Mathematics and Educational Psychology: Construction of Learning Environments

Cesare Fregola

Abstract When primary school children learn mathematics, highly complex phenomena occur. These phenomena have been studied in various disciplinary contexts and are organized in a complex and interdisciplinary synthesis, of which references can be found within the evolution of neurosciences, and the psychology of learning as well as experimental psychology. These disciplines are all valuable resources to refer to when researching and experimenting ways to create, plan and realize mathematics learning environments. Particularly for mathematics, these environments aim to facilitate the process of abstraction, stimulate the capacities and abilities that are necessary when entering the realm of mathematics, understand its characteristics, develop and make it possible to develop the skills required to be able to master its language and its uses, and, above all, the motivation to learn. Video games, if conveniently used, can represent learning environments. This essay proposes some reflections that are the result of research and experimentation based on the prerequisites described here. The central focus is mathematics, its prerogatives, and *thought and action in teaching* when it is integrated with the exploration of simulation games and video games, which are an integral part of a digital native's daily life. Just as mathematics is embedded in real-life, art, and science, so are the laws of learning hidden in actions, thought and emotions. With careful observation of children playing video games, it was possible to discover a combination of abilities and skills which are made explicit and are described in this essay.

Keywords Learning environments · Simulation games · Dimensions of mathematics learning · Gagné's hierarchy of learning

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A Communication Learning Environment for Mathematics and Mathematics Teaching

The ubiquity of mathematics in nature, art, science, and music, is often found in many publications. Galileo's passage on the *Language of The Book of Nature*, which appears in one of his works, is one of the most significant observations that reveals the connection and link between mathematical language and what it expresses and is capable of expressing:

> The essence of the world (…) is written in this grand book (and I mean the universe), which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.

However, despite the fact that mathematics is such an intrinsic part of reality, it is not always easy to recognize it at first sight, or to be aware of it every time it appears. In order to be able to see mathematics and master the salient aspects that characterize it as a language capable of expressing concepts, patterns, structures, and both simple and complex phenomena, it is necessary to have keys at one's disposal with which one can open the doors to its world. According to Devlin (2007, p. 195), mathematics manifests in at least two ways, a natural one and an abstract one.

It can be hypothesized that *natural mathematics* lays the foundations for *instinctive, unconscious knowledge*, recognizable to those who are able to describe it in as much as they have studied it, have worked out its patterns, discovered some of its functions and formalized them within the context of a mathematical framework. Tim Pennings, mathematics professor at Hope College, Michigan[1], while observing his pedigree dog retrieve a ball thrown into the water from the beach, was able to notice that the strategies used by the dog followed a process that connected the running speed on the beach with the swimming speed. In fact, the stretch of beach that the dog covers before entering the water is the result of a complex elaboration, a characteristic of infinitesimal calculus when it is used to achieve something in *minimum time*! Thus it is discovered that the dog does not run in a straight line, and this is also true for basketball players who run after the ball as it drops, following a path that is unconsciously driven by an elaboration process which has been studied and formalized in *abstract mathematics*, and added to its theoretical foundations. It can be claimed that there is a type of mathematical knowledge that is used unconsciously and intuitively in a number of different situations, in all kinds of social and work contexts and in the interpretation of natural phenomena, as well as in organizational, technological and artisan processes. The examples arguing the case such as *instinctive, unconscious mathematical knowledge*, also involve humans. The key to understanding this *knowledge* requires a grasp of the language, of that very knowledge, that mathematicians have created a code for, within the context of their discipline.

[1] Compared to Devlin (2005), p. 13.

When instinctive mathematics is used and applied intentionally, a specific language is required, with patterns and schemes that characterize and give form and structure to the second type indicated by Devlin, namely, *abstract mathematics*. This is the type of mathematics that mathematicians deal with.

At this point things get complicated, because the essence of mathematics as part of human culture and its scientific evolution have made it possible to answer the question: "What is mathematics?" (Courant and Robbins 1972; Kline 1985; Maracchia 1975), in different epochs, starting with conceptions and convictions, as well as the results obtained from research on its foundations with the answers changing several times over the course of history (Devlin 2003).

It is appropriate to remember that in Gödel's 1930 work, he redefines the expectation of finding certainty at the base of mathematics (Lolli 2002). The various hypotheses of being able to lay the foundations of the discipline by finding a rigorous, formal justification, along with the definitions and deductions on which it is based, are refuted.

> This is the story today (…). The teaching programmes or publications that reduce it to a form of logical reasoning are unilateral and wrong. But neither is it possible to present mathematics as a science that reveals the secret and simple structures of nature, such as the language of natural laws, since we now know that the relationship between mathematics and nature is much more indirect and complicated, and that nature is not as simple as was once believed. On the other hand, if mathematic representation cannot always simplify the complexity of nature, in what way is it useful? One might as well describe facts using ordinary language. This accumulation of problems makes it even more difficult to answer the question about the nature of mathematical objects. (…) However, over the centuries mathematics has come a long way, and it could not be more different from how it was in Euclid's day…. (Isdrael and Millán Gasca 2012, p. 22)

A dear friend of mine, who directed scientific documentaries, suggested making a distinction between the knowledge required to build a film camera (which involves the expert engineers who design it, the technicians and finally, the different types of factory workers), from the knowledge required to make film documentaries. I have often thought that the level of knowledge concerning a 'film camera' is an asset, but I remain in doubt as to how much basic and useful know-how is required to manage this knowledge, and the actual skills required to make successful documentaries in terms of quality, effectiveness and efficiency.

Leaving metaphors aside, mathematicians increasingly agree on the differences between mathematics, its epistemology and its application, while their views differ greatly in the matter of the teaching methods and educational psychology skills that may characterize the process of teaching and learning mathematics. Obviously, it is necessary to take into account the type of school being addressed; but *teaching* mathematics today, across the board, means guiding students through a discovery of the world of numbers, forms and space. This is needed in order to shape logical reasoning, stimulate critical thinking and arouse elements of awareness, guided by processes that give direction and structure to the mind so as to stimulate the skills of analysis, synthesis, coding, decoding and transcoding, which the study of mathematics requires. If, on the one hand, the process of abstraction, generalization and transfer underlie the construction of mathematical language (Resnick and Ford 1981), it is precisely this construction that requires the activation of metacognitive and socio-relational processes (Fregola 2010a) that, in turn, can lead to the capacity for problem solving, and making decisions in uncertain situations, which are the hallmarks of the social complexity of these times.

