Advances in Mathematics Education

Angelika Bikner-Ahsbahs
Christine Knipping
Norma Presmeg *Editors*

Approaches to Qualitative Research in Mathematics Education

Examples of Methodology and Methods

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Editors
Angelika Bikner-Ahsbahs
Christine Knipping
Faculty 3 of Mathematics and Computer Science
University of Bremen, Bremen, Germany

Norma Presmeg
Mathematics Department
Illinois State University
Normal, IL, USA

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Preface

Over the last three decades, a variety of qualitative research methods have emerged within mathematics education. In 2003, two volumes of ZDM were dedicated to such methods in mathematics education with a focus on interpretative research, to promote a discussion about qualitative methods. Those two volumes necessarily presented only a selection of the range of research available. This book provides a different selection, including chapters based on research since 2003 and research approaches not included in the ZDM volumes, with some overlap in areas of particular importance. It continues the discussion, bringing additional depth and variety and including the close relationship between theory and methodology.

In his book on doing qualitative research, Roth (2005) describes how participating in research practice helps students to understand methodologies in a much better way than general how-to-do descriptions are able to achieve (see also Roth 2006). Given that doing research is more than can be written down as a procedure or as a description, how can a book offer an in-depth insight into such a methodologically enriched process?

In handbooks on research in science and mathematics education (English 2002; Kelley et al. 2008; Lester 2007; Kelly and Lesh 2000), we find chapters on methodological considerations (Cobb 2007; Cobb and Gravemeijer 2008; Lesh 2000, 2002; Schoenfeld 2002, 2007; Silver and Herbst 2007), but detailed descriptions on how methodologies are substantiated in a specific project, how they are implemented to investigate a research question, and how they are used to capture the research objects are normally missing. One exception is a monograph edited by Teppo (1998). Therein scholars have outlined general descriptions of methodologies that they illustrated by examples from their own research. For example, Goldin (1998) described task-based interviews on problem solving this way, and Pirie (1998) exhibited her search for a methodology and her decision-making process in research, concluding as follows:

> …this choice of methodology, should not be undertaken hastily. We must review imaginatively the range of possible approaches to answering our research questions. One approach may at first sight appear seductive, but it is in the details that the connections between questions and successful explorations lie. (p. 96)

The editors of this book share Pirie's and Roth's views and have looked for a way to make such a search for adequate methodologies easier to accomplish by documenting and offering insight into a variety of different methodologies and how each of them can be used in research. However, this was not the only reason for publishing this book. We also felt that like every research discipline, scholars in mathematics education also should communicate their new developments in research methodologies and make them accessible to others in order to sustain a critical debate about methodologies in our field. This is especially demanding for qualitative methodologies because they are deeply intertwined with the respective research objects and research goals. To solve this problem, we have chosen a format that devotes to each research methodology one part of the book. Each part includes both a description of the theoretical and methodological underpinnings of the research approach and a concrete research example of how the approach is used in practice. Some parts describe the underpinnings and the example in two separate chapters, while others take an integrated approach. This structure means the reader can use the book also as an actual guide for the selection of an appropriate methodology, on the basis of both methodological depth and practical implications. The methods and examples presented are not intended as procedures to imitate, but rather they illustrate how different methodologies come to life when applied to a specific question in a specific context.

The exception to this structure is Part XI which presents three alternate approaches to design-based research. It illustrates how cultural and institutional contexts may not only require distinctive and sophisticated methodical adaptations, but also can imply fundamentally different methodological and theoretical underpinnings. Design-based research in the tradition of Realistic Mathematics Education in the Netherlands, didactical engineering in the French didactical culture, and conducting educational design research in the US context to support system-wide instructional improvement demonstrate substantially distinct understandings of design-based research. The theoretical underpinnings described and the examples of the three contributions in this part illustrate this.

Many of the methodologies presented in this book are also used outside mathematics education, but the examples provided are chosen so as to situate the approach in a mathematical and educational context. Some of the methodologies are well known in mathematics education, while others provide innovative approaches to research that readers may not have encountered previously. The contributors come from a wide range of backgrounds within and outside mathematics education, including both experienced and new researchers.

In the first part, Anne R. Teppo provides an introduction to grounded theory as a methodology, beginning with Glaser and Strauss's seminal work in 1967. A clear layout of basic ideas and methodical principles allows the reader to establish a fundamental understanding of essential methods of grounded theory. Teppo's further discussion of variations of this approach by second-generation researchers then insightfully reveals the underlying, and sometimes diverging, methodological perspectives of grounded theory approaches. In the second chapter of Part I, Maike Vollstedt illustrates in the context of mathematics education how such a perspective can shape the way a grounded theory is developed methodically. Based on Strauss and Corbin's (1990/1996) outlines of grounded theory, she constitutes the concept of personal meaning from interview data, collected in Germany and Hong Kong in an intercultural study. Through a pragmatic interpretation of theoretical sampling, comparing codes, and using a coding paradigm, Vollstedt identifies different types of personal meanings and describes conditions of their emergence, which constitute significant elements of a theory of personal meaning in mathematics learning.

In the second part, Götz Krummheuer, Christine Knipping, and David Reid offer two different perspectives on reconstructing social interaction and argumentation in mathematics classrooms, both following Toulmin's theory of argumentation. For Krummheuer, argumentative learning is the essential research agenda, and so methodical analyses of students' participation in collective argumentation are central in his approach. Goffman's idea of decomposition of the speaker's role is a key element in this. While Krummheuer focuses on elementary classrooms and locally developed arguments, Knipping and Reid contribute a "global" model of argumentation, based on empirical research in secondary classrooms. As their focus is reconstructing entire proving processes in the mathematics classroom in this context, comparative methods that allow description of the "gross, anatomical structure" and rationale of the emerging global arguments are essential. Both chapters provide examples to illustrate the methodologies.

In the third part, Angelika Bikner-Ahsbahs shows how the construction of ideal types can be used as a methodological principle of theory construction. She first explains the underlying idea of ideal types, different kinds of these, and their role in theory development. She then illustrates methodical principles of ideal type construction and demonstrates how different heuristics for generating these can ground an emerging theory in empirical contexts. In her second contribution to this part, Bikner-Ahsbahs discusses an example of the ideal type reconstruction of epistemic processes in so-called interest-dense-situations. Key features of structures of these situations are singled out using an approach divided into four steps. Based on these characteristics, several ideal types are construed, providing theoretical insights into the dynamics of epistemic processes.

In Part IV, Luis Radford and Cristina Sabena present a methodology based on a Vygotskian perspective on semiotics. They describe the Vygotskian semiotic approach in terms of an interrelated triplet of principles, methodology, and research questions and refer in particular to two methodological constructs: the semiotic node and the semiotic bundle. In the second half of their chapter, they illustrate the semiotic approach with an example of the analysis of pattern generalization in classroom activity. The research reported in the example, concerning the role of words, gestures, and rhythm in the process of becoming aware of mathematical relationships, contributed to the development of the semiotic approach when unexpected data required the transformation of the theory, methods, and research questions.

In Part V, Tommy Dreyfus, Rina Hershkowitz, and Baruch Schwarz present Abstraction in Context (AiC) as a theoretical-methodological approach for researching students' knowledge constructions. Emergence of constructs that are new to students is investigated, taking into account the particular learning environments and their specific mathematical, curricular, and social components. The authors are especially interested in an integral approach that allows the study of learners' processes of constructing abstract mathematical knowledge, within a methodology based on the AiC theoretical framework. The main methodological tools of AiC are three observable epistemic actions: Recognizing, Building-with, and Construction. A specific example illustrates how these actions and AiC as a theoretical-methodological approach can be applied in a methodical way in research.

In Part VI, the networking of theories is proposed as a methodology by Ivy Kidron and Angelika Bikner-Ahsbahs. Both authors discuss and demonstrate how engaging different theoretical frameworks and models can allow for a more comprehensive understanding of concepts and phenomena on the one hand and the theories involved on the other. The authors argue that this can be done in a strategic, methodological way. Networking strategies and cross-methodologies are presented and illustrated briefly by research examples in the first chapter of this part. In the second chapter, one research example on combining the theory of Abstraction in Context, a cognitive approach, with the theory of interest-dense-situations, a social approach, pictures how the networking process is accomplished. The authors demonstrate how bringing these two perspectives together offers methodologically new ground for gaining insights into students' epistemic processes when learning mathematics.

Part VII offers a methodology for studying classroom processes of teaching and learning over significant spans of time. In the first sections of their chapter, Geoffrey B. Saxe, Kenton de Kirby, Marie Le, Yasmin Sitabkhan, and Bona Kang present a conceptual framework for understanding the reproduction and alteration of a common ground in classroom communities through time. This framework incorporates analyses at collective and individual levels, looking for collective norms and the function of individuals' use of representations. Later in the chapter, a 19-lesson sequence on integers and fractions is introduced as an example of design research based on the conceptual framework presented earlier. The organization of empirical analysis based on this framework is described. The empirical methods and techniques presented illustrate the innovative potential of this multilevel analytic approach, which is further discussed in the conclusion of the chapter.

Qualitative methodologies are the main focus of this book; however, in Part VIII, Udo Kelle und Nils Buchholtz point to the limitations of a purely qualitative approach. They critically review the continuing dispute about qualitative and quantitative research methods that overshadows research in mathematics education. Both authors question the restriction to either quantitative or qualitative methods, which they find particularly striking in research on teacher knowledge. They argue how a "mixed methods design" can enrich educational research in this domain. Based on data and results from an empirical study on a teacher training program in mathematics, they demonstrate how a mixed methods approach can mutually validate qualitative and quantitative findings.

In Part IX, a mixed methods approach to text analysis, "Qualitative Content Analysis," is introduced. This approach is well established within the social sciences, but it has only recently been applied within mathematics education. In the first chapter of this part, Philipp Mayring describes the theoretical background and methodical procedures of this approach to text analysis. He concludes by comparing these procedures with similar techniques in other methodological approaches, reflecting on strengths and weaknesses of each approach. In the second chapter of this part, Björn Schwarz addresses an example for applying qualitative content analytic methods in a study on professional competence for future mathematics teachers. First, he substantiates why this methodology was implemented into the study and then describes how this was done, while demonstrating the added value of involving inductive and deductive procedures of this methodology.

The idea of validation is critically reflected on in Part X. Ida Ah Chee Mok and David Clarke argue that methodologies required by cross-cultural comparative research are poorly served by the use of triangulation as a mechanism of convergence, but benefit from a wider understanding of triangulation that involves complementary accounts instead. Their argument is illustrated by examples taken from the Learner's Perspective Study (LPS), which examined patterns of participation in the mathematics classroom in 18 countries. Mok and Clarke offer a more in-depth look into how different forms of triangulation, including what they call cultural triangulation, are able to portray the variation of real class activities, by means of the description of two studies, namely, a study on lesson structures of classrooms in Hong Kong and Shanghai and a cross-cultural comparison of learning tasks.

Part XI, on design research as a research methodology, is divided into three chapters. In the first chapter, Arthur Bakker and Dolly van Eerde offer an introduction to design-based research as a specific kind of this methodological approach in realistic mathematics education, reviewing key features of it and how validity and reliability are interpreted. Illustrating and reflecting the pivotal methodical steps and the role of the theory of design-based research, they close with an example from statistics education.

In the second chapter of this part, Michèle Artigue considers didactical engineering in the French tradition as a case of design research. She describes the evolution of didactical engineering, its characteristics as a research methodology, and its close connections with the development of the theory of didactical situations. Current developments within this design culture are described, in particular the integration of a design element into the anthropological theory of didactics, and second-generation didactical engineering. Specific examples are used to illustrate methodological principles.

Educational design research can be employed at different levels, from the design of a single task to longer sequences of classroom activity and beyond. In the third chapter of Part XI, Erin Henrick, Paul Cobb, and Kara Jackson describe educational design research in the context of school system-wide instructional improvement. They expound the theoretical framework for design research at this level and its research focus on wide-scale instructional improvement. The authors discern that design studies at this level are interventionist in nature, and they describe how researchers address both the complexity of educational settings and the problems that various participants in those settings encounter as they endeavor to make improvements. Examples drawn from the MIST study (study on Middle-School

Mathematics and the Institutional Setting of Teaching), as one of the few design studies conducted at this level, are used to illustrate these points.

Taken together, these 11 parts provide a systematic account of a variety of directions in which qualitative research in mathematics education is moving, through an analysis of the essential interaction between theoretical and methodological aspects of this research. In each case, a description of a pragmatic example in which the methodology has been used brings these considerations to life, thus adumbrating ways in which certain methodologies bring certain issues to the fore. We summarize the connections between the parts in Part XII. The account is of necessity incomplete: As research in mathematics education continues to evolve, so do the tools with which researchers investigate their questions. Even as a snapshot of current research, the account is incomplete, because we have chosen to highlight developments in qualitative methodologies, with only a small glimpse into their interactions with their quantitative counterparts. However, the usefulness of this book lies in the juxtaposition, with practical examples, of accounts of theoretical and methodological aspects of qualitative mathematics education research that, taken together, illustrate the current state of the art.

Bremen, Germany

Angelika Bikner-Ahsbahs
Christine Knipping
Norma Presmeg

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Contents

Part I
Grounded Theory Methodology

Chapter 1
Grounded Theory Methods

Anne R. Teppo

Abstract The essential methods of grounded theory research, beginning with Glaser and Strauss's seminal work in 1967, are described. These methods include concurrent data collection and analysis, coding of data into concepts and categories, the use of interpretative frameworks, theoretical sampling, memoing, and the integration of categories into grounded theory. Variations in methods developed by second-generation grounded theory researchers are presented in the contexts of their methodological perspectives.

Keywords Grounded theory

1.1 The Development of "Grounded Theory"

Sociologists Barney Glaser and Anselm Strauss set out in their book *The Discovery of Grounded Theory* (1967) to describe a set of research methods that grew out of the authors' collaborative, qualitative study of the interactions between hospital staff and dying patients. Their particular research approach ran counter to the then prevailing social science techniques that focused on theory verification. Instead of using theory at the beginning of research to direct data collection, Glaser and Strauss's method begins with joint data collection and analysis in order to generate theory that "emerges" and is grounded in empirical data; theory that will "fit the situation being researched, and work when put into use" (Glaser and Strauss 1967, p. 3). Bryant (2009, para. 2) notes that the *Discovery* book "was first and foremost a manifesto, seeking to present a genuine alternative to the dominant quantitative agenda of the time."

The study of dying patients utilized a method of comparative analysis that was a standard tool in qualitative social science research in the 1960s. However, Glaser and Strauss (1967) developed this method further. Their purpose in using the technique went beyond creating rich descriptions *of* data to that of generating theory

A.R. Teppo (✉)
Independent Scholar, Livingston, MT, USA
e-mail: arteppo@gmail.com

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from data. The end result was a systematic set of techniques labeled the "constant comparative method" (Glaser 1965). Only later did their approach become known as "grounded theory" (Strauss 1991).

The publication of *The Discovery of Grounded Theory* represented the authors' first attempt at articulating their method. As Glaser (1998, p. 14) explains, "It took a lot of thought for Anselm [Strauss] and myself to figure out the 'Discovery' book." However, their book, while it introduced the techniques, did not provide extensive details on how to actually conduct similar research in the field.

Glaser and Strauss did not collaborate again after completing the study on dying and writing a set of four books related to this research. Throughout the years, as Glaser and Strauss continued to refine specific aspects of their methods through their work mentoring doctoral students, they developed separate variations of the procedures. The first book to clarify and further explain the methods of grounded theory was Glaser's 1978 book *Theoretical Sensitivity*. Strauss, in collaboration with Juliet Corbin, outlined his version of grounded theory in the textbook *Basics of Qualitative Research* (1990), which has since been revised through a third edition (Corbin and Strauss 2008). Anselm Strauss died in 1996, while Barney Glaser continues to put out books and readers on grounded theory through his publishing company, Sociology Press.

Anselm Strauss and Barney Glaser constitute the first generation of grounded theorists. Through their mentoring of a cadre of doctoral students, they laid the foundation for a second generation of researchers, who have subsequently gone on to refine, extend, and develop variations of the method that reflect changes in the qualitative research paradigm over the last 40 years (Morse 2009).

There are presently four seminal forms of the grounded theory method (Birks and Mills 2011); that espoused by Glaser and articulated through his writings from *Theoretical Sensitivity* forwards; the methods outlined by Strauss and Corbin in their 1990 through 2008 editions of *Basics of Qualitative Research*, a Constructivist perspective associated with Chamaz's work (2000, 2006, 2009), and an approach based on Situational Analysis (Clarke 2005). Further discussions of the different theoretical perspectives taken by these researchers are presented in Sect. 1.8 at the end of the chapter.

It is recommended that this chapter be read in parallel with Maike Vollstedt's chapter (Chap. 2) that details the use of grounded theory methods in an empirical interview study. Throughout the chapter references will be made to specific sections described by Vollstedt that illustrate the topic under discussion.

1.1.1 Overview of Research Processes

Within the variations in grounded theory research that exist today, there is a set of *essential methods* that characterizes all such research (Birks and Mills 2011). This set includes constant comparative analysis, open and intermediate coding, theoretical sampling and saturation, theoretical integration of codes and categories, and memoing. Additionally, a crucial aspect of this research is the concurrent and continuous nature of data generation and analysis.

Initially, data, such as interviews or field notes, are conceptually coded through constant comparison. As codes are generated, categories are also created to express commonalities among groups of codes. As the analysis continues, decisions of where to select more data (theoretical sampling) are directed by key ideas about the data that emerge through the constant comparison of codes and categories. Coding of new data into codes and categories continues towards the goal of identifying a core category that can account for the majority of the participants' behavior in the substantive area. At this stage, more abstract categories that express connections between the lower-level, substantive categories begin to emerge. These higher-level categories lead to the development of grounded hypotheses that explain relations among observed aspects of the area of study. Throughout the analytic process, memos are constantly being written to capture ideas and thoughts about codes and categories, relationships among concepts, emerging theory, and potential directions for further sampling.

The processes of coding data, abstracting concepts into categories, and theoretical sampling are on-going and interactive as the researcher continues to cycle through these steps towards the goal of developing theory grounded in the data. Constant comparison uses inductive reasoning to abstract concepts and categories from patterns identified in the data, and hypotheses are deduced from these patterns that suggest explanations about what is going on in the substantive area. The cycle continues as further data collection and analysis test the validity of the emerging themes.

Theoretical saturation is reached when no new data or coding produce any additional useful material. At this point, the process begins to integrate the categories and their properties into grounded theory. The memos, which have been continually recording the conceptualizations of the research process, are compared and sorted (either manually or by computer) according to how they relate to each other. The writing of the final product is aided by the information derived from the sorted memos; and the particular grounded theory that is developed is legitimized by how well it fits the substantive area, works to explain observed behavior, and has relevance to practitioners in the field. (See Chap. 2, Sects. 2.1 and 2.2, for a discussion of her decisions to use grounded theory methods to investigate a particular area of interest.)

1.2 Place of Literature Review in Grounded Theory

In contrast to research designed to verify theory derived from the literature, grounded theory studies do not begin with a formal literature review. Glaser (1998) stresses that reading other studies beforehand about the substantive area may lead the researcher to "see" what is not there rather than what actually is. Also, once data generation begins, the researcher may find that the categories created about the substantive area are different from what might have been postulated from any preconceived body of literature, and, therefore, an extensive preview may not be relevant to the final focus of the research.

However, it is impossible for the researcher to enter the field with an empty mind. He or she always brings a set of experiences and professional knowledge to the endeavor. In fact, it is this background that forms the basis for the researcher's sensitivity or "ability to see what is in the data" (Bryant 2009, para. 96). Birks and Mills (2011) suggest a "limited and purposive preliminary review" that can help orient the researcher to the general area of study, as well as provide some initial sensitivity towards conceptualizing the data.

As the research progresses, a literature search can provide an additional source of data for locating similarities and differences with the study's grounded categories. Such comparisons can enlarge the scope of the emerging theory, increasing its relevance to a larger set of conditions. A literature review conducted during the final stages of the research can be used to indicate how one's emerging theory fits into what has already been published in the field. The literature may confirm the researcher's developed theory or his or her theory may extend or go beyond that previously published.

1.3 Data Analysis: Open Coding

Grounded theory analysis uses the technique of constant comparison to render the data into codes and categories that reflect layers of abstraction based on phenomena and relations observed in the data. During initial coding, incidents, events and items of interest are identified and labeled with code names that reflect a particular conceptual aspect of each of these phenomena. As analysis continues, codes having similar attributes are grouped together into categories representing a higher level of conceptual abstraction. A second phase of analysis, sometimes identified as "intermediate coding" (Birks and Mills 2011), focuses on linking categories and subcategories together and articulating the relations among them. Section 1.3 discusses the first level of open coding. Intermediate coding, and the use of a coding paradigm within this level of analysis, are addressed in detail in Sect. 1.4.

Open coding starts as soon as the first set of data has been generated. This coding process consists of two analytic, meaning-making procedures, (1) asking questions of the data and (2) constantly comparing incidents. The goal of this process is to conceptualize the data into a collection of codified phenomena, or "substantive codes" that abstractly identify particular aspects of the empirical area study.

The field notes, observations, or documents, etc. are "fractured" into identifiable fragments, or *incidents*. These discrete parts, which may consist of a word or phrase, a complete sentence, or possibly a whole paragraph, are labeled or *coded* by asking *sensitizing questions* such as "What is going on here?" or "What are the actors doing?" in order to identify concepts that stand for particular incidents.

The researcher codes as many incidents as he or she can, using *constant comparison* (in addition to asking questions) to classify data on the basis of similarities and differences. Each newly identified fragment is compared to those already coded. Similar phenomena are given the same code name and new names are developed for previously unidentified aspects of the data. New incidents are also compared to those previously coded as the same concept to check for conflicts in the conceptualization of the represented phenomenon. Similar codes are grouped together to form *categories* that abstract properties common to the collected codes. In addition, *memos*, written during the coding process, note the original data on which each code was based and record researcher thoughts about salient properties linked to the conceptualization of that particular piece of data. (See Chap. 2, Sect. 2.5.3 for examples of open coding.)

1.4 Memoing

Memos are continually written during the on-going data collection and analysis cycles in order to record ideas and insights, *as they are triggered* by particular aspects of data, or by comparisons or "conflicts" in the developing line of thought. Memos provide "moment capture" (Glaser 1998), enabling the researcher to concretize a fleeting idea as it occurs. They are also "the running logs of analytic thinking" (Corbin and Strauss 2008). "Memos are not so much about specific incidents or events, but about the conceptual ideas derived from these. It is the denoting of concepts and their relationships that moves the research from raw data to findings" (Corbin and Strauss 2008, p. 123).

Throughout the research process, memos create an *audit trail* of the development of the analyst's thinking and the direction of theoretical sampling. They provide transparency to the research process; from initial concepts identified during open coding; through the growth of categories, their properties, dimensions, and relations that are generated during intermediate coding; to the final integration of ideas into theory. (See Chap. 2, Sects. 2.5 and 2.5.3, for examples of the type of information noted in her memos.)

Birks and Mills (2011, p. 55) emphasize the importance of memo writing as a method for keeping "accountable for your actions and decisions as the researcher facilitating" the research process. Memos provide opportunities to note instances where personal bias enters the analytic process (Corbin and Strauss 2008). Such circumstances may arise when the researcher becomes aware of an inconsistency or incongruence between the participants' and his or her interpretations of particular phenomena or events. Memoing about these contradictions brings the bias to the fore and promotes conceptualization that is more accurately grounded in the data.

1.4.1 Writing Memos and Using Diagrams

Glaser (1998) advocates the use of unstructured memos. Writing in correct English or with proper grammar is not important; the researcher should feel free to express his or her ideas in whatever form is comfortable and promotes the outpouring of ideas. A lack of restrictive rules and attention to form helps writers overcome any writer's block that may impede this essential analytic process.

At the same time, there are organizational precepts that help researchers manage and retrieve the ever-growing number of memos that accumulate as analysis progresses (Corbin and Strauss 2008). At the very least, the researcher should date each memo, create a heading, and indicate the document and raw data upon which the memo is based. It is also useful to include short quotes from an interview or segment of the original field note to remind the researcher of the data that generated the idea behind the memo. These data can be used later to illustrate aspects of the grounded theory in the final written product.

Besides memos, diagrams can also be used to support the on-going analysis (Corbin and Strauss 2008). Diagrams, which are visual, are naturally more abstract than raw data and, as such, they promote thinking at a conceptual, rather than at an empirical level. Diagrams are particularly useful in showing relationships between concepts. (See Chap. 2, Figs. 2.3 and 2.4, for examples of diagrams used to display relations between categories.)

Glaser (1998), however, cautions researchers about an over-reliance on diagrams as a way to explicate grounded theory. While the diagram may visualize relationships, it is in the write-up, or text, where the meaning of the relationship is made explicit. A diagram is "an aid to comprehending the meaning of the written theory. It is not a theory in and of itself" (p. 169).

1.4.2 Using Computer Programs

Various software programs have been designed to help *manage* qualitative data – allowing the researcher to store, search for, retrieve, and organize research artifacts such as interview data, codes, categories, and memos. However, these programs should be regarded as tools that facilitate, rather than replace the researcher's analytic thinking processes (Corbin and Strauss 2008; Birks and Mills 2011). It is the researcher that must decide how data are to be coded and creatively determine the meanings that emerge from the constant comparison of data to codes and categories.

In the third edition of *Basics of Qualitative Research* (Corbin and Strauss 2008), Corbin presents examples of data management and analysis that were generated using a particular computer program. These data and analyses are also available online to enable the reader to "work live" with the data and practice coding techniques. (See Chap. 2, Sect. 2.5.3, for examples of computer-assisted coding using the same program.)

Computer programs also facilitate the creation of an audit trail to help keep track of the researcher's analytic progress. The software makes it possible to organize, reorganize and diagram connections between codes and categories in many different ways as theoretical sampling and constant comparison continue. Memos, linked to these actions, help trace the researcher's reasoning during the continuous and concurrent analytic and conceptualization processes. In addition, the software program's ability to quickly access memos, codes, and raw data greatly enhance the write-up phase of the research. The writer can easily call up particular incidents from the raw data to use as illustrations and access memos and diagrams that describe specific relationships among categories.

1.5 Intermediate Coding and the Use of a Coding Paradigm

Since Glaser and Strauss first wrote *The Discovery of Grounded Theory* (1967), these authors, as well as the next generation of grounded theory researchers, have more fully articulated, described, and refined the basic grounded theory methods (Birks and Mills 2011). While the essential stages and processes have remained constant, different authors have, in some instances, employed different names to identify similar methods or to distinguish particular techniques. In the following discussions that outline essential aspects of ground theory, variations in techniques and terminology will be noted to clarify similarities and differences among these different authors.

Intermediate coding is the second coding phase. During this stage, coding becomes more focused as the researcher identifies a particular analytic direction. Categories are integrated as relationships among categories and sub categories are identified and the properties of categories become more fully developed. Data that were originally fractured into substantive codes are now put back together at a more abstract conceptual level in order to begin to synthesize and explain phenomena identified in the data (Birks and Mills 2011).

Various authors recommend different methods to focus the researcher's attention during this phase of coding. Glaser and Holton (2004, para. 55), using the term, "selective coding," suggest the researcher restrict coding comparisons to those "variables that relate to the core variable" in significant ways that can lead to the development of a grounded theory. Charmaz (2006, p. 58) also emphasizes a more selective approach to coding at this stage. Her "focused coding" describes using those "initial codes [that] make the most analytic sense to categorize your data incisively and completely." Once such codes have been selected, both authors emphasize the use of constant comparison to develop significant analytic categories and relations.

Strauss (1987, p. 32) describes a technique of more focused coding that operates in conjunction with a particular *coding paradigm. Axial coding* "consists of intense analysis done around [the 'axis' of] one category at a time, in terms of the paradigm items (conditions, consequences, and so forth)." In contrast to the other procedures described above, the inclusion of a coding paradigm that directs the researcher's analytic focus provides a more structured approach designed to develop analytic categories aligned explicitly within a particular social science perspective. While the term "axial coding" is used in Strauss and Corbin's 1990 and 1998 texts, the third edition (Corbin and Strauss 2008) places decreased emphasis on using this label to identify the processes of intermediate coding directed by a particular coding paradigm.

1.5.1 Heuristic Concepts

In this section the nature of the kinds of questions that might be asked of the data and the types of interpretations drawn from comparisons of codes and categories are examined more closely by considering the role that *coding paradigms* or "heuristic concepts" play in "the interpretation, description and explanation of the empirical world under study" (Kelle 2005, para. 31). A coding paradigm, while perhaps implicitly invoked during open coding, provides a particular theoretical perspective and set of heuristic concepts that structurally guide researchers as they begin to code for specific categories and identify relationships among categories.

As Kelle (2005, para. 39) notes, a crucial characteristic of a particular set of heuristic concepts is that it has "limited empirical content." That is, "heuristic categories cannot be used to construct empirically contentful propositions without additional information about empirical phenomena. This makes them rather useless in the context of [developing hypotheses in verification studies], but it is their *strength in the context of exploratory, interpretative research*" (emphasis added). Importantly, the use of low empirical content heuristic concepts makes it more difficult for a researcher to force data to fit pre-specified categories. The heuristic concepts, rather, provide "a theoretical axis or a skeleton" (Kelle 2005, para. 40) around which substantive data are coded to create categories and grounded theory. Blumer (1969, pp. 147–148; quoted in Clarke 2005, p. 77) also notes the value of what he terms *sensitizing concepts* in framing the direction of analysis: "Whereas definitive concepts provide prescriptions of what to see, sensitizing concepts merely suggest directions along which to look." At the same time, the researcher must also be aware that the structure that a particular heuristic "lens" provides may preclude the researcher from noticing other relevant phenomena.

Being able to identify for oneself an appropriate set of heuristic concepts may be problematic, however. Glaser (1998) recommends "reading vociferously" in other substantive areas within the professional domain of the research study in order to build *theoretical sensitivity* and accumulate a repertoire of *theoretical codes*. Yet the novice researcher remains at a disadvantage and may need a more ready-made coding paradigm with an explicit structure and set of procedural rules to move beyond the initial steps of coding and category construction in order to build grounded theory (Birks and Mills 2011). (See Chap. 2, Sect. 2.3 for her discussion on theoretical sensitivity and the identification of sensitizing concepts that were appropriate to her research question; also Sect. 2.5.3.)

1.5.2 Coding for Process

This section uses the coding paradigm developed by Anselm Strauss and Juliet Corbin (Strauss and Corbin 1990, 1998; Corbin and Strauss 2008) to illustrate how a set of heuristic concepts, framed within a specific theoretical tradition, provides analytic structure during intermediate coding. The example highlights the ways in which this structure uses a particular disciplined perspective to guide the construction of grounded theory.

Corbin and Strauss (2008, pp. 98–100) focus their coding procedures, intended to investigate complex social behavior, around the theoretical construct of *process* – defined as "a sequence or a series of actions/interactions/emotions taken in response to situations or problems, or for the purpose of reaching a goal as persons attempt to carry out tasks, solve certain problems, or manage events in their lives." This notion of process, when framed in terms of *relational* categories, also provides a heuristic coding device. That is, individuals (or groups, etc.) *respond*, in a goal-oriented way, to particular contexts or events *with* actions, interactions, and/or emotions *that result* in specific consequences. Thus, analyzing data for process is a way to "capture the dynamic quality of inter/action and emotions" (Corbin and Strauss 2008, p. 98).

Additional heuristic concepts that identify and deal with complex social process are operationalized through the structure given in the Conditional/Consequential Matrix (Corbin and Strauss 2008, pp. 93–95). The Matrix consists of a set of concentric circles, each representing a level of social interaction, moving from the outer-most macro level (representing international or global conditions) through intermediate levels (such as organizational or institutional) to the micro level where the action/interaction/emotional responses are located. Conditions at any level may affect participants or organizations at any other level, moving both inwards and outwards across the Matrix. Importantly, the specific entities that constitute the conditions and consequences at each level of the Matrix are not pre-determined, but must "emerge" from the area of investigation. In addition, the levels considered for analysis are determined by the "type and scope of the phenomenon being studied" (p. 94). Thus, while providing structure, the Matrix and its constituting concepts focus analysis within the particular paradigm in ways that allow the researcher to construct empirically grounded theory that explains the phenomena under study.

To actualize the coding paradigm, Corbin and Strauss (2008, p. 10) suggest some of the following prompts when analyzing for process; "What is going on here? What are the problems or situations as defined by participants? What are the structural conditions that give rise to those situations? How are persons responding to these though inter/action and emotional response?" Answers to these questions focus the direction of intermediate coding. Particular incidents or pre-coded concepts become abstracted and further categorized as conditions, others as consequences, etc., and the specificity of the categorizations reflects the particular *meaning* that participants assign to experiences (either reported on or observed) in the substantive area.

Although the procedural steps outlined above appear to be highly prescriptive, Corbin and Strauss (2008) stress that they are not intended to be "a recipe for doing qualitative research." Individual analysts must always adapt particular methods to fit the realities of their own work. Equally important, a researcher should carefully consider how any given coding paradigm aligns with his or her research goals before considering its application as a viable research technique. (See Chap. 2, Sect. 2.5.2, for her description of the creation of a research-specific coding paradigm, and Sects. 2.5.2 and 2.5.3 for examples of "axial coding.")

1.6 Delimiting the Study

As ongoing data collection, and open and intermediate coding progress in parallel, the researcher begins to identify key categories, deepen descriptions of their properties and identify relations among these categories. Instead of continuing to collect *all* data available in the field of study, the researcher can now begin to purposively seek out only those data that have the potential to further inform the development of these salient categories and their properties. The researcher begins to direct and delimit the work through *theoretical sampling* and the development of a *core category* until the coding process yields category *saturation.*

1.6.1 Theoretical Sampling and Saturation

Having a research purpose of theory generation, rather than that of theory verification or rich description, establishes a different set of criteria for the type of data that are collected. It is not necessary to collect *all* available data, or those that are considered *representative* of a general population in terms of certain properties. Rather, the initial groups or situations from which data are to be collected are chosen, not on the basis of existing theory, but because of their potential to generate theory about the substantive area under study. Once categories begin to develop through ongoing data generation and analysis, further data collection, through *theoretical sampling*, is directed by a search to learn more about these categories. (Chapter 2, Sect. 2.4 uses the term "chronological parallelism" to describe this ongoing, concurrent process of data collection, analysis and the development of theory.)

Theoretical sampling can be directed by questions such as, "*What* groups or subgroups does one turn to *next* in data collection? And for *what* theoretical purpose?" (Glaser and Strauss 1967, p. 47, their emphasis). Corbin and Strauss (2008) characterize theoretical sampling as "concept driven." For theory building, further data collection is not about persons; decisions of what and where to sample next relate to *concepts*. (See Chap. 2, Sect. 2.4, for her criteria for theoretical sampling.)

On-going, simultaneous data collection, coding and category analysis lead to refinements in existing categories and their properties and to further formulation of the emerging theory. This process, in turn, informs the direction for further theoretical sampling. "Data collection never gets too far ahead of analysis because … the questions to be asked in the next interview or observation are based on what was discovered during the previous analysis" (Corbin and Strauss 2008, p. 145).

Theoretical sampling should also seek for *variability* in data. Comparisons for similarities and differences across sites, as well as persons, promote density and depth in a concept's dimensions, properties, and its relations to other concepts. Glaser and Strauss (1967) suggest explicitly sampling for different kinds of data or using different techniques of data collection, creating *slices of data*. The variety offered by different slices of data provides analysts "different views or vantage points from which to understand a category and develop its properties" (p. 65).

Qualitative data may be collected from many different sources. Field evidence may, for example, consist of observations or interviews. Another fruitful source is written material and documentary data, such as letters, biographies and autobiographies, speeches, etc. The library provides an excellent source of documentary material and may be theoretically sampled for concepts derived from analysis, just as with any other research site. Secondary analysis can also be carried out on interviews or field notes previously collected by another researcher. Additionally, theoretical sampling may point the researcher back to previously collected and analyzed data in order to reexamine old data in light of further insights developed through later analyses.

Theoretical sampling ends when categories have reached *saturation*. At this point, no new data will yield additional useful information about the properties of any of the categories. Evidence of saturation of particular categories is also indicated by the presence of "interchangeable indicators" that refer to particular incidents all coded for the same category (Glaser 1998). In such cases, the particular evidence may be changed without affecting the conceptualization of the category.

1.6.2 Core Category

As theoretical sampling and analysis continue, one or more of the developing categories will emerge as a key representative of important aspects of the phenomena under study. These categories form the nucleus of the emerging theory, guide further data collection, and become the most saturated categories under additional theoretical sampling. At this point a *core category* is identified that "appears to have the greatest explanatory relevance and highest potential for linking all the other categories together. … [and to] convey theoretically what the research is all about" (Corbin and Strauss 2008, p. 104). (See Chap. 2, Sect. 2.5.4 for an example of core category selection.)

1.7 Theoretical Integration

Grounded theory does not consist of a set of dense descriptions of the phenomena under study, nor is it merely a list of well-developed categories or findings. "It is the overall unifying explanatory scheme that raises findings to the level of theory" (Corbin and Strauss 2008, p. 104). The scheme provides the "cohesiveness of [grounded] theory" in terms of an "overarching explanatory concept … that, taken together with other concepts, explains the what, how, when, where, and why of something" (ibid., p. 55).

The final phase of analysis that leads to the development of grounded theory involves processes of theoretical integration. Central to these processes are the identification of a core category, the achievement of theoretical saturation of this and other important categories, and the set of analytic memos that have been continuously generated throughout all phases of the research (Birks and Mills 2011). Strategies that can be used to facilitate theoretical integration include using theoretical codes (Glaser 1998), selective coding (Strauss and Corbin 1990), writing a story line (Birks and Mills 2011; Strauss and Corbin 1990), and sorting memos (Glaser 1998; Corbin and Strauss 2008). These techniques help the researcher identify and articulate the nature of the abstract relationships that connect the core category with other important categories; and, ultimately, integrate the categories and relationships into a coherent conceptual explanation of a particular aspect of the substantive area of study. It is time now to integrate the pieces. "You have fractured a story descriptively and are now putting it back together conceptually" (Glaser 1998, p. 194).

Glaser (1998) describes the final theory-building phase in terms of the identification of "theoretical codes." In contrast to "substantive codes," which consist of the categories and properties abstracted from the substantive data, theoretical codes "conceptualize how the substantive codes may relate to each other as hypotheses to be integrated into the theory" (Glaser and Holton 2004). Theoretical codes are formal concepts drawn from existing theory related to or tangential to the researcher's field of study. To be appropriate for use in the integrative phase of grounded theory-building, these theoretical codes must be at an appropriate low-content level, and must earn their way into the analysis as having "emerged" from the data "as much as substantive codes" (Glaser 1998).

Strauss and Corbin (1990) use the term "selective coding" to identify a set of processes leading to theoretical integration. Central to these processes are the identification of a core category and the orderly development of relationships to other categories. (See Chap. 2, Sect. 2.5.4, for an example of selective coding leading to the identification of a core category.)

Corbin and Strauss (2008), while no longer using the term "selective coding," continue to place the selection of a core category within the theory building phase of the research. As with their earlier text, they also suggest that authors write a story line as a way to start thinking about the integration process. This story usually consists of a few sentences that describe "what the research is all about." The authors suggest the question, "what seems to be going on here?" as a useful prompt to facilitate the flow of ideas (Corbin and Strauss 2008, p. 107).

1.7.1 Sorting Memos

The collection of memos that has been accumulating since the beginning phase of open coding provides an important resource for theoretical integration. While early memos may be merely informal descriptions, later memos will reflect a more mature perspective, "generally becoming more summary-like, abstract, and integrative"

(Corbin and Strauss 2008, p. 108). Thus, when it is time to begin writing up the theory, the "discussions in the memos" provide a summarization and suggest "major themes" for the writing process (Glaser and Strauss 1967).

The process of *sorting memos* can facilitate the organization and structuring of the final integrated theory. Glaser (1998, p. 189) describes this process as a form of comparative analysis in which memos are sorted into piles on the basis of how they relate "theoretically and substantively to other memos" (p. 189). As integration begins to emerge, it may take several iterations of sorting, comparing, and resorting before all the memos fit into an emergent theory. Corbin and Strauss (2008, p. 108) note that, if memos have been written within a computer program, they can be retrieved and sorted electronically in many different ways "until a logical theoretical structure is constructed." The sorted memo piles then form the outline for the final written product, where piles may represent separate chapters, sections of a chapter, or paragraphs of a book or paper.

1.7.2 Validating the Theory

At the end of the research process, it is important to validate the emergent grounded theory. Corbin and Strauss (2008, p. 113) note that, while "theory is constructed from data, … by the time of integration, it represents an abstract rendition" of these data. Therefore, the researcher must be sure to compare this "abstraction" against the raw data to ensure it fits and is able "to explain most of the cases." Alternatively, the researcher may ask participants in the field to read what has been written and give their perceptions of the fit.

Glaser (1998, pp. 18–19) defines the criteria for judging grounded theory in terms of fit, workability, relevance, and modifiability. *Fit* addresses the need for a concept to "adequately express the pattern in the data" for which it was created. *Workability* refers to the theory's ability to "sufficiently account for how the main concern of the participants in a substantive area is continually resolved." *Relevance* indicates that the theory does indeed deal "with the main concern of the participants involved." *Modifiability* reflects the fact that grounded theory is "never right or wrong;" it has the ability to be continually modified as new data are introduced. "New data never provides a disproof, just an analytic challenge."

1.8 Interpretive Frameworks

This section briefly examines the variations in essential grounded theory methods that have been developed over two generations of active researchers in terms of how these variations reflect different interpretative frameworks or sets of philosophical assumptions. Such an examination is informative for the beginning researcher since "the methodology subscribed to influences the analysis of the data" (Birks and Mills

2011, p. 4). Being able to ground a study within a particular theoretical framework also enables the researcher to justify claims about the nature of the data, articulate his or her position as a researcher in the field, and defend the legitimacy of the knowledge produced in the form of grounded theory (Bryant 2009).

To begin at the beginning, the description of grounded theory research, as laid out by Glaser and Strauss (1967), in *The Discovery of Grounded Theory* should be considered as a set of *methods* rather than as a *methodology* based on a particular philosophical or theoretical perspective. That is, while the book describes the procedures used to carry out the research, the authors did not explicitly situate these processes within a set of principles that determine the ways in which the methods are to be used and interpreted (Bryant 2009). Questions of ontology (the study of the nature of reality) and epistemology (the nature of justifiable knowledge) were not openly addressed.

Glaser and Strauss conducted their investigation within the prevailing post-positive research paradigm, at a time when the predominate perspective in qualitative research was that, while reality was assumed to exist, it could only be imperfectly perceived, and that the researcher was expected to be a passive, objective observer. Reflecting this perspective, Glaser and Strauss (1967) viewed data as something to be "collected," stressed the importance of open-mindedness in not engaging with relevant literature before entering the field, and characterized concepts and theory as being "discovered" or as "emerging" from the data.

Since then, the second generation of grounded theorists has endeavored to position the essential methods of grounded theory within the more recent post-modern turn. For example, Charmaz (2006) bases her version of grounded theory research within the constructivist perspective, which takes a relativist position. That is, reality is locally constructed and the researcher is seen as an active participant in the joint construction of data with those in the research site. Analyses are viewed "as interpretative renderings not as objective reports or the only viewpoint on the topic" (Charmaz 2009, p. 131).

Glaser, however, has continued in his writings to avoid espousing a particular theoretical paradigm in the belief that doing so restricts the broad potential of grounded theory. His use of a language of "emergence" in the processes of data collection and analysis has led others to situate him "as a critical realist researching within the post-positive paradigm" (Birks and Mills 2011, p. 5).

1.8.1 Pragmatism

Bryant (2009), noting the lack of theoretical grounding in the early writings of Glaser and Strauss, has proposed re-interpreting their methods within the pragmatist tradition, particularly that expressed by the contemporary pragmatist John Dewey (1859–1952) and the neopragmatist Richard Rorty (1931–2007). By doing so, "clear and concise criteria for developing and evaluating" the research techniques can be addressed (ibid., para. 60), including epistemological issues, such as "where codes, categories, concepts and theories come from, and the processes involved in their derivation and articulation" (ibid., para. 68).

For the pragmatist, knowledge is viewed as instrumental; a "tool" that is judged not in terms of "its universal validity, but [in] its usefulness in a specific context" (Bryant 2009, para. 72). Thus, grounded theory that has *fit*, *grab*, or *works* can be seen as meeting the pragmatist's criteria of a systematically generated, explanatory hypothesis. Theories are also regarded as provisional and can be altered upon further inquiry (Hookway 2008).

Pragmatists consider that, "our ability to think about external things and to steadily improve our understanding of them rests upon our experience" (Hookway 2008, p. 16). This view supports the idea of the grounded theory researcher as being an active participant, rather than an objective receiver of external stimuli. In addition, such a perspective supports the notion of *theoretical sensitivity*, or the way in which we "see the data."

The pragmatist tradition is especially appealing for grounded theory methods employed in practice-led disciplines in its emphasis on the relationship between theory and practice. Because theories are judged in terms of their utility within specific contexts, they have direct relevance to those affected by the situations under study. Here, Glaser's (1998) criterion of *workability* and *relevance* are particularly apt.

1.8.2 Corbin and Strauss Circa 2008: Pragmatism and Symbolic Interactionism

Corbin and Strauss did not locate their grounded theory procedures within a particular interpretative framework until the publication of the third edition of *The Basics of Qualitative Research*. Corbin and Strauss (2008) note that while much of the philosophical position described in the Introduction to this book reflects the position taken by Strauss in his book *Continual Permutations of Action* (1993), in the time that has passed since then, Corbin has also left her stamp on this exposition.

The basic assumptions of Corbin and Strauss's methodological foundation are derived from pragmatism and symbolic interactionism. Symbolic interactionism, as articulated by Blumer, states that "people act toward things based on the meaning those things have for them, and these meanings are derived from social interaction and modified through interpretation" (Society for the Study of Symbolic Interaction 2011). Further foundation for Corbin and Strauss's (2008, p. 2) perspective is drawn from Dewey's and Mead's assumption that "knowledge is created through action and interaction." Action and interaction also occur within social complexity. As Dewey states, "Neither inquiry nor the most abstractly formal set of symbols can escape from the cultural matrix in which they live, move and have their being" (as cited by Corbin and Strauss 2008, p. 3).

Corbin also recognizes the influences on her own theoretical perspective of contemporary feminists, constructivists and postmodernists; in particular, the relativist position of constructivists in which meaning and knowledge are co-constructed by the researcher and the participants. Related to feminist thinking, Corbin notes that "we must be self-reflexive about how we influence the research process and, in turn, how it influences us" (Corbin and Strauss 2008, p. 11).

1.8.3 Constructivist Grounded Theory

Methodologically, constructivist grounded theory takes the position that individuals' perceptions of reality and the meanings they ascribe to their experiences are constructed through human activity within particular contexts and social environments. This perspective affects the nature of the relationship between the researcher and the participants in his or her study, how data are perceived and the methods by which they are generated, and emphasizes the importance of a self-reflexive stance for the researcher throughout the research process.

The interview is considered an important method in constructivist research. It is a situation in which data are not "collected" but "generated," and facts are not "discovered" but, rather, meaning is co-constructed between the researcher and the informant. "Interviews are not neutral, context-free tools; rather, they provide a site for the interplay between two people that leads to data that is negotiated and contextual" (Birks and Mills 2011, p. 56).

It is impossible for the researcher to maintain the role of an unbiased, objective observer in any part of the research process, not just during data generation. The researcher's biases and subjectivity enter into all phases of analysis. Reflexive memoing can help the researcher understand "the multiple perspectives of multiple participants" including that of him or herself (Charmaz 2009, p. 132). Such self-reflection is necessary since we are "part of our constructed theory and this theory reflects the vantage points inherent in our varied experiences, whether or not we are aware of them" (Charmaz 2006, p. 149).

1.8.4 Situational Analysis

The variations in grounded theory methods presented above focus on social science research that investigates processes involving "'the knowing subject' as centered decision maker" (Clarke 2005, p. xxix). However, such analytic approaches do not fully meet the needs of mathematics education research, in which *mathematics*, the subject matter, should be an integral part of any research study. *Situational analysis*, developed by Adele Clarke (2005), offers a promising perspective and set of analytic tools for researching the messiness and complexity of mathematics classroom teaching and learning, in relation to a particular topic of study.

