circle, but rather with other curves such as the conchoid curves, ellipses and cycloids (A VII 4 No. 24, No. 28, No. 29). Perhaps Leibniz coined the term after he noted a theorem of J. De Witt[34] as "ellipsis character" (A VII 4 No. 28, 502).

2.4 Barrow's Prop. XI, § XIX, and Leibniz's Theorem

In the *Lectiones geometricae*, Barrow demonstrates some rules, which are equivalent to rules of differential calculus: for example, VIII, § IX (quotient rule); IX, § XII (product rule)[35]. However, the texts in which Leibniz himself derives these rules (A VII 5 No. 51_2, No. 70, No. 89), do not seem to depend immediately on Barrow's publication. Although Barrow's text contains equivalents to the quotient rule and the product rule, the two rules are not formulated explicitly. The former theorem is a rule for determining geometrically the tangent of a curve if the tangent of the curve with reciprocal ordinates is known. The second is embedded in a more general theorem on the construction of tangents of curves that form the geometric means.

When Leibniz records the first example of a simple case of the product rule of differentiation in his new notation on 27 November 1675, he calculates with infinitesimal differences and remarks: "Now this is a really noteworthy theorem and a general one for all curves. But nothing new can be deduced from it, because we had already obtained it."[36] Apparently he immediately realized that the statement is equivalent to another one he had used since the spring of 1673, expressing the relations of the area of a segment of a curve and its complement to the circumscribed rectangle[37], in modern notation $xy = \int ydx + \int xdy$. It is noteworthy that Leibniz examines in this text the relationship between integration and inverse tangent method, and even in the same paragraph carries out transformations that are based on the equality of the ratio of the infinitesimal quantities dx and dy with the ratio of the subtangent t and the ordinate y. But he obtains the result directly from the investigation of sums (integrals) and differences of abscissas and ordinates of curves.

The situation is similar with his derivation of the quotient rule: In the spring of 1673 Leibniz calculates the differences of the terms of the sequence $\frac{1}{a^2}$ (A VII 3 No. 13, 160), in August 1673 the differences of the ordinates of the hyperbola $y = \frac{a^2}{c+x}$ (A VII 4 No. 40_2, 683). Again, the quotient rule is the result of a calcula-

[34] Witt (1659), especially book I, prop. 18, 224.

[35] See Breger (2004, 199–200).

[36] Leibniz (1920, 107); "Quod Theorema sane memorabile curvis omnibus est commune. Sed nihil novi ex eo ducetur, quia adhibuimus jam." (A VII 5 No. 51_2, 365.).

[37] See A VII 4 No. 10, 136; A VII 4 No. 40, 690 and 705; more detailed in A VII 3 No. 40, 578–579 (October 1674—January 1675).

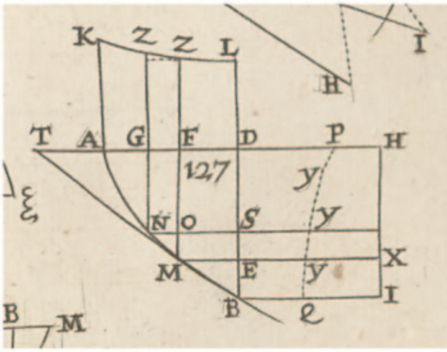


Fig. 6 Barrow's Fig. 127

tion of differences: Leibniz subtracts fractions that express neighbouring ordinates (A VII 5 No. 70, 506; A VII 5 No. 89, 593–595). The consideration of tangents is irrelevant to this.

But there is a topic based on the consideration of tangents that is of interest for the invention of differential and integral calculus: the discovery of the equivalence of solutions for the inverse tangent problem and the problem of the determination of the area under a curve. This is what is often called the geometric form of the fundamental theorem of differential and integral calculus. N. Guicciardini, in a review of the volumes A VII 4, and VII 5, recently called attention to a possible influence of Barrow on Leibniz in regard to this issue[38].