These elements form the basis for self-efficacy (Bandura 2000) and can contribute to autonomy from self-limiting schemes that were automatically learned in childhood when a set of convictions is formed about oneself, others, and the world (Berne, Montuschi, and Fregola). As is well known, *self-efficacy* is the ability to carry out adequate and appropriate actions for specific purposes and enables a person to anticipate how to behave in order to reach set goals. Self-efficacy is based on the personal conviction of knowing how to use one's abilities efficiently, by putting oneself in the condition to act in a transformative way (Olmetti Peja 2007, p. 46). *Autonomy* implies a fundamental passage, namely, that of being, and being able to perceive oneself, as an active agent capable of making choices and decisions, elaborating strategies that form the basis of behaviour that facilitates the possibility of relating with the outside world, from an evolving perspective which focuses on the person's relationship to his/her own learning…of mathematics (Fregola 2011).

In this respect, the words of Emma Castelnuovo (1964), the academic who was a forerunner in Italian mathematics teaching, are still valid today. She wrote:

> Philosophical doctrines, pedagogic research, psychological investigation and social issues have led to statements concerning fundamental principles on general teaching doctrines that cannot be ignored if one is to take a serious approach to teaching

and she continues the preface to her book by specifying that if, on the one hand, a maths lesson is:

> (…) usually boring … the young people who come out of our secondary schools today very often have the idea that mathematics, on the one hand, consists of pure mechanical processes, and on the other hand, is a perfect construction that is, at this point, complete, so they wonder whether there are still discoveries to be made in this discipline.[2] (p. 1)

Thus, when researching the answer, "what mathematics is", and taking into account Devlin's views, there are at least three different directions that can be taken in educational research.

1. the rediscovery of natural mathematics;
2. the reconstruction of mathematical language and the definition of the minimum levels of formal mathematics knowledge; and
3. the reconnection between these two worlds so that another type of knowledge can arise, which leads to developing awareness and understanding of the mathematics that is hidden in nature, technology, human behaviour and mastering mathematical skills that are abstract or generalized and organized in a formal structure which enables it to be used in a variety of contexts and for studying, for people for whom it has central importance in their working lives and in research.

[2] Little more than half a century has passed since then, and the fact that these affirmations can still be applied to present-day students, leads one to wonder about the effectiveness of educational psychology research on the mathematics teaching-learning process.

These points can be shared with mathematicians and professional communities who, out of necessity or choice, must have access to, and occupy partly or completely, this mathematical ‘space’.

This paragraph aimed to highlight the fact that the educational matter for the learning of mathematics is still an opened theme that became more and more complex. Some relevant considerations on this subject open a question on which the subsequent paragraph will be articulated.

Can a Place Exist for a Systemic Mathematical Teaching Approach?

Educational research and educational psychology (Perini 1997) have developed considerably (Camaioni and Di Blasio 2007), and as a consequence, teaching has found fertile terrain for creating its own theoretical foundations. “Maths teaching should both refer to and encapsulate the entire body of research *on disseminating mathematical… knowledge,* combining at the same time the art of teaching and the scientific studies on this art” (Brousseau 2008, pp. 40–48). According to Bruno D’Amore (1999), mathematics teaching cannot be reduced to a simple application of disciplines such as psychology, sociology, pedagogy, linguistics or history, even if it must have a connection with these (Heritage 2007; Heritage et al. 2009). One of the reasons of this inefficacy in the integration between the various disciplinary sectors is because teaching mathematics requires a solid mathematical background which allows for the necessary reflection on the theories studied. In fact, when we talk about teaching mathematics we talk about improving the quality of teaching, but the mathematical contents are still at the heart of the matter as well as the teaching techniques, along with the ploys used by experienced teachers that, through trial-and-error, become established practices and confirm the idea that teaching mathematics is an art, even though it has given rise to interesting results (D’Amore 1999, p. 31).[3]

There is another element to be taken into consideration. Thanks to developments in neuroscience, it is possible to assume that there is a certain similarity between *natural mathematics* and the world of *learning*. In fact, as far as we know, the laws of learning are often set in motion regardless of the level of knowledge. Slowly but surely, models are used that allow one to understand some processes thanks to which knowledge, memory, attention, motivation and emotions can be developed. But there are still few experimental indications on how this meta-knowledge may be used to organize, facilitate or direct learning in general, and in particular, learning mathematics.

To paraphrase Rivoltella (2012), one may say that an interesting contribution to the development of mathematics instruction can be found in the relationship

[3] Bruno D’Amore (1999) proposes an analysis of possible interpretations of mathematics teaching by moving in the direction of a theory of teaching mathematics itself.

between neuroscience and knowledge technologies which, nevertheless, must be connected with themes and motives and with teaching activity.[4] An example of this can be found in studies on numerical intelligence (Lucangeli et al. 2007), thanks to which it is possible to detect learning paths which integrate the results of research on cognitive and neuropsychological patterns with traditional teaching methods (Lucangeli et al. 2011).[5]

It is useful to our argument to remember that, in 1905, the psychologist Binet proposed an intelligence model with a procedure which was based on a comparison between the mental age and the actual age. In 1912, the psychologist Stern perfected this method by introducing the concept of intelligence quotient (IQ) as the relationship between the mental age and actual age, multiplied by 100. The American psychologist Howard Gardner (1989, 1995, 1999) distinguishes nine types of intelligence and, among these, proposes a logical-mathematical intelligence, which he defines as the human mind's capacity to resolve any problem by following the principles and rules of Logic. It is characterized above all by the ability to discover, invent and reason, and proves to be particularly useful in many human activities, both intellectual and pragmatic (Olmetti Peja 2007). This completes the possibility of understanding a person's internal aspects, which would allow those who work in the field of mathematical instruction to draw on knowledge that could contribute significantly to the devising, planning and management of mathematical learning environments. This would take into account not only the interpretive models that come from mathematics, but also models that can provide indications about what could be defined as instinctive meta-knowledge that stems from the workings of our mind and gradually reveals itself.

The hypothesis that motivates our work is that it is possible to place the relationship between the student and his/her own learning at the heart of the teaching-learning process, in so far as *the encounter with knowledge activates processes that border on intention and surprise* (Fregola 2011, p. 104).

Philosophical doctrines cannot ignore the needs and characteristics of digital natives (Prensky 2001) and the evolution of virtual epistemology (Lévy 2005), that introduces the need for further research (see Fregola 2010a) into how natural mathematics and the world of abstract mathematics are both separate and connected.[6]

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[4] Compared to previous paragraph.