Adele Clarke (a doctoral student under Anselm Strauss) developed, beginning in the mid 1990s, her methodological approach as a way to extend and go beyond the analytic heuristics of traditional grounded theory (Clarke 2005). Situational analysis reflects the postmodern assumptions that all knowledge is socially and culturally situated and that situations are complex, messy, and interrelated. Further, inquiry is directed towards examining the "relations of knowledges to the sites of their production and consumption practices" (Clarke 2005, pp. xxxv, xxxiv), and that there can be "simultaneous 'truths' of multiple knowledges" (p. 19). Her methodology draws on social interactionism and constructionism, and also incorporates aspects of Foucault's notions of discursive fields, and ideas developed in action-network theory.

Situational analysis recognizes "the analytic importance of the nonhuman" (Clarke 2005, p. xxxiv). Within situations, nonhuman and human elements are involved in processes of "co-construction and co-constitution," and the nonhuman elements "structurally condition the interactions … through their specific material properties and requirements" (Clarke 2009, p. 203).

Foucault's emphasis on "how discourses are produced and how we are constituted through them" forms an integral part of Clarke's (2005, p. 147) approach. She draws on Foucault's concept of "discursive practice," which describes processes of action and change in terms of how "ways of framing and representing linguistic conventions of meanings and habits of usage together constitute specific discursive fields" (p. 54). Discourses also include disciplinary elements that, as formulated by Foucault, represent a "series of organizing practices that produce the rules through which individuals … make themselves up as subjects" (p. 56).

Clark's methodology is designed to address "the situation" as the basic unit of analysis and to consider the complexity inherent in such a unit.

> The fundamental assumptions are that everything *in* the situation *both constitutes and affects* most everything else in the situation in some way(s). … People and things, humans and nonhumans, fields of practice, discourses, disciplinary and other regimes/formations, symbols, technologies, controversies, organizations and institutions–each and all can be present and mutually consequential (Clarke 2009, pp. 209–210).

As a way to "empirically" construct the situation of inquiry from "multiple angles of perception" and understand "its elements and their relations," Clarke (2005, pp. xxii, 72) developed a form of "cartographic situational analysis," or set of mapping strategies. Briefly, the three types of maps and analyses are (p. 86):

1. *Situational maps* as strategies for articulating the elements in the situation and examining relations among them
2. *Social worlds/arenas maps* as cartographies of collective commitments, relations, and sites of action
3. *Positional maps* as simplification strategies for plotting positions articulated and not articulated in discourses.

Situational and relational maps should not be considered theory as such. Rather they provide "a systematic, coherent, and potentially provocative way to enter and memo the considerable complexities of a project" (Clarke 2005, p. 103). These maps spark deeper analyses, raising questions to be addressed and suggesting areas for further theoretical sampling. Over the course of the study, the researcher may construct many different situational maps and consider different sets of relations as the focus of the study is identified and particular elements are considered for closer scrutiny.

1.9 End Comment

The information presented in this chapter represents a cursory introduction to grounded theory methods. It is recommended that readers interested in using this approach to research find and work with an experienced grounded theory mentor. However, for those in a "minus-mentoring" situation (Glaser 1998), a good place to start is to deeply mine the original sources from which the material in this chapter was derived. Many of the authors cited, such as Clarke (2005), Charmaz (2006), Corbin and Strauss (2008), and Birks and Mills (2011) include extensive examples from actual grounded theory studies to illustrate particular techniques. See also Chap. 2 by Maike Vollsted in this volume.

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Chapter 2
To See the Wood for the Trees: The Development of Theory from Empirical Interview Data Using Grounded Theory

Maike Vollstedt

Abstract The way from empirical interview data to the development of theory is illustrated with reference to an intercultural study. This study was located in the field of mathematics education and focused on the development of a theory of personal meaning. Starting from only a rough understanding of what personal meaning might be, interviews were conducted with students from lower secondary level in Germany and Hong Kong. Due to the setting of the study in two cultures, a pragmatic interpretation of theoretical sampling had to be taken so that as much data as possible was collected to choose from throughout the analytical process. Data analysis followed grounded theory according to Strauss and Corbin (Basics of qualitative research: Grounded theory procedures and techniques . Newbury Park: Sage, Grounded theory: Grundlagen qualitativer Sozialforschung [Basics of qualitative research: Grounded theory procedures and techniques]. Weinheim: Beltz; see also Chap. 1). Therefore, different types of codes (in-vivo, empirically developed, and conceptual) as well as different types of coding (open, axial, and selected) were the result of constant comparison and writing memos. By comparing codes and using a coding paradigm, categories and concepts were developed so that the theory of personal meaning started to evolve from the data. The results of the analyzing process were an empirically grounded theory of personal meaning consisting of 17 different kinds of personal meaning on the one hand and an underlying theoretical framework that describes the surrounding conditions of the construction of personal meaning on the other hand.

Keywords Grounded theory • Personal meaning

In the previous chapter, Teppo gives an introduction to grounded theory and its development into different specifications of the grounded theory methods. In the first section she especially focuses on the four different lines of development of the

M. Vollstedt (✉)
Department 3 of Mathematics and Computer Science, Bremen University, Bibliothekstrasse 1, 28359 Bremen, Germany
e-mail: vollstedt@math.uni-bremen.de

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theory of grounded theory in the different schools following the two founders Strauss and Glaser respectively. The prevalent form of grounded theory used in Germany is the one elaborated by Strauss and his disciple Corbin as presented in their 1990 book *Basics of Qualitative Research*. Hence, I also followed their approach in my study so, accordingly, this article provides an example of the application of grounded theory to mathematics educational research following Strauss and Corbin (1990). As I actually worked with the German translation from 1996 of their 1990 book, I will always give both references throughout this text.

The empirical interview study presented here was carried out in Germany and Hong Kong (see Vollstedt 2011b). The aim was to find out and describe what is personally meaningful for the students when they learn mathematics or engage in mathematical problems in a school context and, thus, develop a theory of personal meaning (German: *Sinnkonstruktion*). The resulting theory about personal meaning was supposed to be laid out by different kinds of personal meaning. In the process of data analysis, I followed grounded theory methods according to Strauss and Corbin (1990, 1996). Hence, I also adopted their guidelines for the research process as well as their terminology.

When starting an empirical (interview) study, data often look very confusing and seemingly unrelated. One usually cannot see the wood for the trees at the beginning of data analysis. Therefore, we need a tool to detect a structure in the data that can be further worked out. Following grounded theory is a good possibility to finally see the wood for the trees—i.e. to develop an empirically grounded theory—as it combines methodological as well as methodical aspects (see Chap. 1) that provide guidelines throughout the research process.

This article may in some places diverge from Teppo's (see Chap. 1) description and terminology as she gives a review of the different streams of grounded theory in its different seminal forms. In contrast, I concentrate on one specific line of grounded theory. Nonetheless, it is recommended to read this illustrative chapter of the part alongside the previous chapter of this book as I will often draw back on the methodological basis laid out by Teppo.

2.1 Background and Focus of the Study

The study presented here was embedded in the Graduate Research Group on Educational Experience and Learner Development (German: *Bildungsgangforschung*) at the University of Hamburg, financed by the German Research Foundation DFG. The group's research focused on the question how children, adolescents and young adults act in situations of learning and instruction, how they interpret their learning tasks, and what can be done to encourage their educational development. Hence, in a school context, research on Educational Experience and Learner Development is primarily (empirical) research in teaching and instruction. The emphasis is placed on the perspective of the learners and their development. At the time I was member of the Graduate Research Group, we were especially investigating the role of meaning for learning and educational development.

Vinner (2007, p. 6) points out that humans have a "need for meaning" and that meaningful life and meaningful learning might have the same origin although they seem to be different concepts. If meaningful learning is a special case of "man's search for meaning" (ibid.), this specific human attitude does not disappear before entering the classroom. Meaning is also sought inside the classroom when students engage in learning and dealing with subject contents. Therefore, the question of meaning is posed time and again by students when they are learning mathematics. The demand for meaning in (mathematics) education has been detected for many years. Hence, meaningful learning has been identified as one of the major goals of education (ibid.). Consequently, one of the challenges posed also—if not especially—for mathematics education is to find convincing answers to the questions of meaning. In addition, if the aim is to make the learning of mathematics meaningful for the students, we need to ask what is meaningful to them rather than to impose some kind of meaning on them, which might be meaningful from a normative perspective but does not prove to be personally meaningful.

There is no commonly accepted interpretation of the term *meaning* in the field of mathematics education. The diversity of concepts is due to a mixture of philosophical and non-philosophical interpretations as the collection of articles of the BACOMET-group shows (Kilpatrick et al. 2005). Howson (2005, p. 18) convincingly distinguishes between two different aspects of meaning, "namely, those relating to relevance and personal significance (e.g., 'What is the point of this for me?') and those referring to the objective sense intended (i.e., signification and referents)". Hence, "[e]ven if students have constructed a certain meaning of a concept, that concept may still not yet be 'meaningful' for him or her in the sense of relevance to his/her life in general" (Kilpatrick et al. 2005, p. 14). Here, the mathematical meaning is obviously not interchangeable with the philosophical kind of meaning the student relates to his/her life.

As my study was embedded in the Graduate Research Group, I focused on the student's perspective. I therefore concentrated on Howson's first aspect of meaning and asked for the kinds of meaning that relate to the individual's relevance in the context of learning mathematics. To emphasize the focus of the learner's perspective over the, as Howson terms it, objective sense, I picked the term "personal meaning" instead of "sense-making" to denote the concept. By doing so I am also aware that subject-inherent sense-making sometimes also may be personally meaningful for the students. Accordingly, I did not look at what might be meaningful from a normative or domain-specific perspective, but—on the contrary—I investigated the aspects the students judge to be meaningful for them. As Kilpatrick, Hoyles and Skovsmose pointed out (see above), these do not necessarily have to (but may) be the same.

2.2 Realization of the Study

At the beginning of a study following grounded theory, there is no completed theory but—on the contrary—an open field of study whose relevant aspects become clearer and clearer throughout the research process. This was similar in my study. Prior to it, there was neither a developed theory about what personal meaning in a school context is, nor any empirical results about how personal meaning is constructed in a school context, nor any different kinds or types of personal meaning. The field of research was untilled except for a very rough understanding of personal meaning as described above. Therefore, the decision for reconstructive methods was reasonable—especially as the concerns of reconstructive studies are to understand a certain phenomenon better and to generate new theory that is empirically grounded (Jungwirth 2003).

To get a clearer glance at what is meaningful for the students in their learning processes, I conducted my study in two different learning cultures, Germany and Hong Kong. This decision offered the possibility of getting a sharper view on my own learning culture by being contrasted with a different setting I was not acquainted with. Stigler and Perry (1988, p. 199) describe this with respect to teaching practices as follows:

> Cross cultural comparison […] leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, we tend not to question our own traditional teaching practices and we may not even be aware of the choices we have made in constructing the educational process.

Similar to the teaching practices, we do not question our own beliefs and about teaching and learning when we do not reflect them against the background of another culture. Looking at another teaching and learning culture, thus, offers the possibility to reflect aspects that have been taken for granted beforehand and so to get a clearer picture of one's own culture, too. Hence, conducting a comparative study in two different cultures gives us a deeper understanding of our own teaching and learning culture (Jablonka 2006; Kaiser et al. 2006). Accordingly, it is a methodological tool to see the characteristics of both cultures more clearly. My study was conducted in Germany and Hong Kong being representatives of the Western and the Confucian Heritage Culture.[1]

One aim of the study was to develop a theory of personal meaning from empirical interview data. The theory is elaborated by means of the reconstruction of different kinds of personal meaning in the context of academic learning of mathematics.

The study is based on 34 guided interviews conducted in Germany and Hong Kong with students from lower secondary level. At the time they were interviewed, the students were 15 or 16 years old respectively. Seventeen students from each country participated in the study; all attended the highest school type in the respective educational system. In Hong Kong, I collaborated with schools that use English as medium of instruction. It was, thus, possible to conduct the interviews in English. The guided interviews lasted for about 35–45 min and began with a sequence of stimulated recall (Gass and Mackey 2000). This means that the students watched a five- to ten-minute video sequence of the last lesson they attended. Their task was

[1] I also investigated the role of the students' cultural background for the construction of personal meaning by comparing the results of the students from Germany and Hong Kong. As this part of the project is not related to the application of grounded theory, it will not be reported in this chapter in detail (for further information see Vollstedt 2011b).

to reflect on and verbalize the thoughts they had during the lesson. The subsequent interviews then tackled various topics that were assumed to be related to our understanding of personal meaning (see below). This understanding was at that time quite broad and not yet focused. The intention was to come as close as possible to the aspects related to learning mathematics which are personally meaningful for the students in a school context. Students were for instance asked about their associations of the words *mathematics* and *mathematics lessons* and about the characteristics of a good lesson. They were interrogated about their beliefs with relation to mathematics, mathematics lessons and their learning of mathematics as well as about their feelings, their learning strategies, their goals etc. In addition, they were asked about their preferred learning conditions and the reasons why they learn mathematics, whether they see a relation between mathematics and their lives, and whether they might need mathematics for their dream job. All these questions were supposed to give information about aspects that might be relevant for the construction of personal meaning.

The decision to analyze the data in a coding process is made for methodological reasons as well as for reasons of content. From the methodological perspective, coding is a core element for the development of a theory which is grounded in empirical data. To break up and to continuously compare the data is equally constitutional for the development of a grounded theory as well as for the development of codes throughout the analytical process. Thus, relations between phenomena can be detected in the data; phenomena can be distinguished and sharpened. Thereby, the aim of this comparative analysis is to use descriptive categories to come to analytical concepts so that the relations between phenomena can be explored and clarified (Tiefel 2005; see Chap. 1).

Additionally, in my study, there was also a content argument for the coding analysis as personal meaning can be understood as an individual psychological construct. It can be revealed by character traits and individual attitudes from which one can draw conclusions on the kinds of personal meaning preferred by the interviewed students. Thereby, it is of no importance at which time in the interview the utterance was made as long as the incidents mentioned were considered to be relevant for the development of the theory. Therefore, the sequentiality of the interviews can be neglected so that I chose a coding procedure instead of a sequential analysis method for this study. Coding thereby is characterized as a process of continuous comparison of phenomena, codes and categories with the aim of reaching analytical concepts which explore and clarify relationships between phenomena via descriptive categories (Tiefel 2005; see Chap. 1).

As the data of this study were collected to develop an empirically grounded theory, I decided to use grounded theory following Strauss and Corbin (1990, 1996). I chose their approach because they offer the most concrete guide to the grounded theory method that was available in Germany at the time the study was carried out. The authors point out that their outline of this method is not to be adhered to rigidly but it can be used rather as guidance for the research process (ibid.). Yet, this may not be understood as the permission for undirected interpretations. The guidelines given are more than just an enumeration of recommendations as they mark some operations as obligatory. A coding procedure and the writing of analytical memos for instance are among these (Strauss 1987; Strübing 2004; see also Chap. 1).

The following passages give a more detailed introduction to the different decisions made throughout the research process with concrete examples from my study. The main focus thereby lies on the different ways of coding.

2.3 Theoretical Sensitivity and Sensitizing Concepts

In a study following grounded theory, there are no hypotheses to be tested nor is there a fully developed theory of the research field. In return, grounded theory postulates a high level of theoretical sensitivity of the researcher. According to Strauss and Corbin (1990, p. 42), only this "attribute of having insight, the ability to give meaning to data, the capacity to understand, and capability to separate the pertinent from that which isn't […] allows one to develop a theory that is grounded, conceptually dense, and well integrated". To come nearer to our object of research, we need sensitizing concepts (Flick 2005) which are influenced by theoretical prior knowledge. Hence, researchers do not enter the field of study as *tabula rasa* as the approach of grounded theory is often misunderstood (Strübing 2004; see also Chap. 1, Sects. 1.2 and 1.5.1, for the place of literature review in grounded theory).

Strauss and Corbin (1990, 1996) explicitly mention literature, particularly technical literature, as one source of theoretical sensitivity. Other sources are professional and personal experience as well as the intensive interaction with the data throughout the analytical process. In my case, it seemed reasonable that personal meaning is somehow related to or influenced by concepts from educational psychology like the basic needs for autonomy, competence and social relatedness (Ryan and Deci 2002), personal or situational interest (Krapp 2002), concepts from mathematics education like mathematical beliefs (Op't Eynde et al. 2002) or mathematical thinking styles (Borromeo Ferri 2004), and concepts from educational experience and learner development like developmental tasks (Havighurst 1972; Trautmann 2004). These concepts therefore were taken as sensitizing concepts into the analytical process. As Teppo (Chap. 1) points out, a review of related literature can also provide links to which the newly developed theory can be adhered.

2.4 Interdependence of Data Collection, Analysis, and Development of Theory

According to Strauss and Corbin (1990, 1996), a grounded theory is developed from the study of phenomena occurring in the respective field of research. The data collected need to be analyzed systematically to discover, develop, and verify the theory.

> Therefore, data collection, analysis, and theory stand in reciprocal relationship with each other. One does not begin with a theory, then prove it. Rather, one begins with an area of study and what is relevant to that area is allowed to emerge. (Strauss and Corbin 1990, p. 23)

Strübing (2004) describes this close interdependence of data collection and analysis as functionally dependent and chronologically parallel. None of these processes is thereby understood as final; even the theory developed at the end of the researching process is characterized by tentativeness as it can be further developed in future research projects. The research process in the course of developing an empirically grounded theory then is iterative and circular (Strübing 2004; see Chap. 1). Please note that the procedure is repetitive and circular—but not the theory which is developed in this process.

This close interaction of data collection, analysis, and development of theory is also reflected in the procedure of data collection and selection of cases that are to be analyzed. The strategy used in grounded theory for this procedure is called *theoretical sampling* (see Chap. 1, Sects. 1.4 and 1.6.1). This term should not be confused with representative sampling as it is used in studies with large sample sizes opting to test hypotheses. According to Strauss and Corbin (1990, p. 177), theoretical sampling is "sampling on the basis of concepts that have proven theoretical relevance to the evolving theory". This means that the concepts are relevant with respect to the developing theory as they repeatedly occur in the data, or, on the contrary, are notably absent when comparing the incidents (ibid). In order to note which concepts are relevant, theoretical sensitivity is needed, i.e. sensitivity to recognize relevant indicators in the data. As sensitivity increases over time, it is possible that previously analyzed data must be recoded with the additional knowledge gathered in the analytical process (ibid.; Chap. 1, Sect. 1.6.1). Therefore, two aspects characterize theoretical sampling: chronological parallelism of data collection, analysis, and development of theory on the one hand, and a certain influence of the developing theory on the data collection on the other hand.

Chronological parallelism of data collection, analysis, and development of theory is difficult to realize in a study that is carried out in two cultures. If the demand for chronological parallelism is, however, applied not to the collection of new data but to the choice of which cases are to be analyzed from an assorted pool of data, it still can be satisfied. This is also in line with the argumentation of Strauss and Corbin (1990, p. 181, original emphasis), who argue that "*one can sample from previously collected data, as well as from data yet to be gathered*". Following this interpretation of theoretical sampling, I collected as much data as possible in both countries by having interviewed every student who volunteered. By this means, I generated a data set of 17 interviews per country. In addition, I kept the videotapes of all lessons I attended as well as the teaching materials used. Although I was interested in the personal view of the students on their learning process of mathematics, I wanted to be able to draw back on these materials if necessary throughout the analytical process. Further, at the time of data collection, I took field notes. The field notes concentrated on my experiences within the foreign culture, kept track of my understanding of the Hong Kong school system as well as the information I got about the teachers, and noted down

Example 2.1 Field note taken on April 4, 2006 in Hong Kong

> **Wah Yan College, class 4D/4C, Mr. Ng (approx. mid-thirties)**
> - 12 years of teaching experience
> - School is in fact CMI (Chinese as Medium of Instruction), but from Secondary level on, 3 subjects are taught in English → all are natural sciences!
> - Headmaster is one of the authors of the schoolbook that is used in class
> - Filmed lesson
> - Immediately, several students volunteered for the interviews! (It probably helped that I had my fingers crossed?!)
> - School was founded in 1999, hence everything is quite new
> - Class is better performing than average
> - Today directly interview with Camryn (she was addressed by Mr. Ng before class; he said she does not like his way of teaching so talking to her might be interesting for my study. She denies this. We'll see.
> - Sequence of stimulated recall: Introduction to direct variations (more or less ex-cathedra teaching)
> - Got a copy of the teacher's version of the book together with a seating plan
> - The teacher's version of the book could theoretically be read out in class exactly the way it is; Mr. Ng does not do this
> - Solutions are printed in red next to the question (lighter shade of grey in the copy)
> - The students' version of the book is similar to the teacher's but with solutions at the end of the book

some experiences from the interviews. Example 2.1 above gives an idea about what these notes looked like. As the analysis proceeded, it turned out to be not necessary to come back to the additional material as the interviews proved to be a very rich source with respect to the focus of my study.

After having collected so much data, one might be overwhelmed by it and it is a challenge to decide where to start the analysis. What should I begin with to find a way through the material? Or, with reference to the title of this chapter: I see a large conglomerate of bigger and smaller plants in front of me that I'd like to explore. But I can't walk through them to understand them—there is too much thicket, bushes and fern. Where and how should I start to find a way through them?

I chose to start with the analysis of interviews according to certain considerations. When listening to the mp3-files after the data collection, I wrote recapitulatory memos that summed up the topics that were talked about in the interviews. I always tried to keep the formulations as close as possible to the ones used by the interviewees. These memos were the first step towards a detailed transcription and also served as its basis. Therefore, I also stuck to the grammatical mistakes. As a whole recapitulatory memo is too long to be presented here, Example 2.2 below gives an excerpt from the interview with William, a student from Hong Kong to

Example 2.2 Excerpt from the recapitulatory memo of William's interview

Time	Main aspects
[…]	[…]
27:12	*Anything of special interest in lesson?* Not much, only doing the exercises. It's quite fun, solving the formula. Discuss with my classmate, knowing what is [quarter].[a]
29:58	*Anything interesting in topic?* Drawing a graph to find the median is quite fun. Because drawing a graph, although it's complicated, but the graph is very beautiful and it's very easy to find some information. So, it's very interesting and attracts me.
31:53	*Associations math?* - Receipts: I like to calculate whether it's correct. It's very interesting. - Sudoku: It's about numbers and logical thinking. - Triangles: Calculating angles is fun and interesting. - Economy: It's always about math. - Computers: Are a calculator. - Time: When I listen to music, it's counting the time; when I sleep I calculate whether I can sleep how long; prepare my timetable. - Volume: Bathing—I like to turn on the tub and to […] the volume, although it's very difficult.
35:31	*Like math?* Most certainly. It's interesting; the logical thinking lets me feel excited. I feel happy after having finished calculating a formula. I like math lessons very much because it's the place, the time I can interact with the math very much. The knowledge of math is very wide. Sometimes it's difficult, but I'm keen on that. Because if I understand that, I get more things in the mind and brain and I feel great at that time. I don't like using a calculator. Using a calculator is fast, but there isn't a feeling of success, so I like calculating by myself. *Do you also do it in class?* Yes, I try. If there're too many numbers, I use the calculator. But if there are less numbers, I do it by myself.
38:52	*Associations math lessons?* Math teacher: She is funny, enjoyable because everything is new. Happy: We can freely talk: In some other lessons teachers don't like us to talk, but in math we can discuss. Interesting, enjoyable: No need to remember things, not like history, geography: just calculating, observation of the graph. It's easier, interesting. If you listen clearly, you can do your exercises easily. You only need to remember the formula. Most of math lessons is recess. After I go out of math lessons, I feel very happy and have […] confidence, maybe because of the logical thinking I do for the questions.
42:50	*Like math lessons?* Yes, I like it very much. One reason is: Ms. Ting is very funny, interesting. Her talking to us is sometimes some jokes. Imagine, I solve a formula, I can […] confidence, increase myself.
[…]	[…]

[a]Expressions in square brackets were not perfectly understandable

illustrate what these memos looked like. The sequence is taken from the beginning of the guided interview following the stimulated recall. We shall have a more detailed look at the mid part of this excerpt below.

The interviews were selected for analysis with reference to these recapitulatory memos. The first interview was chosen due to the personal characteristics of the student; the successive interviews then were chosen either in minimal or maximal contrast to the students analyzed beforehand with respect to the characteristic under consideration. To be more precise, I started the analytical process with William, a very high-performing student from Hong Kong, who wanted to be challenged in his mathematics lessons (see above). The interview with William was exceptional as it was very long compared to the other interviews and, judging from the first impression deduced from the recapitulatory memo, it was very detailed and provided lots of examples William used to undermine his thoughts. Due to this richness, I felt confident that it was a good interview to start with.

William's classmate Vincent was similar with respect to his wish to be challenged in a mathematics lesson so that I analyzed his interview secondly. By this minimal contrast, it was possible to sharpen the concepts that were developed so far and get some more ideas about how they are conceptualized. In addition, new concepts that were not present in William's interview could be developed.

The third analysis dealt with Alban's interview, a low-performing student from Hong Kong who was afraid to fail and to lose his face. This case formed a maximal contrast to the first two with respect to the level of the students' achievement. Hence, the concepts could be deepened again concerning their scope and new concepts were developed. Following this procedure, I first analyzed all interviews from Hong Kong before I proceeded with the German interviews. By this means, I could guarantee utmost sensitivity to the data as I did not apply concepts that were developed from a person with Western cultural background in the context of Western lessons to ways of learning in a Confucian heritage culture. Rather, the concepts were developed from Confucian heritage data and later refined with Western data.

Throughout the analytical process, the sensitivity towards the concepts under consideration grows as more and more concepts are developed (see Chap. 1, Sect. 1.3). To ensure that also concepts could be applied to interviews that were analyzed at the beginning of the analytical process, some of the interviews were coded again. By doing so I was able to tag codes to phenomena that otherwise would have been overlooked, as I was not sensitive enough for them in the first coding cycle.

Finally, theoretical saturation was reached (see Chap. 1, Sect. 1.6.1): In the course of the analytical process of the last two interviews, no new categories were developed and the relationship between the categories seemed well established and validated (Strauss and Corbin 1990, 1996). Hence, I did not collect more data but decided to write down the theory as it was developed up to this point. As mentioned above, this does not mean that the theory is unchangeable—on the contrary: Although the theory of personal meaning may be corroborated by future research, it may well be the case that it can also be elaborated or extended further.

2.5 Data Analysis

When we think about our data as the thick and indistinguishable conglomerate of trees, thicket and bushes again, the coding procedure in grounded theory is our tool to bushwhack deeper and deeper into it. To be more precise, we can distinguish between different kinds of coding steps. Teppo (Chap. 1, Sects. 1.3 and 1.5, with reference to Birks and Mills 2011) differentiates between open and intermediate coding. Strauss and Corbin (1990, 1996) on the other hand discriminate three different types of coding: open, axial, and selective coding.[2] They also state that the decision for different types of coding is artificial and can hardly be made transparent in a coding process. Due to the circular design of the research process (ibid), coding is not necessarily linear. It alternates in particular between open and axial coding (ibid). Accordingly, the analytical process is marked by inductive and deductive thinking: The continuous interplay between deductive assumptions concerning the relationship between phenomena and the attempt to verify it with reference to the data is constitutive for the groundedness of the theory in empirical data (ibid).

This oscillating process is supported by analytical memos and diagrams. They reflect the analytical process and the relationships between the concepts in written analysis protocols or graphical representations respectively (ibid). Abstract thoughts about concrete data can be recorded so that they are prepared for verification or falsification respectively in relation to the material. In line with constant comparison of passages and concepts or categories while coding the data, the production of memos and diagrams is another essential element for the development of an empirically grounded theory (see Corbin and Strauss 1990; see Chap. 1, Sects. 1.1.1 and 1.4). In this study, I wrote recapitulatory memos for every person to keep a synopsis of every interview (see above) and analytical memos for every code to refine the description more and more over time (see below). In addition, I attached memos to certain passages from the interviews that brought up questions that I thought might be answered later on in the coding process. Diagrams were developed to graphically represent the relationships between different levels of codes in the process of axial coding (see below).

Several people were involved in the coding process. Primarily, I worked together with research students. Thus, we were able to develop codes consensually as well as independently. The codes that were developed individually or collaboratively could therefore be discussed intensively. At the beginning of the coding process, there was no code system that could have been applied. Therefore, the first codes were generated consensually. To achieve this, some interviews were analyzed collaboratively so that the developed concepts could intensively be discussed in little sections. We started in very great detail so that soon a great number of concepts was developed. Subsequently, the following interviews were analyzed independently so that the results were compared afterwards. The findings showed that basically we tagged the same contents with codes so that the same phenomena were labeled as categories.

[2] Teppo (see Chap. 1, Sect. 5) groups axial and selective coding under the term intermediate coding.

However, differences occurred whether the respective phenomenon rather belonged to the realm of personal meaning or whether it described a precondition that influences the construction of a personal meaning. This discussion led to a more precise description of the categories as well as a stronger awareness that we have to make the distinction between personal meaning in contrast to its preliminaries. Please note that categories were developed with respect to several interviews, i.e. categories do not describe phenomena that are special for a certain student.

Due to reasons of efficiency and scarce resources, I had to code the majority of the interviews on my own. However, when I came to sections in the interviews that seemed to be not straight forward, I sought the discussion with people who have been involved in the project for some time. Also, the progress of the analytical process was discussed time and again with my colleagues in research colloquia where the whole working group attended, or smaller meetings with my supervisor or just a few colleagues.

From the technical side, the study was carried out with the help of the software MAXQDA (1989–2013). The program can be downloaded from http://www.maxqda.com/. The full version is subject to licensing, the demo version can be tried out for 30 days for free. MAXQDA has been developed specifically to analyze qualitative data and offers a wide range of methods for analysis. Among other features, codes can be organized into a hierarchy and complex inquiries can be made about the coded data to work out connections and differences between the codes.

2.5.1 Open Coding

The data that were analyzed in this study consisted of two different groups of texts: the transcribed interviews with students from Germany and from Hong Kong. I started the analysis with the interviews from the Hong Kong data set to encounter them as unbiased as possible and with a great theoretical sensitivity (see above). Hence, the category system was developed with reference to the Hong Kong interviews and it was adapted and further developed with the help of the German data. I tried to keep the influence of the Western perspective on the Hong Kong data as little as possible.

Although the three different types of coding do not occur sequentially (see above), open coding usually is the first approach to the data. Sensitizing questions and constant comparison are core elements of this coding step (See Chap. 1, Sect. 1.3, for a detailed description of open coding). Strauss and Corbin (1990, 1996) use the terms *concept* and *category* to denote a phenomenon that is categorized and conceptualized by assigning it to one code on the one hand and, on the other hand, concepts of higher order, i.e. concepts that are subsequently compared again so that they can be grouped to more abstract concepts.

The name of codes, concepts, and categories can be derived in different ways. Firstly, there are codes that are developed *in vivo* (Strauss and Corbin 1990, 1996). These codes get their names directly or with only little variation from the data.

The concepts are directly mentioned and named by the interviewee. Secondly, there are codes which are also developed from the data and which are named by the researcher in the course of the analytical process. Thirdly, codes can be related to technical literature applied to enhance theoretical sensitivity (ibid). In this case, theoretical concepts that are relevant for the research question and, hence, that are part of the theoretical background of the study are assigned to the data. Their names are taken over; these names mark the relevance of the theoretical concept for the theory. These codes are called *conceptual codes*. The denomination of codes, concepts, and categories is preliminary at first and may be changed in the course or further analyses. Examples of the three different kinds of coding are presented in the illustrative part of this section below.

With reference to our forest metaphor, open coding helps us to name the different kinds of plants and maybe animals we come across on our way through the conglomerate of trees and thicket. The result is that they are not so indistinguishable anymore. We begin to understand what we are exploring.

2.5.2 Axial Coding

In her overview on intermediate coding, Teppo (see Chap. 1, Sect. 1.5) gives some introduction to axial coding as well as the use of a coding paradigm. She also describes selective coding according to Glaser (2004) in this subsection as a way to focus the researcher's attention on this part of intermediate coding. Strauss and Corbin (1990, 1996), on the other hand, differentiate more strongly between axial and selective coding as separate steps in the analytical process. Therefore, this section will discuss the application of axial coding, whereas selective coding is presented below.

Following Strauss and Corbin (1990, 1996), axial coding is the second step in the coding process. They suggest investigating the following elements to work out the relations between the categories with the help of a coding paradigm (see Chap. 1, Sect. 1.5): causal conditions, context, intervening conditions, action/interaction strategies, and consequences. Strauss and Corbin perceive the coding paradigm as obligatory element of a grounded theory in contrast to the elements used. Therefore, Tiefel (2005) for instance adapted the coding paradigm to her study with respect to a theoretical framework of learning and education. Both versions, i.e. the one by Strauss and Corbin as well as the one by Tiefel, however, seemed of little use for my study so that I also adapted the coding paradigm to come to one that matches my study better. I assumed that there are certain personal preliminaries like the student's personal traits or his/her personal background that might influence the construction of personal meaning. In addition, the kind of personal meaning constructed by the student might influence the student's actions or judgments. Therefore, I analyzed the phenomena with respect to their preliminaries and consequences in the course of axial coding. The results were recorded in theoretical memos and diagrams. Thus, the different kinds of personal meaning, which were developed as main categories, could be theoretically refined and contextually condensed.

The development of main categories from categories works differently than the development of categories from concepts in the course of open coding. In open coding, concepts were related with reference to their content. Similar phenomena were collected in categories of different levels of abstraction. In axial coding, we look for relations between categories and concepts that are proposed by the interviewees themselves. Hence, relations are established between a category (the main category) and other categories or concepts (the subcategories). The differentiation between main categories and subcategories therefore lies on another analytical level than the relation between categories and concepts.

When thinking about our trees metaphor, with axial coding we now begin to understand the relationship between the different plants and trees. Anemones, for instance, are little flowers that widely grow in the undergrowth and underneath trees. They only blossom in springtime when the trees do not yet have strong leaves as they are in the need of much light. The "structure" of the trees and other plants becomes clearer and clearer—especially concerning their relations.

2.5.3 Exemplary Illustration of Open and Axial Coding Using Memos and Diagrams

Before I continue with selective coding, I illustrate the open and axial coding processes with the help of an extract from the interview with William, the student from Hong Kong we already met above in the illustration of the recapitulatory memo. I also show how memos and diagrams can help in the analytical process and how they were used in the course of the analysis. Please note that my interpretation is just one possible interpretation and that other interpretations may also be valid. Especially with a focus on another research question, one might come up with quite different concepts and categories.

To understand the section chosen a bit more easily, consider the following information: The extract quoted below was preceded by the stimulated recall about a section of his last mathematics lesson in which the class learned about the median. In the part of the interview from which the section was taken, the questions dealt with the student's attitude towards mathematics and mathematics lessons. In the interview with William, I started with the question about his associations with the word "mathematics", which was followed by the section below. Questions about his associations with the word "mathematics lesson" and whether he liked mathematics lessons then succeeded (see above). It could be reconstructed from these and other parts of the interview that William liked mathematics lessons very much and he was eager for mathematical knowledge. He therefore wanted his teacher, Ms. Ting, to arrive more quickly at the classroom after the bell rang so that the lesson could start earlier and that they could learn more in a lesson.

1. Interviewer: Do you like mathematics?
2. William: Oh, certainly.
3. Interviewer: Ya?

4. William: Because … I say it's interesting, the logical thinking is … let me feel … exciting
5. … becau- … I feel successful after I finish … calculating a … formula … also feel … (3 sec)
6. happy, happy because it's quite … (5 sec) I feel successful also … (2 sec) when I'm
7. … (3 sec) I like the mathematics lesson very much because … this the … the place, the
8. time I can interact with the mathematics very much (2 sec) becau- I don't know … the
9. … knowledge of the mathematics is very wide so … learning it is … although is, maybe
10. sometimes is difficult but … I'm keen on that because … if I understand that … what is
11. that thing about … (14 sec) I get … I get more more more things in the mind and in the
12. brain, so … (3 sec) I don't know that word is in English but … maybe I try to use another
13. word to explain to you, … the knowledge come into your brain and you feel more, you
14. get more information and get more knowledge and feel great at that time … (3 sec) I
15. don't know that word, sorry.

The excerpt presented starts with the question whether William likes mathematics. He confirms this question and stresses it explicitly with "certainly" (1–2[3]). From this utterance, we can reconstruct a positive attitude towards mathematics. Therefore we can generate the code *positive attitude towards mathematics* and so develop our first concept. To remember later on in the coding process which incidents we wanted to denote with this code, we should write a code memo containing a description of the phenomenon labeled with this code and possibly give an example of an utterance which might stand exemplarily for this code. Although it often seemed straightforward what the code was about judging by its name, it later on frequently turned out wrong in my study. One day I was really sure about what concept I wanted to denote with a certain code and thought that writing a memo would take too much time. Then, a couple of days later I was cross with myself for not having written a memo. It is often difficult to draw the lines between two codes when in doubt whether to add a new interview line to an existing code or whether to create a new one. When you cannot refer back to a definition in a memo, things turn out even worse.

[3] In the original interview, the transcript lines were numbered differently. There, every speech act was labeled with one number, i.e. this section was enumerated with 132–135. To make it as easy as possible to follow the coding process, I chose here to number every line as presented above in order to find the different bits labeled with codes more easily.

Code memos should be kept up to date. They will become more and more explicit over time when we come across similar incidents, which also belong to a certain concept, or—even more precisely—when we detect utterances in the data that just do not belong to the code. It is also helpful to expand the information collected in the code memo and make notes about these concepts that are close to the one explained. Therefore, memos get more and more detailed over time. With reference to the code developed above (*positive attitude towards mathematics*), at first, I just noted down that the interviewee mentions something positive about his or her attitude towards mathematics. The illustrative line taken from the interview helps to get a better understanding of the code when referred back to later on in the analytical process. When more and more passages were coded, the information was enriched, and more illustrative examples were added. For instance, students did not only generally talk about liking mathematics or certain fields of mathematics (e.g. geometry) but they also like mathematics for its difficulty and because they are challenged by it. I also noted down that the concepts labeled with this code referred to mathematics and not to the activity of doing mathematics (like problem solving) or to mathematics lessons. These instances belonged to other codes.

After William's short answer, the interviewer replies with a confirmative "Ya?" (3) and William elaborates more on his attitude towards mathematics. He relates it at first to his interest in mathematics: "I say it's interesting" (4). Here, we can use a code from our sensitizing concepts that we read about in technical literature: We can link this utterance to the concept of personal interest (e.g. Krapp 2002). Again, we develop a code (*personal interest in mathematics*) and write a code memo as explained above. Due to the succeeding utterance ("the logical thinking is … let me feel exciting", 4), one can argue whether William's interest results at least partly from his excitement to think logically. Therefore, in the code memo of *personal interest in mathematics* we can add this idea so that later on in the coding process, we can check whether this relation is made more explicit by other interviewees or whether we can find other incidents which suggest this relation. In addition, we can attach an analytical memo directly to this incident in the interview (i.e. next to the transcript line) with the idea that there might be a relation between William's personal interest in mathematics and his excitement about logical thinking. These ideas and codes about a relation between personal interest of the student and a positive attitude towards mathematics are very first ideas of axial coding as we think about the relation of two concepts that lie apart from the grouping of similar concepts in one bigger category. Thus, we can see that the discrimination of the three different types of coding is artificial as at least open and axial coding interact to quite some extent.

William's excitement about logical thinking, however, seems to be another phenomenon. It shows that William enjoys when he can think logically. We can develop a new code *enjoyment of logical thinking* and write a code memo respectively. The name of the code is partly inspired by the interviewee's formulation, i.e. it is partly coded *in vivo*. William then links the enjoyment of logical thinking to the feeling of success after having finished his calculation and the application of a formula (5). At this instance, again, we can generate a code (and write a corresponding memo) that comes from a sensitizing concept, i.e. the experience of competence as formulated in self-determination theory by Deci and Ryan (2002). We call our code *experience of competence by successful calculation*. Then, William tells the interviewer that he also feels happy when he is successful with the calculation (5–6). Hence, William also links the experience of competence due to his successful calculation with enjoyment so that we get another code: *enjoyment of experience of competence*. Now we realize that we had a similar code beforehand, the *enjoyment of logical thinking*. Thus, we can now generate a broader code that embraces two codes: the category *enjoyment* with the two subcategories or concepts *enjoyment of successful calculation* and *enjoyment of logical thinking*.

After some stammering containing half sentences which cannot be clearly linked or interpreted ("… (5 sec) I feel successful also … (2 sec) when I'm … (3 sec)" (6–7), William further elaborates on his attitude towards mathematics lessons. He explains that he likes his mathematics lessons very much as they provide the time when and the place where to interact with mathematical contents (7–8). William therefore shows a *positive attitude towards mathematics lessons*. Again, we can combine two concepts in a category: *positive attitude towards mathematics lessons* and *positive attitude towards mathematics* can be interpreted as two subcategories of *positive attitude*. In addition, William seems to enjoy interacting with the mathematics (8), i.e. we have our third subcategory of *enjoyment*: *enjoyment of active engagement with tasks*.

Then, William goes on and states that "the knowledge of the mathematics is very wide so … learning it is … although is, maybe sometimes is difficult but … I'm keen on that" (8–9). So, although it is sometimes difficult to understand, William likes to learn more about mathematics. Hence, he does not shy away from difficult topics; on the contrary, it seems that he likes to be challenged by mathematics ("I'm keen on that", 10). Thus, we can develop a new code together with its memo: *enjoyment of challenge by difficult mathematics*.

In William's last longer utterance he obviously has problems in formulating his thoughts. We can tell this from the long pause of 14 seconds in line 11, as well as the fact that he addresses his formulation problems. Still, his thoughts are understandable so that we can interpret them. In this section he makes a connection between understanding and knowledge: "if I understand that … what is that thing about … (14 sec) I get … I get more more more things in the mind and in the brain, so […] the knowledge come into your brain and […] you get more information and get more knowledge and feel great at that time" (10–14). In William's opinion, understanding of the topics seems to be a precondition for education and for knowing more, probably even for becoming more intelligent. He seems to value the broadness of the mathematical body of knowledge and it is his aim to get more knowledge. In addition, he also feels great when he learns more (13–14). Therefore we can generate the codes *eagerness for knowledge* and *enjoyment of knowledge* (again as a subcategory of *enjoyment*) together with their memos.

Another instance of *enjoyment of knowledge* can be reconstructed from William's utterance that he feels great when he gets more information and when "the knowledge come into your brain" (13). William's eagerness to know more combined with

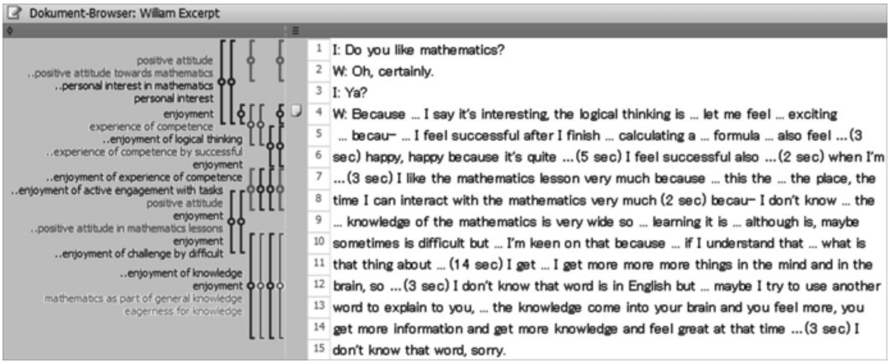

Fig. 2.1 Coded excerpt from William's interview (Screenshot taken from MAXQDA)

his emphasis of the broadness of mathematics suggests that he values mathematics as a part of general knowledge that is to be aspired. Thus, a new code can be *mathematics as part of general knowledge*.

When applying these codes and the analytical memo about the connection between logical thinking and personal interest in mathematics to this section using the software MAXQDA the coded passage looks as presented in Fig. 2.1 above.

To recapitulate, in this interview excerpt we learn something about William's personal attitudes as well as instances that are important for him in the context of learning mathematics. He shows the belief that mathematics may sometimes be difficult and that mathematics lessons provide the conditions in which he can actively engage with mathematical contents. He has a positive attitude towards mathematics and he is interested in the subject as well as the contents. He likes to think logically and to be challenged by difficult topics. Finally, he is eager to learn and wants to develop himself.

Correspondingly, when we subsume our findings from this interview excerpt, we come up with the following (preliminary) list of codes as presented in Fig. 2.2 (given in alphabetical order).

For axial coding we now need to relate categories and concepts on a different level. As described above, I made changes in the coding paradigm, as the elements proposed by Strauss and Corbin (1990, 1996) did not match my research question. To elaborate the different kinds of personal meaning, we need to relate those aspects that are personally meaningful with those which are preconditions and consequences.

When we have a closer look at the categories developed so far in the course of the analytical process, we realize that *eagerness for knowledge*, *personal interest in mathematics*, as well as *positive attitude towards mathematics* or *mathematics lessons* denote elements of William's character. They signify features belonging to his personal traits. Therefore, they are elements of the preliminaries William brings to the process of constructing personal meaning. On the other hand, a closer look to the categories grouped beneath *enjoyment* shows us that we need to distinguish

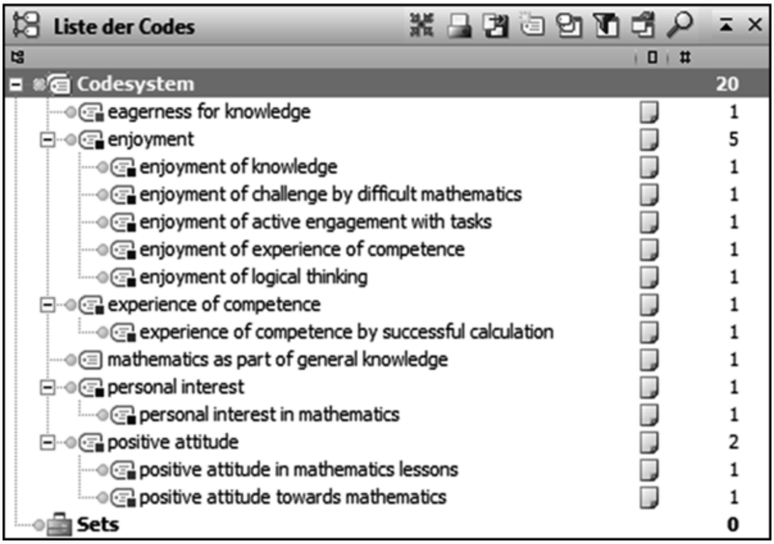

Fig. 2.2 List of codes in alphabetical order (Screenshot taken from MAXQDA)

between the enjoyment itself and the source from which the enjoyment origins. Consequently, the sources are manifold, but they all share the same consequence: the experience of enjoyment. In other words: The phenomena described by the source of enjoyment are personally meaningful for William—provided that he is able to realize them. He then enjoys the learning of mathematics or dealing with mathematical contents.

We can deduce two main statements from these findings: The first one is that the theoretical framework, which relates personal meaning to the surrounding conditions of its construction, becomes clearer and clearer. We now know that we need to distinguish between preliminaries, elements that relate to personal relevance, and consequences. In the course of the analytical process, this model was again refined until the theoretical framework as presented in Fig. 2.3 was developed. With respect to preliminaries, we distinguish between personal background (e.g. cultural and socio-economic background, age, and gender) and personal traits. The latter can be specified in more detail with the help of concepts that are determined in educational psychology (e.g. interest, motivation, and self-efficacy), mathematics education (e.g. mathematical beliefs and thinking styles) or concepts from the didactics of Educational Experience and Learner Development (denoted as *Bildungsgang-didactics* in Fig. 2.3) like developmental tasks.