Barrow, who was the first to publish such a theorem, put the two reciprocal statements into two separate theorems (*Lectiones geometricae* X, § XI, 78 and XI, § XIX, 90–91). The second is illustrated by his Fig. 127 (Fig. 6):

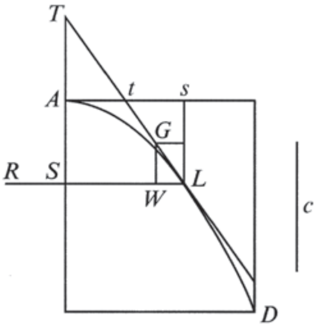
"Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, TF: FM = R: FZ. Then the space ADLK is equal to the rectangle contained by R and DB."[39]

The subtangent TF of AMB is to the ordinate FM of AMB as a constant R to the ordinate FZ of KZL. So we have $FZ = R \times FM/TF$. If $FM = y$, then results $FT = y/y'$,

[38] Guicciardini (2010, p. 546): "The marginalia to Isaac Barrow's *Lectiones geometricae* (1670) are particularly noteworthy, since Barrow's lectures would have provided Leibniz with a geometric expression of the so-called fundamental theorem of the calculus." Nauenberg (2014) argues for an influence of Barrow's *Lectiones geometricae* on Leibniz in the case of this theorem on the basis of an analysis of Leibniz (1693). Unfortunately Nauenberg does not investigate any of the manuscripts from 1673 (published in A VII 4) nor does he notice occurrences of similar statements in the manuscripts from 1674 to 1676 (e.g. A VII 5 No. 26, 203–204, and A VII 5 No. 49, 348).

[39] Barrow (1916), 135; "Porro, sit curva quaepiam AMB, cujus axis AD, & huic perpendicularis BD; tum alia sit linea KZL talis, ut sumpto in curva AB utcunque puncto M; & per hoc ductis rectâ MT curvam AB tangente, rectâ MFZ ad DB parallelâ (quae lineam KL secet in Z, rectam AD in F) datâque quâdam lineâ R; sit TF . FM :: R . FZ; erit spatium ADLK aequale rectangulo ex R, & DB."

Fig. 7 A VII 4 No. 40_3, 692



and thus $FM/TF=y'$ and $FZ=R \times y'$. The curve *AMB* is therefore an antiderivative of the curve *KZL*. Barrow proved this as follows:

"For, if DH=R and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have NO : MO = TF : FM = R : FZ; NO . FZ = MO . R, and FG . FZ=ES . EX. Hence, since the sum of such rectangles as FG . FZ differs only in the least degree from the space ADLK, and the rectangles ES . EX form the rectangle DHIB, the theorem is quite obvious."[40]

Leibniz formulates an equivalent theorem in August 1673:

"Let *LD* be a curve [Fig. 7], its sine (i. e. the ordinate perpendicular [to the abscissa *AS*]) *SL*, abscissa *AS*, tangent *TL*, characteristic triangle *GWL*. And if it happens that *ST* is to *SL* or *GW* is to *WL* as a certain constant straight line [*c* is to] the corresponding sine *SR* of another curve with the same axis (i. e. the same abscissa), then any portion of the area below the other curve, cut off by its sine, will be equal to a rectangle formed by *SL* in *c*. The demonstration of this is very easy: $\frac{TS}{SL} = \frac{GW}{WL}$ or $\frac{g}{w}$ from construction; $= \frac{c}{RS=r}$ from presupposition, therefore $cw=rg$. This suffices as proof for those understanding these matters."[41]

The theorems of Barrow and Leibniz differ mainly by the fact that Leibniz already in the formulation of the statement introduces the infinitesimal characteristic

[40] Barrow (1916, p. 135); "Nam sit DH=R; & compleatur rectangulum BDHI; tum assumptâ MN indefinite parvâ curvae AB particulâ ducantur NG ad BD; & MEX, NOS ad AD parallelae. Estque NO . MO :: TF . FM :: R . FZ. Unde NO×FZ=MO×R; hoc est FG×FZ=ES×EX. ergò cum omnia rectangula FG×FZ minimè differant à spatio ADLK; & omnia totidem rectangula ES×EX componant rectangulum DHIB, satis liquet Propositum."