[5] Interesting applications of this research can be found in special teaching methods; in particular as an example of integrating the various disciplines (mathematics, psychology, neurosciences, teaching, anthropology). It could be interesting to consult the text by Biancardi et al. (2003), Franco Angeli, Milano.

[6] With reference to the course *Teaching mathematics for integration* that the author holds at the Faculty of Primary Education Sciences at l'Università di L'Aquila for future, special education teachers who must plan individual courses for children with special needs. The aims of the course can be summed up as follows: to provide a program which integrates mathematical knowledge with disciplinary and pedagogical knowledge in mathematics, which can be applied to the principal difficulties in learning and various disabilities; to help teachers acquire a methodology which includes designing, planning, implementing, testing and evaluating mathematics teaching programs, characterized by a process of abstraction, representation and formalization with reference to a logical-mathematical language; and to cultivate an open attitude towards the mathematics teaching-learning process, by getting teachers to experience standard situations that follow the

Pedagogical research could help reduce the risk of establishing teaching-learning models based on convictions that do not allow for innovation, and have not been tested by rigorous research methodology which ensures its validity and measures its effectiveness (Lucisano and Anna Salerni 2002), as well as providing real processes that are efficient and inexpensive and can be realistically applied in schools.

Psychological research*,* that in fact goes much further beyond the general reference to Freud, quoted by the author, given the evolution in research and the constructs available[7] for the teacher's use, while at the same time respecting the pedagogic parameters which guarantee that the child's inner world is protected. This can be achieved by concentrating more on the impact from emotional, affective and socio-relational variables[8], rather than increasing the range of activities.

Social questions in relation to which, the paradigm of complexity (Morin 2001), and liquid modernity (Bauman 2009, 2011), provide interpretive keys which project the analysis of mathematical learning needs into a dynamically evolving sociocultural context. As regards to this context, as well as mathematical knowledge, learning processes could also play an important role. For learning mathematics, in its various phases of the learning and growth cycle, requires and also makes it possible to strengthen, as *organization and thought forms*, models and patterns[9] that are useful for the future, as well as providing personal tools to work with.

The communication learning environment sets out to provide a place where the prerequisites can be defined, starting with integrated learning backgrounds which, in a situational approach, can give direction to the process of conception, planning and realization in mathematics teaching environments[10], within a complex system of mathematics teaching methods (Olmetti Peja 2010). These are structured dynamically by connecting phenomena from multiple sources, through which the perspective of educational psychology is currently being redefined.

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principles of graduality and transcoding, thus helping create learning environments for special needs cases and their teachers.

[7] Mentioned here are some examples referring to themes that have already been intentionally implemented, which guide the process of didactic decision and the organization of the teaching-learning process: multiple intelligences diffusion; self-efficacy; the ego-states, the stimuli and psychology games in class; growth models; the attachment model.

[8] In this regard compare <http://www.eatanews.org/wp-content/uploads/2012/09/ethics-code-feb-13th-edit.pdf>.

[9] In this context, 'pattern' refers to Piaget's use of the term. See Liverta Sempio (1998, p. 150).

[10] An important program was left by the International Commission for the Study and Improvement of Mathematics Teaching (in French CIEAEM, Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques), founded in 1950, among whose members were the mathematician, educationalist and philosopher, Caleb Gattegno, from the University of London, the French mathematician Gustave Choquet (President) and the Swiss Jean Piaget (Vice-President), psychologist and epistemologist. Using updated teaching methods they attempted to establish a connection between three fields of knowledge which, at that time, were evolving rapidly, in the hope that this would contribute to, "creating a society where people would be able to use mathematical reasoning and its tools to act rationally and develop a capacity for critical thinking, both as citizens and future scientists. Such a humanistic perspective in mathematics education should have been a safeguard against both technocratic behaviour and ideological blindness" (*50 anni di CIEAEM: dove siamo e dove andiamo? Manifesto 2000* per l'anno della matematica) (Fregola 2010a, p. 13).

Mathematics Learning Environments

The term ‘environment’ is used in the broad sense of the word here. To be certain, it indicates a place that is either physical or virtual, where the arrangement and positioning of people is determined by the structure of the place. Technology also plays a significant part in this ‘environment’, and in turn is affected by, and has an effect on, how space and time is organized and conceived.

The concept of environment is meant as a *mental place*, defined via the characteristics of the assignment proposed, which requires specific actions, suitable relational methods and an assessment process which not only takes into account the results, but also the support provided by the teacher (scaffolding) and, in a more general sense, the emotional climate and cognitive styles that come into play, and are also an intrinsic part of the learning environment. In this sense, according to Antonietti (2003), the concept of learning environment overlaps with the concept of setting, by integrating the physical elements inherent in the learning process, the planned objectives and strategies used to achieve them, in an organic and coherent system. Inside this environment, the complementary relationship between teacher and pupil takes shape, which is still an asymmetric relationship type (Carletti and Varani 2007).

It is well known that mathematics, more than other subjects, presents difficulties regarding its structural characteristics that often brings out a child’s anxiety about being unable to learn and triggers *the fear of not being good enough*. This manifests itself into the form of underestimating the worth of mathematics itself, the teacher, the teaching method, and above all the child’s ability to learn (Fregola 2010c). Between 2003 and 2009, we carried out field research which involved roughly 180 children from fourth-grade primary school, from eight classes, followed by about 100 children from third grade, from four classes. The aim of the research was to study the Drivers, studied in Transactional Analysis[11]. Emotional Drivers in Transactional Analysis, are automatisms which can be detected through five behavioural profiles that often go unnoticed. They are neither right nor wrong; they come into play without any intentional control on our part. If there is strong emotional involvement which arises from the situation or the objectives at stake, they are more likely to emerge. It is possible, for the most part, to learn and recognize the Drivers and use the functional aspects that characterize them. The five Driver profiles are as follows: be perfect, make an effort, hurry up, be strong, please people. One of the

[11] Transactional Analysis (TA) is a humanistic-existential branch of psychology, introduced by Eric Berne in the 1950s–1960s. TA is a psychological and social theory based on the philosophy of mutual wellbeing and on a construct that involves studying three ego states of the personality, each of which is defined by Berne as a coherent system of thoughts, feelings and behaviours. The ego states are not roles; they are psychological and phenomenological realities. In every person there are three ego states that are defined as Parent, Child and Adult, which are recognizable according to distinct types of behaviour. Transaction is the unit of social exchange in communication that takes place between people’s ego states. The phenomena that emerge in the process of interaction can be read, partly recognized, and acted on with greater awareness and intention, thus leading to a more effective exchange based on the principle of expressing oneself in the best way possible when relating to another. The effectiveness of TA in education and learning is the subject of substantial research (Montuschi 1993, 1997; Fregola 2011). See also: <http://issuu.com/mathetica/docs/semestrale0>.

assets of Transactional Analysis is the terms it uses, as many of them can set off a process of understanding, also due to the fact that some ‘technical’ meanings come from everyday language. For example, it is not difficult to assume that meticulous behaviour, extremely precise use of language when speaking and a rigid and upright posture can be traced to the Driver ‘be perfect’.