The second statement is that the sources of enjoyment detected seem to play a decisive role for the development of a theory of personal meaning, as they are elements that are meaningful to William. Hence, they are the first elements that give us an idea about different kinds of personal meaning. In the course of the further analytical process, the different sources of enjoyment show varying degrees of

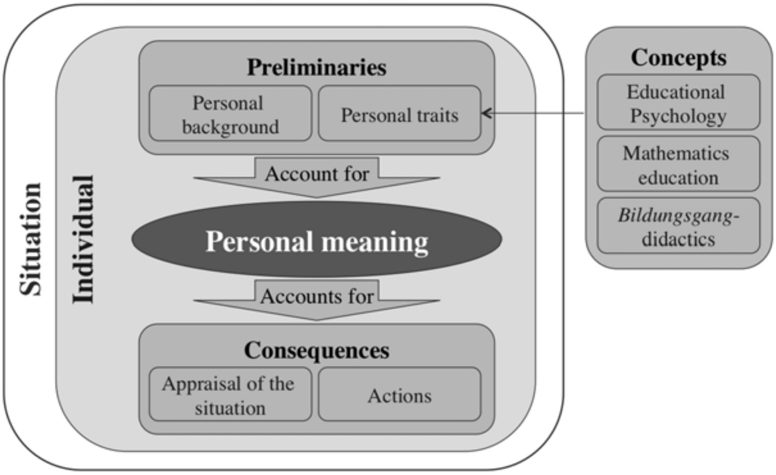

Fig. 2.3 Theoretical framework of personal meaning as developed in the course of the analytical process

relevance for different kinds of personal meaning. One source thereby might be decisive for one kind of personal meaning and also relevant but not central for other kinds. To illustrate this with a more concrete example, let us investigate the idea of *challenge by difficult mathematics* in more detail: At the end of the analyses, this phenomenon that students want to be challenged by difficult topics or tasks proves to be important for the kind of personal meaning *experience of competence* in which it is relevant for the students to experience themselves as competent and successful (see also the need for competence as described in Self-Determination Theory according to Deci and Ryan 2002). One of the personal traits considered as relevant for the construction of this kind of personal meaning is that the student likes to be challenged by difficult mathematics as these contents especially bear the possibility of experiencing competence after they have been successfully solved. The second kind of personal meaning to which *challenge by difficult mathematics* was central is *cognitive challenge*, for which it is the defining element. The final coding paradigm is shown in Fig. 2.4 below. Relevant preliminaries for this kind of personal meaning were a wish for cognitive challenge and that difficult tasks were provided in the lesson so that it was possible for the student to engage with them. Some of the students also are very ambitious and they like competitions with their classmates. Consequences that derive from *cognitive challenge* are for instance that the student can improve his/her achievement and that he/she enjoys the challenge. Hence, the student can experience competence and success. Here, again, the close relationship between some kinds of personal meaning becomes evident.

As only a short excerpt could be shown, it is difficult to clarify the steps of constant comparison in the latter coding process. Hence, from this article it hardly becomes clear how categories become more and more complex and how the 'big idea' of every category arises while the analytical process is proceeding. To cushion this, let me add some general ideas about working with grounded theory. When we

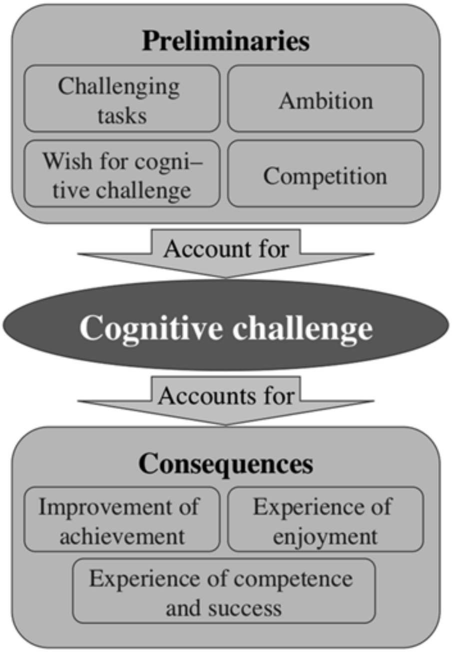

Fig. 2.4 Completed coding paradigm for Cognitive challenge

use the grounded theory method to develop theory from empirical data, our general aim is to discover elements of a theory about our research question in these data. The difficulty is to decide which elements are relevant and how to combine them in such a way that a consistent theory arises. The first thing is that we constantly have to ask ourselves about the more general idea behind what the interviewees say. This means that we have to generalize from the concrete expressions to deduce the more general idea that is relevant for our research question. So, what is behind what the interviewee (or the data in general) tells me? Throughout the research process, these ideas can be linked with each other or—equally or even more interesting—not linked. On the one hand, concepts can be grouped as they denote similar phenomena (subcategories in a category of higher order). On the other hand, concepts can be linked although they do not denote a similar idea. Then, the connection is usually suggested by the interviewees, who combine them in their expressions (axial coding). Here we have to pay attention to the links that can be developed in the analysis and those that cannot be established. Lots of questions arise: Why is that so? Do the categories describe different 'big ideas'? Or do my categories denote facets of an overarching 'bigger idea'? Why can't I put them in one main category? What is missing? And why is it missing? Do I need more (other?) data to answer this question? So here, again, we have to look for the more general idea on category level.

On this level of analysis, we usually keep writing memos over memos to remember all our ideas about combinations of categories and also about links between categories that are not possible and why they are not possible. We formulate hypotheses about them and try to find more evidence or counterevidence with the help of new data or sometimes even data that have been analyzed beforehand. After some time and after the analysis of more data, some links between categories become more and more established as they occur time and again in the data; other links cannot be verified with new data so that we have to dismiss them. You can see that, slowly, a closely-knit net of combinations of categories arises from the data.

2.5.4 Selective Coding

Selective coding describes a procedure similar to axial coding but it is carried out on a more abstract level. The aim is theoretical integration of the developed categories into a consistent overarching theory (see Chap. 1, Sect. 1.7). This means that we are looking for a core category, which is related to all other main categories that were established in axial coding.

Following Strauss and Corbin (1990, 1996), selective coding is the third step in the coding procedure. As Teppo (Chap. 1, Sect. 1.7, with reference to Corbin and Strauss 2008) points out, the questions that have to be answered are "what is the research all about" and "what seems to be going on here". The aim in this analytic step is to find the common thread that runs through the study. Or—in our trees and anemones metaphor—to detect paths that lead the way through all the trees and plants. We finally get to the point of realizing that we are investigating a complex conglomerate of trees, which finally turns out to be a beautiful forest.

When the analytical process in my study came to an end, 17 main categories were developed, that could be described with reference to several subcategories. The main categories cover a broad range from the fulfillment of duty and the wish for cognitive challenge when dealing with mathematics to the experience of social relatedness. So what is their combining element? All these instances are in some way or another important for the students when they are dealing with mathematics. In other words: All phenomena describe aspects or phenomena in the context of learning mathematics at school that are personally relevant for the individual. This relevance makes the phenomena personally meaningful for the students. Hence, when asking the sensitizing questions of selective coding, I decided in favor of the core category *personal relevance*. The different kinds of personal meaning can be characterized as those incidents that are dealt with in the context of learning and dealing with mathematics at school which are personally relevant for the students. With reference to the codes that were developed in the course of the analytical process of the study, this means that the main categories worked out in axial coding describe the different kinds of personal meaning.

Strauss and Corbin (1990, p. 116) define the core category, which is to be developed in this step, as the "central phenomenon around which all the other categories are integrated". It might have been developed in the course of axial coding or it might as well arise in selective coding. The phenomenon being central for selective coding may in some research even be contained in the formulation of the research question (Böhm 2005).

Personal relevance fulfills the assessment factors for a core category suggested by Strauss (1987[4]). The core category *personal relevance* is the central element of the developed theory and can easily be interwoven with the main categories in a close network. This is due to the fact that every main category developed in axial coding describes another kind of personal meaning. Each of these categories therefore categorizes another specification of personal relevance. Every main category, i.e. the different kinds of personal meaning, together with its subcategories describes indicators for the core phenomenon, which frequently occur in the data and form a pattern.

2.6 Going Beyond Grounded Theory

Having reached this point, I came up with a dense grounded theory about personal meaning based on the construction of 15- to 16-year-old students from Germany and Hong Kong when they learn mathematics. I was able to describe 17 different kinds in rich detail. I could have stopped here—and actually the application of grounded theory methods ends here. Moreover, I was interested in the relationship between the different kinds of personal meaning, i.e. the main categories of my theory. Is there some axis they all refer to and according to which they can be ordered? Is there a basic underlying, subject-independent dimension which can be used to work out guidelines or more general criteria to think about personal meaning across different subjects? To answer these questions, I had to think about the different kinds of personal meaning I had worked out from a more general perspective. By doing so, I followed the methods laid out by Kelle and Kluge (1999). The two dimensions I finally came up with were the relatedness towards the individual and the relatedness towards subject contents, i.e. mathematics. I was able to arrange all kinds of personal meaning with reference to these two dimensions. Then, seven different types of personal meaning could be deduced from the arrangement (see ibid). As the typology is not reported here in detail, see Vollstedt 2011a or 2011b for more detail.

The analytical elaboration of the categories finally resulted in a decisive advancement of the theory, which gained more explicitness and density. Furthermore, it is possible to integrate maximum variation of the specifications of the core category *personal relevance* into the theory as can be seen in the development of the typology. By writing down the theory that has been developed from our interview data, we give other people the possibility of also understanding and referring to the theory we worked out.

[4] At the time of writing his introduction to *Qualitative analysis for social sciences*, Strauss (1987) used the term *key category* instead of *core category*. They denote, however, the same kind of category.

2.7 Conclusion

The aim of this chapter was to trace the analytical process of an empirical interview study using grounded theory. To achieve this, an excerpt from one interview with a student from Hong Kong was analyzed and the analytical process was shown in as much detail as possible. It is, of course, not possible to illustrate every little step of the highly complex analytical process with such a short excerpt. Still, I tried to give insight into the different levels of coding as well as to provide examples for the decisions that have to be made throughout the analysis.

To conclude, the basic idea of the development of theory using grounded theory is to get the main ideas behind what the interviewees say (or our data provide), to formulate hypotheses about links between these ideas, and to try to establish or dismiss these links. To finally come to a dense theory that is empirically grounded, a very detailed analysis of the data is necessary. The ideas discovered have to be knit together tightly with the help of empirical evidence. Eventually, we see in the data not only manifold expressions or phenomena but concepts and categories that are strongly interwoven to form a theory about our research question in focus.

In other words: We started our journey with an indistinctive conglomerate of plants, began with a categorization of trees, bushes and animals and finally reached a good understanding of our forest with all its paths, bigger ways and shortcuts through the undergrowth. Having laid out the theory now also puts up signposts to enable other people to enjoy a day in the forest without being lost, and to come back once in a while. Thus, in the end, it is possible to see the wood despite—or precisely because of—all the trees.

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Part II
Approaches to Reconstructing Argumentation

Chapter 3
Methods for Reconstructing Processes of Argumentation and Participation in Primary Mathematics Classroom Interaction

Götz Krummheuer

Abstract This paper presents two methods of analysis of interaction processes in mathematics classes—the *analysis of argumentation* and the *analysis of participation* –, and it furthermore explores the relationship between these methods and their resulting impact on the development of elements of an interaction theory of mathematics learning. The main theoretical assumption of this article is that learning mathematics depends on the student's participation in processes of collective argumentation. On the empirical level such processes will be analyzed with methods that are based on Toulmin's theory of argumentation (Toulmin, SE. (1969). The uses of argument. Cambridge, UK: Cambridge University Press) and Goffman's idea of decomposition of the speaker's role (Goffman, E. (1981). Footing. Forms of talk. ders. Philadelphia: University of Philadelphia Press). Different statuses of participation in processes of argumentation will be considered, which allow a theoretical description of different stages in the process of learning mathematics from the perspective of an interaction theory of mathematics learning.

Keywords Argumentation • Interaction theory of learning mathematics • Participation • Production-design • Social interaction

G. Krummheuer (✉)
Goethe University Frankfurt am Main, Robert-Mayer-Str. 6-8,
60325 Frankfurt am Main, Germany
e-mail: Krummheuer@math.uni-frankfurt.de

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3.1 Introduction

Who did it a different way?[1]

This question of a second grade mathematics teacher might serve as a catchy utterance that highlights my interest in the analysis of argumentation and participation. The teacher was a member of an American-German project funded by the Spencer Foundation (Cobb and Bauersfeld 1995). The research team analyzed videotaped classroom situations, which had been accomplished in the following way: First, there were group work sessions, in which groups of children were supposed to solve the problems of given work sheets. The instructions for the children were that they should solve as many problems as they want and that they have to explain to each other their ideas about the solution. After a group work session, the teacher initiated a whole class discussion, in which the students were asked to present the results of their group work. After the presentation of a few solutions, the teacher generally asked: "Who did it a different way". The intention of this question was that the second graders should present a variety of different ways of solving the given mathematical problems.[2] While analyzing parts of the video sequences, it was striking to me, that the presented ways of solving the problems mainly serve in the interaction as an explanation and justification for the found result. That means they have an *argumentative* function in the sense that the students attempt to *demonstrate* to the class what they did and to *convince* the class that their way of solving the problem is 'ok'. Moreover, in the emerging sequence of presenting different solutions for the same problem, the question arose for me, how inventive and independent were the presented solutions at the end of such a round of classroom discussion.

Methodologically, this insight led to the search for procedures that allow the accomplished argumentation of the participants of a mathematics class to be analyzed and that allow a classification of the originality of their contributions in the series of ongoing presentations of different solving methods. In the following, I introduce an analysis of argumentation and an analysis of participation, each method exemplified by the same two primary mathematics classroom interactions.

3.2 The Concepts of Argumentation and Participation

The two methods in question are embedded in an interaction theory of mathematics learning in everyday classroom situations (Krummheuer and Brandt 2001; Krummheuer and Fetzer 2005; Krummheuer 2013). This theory will not be outlined in detail. Here I only first unfold the general theoretical view in which way the interactive accomplishment of an argumentation is connected with different statuses of students' participation. Secondly, I propose some clarifications about the relationship between the applied different qualitative research methods.

[1] Quoted from Wood (1995).

[2] For more details about this teaching intervention see Cobb et al. (1995).

The general theoretical view concerns the interrelation between a theory of interaction in mathematics classrooms and a theory of mathematics learning. With regard to learning mathematics one usually assumes that the sense of a mathematical argumentation is a *pre*-condition for the possibility of learning mathematics and not only the desired *out*-come. In this sense, learning mathematics is *argumentative* learning. It is based on the students' participation in an "accounting practice" (Garfinkel 1967; p. 1) of evolving explanations and justifications, which are helpful and supportive in the initiation of the students' learning processes in mathematics. Mathematics learning is considered to be "learning-as-participation" (Sfard 2008; see also Krummheuer 2011). In such situations, the participants can generate relatively sophisticated arguments. Of course, it also might regularly occur, that one hardly can find explications of elements of an argument. Both belong to everyday mathematics classroom situations.

In mathematics classroom situations, as Yackel (1995), for example, outlines, "explaining is not an individual but a collective activity" (p. 151). It is also characteristic for any everyday classroom situation that the students contribute to these collective activities in different statuses. Thus, considering the role of argumentation for the learning of mathematics, it is not only of interest

- how an argument accomplished in the course of interaction is structured but also
- how the teacher and the students are involved in its interactive (collective) production.

I treat the first question with the theoretical approach of Toulmin, which I call the "analysis of argumentation". The second issue deals with Goffman's idea of the decomposition of the role of the speaker, which I call the "analysis of participation". In the following I present these approaches by applying two scenes from a first grade mathematics class. These scenes are taken from the video corpus of the project "Rekonstruktion von Formaten kollektiven Argumentierens" (Reconstruction of Formats of collective Argumentation) that has been funded by the German Research Foundation (DFG). Analyses of these scenes were published for the first time in Krummheuer and Brandt (2001) and partly in English in Krummheuer (2007).

In this project we apply different qualitative research methods, which we construe in a hierarchical sequence in order to reconstruct theoretically relevant aspects of the classroom interaction. The basic and initial procedure is always the implementation of the *analysis of interaction* based on ethnomethodological conversation analysis (Schegloff 1982; Sacks 1998; Ten Have 1999). This method serves to reconstruct the process of negotiation of meaning and leads to a reliable interpretation. Due to limitations in space, this method is not outlined in this paper. On the basis of the results of this analysis, several other methods are applied such as the analysis of argumentation and analysis of participation. In the following presentation of two concrete examples, it has to take into account that, due to the hierarchical dependency on the results of the analysis of interaction, an interpretation of the process of negotiation of meaning of these two scenes already exists. Its results will be presented in the following—its deduction however will not be validated (see Krummheuer and Brandt 2001 and Krummheuer 2007).

3.2.1 The Example "Thirteen Pearls"[3]

3.2.1.1 The Transcript

The presented episode has been taken from a lesson in a first grade mathematics class. Within this lesson the additive decomposition of two-digit numbers in the range from 11 to 20 will be treated. The string of pearls, which is referred to in the episode, contains ten black and ten white pearls lined up on a string: ●●●●●●●●●●○○○○○○○○○○. The episode "Thirteen Pearls" begins as follows:

092	T	Yehes, now! I'm .. keen to see what the children say.
093		*holds a string of pearls in the air*: ●●●○○○○○○○○○○
093.1	8:59 h	
094	Marina	I see
095	Franzi	Thirteen
096	*Marina, Franzi, Jarek and Wayne raise their hands*; *some children count while whispering*	
097	T	*Whispering* two, three
098	Goran	Thirteen. *Quickly raises his hand*
099	*Julian, Conny and two other children raise their hands*	
100	T	*Whispering* two, three, four, five fingers I see .. six, seven, eight. *Loud again* Wayne?
101	Wayne	Thirteen
101.1	Marina	*Takes down her arm*
102	T	Or?
103	S	Uhm
104	T	Jarek?
105	Jarek	Uhm .. three plus ten
106	<T	Or Marina?
107	<Marina	*Raises her hand animatedly* ten plus, three
108 109	T	Or? Oh! The children see quite a lot, huh. It's always the same, but they do see quite a lot. Julian
110	Julian	Uhm, el, naw, right eleven, plus two
111	T	Or? Jarek?
112	Jarek	Seven minus zero
113	T	S, seven minus zero?
114	S	Huh?
115	S	Huh?
116 117	T	Let's try it. Come to the front. Seven minus zero?… Jarek has said something, and we have to check that. Come here!
118	Jarek	*Goes to the front*

[3] In the original German transcript we do not use the standard interpunctuation, and denote speaking pauses, raisings of the pinch and so forth. In the English translation we do not use these paralinguistic notations. Word order and tone of voice differ too much between German and English. The original transcripts are reproduced in the appendix.

119 120 121 122	T	*Holds up a string of pearls in Jarek's direction* show us seven minus zero. Show us seven, turn around to the class so that the children can see it and so that one can **compare**. *Holds her own string of pearls up again; she is still showing thirteen* So. **Seven**?
122.1	Jarek	*Silently counts the pearls on his string*
122.2	T	Count aloud!
123 124 125	Jarek	*Counts out on his string holding it up in the air* one, two, three, four, five, six, seven *string of pearls*: .... minus zero *lets go of the end he counted; shows* ...°°°°°°°°°° is thirteen.
125.1	*While the teacher is speaking Jarek returns to his table*	
126 127 128 129 130	T	*Breathing astonishedly* h, ha now I understand, what Jarek did. *Puts her own string away and takes over Jarek's*. He claimed, he started from this side and counted off seven. One, two, three, four, five, six, seven. *Shows it on her string* there he said, minus zero is that. *Shows* ...°°°°°°°°°° does that work?

3.2.1.2 Analyses of the Scene

As mentioned above, at this point I will not perform a systematic analysis of the scene. Instead, while going through the episode, I would like to make some comments on the emergence of the situation of argumentation, which unfolds in the course of interaction in two different ways in the beginning and at the end of the scene.

The Analysis of Argumentation

In the transcript, a first phase can be identified in the lines 92–111. The teacher holds up 13 pearls of a string of pearls, which consists of 20 pearls and asks the students to make corresponding statements. Up to line 108 there seems to be a prevailing content-related interpretation of the teacher's question, which seems to have been stabilized by line 111. In mathematical terms, one could say that it deals with the additive decomposition of the number 13. The participants of the classroom situation would undoubtedly formulate this in a different way. Without any problem, however, one can reconstruct the interaction-pattern "initiation—reply—evaluation" (Mehan 1979), which is typical of the teacher-guided talk in many classroom situations. We see here a type of interaction process evolving, which we call a "smooth course of interaction" ("interaktionaler Gleichfluss"; Krummheuer and Brandt 2001, p. 56).

Before I come to an analysis of the process of argumentation, I will shortly introduce the approach of Toulmin. The contributions of single persons in the interaction with regard to their function within a interactive accomplishment of an argumentation (Kopperschmidt 1989) are viewed independently from their individual intention.

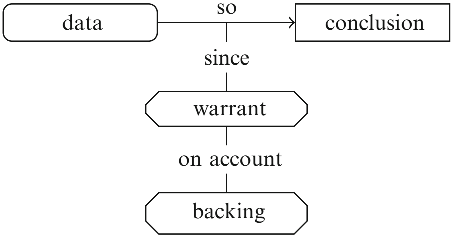

Fig. 3.1 Toulmin's layout of argumentation (In the figures in this paper, Data are enclosed in rectangles with rounded corners, Warrants and Backings in rectangles with angled corners, and Conclusions in plain rectangles)

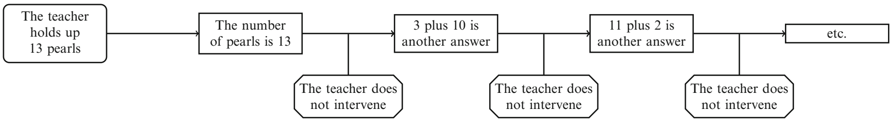

Fig. 3.2 Chain of argumentations

In the following I confine myself to the four central categories of an argumentation in the sense of Toulmin. They are

- data
- conclusion
- warrant
- backing

Toulmin (1969) developed the following graphical layout (see above Fig. 3.1).

The general idea of an argumentation consists of tracing the statement to be proven the conclusion back to undoubted statements ("data"). This relationship is expressed in the first line of the layout and can altogether be referred to as the inference of the argument. Such an inference requires a legitimation. Statements that contribute to this represent the warrant. Of another quality are those statements, which refer to the permissibility of the warrant. Toulmin (1969) calls them "backings". They represent undoubtable basic convictions. Arguments can be chained together in such a way that an accepted conclusion can function again as data for a subsequent new argument.

This is the case at the beginning of the scene. How is the correctness of the decomposition of the number 13 substantiated? Up to line 111 no reasons or justifications are explicitly given and, in the way the teacher asks, are not demanded. Her call for another decomposition ("or") is taken as the warrant that the given answer is correct: she only would intervene if this answer were wrong. With regard to Toulmin's layout, at this point one can identify a chain of argumentations, in which the conclusion of one inference evolves into the data for a new argument (see above Fig. 3.2).

This smooth course of interaction changes with Jarek's answer "seven minus zero" in line 112, which can be seen as the beginning of the second phase of this episode. The accounting practice seems to become more differentiated. Presumably, only when answers are incorrect does the teacher ask for a more detailed explanation.

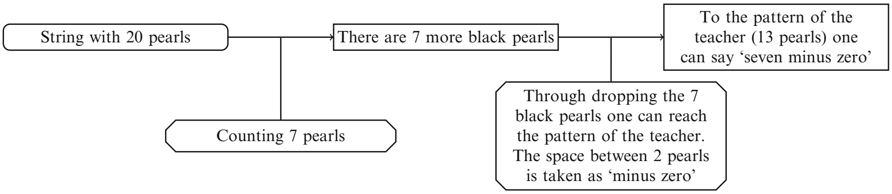

Fig. 3.3 Toulmin scheme of Jarek's argumentation 7–0

Jarek counts the seven black pearls from the black end of the string and while doing so he holds the following eighth pearl in his hand. Then he says minus zero <124>, *lets go of the end he counted; shows* •••○○○○○○○○○○ is thirteen <125>. This is the pattern of the string of pearls, which the teacher was holding up. Most likely, he understands the space between the seven black pearls and the remaining 13 pearls as zero and decomposes in this manner the number 20. Hereby 20 represents the entire string. Mathematically speaking, one could rephrase his statement as $20-7-0=13$, which would correspond to a decomposition of $20=13+7+0$. Because of the previous strict intervention by the teacher, doubts remain about whether this presentation corresponds to Jarek's original approach.

In terms of Toulmin's theory of argumentation, one can see a rather sophisticated argument that Jarek formulates (see above Fig. 3.3).

Referring back to the first phase, we can reconstruct backings in neither of the two phases. This happens often in the discourse of primary mathematics classroom. We call the subset consisting of "data so conclusion, since warrant" as the "core" (Krummheuer 1995, p. 243) of an argumentation. Formally, this core is the "minimal form of an argumentation" (p. 243). The applied warrant provides the ground for an argument that certifies the soundness of all arguments of that type.

The Analysis of Participation

Let me now come to the topic of "participation". An important question of this type of analysis is: What kind of mathematical responsibility and originality has to be ascribed to the students' utterances? With respect to the first phase of the scene, for example, one has to clarify, whether all the answers of the students are based on their own original arithmetical calculations or whether they are rather slight modifications of solutions presented shortly before by another student.

In order to differentiate speakers' utterances according to their responsibility and originality, we refer to Goffman's proposal of decomposing an utterance into the following partial functions:

- the syntactical form with its specific choice of words and its specific formulation (function of formulation) and
- the content-related (semantic) contribution (function of content).

A speaker does not necessarily need to take over responsibility and originality for both functions but only for one or in fact also for none of them. Formally, one can differentiate four cases:

- A speaking person is completely responsible (syntax and semantic) for his utterance: a speaker of this kind of liability is called an "author". He is a person who verbally expresses his own idea in his own words.
- A speaking person claims responsibility neither for the syntactical form nor for the semantic aspect of his statement. A speaking person in this status is called a "relayer".
- A speaking person takes over (almost) identical formulations of parts of a prior utterance and tries to thus express an own, new idea. This status is called "ghostee". He traduces his *own idea* in the words of somebody else.
- A speaking person takes over the idea of a prior utterance and tries to express it with his own words. This status is called "spokesman". Such a person paraphrases the contents of a prior statement with his *own words*.

One can summarize these distinctions as shown in Fig. 3.4 below.

The names of these four categories are taken from Levinson (1988), who further developed Goffman's initial ideas. Except with the category "author", there must be a person's utterance that procedes the momentary speaker. Goffman (1981) calls such a setting "production format" (p 145). This expression has been altered to "production-design", since in the interactional theory of mathematics learning that this article is based on, the notion of "format" is already occupied by the meaning in Bruner's work on language acquisition (Bruner 1982, 1983; Krummheuer 1995).

In the analyses I apply first the analysis of participation and then use these results for an analysis of argumentation pertaining to the same utterances in an interaction. Then the content and formulation of such an utterance can be additionally classified according to the Toulmin-scheme. For example, a student reformulating a warrant in his own words, which had been produced by the teacher earlier would be classified as a *spokesman of a warrant*.

I will apply this production-design to the presented example. We begin with the first phase of the scene trying to reconstruct how much responsibility and originality one can ascribe to the answers of the students. We see that there are three children who present "thirteen" as an answer. They are Franzi <095>, Goran <098>, and Wayne <101>. Then Jarek follows with the answer "three plus ten" <105>, then

	Responsibility for the **content** of an utterance	Responsibility for the **formulation** of an utterance
author	+	+
relayer	-	-
ghostee	+	-
spokesman	-	+

Fig. 3.4 Design of participation

Mariana with “ten plus three” <101.1>, and then Julian with “eleven plus two” <110>. The students who gave these answers could have mentally calculated their solutions in advance. The first, not-trivial answer “3 + 10” by Jarek in line 105 could also be given without any mental calculation, though, through the splitting of the German word “dreizehn” (*three-ten*) into “three and ten”. This is also, additionally, suggested by the fact that earlier the number of visible pearls is already ratified as 13.[4] The following two answers “10 + 3” and “11 + 2” again could have been found by mental calculation, through logical consideration or through a “strategy of minimal change”. The mental activity in the case of 10 + 3 would be to apply the commutative law and in the case of 11 + 2 the strategy of regrouping (the first summand of 10 + 3 is increased by 1 and the second summand is correspondingly decreased by 1). I use the term “strategy of minimal change” to describe a tactical behavior of students, which, without much cognitive exertion, consists of applying a minimal change to an answer which has previously been evaluated as correct, and then to ‘wait and see’ what the teacher thinks of it.

In the course of this smooth course of interaction, it remains unclear in which status the answering students are reacting. Obviously it does not seem necessary for the participants, or at least for the teacher, to make this differentiation. Either it might be clear for them or it might not be important for them or her at this moment., Except for the first answer of Franzi, all the other students act by applying the strategy of minimal change. We can classify Jarek by his answer “three plus ten” as a spokesman: he takes over the syntactic composition of the German word for 13 and paraphrases it by separating this word into its components “three” and “ten”. Under the given assumption of the application of the strategy of minimal change, we understand that Marian in her following answer “ten plus three” is acting in the status of a relayer: She takes over Jarek’s answer “three plus ten” and just reverses the two number words.[5] By analogy, Julian’s status, while presenting his answer “eleven plus two”, can be interpreted as that of a relayer—just repeating Marian’s solution with minimal change—or as that of a spokesman – paraphrasing Marian’s solution- in his own words, again making use of the assumption that he applies the strategy of minimal change.

Bringing together these results the analysis of participation looks as follows (Fig. 3.5).

[4] In contrast to the English word “thirteen“the German word for 13 contains the identical names for 3 (”drei”) and 10 “zehn”. The word for 13 is just the combination of “zehn” and “drei” to “dreizehn”. In colloquial German it is also common to use the word “und” (in English: “and”) for the arithmetic expression “plus”.

[5] As mentioned above, in this concrete situation of a smoothly running course of interaction, the participants by themselves keep the process of negotiation in an ambiguous state, in which it remains unclarified, what the single students might have thought, when presenting their answers. That impacts our results so that we, as analysists of this situation, have to consider a certain vagueness in our interpretation.

function of: speaking person	utterance	*reference to a prior speaker*	argumentative function of the utterance
teacher: author	I'm keen to see what the children say *holds a string of pearls in the air*		presenting the problem (**data**)
Franzi: author	thirteen		presenting an answer (**conclusion**)
Goran: relayer	thirteen	*Franzi*	presenting an answer (**conclusion**)
Wayne: relayer	thirteen	*Goran, Franzi*	presenting an answer (**conclusion**)
Jarek: spokesman	three plus ten	*Wayne, Goran, Franzi*	presenting an answer (**conclusion**)
Marian: relayer	ten plus three	*Jarek*	presenting an answer (**conclusion**)
Julian: relayer/spokesman	eleven plus two	*Marina*	presenting an answer (**conclusion**)

Fig. 3.5 Analysis of participation

In contrast, I will present the analysis of participation, as Jarek argues for his conclusion "seven minus zero" (see below Fig. 3.6).

We can sum up this table in the following way: Jarek offers an entirely new solution, which lets him assume the status of an author. The teacher repeats his answer with a rising pitch in her voice, which is interpreted as her criticism of the solution. She does this in the typical manner of a ghostee: she re-applies Jarek's phrase, but ascribes to it a different semantic meaning: the answer is not right, as Jarek assumedly proposes. Later, the teacher attempts, to weave Jarek into her "format of argumentation" (see Krummheuer 1995) as a spokesman, but fails. Jarek counts seven pearls on the string, as wanted by the teacher. But he transforms her requirements into his own argument, and by this he creates the first warrant. He accomplishes the second warrant, then, in an independent and self-responsible status. In this phase, Jarek acts in a relatively elaborated role and, with regard to the participation in the first discourse, this distinguishes this phase from the beginning of the scene.

function of: speaking person	utterance	reference to a prior speaker	argumentative function of the utterance
Jarek: author	seven minus zero \		presenting an answer (**conclusion**)
teacher: ghostee	seven minus zero /	*Jarek*	The answer is not correct (**conclusion**)
	……..		
Jarek ghostee	*counts out on his string holding it up in the air* one two three four five six seven	*teacher*	counting (**warrant 1**)
Jarek author	•••••••. minus zero *lets go of the end he counted; shows* •••oooooooooo		manual demonstration (**warrant 2**)

Fig. 3.6 Analysis of Jarek's participation

3.2.2 The Example of "Mister X"

In the calculation game "Mister X", the students are supposed to guess a number between 10 and 20, which has been noted down by one student on the backside of the blackboard, invisible for the class. For any suggested number that the students guess, the information whether the number to be guessed is higher, lower or equal, is noted at the blackboard. For this, a large "X" has been drawn onto the board. Left of this X the smaller numbers are noted, and to the right, the bigger numbers.

In the beginning of the scene, the numbers 10 and 12 are written on the left side of X, and, among others, the numbers 18 and 14 are written on the right side of X on the blackboard. 13, the number to be determined, is written beneath X. After the correct number—13—has been called out and noted on the blackboard, the following sequence develops:

3.2.2.1 The Transcript

132.		… Why could it only be thirteen in the end? Nicole?
133	9:24 h	
134	Nicole	Because **all** numbers were already in there, except thirteen

135	T	Nope! **Fifteen** for example
136	Jarek	And sixteen.
137	<T	There's a different reason. There's a different reason. **David**
137.1	< Jarek	Sixteen
137.2	*general restlessness in the classroom, loud coughing*	
138	>T	Because fourteen was too big, David
138.1	>S1	A b c d
138.2	>S2	Man, cut it out
139	T	So **stop**. This is **very** important now. Again
139.1	Efrem	And hundred
140	<David	Because fourteen was too big. *Lays the upper part of his body onto the table.*
141	<T	*Points at the 14* yes. And? *Points at 12*
142	David	The, twelve was too small
143	T	Re**peat** that for us. Efrem, **Efrem**, repeat that for us!
144	T	David just said something **very smart**, you can **keep that in mind**. Who can **repeat** that for us again,
145		what **David** just
146	T	said? *quietly* **Petra**
146.1	9:26 h	
147a	<Petra	Because, 'cause the fourteen was too **big**, an and the
147.1	<T	*points at 14*
147b	>Petra	twelve was too too **small**
147.2	>T	*points at 12*
148	T	And in between there remains only **one**
149	Petra	Thirteen
150	T	*Points at 13* thirteen

3.2.2.2 The Analyses of the Scene

The Analysis of Argumentation

The statement "number 13 was the only possible solution in the end" appears as the conclusion of the argumentation. The first argumentation by Nicole in <134> can be understood in the sense that 13 is the sought after number because all other possible numbers have already been excluded (see below Fig. 3.7).

This is all that is generated with regard to this argument in this section of the scene. We cannot reconstruct the explication of a warrant or a backing. From the view of a theory of argumentation, only the warrant makes it possible to transfer the approval of given data to the conclusion. In other words, only through the warrant do accepted statements become the data of an argumentation. In the present case, the warrant might be very obvious so that it does not appear to be necessary to further mention it. To clarify our own understanding of Nicole's argument, I would

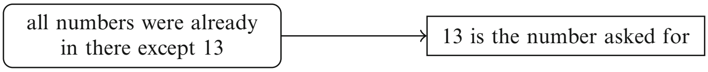

Fig. 3.7 Toulmin scheme of Nina's first argumentation

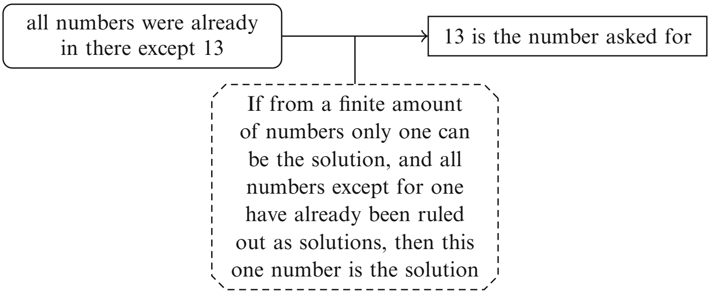

Fig. 3.8 Toulmin scheme of Nina's first argumenation including assumable warrant

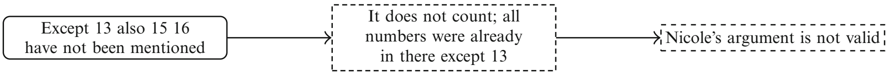

Fig. 3.9 Imagined following argumentation

like to propose an assumable warrant: If from a finite amount of numbers only one can be the solution, and all numbers except for one have already been ruled out as solutions, then this one number is the solution. I do not suppose that students of this first grade can express the warrant with such explicit terms and formulations. Within the graphic layout, I have outlined the warrant with a dashed line in the layout to clarify its origin (see above Fig. 3.8).

One can assume that the first graders within this scene grasp this figure of argumentation. At least during the immediate follow-up statements <135 and 136>, the legitimacy of such a conclusion is not *as such* doubted. Rather, the *data* that Nicole mentioned in <134> is being denied. On the level of its conclusions, the argumentation could be imagined as outlined above in Fig. 3.9.

However, the Toulmin categories shown with a dashed outline are not explicitly formulated. With the statement of the teacher in <137>, there is another reason since she also seems to assume that Nicole's argument has been sufficiently refuted and does not require further comment. With regard to the later exposition of the students' participation in the processes of argumentation, Nicole, the teacher, and Jarek are actively participating.

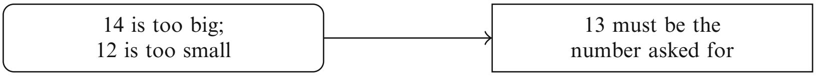

Fig. 3.10 Arguments noted on the blackboard

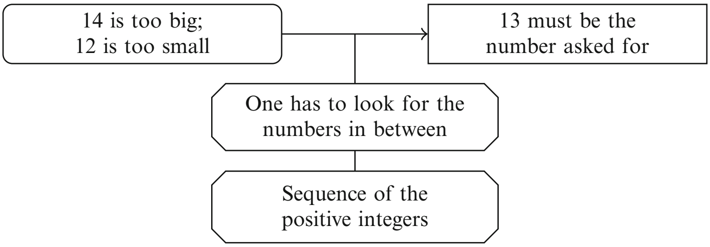

Fig. 3.11 Warrant and backing for 13 as the correct number

From <138> onwards a new argumentation develops. It is expressed in <138, 140 and 141>. In summary, the argument can be understood as follows: 12 and 14 are noted on the blackboard as "too big" and "too small" numbers respectively; they can be referred to as an unquestioned fact. The number in between has to be found. As a layout (see above Fig. 3.10).

In <147 to 148> only a few short objections occur that contain evidence of the warrant and the backing. Thus, only through the conjunction "and", that is formulated by the teacher in <147.1>, is it possible to draw the conclusion from the data added by David. Within this conjunction the warrant is thus implied, which is then further elaborated upon in the succeeding statements. The solution number thus has to lie in between the already stated too big and too small numbers (see above Fig. 3.11).

According to Toulmin, the legitimacy of the warrant, if need be, can be backed by the indication of fundamental conviction. In the given argumentation, one can find a clue to a backing that is to be assumed: The statement "only thirteen in the end" refers to the discrete sequence of the positive integers as a fundamental attribute that justifies this conclusion. The clearness of the solution 13 that is asserted by the initial statement of the teacher is thus backed by the sequence of the natural numbers.

Looking back on the entire argumentation, several aspects seem worth mentioning: Within the scene, two argumentations, or cycles of argumentation are generated that exhibit very different degrees of explication according to Toulmin's categories. The last argumentation is worked out relatively completely. One can recognize here that the teacher, surely driven by a didactical motivation, purposefully contributes to this.

The Analysis of Participation

The teacher opens the argumentation by naming the conclusion that is still to be substantiated. David's statement "because fourteen was too big" <138> is accepted as a first answer. He formulates the statement as a justification and at this point introduces data himself. He thus takes responsibility with regard to the phrasing and content function for the statement. He functions as an author. He repeats this data <140> following the teacher's request <139>. In this repetition he functions as the imitator, whereas the responsibility for the choice of words is to be seen in his first statement: he imitates himself, so to speak. The teacher accompanies his statement with gestures <141> and thereby transfers it into another form. She supports David's idea as a spokesman. With "and" <141>, the teacher demands more data. The second data, twelve is too small <142>, is only stated by David, after the teacher has pointed at the twelve on the blackboard <141>, which has been noted there as "too small". The *teacher* thus introduces the idea of this second data to the process of argumentation. David transforms the gesture, which includes the idea of data, into words. As a spokesman, he thus introduces the idea with a new choice of words.

Petra's statement in <147> is the repetition of David's statements. The teacher had given her this task <145,146>. It is possible to reconstruct her role at this point as a relayer. We understand the teacher's gestured reference toward the two numbers noted on the blackboard in <147.1> primarily as an underscoring of Petra's statement and thus they move within the framework of Petra's act of imitation.

In <148>, the teacher takes up a new idea. She asks for the numbers in between 12 and 14. At this point she functions as the author and with it, constructs parts of a backing, only complete with the naming of the number 13 by Petra in <149> and by the teacher in <150>. It is a little more difficult to classify Petra's status with her statement in <149>. She clearly speaks before the teacher points to the number 13 in <150>. In this respect, she does not just parrot the answer. Her answer resembles the insertion of one word into a completion exercise, in which it is largely clear, what is needed to fill the gap. It has to be considered that the number 13 is noted on the blackboard, and in a way, must be spoken out loud. We usually attribute speakers of such one-word answers the status of an imitator (see below Fig. 3.12).

We summarize the results of the analysis in the following table.

3.2.3 Comparison of the Results of the Analyses of the Two Scenes

The two scenes belong to the same mathematics class and took place only a few days apart, their topics refer to the introduction into the 2-digit-integers from 10 to 20, they are teacher-guided, in both episodes the teacher deliberately stresses the argumentative aspect of explaining a solution that came up. This opens the scope of

speaker: function	statement	*reference to a previous speaker*	argumentative function of the statement
teacher: author	why could it only be thirteen in the end		unambiguousness of the solution number 13. (**conclusion**)
David: author	because fourteen was too big		14 was too big. (**data**)
David: relayer	because fourteen was too big	*David*	
teacher: spokesman	*points at 14 yes*	*David*	14 was too big. (**data**)
teacher: author	and		link to still to be created second data (**warrant**)
teacher: author	*points at 12*		12 was too low. (**data**)
David: spokesman	the . twelve was too small	*teacher*	12 was too small. (**data**)
Petra: relayer	because cause fourteen was too big, an and twelve was too too small	*David*	upper and lower limit (**data + warrant**)
teacher: author	and in between there is only one		sequence of pAositive integers (**backing**)
Petra: relayer	thirteen	*teacher*	sequence of positive integers (**backing**)

Fig. 3.12 Analysis of participation

some reflections concerning the accounting practice and its impact on the conditions of the possibility of learning mathematics.

In the beginning I introduced the notion of “argumentative learning”. The comparison of the results of the analyses of the two episodes is now going to focus on the question: how is the accounting practice interactively accomplished and how are the participants of the class engaged in this process?

Comparing analyses of the two episodes one finds processes of argumentation that are rather differently shaped.

- At the beginning of the example "thirteen pearls", a cycle of subsequent arguments emerges, in which the warrant is a non-subject-related one: The inference is legitimate as long as the teacher does not intervene. From the stance of an observer it is not possible to decide whether the students act according to the strategy of minimal change or presented answers based on personal sophisticated mathematical reasoning. With regard to Toulmin's understanding of an argumentation, the related accounting practice seems rather incomplete: basically the arguments emerging in this practice are of the form "data, so conclusion". The warrants are not content-related. So, it remains rather diffuse, what kind of responsibility and originality the students need to possess in order to successfully participate in this accounting practice. This is true for the students who present a solution as well as for the students who remain all the time in the status of a recipient. The practice seems to be conducted very much as a routine.
- Rountinized accounting practices combined with fairly incomplete productions of arguments can also be reconstructed in "Mister X". Here for certain argumentations only the data are presented, and the rest of the argument remains implicit. The conclusion and the warrant seem self-evident. Different to the beginning phase of "Thirteen Pearls", however the status of the participants proves to be more differentiated. One could suspect that David would not have generated the data of the lower limit, the number 12, without his involvement in the process of interaction with the teacher. It remains unclear, how much David grasps the *logical* necessity of the connection of the two data (higher and lower limit), and therefore the warrant, that is undertaken by the teacher through "and". With the status of a spokesman, at this point David acts in what we perceive as a relatively elaborated form of responsibility and originality. It is different in Petra's case: she is involved in the production of the backing, and in a certain sense, one would be able to assume a deeper understanding at this point. However, she makes her statements in the status of a relayer. This suggests that such a deeper understanding does not necessarily have to exist and that not so much the logic of the matter, but rather the logic of the interaction is dominant. Petra thus participates in a "well-functioning" manner and through this, might also contribute to the presentation of a reasonable argumentation for the silent and watching students.[6]
- Finally, Jarek's contribution to the explanation of his unique solution of the decomposition of the number 13 in Thirteen Pearls reveals two aspects of an elaborated contribution to an argumentation: first, utterances will be delivered which cover the entire core of an argument. Second, the students participate in this production in the status of ghostees, spokesmen and/or authors.

If we would interpret the production of more complete arguments and the participation in the status of ghostee and spokesman as different grades of autonomous

[6] As a completion of our hierarchy of analyses we introduce a recipient analysis (see Krummheuer and Brandt 2001 and Krummheuer 2011).

action within a given structure of interaction, then it would be possible to understand the two analyzed episodes as learning situations in which the students have arrived at different levels of autonomy with respect to their mathematical capacity to solve problems of the given kind. Students who can contribute to the production of a warrant in a common argumentation process know more and understand more of the subject matter than students, who can only produce the data or the conclusion. But this is only one side of the coin: students with the status of a spokesman and/or ghostee can be described as learners, who have already embarked upon a path of more autonomous action. Acting in the status of a relayer can be seen as the very first step in the apprenticeship of applying specific mathematical knowledge in a sensible way. Acting in the status of an author and producing the entire core of an argument autonomously would be taken as an indicator that this student is not in the situation of learning something new (see Krummheuer 2011).

The accounting practice and the production design refer to a sub-arena of everyday mathematics classroom situations that make it possible in a special way to grasp the conditions of making learning possible for active participation of students. In the process of collective argumentation, an argument is generated, and the actively participating student participates in its production in two respects:

- he produces statements that can be allocated to certain categories in the sense of Toulmin, and
- through this he takes up a specific speaking role.

3.3 Some Theoretical Remarks

The typical focus of the combined analyses demonstrated above is on micro-analytic processes that are elicited by the turn-taking mechanisms of each interaction process (Mehan 1979; Schegloff 1982; Sacks 1998). This method of analysis helps to clarify aspects of the social conditions of (mathematics) learning on the interactional level of the local turn-taking organization. From a perspective of mathematics education, interest is sometimes targeted on more complex processes of argumentation than that presented above and on long-term processes of mathematics learning, as over an entire course or over a developmental phase of a child. In this section, I first mention some research that deals with the aspect of participation (Sect. 3.3.1). In the second section I refer to approaches that analyze composite argumentations consisting of several Toulmin-schemes (Sect. 3.3.2). Most of them do not take into account the notion of participation.

3.3.1 Further Research on the "Production-Design" in Mathematics Classes

The notion of participation becomes increasingly crucial the more the theoretical interest moves towards a socio-constructivist conception of mathematics learning. There are several approaches that adhere to this general position, which Sfard (2008) characterizes as "participationism" (p. 76). From this stance, she claims, the metaphor of "learning-as-acquisition" has to be replaced by one of "learning-as-participation" (p. 92). She further develops her argument clarifying that mathematics and mathematics learning emerge in specific forms of discourse, and that a membership is won through "participation in communicational activities of any collective that practices this discourse" (ibid. p. 91). Becoming a member of such a discourse involves the learning of the rules and routines of these discourses. In Sect. 3.2.2 I presented my ideas about this participationist view of mathematics learning based on my research.

Astoundingly, to my knowledge, there is hardly any discussion about research methods concerning the topic of participation. Birgit Brandt and I developed an additional method for the analysis of the recipient design (Brandt 1998; Krummheuer and Brandt 2001; Brandt and Tatsis 2009; Krummheuer 2011). In the work of Brandt and Tatsis (2009) the authors stress the issue of face-saving moments in classroom interaction.