[41] "Sit curva *LD*, cuius sinus (id est ordinata normalis) *SL*, abscissa *AS*, tangens *TL*, triangulum characteristicum *GWL*. Sique fiat ut *ST* ad *SL*, vel *GW* ad *WL*, ita recta quaedam constans [*c* ad] alterius cuiusdam figurae eiusdem axis sinum respondentem (respondentem inquam[,] id est eiusdem abscissae) *SR*, portio quaelibet ab altera figura abscissa per sinum eius, aequabitur rectangulo *SL* in *c*. Cuius rei demonstratio haec est perfacilis: $\frac{TS}{SL} = \frac{GW}{WL}$ vel $\frac{g}{w}$ per constructionem; $= \frac{c}{RS=r}$ ex hypothesi, ergo $cw=rg$. Quod rerum harum intelligentibus sufficit ad demonstrationem." (*Pars III[tia] Methodi tangentium inversae et de functionibus*, A VII 4 No. 40_3, 692–693.) − Neither Gerhardt (1848, 20–22), or Gerhardt (1855, 55–57), nor Mahnke (1926, 43–59), mention this theorem in their investigations of the manuscript.

triangle, while Barrow does so only in the proof. In both cases the ratio of the ordinate to its subtangent is formed without using a terminological designation for the subtangent; in the hypothesis in each case only the tangent is employed, but the tangent no longer occurs in the formulation of the proportional equations. Only in the example immediately following does Leibniz employ his usual term "producta" for the subtangent. In the subsequent conclusion, with which he emphasizes the general validity of the theorem, he writes by mistake "tangent" instead of "producta": "Therefore, the quadrature of all curves can be obtained, whose sines are to a certain constant straight line, as the sine of another known curve is to its tangent; or as the ratio of the sines of the characteristic triangle of a known curve".[42]

All this taken together, could create the impression that Leibniz formulated his theorem on the model of Barrow's [43]. This possibility cannot be completely ruled out, but it is to be noted that the statement $\frac{TS}{SL} = \frac{GW}{WL}$, which is required for the formulation of the theorem, is based on the similarity of the characteristic triangle with the triangle of subtangent, ordinate and tangent, and is already pronounced by Leibniz in the beginning of the manuscript (A VII 4 No. 40_1, 657). He expresses the relationship between ordinate and subtangent in the following way, letting the difference of the abscissas be equal to the unit: "In short, the matter goes back to the following: The straight line *ED*, the ordinate, divided by *ID*, its difference to the preceding ordinate, gives the straight line *ME* [scil: the subtangent]" [44]. Leibniz then refers to this relation in A VII 4 No. 40_3, 689, shortly before the formulation of the theorem. The second half of the theorem $\frac{GW = g}{WL = w} = \frac{c}{RS = r}$ does not follow immediately from the preceding considerations. A similar proportional equation had been obtained by Leibniz in his previous investigations of the characteristic triangle of the circle, the radius playing the role of the constant term *c* (A VII 4 No. 27, prop. 3, 467), but he had not attempted a generalization there. He studies the same example among others again in A VII 4 No. 40_3, 696–697, but his calculations probably were only carried out after the formulation of the theorem. It is therefore likely to remain an open question whether Leibniz found this theorem in the course of his studies on the inverse tangent method entirely independent from Barrow, or whether he had encountered the theorem while reading Barrow's book.

Overall, it should be noted that Leibniz counted Barrow among the great mathematicians in the prehistory of calculus, as is evident by his statement from 1 November 1675 in *Analyseos tetragonisticae pars tertia*: "Most of the theorems of the geometry of indivisibles which are to be found in the works of Cavalieri, Vincent,

[42] "Quare omnium figurarum haberi potest quadratura, quarum sinus sunt ad rectam quandam constantem, ut sunt sinus alterius cuiusdam figurae cognitae ad suam tangentem; seu ratio sinuum trianguli characteristici figurae cognitae."

[43] It should be taken into consideration that both theorems are formulated in analogy to the rectification theorem of Heuraet (1659, 518): Heuraet used the normal of the curve for the proportional equation, Barrow and Leibniz use the subtangent.

[44] "Breviter res eo redit: Recta *ED*, applicata, divisa per *ID*, differentiam ab ipsamet et applicata praecedente, dat rectam *ME*." (A VII 4 No. 40_1, 660); see also Mahnke (1926, 44–46).