Looking at the teaching-learning process of two-digit division calculations and related research, it was possible to detect that through recognizing the children’s Drivers and their own Drivers, the teacher can intervene in order to reduce the negative effects, which manifest as negative emotional states which, in turn, reduce the possibility to organize thinking in an effective way (Fregola 2010b).

For example, the fear of mathematics, the impression of feeling inadequate, the rage towards the teacher, uncertainty, frustration, hostility, and ineffective competition (Fregola 2010a, p. 3).[12] The general hypothesis behind the research is that it is possible to integrate didactic practices that have been consolidated with an approach that takes into account emotional and relational skills, which often go unnoticed by the teachers and students, within a mathematics teaching-learning strategy based on integrating Transactional Analysis in didactic communication models.

It was possible to observe the social dimension that is present in a learning environment, confirmed by the fact that it is “a place where learners may work together and support each other as they use a variety of tools and information resources in their guided pursuit of learning goals and problem solving activities” (Wilson 1996).

Setting Up a Learning Environment[13]

Setting up a learning environment requires keeping various interactive aspects under control, some of which must necessarily be agreed upon with the pupils so that they may become effectively responsible and involved in managing the process. Salomon (1996) systemizes the elements that make up a learning environment:

- physical environment (e.g., spaces available, functional layout of classroom);
- times;
- the participants working inside the environment and relationships which establish the relational and operative climate;
- expectations;
- behaviour, rules, and agreed commitments;
- tasks and activities; and
- tools or artefacts; object of observation, reading, reasoning, manipulation.

These factors must contribute, each one in their own specific way, to organizing environments that should have a series of characteristics, which various authors have attempted to outline (Crismond et al. 2008). Black and McClintock (1996) propose the following key aspects:

[12] Research protocol can be found here: <www.mathetica.it>.

[13] As regards to the content of this paragraph, it is worth noting Laura di Giovanni’s unpublished thesis: *Videogames and learning environments*, written for the Primary Education Sciences degree course (2010–2011).

- observation of artefacts anchored in authentic situations;
- construction of interpretations based on observations, and constructing arguments for the validity of their interpretations;
- materials contextualisation;
- peer collaboration on the same processes;
- cognitive apprenticeship in observation, interpretation, contextualisation;
- multiple interpretations that enhance cognitive flexibility; and
- gaining transferability by seeing multiple manifestations of the same interpretations.

Regarding this matter, Lebow (1993) indicates the following principles:

- fostering personal autonomy and control over learning by supporting self-regulation, and by proposing subject-matter relevant to the learners;
- creating a learning context that supports the development of personal autonomy and relationships;
- embedding the reasons for learning into the learning activity itself;
- supporting learning feedback, promoting capacities and aptitudes which allow the student to take increasing responsibility for the process of reorganizing his/her knowledge; and
- supporting the learners' tendency to engage in intentional learning processes, thus encouraging the strategic exploration of errors.

In particular, the theory of learning provided by constructivist epistemology has helped define an indicative framework which the teacher can intentionally and consciously draw on when he/she is in the position to plan and create a mathematics learning environment. There are times when traditional educational values (replicability, reliability, communication, control) contrast with the primary values of constructivism (collaboration, personal autonomy, generativity, reflexivity, active engagement, personal relevance, pluralism); but it is precisely for this reason that they can provide precious indications when deciding on teaching methods.

Lastly, according to Savery and Duffy (1996), the principles for planning learning environments should:

- anchor all learning activities to larger tasks or problems;
- support the learner in developing ownership for any type of problem;
- design an authentic task;
- design the task and the learning environment to reflect the complexity of the environment;
- give the learner ownership of the process used to develop a solution;
- design the learning environment to support and challenge the learner's thinking;
- encourage testing ideas against alternative views and alternative contexts; and
- provide opportunity for and support reflection on both the content learned and the learning process.

A conception *of learning*, that focuses mainly on the interpretive activity of the subject, emerges from various factors and models, discarding the concept of *truth* in favour of a consensus gained by comparison and dialogue. As far as the mathematical contents are concerned, this aspect must be emphasized so that the learning process—much as it is personalized and focused on the student so as to promote personal autonomy, self-regulation and control over learning—needs to match the specific formalized level required by mathematical language and by the capacity to master elaborative procedures and problem solving. Nevertheless, with *instructional design*, it is very difficult to fully implement the various principles of constructivism. For these reasons, research today aims to interpret constructivism in many different situations, contexts and content domains. The researchers Carletti and Varani (2007, pp. 32–51) believe that not only are the general features that typify learning environments important, but also some strong core factors which are intertwined in various ways, such as the tools used to organize knowledge, group work, metacognitive reflection, the use of technology and a strong focus on the choice of discipline issues and methods of evaluation.

Thus, from the factors that have been highlighted (Carletti and Varani 2007, p. 31), it emerges that the definition of a learning environment that takes into account the characteristics of mathematics[14] requires a vast repertoire of skills and approaches. It requires researching borderline fields of integration between different disciplines; it shifts in the direction of renewed methodological rigour, which is able to synthesize the contributions coming from different science disciplines, starting with its own epistemology, and phenomena that can directly or indirectly affect the process of mathematics teaching-learning. Thus, the choices that govern the conception, planning and use of learning environments require the teacher to focus his/her attention and decision-making process on aspects that have already been altered due to a process of change, which involves the integration between tradition and innovation.

More specifically, all this requires renewed attention to a variety of relative notions, namely:

- learning;
- the role of the learner;
- the social dimension;
- the dimension of organizational and financial resources of the school;
- ways of structuring the task environment; and
- developing self-awareness about the knowledge-building process.