3.3.2 More Complexly Structured Argumentations

I restricted the analysis of argumentation to the notion of argumentative learning thus connecting an interaction theory of mathematics classroom processes with a socio-constructivist theory of mathematics learning. Toulmin's model of argumentation is not restricted to this issue. Within mathematics education Toulmin's approach is often related to the widely spread discussion about proof in mathematics classes. As long as this discussion is focused on primary mathematics education, it usually refers to the notion of "explanation" (Whitenack and Knipping 2001; Yackel 2002; Fetzer 2011). At the secondary school level, the work about students' argumentation, then, is directly related to the idea of a mathematical proof. For example, concerning the Pythagorean Theorem in French and German mathematics classes, the work of Knipping (2003) might the mentioned. At university-level the group around Rasmussen analyzes argumentations of university students during differential equations classes (Stephan and Rasmussen 2002). Hoyles and Küchemann (2002) analyze the quality of logical deduction and mathematical arguments. Pedemonte (2007) is interested in the relationship of the content and structure of arguments from a cognitive point of view. A rather rhetorical research approach about argumentation can be found in Inglis and Mejia-Ramos 2008), whose work focuses on the warrant of an argumentation. Most of these studies analyze more complex arguments that exceed the structure of an iterated argument, as exemplified at the beginning of the scene "Thirteen Pearls". Especially Knipping (2010) reconstructs different patterns for such more complex arguments (see more in Knipping and Reid in this book).

Analyses at this more global level allow comparisons of styles of discourse in mathematics classes and can help to elaborate the relationship between argumentations in everyday mathematics class situation and "pure" mathematical proofs as they are taught in mathematics classes.

Appendix: Transcripts and Rules of Transcription

Thirteen Pearls

92	L	jaha / so \ . ich . bin mal gespannt was die Kinder sagen \
93		*hält eine Rechenkette in die Höhe*: ···oooooooooo
93.1		8:59 h
94	Marina	ach so
95	Franzi	Dreizehn
96	*Marina, Franzi, Jarek und Wayne melden sich; einige Kinder zählen*	
97		*flüsternd.*
98	L	*flüsternd* zwei drei
99	Goran	dreizehn *meldet sich dabei schnell*
100	*Julian, Conny und noch zwei andere Kinder zeigen auf*	
101	L	*flüsternd* zwei drei vier fünf Finger sehe ich . sechs . sieben .
102		acht . + .. Wayne /
103	Wayne	dreizehn \
103.1	Marina	*nimmt den Arm runter*
104	L	oder \
105	S	Ä
106	L	Jarek /
107	Jarek	äm .. drei plus zehn \
108	<L	oder \ . . Marina /
109	<Marina	*zeigt betont auf* zehn plus . drei \
110	L	oder \ oh . die Kinder sehen ganz schön viel ne / . das ist
111		immer dasselbe aber die sehen ganz schön viel . Julian \
112	Julian	äh . öl . nee . doch ölf . plus zwei \
113	L	oder \ .. Jarek /
114	Jarek	sieben minus null \
115	L	sieben minus null /
116	S	häh /
117	S	häh /

118	L	versuchen wa mal \ . komm mal nach vorne / sieben minus
119		null / ... der Jarek hat was gesagt /
120		und das müssen wir mal überprüfen \ komm mal her
121	Jarek	*kommt nach vorne*
122	L	*hält Jarek eine Rechenkette hin* zeig mal sieben minus null . zeig mal sieben / dreh
123		dich mal zur Klasse um damit die Kinder das sehen können
124		und damit man das **vergleichen** kann \ *hält wieder ihre*
125		*eigene Rechenkette hoch; dabei zeigt sie nach*
126		*wie vor dreizehn an* also \ . **sieben** /
126.1	Jarek	*zählt leise die Kugeln an seiner Kette ab*
126.2	L	zähl mal **ganz laut** /
127	Jarek	*zählt an seiner Kette ab und hält sie dabei hoch* eins zwei
128		drei vier fünf sechs sieben *Perlenkette*: ······ . minus null
129		*lässt das abgezählte Ende fallen; zeigt*
130		···ooooooooooo ist dreizehn \
130.1		*Währenddessen die Lehrerin redet geht Jarek an seinen*
131		*Platz zurück.*
132	L	*erstaunt gehaucht* hha jetzt versteh ich \ was hat der Jarek
133		gemacht \ ... *legt ihre eigene Kette weg und übernimmt die*
134		*von Jarek* . der hat behauptet / . der hat von dieser Seite
135		angefangen und hat sieben abgezählt \ eins zwei drei vier
136		fünf sechs sieben \ *zeigt es an ihrer Kette*
137		da hat er gesagt . minus null ist . das . *zeigt*
138		···ooooooooooo geht das \

Mister X

132	L	. wesh-... weshalb konnte es nur die Dreizehn sein \Nicole
133	9:25 h	
134	Nicole	weil aweil **alle** Zahlen schon da drinne außer dreizehn
135	L	nö / . **fünfzehn** zum Beispiel /
136	Jarek	und sechzehn \
137	<L	gibt nen andern Grund \ gibt nen andern Grund \ **David** \
137.1	<Jarek	sechzehn
137.2		*Es herrscht allgemeine Unruhe in der Klasse. Es wird laut gehustet.*
138	<L	weil vierzehn zu groß war - David
138.1	<S1	a b c d

138.2	<S2	Mann, hör aaauf
139	L	so **stop** \ das is jetzt **ganz** wichtig \ nochma \
139.1	Efrem	und hundert
140	<David	weil vierzehn zu groß wa\ *legt seinen Oberkörper auf den Tisch* +
141	<L	*zeigt auf die 14* ja \ . + und / *zeigt auf die 12*
142	David	die . zwölf war zu klein \
143	L	wieder **hol** + das ma Efrem \ **Efrem** \. wiederhol das ma
144		\ . der David hat was **ganz Kluges** gesagt \das könntihr
145		euch **merken** \ wer kann das nochmal **wiederholen** was
146		der **David** gesacht hat \
147		*leise* **Petra** \
147.1	9:26 h	
148.1	<Petra	weil weil die vierzehn zu **groß** war / un und die zwölf zu zu **klein** war \
148.2	L	*zeigt auf die 14* ja \. + und / *zeigt auf die 12*
149	Petra	Dreizehn
150	L	*zeigt auf die 13* dreizehn \ +

Rules of Transcription

Column 1	396	Serially numbered lines
	396.1	Inserted lines
Column 2		Time line
Column 3		Abbreviations for the names of the interacting people
Column 4		Verbal (regular font) and non-verbal (*italic font)* actions
/		Rising pitch
–		Even pitch
\		Falling pitch
,		Breathing space
. .. …		Breaks of. 1, 2 or. 3 s
(4 s.)		Breaks of a specified time span
bold		Accentuated word
s p a c e d		Spoken slowly
(word)		Unclear utterance

(remark)	Remark, offering alternatives to unclear utterances
+	The indicated way of speaking ends at this symbol
#	There is no break, the second speaker follows immediately from the first
<	Indicates where people are talking at the same time
>	The next block of simultaneous speech is indicated by a change in arrow direction

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Chapter 4
Reconstructing Argumentation Structures: A Perspective on Proving Processes in Secondary Mathematics Classroom Interactions

Christine Knipping and David Reid

Abstract This chapter provides theoretical and methodological tools, both to reconstruct argumentation structures in mathematical proving processes and to shed light on the rationales of those processes. Toulmin's functional model of argumentation is used for reconstructing local arguments, and it is extended to provide a 'global' model of argumentation for reconstructing proving processes in the mathematics classroom. Several examples drawn from empirical research are included, illustrating each stage of the methods used. Comparison of argumentation structures reveals differences in the rationale of proving processes in different mathematics classrooms.

Keywords Argumentation • Mathematical proving processes • Comparative methods

4.1 Introduction

This chapter describes a variant of the method of analysis of argumentation processes in mathematics classes described in the previous chapter, see also Krummheuer (1995). It builds on Toulmin's theory of argumentation (Toulmin 1958) and provides a method to reveal the rationality of arguments that are produced during proving processes in secondary level mathematics classrooms. The method allows the

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C. Knipping (✉) • D. Reid
Faculty 3 of Mathematics and Computer Science, University of Bremen, Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: knipping@math.uni-bremen.de; dreid@math.uni-bremen.de

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description of both global argumentation structures as well as local argumentations like those described by Krummheuer.

This chapter suggests a method by which complex argumentations in proving processes can be reconstructed and analyzed. A three stage process is proposed: reconstructing the sequencing and meaning of classroom talk; analyzing local argumentations and global argumentation structures; and finally comparing these argumentation structures and revealing their rationale. The second stage involves two moves, first analyzing local arguments on the basis of Toulmin's functional model of argumentation, and second analyzing the global argumentative structure of the proving process. The first stage and the first move of the second stage are similar to the processes of *analysis of interaction* and *analysis of argumentation* described by Krummheuer. We will describe some differences in our methods in these stages, but we will focus here mainly on the second move of the second stage, analysis of global argumentation structures. To illustrate patterns in the global analysis of the classroom talk a schematic representation of the overall argumentative structure is used.

While there is considerable methodological overlap between the methods described here and those described by Krummheuer, there are some differences arising out of our differing research contexts and interests. Proving processes in secondary school classrooms follow their own peculiar rationale, and our interest is in reconstructing and analyzing the complex argumentative structure of these classroom conversations. Like Krummheuer we see argumentation as both a precondition for learning and a desired outcome. In the secondary classrooms we study mathematical proof is the topic, and so our emphasis is on argumentation as an outcome. We are more interested in *learning* argumentation than in *argumentative* learning.

In addition, we analyze both local argumentations and global argumentation structures, and our global analyses allow comparisons of styles of discourse in mathematics classes. This allows us to consider not only the "classroom culture" in a given classroom, but also to compare classroom cultures and to identify differences and similarities at the level of global argumentation structures.

This chapter is organized as follows. We first discuss not only why an alternative conception of 'rational argument' (see Toulmin 1958), distinct from the one in formal logic, is important for understanding proving processes in the mathematical classroom, but also how such a conception leads to a different reconstruction of arguments found in classroom proving processes. This provides an alternate perspective on arguments in the context of classroom proof and proving. We then describe the theoretical and methodological grounds of Knipping's (2008) method of reconstructing and analyzing argumentation structures, before presenting the method itself. Examples of how we apply the argumentation analyses to real data will illustrate each individual stage of the method. We then compare global argumentation structures reconstructed from a German and a Canadian classroom to show how this method can reveal differences in the rationale of proving processes.

4.2 The Importance of Understanding Proving Practices in the Classroom

Teaching proof is considered to be challenging. Numerous empirical studies have documented that students up to university level have difficulties in recognizing different types of reasoning and producing mathematical proofs (e.g., Harel and Sowder 1998; Healy and Hoyles 1998; Reiss et al. 2001).[1] Students find proof difficult and often do not understand why so much emphasis is put on mathematical proof (Moore 1994). Some research has investigated these difficulties (Chazan 1993; Reid 1995; Pedemonte 2002a, b, 2007) and has explained aspects of the problem from an individual student's point of view. Researchers have offered alternative ways of teaching proof (e.g., Garuti et al. 1998; Mariotti et al. 1997; Jahnke 1978), but only a few have documented students' proving processes in alternative teaching environments (Balacheff 1988, 1991).

Very little research so far has looked at these difficulties with a focus on proving practices in the classroom (Sekiguchi 1991; Herbst 1998, 2002a; Knipping 2003). Such a focus is important for two reasons:

- Teachers' approaches to proof are not guided solely by logical considerations; pedagogical and practical considerations are also important.
- Written proofs, whether produced in classrooms or presented in texts, do not reflect the process of their creation, which is itself worthy of study.

Sekiguchi (1991) explores the social nature of proof in the mathematics classroom using ethnographic methods. He confirms that proof practice in the classroom does not follow "the patterns of formal mathematics", but that pedagogical and practical motivations shape practices in the classroom. He finds various forms of practices called "proof" by the participants, but his research focuses on the standard form for writing proofs in US classrooms, the two-column proof format (Sekiguchi 1991). Herbst (2002b) describes how this proving custom developed at the end of the nineteenth-century, in a historical context where the demand arose that every student should be able to *do* proofs. Herbst helps us to understand that the practice of producing two-column proofs was a reaction to this demand and the difficulty "to organize classrooms where students can be expected to produce arguments and proofs" (Herbst 2002b, p. 284). Thus a way of writing proofs that is peculiar to the school context evolved, not out of logical requirements, but out of a specific historical context, students' difficulties, and teachers' efforts to address the difficulties and challenges of engaging students in proofs.

The two-column proof format is not the only format that can be found in mathematics classrooms. In France and Germany (Knipping 2002, 2003, 2004),

[1] For an overview of these studies see as well Harel and Sowder 1998 or Hanna 2000.

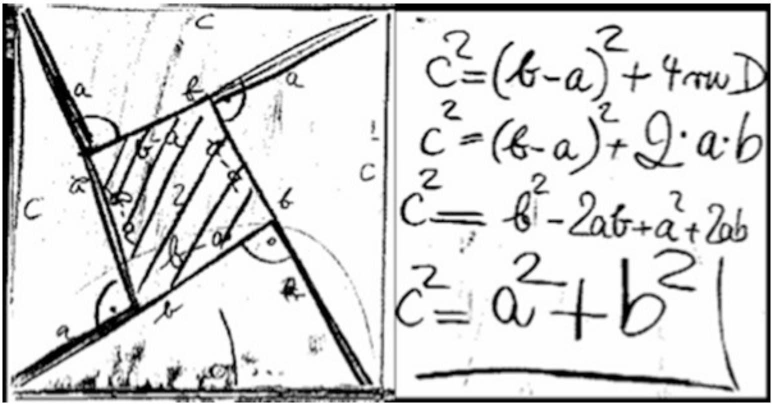


Fig. 4.1 Written proof of the Pythagorean Theorem in a grade 9 class in Germany, 1998. ("4rwD" means *4 rechtwinklige Dreiecke*, i.e., 4 right angled triangles)

other formats for written proofs have evolved. Whatever format is used in presenting a proof, in the classroom context writing proofs is only part of, and more a means to engage students in, proving (or the illusion of doing proofs), as Sekiguchi and Herbst demonstrate. In classrooms the process itself, embedded in the constraints of teaching, seems to be of major importance, but arguments produced in these processes are far different from the written forms that are produced. For example, Fig. 4.1 shows a photograph of a blackboard at the end of a lesson on the Pythagorean Theorem, in a German grade 9 upper stream secondary school ("*Gymnasium*") class. The teacher and probably the students considered this a proof of the Pythagorean Theorem. The proof was the product of processes during an entire lesson. Diagrams like that presented on the left side of the board are classic illustrations of the Pythagorean Theorem and its proof. They are recognized by many people, and can be found throughout the history of mathematics in many cultures. However, our focus in this chapter is on the proof written on the right side of the blackboard and the argument that led to it. The algebraic equations written there are a very condensed form of argument, which leaves out many steps that have been developed in the oral part of the classroom proving process we will discuss later.

Because classroom proving processes are guided by more than logical considerations and because the written proofs that result from classroom proving processes are incomplete as records of those processes, reconstructing classroom proving processes and their overall structure requires a model that acknowledges their context dependence. Research within and outside of mathematics education has addressed this concern that there is no universal way to describe and formulate context dependent arguments. We will discuss this next.

4.3 Approaches to Describing Arguments

4.3.1 The Inadequacy of Logical Analysis for Reconstructing Proving Processes in Classrooms

In mathematics considerable attention has been paid to the nature of proofs. As proofs can be seen as the end product of the work of mathematicians, i.e., as explanations which are accepted by the mathematical community as proofs (Balacheff 1987) one might expect that the process by which proofs come to be can be analyzed on the basis of what proofs are. Further, the final goal of teaching proof is to bring students to an understanding of the logic behind mathematical proofs and to accept the same kinds of explanations as proofs as are accepted by mathematicians. Therefore, the misunderstanding can arise that the analysis of proof teaching in the classroom can be based on logical analysis of classroom arguments. This is not true for several reasons.

First, as discussed earlier, Sekiguchi (1991) and Herbst (1998, 2002a) illustrate that when teachers teach proof they do not follow "the patterns of formal mathematics", but pedagogical and practical motivations shape their practices in the classroom. Their practices in the classroom are a complex response to the challenge of teaching proofs and of engaging students in proving. In these multi-faceted processes classroom conversations with complex argumentation structures occur that might appear "illogical" to a mathematician. Revealing these complex structures is necessary to better understand the complexity of teaching proof and proving, but this cannot be undertaken by means of formal logic alone.

Second, students are in the process of developing logical thinking patterns, and so the thinking they express in classrooms will include many elements which a logical analysis would simply describe as "illogical" but which are nevertheless important to the future development of their thinking. As learning necessarily depends on the students' thinking at the time, a method of analysis that cannot go beyond dismissing it as "illogical" is not helpful. What is needed is a conception of 'rational argument' that does not cut off students' rationality, see also Walton (1989, 1998).

Third, logical rationality, which has historically decontextualized intellectual and practical rationality, is widely questioned by philosophers and historians of science. Toulmin (1990), for example, deconstructs the logical ideal of reason as a historical project of Modernity that we have to appreciate, but that has been overcome by the facts of the twentieth-century science. He argues that "the decontextualizing of problems so typical of High Modernity is no longer a serious option" (Toulmin 1990, p. 201). Instead science came to a "renewed acceptance of practice" (p. 192) and to a reconceptualization of rationality that does not cut "the subject off from practical considerations" (p. 201) in the way that formal logic does. In his earlier work Toulmin (1958) had enquired into different fields of argument and addressed the question of "What things about the forms and merits of our arguments are *field-invariant* and what things about them are *field-dependent*?" (p. 15, emphasis in original). Contrasting arguments in jurisprudence to formal logic he suggests that there is a layout of arguments that is field invariant, but that allows the characterization of arguments related to their context of use. This layout or functional model is the topic of the next section.

4.3.2 Toulmin's Functional Model of Argument

Like Krummheuer, we make use of Toulmin's (1958) functional model of argumentation, but with a different focus as we are interested in argumentation as an outcome of learning. Toulmin's model has the important characteristic that is was developed to reconstruct arguments in different fields, such as law or medicine. As noted above, logical analysis is inadequate for argumentations in mathematics classrooms, which perhaps bear more resemblance to argumentations in other domains such as law, where public discussion of facts and the relationships between them are important. Toulmin's model is intended to be applicable to arguments in any field.

Rejecting a mathematical logical model Toulmin (1958) investigates the functional structure of rational arguments in general. Therefore he asks "What, then, is involved in establishing conclusions by the production of arguments?" (p. 97). Toulmin's first answer is that facts (data) might be cited to support the conclusion. He illustrates this by the following example. If we assert that 'Harry's hair is not black', we might ground this on "our personal knowledge that it is in fact red" (p. 97). We produce a datum that we consider as an evident fact to justify our assertion (conclusion). If this is accepted, this very simple step, datum – conclusion, can represent a rational argument.

But this step, its nature and justification, can be challenged, actually or potentially, and therefore it is often explicitly justified. Instead of additional information, an explanation of a more general style, by rules, principles or inference-licenses has to be formulated (p. 98). Toulmin's second answer addresses this type of challenge. A 'warrant' might be given to establish the "bearing on the conclusion of the data already produced" (p. 98). These warrants "act as bridges, and authorize the sort of step to which our particular argument commits us" (p. 98). In the example above the implicit warrant of the argument is "If anything is red, it will not also be black." (p. 98). While Toulmin acknowledges that the distinction between data and warrants may not always be clear, their functions are distinct, "in one situation to convey a piece of information, in another to authorise a step in an argument" (p. 99). In fact, the same statement might serve as either datum or warrant or both at once, depending on context (p. 99), but according to Toulmin the distinction between datum, warrant, and the conclusion or claim provides the elements for the "skeleton of a pattern for analyzing arguments" (p. 99, see Fig. 4.2). In the following we will use "claim" in cases where data and warrants have not yet been provided, and "conclusion" when they have been.

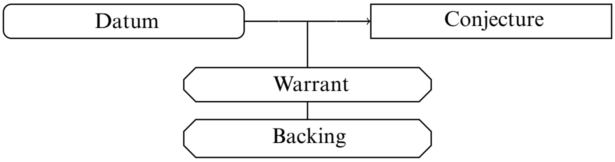


Fig. 4.2 Toulmin model (In our diagrams Data are enclosed in rectangles with rounded corners, Warrants and Backings in rectangles with angled corners, and Conclusions in plain rectangles)

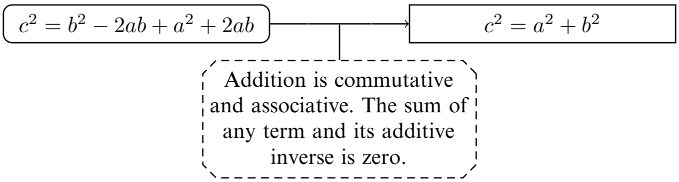


Fig. 4.3 Datum, warrant and conclusion for the final step in the written proof

Toulmin adds several other elements to this skeleton, only one of which will be discussed here. Both the datum and the warrant of an argument can be questioned. If a datum requires support, a new argument in which it is the conclusion can be developed. If a warrant is in doubt, a statement Toulmin calls a "backing" can be offered to support it.

Looking back at the proof in Fig. 4.1, we can analyze the last step in the argument in terms of Toulmin's model (see Fig. 4.3). In it we can see an important characteristic of many arguments: warrants are often left implicit. In this case our awareness that the warrant is not included in the written proof helps us to ask the question "Was the warrant ever stated as the proof was developed, or was it always implicit?", showing once more the importance of analyzing the classroom proving process.

Toulmin states, "The data we cite if a claim is challenged depend on the warrants we are prepared to operate with in that field, and the warrants to which we commit ourselves are implicit in the particular steps from data to claims we are prepared to take and to admit." (p. 100). Therefore careful analyses of the types of warrants (and backings) that are employed explicitly or implicitly in concrete classroom situations, allow us to reconstruct the kinds of mathematical justifications students and teacher together operate on. In particular, the comparison of warrants and backings in different arguments can reveal what sort of argument types are used in proving processes in mathematics classrooms.

For example in Fig. 4.3, we have supplied an implicit warrant based on mathematical properties of addition. In a different context the warrant for this argument might have been geometrical, interpreting 2ab as the area of a rectangle (or two triangles), or syntactical not interpreting the symbols at all, operating on them purely formally. Any of these types of warrants (and backings) could occur in a classroom and indicate the *field* of justifications in which the students and teacher operate.

Other researchers (e.g., Inglis et al. 2007) have made use of other elements in Toulmin's model, including "modal qualifiers" and "rebuttal". Many arguments do not establish their conclusions with complete certainty, and in such arguments we find qualifiers like "probably" and "possibly" as well as rebuttals that identify cases where the conclusion does not hold. Inglis, Mejía-Ramos and Simpson consider the arguments of postgraduate university students in mathematics and find that modal qualifiers play an important role in their mathematical argumentations. In our work in schools, however, we have found that the mathematical argumentations produced are quite different from what advanced mathematics students produce, and we have not found it necessary to make use of any elements in the Toulmin model beyond data, conclusions, warrants and backings. We have added one element, however, which we call "refutation". A refutation differs from a rebuttal in that a rebuttal is local to a step in an argument and specifies exceptions to the conclusion. A refutation completely negates some part of the argument. In a finished argumentation refuted conclusions would have no place, but as we are concerned with representing the entire argumentation that occurred, it is important for us to include refutations and the arguments they refute, as part of the context of the remainder of the argumentation, even if there is no direct link to be made between the refuted argument and other parts of the argumentation. Aberdein (2006) proposes extending Toulmin's rebuttal element to encompass refutations, but for our purposes we prefer to limit rebuttals to Toulmin's original role, of specifying circumstances where the conclusion does not hold.

An important way in which we have used the Toulmin model that extends it significantly, is our application of it not only to single steps in argumentations, but also as a tool to explore the global structure of an argumentation. In the next section we will describe this distinction in more detail.

4.3.3 Local and Global Arguments

Toulmin (1958) notes "an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one" (p. 94). Toulmin's aim is to explore the fine structure, but in considering classroom argumentations both argumentative forms must be reconstructed. Toulmin's model is useful for reconstructing a step of an argument, which allows us to single out distinct arguments in the proving process (for example as in Fig. 4.3). We will call these "argumentation steps" or *local arguments*. But it is also necessary to lay out the structure of the argument as a whole (the anatomical structure), which we will call *global argument* or the argumentation "structure" of the proving process.

Global arguments in classrooms can be quite complex (as will be shown later). The written proof in the right hand side of Fig. 4.1 provides a simple example. As we noted earlier, a single step of that proof is shown in Fig. 4.3. The global argument presented on the blackboard, reconstructed as a chain of argumentation steps is shown in Fig. 4.4. The final conclusion ($c^2 = a^2 + b^2$), a formulation of the

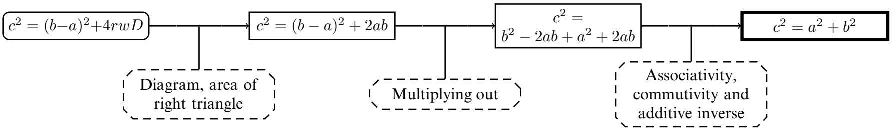


Fig. 4.4 Functional reconstruction of the written proof presented in Fig. 4.1 (The target conclusion is outlined with a thicker line than conclusions that are recycled as data for subsequent steps)

Pythagorean Theorem, is the target conclusion of the global argument. The argument can be reconstructed as a simple chain of conclusions beginning with a datum "$c^2=(b-a)^2+4$rwD" that has been taken from the drawing on the blackboard. This datum leads to a conclusion: $c^2=(b-a)^2+2ab$, but no warrant is explicitly given to support this inference. The information in the diagram (adjacent sides of the right triangle are a and b) and implicit calculations of the area of the four right triangles implicitly support this claim. The next two steps are also based on implicit warrants. In Fig. 4.4 we have reconstructed possible implicit warrants for each step; they are marked by a box with a dashed line. Note that the statement "$c^2=(b-a)^2+2ab$" is not only the conclusion of one step but also the datum of another. Finally the target conclusion: $c^2=a^2+b^2$ is established.

This type of argument can be characterized as a chain of statements, each one deduced from the preceding one on logical and mathematical grounds. This has been described by Duval as "Recyclage" (Duval 1995, pp. 246–248) Once a statement has been established as a conclusion it functions as a datum, an established true fact, in the next step. Aberdein (2006) calls this way of combining single steps "Sequential" and he describes four other ways steps could be combined. As we will see in the following, our empirical research on classroom argumentation provides examples of Aberdein's ways of combining steps, as well as other ways.

4.4 A Method for Reconstructing Arguments in Classrooms

As mentioned above, for reconstructing arguments in classrooms a three stage process is followed:

- reconstructing the sequencing and meaning of classroom talk (including identifying episodes and interpreting the transcripts);
- analyzing arguments and argumentation structures (reconstructing steps of local arguments and short sequences of steps which form "streams"; reconstructing the global structure); and
- comparing local argumentations and comparing global argumentation structures, and revealing their rationale.

Each of these stages is illustrated in the following, but the emphasis is on the second stage. We will do so by discussing episodes of the proving process that lead to the written proof of the Pythagorean Theorem that we have presented above (see Fig. 4.1). The teacher (T) in this class will be referred to as Mrs. Nissen, references (e.g. <6–21>) indicate lines of the transcript of this lesson. "N5" indicates this is taken from the fifth lesson observed in Mrs. Nissen's class. Additional data and analysis can be found in Knipping (2003).

The choice of what part of the lesson to analyze was based on the participants' own identification of what classroom conversations were seen as being proving, through explicit labeling of them as such. The protocols and transcripts show that it is generally the teacher who labels a proving phase.

4.4.1 Reconstructing the Sequencing and Meaning of Classroom Talk

In the following we will first describe how we reconstructed sequencing and meaning of classroom talk on proof and proving. This corresponds to the *analysis of interaction* that begins Krummheuer's research method, and it serves the same function, to establish the 'text' the argumentations use. However, this stage of our method has a different focus, the identification of the flow and sequencing of topics and the reconstructing of the meanings of individual utterances in terms of their role in argumentation.

4.4.1.1 Layout of Episodes

The first step is dividing the proving process into episodes. This means that the general topics emerging in the classroom talk are identified and their sequencing is reconstructed. This allows one to get an overview of the different steps in the argumentation. Proving process in classrooms can occur over long periods of time, from 20 to 40 min or longer. Laying out different episodes of the process helps to make the argumentations in these episodes more accessible to analysis. Once the flow and sequencing of the emerging topics is made visible the reconstruction of the arguments can start. For example, the following topic episodes could be identified in lesson N5:

1. Sketch of the proof diagram <6–21>
2. Goal of the proof <21–28>
3. Meanings of a^2, b^2, c^2 <28–69>
4. Calculating sub-areas of c^2 <69–100>
5. Sascha's Conjecture <101–129>
6. The area of the right triangles <129–155>
7. A mistake on the board <156–175>
8. Transforming the equations found <175–200>

Mrs. Nissen starts the lesson by sketching a drawing on the blackboard (episode 1, see also Fig. 4.1). The class then determines the goal of the proof of the Pythagorean Theorem (episode 2) and discusses the meanings of a^2, b^2, c^2, expressions that are used to state the Pythagorean Theorem and that are related to the drawing (episode 3). In episode 4 the teacher asks the students to calculate the sub-areas of the big square c^2. Sascha supposes (episode 5) that any two triangles form a square, but his conjecture gets refuted by his peers and the teacher. Instead the class calculates the area of two right triangles in a general way (episode 6). By accident the teacher writes in the second line $(b-a^2)$ on the board, but a student points out the mistake which the teacher gratefully corrects into $(b-a)^2$ (episode 7). Together, the teacher and the students transform the equations found and deduce $c^2 = a^2 + b^2$ (episode 8).

4.4.1.2 Turn by Turn Analyses

Argumentations in classroom processes are mostly expressed orally and by a group of participants. Generally arguments are produced by several students together, guided by the teacher. As Herbst showed (Herbst 2002a), it is the teacher who mostly takes responsibility for the structure and correctness of the argument, but students contribute to the argument, so there is a division of labour in the class. Argumentations are co-produced; the teacher and the students together produce the overall argument. Their turns are mutually dependent on each other; their public meanings evolve in response to each other. The argument forms in relation to these emerging meanings. So, in order to reconstruct the structure of an argument first the meanings of each individual turn put forward in class have to be reconstructed. As Krummheuer and Brandt state:

> Expressions do not a priori have a meaning that is shared by all participants, rather they only get this meaning through interaction. In concrete situations of negotiation the participants search for a shared semantic platform. [*Äußerungen besitzen "*a priori *keine von allen Beteiligten geteilte gemeinsame Bedeutung, sondern erhalten diese erst in der Interaktion. In konkreten Situationen des Verhandelns bzw. Aushandelns wird nach einer solchen gemeinsamen semantischen Bedeutungsplattform gesucht"*] (Krummheuer and Brandt 2001, p. 14, our translation).

Because meanings emerge through interaction, reconstructing meanings necessarily involves some reconstruction of the process by which they emerge. Generally statements of classroom talk are incomplete, ambiguous and marked by deictic[2] terms. Deictic terms are replaced as much as possible in the reconstruction of the argumentation. For example, in Excerpt 4.1 the term "das" ("that") can be replaced with '$(b - a)$' because its meaning is apparent from earlier utterances.

Excerpt 4.1 Example of an ambiguous utterance marked by a deictic term

89	Jens: *Das* ist also die eine Seitenlänge von diesem mittleren Quadrat da. Aber trotz #	Jens: So *that* is the side length of that middle square there. But, #

[2] In linguistics, a deictic term is an expression, for example a pronoun, that gets its meaning from its context. The meaning of "this" depends on what is being pointed to. The meaning of "I" depends on who is speaking. In philosophy the word "indexical" is used to express the same idea.

Because the focus of the analysis is the argumentative structure of the classroom talk the reconstruction of the meanings of statements in the turn by turn analysis must consider the argumentative function of the statements: datum, conclusion, warrant, etc. These functions will be identified in the next step of analysis. Utterances are primarily reconstructed according to their function within the collectively emerging argumentation, not with respect to subjective intentions and meanings as in interaction analyses.

4.4.2 Analyzing Arguments and Argumentation Structures

In the following we will describe in detail the moves in the reconstruction of local arguments, then of intermediate argumentation streams, and then of global argumentation structures. This method for reconstructing arguments, argumentation streams and argumentation structures was developed by Knipping (2003, 2008).

4.4.2.1 Functional Reconstruction of the Argumentation

Recall that arguments have in Toulmin's model a general structure of data leading to conclusions, supported by warrants, which in turn can be supported by backings. Statements are characterized as having different functions within an argumentation, and functional analysis can help to reveal the structure of the argumentation.

Analyzing students' and teachers' utterances in the class according to this functional model allows us to reconstruct argumentations evolving in the classroom talk. In our analyses only utterances that are publicly (in the class) accepted or constituted as a statement are taken into account. The teacher's attention to some utterances and deferment of others can play a major role in this. This is not surprising given Herbst's findings that in general only the teacher takes responsibility for the truth of statements (Herbst 2002a). Where alternative argumentations or attempts at an argument are publicly acknowledged, they are also considered in our analyses, although the focus is on the main stream of the argumentation. The issue of alternative argumentations will be discussed in more detail in Sect. 4.4.3, where types of argumentation structures are compared.

In Knipping's (2003, 2004) analyses of classroom processes focusing first on conclusions turned out to be an effective step in reconstructing argumentations. It is helpful to begin by identifying what statement the participants are trying to justify, the claim that will gain the status of a conclusion by their argument. So, before actually analyzing the complete argument we look for conclusions and claims. The following example illustrates such a functional reconstruction of a conclusion, or in fact two conclusions.

The following excerpt marks the beginning of the proving process in class. The teacher has sketched the drawing presented in Fig. 4.1 on the blackboard and seeks to develop a proof of the Pythagorean Theorem together with the class. As yet, no

Excerpt 4.2 Transcript of the argumentation in episode 4 of lesson N5

89	Jens: Das ist also die eine Seitenlänge von diesem mittleren Quadrat da. Aber trotz #	Jens: So this is the side length of that middle square there. But, #
90	L: # Also, Adam sagt, hier soll ich immer *b* minus *a* dran schreiben, oder soetwas ähnliches.	T: # So, Adam says, I should write *b* minus *a* on here, or something like that.
91	L: Er sagt diese vier Seiten sind alle *b* minus *a* lang. Wie kommt er denn darauf? *b* minus *a* lang.	T: He says that these four sides are all *b* minus *a* long. So, how did he get that? *b* minus *a* long.
92	L.: Srike.	T: Srike.
93	Srike: Wir haben ja die eine Seitenlänge vom Dreieck *b*, eh die Kathete. Und dann ist die, also	Srike: Well, we know the length *b* of one side of the triangle, in fact the adjacent side. And then this is therefore
94	vom anderen Dreieck die andere Kathete a, die wird ja davon abgezogen und dann ist das,	the other adjacent side *a* of the other triangle, which of course gets taken away from it and then that is
95	was übrig bleibt dieses *b* minus a.	what is left, this *b* minus *a*.
96	L: Einverstanden? Immer wird von der grünen Strecke ’ne gelbe abgeschnitten und es bleibt	T: Agreed? A yellow segment is always cut off the green, and only the remainder
97	von *b* nur noch der Rest über. *b* minus *a*. Minus *a* heißt das gelbe weg. So *b* minus *a*, ha.	of *b* is still left. *b* minus *a*. Minus *a* means remove the yellow. Thus *b* minus *a*, ha.

written proof has been developed. The teacher asks the students to interpret the given drawing. Jens starts (Excerpt 4.2).

Previously Adam has offered $(b-a)^2$ as a label for the inner square in the figure that is on the board (see Fig. 4.1). Jens, after first being confused about this expression, makes a connection with the side length of the middle square (89). Given this context we interpret the “das/that” as “$(b-a)$”. The teacher reinforces Jens’s interpretation and endorses Adam’s earlier claim (90–91). Jens does not provide any data or warrant for his claim, on the basis of the drawing on the blackboard he seems to consider this as a “matter of fact”, a datum. The teacher asks for an explanation, “How did he get that? *b* minus *a* long.” (91), and reinterprets this “matter of fact” as a claim that needs justification. Srike provides a justification by relating $(b-a)$ to the difference of the lengths of the two legs of the right triangle in the drawing and explains why $b-a$ is the difference in length.

Although Jens proposed his statement as a datum, its status depends on its role in the public argumentation, not on the intention of its proposer. The teacher expects an explanation and Srike provides one, but for a different conclusion. However Srike’s explanation is accepted as justifying both her conclusion and Jens’s original claim. In this situation a single argumentation has two conclusions (see Fig. 4.5).

Srike’s argument is based on the data that “*b* and *a* are the legs of the triangle” (93/94) and that “*b* is the length of one leg” (93). She argues that because “*a* is cut off *b*” (94–97), “$(b-a)$ is the difference of the lengths of the legs” (95–97).

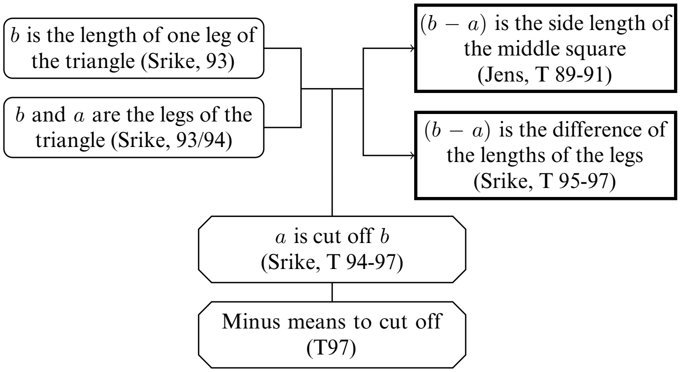


Fig. 4.5 Functional reconstruction of the argumentation in episode 4 of lesson N5, example of a step of an argumentation

The teacher reinforces Srike's attempt of an explanation, confirms her conclusion (94–97) and supports her warrant by the backing "Minus means to cut off" (97). In the given episode the teacher and the students together construct an argument that, with the help of the Toulmin pattern, may be reconstructed as in Fig. 4.5. Our representation is somewhat different, but this argument provides an example of what Aberdein (2006, p. 214) calls a "linked" layout, where two or more data are included. Aberdein did not consider the possibility that two or more conclusions could follow from one step, as occurs here.

It is interesting that in this case the warrant and the backing are given explicitly. Typically, reconstructed arguments in secondary level classroom proving processes are often incomplete, as was the case with the written proof analyzed above. Krummheuer notes that in primary level classrooms, the backing is often not mentioned explicitly, and he calls the data, warrant and conclusion the "core" of an argument. Krummheuer also encounters cases where the warrant is also not given, as in Srike's argument. In such cases the warrant can usually be assumed or taken as implicit, as the transition from datum to conclusion must be justified somehow. In our argumentation analyses we usually do not add implicit warrants, but leave them implicit in the reconstruction. This is meant to illustrate the implicitness of both the argumentation and warrant. This allows the comparison of the degree of explicitness in different argumentation structures. In cases where we do want to talk about an implicit warrant we place it in a dashed box (as in Fig. 4.6).

We have occasionally come across arguments where the datum has been left implicit. In such cases the warrant is present, however, so in the reconstruction the datum is left implicit, and the argument consists of the warrant and the conclusion (see Knipping 2003). As for the conclusions of arguments we have found that these are often formulated as questions. In the reconstruction of the argument these are represented by statements, so that their grammatical form is no longer visible, but their function in the argument is clearer. In the descriptions and comments on the argument this is noted and discussed.

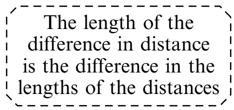


Fig. 4.6 Example of the representation of an implicit warrant, episode 4 of lesson N5

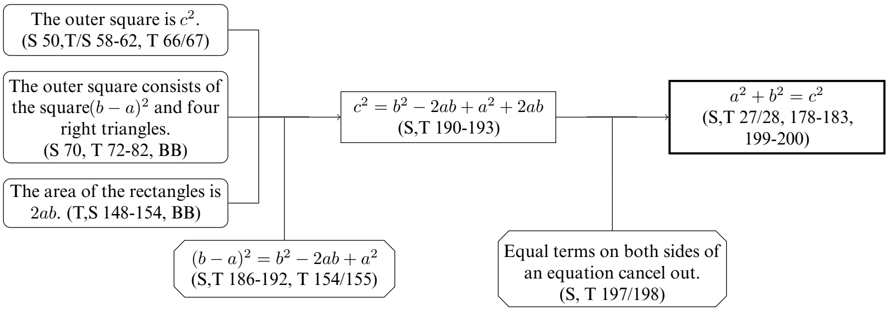


Fig. 4.7 Example of the second half of the argumentation in lesson N5. "BB" means black board. Transcript line numbers refer to Knipping (2003)

In cases where statements are questioned or doubted they are often justified in more than one argumentation step, as in Krummheuer's "chain of argumentations". We call a chain of argumentation steps by which a target conclusion is justified an "argumentation stream" (AS). The target conclusion (the final conclusion of the argumentation) is often marked by the teacher as a goal of one stage within the global argumentation. The example in Fig. 4.7 illustrates the last argumentation stream of the proving process in lesson N5. It involves both "linked" and "sequential" ways to combine steps.

4.4.2.2 Reconstructing the Argumentation Structure of Proving Processes in Class

As discussed earlier, the functional model of Toulmin, which is helpful for reconstructing argumentation steps and streams, is not adequate for more complex argumentation structures. Analyzing proving processes in classrooms requires a different model for capturing the global structure of the argumentations developed there. Knipping developed a schematic representation in order to illustrate the complex argumentation structures of this type of classroom talk, which we will present in the following.

The argumentation structure of a classroom proving process is generally complex. Argumentation streams can be parallel, as well as nested into each other. For example, if the argumentation builds on certain statements more than on others, these will be justified and explained in more detail, leading to multiple arguments in support of them embedded within the larger argument to which they are important.

To address this complexity Knipping developed a schematic representation that allows the description of argumentations at different levels of detail. This approach differs from Aberdein's (2006) as he reduces the complexity of the argumentation by a process of folding that results in a single step that includes all the assumptions (initial data and warrants) of the full argumentation, but which hides the relationships between these assumptions. Knipping's approach also differs from that taken by van Eemeren et al. (1987) who developed two different ways of representing the structures of everyday written argumentations, in that she makes the role of warrants more visible. We will illustrate below how Knipping's method makes the global argumentation visible while preserving the relationships in the local steps.

In the schematic representation shown in Figs. 4.8 and 4.9 all statements in the overall argumentation are represented by rectangles, circles and diamonds. The different symbols not only represent the different functions of the statements (datum, conclusion, warrant) but also the status that the statements have within the global structure of the argumentation. For example, the target conclusion is represented by a black rectangle. White rectangles represent target conclusions of intermediate stages within the global argumentation; they indicate end points of stages. These can become starting points, therefore data, in the next stage of the argumentation. Three statements (the three white rectangles in Fig. 4.8) have the status of data in the argumentation stream shown, but they are at the same time conclusions of earlier argumentation streams. Once their truth was established they became data for a subsequent argumentation stream, the one being presented in Fig. 4.8. Conclusions or data which do not have the status of an intermediate target are represented by circles. Warrants and backings are symbolized by diamonds.

In Knipping (2003) the overall structures of argumentations were analyzed by means of such schemes. She compared argumentation structures in different proving

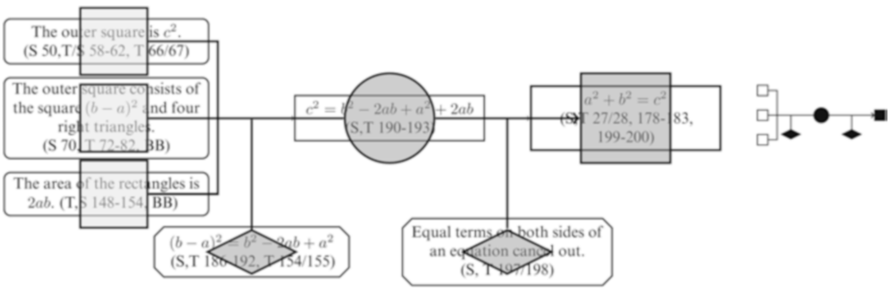


Fig. 4.8 The method of reconstructing a global argumentation

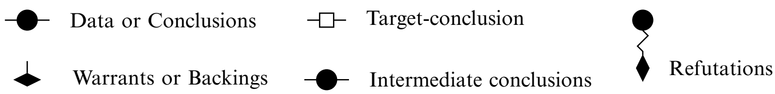


Fig. 4.9 Symbols used in argumentation structure diagrams

processes on the basis of the schematic representations and attempted to reconstruct their peculiar rationale. In the next section we will use some results of our research to illustrate the utility of this method for describing complex argumentations and their rationale. The processes that we have studied occurred in junior high school contexts where proof was an explicit goal of the lessons observed (see Knipping 2003; Reid and Knipping 2010).

4.4.3 Comparing Global Argumentation Structures

There is no established theoretical framework for investigating classroom proving processes. Therefore no model for the explanation of these processes can be formulated before researching the empirical field, but still a sound methodology of discovery is necessary. Analyzing argumentation in the primary mathematics classroom, Krummheuer (2007) considers comparison as a methodological principle that provides a reliable method of this sort and that can give "direction to a novel theoretical construction" (p. 71). This methodological principle underlies the comparisons that we undertake in our research on proving processes in classrooms (Knipping 2003, 2008; Reid and Knipping 2010).

As with Glaser and Strauss (1967), for Krummheuer comparative analysis represents a central activity that allows empirical control of the heuristic generation of theory. In this approach comparisons occur continuously, "the comparison of interpretations of different observed parts of reality represents a main activity on nearly every level of analysis: from the first interpreting approach to the later more theoretical reflection" (Krummheuer 2007, p. 71, describing Strauss and Corbin). The aim of these comparisons is "conceptual representativeness" (see Strauss and Corbin 1990) that is to ground theoretical concepts within the data. This concept differs from the one in quantitative research, where representativeness on the level of the sampling is the goal.

Knipping (2003) compares argumentations at two levels. Local argumentations are compared by analyzing and classifying the warrants (and backings) used according to the *field* of justification they belong to. Global argumentations are compared according to their overall structures. Here we will focus on comparing global argumentation structures, to show how they reveal elements of the rationale underlying the proving process. Two types of structures from our research will be used here as examples. We call them *source*-structure and *spiral*-structure (See Reid and Knipping 2010 for more detailed descriptions. Knipping 2003 and Reid and Knipping 2010 also describe two other structures. The four structures identified so far in our work only begin to describe the variety possible in classroom proving processes.). While the structures themselves are grounded in the data, the metaphors we use to describe them reflect our later interpretations.

4.4.3.1 Source-Structure

In proving discourses with a source like argumentation structure, arguments and ideas arise from a variety of origins, like water welling up from many springs. This is illustrated by Fig. 4.10, which is the global argumentation structure for lesson N5 from Mrs. Nissen's class (Knipping 2003). The structure has the following characteristic features:

- Parallel arguments for the same conclusion (AS-1 and AS-2).
- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (AS-8).
- The presence of refutations in the argumentation structure (AS-3, AS-6).

By a *parallel argumentation* we mean argumentation streams in a proving process in which different arguments can be found supporting the same conclusion. This happens, for example, if substantially different arguments are produced for the same conclusion. Both AS-1 and AS-2 support the conclusion that the side of the outer square is *c* (see Fig. 4.1).

Several argumentation streams can support a single conclusion without being parallel. The conclusion of each one can act as a datum for a subsequent step that requires the data from all of them. We are especially interested in noting cases where the data in a linked argumentation step are conclusions from a number of

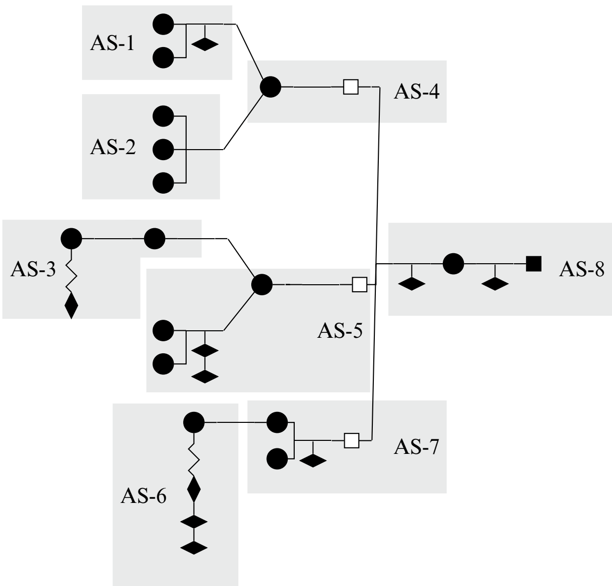


Fig. 4.10 *Source*-structure

argumentation streams. For example, the conclusions of AS-4, AS-5 and AS7 act as data for AS-8. Details of these three conclusions and AS-8 are shown in Figs. 4.7 and 4.8.