Wallis, Gregory and Barrow are immediately evident from the calculus"[45]. Beyond this, Leibniz doesn't seem to have acknowledged any influence of Barrow's writings on his discovery of the calculus, neither then nor later. In 1686 he places Barrow alongside James Gregory in *De geometria recondita* when he writes: "These were followed by the Scotsman James Gregory and the Englishman Isaac Barrow, who in famous theorems of this kind advanced science in a wonderful way."[46]

3 Part II: The Reception of Pietro Mengoli's Work on Series by Leibniz (1672–1676)

Probably in September 1672 Leibniz, in a discussion with Christiaan Huygens, expressed his opinion that he possessed a general method for finding the sum of infinite series. The basis for this was his realization that the terms of a monotonously decreasing zero-sequence are equal to the sums of the differences of the following terms. Huygens tested the mathematical abilities of Leibniz by proposing to him that he find the sum of the series of reciprocal triangular numbers, a result he had found himself some years before, in 1665, but had not published[47].

After a few futile attempts, Leibniz, within a short space of time, achieved success: he found out that the reciprocal triangular numbers are the doubled differences of the harmonic series (A VII 3 No. 1 and 2). In addition he was able to calculate the sum of the higher reciprocal figurate numbers by means of this method, since these can be obtained as triples, quadruples etc. of the iterated differences of the harmonic series. Already before the end of the year 1672 Leibniz had prepared a paper (A III 1 No. 2) for the *Journal des Sçavans*, but unfortunately at that time publication of the journal was interrupted for more than a year.

3.1 Indirect Reception

In January 1673, Leibniz travelled with a diplomatic delegation of the court of the prince elector of Mainz to London[48]. On 12 February 1673, he visited Robert Boyle and met the mathematician John Pell. During a conversation with Pell, Leibniz mentioned his difference method. Pell declared that such a method already had already been found by François Regnauld and had been published, in 1670, in a book

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[45] Leibniz (1920, 87); "Pleraque theoremata Geometriae indivisibilium quae apud Cavalerium, Vincentium, Wallisium, Gregorium, Barrovium extant statim ex calculo patent" (A VII 5 No. 44, 313.).

[46] "Secuti hos sunt Jacobus Gregorius Scotus, & Isaacus Barrovius Anglus, qui praeclaris in hoc genere theorematibus scientiam mire locupletarunt." (Leibniz (1686, 104; *GM* V 232.)).

[47] Huygens (1888–1950, vol XIV, 144–150).

[48] See Hofmann (1974, 23–35).

by Gabriel Mouton[49]. Leibniz was able to consult the book the following day when visiting Henry Oldenburg, the secretary of the Royal Society. He wrote immediately a short defence and a presentation of his method for the Royal Society, in which he also stated his results of the summation of the reciprocal figurate numbers (A III 1 No. 4, 29). On 20 February 1673 he submitted a request for becoming a member of the Royal Society, including a paper with his results (A III 1 No. 7_2). This letter was read at the meeting of the Royal Society on 1 March 1673.

Since Leibniz was already back in Paris on 8 March 1673, he must have left London about two weeks earlier, one week before the meeting took place during which his paper was read. Shortly before his departure from London, Leibniz must have been informed about a certain reaction to his short defence, because in a letter dated 8 March 1673 he asked Oldenburg for more detailed information on Pell's comments. This was the occasion on which he stated the priority of Mengoli concerning the summation of the reciprocal triangular numbers (A III 1 No. 9, 43; OC IX No. 2165, 491). Leibniz received this fuller account of Pell's reaction in a large letter from Oldenburg dated 20 April 1673, which also informed him of his successful election into the Royal Society (A III 1 No. 13; OC IX No. 2196, 2196a, 2202): Oldenburg writes that John Collins told him that Mengoli's result had been published in his book entitled *De additione fractionum sive quadraturae arithmeticae*, Bologna 1658 (recte: Mengoli 1650). There Mengoli indicates that he found the sum of the reciprocal figurate numbers, but failed−as he himself admitted−to find the sum of the series of the reciprocal square numbers and the sum of the harmonic series (A III 1 No. 13_2, 60; OC IX No. 2196, 557). Leibniz replies to Oldenburg on 26 April 1673 (and a second time on 24 May, erroneously believing that his first letter gone missing), writing that he has not yet been able to consult Mengoli's book (A III 1 No. 17, 88, and No. 20, 92–93; OC IX No. 2208, 596 and No. 2233, 648). Since Leibniz also assumed that Mengoli had only found the sums of finite series, Oldenburg makes clear (again with the help of Collins) that Mengoli had actually found the sum of the infinite series of the reciprocal figurate numbers and had demonstrated that the harmonic series cannot be summed, since it exceeds any finite value (A III 1 No. 22, 98; OC IX No. 2238, 667)[50].