These aspects provide the opportunity to support the intentions that drive the learning process by organizing activities that contain the reasons for learning, boost intrinsic motivation and encourage exploration in the student's own learning growth. This can give rise to remarkable results; learning mathematics enhances cognitive, metacognitive and affective abilities, which encourage reflection and monitoring of the material learned and the methods used to promote a sense of self-efficacy and autonomy (Fregola 2011).

[14] In the introduction to their book, *Pensare in Matematica*, Isdrael and Millán Gasca (2012), write: "…teaching base concepts in elementary form requires mastering their subtleties and the countless difficulties that have been addressed over centuries of reflection and elaboration. What is directly taught to children may seem like nothing much in terms of the amount of concepts and methods used, but the clarity and effectiveness of the teaching comes from a background of in-depth understanding that, even though it remains behind the scenes, plays a decisive role".

Simulation Games and Learning Environments

Angela Piu writes:

> Planning a simulation game for mathematics learning in primary school may involve preparing an accurate representation or model from real life that presents a problematic situation which can lead the children to activate a process of construction and discovery guided by mathematical concepts, rules or structure. The interaction with the real life model takes place by assuming roles that require carrying out specific actions, activating behaviour that is coherent with the context of the task in question and manipulating material that has been organised according to the rules and aims of the game. (2010, pp. 112–130)

In a simulation game, a scenario is constructed by reducing the complexity of the real-life situation to its salient aspects, and the characters are established along with some rules. The scenario changes continuously as the game evolves. The participants interpret the characters and own objects which are useful for that context. The roles that are interpreted and the objectives to be reached are based on predefined rules that indicate the limits and the amount of freedom that the child can work with without violating the rules of the game. The strategies represent a combination of different possible solutions and moves with the players, who are immersed in the situation and 'identify' with the part that can achieve their aim. The rules guarantee that the game unfolds in a 'coherent' fashion and establish the type of moves that can be made. However, as the game is carried out, the players' capacity to interpret the role emerges, which also brings out unwritten socio-relational rules that are shared or pertinent to the needs of that given context. The phases and actions are organized as the simulation develops and have been established in the rules beforehand, as already explained.

The representation of different situations and role changes, instead, allow for the exploration of new reasons and perspectives, helping to change points of view and attitudes.

The Component Parts of a Simulation Game Project

Planning and setting up a learning environment where learning mathematics takes place is determined and defined by various elements, namely: the educational aims, the learning objectives and nature of the content along with the specific characteristics of the students and the schools' organizational constraints.

When planning and setting up a simulation game, due to the nature of its component parts, one requires his/her own nature to take a creative approach which must be developed within the context of a complex methodological background. This explores the educational psychology models that can lend rigour and legitimacy to planning and operative decisions, which also take into account the characteristics of relational interaction (Fig. 1). Here are some examples:

- there is a difference if the specific contents refer to *understanding* a mathematical concept, rule or structure, rather than *memorizing* it, which means that there is *knowledge* (Bloom 1983) about how they are expressed specifically in formal mathematics language;

Fig. 1 These pictures were used with children from the 2nd year of primary school as a stimulus to get them to distinguish between regular and irregular images during a simulation game, whose purpose was to introduce the concept of area and surface area, space and volume

- there is a difference between whether one intends to *apply* the concept, rule or structure in problem solving situations or have them recognized in real contexts instead;
- there is a difference between a simulation game that involves using the computer and one that uses a real-life situation with concrete materials or materials that have been set up for the task beforehand; and
- there are differences between a game that is designed for a single student and one that involves working in a group (Gentile 1998, 2000, 2008).

Despite the variety that exists among the different simulation games, they are all based on certain specific elements.

1. **Aims:** they represent the perspective for educational values, for general cognitive skills and abilities that one sets out to foster and develop in the pupils.

2. **Learning objectives:** the cognitive performance that pupils must be able to produce in response to mathematical contents which refer to concepts, rules, procedures, processes, structures, and models.
3. **The simulation model:** The definition of a model gives necessary structure to the chosen simulation.
4. **The setting:** the context in which the simulation takes place, namely, where the dynamics of the simulation are carried out and worked on by those taking part. It is used to establish the 'reality space' in which the action takes place, which helps to give a clearer definition to the roles assumed by the protagonists, since it helps to make the essential relationships and characterizations of the roles more explicit.
5. **Subject area or problematic situation:** the subject or the problematic situation being addressed, which the participants in the simulation activity have to tackle.
6. **The purpose of the game:** the operative goal that the participants can reach through the actions and strategies they will adopt and put into practice.
7. **The roles:** the participants, as protagonists and actors, who will have to interpret various circumstances by tackling the well-structured and sometimes complex situations provided by the games. The roles played are exactly those mentioned, a part of the structure of the game, which are assumed by the participants. These can be assigned—rigidly defined by rules and objectives from the very beginning—or they can be functional, which means that they take shape as the action unfolds on the basis of a general indication of the objectives, so they are liable to change during the course of the game.
8. **The documents:** used to provide information clearly, with reference to:

   - the roles to be played and possible introduction of other roles;
   - the setting;
   - the principal problem and any other difficulties that may come up during the simulation;
   - rules of conduct, how much power they are allowed, strategies that are not permitted; and
   - time and spaces.

9. **The materials:** can be used in the manipulation activities in the simulation game and function as intermediaries between reality and the world of mathematics. As well as providing a concrete reference for mathematical concept, they are also concrete models which are more abstract than the perceived situation, and less abstract than formal symbols (AA VV 1965; Dienes 1971; Post 1971). The materials can be taken from already existing materials or can be implemented or created especially for the game, in cases where suitable materials for the simulation game are not commercially available. If one decides to create prototypes of materials, it is necessary to pay attention to the constraints imposed on producing these materials, in terms of the time it takes to produce them, technological factors, and human and financial resources.