There are two refutations within this argumentation structure. In AS-3 Maren, in the process of describing the area of the outer square c^2, assumes that the area of the inner square is b^2. The teacher contradicts her, refuting Maren's suggestion visually. The teacher then develops together with the class an argument that the side length of the inner square is $b-a$ (AS-5) and therefore the inner square's area must be $(b-a)^2$. Sascha's conjecture (AS-6) is the other example of a refutation. Sascha claims that two of the triangles form a square which the teacher refutes by having the class put together the cut-out triangles. However, she says "I really like your idea, I think ideas that lead to the right result in detours are wonderful" giving value to Sascha's conjecture even though she refuted it.

The source-structure is also characterized by argumentation steps that lack explicit warrants. Argumentation steps without explicit warrants are evident in AS-2, AS-3, AS-4 and AS-5. While this also occurs in other types of argumentation structures it is more frequent in the source-structure. We speculate that this is because of the encouraging in the source structure of conjectures, which may be offered with some supporting data, but which are not further developed. A similar phenomenon occurs in another structure we have identified, the *gathering*-structure (Reid and Knipping 2010). That structure involves the gathering of a large amount of data to support several related conclusions. Again the emphasis is on collecting information (data and conclusions) rather than on the connections between them. In the other structures we have studied (see below) there is much more emphasis on the transitions, and so there are more explicit warrants.

We noted above that the source-structure has several characteristic features, which we have described in the argumentation structure of Mrs. Nissen's lesson 5. In another classroom we have observed lessons that have another structure, the spiral-structure, which shares the same characteristic features, but differs in the way they occur in the global argument.

4.4.3.2 Spiral-Structure

In a proving process with a spiral argumentation structure the final conclusion is proven in many ways. First one approach is taken, then another and another. Each approach can stand on its own, independent of the others. The global argumentation structure depicted in Fig. 4.11 shows a spiral argumentation structure from Mrs. James's grade 9 (age 14–15 years) classroom in Canada (see Reid and Knipping 2010). The class was trying to explain why two diagonals that are perpendicular and bisect each other define a rhombus (see Fig. 4.12). The students had discovered and verified empirically that the quadrilateral produced is a rhombus using dynamic geometry software and the proving process led by the teacher was framed as an attempt to explain this finding using triangle congruence properties.

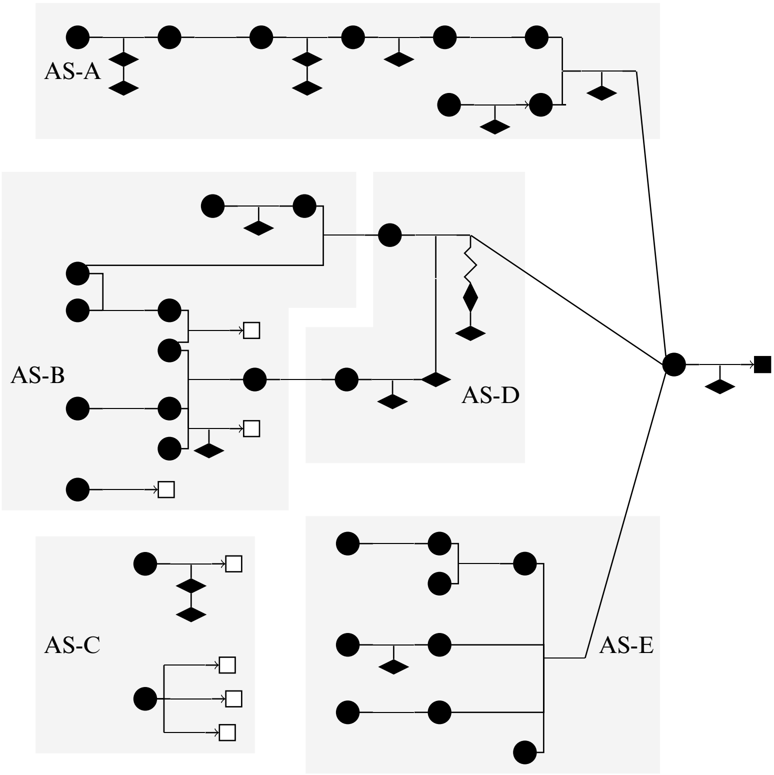
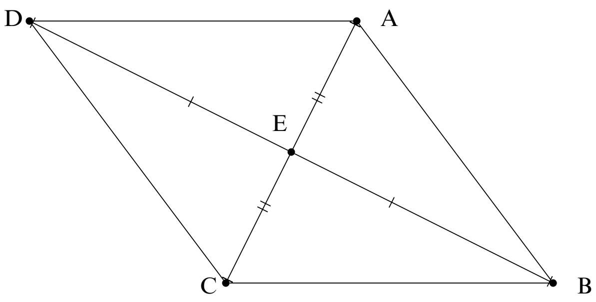


Fig. 4.11 *Spiral*-structure

Fig. 4.12 Diagram from Mrs. James's classroom (reconstruction)



Several features characteristic of the spiral argumentation structure are evident in Fig. 4.11:

- Parallel arguments for the same conclusion (AS-B, AS-D, AS-E).
- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (the final conclusions of AS-B and AS-E).
- The presence of refutations in the argumentation structure (AS-D).

In the argumentation structure there are three parallel arguments AS-B, AS-D, and AS-E. They all lead to one conclusion, that the four sides are congruent, which acts as the datum for the final conclusion that the quadrilateral is a rhombus. In AS-B the congruency of the sides is shown by showing that the four triangles formed by the diagonals are congruent. In AS-D a student offers an alternative argument, based on the idea that the quadrilateral cannot be shown to be a square (see below). This argument is listened to attentively by the teacher, who eventually refutes it. Finally, in AS-E the teacher offers an alternative argument implicitly based on using the Pythagorean Theorem instead of triangle congruency to establish that the four sides are equal.

In the argumentation structure from Mrs. James's classroom there are two argumentation streams that involve steps that have more than one datum, which are in turn conclusions of argumentation streams. In AS-E four data that are conclusions of short arguments combine to establish that the sides are congruent. The teacher assigns two arbitrary numbers 10 and 3 to the legs of triangle AEB (see Fig. 4.12) and this leads to the conclusion that the length of AB is $\sqrt{10^2+3^2}$. The Pythagorean theorem is used implicitly as the warrant. The same procedure is then used to find the length of BC, and to conclude that AB = BC on that basis. A third argument leads to the conclusion that CD is also $\sqrt{10^2+3^2}$. The final datum is the fact that all the triangles are right triangles. These four data are taken together to conclude that the four sides AB, BC, CD, and DA are congruent. In AS-B two data are used to establish the conclusion that the sides are congruent. The first is the culmination of a chain of data/conclusions leading to the conclusion that the four triangles AEB, BEC, CED, and DEA are congruent. The second is the conclusion of a shorter argument, establishing that the sides of the rhombus are corresponding sides of the congruent triangles.

The single refutation in this argumentation structure occurs in AS-D. Unlike the refutations we have discussed in the source-structure, above, what is refuted is not a datum or a warrant, but rather the applicability of the warrant to the argument (see Reid et al. 2008 for details. Verheij 2006 discusses analogous types of rebuttals.). A student, Kaylee, asserts that the diagonals must define a rhombus because in order to define a square the diagonals would have to be the same length. Mrs. James does not refute the fact that if the diagonals were the same length, as well as being perpendicular and bisecting each other, they would define a square. Instead she points out that other quadrilaterals might also be possible that have not been considered and excluded. Here the refutation is directed at the warrant, but does not refute it (as it is correct). Instead it suggests that the warrant is insufficient in the logic the teacher expects mathematical arguments to follow. Although she refutes Kaylee's argument, Mrs. James values it, commenting, "You're on the start, but I'm not sure that you've clinched it. I'm not sure you've got that final part, but … you're three quarters of the way there my dear."

While representing the argumentation structures of proving processes is useful as a way of identifying and structuring the important elements in it, we gain more from comparisons of argumentation structures. For example, Knipping (2002, 2003,

2004) found that in the six classrooms she observed, the proving processes either had argumentations with the source-structure, or another structure that she calls the *reservoir*-structure. The reservoir-structure differs from the source-structure in many ways. Most notably, in the reservoir-structure the reasoning sometimes moves backwards in the logical structure and then forward again. Initial deductions lead to desired conclusions that then demand further support by data. This need is made explicit by identifying possible data that, if they could be established, would lead to the desired conclusion. Also, because transitions are a focus in the proving process, the reservoir-structure has more explicit warrants.

The different structures in the lessons Knipping observed revealed interesting differences in the nature of proof teaching in the two contexts in which they were found. In the next section we will compare the source-structure with the spiral structure.

4.4.3.3 Comparing Source-and-Spiral Argumentation Structures

Both the source-structure and the spiral-structure were observed in proving processes in which the teacher took a prominent role in guiding the process. Arguments were co-produced by teachers and students, but the teacher was in control of the emerging overall structure. Therefore it is not surprising that these argumentation structures have several similar characteristic features including parallel arguments, argumentation steps that have more than one datum, the presence of refutations, and argumentation streams that do not connect to the main structure. However, they differ in how these features play out in the global structure.

One of the main distinctions between the spiral-structure and the source-structure is the *location* of the parallel arguments. In the source-structure the parallel arguments occur at the start of the proving process (AS-1 and AS-2 in Fig. 4.10). The teacher invites input at this stage, but once the basis for the proof is established, the teacher guides the class to the conclusion through an argumentation that no longer has parallel arguments. In the spiral-structure, however, the conclusions of the parallel arguments are almost the final conclusion in the entire structure. In fact, two of the three parallel arguments in Fig. 4.11 (AS-B and AS-E) could stand alone as proofs of the conclusion. Having proven the result in one way, the teacher goes back and proves it again in a different way. And she values students' attempts to prove the conclusion using other approaches.

The source-structure and the spiral-structure differ also in the kinds of refutations they involve and in the inclusion or omission of warrants. Recall that in the source type argumentation structure shown in Fig. 4.10, the refutations are refutations of data. When interpreting the figure on the blackboard students propose statements that the teacher refutes. In contrast, the refutation in the spiral type argumentation structure shown in Fig. 4.11 is a refutation of an argument. The data and warrant are accepted but their adequacy to justify the conclusion is refuted. In the source structure we see also many steps that omit warrants, while most steps in the spiral structure include warrants. The notable exceptions are in AS-A and AS-E. We will discuss some possible reasons for this later.

Although the source and spiral argumentation structures have the same characteristic features, they differ in the placement of the parallel arguments and in the nature of the refutations they include. These differences result in fundamentally different argumentation structures. It is through comparisons of structures that such differences become apparent. While reconstructing argumentation structures and comparing them allows us to identify important features and differences, it does not explain why they occur. To try to explain these differences, we need to return to the data and consider the nature of the local arguments that make up the global structure.

In the proving process in Mrs. Nissen's classroom, the figure on the blackboard is the starting point. Almost all the steps in the argument consist of establishing data that is depicted in the figure. For example, the part of AS-5 shown in Fig. 4.5 establishes that the side of the inner square is $b - a$. The three sub-conclusions of AS-4, AS-5 and AS-7 are all present implicitly in the figure, and unpacking that data is the main focus of the proving process. This explains the focus on data, which we see in the lack of warrants and in the refutation of inaccurate data. Once all the necessary data has been unpacked, the final steps of the argument are straightforward and algebraic. They are what is written on the blackboard at the end, as the written proof, and notably, the warrants that are expressed verbally in the class are not included when the proof is written down. Again the focus is on the data, rather than on the arguments and their warrants. When we consider the nature of the omitted warrants this makes sense, as they are either visual (based on features of the figure on the blackboard) or algebraic procedures well known to everyone in the classroom. In the first case it is hard to imagine how the warrants could be formulated in words or symbols, and in the second case the teacher is simply modeling the standard mathematical practice of omitting from proofs any warrants that the reader can be expected to provide.

In Mrs. James's lesson, the argumentation begins with the given data: the diagonals are perpendicular and meet at their midpoints. From this data the figure is constructed. There are not many explicit warrants in this argumentation stream (AS- A). As in Mrs. Nissen's lesson, if the warrants were made explicit they would either be visual or refer to conventions well known to the students. In the other streams, however, there is a focus on the arguments themselves and warrants become more explicit. Duval's "recyclage" is evident in AS-B where almost all the data are recycled conclusions of previous steps, and in which most steps have explicit warrants. In AS-D we see the focus on the argument in the refutation of it, as opposed to the refutation of data in the source structure. Even in AS-E, where there are many omitted warrants and a configuration similar to the source structure overall, with the final conclusion depending on more than one datum, the fact that the teacher offers this alternative approach to establishing the final conclusion indicates a focus on arguments rather than data and conclusions.

Examining the argumentation structures in these two classrooms allows us to describe their characteristic features, and by comparing them we can understand the different ways these features occur. We see the parallel arguments, refutations and omitted warrants in both, but we see these features occurring differently. Looking more closely at the features of the local arguments helps to explain these differences, and reveals an importance distinction between the rationales of the proving processes taking place. In Mrs. Nissen's class we find in the local arguments a focus on interpreting the given figure. The activity is essentially one of unpacking the data in the figure and expressing it verbally. It is not clear how this could be transferred to proving another theorem, unless a similar complex figure were provided. We suspect this is inevitable in a class focussing on the Pythagorean Theorem. Historically, mathematics educators have struggled with the problem of using mathematically significant theorems as a context for learning proving. The proofs of such theorems are usually sufficiently complex that it is unreasonable to expect students in schools to discover them, unless the teacher provides so much guidance that the students' contributions are limited to activities such as unpacking a diagram, as in Mrs. Nissen's class. Herbst (2002b) describes how this struggle affected the evolution of proof teaching in the US.

In contrast, in Mrs. James's class the focus is more on proving. The result itself is relatively uninteresting, but the recycling of conclusions as data, the provision of warrants, the fact that the same result can be proven in different ways, and bringing different prior knowledge to bear, are all important. Student contributions are valued, even when flawed, and the argumentation, especially in AS-B, served as a model for the students when proving similar claims in subsequent lessons.

The source structure and the spiral structure are interesting to compare because they have many characteristic features in common, including parallel arguments, argumentation steps that have more than one datum, refutations, and unconnected argumentation streams. There are differences in how these features play out in the global structures, however, and to explain these we focus again on local arguments.

4.5 Conclusion

Toulmin's functional model of argument allows us to reconstruct arguments in mathematics classrooms not only at the local level, as Krummheuer does, but also at the global level. By examining argumentation structures (the metaphoric anatomies of proving processes) we can describe their characteristic features, and by comparing structures we can understand the different ways these features occur in an argumentation structure. To better understand the differences we observe, and to shed light on the rationales of the proving processes we return to the local arguments, to physiology in our metaphor. Attention to both the local and the global levels are essential to understanding proving processes in the classroom.

Toulmin's model can also be used to look at both argumentation for learning and learning of argumentation. Krummheuer's work shows how it is applied in primary classrooms to study mathematical argumentation as a pre-condition for learning mathematics, that is, *argumentative learning*. We focus instead on mathematical argumentation as the desired outcome of learning in secondary level classrooms, and hence on learning of argumentation.

Krummheuer's use of the Toulmin model allows him to examine argumentation in a single classroom very closely, and to relate argumentation to participation. Our approach instead provides a more global picture of the argumentation, permitting comparisons between classrooms. Of interest for future research is the combination of these foci, looking at how argumentative learning comes into the learning of argumentation.

We are also interested in bringing together research on argumentation and research on the emergence of disparity in achievement in mathematics classrooms (Knipping 2012). Sociological research in mathematics education (Lubienski 2000) has shown that students of different social backgrounds participate differently in classroom activities, and have different success in school mathematics. We have used concepts from Bernstein (2000) to describe some mechanisms related to the emergence of disparities in mathematics classrooms (e.g., Knipping and Reid 2013a; Knipping et al. 2011). We are only beginning to develop methods for researching argumentation together with social disparity (see Cramer 2014). Krummheuer's methodological integration of research on argumentation with participation should be useful in this context as this allows for the analysis of situations in which participation in argumentation leads to different levels of autonomy and different roles in the classroom.

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Part III
Ideal Type Construction

Chapter 5
Empirically Grounded Building of Ideal Types. A Methodical Principle of Constructing Theory in the Interpretative Research in Mathematics Education

Angelika Bikner-Ahsbahs

Abstract The central question of this article is: How can the development of ideal types contribute to the empirically based construction of theories in the interpretive research of mathematics education? First we specify and localize the theoretical understanding used and then clarify the term 'ideal type' distinguishing between three kinds of ideal types: the ideal type of action, the personal, and the situational ideal type. With the help of examples from empirical research, we show how the construction of ideal types can be used as a methodical principle of theory construction. In so doing, common features and different heuristics of empirically-based theory construction are reconstructed.

Keywords Theories • Ideal type construction • Interpretative research

5.1 Introduction

Theory construction is always the construction of something new, and therefore not directly open to methodological approaches (cf. Kelle 1997, p. 182). In this respect, methods of qualitative analysis of empirical data can serve as heuristics for theory construction. In this article I show how the development of ideal types can be regarded as a methodical principle which "points the way" for an empirically-based theorizing (cf. Weber 1922, p. 190, own translation). Selected examples of empirical research will illustrate a number of principles and heuristics of theory construction in the development of ideal types. But first of all, I put forward a theory concept

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A. Bikner-Ahsbahs (✉)
Faculty 3 of Mathematics and Computer Science, University of Bremen,
Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de

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which fits this kind of qualitative research. I then go on to clarify the term *ideal type* and its relevance for interpretive understanding in everyday situations and in science.

5.2 Theories and Their Significance

A look at the encyclopedia of epistemology (Seiffert and Radnitzky 1989) reveals there can be no consistent meaning of theory. This encyclopedia distinguishes three different theory concepts: "*Theory* as a scientific doctrinal system without considering the underlying methods or its object" (Seiffert and Radnitzky 1989, p. 368, own translation),[1] "Theory […] as assured knowledge arising from the interaction of experience and thinking—determined in accordance with the methods described in the theory or methodology of the inductive sciences" (ibid., p. 368, own translation)[2] and "Theory in contrast to practice"[3] (ibid., p. 368, own translation). The aim of interpretive empirical research is the data-based construction of theory. These empirically based theories cannot belong to the first sort. But they cannot belong to the second sort either, as these theories are not obtained inductively but abductively[4] (Kelle 1997, p. 161ff.; Beck and Jungwirth 1999; Voigt 2000). Subsumption under the third category seems possible, but being a quite common category it does not formulate an adequately differentiated approach to the meaning of theory and, hence, does not offer a classification of theories that assists in building a suitable theory concept.

Mason and Waywood apply their distinction of theory levels exclusively to the research of mathematics education. They distinguish between "background theories", "implicit theories" and "foreground theories" (Mason and Waywood 1996, p. 1056).

"Background theories" are general theories on problems met in mathematics education, so-called "theories of mathematics education" (Mason and Waywood 1996, p. 1058). They involve a characteristic language and they fix the general framework for adequate research questions and aims. They determine which topics may become objects of research and which methods will be accepted. They are

[1] German expression: "*Theorie* als wissenschaftliches Lehrgebäude, ohne Rücksicht auf die Methode(n), mit denen es gewonnen wurde oder auf seinen Gegenstand."

[2] Germen expression "*Theorie* […] als gesichertes Wissen, das aus dem Zusammenwirken von Erfahrung und Denken—und zwar nach ganz bestimmten in der Theorie bzw. Methodologie der induktiven Wissenschaften beschriebenen Methoden—entsteht."

[3] German expression: "Theorie im Gegensatz zur Praxis."

[4] This means the problem we are faced with when our observations cannot be explained by known theories. Explaining an observation by the construction of a new theory is done by abduction: the use of newly created theory elements on a trial basis which would explain unexpected or unsatisfactorily explained observations. But this, of course, does not say anything about the quality of such theory elements. Only a process consisting of examinations and modifications of the theory elements will as a rule lead to their acceptance.

based on a certain world view, e.g. whether the world is taken as objectively given or rather interpreted subjectively, whether the behavior of the actors and the data about the actors in the field can be seen independently from the context or whether mathematical knowledge can be seen as being constructed individually, socially, or rather accessed through cultural acquisition. Such theoretical presumptions may be implicit. It is the aim of many—but not all—research directions to make this background as explicit as possible, and consequently communicable. Even background theories that are not made explicit eventually do become visible, e.g. in methodical research procedures, in the presentation of results, and by the kind of results themselves.

"Implicit theories" are normally not explicitly available, yet they still find a place in the research process. For example, in their respective research domains, scholars could have an implicit theory of knowledge development, of suitable research behavior, of what constitutes the correct representation of research results or research products, of what research is, could be or means, of the role theories play within this research process, and of the notion of theory itself. As a consequence implicit theories are everyday theories of scientists. They do not necessarily have to be commensurate with background theory but do nevertheless influence the research process. This becomes apparent for instance from a study by Maier and Beck (2001) on the development processes of theories, which as a by-product illustrates the nature of such implicit theory through feedback processes with the authors (p. 42ff.). The aim of their study, though, was not to reconstruct implicit theories, but rather to reconstruct the development processes of so-called "foreground" theories, which can be compared with object related theories—theories of medium scope on a research object that is exactly localized. (Kelle 1997, p. 280; Krummheuer and Brandt 2001, p. 198ff.).

"Foreground theories" are explicit theories relating to research objects which have been developed within a research area and which describe, explain or predict facts. Mason and Waywood characterize these theories as "theories within mathematics education" (Mason and Waywood 1996, p. 1060) which in the research process are applied, extended, consolidated and developed, and that may have different functions. Explanatory theories offer to explain how and why something has happened or has been observed. Descriptive theories offer a theoretical frame of ideas with whose help an action can be observed and traced in a certain way (cf. Maier and Beck 2001, p. 43). Predictive theories indicate what may be observed under a certain set of conditions.

The theoretical background assumptions which I refer to in this article are those of interpretive education research as developed in the German-speaking research within mathematics education over the past 30 years. (cf. Jungwirth 2003). Their assumptions are rooted in selected works on symbolic interactionism, grounded theory, objective hermeneutics, and ethnomethodology and phenomenology (Beck and Maier 1993, 1994a, b; Beck and Jungwirth 1999; Helle 2001; Krummheuer and Voigt 1991; Krummheuer and Naujok 1999; Maier and Voigt 1991, 1994; Maier and Beck 2001). It is a basic assumption of interpretive research that the world has already been pre-interpreted by the people who live in it, and that empirical research investigates a question on the basis of reconstructions of these interpretations—consequently on the basis of second-order interpretations. The paradigmatic basis is formed by "orientation to change" (Ulich 1976, p. 26ff., own translation). This means that we do not primarily look into the features a person or a situation *has*, i.e. *possesses,* but how such features *evolve*, how they *are established*, how they *develop*, how they *are stabilized* or *disrupted*.

A wider consensus in interpretive research is the hypothesis that a person's interpretations influence his or her own behavior in a crucial way (cf. Helle 2001, p. 57). Whether these interpretations are understood as creative new constructions or as reconstructions of already preformed social knowledge or social structures cannot be determined in the context of mathematics education research. Both are permitted, as are conceptions connecting the two. Papers on hermeneutic sociology of knowledge, for example, criticize strict objective hermeneutics with its basic assumption of an all-determining existence of latent objective meaning structures, because the theory that human behavior is completely determined by social structures leaves no place for new constructions; neither in science—where this theory should be applied as strictly as in the area of research—nor in the research area itself (cf. Schröer 1994, p. 10ff.). They therefore propose to start from a complementary conception, i.e. from reconstructions of social structures on the part of the protagonists, on the one hand, and the possibility of creative new constructions on the part of individuals, on the other (Schröer 1994).

How do "foreground" theories actually develop within interpretive research of mathematics education, and how are they characterized?

5.3 The Notion of Theory in Interpretative Mathematics Education Research

In general, interpretive mathematics education claims to make clear its theoretical frame and the background theories used. Following the review of the state of discourse by Maier and Beck (2001), the particular notion of theories, understood as "foreground-theories", develops with all its functions and meanings intertwined with research practice, too. To explore this notion of theory within interpretive research in mathematics education, Maier and Beck investigated 20 reports of empirical research with regard to the process of theory development, complemented by feedback from the authors. It was their aim to reconstruct the genesis of theories in this research area in order to develop an initial approach for an empirically founded "theory of theory development" and thus also to arrive at a conceptual clarification of "theory" in interpretive research within mathematics education. Following the study of Maier and Beck, theory is understood as follows:

> […] an intellectual construction by means of which a more or less greater sphere of reality may either be described consistently and systematically or is understood comprehensively and in a differentiated way (where the starting point of view is always selective). Understanding cannot strictly be separated from explaining. (Maier and Beck 2001, p. 43, own translation)

In addition to this, Beck and Maier draw distinctions among "foreground"-theories, according to which interpretive research on mathematics education can be theory-guided or theory-developing (Beck and Maier 1994b). Theory-guided empirical research works out existing theories for a defined research area. This kind of research claims to come to a better understanding of a determined part of reality on the basis of given "foreground" theories. Theory-developing empirical research, however, claims to construct new theory features. According to the categorization by Beck and Maier, theory-developing research can take place either in the "category developing" or "systematic-extensional" way. But I think this differentiation falls short because the feature *systematic-extensional* actually characterizes the research process, which is in fact systematic—in the sense of methodically reflected—and extensional in the sense of an extensive variants genesis. The feature *categories developing* is oriented on the product, namely the evolving categories. And on top of that, the category *developing research* may also take a systematic-extensional course. Products of developing research that do not consist of categories as a rule include reconstructed regularities or patterns. Therefore, I propose the distinguishing of the features *category developing* and *pattern reconstructing*. In the following section, I present three examples of theory-constructing research in order to clarify the distinction between the *categories developing* and *pattern reconstructing* interpretations.

5.4 Theory-Developing Research

A strong focus in empirical research within mathematics education is on learning processes in the context of ordinary mathematics classes. One problem of empirical research in ordinary mathematics classes is the complexity of their objects. Theory features which describe or explain specific phenomena of this field therefore often concentrate on clearly defined parts of mathematics classes that are analyzed in detail on the basis of a limited amount of data (Beck and Maier 1994a, b). The aim of such analyses is extending interpretations beyond the respective intentions and situations (Beck and Maier 1994a, p. 44ff.). End stages of systematic interpretation sequences which often take place in groups are hypotheses of interpretation (Beck and Jungwirth 1999) which initially register typicalities (Beck and Maier 1994a, p. 64) and later bring these typicalities together in a theoretical but empirically founded way to obtain theory elements. This is how Voigt—on the basis of transcripts of inquiring-developing mathematics classes—develops interaction patterns and routines which describe how teachers and learners force each other into action and thereby create a bond. Voigt develops a description scheme by which the emergence of inquiring-developing teaching sequences can be understood (Voigt 1984a, b). Later on Voigt complements this view of mathematics lessons which is restricted to social interactions by a more content specific perspective by asking how themes are socially constituted in class (Voigt 1995).

Krummheuer's objective is to develop a theory of learning in mathematics classes from the perspective of social interactions. In his research he tries to find out what it takes to achieve learning mathematics in a process of social interactions, and how this can be described. To this end, i.e., he transfers Bruner's format concept developed in language-acquisition research to the learning of mathematics in argumentative learning processes and develops related formats of argumentation. Argumentation formats are exemplary social interactions, directed at subject learning, that do not necessarily keep learners in dependency like interaction patterns, but by means of which the learners may experience an increase in independence (Krummheuer 1992). On the basis of argumentation and participation analyses which follow on analyses of social interaction, in collaboration with Brandt Krummheuer extends his research from a dyadic to a polyadic point of view of social interactions and reconstructs features of "interactive thickening"—i.e., teaching situations at elementary school level, where fruitful learning conditions for argumentative learning come together in polyadic learning processes (Krummheuer and Brandt 2001).

Voigt and Krummheuer investigate specific phenomena in ordinary mathematics classes in order to describe them with the help of a concept scheme, or to make them comprehensible in their regularity. The concepts used are partly borrowed from other theories and adapted to the special conditions of the respective research, or newly formulated according to the subject in question. That does not mean that in this case an exclusive categorization in existing theories is effected. According to Maier and Beck (2001, p. 42ff), Voigt rather describes this process as a reciprocally stimulating and forward-moving process of abductive constructions on the basis of empirical data, and set in relation to existing research and theoretical approaches where some ideas have only been recognized in connection with the literature subsequent to the theory construction processes. Main features of this research are abductively gained theory elements. For example, via the acquiring process pattern Voigt can explain how specific "seemingly open" processes in ordinary mathematics classes are steered and subsequently lead to the acquisition result desired by the teacher (Voigt 1984a). Via the concept of interactive thickening, Krummheuer and Brandt are able to identify situations in classes which have significant learning potential (Krummheuer and Brandt 2001).

In his research on how pupils understand teachers' instructions and explanations in ordinary mathematics lessons, Maier (1995) presents a slightly different theory. Together with his working group, on the basis of systematic-extensional interpretations he develops a category system consisting of two characteristic dimensions: a mental and a modal dimension. These are then analyzed in their respective different forms. According to Maier, pupil understanding entails a mental feature, that is an intellectual construction, which may be distinct in a conceptual, process-related, relational, argumentative, elaborative, and reflexive way; and a modal feature which characterizes the manner of pupil understanding as implicit or explicit on the one hand and as verbal or symbolic language understanding on the other. Subsequently, an evaluation model of pupil understanding is developed and validated with additional data. The linking of category system and evaluation model eventually results in a theory offer which can be used as a basis for an analysis of understanding products in dependence on the subject's standards, teacher intentions, and the offers of meaning, and which can also explain digressions from what has been understood from the lessons' meaning offer (Maier 1995).

Representative of other work in this field, these three short summaries of empirical research on interpretive research in mathematics education describe two different types of theory-constructing empirical research: one is the category-developing research, which organizes the research field with help of a feature space, and the other, a pattern-reconstructing research, which reconstructs patterns to describe typical phenomena in this field.

A common feature of all theory-constructing work is that it does not only reconstruct what a particular teacher means in a particular situation, but rather that it works out—independently of situation—what is typical of a particular situation with regard to a specific problem and puts aside everything else, i.e., it is performed on a level of abstraction (Schreiber 1980). According to the above-mentioned research, the identification of typical features may happen in two quite different ways. Maier organizes the interpreted comprehension processes by means of characteristic dimensions and so obtains an organizational framework that can be used for the analysis of comprehension processes, on the one hand, but also for meaning offers during lessons on the other. An analysis of individual cases enables him to get to the bottom of influences, especially when the profiles of meaning offer and pupil understanding do not resonate. In contrast, Voigt reconstructs the characteristic features of particular process patterns, i.e., he sees these patterns into real processes and works out typical features as patterns on the basis of empirical data. This way he obtains *characterizations of ideal types* (I explain this expression in the following passage), of particular processes, however without calling the research process *empirically based construction of ideal types. Interactive thickening* and *argumentation formats* can also be regarded as ideal type descriptions. With the aid of these ideal type characterizations, Voigt and Krummheuer are able to conceive complex teaching phenomena. Now two questions arise:

- What is meant by an ideal type characterization (of a situation)?
- How can ideal type construction and ideal type characterizations be used specifically to construct concepts and develop theories in mathematics education?

5.5 Looking Back: The Roots of the Ideal Type Concept

The ideal type concept comes from Max Weber. Connected to this concept was the attempt to develop a conceptual tool for an interpretive sociology allowing selective conceptual descriptions of social processes that are embedded in complex contexts of meaning and therefore may be explained complexly in their complexity (Weber

1921/1984, p. 38; 1922/1985, p. 194ff.; 1949, p. 93ff.). "But what was meant and what can be meant by that *theoretical* concept can be made unambiguously clear *only* through precise, ideal-typical constructs" (Weber 1949, p. 95)[5] (emphasis included by the author according to the original text). As Weber well notices, this runs the risk of becoming out of touch with reality and therefore he requires a link with reality. According to Weber, an ideal typical construct (which I call an ideal type concept) is a concept by means of which social circumstances may be examined, but which itself only roughly describes real circumstances because "Only those rational constructions are social types of real events which can be observed in reality by at least some kind of approximation […]"[6] (Weber 1921/1984, p. 29, own translation). In consequence, ideal types are constructions of real events and thus approximations of reality: in their pure form, though, they merely represent constructs of ideas. According to Weber: The ideal type

> […] is a conceptual construct (Gedankenbild) which is neither historical reality nor even the 'true' reality. It is even less fitted to serve as a schema under which a real situation or action is to be subsumed as one instance. It has the significance of a purely ideal limiting concept with which the real situation or action is compared and surveyed for the explication of certain of its significant components. Such concepts are constructs in terms of which we formulate relationships by the application of the category of objective possibility. By means of this category, the adequacy of our imagination, oriented and disciplined by reality, is judged. (Weber 1949, p. 93)[7]

The ideal type is gained

> […] formed by the one-sided *accentuation* of one or more points of view and by the synthesis of a great many diffuse, discrete, more or less present and occasionally absent *concrete individual* phenomena, which are arranged according to those one-sidedly emphasized viewpoints into a unified *analytical* construct (Gedankenbild). In its conceptual purity, this mental construct (Gedankenbild) cannot be found empirically anywhere in reality. It is a *Utopia*. (Weber 1949, p. 90)[8]

[5] German expression: "Was aber unter jenem t h e o r e t i s c h en Begriff gedacht w i r d und gedacht werden kann, das ist n u r durch scharfe, das heißt idealtypische Begriffsbildung eindeutig klar zu machen" (Weber 1922/1985, p. 196).

[6] German expression: "Nur solche rationale Konstruktionen sind soziale Typen realen Geschehens, welche in der Realität wenigstens in irgendeiner Annäherung beobachtet werden können […]" (Weber 1921/1984, p. 29, own translation).

[7] German expression: "[…] ist ein Gedankenbild, welches nicht die historische Wirklichkeit oder gar die <<eigentliche>> Wirklichkeit i s t, welches noch viel weniger dazu da ist, als ein Schema zu dienen, in welches die Wirklichkeit als E x e m p l a r eingeordnet werden sollte, sondern welches die Bedeutung eines rein idealen G r e n z b egriffes hat, an welchem die Wirklichkeit zur Verdeutlichung bestimmter bedeutsamer Bestandteile ihres empirischen Gehaltes g e m e s s e n, mit dem sie verglichen wird. Solche Begriffe sind Gebilde, in welchen wir Zusammenhänge unter Verwendung der Kategorie der objektiven Möglichkeit konstruieren, die unsere, an der Wirklichkeit orientierte und geschulte P h a n t a s I e als adäquat b e u r t e i l t" (Weber 1922/1985, p. 194).

[8] German expression: "[…] durch einseitige S t e i g e r u n g e i n e s oder e i n i g e r Gesichtspunkte und durch Zusammenschluß einer Fülle von diffus und diskret, hier mehr, dort weniger, stellenweise gar nicht, vorhandener E i n z e l erscheinungen, die sich jenen einseitig herausgehobenen Gesichtspunkten fügen, zu einem in sich einheitlichen G e d a n k e n bilde. In seiner begrifflichen

Ideal types are indeed understood as constructs of ideas, but as per Weber ideal type construction is oriented towards a "potential lifeworld idea"[9] (Weiß 1975, p. 74, own translation), which exists in single components in reality (potentially) and towards which acting in life practice is already oriented. (Weber 1922/1985, p. 190; cf. 1949, p. 89) According to this, an ideal type is a construction that takes up elements from reality. In order to anchor ideal type construction in an empirical way, Weber requests meaning adequacy and causal adequacy. Meaning adequacy means that the ideal type has to be seen in a meaning context of life practice, and causal adequacy means there has to be a real chance that an acting type—only those ideal types are meant by Weber—could develop in reality approximately in the described way. Therefore a good basis for action must be evident for its development (Weber 1921/1984, pp. 29, 27ff.; Weiß 1975, pp. 61, 68).

Weber does not use ideal types to file reality in them, but as idealized constructions (resulting in concepts), making use of them to examine real divergences from these constructions (Weber 1921/1984, pp. 22, 13). His focus on scientific research is not the construction process of ideal types but the ideal type analysis, "for example when explaining *panic on the stock exchange* at first it is advisable to find out how action would have been developed in the absence of influence by irrational affects and afterwards those irrational components are noted as *disturbances*"[10] (Weber 1921/1984, p. 21, own translation, emphasis included by the author according to the original text).

Weber's ideal type analysis is based on his idea about sociology as a science to understand social acting: "Sociology […] means: a science that wants to understand interpretively social acting and therefore explain causally its course and influences. *Acting* is meant to describe a human behavior (it doesn't matter whether outwardly acting, neglecting or tolerating) if and inasmuch as the acting person or persons associate a subjective *meaning* with it. But *social* acting is called an acting which is related to *other persons'* behavior in its meaning meant by the acting person(s) and thus orientates in its course"[11] (Weber 1921/1984, p. 19, own translation, emphasis included by the author according to the original text). Consequently today, a

Reinheit ist dieses Gedankenbild nirgends in der Wirklichkeit empirisch vorfindbar, es ist eine U t o p i a […]" (Weber 1922/1985, p. 191).

[9] German expression: "potentiell lebensweltlichen Idee" (Weiß 1975, p. 74).

[10] German expression: "z.B. wird bei einer Erklärung der >> Börsenpanik << zweckmäßigerweise zunächst festgestellt: wie ohne Beeinflussung durch irrationale Affekte das Handeln abgelaufen wäre, und dann werden jene irrationalen Komponenten als >> Störungen << eingetragen" (Weber 1921/1984, p. 21).

[11] German expression: "Soziologie […] soll heißen: eine Wissenschaft, welche soziales Handeln deutend verstehen und dadurch in seinem Ablauf und seinen Wirkungen ursächlich erklären will. >> Handeln << soll dabei ein menschliches Verhalten (einerlei ob äußerliches Tun, Unterlassen oder Dulden) heißen, wenn und insofern als der oder die Handelnden mit ihm einen subjektiven Sinn verbinden. >> Soziales << Handeln aber soll ein solches Handeln heißen, welches seinem von dem oder den Handelnden gemeinten Sinn nach auf das Verhalten anderer bezogen wird und daran in seinem Ablauf orientiert ist" (Weber 1921/1984, p. 19).

fundamental feature of a sociology of understanding entails understanding via reconstructing analysis of social acting from the subjects' perspective.

According to Weber meaning comprehension is a feature of daily-life orientation, and the understanding of meaning comprehension is an orientation feature of methodical understanding of interpretative sociology. Weber distinguishes two interpretations of meaning (*Sinn*), by which he means the real subjective supposed meaning of an acting person or a group of acting persons, or a subjectively meant meaning of an imagined acting type. The subjectively intended meaning is always a factual meaning that determines action (cf. Weiß 1975, p. 59). That means that an acting person does not necessarily have to be absolutely aware of his or her motive for acting: motives result from the meaning relationship: "A motive is a meaning relationship that presents itself to the acting or observing person as a meaningful reason for a certain behavior"[12] (Weber according to Weiß 1975, p. 48, own translation). Therefore, identifying motives for acting can be approached by the observer through reconstructing meaningful reasons.

Following Weber, real acting is accompanied by only vaguely determined acting reasons, and may therefore only be construed in an understanding way. He does not perceive this aspect as a fault in sociological research that has to be avoided, but as a specific feature which is distinctly different from the approach of concentrating on functional orders met in natural science:

> In the realm of social constructions […] we are indeed able, *beyond* the mere observation of functional contexts and rules, to achieve something that is eternally beyond the reach of >> natural science <<: we are able to >> understand << the behavior of the *individuals* involved, whereas we cannot, for example, understand the behavior of cells, which can only be seized functionally and be determined according to its procedure"[13] (Weber 1921/1984, p. 32ff., own translation, emphasis included by the author according to the original text)

Understanding may now be on one hand "the construed grasping of the really intended" (own translation, ibid., p. 25) meaning

- "of an average or approximately" (own translation, ibid., p. 25) intended meaning in a number of cases or on the other hand
- "of the scientifically constructed (>> ideal type <<) sense or meaning relationship for a pure type (ideal type) of a frequent" (Weber 1921/1984, p. 25, own translation, emphasis included by the author according to the original text).

Weber distinguishes between two kinds of understanding. Current understanding, which means immediate understanding, and explaining understanding, which

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[12] German expression: "Ein Motiv heißt Sinnzusammenhang, welcher dem Handelnden oder dem Beobachtenden als sinnhafter Grund eines Verhaltens erscheint" (Weber according to Weiß 1975, p. 48).

[13] German expression: "Wir sind ja bei sozialen Gebilden […] in der Lage: über die bloße Feststellung von funktionalen Zusammenhängen und Regeln hinaus etwas allen >> Naturwissenschaften << ewig Unzugängliches zu leisten: eben das >> Verstehen << des Verhaltens der beteiligten Einzelnen, während wir das Verhalten z.B. von Zellen nicht verstehen, sondern nur funktionell erfassen und dann nach Regeln seines Ablaufs feststellen können." (Weber 1921/1984, p. 32ff.)

means construing the understanding of a motivation-based meaning relationship, construing of actions by indication of acting reasons from a meaning relationship. Ideal types describe possibilities of acting but not actual acting determined in advance.

According to Weber, ideal types are not theories in themselves. "This procedure can be indispensable for heuristic as well as expository purposes. The ideal typical concept will help to develop our skill in imputation in research, it is no "hypothesis" but it offers guidance to the construction of hypotheses. It is not a description of reality but it aims to give unambiguous means of expression to such a description" (Weber 1949, p. 90).[14]

Ideal types are according to Weber methodical tools and not theory components of interpretive sociology. Weber does not properly formulate the methodical procedure for constructing ideal types; his descriptions remain vague. He does, though, indicate features and criteria for ideal type construction, which Gerhard uses later as guidelines for the development of a data-based ideal type construction, namely, the process-structure analysis (Gerhard 1991a, b).

It is only Schütz who takes up Weber's ideas of interpretive sociology and works them out on the basis of phenomenological observations (Schütz 1932). In so doing, he shows that the notion of ideal type construction is not only a method of scientific research, but may also be seen as tool for everyday understanding.

5.6 Ideal Type Construction: Method of Everyday Understanding

Starting from the general thesis of intersubjectivity that postulates the experiences of another person are structured in the same way as one's own experience and that, on the other hand, another person's experience takes place concurrently with own experience, Schütz describes from a phenomenological perspective how the social world may be thought to be constructed layer by layer from the experience of a single person (Schütz 1932). Schütz builds on Weber's conceptualization, specifying it and—according to Gerhard—remedying three of its weak points:

– Although taking meaning orientation in daily life as a starting point, Weber fails to explicate how meaning construction takes place.
– Weber leaves open the question of how it is possible to understand the *other*.
– Weber does not carry out any detailed differentiation between subjective and objective meaning in the sense of the subjectively-meant meaning of another

[14] German Expression: "Für die F o r s c h u n g will der idealtypische Begriff das Zurechnungsurteil schulen: er ist keine >> Hypothese <<, aber er will der Hypothesenbildung die Richtung weisen. Er ist nicht eine D a r s t e l l u n g des Wirklichen, aber er will der Darstellung eindeutige Ausdrucksmittel verleihen" (Weber, 1922/1985, p. 190).

person or the probable meaning or context of meaning supposed by an observer. (cf. Gerhard 2001, p. 407ff.)

A central result of Schütz's (1932) analyses is the realization that acting is always oriented to a drafted plan. Drafts for action are based on interpretive schemata resulting from previous experiences. They are based on *because* motives and are oriented towards *in-order-to* motives. The meaning orientation of action is firmly anchored in such motivation contexts. According to Schütz, courses of action may be anticipated and new experiences may be structured with help of experience based patterns sequences. In this way, an interpretation background for further experiences is gradually built up layer by layer.

Schütz assigns two completely different worlds to Weber's differentiation between current and explanatory understanding. Understanding in every-day life is always current understanding, whereas understanding in sociology is explanatory understanding (Schütz 1932, p. 29; 1967, p. 31). Moreover, understanding another person is in both worlds always an understanding of the *other* ("Fremdverstehen"). Understanding of the *other* is made possible as a result of activating interpretation patterns that correspond with those activated in others. According to Schütz, a meaning context may only be approximately reconstructed by another person. Schütz equates an *objective meaning* to a meaning context that can be understood detached from current time and action: "Nevertheless, it is exhausted in the ordering of the interpreter's experiences of the product within the total meaning-context of the interpretive act. (Schütz 1967, p. 134; cf. 1932, p. 152). Whereas subjective meaning is always tied to the relevance of the here and now that means

> […] that every interpretation of subjective meaning involves a reference to a particular person. Furthermore, it must be a person of whom the interpreter has some kind of experience (Erfahrung) and whose subjective states he can run through in simultaneity or quasi-simultaneity, whereas objective meaning is abstracted from and independent of particular persons.[15] (Schütz 1967, p. 135)

Thus, the subjective meaning of another person is made experiencable through a shared "we-relationship" ("Wir-Beziehung") (Schütz 1932, pp. 132, 183; 1967, p. 164). The more I know about the other person, the more comprehensively I can learn about the subjective meaning of the other—albeit only in approximation. A we-relationship develops from a life together in a shared meaning context in "simultaneousness" (Gleichzeitigkeit) (Schütz 1932, p. 151; 1967, p. 134). Schütz distinguishes between two experimental spheres: the world of one's consociates and the world of one's contemporaries. The world of consociates consists of those persons with whom we are in a we-relationship. According to Schütz, the world of contemporaries is a limit concept (Schütz 1932, p. 197; 1967, p. 176f). "In particular, the greater my awareness of the we-relationship, the less is my involvement in it, and the less am I genuinely related to my partner" (Schütz 1967, p. 167; cf. 1932 p. 185).

[15] German Expression: "[…] dass jede Deutung des subjektiven Sinnes seines Erzeugnisses auf ein besonderes Du verweist, von welchem der Deutende Erfahrung hat und dessen aufbauende Bewußtseinsakte er in Gleichzeitigkeit oder Quasigleichzeitigkeit nachvollziehen kann, indessen der objektive Sinn von jedem Du losgelöst und unabhängig ist" (Schütz 1932, p. 152f).

Life in the world of consociates means that motives and actions may be aligned in interaction with the consociate, that meaning constructions may be coordinated with the other. That is already different for an observer in the world of contemporaries. “When I start asking questions of the person observed, I am no longer a mere observer”[16] (Schütz 1967, p. 174).

For the observer of a person or a we-relationship, it is only possible to infer an interpretation on the basis of own interpretation patterns.

Besides near-relationships, everyone enters into social relationships of great distance. For example, school pupils have such a reserved relationship to teachers who do not teach them. This relationship is one of a world of contemporaries. Consequently, my world of contemporaries encompasses all persons with whom I do not share a life context but to whom I have a relationship all the same. Meaning contexts in the world of contemporaries are reconstructed on the basis of interpretation patterns that have been built in consociate world relationships. Construction of meaning in the world of contemporaries is always construction of objective meaning, and that derives from ideal-type construction. “However, it is due to this very abstraction from subjective context of meaning that they exhibit the property which we have called their *again and again* character. They are treated as typical conscious experiences of *someone* and, as such as basically homogeneous and repeatable”[17] (Schütz 1967, p. 184).

This means that repeating or fundamentally repeatable procedures become established as *procedure type* in consciousness. Typical acting is presumed to have invariant “in-order-to” and “because” motives. Thus, a *personal ideal-type*, i.e. a state of consciousness, is constructed which we may presume has respective “because”- and “in-order-to” motives as its subjective meaning context. (Schütz 1932, p. 213ff.; 1967, p. 188ff.). Hence, a procedure type presents the objective meaning context for a personal ideal-type. It is the similar and always recurring of the other that leads to an ideal-type construction.

> This synthesis is a synthesis of recognition in which I monothetically bring within one view my own conscious experiences of someone else. Indeed, these experiences of mine may have been of more than one person. And they may have been of definite individuals or of anonymous >> people <<. It is in this synthesis of recognition that the personal ideal type is constituted. (Schütz 1967, p. 184; cf. 1932, p. 206)

“The contemporary alter ego is therefore anonymous in the sense that its existence is only the individuation of a type, an individuation which is merely supposable or possible” (Schütz 1967, p. 194f.; cf. 1932, p. 218).