The exchange of letters between Leibniz and Oldenburg was interrupted thereafter for 1 year; Leibniz then writes to Oldenburg in July 1674, in order to inform him of the progress he made in the construction of the calculating machine and of his new results concerning quadratures (circle series, cycloid segment: A III 1 No. 30; OC XI No. 2511). Mengoli is not mentioned again until Leibniz's letter to Oldenburg of 16 October 1674, in which he reports on the number theoretical controversy between Jacques Ozanam and Mengoli concerning the six-square problem (A III 1 No. 35 128–129; OC XI No. 2550, 98–99). Mengoli had sought to prove that the problem posed by Ozanam was unsolvable and published this proof, unaware of the serious errors it contained. Ozanam subsequently took delight in humiliating Mengoli by reprinting his flawed proof together with his own successful numerical

[49] Mouton (1670, 384).

[50] For accounts of Mengoli's results and methods see Giusti (1991), Massa (1997), Massa Esteve (2006), Massa Esteve/Delshams (2009) .

solution to the problem[51]. Leibniz who—unlike James Gregory—apparently was not able to solve the problem himself does not seem to have disregarded the mathematical abilities of Mengoli because of this error. In his copy of Ozanam's final flyleaf he only points to the places of error (A VII 1 No. 39, 236–237) and in a sheet enclosed he simply records the mere facts: "Mengoli was wrong, and the example shows that it is possible"[52]. In one of his designs for an international science organization, *Consultatio de naturae cognitione* [1679], Leibniz mentions Mengoli among the scholars, whose cooperation he desires (A IV, 3 No. 133, 868).

Evidently, Leibniz did not get access to Mengoli (1650) during his stay in Paris. But, when he visited London for a second time in October 1676, he made excerpts relating to Mengoli from the correspondence between James Gregory and John Collins. In the sections copied by Leibniz there is a passage on Mengoli's proof of the divergence of the harmonic series, characterized by Leibniz in a marginal note as "ingenious" ("ingeniose", A III 1 No. 88_2, 486–487).

3.2 Leibniz's Excerpts from Mengoli's Circolo

It has been known since the 1920s that Leibniz in April 1676 finally had the opportunity to study Mengoli's book *Circolo* (1672) and that he made extensive excerpts from this work[53]. According to the *Catalogue critique* of the manuscripts of Leibniz (Rivaud (1914–1924), quoted as Cc 2), the first part of these three excerpts (Cc 2, No. 1383 A, 1383 B, 1384) is missing, but probably this missing item is at least partly identical with the manuscript LH 35 XII 1 fol. 9–10 (=Cc 2, No. 1398, 1400, 1401), entitled *Arithmetica infinitorum et interpolationum figuris applicata*, and printed in A VII 3 No. 57_2. This had formerly been located together with the excerpts by Leibniz (see A VII 3 No. 57_1) and had been removed to different place within the collection of Leibniz's manuscripts at an unknown date before the end of the 19th Century.

In *Arithmetica infinitorum et interpolationum figuris applicata* Leibniz essentially discusses the triangular tables of Mengoli (1672, 3–10), and tries to find a method for the computation of the partial sums of the harmonic series. With the help of these tables Mengoli determines by interpolation areas of curve segments, something he did already in Mengoli (1659). The values in these tables represent

51 Mengoli suffered hard from this failure as is obvious from his letters to A. Magliabecchi vom 1 June 1674 and to A. Marchetti from 2 June 1674, published in Mengoli (1986, 41–44).