10. **Materials for the role-play:** specific tools may be used during the role-play (e.g., maps, relief maps, posters, charts, cards) in order to give the participants all the necessary information provided by direct experience. They can be used to represent the simulation and, as the activity unfolds, to show the effect produced by different decisions taken during the game.
11. **Assessment system:** includes assessment criteria on the basis of which a 'score' is applied to the various results produced by the participants' actions during the game. It is virtually a method which shows the results of most of the decisions made during competitive games or games with a lot of restrictions, where the rules carry a lot of weight.
12. **The final discussion (or debriefing):** the concluding summary stage. After the activity has taken place, the final discussion makes it possible to put the simulation in the right perspective, it ensures that the experience is made good use of and is brought into full awareness. The moment of discussion is a key element in that it allows the participants to discuss the results, to compare different opinions and analyze the results achieved and actions taken, so as to analyze, systemize and generalize the contents and mathematical processes which were dealt with during the game. The debriefing can be organized in a number of different ways, from an informal discussion to a structured one, to other forms of reports or written comments about the experience.
13. **Assessment:** the gathering and analysis of data which allows one to establish whether, and to what extent, the set goals were achieved, namely: whether the activity brought about the changes (as regards to knowledge, attitude, and abilities), which were originally proposed in the list of objectives. It also concerns ways of checking the simulation process through using checklists that provide the process descriptors and indicators of single items of knowledge, abilities and skills that portray the concepts, rules, processes, structures and models that one aims to reveal and define in mathematical language. Another object of assessment can be the emotional aspect, the motivation in terms of participation and involvement.

Are Video Games Learning Environments for Mathematics?

It is complicated to make generalizations about this issue in that there is a wide range of video games available on the market and web portals and, to be sure, some of them are for didactic use. Nevertheless, it is evident that the aim of video games is to entertain and there is no explicit indication that one can take into consideration, tout court, a deliberate focus on learning[15], at least as far as formal scholastic

[15] It is interesting observing that the Anglo-Saxon neologism *edutainment* is a fusion of the two words 'educational' and 'entertainment', and expresses the fundamental principle of video games that can enable *learning through playing.*

learning is concerned. During our research, we established that there are at least two functions that can be assigned to video games: (a) defining a learning environment for mathematics concerning calculations, counting, exploring space, problem solving, and decision-making processes; and (b) defining learning environments which concern *learning processes* that are useful for learning mathematics.

From a structural and functional point of view, video games meet many of the requirements that are necessary to be used as learning environments. But before they can be used in this way, they must be inserted into a design process specifically constructed with explicit aims in mind in order to become a tool that informs the students and makes them aware. It is common knowledge that students spend a lot of their time playing video games; it is an activity that challenges and provokes them, is highly stimulating and allows them to search and create problem solving strategies and decisions, and above all, gives structure to the time spent playing.

We asked ourselves what motivated students to spend time on such an exclusive and engrossing activity as video games[16]. Caillois (2000), referring to the game in itself, maintains that people only play these games if they want to, when they want to, and for as long as they like, and that if they were obliged to take part in a game it would immediately cease to be one, in that it would turn into an obligation. Moreover, it would lose its fundamental characteristics, namely, the fact that the players devote themselves to the game spontaneously and for the sole purpose of enjoyment, having the freedom, every time, to choose rest or a productive activity. This observation leads one to the reflection that if video games are to keep the specific qualities of the two possible functions for developing learning, it is necessary to create a relational approach that does not lead to the paradox "be spontaneous" (Wazlawick 2008). The order indicated by the imperatives "you must!" and "Be!" is incompatible with the concept of spontaneity. If, for example, the teacher instructed students to play video games "to develop problem solving skills", it would be asking them to *obey* by being spontaneous. This natural and legitimate behaviour would turn into a sort of paradox that would diminish the magic and playful aspects of the game. Thus, integrating video games into a learning environment requires using skill that comes from a methodology which regulates forms of interaction and relational exchanges between the students.

Caillois subdivides traditional games on the basis of four broad categories, according to the level of competition present, along with chance, mimicry or vertigo that he calls, respectively: Agon, Alea, Mimicry, and Ilinx.[17] Maybe some other categories are necessary, for those games that can restore the sense of feeling able to make decisions, resolve, learn and coordinate emotional, cognitive, metacognitive,

[16] To study this interesting new field more deeply, compared to Aarseth (2001).

[17] The Latin word *Agon* is the spirit of competition; *Alea* indicates the game of dice. The players are completely passive in that victory is only a matter of destiny; there is no ability skill, patience, or training involved. *Mimicry* (the mimicry of insects) for the author, this can be found in man's love of disguising himself, dressing up, wearing a mask, and playing a part. Games involving illusion or an imaginary aspect come under this category. *Ilinx* is the last type of games which are based on the quest for a sense of vertigo.

psychomotor and relational aspects in one single environment, that provides immediate feedback and allows one to connect the results with one's own inner dialogues. In fact, the motivation for playing video games, their limits and strong points, the risks and opportunities that they bring to the learning process and educational processes, depend both on the type and characteristics of the video games, and on the context and the relational contract that the teacher…and the parents can propose (Mangia 2009).

All of this provides a whole range of possible explanations, as well as the search for new explanations concerning phenomena which as yet have no references about the outcomes, effects or results that using video games can have on the processes of mathematics learning. As a matter of fact, most video games, a little like the traditional game Snakes and Ladders[18], entail a process where a goal must be reached, full of challenges to overcome that call for reflection, memory, the discovery of new knowledge, along with further questions which one must find the answers to. Very often, the places and times in which the characters move are reconstructions of geographic places and historical periods, futuristic and fantasy settings. During a video game session the player's logic skills, which are needed to move through the maze-like stages of the game, are continuously stimulated, representing a real stimulus for the mind (Gardner 1999, 1995, 1989). Children can perform a complex activity which, on reaching the final goal, "provides rapid and immediate gratification, which serves to boost self-esteem and self-confidence" (Maragliano et al. 2003).

Some Skills and Abilities Which Are Covertly Developed and Stimulate the Mathematical Mind

Observing children play a number of video games that have some connection, directly or indirectly, with mathematical content, it was possible to notice how, during a video game session, many factors come into play involving skills and abilities that belong to the mathematics learning process.

Here is a suggested list that is still in the development phase:

- ability for abstract thinking;
- ability to make generalizations;
- sense of logic;
- critical thinking skills;
- ability to analyze;
- ability to synthesize information;
- coding, decoding, transcoding abilities;
- transfer skills;
- problem solving abilities;
- capacity to make choices; and
- capacity to make decisions in uncertain situations.

[18] Compared to Marrone (2009).

For example, many video games and simple simulation games[19], which gradually become more complex in terms of mental calculation and calculation skills[20], can be found on the Internet. Depending on the objectives that determine the choice of video game, it is possible to focus on one or more abilities or skills. The video game in Fig. 2 was used to stimulate analysis skills and to develop abilities for coding, decoding, transcoding numerical symbols, and correctly solve arithmetic operations.