These pre-experienced ideal-types form an interpretation pattern not only for meaning constructions in the world of contemporaries but also for meaning context

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[16] German Expression: “Das Du ist für den Beobachter als Beobachter wesensmäßig unbefragbar” (Schütz 1932, p. 193).

[17] Schütz writes: “[…] weil sie aber losgelöst von dem subjektiven Sinnzusammenhang, in dem sie sich konstituieren, betrachtet werden, weisen sie die Idealität des “Immer wieder” auf. Sie werden als typisch fremde Bewusstseinserlebnisse erfasst und sind als solche prinzipiell iterierbar” (Schütz 1932, p. 206).

in the world of consociates (Schütz 1932, p. 218; 1967, p. 192). If a personal ideal-type exists, then this ideal-type is connected with a procedure type. This way a typical acting in the world of contemporaries in a “Du”-relationship is expected when another person is seen as representative of a personal ideal-type. If for example a person is looking through a stack of exercise books in front of her, then one may assume it is a teacher correcting pupils’ work. But if we know that the person teaches mathematics, we may then suppose that she is correcting mathematics work and checking students’ solutions for mathematical problems.

According to Schütz, there is in everyday life a stock of ideal-type constructions as an explanatory basis for explanations of both the world of contemporaries and the world of consociates. These two worlds are not at all to be seen as dichotomous. Starting from the world of consociates, interpretation patterns build up with increasing distance and become ideal-type constructions (Schütz 1932, p. 221; 1967, p. 194).

Observation in the world of contemporaries—the original task of social sciences—can only be effected on the basis of interpretation patterns in the world of contemporaries and standardized “in-order-to” and “because” -motives as typifying constructions. According to Schütz, these constructions must not be confused with the world of consociates. Furthermore, the ideal-types constructed by scholars are not necessarily ideal-type constructions of the protagonists themselves. Therefore, the meaning adequacy and causal adequacy of ideal-type constructions must be ensured, or as Schütz puts it: “This postulate states that, given a social relationship between contemporaries, the personal ideal types of the partners and their typical conscious experiences must be congruent with one another and compatible with the ideal-typical relationship itself” (Schütz 1967, p. 206; cf. 1932, p. 235).

Thus, the constructed ideal-types must not contradict the scholar’s experience background (meaning adequacy) (Schütz 1932, p. 267ff.; 1967, p. 234ff.), and they must, as already required by Weber, be found approximately in the social world to serve as a rational background (causal adequacy) (cf. Schütz 1932, pp. 256ff., 262ff.; 1967, p. 229ff.).

Looking back, Schütz questions whether social collectives may be classified in an ideal-type way. He subsequently rejects such a postulate, pointing out that one may not impute a subjective meaning relationship to a collective. But he does admit that such ideal type constructions may be thought of as patterns of high anonymity which suppose the layer-by-layer constitution of ideal-types in everyday life to be given and do not ask for a subjective meaning context (Schütz 1932, p. 277; 1967, p. 242). Following this, it becomes clear that Schütz reduces the meaning constitution of the social world to subjective meaning constructions; that he allows ideal-types but does not explain how the construction of those ideal-types may come about.

A critical point in Schütz’s work on ideal-type construction in everyday life refers to the question how ideal-types can become an intersubjective part of the world of contemporaries when they are construed exclusively in peoples’ minds (cf. Srubar 1979, p. 43). Srubar explains that in Schütz’s analysis the construction of ideal-types is based on the fundamental repeatability of actions and that is founded on experiences in the world of consociates. But how are ideal-types constituted as carriers of intersubjective meaning in everyday life? Srubar suggests to distinguish two times, the *constituted* time, that is the individually and subjectively experienced time which may be shared in the consociate world, but not in the world of contemporaries, and the *produced* time which is experienced in a world of contemporaries, which "[…] is fixed by the course of social processes" (Srubar 1979, p. 54, own translation).[18] Srubar does not conceive produced time as measurable physical time; but what is exactly meant by produced time is more paraphrased than fixed precisely. He perceives produced time as a time-related experience background which is fixed by regular social (action) processes in the world of contemporaries. "It [the produced time] affects the sphere of everyday reality as an anonymous outer constraint" (ibid., p. 54, own translation),[19] which can, though, also be changed. The daily rhythm which is marked by work, e.g. regularities in everyday school life and the regular order of a mathematics lesson, form produced time as a time-related experience background for individual time experience. Hence, repetitions of sequences of events may not only be experienced in the constituted time but also in the produced time, whereby the latter may be experienced intersubjectively. While working on new material in a mathematics lesson the process sequence *presentation of problem—activities of pupils—collecting and assessing of results—setting of homework* can be influenced by previous lessons and as produced time fix the frame for individual experiences. Jointly experienced, socially fixed or collectively produced courses of events or processes may thus become an ideal-type framework of experience of the world of contemporaries which is—up to a certain degree—shared intersubjectively but also individually fixed. Repeating processes in the social world may be tied together with procedural types and personal ideal-types as a typical frame experience. In the abovementioned introductory lesson, a personal ideal-type could describe a female pupil who takes up the mathematical problem, works on it, produces solutions, gets marks and writes her homework in her exercise book.

Scientific constructions of ideal-type situations from repeated social actions in teaching practice could be equivalents of procedural types of a collective character in the world of contemporaries, as these social actions are fixed from outside and are at the same time repeatedly produced in a collective way. Such ideal-type situations, in the following passage also called situational types, which means situation-related ideal-types or ideal-type characterizations of situations, would describe typical not only sporadically appearing conditions for possible procedural types.

Now, if it is true that people shape their experiences and acting in the worlds of consociates and contemporaries by creating ideal types and that produced time fixes the frame for constituted time, i.e. for constructions of subjective meaning, then it absolutely makes sense to ask for situational ideal types in teaching practice, to reconstruct them systematically and to use them as a means of constructing theory. As a consequence, it would be possible to describe and explanatorily understand

[18] German Expression: "durch den Ablauf sozialer Prozesse gesetzt wird" (Schütz 1932, p. 235).

[19] German Expression: "Sie [die produzierte Zeit] wirkt sich im Bereich der Alltagswirklichkeit als anonymer äußerer Zwang aus" (Schütz 1932, p. 235).

typical sets of conditions. Such situational ideal types form pre-structured offers for teachers because they provide identification patterns for a teaching practice oriented to worlds of consociates/contemporaries, they provide understanding complexes, and offer sensitizing orientation possibilities for practical acting. It is not intended that teachers implement these theory components in practice, but to encourage them to use the pre-structured situational ideal types as guidelines for practical acting in classes. However, it may sometimes be necessary to avoid an orientation in the direction of a situational ideal type. Theories which have been constructed with help of such situational ideal types serve as a sort of spectacles through which teachers may observe their teaching as a collective acting structure, and, based on this background, they may develop their own meaning constructions and create their own theory-led acting.

In the following, I describe how ideal types may be constructed based on data, and with help of examples illustrate how the results of these construction processes can contribute to the development of theories in mathematics education.

5.7 Empirically Based Ideal Type Construction: A New Beginning

As Gerhard (2001) describes, Weber does emphasize the significance of ideal type construction for empirical sociology, yet he gained ideal types himself from his historical constructions, and not in a methodically controlled way from selectively collected data. The data-based ideal type construction and analysis, “die sich heute auf Weber beruft, betritt Neuland, das Weber zwar entdeckt, aber nicht in seinen eigenen Projekten empirischer Sozialforschung erkundet hat. Webers eigene Arbeiten, die Idealtypen verwandten, waren bekanntlich historische Rekonstruktionen [which nowadays refers to Weber, enters unknown territory that had been discovered by Weber but that he has not explored in his own projects of empirical social research. Weber’s own works using ideal types were—as everybody knows—historical reconstructions]” (Gerhard 2001, p. 11, own translation).

With help of a data-based ideal type analysis, Gerhard investigates “patient-careers” (Gerhard 1991a, b) of persons with a kidney disease. It justifies this kind of biography research with its historical relation, because on the one hand biographies are embedded in historical current events, and on the other hand biographical histories as historical processes depend on a time structure of their own. Both justify a use of ideal types following Weber, particularly as the necessary individual case analyses necessitate investigation from the perspective of the acting persons themselves (Gerhard 1986, p. 63). Gerhard explicitly carries on with Weber’s work, and following Weber’s instructions she works out a method of data-based ideal type analysis, process structure analysis, which consists of four steps.

- *Case reconstruction and case contrasting*: first, the biographical data are reconstructed as individual cases (case reconstruction), and afterwards a link is established on the principle of minimum and maximum contrasting.

- *Determination of pure cases*: In a second step, pure courses are gained theoretically and empirical cases are identified which represent these pure cases as well as possible. These empirical cases are the so-called prototypes.
- *Comprehending of individual cases*: Following Weber, Gerhard confronts the individual cases with the ideal type in order to work out each individual case's own work in comparison to the ideal type and to comprehend the events in an explaining way.
- *Comprehending of structures*: the fourth and final step is about comprehending structures, i.e. the question which social structures cause certain events.

5.8 Ideal Type Construction in Research of Mathematics Education

Ideal type construction is not a new phenomenon in mathematics education. Strunz (1968) describes four ideal type approaches to mathematics (Strunz 1968, p. 313ff.), and Kaiser gains ideal type descriptions of English and German mathematics lessons from an ethnographically oriented study on mathematics classes in both countries (Kaiser 1999). A methodically transparent ideal type construction based on the idealization of empirical data as a means of theory construction was not taken up in interpretative research of mathematics education until recently.

Concentrating on the aspect of ideal type construction, initially I describe Knipping's (2003) study and make clear what has to be understood by an idealization of ideal type construction. Taking this as an example, I want to show an ideal type construction's function to show the direction to hypothesis generation. It will become clear that the process of ideal type construction may be understood as a first step to theory development that is an ideal type concept construction, and that with these ideal types a further analysis of empirical data that continues the process of theory development is possible. Subsequently, I present various forms of ideal type construction taken from my project *Mathematikinteresse zwischen Subjekt und Situation* [Interest in mathematics between subject and situation] (Bikner-Ahsbahs 2005; Bikner-Ahsbahs and Halverscheid 2014) and describe how with the help of ideal type construction sub-theories may be developed and in the end brought together to a more comprehensive theoretical view, a first approach to a subject-related theory of interest-dense situations.

5.8.1 Ideal Type Construction by Idealizing of Prototypes

Knipping (2003) compares teaching processes of German and French mathematics lessons on proof processes of Pythagoras' theorem by analyzing the contexts of the proof processes and by analyzing the argumentation processes. Here she orients towards Gerhard (see also Kluge 1999; Gerhard 1991a, b, 2001).

First, proof types and task types are gained from the context analyses of the individual cases. These form the basis for a comparing context analysis of the cases' proof processes. By case comparison and case contrasting, Knipping comes to a case cluster. She chooses prototypes for each of the two groups that represent them best. The comparison of cases with the respective prototypes leads to a further idealizing of these prototypes in separation from the respective case regarding the functions of proof processes. The ideal type constructions of the proof contexts of *realizing that* and *giving reasons for* are based on this analysis. In a methodically totally analogous way, via local and global argumentation analysis of oral and written proof discourses she gains two ideal type characterizations of proof discourses, namely *watching interpretation* and *public explanation*. Despite independently effected context and argumentation analyses, interestingly the same distribution of groups are yielded in those cases.

Knipping analyzed a series of cases, arranged them and chose prototypes for the respective groups. Comparing analyses of prototypes to the cases leads to an idealization of prototypes—and this is where Knipping obtains characteristic feature structures. She uses the respective feature structure as fulfillment norm for a constructed case. This way, for each group she obtains a pure case, which may be described approximately by the real cases, but where she at the same time disregards consistency distracting features. Idealizations to construct ideal types are effected on the base of empirical cases, i.e. "effected in a way

- that a constructed *pure* case is added to an amount of accepted real cases,
- that the constructed case may be described approximately by the real cases,
- that consistency distracting parts are left outside the new total amount" (Bikner-Ahsbahs 2005, p. 86; see also Schreiber 1980, p. 46).

From the cluster results, Knipping obtains two pairs of amounts of accepted real cases where the respective pure case is added. The total amounts are only observed regarding the main features.

In her analyses, Knipping works out a number of features from proof contexts and proof processes and uses them as base for the construction of ideal types. The respective feature contexts and the constructed ideal type characterizations may be regarded as first components of a theory on proof processes in teaching practice, because by ideal type descriptions typical features of proof processes in mathematics lessons are conceptualized.

But how beyond this can these ideal types, as Weber formulated, "[…] offer(s) guidance to the construction of hypotheses." (Weber 1922, p. 190; 1949, p. 90) and in this way lead to the construction of a broader theory?

A basis for that may be, e.g., the empirical distribution of cases on the four groups which have been led to characterization of ideal types, because interestingly in Knipping's research (2003) the observed German classes regarding the context excel in an ideal-type way by the function of *recognizing that* and regarding the proof discourses by *clearly interpreting,* whereas the observed French lessons may be characterized regarding the contexts by the function of *explaining why* and regarding the proof discourses by a *public explaining*. Exactly this fact may encourage the construction of hypotheses or formulation of further questions which might point the way for a theory development of proof processes in mathematics lessons. It is very much the question, e.g. whether both types of proof contexts may be structurally interlocked with both types of proof discourses, and what might be the consequences. Does, e.g., the fixing of the context alignment suggest a determined way of argumentation? If yes, why? Or formulated differently: which structural features of proof contexts and proof discourses lead to such a dichotomy?

To investigate this question in an empirically based way we could, to start with, of course search selectively for examples to prove the opposite, i.e. for cases that do not match this dichotomy. In the event such cases cannot be found, then it would seem to suggest that proof contexts and proof discourses should indeed be observed as being closely interlocked with each other. When such cases *are* found, then more exact analyses of those so-called counter-examples might on the backdrop of the ideal types gained by Knipping contribute to clarifying why there is a dichotomy in the cases of Knipping's research or why these so-called counter-examples drop out of dichotomy.

Only via a thorough working out of the four ideal types was Knipping able to lay the foundations for a theory on proof processes in mathematics lessons, because on this basis it is now possible to formulate questions to point the way for further theory development.

Knipping's example makes clear that on the one hand theory components slip into the construction of ideal types, and that on the other hand (dichotomous) distributions of empirical cases may be there that prompt further questions and initiate a search for counter-examples. And that, in turn, contributes to arriving at a theoretical understanding of determined facts. Hence, ideal type characterizations present concepts that are gained empirically, that condense theoretical understanding, and point the way to a further theory development on the basis of empirical data.

5.8.2 Ideal Type Construction: Principle of Factual Theory Construction

My research project *interest in mathematics between subject and situation* (Bikner-Ahsbahs 2005) explores teaching situations that are in a determined way suited to support interest development in mathematics lessons and researching how these situations develop. On the basis of research results from the psychologically oriented interest research linked with empirical analyses of teaching sequences, I found features of situations that may describe situations of a special interest quality. I call such situations *interest-dense*. Interest-dense situations are given when the learners become involved in solving mathematical problems (*to be involved in the situation*), when one after another they construct continuing deepened meanings (*knowledge dynamics of the situation*), and when the meaning or the significance of the situation has to be searched for in mathematics (*mathematical significance of the situation*). A situation possesses interest density when it is interest-dense (Bikner-Ahsbahs 2005; Bikner-Ahsbahs and Halverscheid 2014).

The data stem from video recordings of the complete lessons of a sixth grade class over the course of half a year. Among these data, interest-dense situations are identified and analyzed regarding the geneses of interest density. Central principle of this analysis is case comparison and case contrasting. Thematically completed teaching episodes serve as cases, whereby above all non-interest-dense situations are also included.

With the help of interpretative sequence analyses on three levels, the level of social interactions, the level of epistemic processes and the level of value assignment or value construction, three sub-theories of interest-dense situations are gained and subsequently brought together. An essential stage of the four theory stages is the construction of ideal types. Basic theoretical insights are condensed to situational ideal types, and these ideal types present the conceptual base of the subsequent theory. In the process, ideal-type construction is not understood as method but rather as methodical principle. The respective process of ideal-type construction is adapted to the subject's specific features and the available data.

5.8.3 A Model of Polar Ideal-Type Construction

On the level of social interactions, initially scenes are chosen where the construction of an interest-dense situation is successful. These scenes are compared to each other and to scenes that start similarly but where the construction of interest density fails. From that, features are gained that may particularly hinder support of the construction of interest density. These features enter the construction of two ideal-type interaction structures, the expectation[20]-dominant interaction structure and the expectation-recessive interaction structure. These interaction structures are developed in a cyclic process of comparative analyses. Each analysis series consists of the following four stages:

- Individual case analyses, i.e. analysis of single interest-dense scenes.
- Comparing analyses of interest-dense scenes.
- Comparison of a genesis of an interest-dense scene with a similarly starting scene where the genesis of interest density fails.
- Construction of an interpretation hypothesis according to the question: What actually contributes to the success or failure of interest density construction?

From the ideal-typical vantage point, the expectation-dominant interaction structure hinders the construction of interest density, whereas the expectation-recessive interaction structure is connected to successful genesis of interest-density.

In addition to these ideal type interaction structures, a feature space is subsequently constructed—in literature this action is called subscription (Kluge 1999, p. 61). The corresponding features are the *teachers' behavior* with the expressions *expectation-run* and *situation-run* and the *students' behavior* with the expressions

[20] By expectation we here always speak of mathematical-in content teaching expectations.

expectation-dependent and *expectation-independent*. At the same time, expectation-dependent students' behavior and expectation-run teachers' behavior interlock to an expectation-dominant interaction structure with the function to (re-)produce the textual teaching expectations; and expectation-independent students' behavior and situation-run teachers' behavior combine to an expectation-recessive interaction structure with the function to produce mathematics-related student contributions from the students' perspective.

As in Knipping's research, a dichotomous distribution of the investigated episodes may here be found, too. However this distribution is based on a dichotomous selection of the scenes. The process of theory development is in this case stimulated by the fact that the free parts of the feature space are filled by empirical cases, and the respective scenes are analyzed regarding a possible genesis of interest density.

These analyses lead to a description of both interaction structures as states of an interactive balance from which the other interaction forms seem to develop. This balance metaphor now clarifies an essential difference between the two interaction structures: the expectation-recessive interaction structure might be described as state of an unstable, and the expectation-dominant interaction structure as state of a steady balance, because the expectation-recessive interaction structure is, in contrast to the expectation-dominant one, a fragile interaction form that might easily be disturbed in its flow. If, e.g., the teacher's behavior changes from a situation-run to an expectation-run teacher's behavior, then students mostly adapt themselves immediately: An expectation-dominant interaction structure is created that even by a sudden change in the teacher's or students' behavior is not limited essentially in its flow. The reason for this stability probably lies in the partially implicit but secure knowledge that the (re-)production of content-related teaching-expectations leads (certainly) to a solution of the problem, whereas a focus on production of students' ideas may be problem-oriented, but it is not at all sure that the solution of the problem comes nearer. That way a permanent adjustment and adaptation of teacher and student to each other is necessary in order to maintain this interaction form. The expectation-recessive interaction structure is consequently fraught with many more imponderables and uncertainties than the expectation-dominant interaction structure, and that makes up its fragile character.

As non-interest-dense situations, from an idealized vantage point, may also develop expectation-recessively, the emergence of the expectation-recessive interaction structure may at best be considered as a necessary but not at all sufficient criterion for the creation of an interest-dense situation. This result leads now to the question of further creation conditions of interest-dense situations and motivates an analysis on either level.

The method of polar ideal-type construction is oriented to the construction of so-called extreme types. It is the obvious thing when constitution conditions of polar situations are demanded. In the following examples, ideal types are constructed in the perspective of epistemic processes and in perspective of valuation constructions. As in this construction process no polar scenes can be found in the data, according to Gerhard (2001) in these cases ideal types are constructed by forming groups.

5.8.4 Construction of Epistemic Action Types

In the data there are two different types of interest-dense situations. Ad hoc interest-dense situations are initiated by students. The genesis process of interest density is in this respect not directly accessible and may also not be reconstructed on the basis of video recordings. Generative-interest-dense situations are teacher-initiated and reconstructable in their genesis process. The analysis of generative-interest-dense situations seems initially to be more fruitful for theory development because the course of recognition development is understandable. The following analysis of ad hoc interest-dense situations on the basis of acquired results leads to a deeper understanding of generation conditions of interest density.

So recognition does show and develop in shape of signs—e.g., words, sentences, diagrams—but the epistemic step behind it arises from the meaning of signs, and the recognition development in an interaction process takes place in a continuing interpretation process. To show this in the analysis of investigated teaching units, we differentiate between signs, intended sign significance and interpreted sign significance.[21]

The analysis starts with an interpretative sequence analysis (Beck and Maier 1994b; Krummheuer and Voigt 1991; Krummheuer 1992; Krummheuer and Naujok 1999). From this, the insight is gained that essential parts of the epistemic processes may be described with the epistemic actions of gathering and connecting meanings and structure-seeing. On the basis of the analysis, the investigated episodes are presented with the help of further signs condensed to a manageable degree and compared with each other. A grouping of generative-interest-dense situations in minimum and maximum distinctions leads to three groups whose typical phase profiles are constructed and presented with help of pictographs. By case comparison and case contrasting the common features are afterwards elaborated from an idealizing vantage point and a typical profile of each group (re-)constructed. This results in three action types: the *graded,* the *helical,* and the *merging type*.

The characterizations *graded, helical* and *merging* describe how social construction of new meanings happens. In the case of the graded phase type, e.g., this happens (usually) in (three) stages, i.e. initially meanings, examples, e.g., are gathered (stage 1), then they are linked to each other or to other meanings (stage 2), and finally structures are made out of the examples and descriptions on the board or in the exercise book (stage 3). Prototypical situations here are chosen to describe ideal-type phases and not, as Knipping did, used as means of ideal-type construction.

The procedure to construct epistemic phase types, the *semiotic sequence analysis* (Bikner-Ahsbahs 2006) consists of three steps:

- Sequential reconstruction of recognition processes
- Comparison of condensed phase descriptions
- Comparison of pictographs

[21] Peirce's triadic sign concept (Peirce 1897; CP 2.228 according to Hoffmann 2005, p. 40ff.; CP 2.308 according to Nöth 2000, p. 62) which distinguishes among object, sign, and interpretant was used and adapted to analyzing social interaction in class (Bikner-Ahsbahs 2005, p. 66ff.)

How does the further process of theory construction take place? I confine myself to a few aspects.

Generative interest-dense situations are compared with ad hoc interest-dense and non-interest-dense situations. Here it is shown that certainly all interest-dense situations lead into phases of structure-seeing, but that the epistemic processes in ad hoc interest-dense situations do not take an exemplary course, and that is a clue to a very narrow acting scope for the teacher. What are the consequences? The condition of interest density in ad hoc interest-dense situations may obviously only be maintained if the teacher acts situation-run, i.e. he or she is prepared to face the situation.

Epistemic action types, too, may be understood as condensed conceptual descriptions of first theory components, namely in the different ways epistemic processes are constructed. Here, the three epistemic actions play a central role. The final comparison of phase types among themselves and with non-interest-dense situations finally leads to the central question as to which features of epistemic processes might in a special way be responsible for the fact that creating interest density fails or succeeds.

Non-interest-dense situations differ from interest-dense situations in the fact that epistemic processes are extensively limited to gathering and connecting meanings. This raises the question why structure-seeing in interest-dense situations does not normally take place distinctly, and is mostly not achieved at all. On the whole three reasons are reconstructed.

- The range of the collected examples is in non-interest-dense situations narrower than in interest-dense situations, i.e., the gathering and connecting phases turn out to be an insufficient basis for structure recognition.
- The construction of an interest-dense situation often gets stuck because there is nobody who might be able to anticipate the mathematical-theoretical direction from the perspective of students' comments and to offer support. That means in many cases that especially the teacher ignores or rejects productive ideas that might lead to structure-seeing, or he stops the process in an expectation-run way. This is exactly not the case in interest-dense situations.
- Finally, finding something new means in many cases having to express what is new with an insufficient vocabulary. This leads occasionally to apparent contradictions or tricky situations that may be remedied through the teacher's verbal support.

5.8.5 Construction of Production Types

Basis for the construction of a third sub-theory is the question how a situation attains mathematical significance, and in which way this contributes to a genesis of interest-dense situations. First of all I clarify what makes up mathematical significance.

The mathematical significance of a situation is observed as a potential to initiate interest relations. By reconstructing processes of value construction, a condition structure is found that may be seen as an essential structure for the production of mathematical significance in the various situations.

Like the previous processes of ideal type construction, this process starts with individual case analyses of interest-dense situations and their comparison. They show that all interest-dense situations lead to assessment situations. Assessed are mathematically substantial ideas, and therefore the performances of all involved idea producers and vice versa. Here, acknowledgement of an idea means at the same time acknowledgement of the idea producers' performance, and vice versa. We therefore may assume an interdependent value-related relation between idea and idea producer. The question now is how these value-related relationships are constructed as these assessment situations are approaching. Analyses of previous processes show that the explicit assessments at the end of interest-dense situations are not at all the only value attributions. They are preceded by already interactively produced or implicitly expressed value constructions during the interaction process. Most of the interactively produced value attributions arise from a process of mutual increasingly emotionally loaded value assigning reactions. Now the question is: which teaching features enable this? And why do interaction processes concentrate, bundled up like that, on the development of mathematically substantial ideas, the idea products.

On the whole, it becomes clear that the group of values assists to produce mathematically substantial ideas of its own, to bundle up the collective interaction process and get it going. Its basis is a kind of implicit social contract: the students produce their own ideas for the mathematical questions, and the teacher organizes the lesson in such a way that their own idea productions are possible. But what does this kind of lesson organization look like? Analyses show that two different kinds of moderate indistinctnesses are basically accepted in these processes: *moderate interpretation indistinctness* and *moderate participation indistinctness*. The acceptance of interpretation indistinctnesses enables students to proceed more easily from previous contributions with their own individual ideas and to expect acceptance of their ideas; and the acceptance of moderate participation indistinctness enables students to spontaneously voice their opinion, to explicate emotionally assessing relations, and to maintain their point of view concerning contents and processes without having been explicitly asked to do so by the teacher. The formal discussion runs in parallel but takes up spontaneous remarks. These conditions represent promising prerequisites for developments of value-related relations to mathematical contents, because the students may voice spontaneously and free of suppression their own ideas, suggest further ideas, take up others, develop and improve, and thus become active participants in the production of mathematically consistent and applicable ideas. At the same time, they have the chance to present themselves and their relation to the subject in an emotionally assessing way and visible for everybody. Now the question is how these relations become apparent and how this is linked to the respective situations?

For this purpose, all interest-dense situations are compared to each other regarding minimal and maximal differences in respect of the gained findings. The result leads to four groups. By an idealizing comparison within these groups and between these groups, four ideal type production methods of mathematically substantial ideas, the production types, are reconstructed, which describe a frame for different person-object relations and varying conditions for students' participation. These production types are the *idea competition*, the *innovational idea production*, the *experts show* and the *quality control*. In a process of quality control, e.g., the simplified version of the mastered division rules for fractions is tested: At first an intensive search is conducted for examples to prove the opposite, and afterwards the new operation is compared to other operations, e.g., the multiplication rule, and tested once more before being accepted. In quality control, therefore, an idea regarded as substantial is being tested in-depth for quality. The students involved may present themselves as experienced inventors of examples, counter examples, questions and provoking remarks, as clarifying searchers for answers, or as distributors of quality seals.

The production types now describe various qualities of mathematical significances, i.e. to initiate different potentials of interest relations. A retrospective comparison of empirical cases with the production types now shows that there are prototypes for the production types, but that the different empirical cases of interest-dense situations do differ considerably and vary considerably from the production types. Interest-dense situations certainly refer to qualities of mathematical significances in the shape of special production ways, but the potential to initiate interest relations is even in interest-dense situations not as a rule completely exhausted. Theoretically, the potential would be exhausted if the specific idealized way to produce a mathematical idea was realized (at least almost).

The difference between interest-dense situations and non-interest-dense situations is that in non-interest-dense situations the specific ways of producing substantial mathematical ideas do not occur. One hindrance to developing such production types is, e.g., effected by not accepting interpretation indistinctnesses, but each single contribution has to be specified before the interaction process may continue. This disturbs the idea flow considerably, and hinders students from contributing to the development of an idea which they would otherwise be able to do.

The various production types now represent a differentiating offer for students to contribute to producing mathematically substantial ideas. This is shown, e.g., by the fact that in situations of different production ways, different pupils contribute to the process. An evaluation of the results against the backdrop of psychological interest theories suggests that by an active participation of students in interest-dense situations interest is really developed or further developed. But up to now this is not answered by the theory that has been developed. This can only be clarified by a further theory development in the direction of an interlocking with individual parallel processes of interest development.

5.8.6 Ideal Type Construction with Ideal Types

In the last step of theory development, the results of the three analyses are brought together. We ask for the context of social interactions, epistemic processes and the production methods regarding genesis and stabilization of interest-dense situations, what characterizes interest-dense situations, and how is the interest supporting quality expressed therein?

An expectation-recessive interaction structure is a necessary condition for the development of interest-density on the whole. The teacher must be able to focus immediately on ad-hoc interest-dense situations so that the state of interest-density is kept. Whereas, concerning generative-interest-dense situations the teacher has leeway. Generative-interest-dense situations now divide into three different, typical course types, the graded, the helical and the merging type. The course types finally differentiate further into three possible production types. But the empirical data indicate that not every production type may develop in every course type, e.g., the expert show only appears in the graded or merging type and not in the helical type, whereas innovational idea production may only be observed as helical type (Table 5.1). Are such results features of the observed class, or is the connection of determined course and production types fundamentally not possible? Which structural connections between course and production may be possible? What does the connection between course and production actually entail?

If we now combine the epistemic action types with production types in a cross tabulation and assume an expectation-recessive interaction structure, then we obtain an overview of possible ideal-types of interest-dense situations which may be characterized by typical descriptions of students' participation. An idea competition, e.g., appears either as graded or as helical process, and this is expressed in different action forms. Table 5.1 shows the results of ideal type construction by ideal types.

We get 12 fields of possible ideal types of generative-interest-dense situations that are characterized by typical action-participations. This result is a step towards a development of a typology of interest-dense situations. But the available data are

Table 5.1 Types of generative interest-dense situations

Epistemic action types / Production types	*"helical"*	*"graded"*	*"merging"*
"innovational idea production"	critical-struggling co-construction	?	?
"idea competition"	reflexive-competing-examining construction	moderately-competing construction	(prepared-merging construction)
"experts show"	(ad-hoc expert presentation)	Categorized presentation and re-construction	merging new- and re-presentations
"quality control"	?	critically testing new- and re-construction	(merging expert's presentations)

Legend: ? uncertain, empirically not verified ideal type
( ) possible, empirically not verified ideal type

not sufficient to fill all the fields empirically or even just to answer the questions whether all fields may be filled empirically, which of the fields remain (almost) empty, and which of them could be grouped to a common ideal type in a meaningful way. On the basis of the existing data, the step may be thought over theoretically but not carried out empirically.

5.9 Summary and Conclusion

Scientific ideal type construction in the interpretative research of mathematics education may be understood as an idealized interpretation of patterns in the world of contemporaries. The examples of empirical research show that in this way complex facts of the world of contemporaries may be conceptualized. The theory development process involving empirically-based ideal type constructions entails two phases reciprocally building on each other: ideal type construction with the aim of interpretative evaluation of empirical data. The results are ideal type characterizations that condense the evaluation results conceptually. These ideal types now present the conceptual base for a further theory development. How we proceed in detail methodically is not stated because ideal type construction is not an evaluation method, but a methodical principle that supports empirically based theory construction. The specific methodical procedure to be taken as a basis for ideal type construction and theory construction has to be developed individually in the respective research process itself. Here, there are common features and principles of different heuristics, depending on the individual situation and available data. Common features are the functions of ideal type constructions, on the one hand to condense theoretical insights in ideal type characterizations into concepts (in order) to constitute, on the other hand, a basis which shows the direction for theory construction. A common principle of empirically based ideal type construction is case comparison and case contrasting. The choice of heuristics for a theory construction on the basis of empirically based ideal type construction is object and data dependent. Therefore there may be many and very different heuristics, as the following list of heuristics used in empirical research with the described examples shows:

- Is the construction of personal ideal types, of action types or situational ideal types useful?
- Is there a polar situation?
- Is a grouping procedure appropriate?
- Are there appropriate prototypes?
- May a feature space be (re-)constructed?
- May the fields of a feature space be empirically filled?
- Which contexts may be expected from it?
- How may the cases be distributed among the ideal type groups?
- Which structures are taken as a basis for such distributions?
- Do these cases consist of long processes so that they should be presented in a condensed way?

- Is there a structural context of already constructed ideal types?
- Is it practical to construct new ideal types by already existing ideal types?
- Is it practical to summarize ideal types to one type?

The abovementioned examples show that situational ideal types may conceptualize complex teaching situations in their typicality. We therefore hope that by means of theory development by ideal type construction we may come to teaching theories of mathematics education which are able to describe the complex contexts of mathematics lessons in a theoretically adequate way. In addition, situational ideal types provide recognition patterns for typical teaching situations and may therefore become a basis for teaching analyses in science and teacher training. Theory construction by means of ideal type construction could therefore become a fruitful methodical principle of interpretative research in mathematics education and might contribute to the development of theories in mathematics education by analyzing teaching practice for teaching practice.

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Chapter 6
How Ideal Type Construction Can Be Achieved: An Example

Angelika Bikner-Ahsbahs

Abstract This chapter of the part presents an example of an ideal type construction of epistemic processes focusing on interest-dense situations. The ideal type construction consists of four steps. The first step starts with reconstructing empirical cases of epistemic processes by data aggregation; it yields to pictographs that represent the investigated episodes in terms of phase structures. In the second step, these structures are grouped according to high homogeneity within and heterogeneity between the groups to shape the base for disclosing the situational key features of each group. In the third step these key features are used to create ideal types of the epistemic processes as "pure cases". Finally the paper illustrates the fourth step describing how these types may contribute to gaining theoretical insight into the dynamic of the epistemic processes investigated.

Keywords Interest-dense situations • Ideal type construction • Epistemic process • Interpretive research

6.1 Introduction

In this chapter, I present an example about ideal type construction to elucidate in what way and what for this methodology may be used in research. The example is taken from the project "Math interest between subject and situation" (Bikner-Ahsbahs 2003, 2005, 2006). It has already been used to illustrate the additional value of building ideal types for reconstructing students' ways of coming to know and the researcher's way of constructing knowledge of the very same research process in Bikner-Ahsbahs (2008). The current paper also shows how ideal type construction assists in structuring such epistemic processes but it focuses more on the methodical steps, specifically on showing the added value of the ideal type's function of leading the way to theorizing (cf. Weber 1922, p. 190; 1949, p. 90; see Chap. 5). Two different forms of theorizing are presented revealing new results about the specificity of the interest-dense situations investigated.

A. Bikner-Ahsbahs (✉)
Faculty 3 of Mathematics and Computer Science, University of Bremen, Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de

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Interest-dense situations may occur in mathematics classes when students fulfill three conditions while working collectively on a mathematical problem: the students are deeply involved in solving the problem; they deepen their insights by progressing and further constructing mathematical meanings; and they either explicitly or implicitly value highly the mathematics they are working on. An interest-dense situation can either be initiated ad-hoc by interest on the part of the student or, alternatively, generated by the behavior of the teacher: the latter is called a 'generative interest-dense situation' and can be achieved when the teacher guides the epistemic process towards transferring the responsibility of constructing knowledge to the students. In all interest-dense situations, the teacher is guided by the situation of student knowledge construction and not by his own content expectations (cf. Chap. 5, Bikner-Ahsbahs & Halverscheid 2014).

In the example I address the empirically based ideal type construction of epistemic processes in generative interest-dense situations. The data source consists of the transcripts of all the generative interest-dense situations that occurred in the course of the abovementioned project (Bikner-Ahsbahs 2005). Methodological background is an adapted version of the empirically grounded ideal type construction that Gerhard (1986, 1991, 2001) has worked out by grounding it on Weber's (1949/translated from Weber 1922; Weber 1921) description of ideal types. The methodology in this example will be based on a process of grouping cases (cf. Kluge 1999, p. 26ff.); it follows four steps (cf. Chap. 5):

1. *Re-constructing the cases*: the epistemic process of each situation is reconstructed and aggregated twice: by building a condensed process diagram, which then is compressed into a phase diagram represented by pictographs.
2. *Grouping the cases according to maximum homogeneity within each group and heterogeneity among the groups*: The phase diagrams are compared, contrasted, and grouped as described.
3. *Building ideal types as pure cases* through idealizing main features of the groups, and at the same time disregarding less important aspects.
4. *Building theoretical knowledge* by comparing the ideal types among each other and analyzing the empirical cases against the background of ideal types.

The first step reflects the way data are approached and phenomena are identified. Before doing so in the example to be presented, I will portray some additional methodological and theoretical considerations of the abovementioned project.

6.2 Methodological and Theoretical Considerations

The background theory (Mason and Waywood 1996) of the project is interpretive research of social interaction in classrooms. Interpretive research assumes that the social everyday world is constituted by the individuals' interpretations of their own and others' actions and mutually orienting their acting towards those of others (cf. Treibel 2000, p. 113).

> According to this assumption, mathematical meanings emerge through interpretations of actions. These interpretations orientate themselves mutually towards the other participants' actions and interpretations. Thus, mathematical meaning is a product of social interaction. Primarily, it is part of the interaction space and not of an individual. These meanings are the sequential steps which together assemble the process of social constructions of mathematical meanings. (Bikner-Ahsbahs 2006, p. 161)

If they are accepted, these mathematical meanings are perceived as knowledge elements, being viable in the local situation. In this sense, knowledge is of situated existence; it may be further developed, and changed later, and sometimes it may even be false. This is what is meant by the expression *a mathematical meaning is taken as shared through a process of negotiation* (see Krummheuer 1992, 1995).

Through re-interpreting the actions and utterances of the students, a researcher is able to reconstruct the epistemic process. In this paper, this is done through the use of Peirce's sign concept in a way that is adapted to the conceptualization of knowledge construction within social interaction. According to Peirce, a sign z is always related to an object o and an interpretant i: "A sign … is something which stands to somebody for something in some respect" (Fig. 6.1) (Peirce; see Zeman 1977, p. 24). Hence, the interpretant provides a special view on the object given by the sign.

Originally, a sign is meant to mediate between object and interpretant, which itself is a sign. However, a person who sets the sign often has another object in mind than the person who interprets this sign. This divergence can be grasped by Peirce's distinction between the "dynamic" and the "immediate" object (Hoffmann 2005, p. 51). The dynamic object is a kind of limit concept made up of all adequate interpretations possible. The immediate object is the one the actual interpretant refers to: for example, the mathematical object that a learner speaks of in a learning situation, which may be different from that of the mathematical culture. In the first step of ideal type construction, the development of immediate objects is re-constructed—and thus results in a description of the epistemic process. Such a process will now be illustrated by a specific example following the four above-mentioned steps.

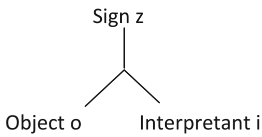

Fig. 6.1 Peirce's triadic sign relation (cf. Hoffmann 2004, p. 198)

6.3 Example: Constructing Ideal Types of Epistemic Processes

6.3.1 Step 1: Re-constructing the Cases Illustrated by an Epistemic Process as a Case for Ideal Type Construction

In a grade 6 class the teacher poses a prison story: "In a prison there are as many officers as there are cells. All the prison cells and the prison officers are numbered; every prison officer passes all the cells and makes a cross on the door if the cell number is divisible by his own number. The prisoners with cell doors that have exactly three crosses are set free. Which numbers are meant?" (cf. Bikner-Ahsbahs 2006, p. 164). The teacher has written a table on the blackboard (see Table 6.1) in which the class had explored the task situation making crosses as the officers do. The students have written this table down in their paper notebooks.

The solving of this task was taken as the initial interest-dense situation to be analysed in order to reconstruct its epistemic process. This initial analysis served as a starting point for structuring the entire analysis process for all the cases. This first analysis provided an epistemic action model that consists of three epistemic actions and was to serve as the main tool for analyzing the subsequent situations. The tool was checked, revised, and further developed by additional analyses. In this chapter, I cannot show all the aspects that appeared in this process: we will focus solely on the process of ideal type construction. Since this initial analysis determined a great deal of the subsequent analysis process, it will be used to illustrate how the re-construction of the cases was worked out.

By means of analyzing the following first three utterances we clarify how Peirce's sign concept is used for analyzing data (see Fig. 6.3). Then, two subsequent scenes are presented to illustrate the reconstruction of the epistemic process and its main epistemic actions: gathering and connecting mathematical meanings and structure-seeing. Using symbols (Table 6.3), the epistemic process is condensed in a diagram (Fig. 6.5), and reconstructing this case is completed by building its phase diagram represented by pictographs (Fig. 6.6).

6.3.1.1 Approaching the Empirical Case with Peirce's Sign Concept

[See transcription key in the appendix.]

1. T: well now let's look, at o-u-r- table', which n-u-mbers have THREE crosses (..)
2. Sven: shall we write' them down' the numbers'
3. T: yeah we want to NAME them first' which [numbers] we'll FIND those we want to SEEK ' em together now here (.) Marcus. (cf. Bikner-Ahsbahs 2005, p. 205; 2008 p. 111)

Table 6.1 Cell numbers, the officers' numbers, and their crosses (Bikner-Ahsbahs 2006, p. 163)

	Cell no.										
Officer no.	1	2	3	4	5	6	7	8	9	10	Up to 20
1	x	x	x	x	x	x	x	x	x	x	
2		x		x		x		x		x	
3			x			x			x		
4				x				x			
5					x					x	
6						x					
7							x				
8								x			
9									x		
10										x	

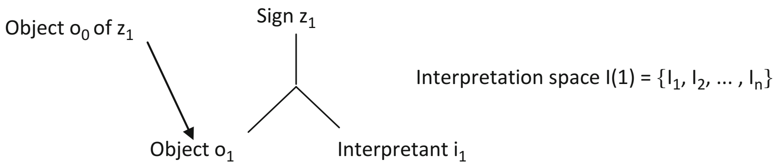

Fig. 6.2 Triadic division of one interaction (cf. Bikner-Ahsbahs 2005, p. 198; 2008, p. 111)

The first sign z_1 = "well now let's look, at o-u-r- table', which n-u-mbers [cell numbers] have THREE crosses (..)" is the statement made by the teacher in line 1. He wants the students to "look at our table" and find the (cell) numbers with three crosses. The object o_0 of z_1 is the set of all these numbers in the Table 6.1 (Fig. 6.2).

The interpretant of the first sign z_1 is Sven's response: i_1 = "shall we write' them down' the numbers'". The interpretation space I(1) of i_1 contains all possible interpretations I_k (k = 1,…,n). Sven might have understood that the teacher wants him to write down the numbers. On the other hand, Sven probably does not know what to do since he finishes his answer with raising his voice. He might be uncertain and wants to know exactly what the teacher expects. Such an open kind of question is probably unusual and therefore confuses him; he seems disorientated. The immediate mathematical object o_1 concerning the I_k, is "dealing with the numbers with three crosses". i_1 now is taken as the next sign z_2 and o_1 is the guideline object. The next interpretant i_2 is the teacher's response. Its interpretation space I(2) might include that the student needs clarification. His affirmation: "yeah" indicates some understanding, and then a definition is given in respect of what dealing with the numbers with three crosses mean: naming them, finding them, and more immediately "we wish to SEEK 'em together". Dealing with the numbers (o_1) is now clarified as "o_2: seeking, finding and naming the numbers orally altogether". In the following scene students begin to offer examples.

6.3.1.2 The Epistemic Actions of *Gathering* and *Connecting* Mathematical Meanings

Scene: "the FOUR". (cf. Bikner-Ahsbahs 2005, p. 207)

4 Marcus: four
5 T: the FOUR' (.) mhm
6 /S: yes
7 T: (writes 4 on the board) Lisa
8 Lisa: six
9 T: six. (begins to write)
10 /S + S: n-o-o
11 /S + S: n-o-o
12 /S + S: she s got FOUR already
13 /T: she s got four I hear'
14 S: yes

In the project, this scene has been analysed by means of Peirce's sign concept. Figure 6.3 shows how broad interpretation spaces can be at the beginning, and how they converge to one interpretation. In line 4, Marcus is offering the vague proposal that 4 is a possible object (as a number with three crosses) (4). This offer is questioned (5) and therefore a possible example, it then is confirmed (6) and finally accepted (7) as an example. Then Lisa proposes 6 as another example (8), but 6 is rejected (10, 11) and therefore it becomes a possible counterexample which is substantiated and finally (12–14) confirmed. In similar ways, the numbers 1 and 9 are *gathered* as examples and other numbers are *gathered* as counterexamples.

The gathered numbers are now written on the blackboard. The teacher also asks the students to enter the officers' numbers in the new table (see Table 6.2), i.e. he asks them to *connect* both kinds of numbers. This is easy for those numbers that were already in the Table 6.1, but more difficult for those which are missing. This begins with number 49, which does not appear in the given table (Table 6.1). The students cannot just read the officers' numbers: they have to connect the features of the number 49 to those they have gathered before. Therefore, 49 is intensively investigated and finally approved as a number belonging to those with three crosses, because the officers' numbers are 49, 7 and 1. Some of the students have meanwhile enlarged the original table. What they found was now completed in the table at the blackboard (Table 6.2).

What we find here are two different kinds of epistemic actions which constitute progressing in the epistemic process: *gathering* and *connecting* mathematical meanings. Through gathering, the students provide the material which then they work with, in that they try to find connections among them; for example, how the officers' numbers are connected with the cell numbers. In the next section the epistemic action of structure-seeing and the way it appears will be illustrated by another scene in the solving process.