52 "Erravit Mengolus, idque possibile esse docet… exemplum" (A VII 1 No. 40, 241). − The affair has been studied in detail by Nastasi/Scimone (1994) . Leibniz had communicated Ozanam's problem in the aforementioned letter from 8 March 1673 to Oldenburg (III 1 No. 9, 42; OC IX No. 2165, 490–491). His own contributions from 1672 to 1676 and some of the material by Mengoli and Ozanam is published in A VII 1 No. 37–40, 42–44, 49–52, 55–61, 93, 96–100; see also Hofmann (1969).

53 A VII 6 No. 13, 113–131; the excerpts are recorded in Cc 2 and are mentioned in Mahnke (1926), 8 n. 7.

Fig. 8 Leibniz's harmonic triangle (End-1673—Mid-1674), A VII 3 No. 30, 337 (detail)

$$
\begin{array}{ccccccccccc}
 & & & & & \cdots & & & & & \\
 & & & & \frac{1}{5} & & \cdots & & & & \\
 & & & \frac{1}{4} & & \frac{1}{20} & & \cdots & & & \\
 & & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \cdots & & \\
 & \frac{1}{2} & & \frac{1}{6} & & \frac{1}{12} & & \frac{1}{20} & & \cdots & \\
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots \\
 & \frac{3}{2} & & \frac{5}{6} & & \frac{7}{12} & & \frac{9}{20} & & \cdots & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

special values of the beta function in today's terminology[54]. Already after reading the first pages of the work of Mengoli, Leibniz became convinced of the truth of the statements made by Oldenburg (or Collins) in regard to the results of Mengoli concerning the summation of the reciprocal figurate numbers. In addition he was able to recognize that Mengoli had already been in possession of the harmonic triangle, used by Leibniz in several different forms and arrangements since the end of 1672 (e.g. A III 1 No. 2, 9; A VII 3 No. 53_2, 710) and for which Leibniz coined the expression "harmonic triangle" in his manuscript *De triangulo harmonico* (A VII 3 No. 30, 337) between the end of 1673 and the middle of 1674[55]. There Leibniz arranges the terms starting from a horizontal line with the terms of the harmonic sequence and indicates the differences of two neighbouring terms in the lines above, the sums in the lines below (Fig. 8).

Mengoli uses brackets in writing the denominator of a fraction behind the nominator (Fig. 9). Leibniz in his excerpt reproduces this arrangement, but (as in his other manuscripts) uses the common notation for fractions (Fig. 10).

The excerpts from Mengoli's *Circolo* consist of a large sheet with three triangular tables (Mengoli (1672, 16, 19 and 7)), which was later folded (A VII 6 No. 13_1, 113–120), and a folded sheet that bears the title *Pars secunda excerptorum ex Circulo Mengoli et ad eum annotatorum* (A VII 6 No. 13_2, 120–131), containing a text primarily concerned with the circle calculation of Mengoli, starting from page 23 of *Circolo*. This part of the manuscript is, however, partly damaged and can only be deciphered with difficulty; nonetheless, it can can be said that Leibniz reconstructed step by step the most important stages of Mengoli's argumentation. Only in two places did the Italian text cause difficulties for him, with the result that he did not attempt to provide a Latin paraphrase of the content, but instead quoted the

[54] See M. Rosa Massa Esteve (2006); a detailed account of the content of Mengoli (1672) is provided in Massa Esteve/Delshams (2009) .

[55] See A VII 3 No. 30, 336–340, and Probst (2006).