Ability: to resolve operations correctly and quickly
Content: rapid mental calculation
Age: 7 years and upward
Speed: slow normal fast
Result: from one figure from one or two figures
Operations: …. all ……

Fig. 2 The wolf and the hare

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[19] Some games have been included in an interesting paper Nesler (2007).

[20] See, for example: <http://www.matematicamente.it/giochi_e_gare/gioca_con_la_matematica/lupo_e_lepre%3A_calcolo_mentale_rapido,_7%2B_anni_200804113058/>.

The ways in which these abilities and skills are expressed during play are many and varied. Some of them can be observed directly, others indirectly, through intercepting variables which makes it possible to understand the specific abilities or skills being used, in any case, it is possible to pin down the indicators which help the teacher to recognize them. By indicating one or more of these skills or abilities as the specific aim of the game, it is possible to intentionally focus on the function that the game itself can provide in the learning process.

In our opinion, so that a teacher may intervene to influence the learning process without "breaking the spell of playing", two conditions are necessary. First, because it is necessary to have experience in the psychology of learning which allows one to draw on theoretical references that can guide the decision-making process for teaching and the actions taken as a consequence. Secondly, in order to carry out intentional observation, it is necessary to avail oneself of specific tools that are provided by research methodology and experimental pedagogy. However, suitable training is needed in order to use them properly, which does not come from knowledge but from specific experience in this field. For example: the *ability to analyze* can be measured by observing the child during play, he/she identifies the information present in different formats and languages and uses them to structure a choice, to reach a goal provided by the game model. He/she compares the different alternatives available by making appropriate assessments in terms of the relationship between cost/benefit, efficacy/efficiency, the effectiveness of actions taken in order to complete the model and go on to the next one.

Measuring the *ability for abstract thinking* requires a more complex definition of indicators. An approach we were able to experiment with, related to a transcoding pattern, is given in the following:

- Level 1: phrases are used with some references to the concept, rule, structure;
- Level 2: some examples or phrases are used that refer to the concept, rule, structure in order to explain it;
- Level 3: a connection is established between the concept, rule, structure, by relating it to a sensory-motor level: actions, manipulation of everyday objects;
- Level 4: the concept, rule, structure is referred to using references to the formal process that may represent them; and
- Level 5: the concept, rule, structure is referred to using mathematical language and an explanation is provided in student's own words.

Is Gagné's Hierarchy Still Relevant Today?

A contribution towards educational programming, which entails the use of video games, can be found through examining and carrying out an integrated reworking of Gagné's hierarchy (1973). The original definition given in Gagné's hierarchy of learning is a set of specified intellectual capabilities that have a relationship to each other, and these possible relationships are highlighted by using a specifically designed learning task.

Every hierarchical block contains the description of a capability and the formulation of every capability is expressed in behavioural terms (i.e., it indicates, as precisely as possible, what type of behaviour must be expressed by the student who is using *that particular skill*). For Gagné it is not enough to define the abilities and knowledge to be put into practice; it is necessary to identify the relationship order between them which allows for optimal progression that gives structure and form to the teaching process. In a traditional learning environment, task analysis refers to the possibility and prospect of subdividing a subject and a wider-ranging set of objectives with a series of intermediary activities which are more limited and more easily transferred to educational procedures. In an environment where video games are used, one possible way of developing this analysis is by resorting to a retrospective construction of a learning hierarchy that represents a 'network' of concepts, rules and structures that are found in video games. The skills are considered to be hierarchically connected if, while learning one of them during a specific task, the one that is deemed 'easier' produces a positive transfer for the one that is considered more difficult. According to another interpretation, two skills are considered to be hierarchically connected if, by being able to use one of them during a task, the more complex one, one can use the other one that is considered less complex. In either case, the key elements can be attributed to positive transfer, along with the fact that one skill is needed in order to acquire another one. Leaving aside the nature of a video game, it is evident that the child's immersion in the game moves along two axes that we have defined the analysis/synthesis axis and the problem solving/decision-making axis. Gagné's hierarchy has provided a useful support tool to help define the beginnings of a broader hierarchy, which has been integrated by our observations and can serve as a guide to pinpointing the capabilities that are used during a video game.

It is true that we focused on video games, but the real purpose is to study video games in order to connect the affinities and isomorphism between playing video games and educating a mind that generates mathematical thinking and, as we say in the mathematics field….vice versa.

The most important aspect to underline here is that the levels of Gagné's hierarchy were formed in relation to the learning theories around at that time. Gagné's model aimed to integrate learning theories with the choice and definition of teaching strategies in a way that allowed for the variables[21] investigated by every theory, which have an effect on the decision-making teaching procedure. However, Gagné's model does not explicitly consider the metacognitive processes, the relational dynamics of affective and emotional variables that can exist in a communication learning environment. This is also due to the way that research has evolved which, in

[21] This refers, in particular, to the variables identified by Bloom (1972, 1979), when setting out his theory of scholastic learning: cognitive input capacities, affective characteristics and quality of education. The quality of education is the variable that is closely linked to teaching skills, which allow the teacher to contribute to and encourage learning by using the input variables and the capacity to analyze and plan the teaching.

the early 90s, had not yet started to define certain constructs related to sociocultural variables and Gardner's[22] multiple intelligences. For this reason we integrated into our work some dimensions relating to the study of variables which, even though they have always been part of the communication learning environment, were not as yet observed by using theories of reference that were only introduced more recently. At the same time games and gaming (Huizinga 2002) did not have virtual environments and the natives were not yet *digital natives*.

The following shows the key aspects of intellectual abilities indicated in Gagné's hierarchy in a list that does not claim to cover everything, but indicates, in particular, some examples of basic mathematics terms. Tables are a tool and can be used as a guide in the design of a video game as a learning environment. In fact video games activate factors that remain hidden. Teachers/students are not always aware of the function that these factors have in the learning teaching process.

Intellectual Dimension

- **Discrimination:**[23] distinguishing between different parts of the environment, colours, shapes, different sizes, measurements, structures and distances.
- **Concrete concepts:**[24] analytical skills, recursiveness, consistency, observation skills, classification skills, recognizing and using quantities, understanding and using [geometric shapes, different sizes, measurements, spatiotemporal indicators].
- **Regole:**[25] calculating skills, capable of using mathematical language, spatial perception organization skills; understanding and using symbolic language, and understanding the semiotic language proposed in the game.
- **Higher-order rules/problem solving:**[26] ability to make abstractions, capacity to synthesize, capacity to make choices, strategies and learning through trial-and-error, logical thinking skills, resolving problems by using problem solving strategies, knowing the interface and purpose of the game, asking questions, knowing how to analyze and resolve problems autonomously.