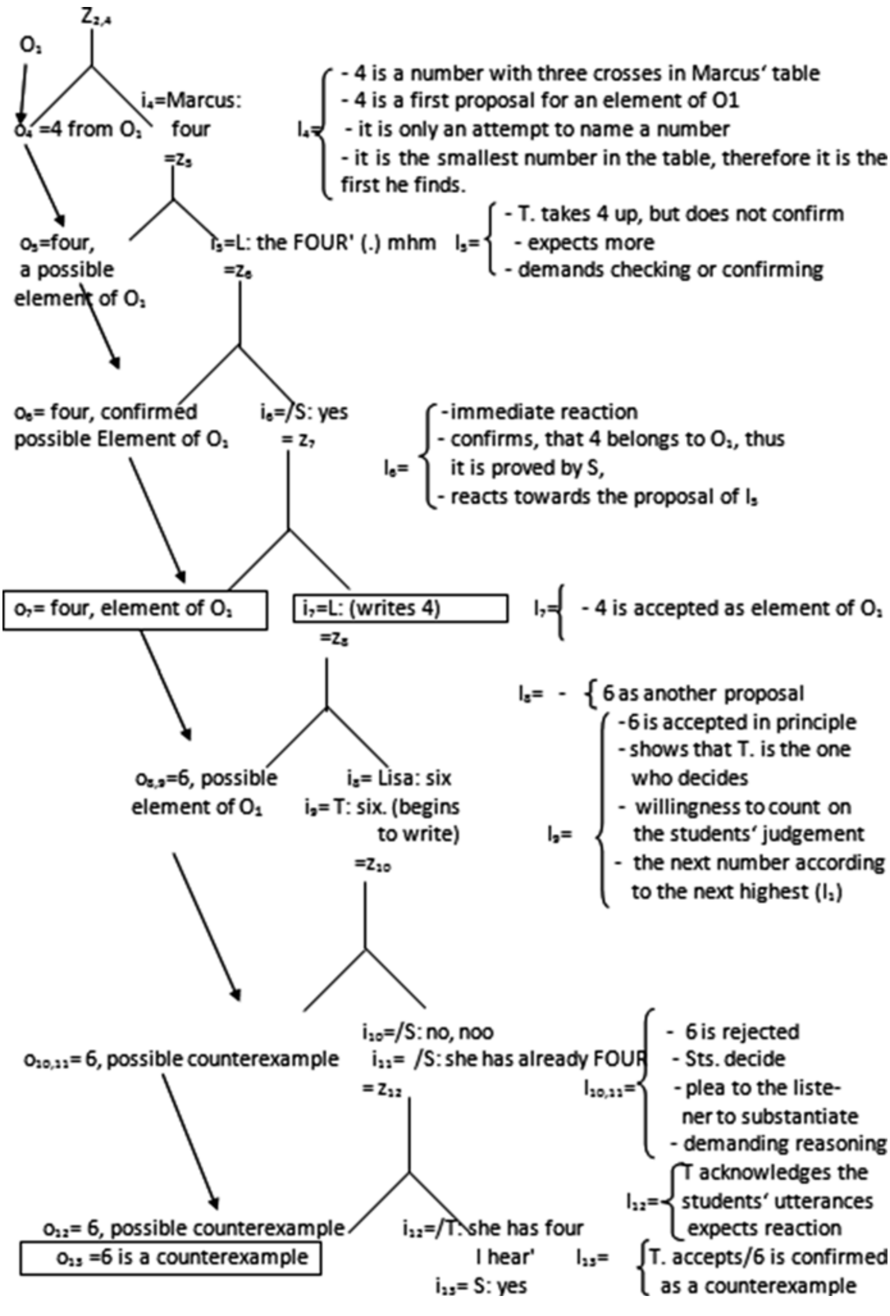

Fig. 6.3 Analysis diagram of the scene "The FOUR" (cf. Bikner-Ahsbahs 2005, p. 208; 2008, 114)

Table 6.2 The second table on the blackboard (cf. Bikner-Ahsbahs 2005, p. 214; 2008, p. 121)

Cells with three crosses of officer number	4	9	25	49
	1	1	1	1
	2	3	5	7
	4	9	25	49

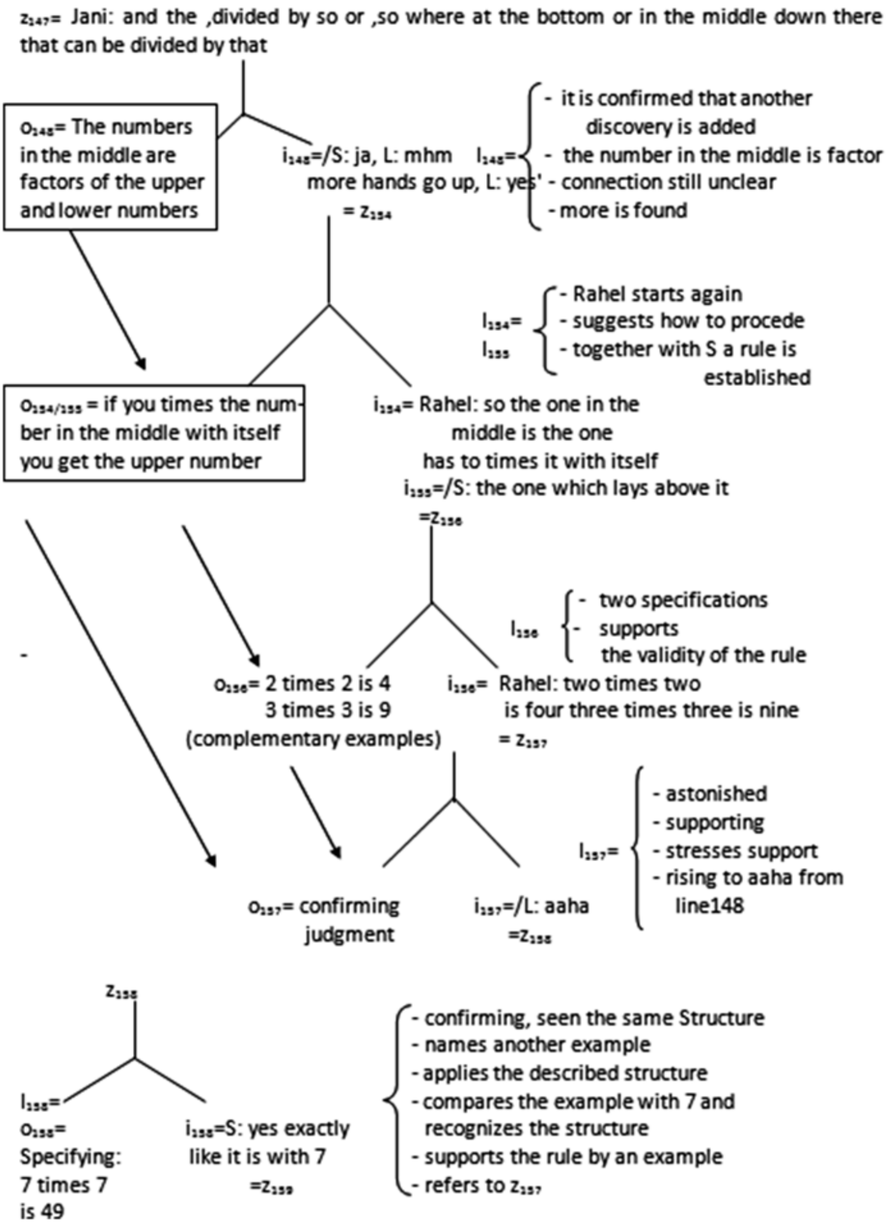

Fig. 6.4 Analysis diagram of the scene "the middle times itself" (cf. Bikner-Ahsbahs 2005, p. 215; 2008, p. 118f.)

6.3.1.3 The Epistemic Action of *Structure-Seeing*

Working in groups, the students now address the teacher's question as to what can be discovered in the given new table (Table 6.2). Eagerly, the students begin to seek connections. "there is always the number one", "the number at the top and below are the same" are two discoveries which the teacher immediately confirms. In line 147 of the next scene, the situation changes because the teacher now stops checking the results and reduces his participation. This leaves room to the students to take over responsibility for the ensuing discourse (see Fig. 6.4).

Scene: "the middle times itself" (cf. Bikner-Ahsbahs 2005, p. 214)

147 Jani: and the, divided by so or, so where at the bottom or in the middle down there that can be divided by that
148 /S: yes
149 L: mhm
150 S: yes
151 (further hands shoot up)
152 (fingers snapping)
153 L: yeah'
154 Rahel: so the one in the middle is the one has to times it with itself
155 /S: the one above
156 /Rahel: two times two is four three times three is nine
157 /L: aaha.
158 S: yes exactly like it is with seven.
159 T: if you times the one with itself then you get the number at the top and the one at the bottom, mhm.

From line 130 on, Anne and Josa first observe a regularity: there are the same numbers at the top and at the bottom. This regularity is just a connection. Later, *structure-seeing* occurs on two occasions. Jani (147) realizes that the numbers at the top and the bottom are divisible by the ones in the middle. This immediately is confirmed by her classmates and the teacher (148–150). Then Rahel initiates a change of view and together with another student creates the rule: if one times the middle number with itself then one gets the number at the top (cf. 154). Jani's broken language in line 147 indicates that the law is still emerging, whereas Rahel creates the rule fluently. Immediately, the students check and specify Rahel's rule (155, 156 and 157). This specification begins with the smallest number 2 and is then done with two subsequently bigger numbers 3 and 7; hence, this process could in principle be continued. Then the teacher takes up the rule and expresses it in more general terms (159). He finishes with agreement.

During the initial analysis, the three main epistemic actions *collectively gathering meaning*, *connecting meaning* and *structure-seeing* were carried out. Gathering meanings refers to single features such as examples, or counterexamples. Connecting mathematical meanings refers to a small number of examples being related to each other. The structure, that now appears is a (new) entity made up of relationships

Table 6.3 Codes for epistemic actions (cf. Bikner-Ahsbahs 2006, p. 165)

Gathering meaning:	•
Connecting meaning:	
Structure seeing:	
Structure seeing and making them more concrete:	
Structure seeing and reasoning:	
Teacher actions: 	initiation
Student actions: 	gathering examples, counterexamples

within a set of mathematical knowledge elements. Signs, indicating structure-seeing, refer to a structure or a set of examples (even potentially infinite), which all share the same structure (Rahel's rule).

6.3.1.4 Representing the Course of the Epistemic Process

These epistemic actions are now represented by codes (Table 6.3), offering a brief overview of transcripts comprising more than a thousand lines by condensed process diagrams. Figure 6.5 represents the course of the epistemic process of the episode described above as example.

In the scene above, the teacher initiates gathering examples and counterexamples for numbers with three crosses as a collective action in a process of social interaction. The examples are 4, 9 und 25, and as counterexamples the students found 6 and 10, but also 16 and others.

In the condensed process diagram (Fig. 6.5) we can observe three different phases, all of which are initiated by the teacher. The first phase (phase I) consists mainly of gathering actions: examples and counterexamples are gathered to which the teacher reacts by understanding what the students relate. The second phase (phase II) follows after a writing process. It mainly consists of connecting actions. The third one (phase IIIa) mainly comprises seeing, concretizing and specifying structures. It is interrupted by a short situation in which the teacher creates a factor diagram of the numbers in question and tries to push the epistemic process. This factor diagram forms the basis for continuing the discourse about numbers with three crosses in phase IIIb, and results in the insight that these numbers all are squares of prime numbers. This insight is a result of making conjectures about the features of the numbers which were tested by means of finding examples and counterexamples, and could ultimately be explained.

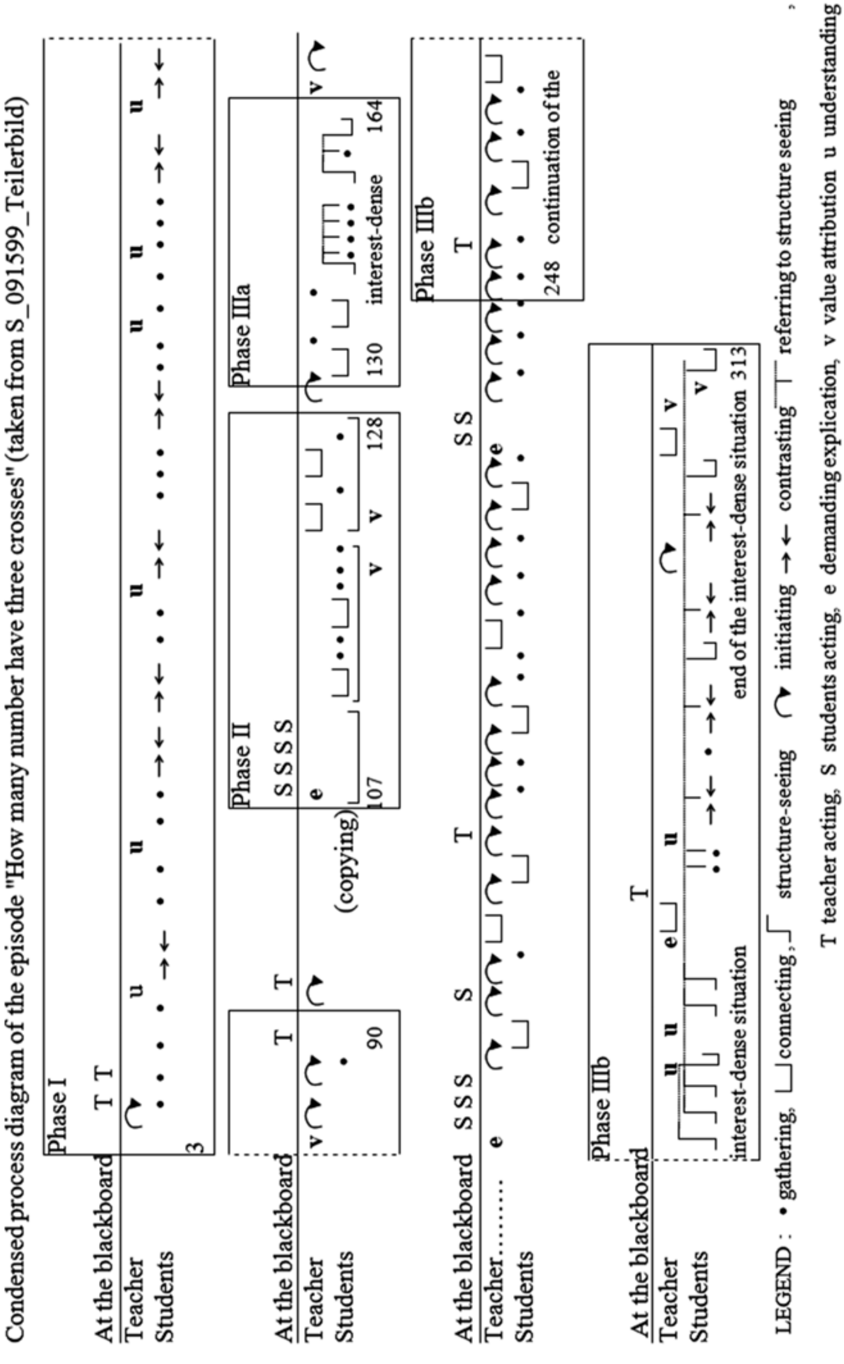

Fig. 6.5 Condensed process diagram (cf. Bikner-Ahsbahs 2005, p. 201; 2008, p. 127)

6.3.1.5 A Pictograph Representing the Phase Structure

The first step of data aggregation led to a condensed process diagram (Fig. 6.5); the second led to a phase structure represented by a pictograph (Fig. 6.6). In every phase the activity is mainly built by the actions represented in the pictograph in Fig. 6.6. In phase I, the students gather examples and counter examples of numbers with exactly three crosses. In phase II, the examples are connected with each other and with other pieces of knowledge. During phase IIIa, the first pattern rules are found. This phase is interrupted by a phase in which the teacher initiates small pieces of knowledge. In the final phase IIIb, further patterns and laws are sought, identified and named, but also validated and substantiated. The learning episode results in finding that the numbers with three crosses are exactly the squares of prime numbers.

This phase structure describes an epistemic process sequenced, or graded, by the actions *gathering meaning*, *connecting meaning*, and *structure-seeing*, and every phase is initiated—in this case by the teacher. Gathering actions provides the material for further work, thus preparing for connecting actions. Connecting actions establish relationships, which in turn provide the material for further work in that these relationships undergo a re-structuring. In this way, structure-seeing is prepared. The structures are then checked and validated.

6.3.2 *Step 2: Grouping the Cases*

Every interest-dense situation is investigated in the way described in step 1, leading to different pictographs. These pictographs aggregate all the information from the analyses carried out beforehand. In this step, they are compared and contrasted, and finally classified into three groups of different process structures. Group 1 (Fig. 6.7) is distinguished from the others in the way epistemic actions lead to structure-seeing. This appears in a *graded* way, with phases that mainly consist of just one kind of epistemic action, and each phase is initiated in one way or another. The

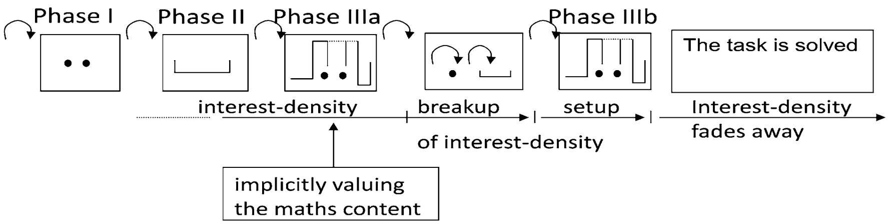

Fig. 6.6 Pictograph of the phase structure (cf. Bikner-Ahsbahs 2005, p. 222; 2008, p. 129)

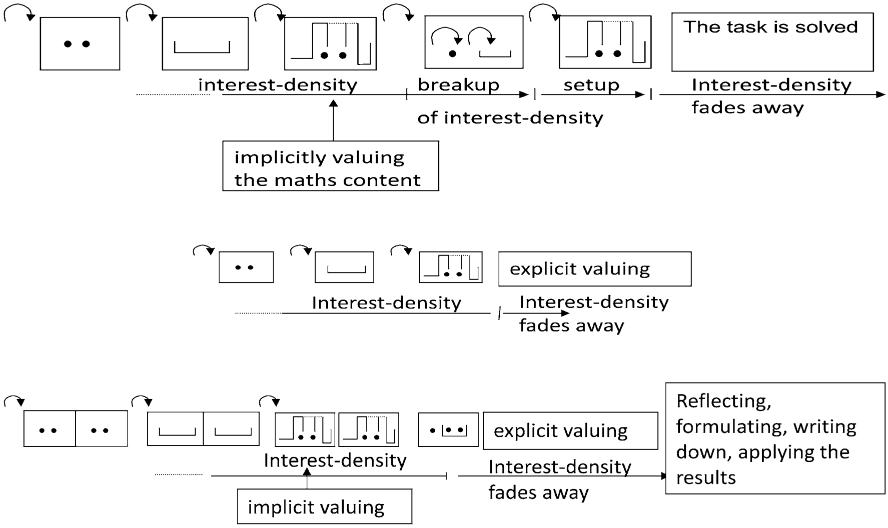

Fig. 6.7 Pictographs of the phase structures of group 1 (cf. Bikner-Ahsbahs 2005, p. 222; 2008, p. 132f)

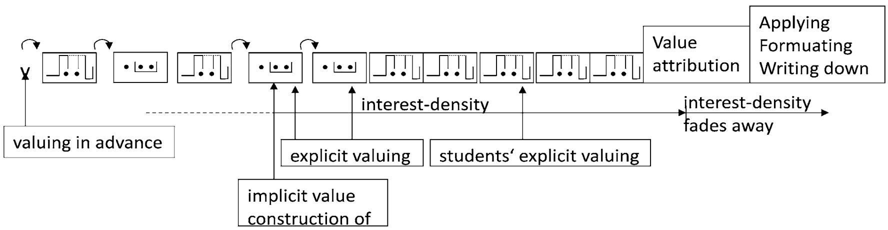

Fig. 6.8 One example pictograph of the phase structures of group 2 (cf. Bikner-Ahsbahs 2005, p. 247–248; Bikner-Ahsbahs 2008, p. 133)

second pictograph of Fig. 6.7 is a prototype of this group, although there may be a variety of different forms. A common feature of group 2 (Fig. 6.8) is that gathering and connecting are intertwined, and ideally structure-seeing is not initiated but seems to be a consequence of the students' deep involvement in the process in which they gather mathematical meanings and spontaneously connect them up, i.e. they approach structure-seeing in a *helical* way. Group 3 (Fig. 6.9) is different from the other two in that there is a separated preparation of groups or individuals working on the task before the different perspectives are merged. During this merging process, the epistemic actions do not have a clear structure: they may lead to structure-seeing or even start with structure-seeing.

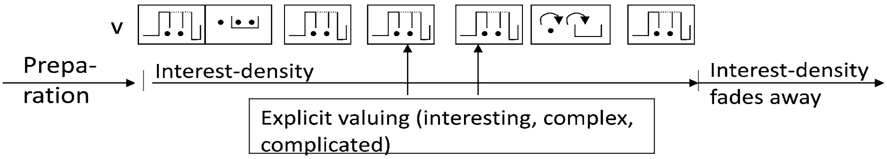

Fig. 6.9 One example pictograph of the phase structures of group 3 (Bikner-Ahsbahs 2005, p. 254; 2008, p. 135)

6.3.3 Step 3: Building Ideal Types

We now reconsider the groups and the underlying results of the reconstruction of the cases, and we idealize their features solely in respect of the specific dynamics of the epistemic process. This way we build ideal types as situational types with a precise rationality of the dynamics of the epistemic process in mathematics classes (cf. Bikner-Ahsbahs 2005, p. 195ff.). Key tools are the three epistemic actions which were reconstructed during the process of analysis of the cases.

The grade-structured type (idealized from group 1): Whenever an epistemic process starts, it begins by the teacher initiating a process of gathering mathematical meanings. The students then join in all together until no more mathematical meanings that are relevant for solving the task can be found and the students get stuck. Then the teacher initiates the connection of mathematical meanings in the way they are needed to solve the task or answer the question. This act of initiating offered by the teacher prepares the basis for structure-seeing. However, the phase of structure-seeing is likewise initiated by the teacher; for example, by asking a typical question such as "what pattern can you find?", and can start via connecting actions. The dynamic of this epistemic process is graded and stabilized by a clear expectation about the kind of epistemic action within every phase, and by the teacher, who organizes the phases by suitable acts of initiating. In every phase, the way the students participate is determined, but within the respective phase the students are free to follow their line of thought in the respective epistemic action. In a graded type, teacher's and students' actions are clearly determined and directed at answering the question or solving the task. Structure-seeing is prepared stepwise. First, single knowledge elements are gathered and then connected. This results in the experiencing of relationships, which in the third phase are looked at with the aim of recognizing patterns. (cf. Bikner-Ahsbahs 2005, p. 223f.)

The helical-structured type (idealized from group 2): The dynamics of the epistemic process in this type are different from the graded one. The starting point is also a question or a task in which the students become involved. They begin to gather mathematical meanings, which they immediately intertwine with connecting actions. Gathering-connecting actions assemble and activate more and more mathematical meanings. Initiation is not needed since activities of this kind may be repeated and, in the course of being repeated, are further developed until the students are able to see structures that are validated, and proven. These structures may provide new material, which in turn is investigated again by gathering-connecting actions leading to seeing further structures. This recursive process of gathering-connecting action stabilizes the dynamics of the epistemic process and provides new material, which again may form the basis for a subsequent phase of structure-seeing. Thus, a *helical-structured* type of epistemic processes is shaped. From the beginning, relationships are built and intensively broadened and re-structured until entities of relations appear as structures to the students. Since all the students have access to the same material they all are able to think along with this process and re-construct the structures their classmates have pointed out. (cf. Bikner-Ahsbahs 2005, p. 249f.)

The merging-structured type (idealized from group 3): The students start by working alone or in small groups. This is organized by the teacher either as homework or during the lesson in school. Afterwards, the insights developed by the students are presented and allowed to flow together. The students are expected to present their lines of thought, question the other presentations, and discuss the results altogether. As in the other types, the epistemic process finishes with a phase of seeing, validating and proving structures. When the students come together they exchange the knowledge they gained, which may start by presenting observed structures or lead to structure-seeing in the course of presentation and reflections. (cf. Bikner-Ahsbahs 2005, p. 256f.)

In the ideal types, all the investigated situations pass through gathering and connecting actions and lead to phases of structure-seeing. This seems to be made possible by allowing the students to gather and connect mathematical meanings in their own time and for as long as they need. It takes only one student to recognize a structure for the other classmates to be able to catch on and immediately reconstruct the structure. Hence, structure-seeing is a result of a social process of constructing knowledge. The basic assumption underlying the three ideal types is that students are engaged in the specific dynamics of the epistemic process and behave according to its affordances. Structure-seeing, testing and validating the structure do not occur before gathering and connecting actions are somehow theoretically saturated in respect of the given task or question.

6.3.4 Step 4: Building Theoretical Knowledge

Following Weber (1922, p. 190; 1949, p. 90), the ideal types are used as theoretical constructs in step four. They reflect real learning episodes, but which themselves cannot be regarded as real because they are considered to appear according to a specific pure rationality (see Chap. 5). However, they may guide the understanding of a specific situation. For example the real episode "numbers with three crosses" can be compared and contrasted with the graded type as it refers to the same group. Its three phases are orchestrated by tables. The first table reflects the story, and is explored by gathering actions. Building a new table that consists only of the numbers with three crosses and their factors is a new step that calls for connecting actions. This is initiated by the teacher. Finally, one has to take a step back and look at the new table to find patterns. The teacher's initiation in this point helps the students to adapt their views and understand the structure of the table. However in phase IIIa of the investigated real situation, the students still did not realize that the three crosses appear only with squares of prime numbers. Therefore, the third phase in the epistemic process was interrupted because the teacher directed the students towards the solution by the use of the factor diagram which they had learnt before. Thus, the students produced what the teacher had in mind. However, when the teacher asked the students to check whether the squares of further (prime) numbers were exactly the numbers with three crosses, the students were lost. When the teacher stopped directing the students to proceed the way he wanted them to, but left room for exploration, the students were able to see the results as a new structure in phase IIIb. Only then were they able to check it with further numbers and then prove the rule by the use of a factor diagram. This difference to the graded-structured type supports the theory building process because it gives evidence why the rationality of the graded type was not followed and the final result was not understood: just forcing the students to produce the result the teacher wanted did not lead to any in-depth insights, because gathering and connecting meanings were not yet saturated for structure-seeing.

We now follow another direction in building theory and consider all three ideal types looking for common categories or underlying dimensions (cf. Kluge 1999, p. 93ff.). In the *grade-structured* type, the interaction is organised by the teacher initiating the phases. The students act according to the teacher's directions, but within the phases they are (*locally*) free to follow their own line of thought. The *helical-structured* type is not organised by the teacher, but (*globally*) self-regulated by the students throughout the whole process. In the *merging-structured* type, the organisation and the kind of interaction is even planned beforehand by the teacher. Only in the preparation phase are the students free to follow their own line of thought. In the merging phase, the students are supposed to present their thoughts, ask questions, and discuss their results. In the graded type, the teacher's planning is reduced to preparing the material to be worked with and becoming clear about his three-step initiations. However, exactly at which point in time teacher initiations are needed is not clear beforehand: it depends on the epistemic process, and it has to be decided locally. Therefore, we can distinguish three different forms of organization which seem to be deeply linked to dynamic forms of fruitful epistemic processes. They are shaped by two dimensions: on the one hand, students' self-regulated epistemic actions, that can be *local* (graded type), *global* (helical type) or are *merged*—appearing globally in the preparation phase and locally during the presentation (merging type) and, on the other hand, the teacher's inducing of (specific) epistemic actions, that can be *left aside* (helical type), that can be *organised locally* (graded type), or *globally* (merging type).

6.4 What Can Be Learnt from This Example?

First of all, ideal types are limit constructs, i.e. they are theoretical concepts offering a clear and precise rationality that does not exist in reality. However, ideal types reflect specific aspects of the empirical world which serve as a means for investigating data and result in building theoretical elements. This may happen on two levels:

either ideal types are used to more deeply understand the single cases, or all the ideal types are used to explore whether there is a feature space in which they can be embedded. As described in the previous chapter, this feature space is relevant because it may disclose ideal types that have not yet been reconstructed or dimension to describe epistemic processes. Hence, it may offer predictions or structure a specific area of investigation.

The example described in this paper also shows how the ideal types are used to deepen understanding of the empirical cases. For instance, the epistemic processes of the prototypes of the graded type are all shaped by a specific way of dealing with a task. Gathering is enacted by exploring a situation given by a task like collecting numbers with three crosses from the table. Connecting actions are enacted by building a sub-situation from the previous one, like building a new table that only consists of the numbers with three crosses and their factors. Both situations are still deeply connected to the original story of the task. In the second phase, decontextualization may begin naturally, but when the teacher asks students to look for patterns it is initiated explicitly. Thus, the three-step initiation of the graded type may offer a teaching heuristic of how to shape a fruitful epistemic process towards structure-seeing.

Appendix: Transcription Key

S(s), T	Student(s), teacher
EXACT	Emphasized or with a loud voice
e-x-a-c-t	Prolonged
exact.	Dropping the voice
exact´	Raising the voice
,exact	With a new onset
(.),(..)…	1, 2 … sec pause
(....)	More than 3 sec pause
(*gets up*)	Nonverbal activity
/S	Interrupts the previous speaker

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Part IV
Semiotic Research

Chapter 7
The Question of Method in a Vygotskian Semiotic Approach

Luis Radford and Cristina Sabena

The critical issue, then, is method

Vygotsky 1987, p. 45

Abstract In this chapter we present the main ideas of an educational Vygotskian semiotic approach, emphasizing in particular some crucial questions about its methods of inquiry. We resort, on the one hand, to Leont'ev's (1978) work on activity, and, on the other hand, to Vygotsky's cultural psychology. Considering a theory as an interrelated triplet of "components" (P, M, Q), where P stands for principles, M stands for methodology, and Q for research questions, in the first part of the chapter we present a brief sketch of the Vygotskian semiotic approach through the lenses of the aforementioned components. We refer in particular to two methodological constructs that have been built to account for multimodal sensuous actions: the semiotic node and the semiotic bundle. To illustrate the semiotic approach, in the second part of the chapter we discuss an example from a classroom activity concerning pattern generalization. This example constituted an important step in developing the semiotic approach under consideration. The example is about the role of words, gestures, and rhythm in the students' process of objectifying (i.e., noticing or becoming aware of) mathematical relationships. We discuss how a "crude fact" that was not anticipated led to a transformation of the theory, and in particular its methods and research questions.

Keywords Vygotskian semiotic approach • Semiotic bundle • Semiotic node

L. Radford (✉)
École des sciences de l'éducation, Université Laurentienne, Sudbury, ON, Canada

Faculty of Humanities, School of Education, Manchester, UK
e-mail: lradford@laurentian.ca

C. Sabena
Dipartimento di Filosofia e Scienze dell'Educazione, Università di Torino,
Via Gaudenzio Ferrari, 9/11, 10124 Torino, Italy
e-mail: cristina.sabena@unito.it

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7.1 Introduction

Mathematics education researchers resort to procedures to describe the phenomena they investigate and also to make claims about their objects of study. Naturally, since the procedures allow researchers to argue and reach conclusions, their "appropriateness" is of paramount importance: to a large extent, the cogency of an investigation depends on the persuasiveness of the procedures.

One of the most vigorous debates in the history of mathematics education has consequently been the one revolving around the *nature* of these procedures. Should we understand these procedures in the sense of the natural sciences? Or should we rather understand them in the sense of the social sciences? The choice is not simple. It entails adopting a view of the nature of the *phenomena* dealt with. In other words, the question about the nature of the procedures goes beyond the procedures themselves.

In the first case, procedures are generally understood as based on, or following, *models* of scientific practice. The testing of the models—e.g., models of didactical situations—and replicability of results become central questions. Naturally, within this approach, it is assumed that educational phenomena are amenable to be modeled. That is, there are some regularities that remain constant in the observed phenomena; furthermore these regularities can be grasped (even if only approximately) if the proper tools are employed. In this case, one of the tasks of mathematics education is to grasp such regularities (Brousseau 2005).

In the second case, procedures are not understood as models. This is the view that the social sciences—at least in some of their recent trends (e.g., Atkinson and Hammersley 1994; Shweder and LeVine 1984)—tend to adopt. The nature of the social phenomena is considered to be non-amenable to be modeled or factored out in terms of controllable variables. Deeply sensitive to their context (social, cultural, historical, etc.), social phenomena (which includes educational phenomena, e.g., teaching and learning) are assumed to be messy by nature. You may try to remove the redundant, the apparently unnecessary, the fuzzy, and what will remain will still be redundant and fuzzy, not because you did not do your job well, but because it is the phenomenon's real nature.

By virtue of their radical differences, both research paradigms convey different ideas of the searchable and have recourse to different procedures or methods. The semiotic approach that we discuss in this chapter belongs to the social science paradigm. As such, it conceives of the educational phenomena as messy and context sensitive. Its claims are not backed up by some immutable laws whose existence is asserted by a confrontation of the laws and empirical facts. Rather, general assertions are sustained by actual references that may guide further action through a reflective stance.

The focus of the semiotic approach that we discuss in this paper is on the phenomenon of teaching and learning—a phenomenon embedded in the idea of classroom activity.

In the sense that the semiotic approach does not aim to uncover hidden laws behind teachers' and students' actions, the approach could be said to be interpretative. But it is more than that. We do not register the educational phenomenon in order to offer plausible interpretations of it. Although we provide interpretations, we also design the classroom activities, and by designing them, we alter and transform the manners in which teaching and learning co-occur.

In this chapter, we present the main ideas of our semiotic approach, emphasizing in particular some crucial questions about our methods of inquiry. In Sect. 7.2 we discuss the concept of method as a central problem of scientific inquiry. We draw on Vygotsky's idea of method—an idea that tormented him throughout his short life and to which he continuously returned as he moved progressively away from the influence of reflexology and instrumentation to a more encompassing view of humans and the human mind. In Sects. 7.3 and 7.4 we present the theoretical underpinnings of our semiotic methodology, which we illustrate in Sect. 7.5 through a classroom example.

7.2 Method as the Central Problem of Scientific Inquiry

In trying to provide scientific accounts of human phenomena, a method of enquiry has to be devised. "Finding a method," Vygotsky (1993) says, "is one of the most important tasks of the researcher" (p. 27). Now, it would be a mistake to think that methods precede the inquiry or research that they are supposed to support. In general, a method "is simultaneously a prerequisite and product, a tool and a result of the research" (Vygotsky 1993, p. 27).

This concept of method, as simple as it may appear, only makes sense within a theoretical general view of *what* is studied and *how* it can be studied. For Vygotsky, who followed Marx's Hegelian view of reality, both the object of study (reality) and the manner in which it can be studied are always in motion. They come to form a dialectical unity where the components affect each other in a dynamic way. It is hence unimaginable that a method could precede in its entirety the investigation, which is in itself an activity in continuous movement.

But a method is more than something that comes into existence in the course of research. Method, as Vygotsky understood it, is not the mere systematic application of a set of principles. Nor is it simply a way of doing something—a technique. Method comes from the Greek *methodos*, a word made up of *meta*—"after"—and *hodos*—"a traveling"—meaning hence "a following after" (Online etymology dictionary 2013). A method's main characteristic is to be inquisitional and reflective, that is, a philosophical practice. It is in this non-instrumentalist Vygotskian sense that we understand method here and that method can be said to be at the heart of a theory.

Let us notice that the sought object in the "following after" of a method is not merely something that is there, waiting to be discovered. By asking questions—research questions—theories fabricate those objects. They also fabricate the evidence that shows the objects in accordance with the procedures that theories follow in their persuasive endeavour.

This does not mean, however, that theories fabricate their objects and methods as they wish. This would amount to a blunt and self-defeating relativism. What it does mean is that methods are rooted in theoretical principles that convey worldviews.

Let us give a short example. In his *Genetic Epistemology*, Piaget (1970) resorts to methods that, at first sight, seem extremely simple and even merely instrumental: a few objects and a child who is required to answer some questions or to solve some problems in laboratory interviews. The design of the task and the setting into motion of the child's activity (along with the supposedly neutral role of the observing researcher) unavoidably embodies a worldview of human intelligence and its main traits—e.g., that intelligence and its development can be accounted for in terms of problem solving procedures and their underpinning formal logical meanings. Piaget's methods crystalize aspects of a general Western worldview: one in which, since Kant, reason appears as a regulative entity of human experience supplemented with the nineteenth century understanding of evolution. As Walkerdine (1997) notes:

> In the work of Piaget, an evolutionary model was used in which scientific and mathematical reasoning were understood as the pinnacle of an evolutionary process of adaptation. The model viewed the physical world as governed by logicomathematical laws, which came to form the basis of children's development of rationality. (p. 59)

Infused with such a worldview, Piaget's investigative procedures turn to find traces of logical thinking behind the child's action and utterances:

> Piaget examines a child's protocol and picks out the significant underlying propositions (which he can then order in the logical parlance of *p*'s and *q*'s); the mental action reflected in the protocol is a series of operations performed on the propositions. The individual has reached formal operations when he can systematically and exhaustively explore the relations between propositions describing a phenomenon. (Gardner 1970, p. 359)

Piaget's tasks are designed in a way to elicit logical propositions and their combination in the child's actions and discourse. It is in this sense that theories fabricate their objects of investigation and the evidence to sustain their claims.

We should not be led to think though that methods remain caught in their own endeavours and are blind to other possibilities. We have insisted on the fact that methods are not merely instrumental procedures to follow. Methods are part of a reflective, philosophical practice. And as such, they are prone, at least in principle, to continuously examine their results and the worldviews that they purport. There is also another source of change and transformation: since methods embody crystallization of cultural worldviews and since worldviews within a given culture are not homogeneous, methods do not go generally undisputed. Thus, anthropologist Lévi-Strauss criticized Piaget for resorting to a rather artificial methodology:

> What I do ask, and I formulate this question rather naively in ethnological terms, is whether Piaget's research techniques aren't rather artificial in character. His experiments are set up in advance, prefabricated, which does not seem to me to be the best way to understand the mind in all its spontaneity. (Grinevald 1983, p. 84)

Let us summarize. Methods are a central element of scientific enquiry. But methods cannot be reduced to a pure instrumental sequence of steps defined in advance and to be followed blindly. Methods convey worldviews. That is, they make assumptions about *what* is to be known and *how* it can be known. And because what distinguishes a scientific inquiry from other inquiries, we suggest, is its systematic and explicit character, the scientific inquiry has to be as precise as possible about the principles it adopts. These principles wrap already the raw material to be studied with categorial substance—that is, with conceptual categories that already infuse the objects of study with scientific value and understanding. “We study [a] given particular gas not as such, but from a special viewpoint” (Vygotsky 1997, p. 318). This is why

> The material of science is not raw, but logically elaborated, natural material which has been selected according to a certain feature. Physical body, movement, matter – these are all abstractions. The fact itself of naming a fact by a word means to frame this fact in a concept, to single out one of its aspects; it is an act toward understanding this fact by including it into a category of phenomena which have been empirically studied before. (Vygotsky 1997, p. 249)

In previous work (Radford 2008a), to try to better understand theories in mathematics education, and to avoid forgetting the philosophical or reflective nature of their methods, we have suggested that it may be worthwhile to think of theories as dynamic entities composed of interrelated “parts.” These parts are: (1) the principles that are assumed by the theory and that define the spectrum of *what* is to be known and *how* it can be known; (2) the methodology or method (that is, the reflective procedures through which the inquisitive endeavour is carried out); and (3) the research questions that the theory strives to answer or investigate. In short, a theory, we suggest, is a triplet T=(P, M, Q), where P stands for principles, M stands for methodology, and Q for research questions. This analytic description of theories does not mean, as the previous discussion suggests, that the different parts of the theory are independent of each other. They are interconnected and evolve in a dynamic way. Thus, a result may require a new or deeper interpretation for which new theoretical principles have to be elaborated, or it may require the development of new methodologies. A new result may also lead one to ask new research questions.

In the following section, we discuss some aspects of our semiotic approach. We start by addressing the links between semiotics and education.

7.3 A Vygotskian Semiotic Approach

The semiotic approach that we outline seeks to answer questions about teaching and learning. At first sight, it may seem curious to resort to semiotics to answer educational questions. Indeed, semiotics, in its different trends and developments, is not a theory of teaching, nor is it a theory of learning. Semiotics was developed in close relation to phenomenological concerns—e.g. Peirce (1958), Husserl (1970), Hegel (2009), and around questions of language—e.g., Saussure (1916). Where is the connection? Semiotics is a theory of how signs signify. It is a theory of signification. It can provide insights into the manner in which educational practices work, for as Walkerdine (1997) noted, “All practices are produced through the exchange of signs and are both material and discursive” (p. 63).

As a cursory glimpse at a classroom would show, there is indeed a tremendous array of signs (some of them written and oral, but also embodied signs such as gestures and body posture) and artifacts in circulation in a teaching and learning activity. And this would be even more evident in a mathematics classroom, where recourse to concrete objects (e.g., plastic geometric shapes, blocks, etc.) is often made.

Since semiotics is not a theory of knowing or a theory of learning, to be successfully used in education, semiotics has to be *integrated* into an educational theory. This integration cannot be a mere juxtaposition of semiotic concepts and educational ones (Radford 2013a). Since theories are based in theoretical principles and specific methodologies, there is a limit to the integration that can be achieved—for a typology, see Prediger et al. (2008). This integration depends strongly on the *compatibility* of the principles of the theories (Radford 2008a).

The problem of integration of theories does not concern mathematics education only. Vygotsky criticized the efforts made by Luria and other Russian scholars who were attempting to combine Freud's work and Marxist psychology and the contradictions that such an endeavour caused. As a result of a direct fusing of these theories, a series of contradictions appeared. Since these contradictions were unavoidable, they were merely excluded, leading to a strange situation that Vygotsky (1997) summarizes as follows:

> Very flagrant, sharp contradictions which strike the eye are removed in a very elementary way: they are simply excluded from the system, are declared to be exaggerations, etc. Thus, Freudian theory is de-sexualized as pansexualism obviously does not square with Marx's philosophy. No problem, we are told – we will accept Freudian theory without the doctrine of sexuality. But this doctrine forms the very nerve, soul, center of the whole system. Can we accept a system without its center? After all, Freudian theory without the doctrine of the sexual nature of the unconscious is like Christianity without Christ or Buddhism with Allah. (p. 261)

The integration of education and semiotics requires us to be careful so that we do not denaturalize the theories we try to connect. In our case, we resort, on the one hand, to Leont'ev's (1978) Hegelian phenomenological account of knowledge and knowing, and on the other hand, to Vygotsky's cultural psychology. The former provides us with a historical conception of signification from which learning can be defined as a social semiotic process that is always in the making, unsettled and unsettleable. The latter provides us with a psychological account of signs. In contradistinction to Saussure's (1916) and Peirce's (1958) semiotics, Vygotsky's semiotics does not resort to a representational idea of signs. His concept of sign is rather located within his work in special education: a sign is an auxiliary means to organize our behavior. Signs are tools of reflection that allows individuals to plan action. Thus the knot in the handkerchief serves the purpose of a recall that moves the individual into action. The Vygotskian concept of sign provides us with clues to understand the actual processes of teaching and learning.

What follows is a succinct account of the main ideas of the resulting semiotic approach to mathematics teaching and learning.

7.3.1 Knowledge

Grosso modo, there are two main philosophical traditions that have inspired theories of knowledge in the Western World. The first one is the rationalist tradition, epitomized by Kant, in which knowledge is considered to be the result of the doings and meditations of a subject whose mind obeys logical drives—either already there ("within our own soul," as Leibniz (1949, p. 15) used to say) or developmentally (as in Piaget's (1970) *Genetic Epistemology*). The second tradition is the dialectical-materialist one developed by Hegel and Marx, where knowledge is not the result of logical drives but the result of individuals' sensuous reflections and material deeds in cultural, historical, and political contexts. In opposition to the rationalist tradition, in the dialectical-materialist view knowledge is not something that we represent. Actually, knowledge cannot be represented, for knowledge is always in motion. Knowledge is *pure possibility*. It is constituted of culturally and historically encoded forms of reflection and action that, instead of lending themselves to representation, are sources for action (Radford 2013b). Numbers, for instance, are not things or essences to be represented. They are possibilities for action (e.g., to count or to carry out complex calculations).

As pure possibility, knowledge cannot be an object of consciousness. To become an object of consciousness and thought, knowledge has to be set into motion. Knowledge has to be filled up with concrete determinations. And this can only happen through activity—sensuous and material activity. This is what students and teachers do when they participate in classroom activity.

Let us refer to a short example to illustrate these ideas. The example is about pattern generalization.

Pattern generalization is a cultural activity at the heart of many ancient civilizations. The Pythagoreans and the Babylonians, for instance, practiced it, where it started as an endeavour motivated to answer concrete counting processes or sense-making investigations. These endeavours became encoded ways of reflecting and acting that were refined in the course of cultural history (Diophantus, Fermat, etc.).

In contemporary curricula, in particular in the English-speaking countries, pattern generalization appears often as a road to algebra. It is within this pedagogical intention that we have resorted to it.

As an object of knowledge, pattern generalization is not something to be represented. It is something to be known. However, from the students' viewpoint, pattern generalization (in fact all mathematical content to be known) appears, first, as pure possibility (a possibility to do something, to solve some problems or to argue about something). And in order for it to be known, it has to be set into motion. Knowledge has to evolve and to *appear* in concrete practice. By being filled up with some conceptual content, what appears is not knowledge in its entirety, but a concrete instance of it. Hegel (2009) called it the *singular*. We have, then: (1) the *general*, which is knowledge as such (in this case pattern generalization), (2) the *activity* through which knowledge is brought forward or actualized, and (3) knowledge in motion, filled up with conceptual content, that is, the *singular*. Figure 7.1 provides a diagram of these three elements.

Fig. 7.1 The singular as knowledge actualized in activity

What Fig. 7.1 expresses is the mediated nature of knowledge. We do not have access to knowledge but through mediation. As pure possibility, knowledge cannot be fully accounted for by any one of its instances (the singulars). Not even the most perfect triangle reveals the depth of the concept of triangle, not because we will always make unnoticeable mistakes in drawing a triangle or because there would be triangles with other shapes different from the one we drew. The reason is this: The concept of triangle cannot be revealed in its representation, because the concept is not representable. The concept is knowledge, that is possibility, and as such cannot be represented; it can only be actualized in the activity that fills it up with particular conceptual content.

The singular as actualization of knowledge in activity should not be seen as something static or as an end point, but as an *event*. It is rather an "unfinished and inherently open-ended event" (Roth 2013). It is a process—a *semiotic process* through and through. Not only because in the activity that actualizes knowledge and transforms it into an event students and teachers resort to discursive, embodied, and material signs and artifacts, but, overall (and indeed this is the real reason), because in mobilizing signs students and teachers engage in processes of signification. The singular is a semiotic event.

From a semiotic viewpoint, there is something extremely important to understand about the activity that actualizes knowledge. This activity is, essentially, an activity of signification. In fact, the activity through which knowledge is actualized is an activity of conflicting significations. The teacher is aware of the aim of the activity. In our example, the aim (or in Leont'ev's terminology, the *object* of the activity) is to make the students aware of the historically and culturally constituted way of thinking and reflecting about pattern generalization. Before engaging in the activity, the students do not know about such a way of reflecting and thinking—at least not in all the scientific-cultural curricular details. If the students knew, there would not be learning on the horizon. The activity would be an exercise activity—i.e., practicing something already known. The epistemological asymmetry that underpins teaching and learning activity (Roth and Radford 2011) infuses the activity with its inherent contradictions. The idea of contradiction has to be understood here in its dialectical sense, namely as precisely what drives the activity further.

7.3.2 Learning

Now, the fact that the students do not know yet the aim of the activity (e.g., how to generalize a pattern algebraically) does not mean that they cannot engage in the activity. In fact, they resort to what they already know. This is why it is not surprising that, when students engage in algebraic pattern activity, they resort to arithmetic generalizations.

The conflicting significations that are at the heart of the activity can be formulated in the following terms. The aim of the activity (knowing how to generalize patterns algebraically) is dynamically and variously refracted in the students' and teachers' consciousness as the activity unfolds. The conflicting significations move (in a dialectical sense), creating tensions that, at moments, may be partially resolved or intensified. Attenuated or not, these tensions do not disappear. They constitute mobile *wholes* made up of different perspectives and positions that each participant of the activity brings in.

The attuning of inter-subjective perspectives is the requisite for learning to occur. It does not mean that teachers and students have to agree on, say, the manner in which a pattern can be generalized. Attuning refers also to matters of deep disagreement and unresolved tensions.

In previous work we have suggested that learning can be studied through *processes of objectification*, that is "those processes through which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and action" (Radford 2010, p. 3). In light of the previous discussion we want to stress that *acquaintance* does not mean *agreement*. It means *understanding*—a socially responsible and conceptually articulated understanding of something even if we do not agree with it.

7.4 The Methodology of Our Semiotic Approach

We are now in a position to describe the chief elements of our methodology. Because knowledge is pure possibility, for it to become the object of students' consciousness, it has to be set into motion through activity. The first problem is hence the *design* of the activity.

We spend a great deal of time working with teachers designing teaching-learning activities. The curricular goals are taken as the basis of the activities. They are very general—e.g., to think algebraically about pattern generalization, to solve equations algebraically, to think probabilistically, to argue and prove, etc.

These aims are general and need more specification. The specification depends on the curricular requirements. In our research with young and adolescent students about pattern generalization, some of the specifications refer to a focus on functional relationships between variables in figural sequences and the building of formulas for remote terms—using the standard algebraic symbolism or a conjunction of other semiotic systems (Sabena et al. 2005).

The specifications shape the conceptual content of the activity through which knowledge will be instantiated. An a priori epistemological analysis (Artigue 1995) helps us structure the activity: we carefully select the questions and problems and their order in the activity. The first questions are easy, to ensure that students embark in the activity; bit by bit the questions become more and more complex, leading the students to mobilize the mathematical content in depth—for some examples see Radford and Demers (2004), and Radford et al. (2009).

Because leaning is a social phenomenon, classroom interaction is a central element of the activities we design (Radford 2011). Usually, the classroom is divided into small groups. The teacher circulates among the groups and engages in discussions with the students (Radford 2013b). Naturally, it is impossible to predict the manner in which interaction will occur. The activity that mediates and actualizes knowledge is unpredictable. Although planned, this activity is an *event*—something unrepeatable and always new. This is why we see the classroom as a dynamic system going through states out of which the conflicting significations arise.