1

1(2) 1(2)

1(3) 1(6) 1(3)

1(4) 1(12) 1(12) 1(4)

1(5) 1(20) 1(30) 1(20) 1(5)

1(6) 1(30) 1(60) 1(60) 1(30) 1(6)

1(7) 1(42) 1(105) 1(140) 1(105) 1(42) 1(7)

1(8) 1(56) 1(168) 1(240) 1(240) 1(168) 1(56) 1(8)

Fig. 9 Triangular table, Mengoli (1672, 4)

$$1$$

$$\frac{1}{2} \quad \frac{1}{2}$$

$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$

$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$

$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$

$$\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}$$

Fig. 10 Leibniz's harmonic triangle (April 1676), A VII 3 No. 57_2, 736

Italian text verbatim[56]. In some places Leibniz inserted comments. For example, he noted in the excerpt that from the approximation sequences for the proportion of the square to the inscribed circle

$$\frac{3}{2} > \frac{3.3.5}{2.4.4.} > \frac{3.3.5.5.7}{2.4.4.6.6} > \ldots \frac{4}{\pi} \ldots > \frac{3.3.5.5.7.7}{2.4.4.6.6.8} > \frac{3.3.5.5}{2.4.4.6} > \frac{3.3}{2.4}$$

[56] A VII 6 No. 13_1, 129 and No. 13_2, 131; cf. Mengoli (1672), § 105, 39 and § 159, 59.

by forming differences new circle series can be obtained. In the computation of the series

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{6}+\frac{2}{45}+\frac{32}{1575} \text{ etc. and } 1-\frac{\pi}{4}=\frac{1}{9}+\frac{8}{225}+\frac{192}{11025} \text{ etc.}$$

however, minor errors of calculation came about (A VII 6 No. 13_2, 121–125).

It seems that Leibniz did not enter into a further investigation of Mengoli's methods after making the excerpts. In the manuscript of *De quadratura arithmetica*, on which he probably worked until shortly before his departure from Paris at the beginning of October 1676, Leibniz included his results on the summation of the reciprocal figurate numbers and the harmonic triangle without so much as mentioning Mengoli − at least as far as can be established from the extant manuscripts[57]. Up to now no additional documents have been found in Leibniz's manuscripts from his Paris sojourn which provide evidence of further occupation with Mengoli's methods on his part. As far as the later period is concerned, only some manuscripts from 1679 are known where he mentions Mengoli's name. This is of course the same year in which Leibniz probably wrote the *Consultatio de naturae cognitione*, mentioned above. These manuscripts belong to the group around the study *De cyclometria per interpolatione*, dated 26 March 1679, in which Leibniz discusses the results of James Gregory, John Wallis and Pietro Mengoli in circle calculation and tries to find simpler approximation sequences[58]. Only a detailed analysis of these manuscripts will be able to furnish us with more information, but this is a task which will still needs to be carried out. Indeed, since the contents of many of Leibniz's unpublished mathematical manuscripts from his Hanover period are insufficiently known, it is quite possible that still further evidence of his reception of Mengoli will be found in the future.

Conclusion

Barrow and Mengoli were mathematicians who made discoveries and published results—and in some cases also their methods—which Leibniz achieved only years later. Despite appreciating their accomplishments, he evidently never acknowledged any influence of their writings on his own discoveries between 1672 and 1676. He behaved remarkably differently in the cases of Brouncker, Cavalieri, Descartes, Fabri, Fermat, Galileo, Guldin, van Heuraet, Huygens, Pascal, Ricci, Rober-

[57] Mengoli is not mentioned in the draft, which Leibniz took to Hanover, the version of the manuscript which he had left in Paris, intended for publication, was lost later; cf. G. W. Leibniz, *De quadratura arithmetica* (A VII 6, introduction, xxi-xxiv, and No. 51, 606–611).

[58] The manuscript of *De cyclometria per interpolatione* is located in LH 35 II 1 fol. 68–73 between the excerpts from Mengoli fol. 67 + 79, 74,75; fol. 76 discusses the interpolation result of Wallis, fols 77 and 78 contain the sequences for circle approximation and triangular tables from the excerpts from Mengoli's *Circolo*.