Observing the behaviour of differently aged children, while they were playing video games, it was possible to identify some skills/abilities they used. These are often used by children without them being self-aware. The hypothesis is that these abilities are the same ones needed to learn concepts, rules, and mathematical structures. Thus, consequently, they can be stimulated and potentiated with a guided use of video games. We are certainly talking about selected games. The skills/abilities we are referring to belong to the cognitive, psychomotor, metacognitive and socio-relational dimensions of the learning process.[27]

[22] Compared to Footnote 7.

[23] Compared to Gagné (1973), from p. 193.

[24] Compared to Gagné (1973), from p. 210.

[25] Compared to Gagné (1973), from p. 249.

[26] Compared to Gagné (1973), from p. 257.

[27] A relevant contribution has been provided by Laura Di Giovanni and da Maria Eledia Mangia. The idea and the initial findings come from observations I made while watching my sons playing video games.

Cognitive Dimension
Coding-decoding-transcoding skills; making decisions in uncertain situations; analyzing the material proposed in the game; taking prompt action; always proceeding in a consistent manner; experiencing the cause-effect relationship; checking decisions that have been made; working out the rules of the game; understanding and following procedures and rules of the game; setting off inference processes; setting off induction processes; identifying the objective of the game; exploring worlds that are imaginary or distant for the player; narrating video game experiences; collecting and elaborating information through visual activities; managing interdependent variables; managing different information and different types of information at the same time; using selective attention; realizing spatial integration; understanding and mastering depictions of reality; managing images in a two-dimensional space; using serial processes; spotting and recognizing obstacles by understanding their nature; distinguishing the figures from the setting by concentrating on the scene and place where the action unfolds rather than on weak identifying signals provided by symbols, objects; paying attention to the moving images on the screen; analyzing sounds; reading and understanding instructions; letting oneself be led by one's imagination; adopting different points of view; selecting and recognizing objectives in order to reach the next levels; single actions (managing time, organizing space); bodies of knowledge (reactivating previously acquired information and knowledge; explaining the plot of the virtual story; understanding iconic and spatial requests; dealing with symbolic systems).

Psychomotor Dimension
Eye-hand coordination; combined variations of fine motor movements of fingers and hands; combination of different types of information (visual, tactile, proprioceptive, lexical semantic, musical) that generate plans of action which, on reaching a satisfactorily efficient level, are stabilized, represented 'mentally' and memorized; pairing and combination of movements; automization of movements through using conscious control while tactically managing movement; anticipation and Reaction [space] (forwards/backwards, right/left, near/far, inside/outside, above/below, long/short, high/low, wide/narrow, open/closed, large/small); anticipation and Reaction [space] (before/after, at the same time, fast/slow).

Metacognitive Dimension
The video games proposed encourage the player to ask questions and make decisions; they foster planning skills and enable the player to make a retroactive check of the mental route taken during the playing of the game. They develop and increase meta-memory skills, namely, awareness of strategic behaviour and memory systems that are activated during the game; they trigger a self-regulatory process which involves controlling and adjusting strategies on the basis of previous knowledge. From this, one can deduce that the use of video games leads to the development of metacognitive skills such as: identification of the elements that he/she is about to deal with (comprehension); awareness of one's own strategies, own resources and resources available, and possible application thereof (estimation and forecasting); strategy planning (planning); control and supervision of performance (monitoring); results assessment; self-regulatory feedback on game process.

Socio-Relational Dimension
Peer comparison; formulating questions; providing answers; regulating and understanding one's own and others emotional states; narrating the video game experience; asking for help; knowing how to ask for collaboration and cooperation in order to pursue the objective; knowing how to communicate effectively and efficiently.

It has been observed that the presence of a finishing line and multiple difficulty levels are among the main attractions of video games. Moreover, the range of difficulty and the stimulating situations have a motivating effect which activates agency, resilience and the search for increasingly time-saving strategies.

Learning Environments and Education of the Mind Within a Complex System

Roughly 30 years ago, Lucio Lombardo Radice (1986, p. 147) wrote:

> The Wright brothers' biplane, Guglielmo Marconi's crystal radio set, the Lumières' cinema, the ultramodern representatives of the youth of the fathers of that time, are seen by our children as belonging to an archaeological museum. For them Enrico Fermi's wonder from 20 years ago, the atomic pile, has the patina of an ancient monument. Even mathematics, even "divine" and "perfect" geometry have had their foundations renewed, developed and revolutionised, so that excellent formalist teachers, trained in modern teaching 20 years ago, have to study everything from the beginning again in order to come into line with the new programmes for their pupils.

He concludes by highlighting the necessity to plan an education for the mind. Many years have passed since the publication of the book by Michele Pellerey (1983), *Per un Insegnamento della Matematica dal Volto Umano* (*For mathematics teaching with a human face*). I often discover in those pages, between the lines, the outlines for defining and managing learning environments, that, besides having a human face, deal with technology and the web 2.0. As things happen during the interaction between teacher and child, the teacher is often unaware of their behaviour, thoughts, emotions and reference systems that stimulate behaviour, thoughts and emotions which cross the border line between the child's external world and inner world who, in turn, is constructing his/her own frame of reference.[28] Learning environments stimulate learning for abstraction and for immersion, and one could put forward a hypothesis that mathematical language could be considered the beginning of a journey distributed over time, in relation to which the terms 'concrete' and 'virtual', which gradually change form, become the 'frames' of a semantic network (De Keckhove 1991), whose most efficient expression coincides with the child being able to master all of the significant mathematical constructs. In this way, the development of 'mathematical thinking' can be understood as constituting a set of necessary skills for handling mathematical learning in the most varied scopes of application and within a system of self-regulating skills to be used in the child's own learning process. Indeed, the redefinition of the sense of identity and belonging, and as a result, the relationship each person has with their own learning, is already underway (Fregola and Iozzelli 2013).

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[28] In this context, Frame of Reference is seen from a Transactional Analysis point of view. It is a construct introduced by Jaqui Schiff, written up by Viene, which is defined as a structure of associated responses that integrate the different ego states in response to specific stimuli. It provides a person with a perceptive, conceptual, affective system, as well as one of action, and is used to define the Self, Others, the World … the perception of the self that is learned.

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