val, Saint-Vincent, Sluse, Wallis, and Wren. And this is just to name contemporary authors, who were in his view probably the most important for his mathematical development. One possible reason could be that Leibniz with regard to Mengoli and Barrow was always convinced that he had acquired his knowledge independently of them. A similar picture emerges when we look at the authors Leibniz commonly named as sources and predecessors, for also in these cases he showed different attitudes in different issues. For example, in *De quadratura arithmetica* as well as in the draft of a historical introduction to this treatise, he emphasized the originality of his proof of the method of quadrature of the higher parabolas and hyperbolas contrasted with the results of Fermat and Wallis[59]. He appears to behave the same way in his references to James Gregory and Newton. Informed by Huygens that the auxiliary curve which he used for his circle quadrature had already appeared in print in Gregory (1668), Leibniz added a note in his treatise of the circle quadrature of October 1674, emphasizing his independence of Gregory: "I further do not conceal that Mons. Hugens brought to my attention, to wit that Mr Gregory hit upon the anonymous curve I use here, but for a different purpose, and without perceiving that property which served as the basis for my demonstration"[60]. Later, in *De quadratura arithmetica* of 1676, no such remarks can be found. Also in the sections concerning the circle series there is no mention of Gregory, although Leibniz had already been informed in April 1675 of Gregory's identical series by Oldenburg[61]. Furthermore, the same is true for the sine series of Newton, contained in the same letter from Oldenburg, and later reported again to Leibniz by Georg Mohr[62]. Leibniz did not mention it in his treatment of the sine series, while he praised the binomial theorem of Newton, of which he had gained knowledge through another letter from

[59] *De quadratura arithmetica*, A VII 6 No. 51, 588–589. Leibniz cancelled this and other historical sections in the surviving mansucripts of his treatise in order to include them in an ample introduction. There exist several manuscripts, one with outlines of this introduction (A VII 6 No. 39, 427–432), three shorter pieces (A VII 6 No. 40, 433–436; No. 41, 437–439; No. 49_2, 514–518), and an extensive elaboration of the main part, entitled *Dissertatio exoterica de usu geometriae, et statu praesenti, ac novissimis ejus incrementis* (A VII 6 No. 49_1, 483–514; for the remarks concerning Fermat and Wallis see 507). This manuscript is split into two parts preserved in different locations of the Leibniz papers, and both have been published by C. I. Gerhardt separately without recognizing the connection between them (GM V 316–326, and Gerhardt (1891, 157–176, text 167–175). The first part is also printed in A VI 3 No. 54_1, 437–450; a partial translation of the second part is in Leibniz (1920, 186–190), the remarks mentioned are omitted there. The edition of the two isolated fragments has caused misunderstandings. At the end of the text (A VII 6 No. 49_1, 510–514), Leibniz presents briefly the main result of his *Quadratura arithmetica*; Child in Leibniz (1920, 190), declared his incomprehension: "It is difficult to see the object Leibniz had in writing this long historical prelude to an imperfect proof of his arithmetical quadrature, unless it can be ascribed to a motive of self-praise."

[60] "Je ne dissimule non plus que ce Mons. Hugens m'a fait remarquer, sçavoir que Mons. Gregory a touché la Courbe Anonyme dont je me sers icy, mais pour un tout autre usage, et sans s'appercevoir de cette proprieté qui a servi de fondement à ma demonstration" (A III 1 No. 39_2, 169).

[61] A III 1 No. 49_2, 235; OC XI No. 2642, 267.— For Newton's sine series see A III 1, 233; OC XI, 266.

[62] A III 1 No. 80_1, 375; OC XII No. 2893, 268–269; VII 6 No. 17, 162; VII 6 No. 47, 465.

Oldenburg in August 1676[63]. It is not certain if Leibniz discovered the sine series independently, although all the preconditions for him to deduce it in a way analogous to the method he employed for the logarithmic series in proposition XLVII, were given[64]. In *De quadratura arithmetica* he announced the corresponding proposition XLVIII quite ambiguously, relating only to the proof, not to the invention of the series: "Hence a similar rule for the trigonometric regress, or the invention of the sides from the given angles, was not difficult to demonstrate."[65] To sum up, then, from the examples investigated in this essay a common pattern can be established: When Leibniz was convinced that he had discovered a result or a method by himself, he regarded it as his own achievement for which he had no need to acknowledge a debt to any predecessor.

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[63] A III 1 No. 88_5; Newton (1960), 20–32.

[64] See A III 1 No. 89_1, 566, and Hofmann (1957).

[65] "Hinc jam similem regulam pro regressu trigonometrico, seu inventione laterum ex angulis datis, demonstrare difficile non fuit" (*De quadratura arithmetica*, A VII 6 No. 51, 642).

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