The role that we ascribe to the teacher is particularly different from the one we find in most other educational approaches. Indeed, for us, the teacher is not a coach or a guide or a helper or an observer—or worse, someone who transmits knowledge. Her main role is *ethical* (Radford and Roth 2011). The teacher is part of the activity that mediates and actualizes knowledge. She is part of the whole ensemble of classroom consciousness trying to get attuned with each other. Much like the students, she brings to this activity her idiosyncratic way of thinking and understanding mathematics. It is out of the personal efforts of all members of the activity that the activity eventfully realizes the general in the singular.

In coming to understand others and the mathematical task at hand, teachers and students engage hence in activity. They do not engage in a purely meditative manner, but in a sensuous and material way. They resort to a wide range of semiotic systems through which they come to form their intentions and ideas against the background of culturally and historically constituted ways of thinking and acting. In the course of the objectification processes, students and teachers produce multimodal actions. Through these actions complex meanings are formed in an intersubjective way.

Since "the method must be adequate to the subject studied" (Vygotsky 1993, p. 27), to investigate these processes of objectification and signification, we use fine-grained video-analysis.

One or more video-cameras are used to register the teacher's and students' small group activities and classroom discussions. Videos are fully transcribed, and complemented with written materials produced during the activity (students' sheets, field notes by the researcher, etc.). From video and the transcript, episodes are selected, which are helpful in answering the specific research questions (Q) of the study. These episodes are carefully analysed over and over in detail, and confronted with the theoretical assumptions (P).

This kind of analysis is consonant with microethnographic methodologies (Streeck and Mehus 2005), since it "encompasses a collection of techniques and analyses tracing the moment-by-moment bodily and situated activity of subjects engaged in certain events and interactions" (Nemirovsky et al. 2012, p. 294), in which a particular attention is given to "talk, gesture, facial expression, body posture, drawing of symbols, manipulation of tools, pointing, pace, and gaze" (ibid.): they constitute semiotic resources through which the students' and teacher's mathematical activity develops.

Our semiotic approach also allows us to theoretically include embodied means of expression, as semiotic resources in learning processes, and to look at their relationship with the traditionally studied semiotic systems (e.g. written mathematical symbolism). In looking at the different semiotic resources in an integrated and systemic way, attention is paid to relationships, dialectics, and dynamics between them. Some of these relationships may concern different kinds of resources in the same time moment: for example, they may concern co-occurrences of words and gestures.

However, in opposition to pure semiotic approaches and microethnographic methodologies, we are not interested in the semiotic resources per se. We are interested in the manner in which teachers and students resort to the semiotic resources in processes of learning, that, as mentioned previously, we theorize as processes of objectification. The methodological problem for us is, hence, to account for the manner in which the whole range of semiotic resources are used by teachers and students in the course of the social processes of objectification through which students become aware of the cultural logic and meanings of thinking and doing mathematically.

In order to provide description and interpretation of learning as a sign-mediated activity, two methodological constructs have been developed: the concept of semiotic node and the concept of semiotic bundle.

A *semiotic node* is a part of the students' and teachers' joint activity where embodied and other signs from various semiotic systems are put to work together in processes of objectification. In other words, a semiotic node refers to segments of activity where students and teachers bring forward possible mathematical interpretations and courses of action against the backdrop of culturally and historically constituted forms of thinking and doing (Radford et al. 2003). The central idea is that mathematics learning is a reflective activity that involves consciousness. And consciousness, from the dialectical materialist viewpoint we adopt here, is intimately related to our use of semiotic systems and artifacts. In the course of the process of objectification—in particular, in those crucial moments in which the students gain an awareness and understanding of cultural mathematical meanings—"signs play different and complementary roles" (Radford 2009, p. 474). Through the concept of semiotic node we explore focal points of the activity that mediates knowledge and where episodes of objectification occur. Semiotic nodes provide us with relevant segments of the semiotic activity where learning is taking place.

In this sense, semiotic nodes are methodological tools to study learning. Through the teacher's and students' use of various semiotic resources, we can have, methodologically speaking, an idea of the students' reflective learning activity and the kind of interpretations and meanings that the students produce.

The evolution of semiotic nodes provides us with a more general view of the manner in which learning is occurring. To investigate the evolution of semiotic nodes, we have introduced the concept of *semiotic contraction*. A semiotic contraction refers to the reorganization of semiotic resources that occurs as a result of the students' increased consciousness of mathematical meanings and interpretations. Contraction "makes it possible to cleanse the remnants of the evolving mathematical experience in order to highlight the central elements that constitute it" (Radford 2008b, p. 94). Thus, fewer gestures may be required as the students refine their ideas and become more and more conscious of mathematical structures and ideas.

The concept of *semiotic bundle* offers also a synchronic and a diachronic approach to the investigation of learning. Here, the focus is in the evolution of signs. This notion has been elaborated by Arzarello (2006), Arzarello et al. (2009) in order to give account of the multimodality of mathematics learning and teaching processes. The term "multimodality" comes from neuroscientific studies that have highlighted the role of the brain's sensory-motor system in conceptual knowledge and have proposed a multimodal model for brain functioning, instead of a modular model (Gallese and Lakoff 2005). On the other hand, "multimodality" is also used in communication design to speak of the multiple modes we use to communicate and express meanings to our interlocutors: e.g. words, sounds, figures, etc. Within this perspective, a semiotic bundle has been defined as

> a *system of signs* […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p. 100)

Focusing the attention on a wide variety of means of expression, from the standard mathematical symbols (e.g., algebraic representations) to the embodied ones (such as gestures, gazes, and so on), and considering all of them as semiotic resources in teaching and learning processes, the semiotic bundle construct widens the range of semiotic resources that are traditionally discussed in mathematics education literature (e.g., Duval 2006; Ernest 2006).

In order to clarify the notion of semiotic bundle, we can consider for instance the set of words, the set of gestures, and that of written signs (e.g. algebraic symbols) that are used in a certain mathematical activity. The three sets, which are used along the mathematical activity, constitute the semiotic bundle: the interpretation of one kind of resources (e.g. speech) can be fully done only taking into account also the other resources (gestures and written signs). In this sense, the semiotic bundle considers the semiotic resources in a unifying analysis tool. Of course, depending of the needs of analysis, each semiotic set can also be analysed in a separated way. But since different semiotic sets very often intertwine, a global view on them is necessary.

The semiotic bundle can be an analytical tool in order to detect cases of semiotic nodes, when the attention focuses on *synchronic* relationships between signs used to accomplish an objectification process.

Besides the synchronic view, the semiotic bundle offers the possibility of performing a *diachronic analysis*, that is to say of studying the evolution of semiotic resources in the passing of time, and the evolution of their mutual relationships. With this view, genetic phenomena regarding signs may be observed, when some signs are transformed into another kind of signs (e.g., of gestures giving origin to written drawing in pre-algebraic context, see Sabena et al. 2012). A diachronic view has allowed researchers in gesture studies to elaborate the notion of "catchment". McNeill and colleagues identified a catchment when some gesture form features recur in at least two (not necessarily consecutive) gestures (McNeill 2005; McNeill et al. 2001). According to their framework, they interpreted catchments as indicating discourse cohesion, due to the recurrence of consistent visuospatial imagery in the speaker's thinking. In our semiotic frame, catchments may be of great importance since they can give us clues about the evolutions of meanings in students' multimodal discourses and in their objectifying processes (for an example about catchments in structuring a mathematical argument, see Arzarello and Sabena 2014).

To illustrate our methodology, in the remainder of the chapter we show an example that constituted an important step in developing our semiotic approach. It showed us that, as quoted above, a method can be "simultaneously a prerequisite and product, a tool and a result of the research" (Vygotsky 1993, p. 27).

The example is about the role of words, gestures and rhythm in objectifying (i.e., noticing or becoming aware of) mathematical relationships; the analysis has been reported previously (Sabena et al. 2005; Radford et al. 2006, 2007).

During our research activity, we did not anticipate rhythm as playing a subtle and profound semiotic role in mathematics cognition. Watching a video clip over and over within the possibilities of a low motion and frame-to-frame analysis, and focusing on students' words and gestures, we began to notice that rhythm was playing a fundamental role as a semiotic resource in the students' activity. This "crude fact" was theorized through the principles of the theory: we realized that rhythm was a fundamental semiotic means of knowledge objectification. That is, through an apparently unconscious recourse to rhythm, the students started perceiving, behind the mathematical signs, a general mathematical structure.

Dedicated software developed in linguistic research allowed us to carry out a pitch and prosodic analysis to confirm the role of rhythm. To be duly interpreted, the new results required a refinement of the theoretical principles. We gained a new theoretical sensitivity that allowed us to be alert to phenomena that escaped our research lenses before. The methodology of analysis also evolved, with the refinement of both the technical means (e.g. the use of the new software), and a more sensitive research eye. We call the resulting methodology a *multi-semiotic methodology* (Radford et al. 2006), and we illustrate it in the next section, with reference to the specific example.

7.5 Multi-Semiotic Analysis: An Example Concerning Pattern Generalization

To illustrate our semiotic approach, we refer to a classroom activity concerning pattern generalization as a way to approach algebraic thinking.

The data come from a 5-year longitudinal research program, and were collected during classroom lessons that are part of the regular school mathematics program in a French-Language school in Ontario. As described above, lessons are jointly designed by the teacher and our research team. The students spend substantial periods of time working together in small groups of 3 or 4, with the teacher interacting continuously with the different groups. At some points, the teacher conducts general discussions allowing the students to expose, compare, and confront their different solutions.

We focus on a classical pattern problem that Grade 9 students had to investigate in a math lesson. The problem deals with the study of an elementary sequence that is visually depicted (see Fig. 7.2). In the first part of the lesson, the students were required to continue the sequence, drawing Terms 4 and 5 and then to find out the number of circles on Terms 10 and 100. In the second part, the students were asked

Fig. 7.2 The three first terms of the sequence

to write a message explaining how to calculate the number of circles in any term and, in the third part, to write an algebraic formula.

We provide a multi-semiotic microanalysis of the work done on the first and in the second part of the math lesson by one group of students formed by Jay, Mimi, and Rita. Referring to the first part, we illustrate in particular how words and gestures play a crucial role in allowing the students to perceive the terms as divided into two rows. In the course of the students' joint activity, knowledge as pure possibility becomes actualized in the form of a factual generalization (Radford 2003), i.e. a generalization of actions in the form of an operational schema that applies to any concrete term, regardless of its position in the sequence. Referring to the second part, we show how rhythm serves as a subtle semiotic device that helps the students notice a regularity that proved to be crucial to convey a sensuous meaning of mathematical generality.

7.5.1 Words-Gesture Combinations in the Production of a Factual Generalization

At the beginning of the activity, the students count the number of circles in the terms, and realize that it increases by two each time. Then, in order to draw Term 4, they use gestures and speech through which they identify the two rows of the terms and their numerosity as key-elements in the problem solution:

1. Rita	You have five here… (pointing to Term 3 on the sheet)
2. Mimi	So, yeah, you have five on top (she points to the sheet, placing her hand in a horizontal position, in the space in which Jay is beginning to draw Term 4; see Fig. 7.3) and six on the… (she points again to the sheet, placing her hand a bit lower)
3. Jay	Why are you putting…? Oh yeah, yeah, there will be eleven, I think (He starts drawing Term 4)
4. Rita	Yep
5. Mimi	But you must go six on the bottom … (Jay has just finished drawing the first row of circles) and five on the top (Jay finishes drawing the second row)

Although Jay materially undertakes the task of drawing Terms 4 and 5, each student is engaged in the action. In line 1, Rita is not merely informing her group-mates that Term 4 contains a row of five circles. In fact, through a deictic gesture she is suggesting a qualitative and quantitative way to apprehend the next terms. Pointing to a

Fig. 7.3 Mimi's first gesture on line 2

specific part of Term 3, which is given on the sheet, but referring in her speech to Term 4, Rita provides a link between the two terms. Through gesture and speech she is suggesting a specific way to build Term 4. This is an example of a process of perceptual semiosis: a process in which perception is continuously refined through signs.

This grasping of the term is easily adopted by Mimi, and properly described through the spatial deictics "top" and "bottom" (lines 2 and 5). It amounts to shifting from blunt counting to a scheme of counting. This scheme is the first step in the process through which knowledge as pure possibility is endowed with concrete determinations. From something fuzzy and general, knowledge becomes shaped, refined, and specified. It does not become a thing or an object (as in other accounts of objectification). The schema is possibility transformed into action, the result being an open event itself in movement and open to further transformation. In dialectical logic, the schema is an example of the ascent from the abstract to the concrete (Radford 2013b).

In line 2, Mimi's words are accompanied by two corresponding deictic gestures, which allow her to participate in the drawing process and depict the spatial position of the rows in an iconic way. In line 5, Mimi does not make any gestures; rather, her words are perfectly synchronized with Jay's action, almost directing him in the action of drawing: in fact, to complete her sentence with the description of the second row, Mimi waits until Jay finishes drawing the first row of circles.

The gesture-speech combination referring to the spatial location "top" and "bottom" is soon after enacted by Jay to explain why he thinks that Term 10 will have 23 chips and Term 100 will have 203 chips:

6. Jay	Ok. Term 4 has five on top, right? (with his pencil, he points to the top row of Term 4, moving his pencil from the left to the right, Fig. 7.4, left)
7. Mimi	Yeah…

Fig. 7.4 *Left*, Jay's moving gesture (line 6). *Right*, Jays' second gesture (line 8)

Fig. 7.5 Synchronization between the two students' gestures

8. Jay	…and it has six on the bottom (he points to the bottom row using a similar gesture as in line 7, Fig. 7.4, right).
9. Mimi	(pointing to the circles while counting) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. (Pause) […] Oh yeah. Term 10 would have …
10. Jay	10 there would be like …
11. Mimi	There would be *eleven* (Fig. 7.5, left: she is making a quick gesture that points to the air. Jay is placing his hand in a horizontal position) and there would be *ten* (Fig. 7.5, right: she is making the same quick gesture but higher up. Jay is shifting his hand lower down) right?
12. Jay	Eleven (Fig. 7.6: similar gesture but more evident, with the whole hand) and twelve (same gesture but lower).
13. Mimi	Eleven and twelve. So it would make twenty-three, yeah.
14. Jay	100 would have one-hundred and one and one-hundred and two (Fig. 7.7: same gestures as the previous ones, but in the space in front of his face).
15. Mimi	Ok. Cool. Got it now. I just wanted to know how you got that.

Fig. 7.6 Synchronization between the students' gestures

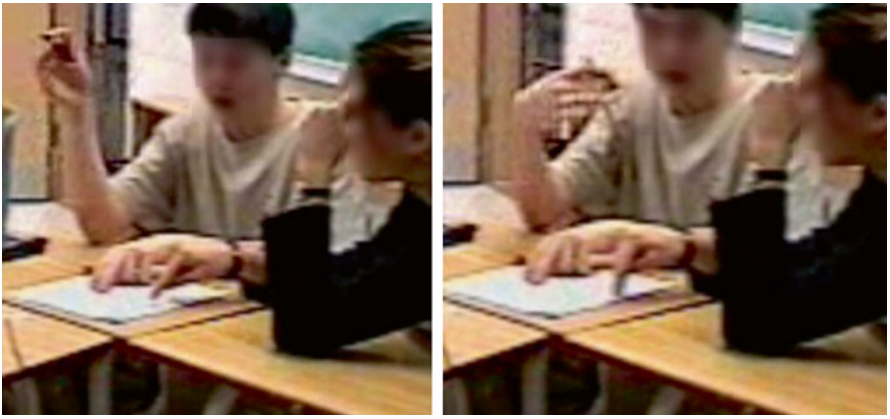

Fig. 7.7 Synchronization between gestures

As we can see in the transcript and related pictures, both Jay and Mimi enact the same gesture-speech combination at different times. The repeated enactment, which shows a gesture catchment, allows the students to shift from the given drawings (representing Terms 3 and 4) to imagined ones (referring to Terms 10 and 100). This shifting is carried out while preserving a certain schema in the grasping of the term, as an important means for accomplishing a factual generalization of the pattern (which can be a first step in the algebraic generalization process).

In Jay's first utterance (lines 6 and 8), the deictic gestures appear endowed with a dynamic feature that clearly depicts the geometric grasping of the term as made up of two horizontal rows. Its goal is to clear away any ambiguity about the referent of the discourse, in order to explain a strategy. Term 4 is perceptively present on the scene, and indeed materially touched by Jay through his pencil. Talking about Term

Fig. 7.8 The detachment of gestures

10, Mimi (line 11) performs two gestures that keep certain specific aspects of those of Jay, that is, one gesture for each row, and the vertical shift. But now, because the referred term is not available in the perceptual field, the gestures are made in the air. Also Jay's last gestures (line 13), referring to Term 100, appear in the air in the space in front of him, as if pointing to the rows of a non visible term. Indeed, if we pay attention to the position of his hands when he refers to the different terms, we can notice a progressive detachment from the sheet (Fig. 7.8).

Furthermore, from the micro-analysis of the video, carried out with slow motion devices, we can detect that Jay is following Mimi's argument so closely that his gestures appear perfectly synchronous with his mate's words and gestures (see line 11 and Fig. 7.5).

The previous episodes show key instances of a process of objectification through which the students become aware of a culturally and historically constituted manner of thinking about sequences. More specifically, through a sensuous coordination of gestures and speech, the students make apparent key traits of Term 100—a term that is not directly perceivable. The tight coordination between gestures and speech takes place in a particular segment of the students' mathematical activity, leading to the objectification of knowledge: it constitutes a semiotic node. In the considered episode, gestures play a specific role in the knowledge objectification: the indexicality of the repeated gestures undergoes a gradual shift from an *existential signification* (referring to Terms 3 and 4, materially present on the sheet) to an *imaginative* mode of signification (referring to Terms 10 and 100).

Notice that the objectifying gestures undergo a process of simplification that involves the loss of movement (along the rows of the term) and a shortening of their duration. A progressive simplification is also evident in the uttered words: from line ten onward, the deictic terms disappear, leaving barely numerical semantic content, organized by the conjunction "and". Even if Terms 10 and 100 are not materially present, the students can *imagine* them very precisely and would be able to draw them; but, having reached a certain stage in the process of objectification, they do not need to specify all the details, and the reference to the form of the term can smoothly remain implicit in their speech. We have referred to this simplification of the students' semiotic activity as a *semiotic contraction* (Radford 2008b).

7.5.2 *Words, Gesture and Rhythm: Refining the Generalization*

The genesis of algebraic generalizations entails the awareness that something stays the same and that something else changes. In order to perceive the general, the students have to make choices: they have to bring to the fore some aspects of the terms (emphasis) and leave some other aspects behind (de-emphasis). In this striving, all the resources at students' disposal may be of great help—even rhythm, with its combination of sound and silence. While we were conducting our video-analysis of the second part of the activity, and were focusing on words and gestures, rhythm came unexpectedly to the fore as another important semiotic means of objectification.

Rhythm creates the expectation of a forthcoming event (You 1994) and constitutes a crucial semiotic device in making apparent the perception of an order that goes beyond the particular terms. It emerged in a moment in which the students were stuck in discussing Mimi's hypothesis that to find out the number of circles in any term of the sequence you need to add three to the number of the term. Since Joy refuses this hypothesis, on the base that it does not hold for Term 100 (where there are 203 chips), Mimi said:

16a. Mimi You know what I mean? Like… for Term 1 (pointing gesture to Term 1) you will add like (making another gesture, see Fig. 7.9)…

To explore the role that digit 3 may play, in line 16a Mimi makes two gestures, each one coordinated with word-expressions of differing values. The first couple gesture/word has an indexical-associative meaning: it indicates the first circle on the top of the first row and associates it with Fig. 7.1 (see Fig. 7.9, left bottom). The second couple achieves a meaningful link between digit 3 and three "remarkable"

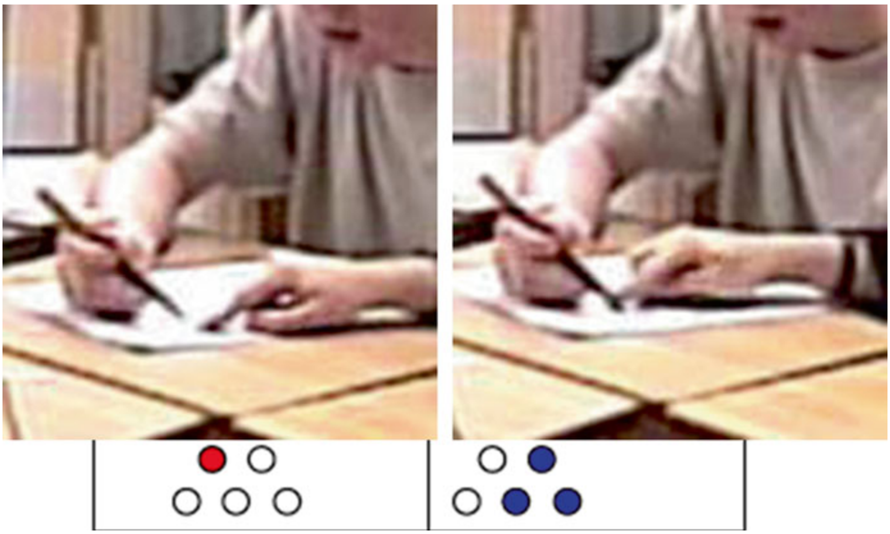

Fig. 7.9 Gestures in line 16a

circles in the term. The resulting geometric-numeric link is linguistically specified in additive terms ("you will add") (see Fig. 7.9, right bottom).

Although Mimi has not mentioned or pointed to the first circle on the bottom row, the circle has been noticed. That is, although the first circle of the bottom has remained outside the realms of word and gesture, it has fallen into the realm of vision. Indeed, right after finishing her previous utterance, Mimi starts with a decisive "OK!" that announces the recapitulation of what has been said and the opening up towards a deeper level of objectification, a level where all the circles of the terms will become objects of discourse, gesture and vision. She says:

16b. Mimi OK! It would be like one (indexical gesture on Term 1), one (indexical gesture on Term 1), plus three (grouping gesture); this (making the same set of gestures but now on Term 2) would be two, two, plus three; this (making the same set of gestures but now on Term 3) would be three, three, plus three.

Making two indexical gestures and one "grouping gesture" that surrounds the three last circles on Term 1, Mimi renders a specific configuration apparent to herself and to her group-mates. This set of three gestures is repeated as she moves to Term 2 and Term 3. The gestures are accompanied by the same sentence structure (see Fig. 7.10). Through a coordination of gestures and words, Mimi thereby objectifies (i.e., notices) a general structure in a dynamic way and moves from particular terms towards a grasping of the general term of the sequence. Notice that, in our interpretation, gestures and words are not uttered once the idea has been formed. On the contrary, the idea is taking place *while* Mimi is gesturing and talking. We move away here from rationalist interpretations where gestures and words would appear and be used after the idea is formed. In other words, communication does not follow understanding and interpretation. Mimi is talking here to her teammates and to herself, at the same time.

In the course of our data analysis, a closer attention to the previous passage suggested that the coming into existence of the refined students' schema is much more than a matter of coordinating word and gesture. There was another important element, concerning the rhythmical way in which words and gestures were performed.

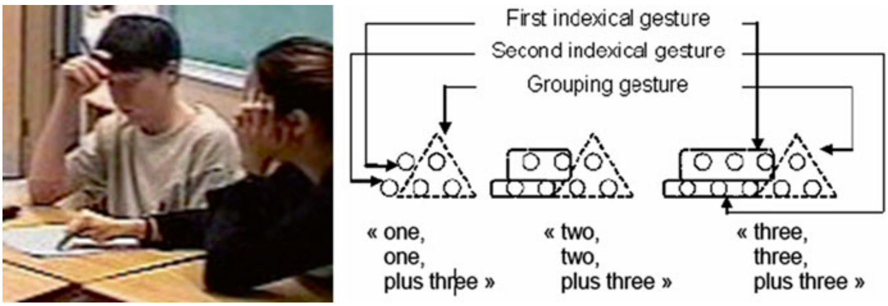

Fig. 7.10 On the *left*, Mimi making the (first) indexical gesture on Term 1. On the *right*, the new spatial perception of the terms as a result of the process of knowledge objectification

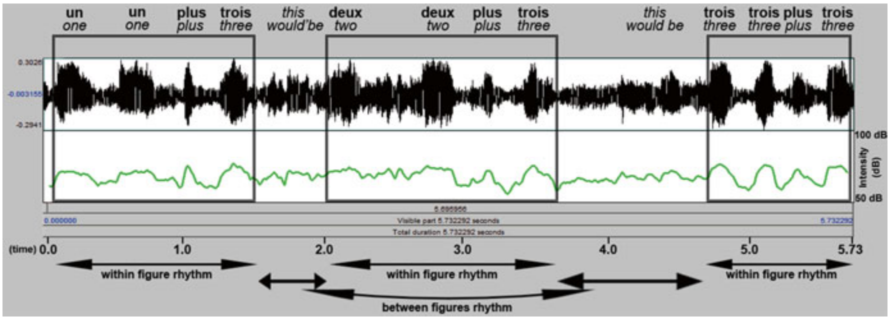

Fig. 7.11 Prosodic analysis of Mimi's utterance conducted with Praat

After listening to the audio recording, to get a better idea of the manner in which the students emphasize and de-emphasize the various features of the terms through rhythm, we conducted a prosodic analysis of Mimi's key utterance in line 16b ("one plus one plus three" etc.). Prosody refers to all those vocal features to which speakers resort in order to mark, in a distinctive way, the ideas conveyed in conversation. Typical prosodic elements include intonation, prominence (as indicated by the duration of words) and perceived pitch.

Our prosodic investigation was carried out using Praat (www.praat.org)—a software devoted to voice analysis. Our prosodic analysis focused on the temporal distribution of words and word intensity. In the top part of Fig. 7.11, the waveform shows a visual distribution of words in time; the curve at the bottom shows the intensity of uttered words (measured in dB).

The waveform allows us to neatly differentiate two kinds of rhythms: within and between terms. The first type of rhythm, generated through word intensity and pauses between words, helps the students to make apparent a structure within each term. In conjunction with words and gestures (the hand performing the same kind of gesture on each term), this rhythm organizes the way of counting. The other type of rhythm appears as a result of generated "transitions" between the counting processes carried out by Mimi when she goes from one term to the next. To generate these transitions, at the lexical level, Mimi uses the same expression, namely "this would be", the semantic value of which indicates the hypothetical nature of the emerging counting schema. At the temporal level, this expression allows Mimi to accomplish a separation between the counted terms. At the kinesthetic level, the transition corresponds to the shifting of the hand from one term to the next. Figure 7.12 provides us with a precise idea of the within and between terms' rhythm.

Using a matrix system of reference a_{ij} for the terms of Fig. 7.12, the data in row 3 indicate that $a_{33} < a_{32}$, $a_{38} < a_{37}$, $a_{313} < a_{312}$, i.e. the data show that the time elapsed between the additive preposition "plus" and the uttered number prior to it is consistently shorter than the elapsed time between the two uttered numbers before "plus". Thus, while the elapsed time between the second "one" and "plus" is 0.360 s (a_{33}), the elapsed time between "one" and "one" is 0.508 s (a_{32}). It is also interesting to note that, in the case of Terms 1 and 2, the elapsed time between "plus" and the

	un *one*	Un *One*	plus *plus*	trois *three*	this woul d'be	Deux *two*	deux *two*	plus *plus*	trois *three*	this woul d'be	trois *three*	Trois *three*	plus *plus*	Trios *three*
1. Intensity (dB)	76.58	77.52	80.04	81.93		78.72	78.61	77.44	80.66		81.73	81.24	77.94	80.38
2. Time (s)	0.157	0.665	1.025	1.348	0.813	2.161	2.798	3.158	3.463		4.793	5.116	5.347	5.633
3. Time (s) between consecutive words		0.508	0.36	0.323			0.637	0.36	0.305			0.323	0.231	0.286
4. Total time (s)	1.191				0.511	1.302				1.035	0.840			

Fig. 7.12 Intensity and time data of Mimi's utterance, as derived from Praat prosodic analysis. *Rows 1* and *2* show the intensity (dB) and time position of words (s), both measured at the middle of the duration of the word. *Row 3* gives the elapsed time between consecutive words. *Row 4* gives the total time of the speech segments

following word is shorter than the time between "plus" and the uttered number before it (i.e. $a_{34} < a_{33}$, $a_{39} < a_{38}$). The rhythmic distribution of words hence suggests that the preposition "plus" does not merely play the role of an arithmetic operation. By emphasizing and de-emphasizing aspects of the terms, it plays a key prosodic role in the constitution of the counting schema.

Note that the temporal distribution of words of the two first speech segments ($0.157 \leq t \leq 1.348$; $2.161 \leq t \leq 3.463$) is quite similar to that of the third speech segment ($4.793 \leq t \leq 5.633$). However, the data indicate that the duration of the latter (0.840 s) is shorter than the duration of the former (i.e. 1.191 and 1.302; see row 5).

The students did not need to go beyond Term 3 to come up with the refined counting schema. One of the reasons for this may be that the generalized structure was recognized during the investigation of the two first terms and the third term hence played the role of verification.

The previous data help us understand the students' mechanisms of emphasizing and de-emphasizing features of the terms. The prosodic analysis sheds light on the articulated ways in which rhythm is used as a semiotic device in the students' phenomenological apprehension of the general. This is why it may be worthwhile to think of algebraic generalization as a process similar to the creation of a sculpture or of a painting. Some elements are brought to the fore; others are left in the back. Both are important, for it is through their *contrast* that one notices what has to be noticed. Rhythm accentuates this contrast in the students' semiotic activity. It heightens the constant and the variable as well as their relationships in the act of generalization.

7.6 Concluding Remarks

In this chapter we discussed some aspects of the methodology of our semiotic approach. Drawing on Vygotsky's idea of method we argued that a method is not an instrument or a mere sequence of actions to be followed. A method is rather a reflexive and critical endeavour—a philosophical practice. As such a method conveys a worldview that provides ideas about the entities or phenomena that can be investigated and how they can be investigated. These ideas are translated into theoretical principles in a particular language and meanings through which research questions can be expressed. This is why methods work in tandem with theoretical principles and research questions and that a theory can be considered as an interrelated triplet of "parts": (P, M, Q), where P stands for principles, M stands for methodology, and Q for research questions.

The principles P should clarify our assumptions and ideas about knowledge and learning. We presented a succinct sketch of them in the first part of the chapter. Following Hegel's dialectical materialism, we suggested that, from the students' viewpoint, knowledge appears as pure possibility. However, for knowledge to become an object of consciousness and thought, it has to be set in motion and filled up with conceptual determinations. This is what teaching-learning activity does. In the course of the activity (in Leont'ev's (1978) sense), knowledge becomes actualized or realized. However, knowledge's actualization is not a thing. Its actualization is an *event*.

In the example that we discussed in the chapter, knowledge is the pure possibility of thinking, reflecting, and solving pattern generalization problems in a cultural and historical manner that has been refined through centuries by previous generations.

From the students' viewpoint, the algebraic manner of thinking about patterns is there, as pure possibility. It becomes actualized as the students engage in sensuous, material activity. In the course of the activity through which knowledge is actualized, knowledge reveals itself and can become an object of consciousness and thought. In our example, its sensuous and material revelation occurred through the formation of a schema. Let us insist on the idea that the schema is not an objectified thing, but an event: the schema is possibility transformed into action, the result being an open event itself in movement and open to further transformation.

Within this context, the account of learning rests on the account of how knowledge is transformed from pure possibility into an object of consciousness. The method is the critical and reflexive endeavour through which this transformation is investigated. Because the activity that mediates and actualizes knowledge into a singular event is an intersubjective, sensuous, and material activity, we trace all signs that intervene in the activity—traditional written signs, but also corporeal signs, such as gestures and posture (e.g. position of the hands and the fingers).

Through fine-grained semiotic analyses we accounted for the manner in which signs signified in the mediating activity. We discussed how we became conscious of the importance of rhythm in mathematics cognition, and how a "crude fact" led to a transformation of our theory, and in particular its methods and research questions. We mentioned in particular two methodological constructs that have been built to help us disentangle the intricacies of multimodal sensuous actions: the semiotic node and the semiotic bundle. The former provides us with a synchronic tool to focus on the manner in which students endow with meaning their actions in coming to discern mathematical relationships and structures in their work. The latter provides us with a diachronic tool to follow the evolution of signs' interrelationships in the course of the activity. For instance, if we carefully analyse the excerpts presented here by looking at the diachronic evolution of the semiotic bundle, we can observe how the catchment develops and how meanings are emerging along with the evolution. This evolution signals the key moments of the students' objectification process in the pattern generalization activity.

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Part V
A Theory on Abstraction and Its Methodology

Chapter 8
The Nested Epistemic Actions Model for Abstraction in Context: Theory as Methodological Tool and Methodological Tool as Theory

Tommy Dreyfus, Rina Hershkowitz, and Baruch Schwarz

Abstract Understanding how students construct abstract mathematical knowledge is a central concern of research in mathematics education. Abstraction in Context (AiC) is a theoretical framework for studying students' processes of constructing abstract mathematical knowledge as it occurs in a context that includes specific mathematical, curricular and social components as well as a particular learning environment. The emergence of constructs that are new to a student is described and analyzed, according to AiC, by means of a model with three observable epistemic actions: Recognizing, Building-with and Constructing–the RBC-model. While being part of the theoretical framework, the RBC-model also serves as the main methodological tool of AiC.

In the first section of this chapter, we give an outline of the theoretical aspects of AiC as background to the description of the elements of our methodology in the second section, and their application to a specific example in the third section. In the concluding section, we close the circle by exhibiting the strong relationship of theory and methodology in AiC as it is mediated by the RBC-model.

Keywords Abstraction in context • Epistemic actions model • RBC+C-model

T. Dreyfus (✉)
Department of MSTE, School of Education, Tel Aviv University,
P. O. Box 39040, Ramat Aviv, Tel Aviv 69978, Israel
e-mail: tommyd@post.tau.ac.il

R. Hershkowitz
Science Teaching Department, Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: rina.hershkowitz@weizmann.ac.il

B. Schwarz
School of Education, The Hebrew University, Mount Scopus, 91905 Jerusalem, Israel
e-mail: baruch.schwarz@mail.huji.ac.il

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8.1 Theory

In this section, we give a summary of the theoretical aspects of AiC. We refer the interested reader to the literature for more extensive treatments (Hershkowitz et al. 2001; Schwarz et al. 2009). Our approach took shape in the course of research that accompanied innovative curriculum development, when questions arose such as "What did students learn and consolidate, and how? What mathematical concepts and strategies remain with them?" A more subtle issue has to do with the particularities of typical mathematical curricula in which ideas fostered in certain activities are solicited as elements in a reorganized structure: archetypical reorganizations are about the passage from the realm of numbers to the realm of algebra, and from algebra to functions and their representations. Although students may learn ideas that designers did not intend to foster, the structures of mathematical curricula impose constraints and open affordances for certain types of learning processes.

The salient characteristics of the mathematical curricula and classroom learning environments in which we study abstraction are as follows:

- Curricula are organized as *successions of activities* and themes proposed along these activities very often *transform* previous ones. Hence curricula express an *intention of continuous transformation*.
- Our theory of abstraction takes into account the particularities of contexts: The "C" in AiC implies that the role of mathematical, curricular, historical and social context is central to AiC. In particular, the social and interactional context may vary considerably according to the teacher's decisions.
- Nevertheless, there is an underlying expectation of students' responsibility for their own learning in an environment that encourages inquiry. Students have the responsibility to report and justify their work and their conclusions to their peers and their teacher, for example during whole class discussions.

In classrooms, abstraction often takes place in interacting small groups of two to four students. Hence, we focus on small groups as well as individuals and two dual issues are central in our approach: On the one hand, we look at what is shared–shared meanings or knowledge to consider what allows groups to continue further constructing knowledge in the same topic together (Hershkowitz et al. 2007). On the other hand, we try to discern agencies as we look at who initiates, how labor is divided and whether responsibilities are taken by different students.

This attention to a special kind of curriculum and to the emergence of learning processes within various contexts led us to theoretical forefathers that belong to different traditions, Freudenthal and Davydov. Freudenthal (1991) provided what many mathematicians have in mind when they think of abstraction. Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular. These insights constitute a cultural legacy that led his collaborators to the idea of "vertical mathematization". Vertical mathematization points to a process that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means by which students construct a new abstract construct. As researchers in mathematics education, we preferred the expression “vertical reorganisation” to the expression “vertical mathematization” to discern between what is intended by the teacher–the mathematization, and what often happens–a reorganisation, which is a radical change which sometimes does not coincide with what is intended. In vertical reorganisation, previous constructs serve as building blocks in the process of constructing. Often these building blocks are not only reorganised but also integrated and interwoven, thus adding a layer of depth to the learner’s knowledge, and giving expression to the composite nature of the mathematics. Sequences of problem situations provide opportunities to capitalise on the new constructs repeatedly, and to turn them into building blocks for further constructing actions, where each construct includes ‘pockets’ of past constructs on one hand, and is itself a potential component for new constructs.

Davydov was one of the most prominent followers of the historical cultural theory of human development initiated by Vygotsky. For Davydov (1990), scientific knowledge is not a simple expansion of people’s everyday experience. It requires the cultivation of particular ways of thinking, which permit the internal connections of ideas and their essence to emerge, thus enriching rather than impoverishing reality. According to Davydov’s “method of ascent to the concrete”, abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It is a dialectical process that ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form. As such, abstraction clearly contrasts with *generalization*, which is an extension to less specific criteria–an extension, which is not more developed and therefore does not require reorganization.

The reference to both Freudenthal and Davydov’s theories of abstraction in AiC expresses an effort to bridge between the cognitive and socio-cultural dimensions: On the one hand, Freudenthal, an eminent mathematician influenced by Piagetian ideas, saw mathematization as a reflective effort done by the student toward a construct foreseen beforehand by the mathematician. Davydov’s approach is socio-cultural. His analyses of abstraction uncover dialectical processes often occurring in interactions between the adult and the student. The bridge between those two perspectives consists of the necessary design effort for providing opportunities for abstraction to occur. The design is accountable for what the mathematician and the mathematics educator intend to “instill”. This design hopefully opens the door for negotiations of meaning in interactions, possibly leading to abstractions as desirable constructs.

AiC adopts the views of vertical mathematization and ascent to the concrete and builds on them to define abstraction as a process of vertically reorganizing some of the learner’s previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. Activity theory proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, AiC follows Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy.

According to activity theory, outcomes of previous activities naturally turn to artifacts in further ones, a feature that is crucial in tracing the genesis and the development of abstraction through a succession of activities. The kinds of action that are relevant to abstraction are epistemic actions – actions that pertain to the knowing of the participants and that are observable by participants and researchers.

We postulate that the genesis of an abstraction passes through a three-stage process, which includes the need for a new construct, the emergence of the new construct, and the consolidation of that construct. We will exhibit the crucial role of epistemic actions for the second and third stages below. The nature of the first stage is well expressed by Kidron and Monaghan (2009) when dealing with the need that pushes students to engage in abstraction, a need that emerges from a suitable design and from an initial vagueness for the learner:

> The learners' need for new knowledge is inherent to the task design but this need is an important stage of the process of abstraction and must precede the constructing process, the vertical reorganization of prior existing constructs. This need for a new construct permits the link between the past knowledge and the future construction. Without the Davydovian analysis, this need, which must precede the constructing process, could be viewed naively and mechanically, but with Davydov's dialectic analysis the abstraction proceeds from an initial unrefined first form to a final coherent construct in a two-way relationship between the concrete and the abstract–the learner needs the knowledge to make sense of a situation. At the moment when a learner realizes the need for a new construct, the learner already has an initial vague form of the future construct as a result of prior knowledge. Realizing the need for the new construct, the learner enters a second stage in which s/he is ready to build with her/his prior knowledge in order to develop the initial form to a consistent and elaborate higher form, the new construct, which provides a scientific explanation of the reality. (pp. 86–87)

As mentioned above, a central component of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner seeing the relevance of a specific previous construct to the situation or problem at hand. Building-with comprises the use and combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. Hence, while the term *constructing* refers to the epistemic action – a process – the term *construct* refers to an outcome of such action. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of his new construct, and the learner's construct is often fragile and context dependent. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to consolidation.

Consolidation is a never-ending process through which students become aware of their construct, the use of the construct becomes more immediate and self-evident, the student's confidence in using the construct increases, and the student demonstrates more and more flexibility in using the construct (Dreyfus and Tsamir 2004). Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities, possibly in the course of a further constructing action. These further activities may lead to new constructs. Hence consolidation connects successive constructing actions and is closely related to the design of sequences of activities.

In processes of abstraction, the epistemic actions are nested. C-actions depend on R- and B-actions; the R- and B-actions are the building blocks of the C-action; at the same time, the C-action is more than the collection of all R- and B-actions that make up the C-action, in the same sense as the whole is more than the sum of its parts. The C-action draws its power from the mathematical connections, which link these building blocks and make them into a single whole unity. It is in this sense that we say that R- and B-actions are constitutive of and nested in the C-action. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognising this construct, at least implicitly. Moreover, a lower level C-action may be nested in a more global one, if the former is made for the sake of the latter. This nested character was observed in the classrooms and in interviews in which we studied abstraction and it substantiated our theoretical tenets according to which the curriculum was intended to afford a continuous transformation of constructs. Given these characteristics, we named the model the *dynamically nested epistemic actions model of abstraction in context*, more simply the RBC-model, or RBC + C model using the second C in order to point at the important role of consolidation. The RBC-model is the theoretical and micro-analytic lens, through which we observe and analyse the dynamics of abstraction in context. We will, in the concluding section, come back to the RBC-model in order to show how the model as a part of the theory interacts with the same model as methodological tool, and hence theory and methodology mutually depend on each other, influence each other and undergo successive refinements.

8.2 The AiC Methodology

In this section, we give a concise description of the methodology of AiC, its elements, its tools and its methods. We decided to keep this description concise since in our opinion, methodological tools and methods – those of AiC as well as those of any other theoretical framework—obtain their significance only through their application to specific cases; our main intention, in this chapter, is therefore to show the application of the methodology to a specific set of data; this application will be exemplified in the next section, accompanied by frequent reference back to the present section.

8.2.1 Design for Abstraction

The importance of design in our theory is considerable. It constrains the kinds of actions participants may carry out, and affords the ones that seem desirable to the designer and teacher. Of course, students' actions are never determined by the design, and there is always a gap between intended actions and actions enacted. But the diminution of this gap is of crucial importance for abstraction to occur. The design is a challenging task and it is often an integral part of the research project (Kouropatov and Dreyfus 2013), or the project is a design-research-redesign project (Hershkowitz et al. 2002). Often also, a particular design that has been successful in some condition is taken over from elsewhere. For example, in a recent study on the tension between the discrete and the continuous in a differential equations course, the design, and even the data, were taken as is from a research study in a different country in which only one member in the team had been involved in the past (Tabach et al. 2014). The difficulty involved in the design originates from the fact that it has to comply with an intention of continuous transformation of constructs. This requires the elaboration of sequences of activities that offer the students opportunities to learn well defined mathematical ideas (e.g., how a solution to a first order differential equation can be approximated by discrete steps and that the approximation improves as the steps become smaller; the notion of integral as an accumulating quantity; that algebra is a tool for justification, etc.). It also requires the elaboration of further activities to apply these ideas as tools in familiar contexts or as tools in contexts that necessitate the elaboration of new ideas. What is common to all these learning aims is that they include adding new connections between students' previous knowledge, hence adding depth to the students' understanding, integrating their knowledge in ways not available to them before. In brief, the design intends to create a didactical hierarchy aimed at vertical reorganization of students' knowledge.

Beyond these very general remarks, designs may considerably vary to afford intended abstractions. However, in several research studies using AiC to study abstraction, some design principles have been articulated: In addition to creating a collaborative situation, some of these principles are triggering a cognitive conflict, asking for hypotheses and providing tools for testing them, and reflective argumentation (Prusak et al. 2012). Also, the sequence of activities often alternates contexts in which small groups work or argue together with teacher-led (reflective) discussions.

8.2.2 A Priori Analysis

In a move pertaining to design but specialized on its epistemological aspects, an effort is then made to foresee trajectories of students' learning: an a priori *analysis* of the activities (e.g., Ron et al. 2010). Assumptions are first made about the previous knowledge of the students. Then possible paths to deal with the activities and answer the questions they raise are examined. The most relevant question asked in the a priori analysis is what knowledge is helpful or even necessary to deal with the task and to complete it to the designer's or teacher's satisfaction. We are particularly interested in knowledge that has not been relevant in previous activities carried out by the students. With respect to knowledge that has not been relevant in previous activities and is helpful or necessary in the current one, we assume that the designer or teacher intended this knowledge to be constructed by the students in the current activity. The aim of the a priori analysis is to identify such elements of knowledge intended by the design, typically concepts or strategies thought of in terms of the content domain of mathematics. It is our working assumption that the new constructs, which emerge for the students when dealing with the task, are in close correspondence with the knowledge elements identified in the a priori analysis, if the design and the a priori analysis are adapted to the learner. Nevertheless, students' constructs are of course to be distinguished from the knowledge elements intended by the design.

The selection of knowledge elements in the a priori analysis has a considerable influence on the a posteriori RBC-analysis that is being carried out later. It can therefore not be left vague or undetermined. Therefore, two or three researchers in a team usually carry out a priori analyses independently, and then differences are ironed out until there is agreement among the researchers. For each knowledge element, we do not only give a definition in terms of the mathematical meaning of the element in the context in which it is being used, but also an operational definition. The operational definition is methodologically important: It fixes under what circumstances the researcher will say that a student is using or expressing a construct that represents this knowledge element. Hence, the operational definition will take into account the definition of constructing, namely that constructing refers to the first time a new construct is expressed or used by the learner. Examples for knowledge elements with their general as well as their operational definitions will be given in the next section.

The result of the a priori analysis is not or not only a list of knowledge elements but rather a structure of knowledge elements, with some elements being contained in (or part of) others, and some being prerequisite for others. Often, there is an overarching knowledge element, within which all others are included. An example for this is the Seals activity (Dreyfus et al. 2001), The Seals activity was intended for grade 7 students, whose previous learning experiences did not include tasks where algebra was used as a tool for justification; and while they had learned and gained experience with the simple distributive law $a(c+d)=ac+ad$, they had never had a need or occasion to similarly expand $(a+b)(c+d)$ (the extended distributive law). The Seals activity presented the students with numerical examples of "seals" of the form shown in Fig. 8.1, and asked them to find as many properties as possible of such seals. After some exploration, they were led to focus on the property that the difference between the product of the elements in the minor diagonal and the product of the elements in the major diagonal is always 12, and asked to justify this phenomenon. This was intended to (and did in many cases) lead student groups to use algebra as a tool for justification and develop the extended distributive law in the

Fig. 8.1 The “seal”

X	X+2
X+6	X+8

process. Hence the a priori analysis resulted in algebra as a tool for justification as the overarching knowledge element and, nested within it, the extended distributive law as a further knowledge element. By the way, this overarching knowledge element was clearly recognized by the researchers only after we analyzed several groups solving the activity and observed difficulties some of them experienced in solving the task.

One aim of the a priori analysis is to focus, at least initially, the researchers' attention on the intended constructs, on which the RBC micro-analysis of students' constructing of knowledge is then based. Sometimes the a priori analysis, as meticulous as it may be, may miss the necessity of certain knowledge elements for performing a task or may fail to predict some of the students' ways of going about a task. In such cases, the a posteriori analysis of the data according to the RBC-model will be used to provide improvements to the a priori analysis. Hence, the a priori analysis serves as a working hypothesis that may later be confirmed or modified by the micro-analysis of the data.

8.2.3 Data Collection and Preliminary Analysis

The setting of a specific research study occasionally influences the decision on which students to focus the data collection. In individual or small group interview studies, data from all students are collected. In studies that take place in classrooms, the potential volume of data is enormous, and it is mandatory to make a choice. We then select a few focus groups or focus students for close observation. Group work of the focus students, as well as whole classroom discussions, are videotaped and audio-taped. Tapes are transcribed, in some cases with intonation signs. The analysis uses all student productions (worksheets), the transcripts, and researchers' field notes. Since meaning making in mathematics is very often achieved in multi-modal channels (Radford 2009), the videotapes are important for example for gestures, facial expressions, as well as to identify in a group of students who wrote or drew what when (for which purpose we have recently started using *Livescribe* pens as well (Hickman and Monaghan 2013). It is very often helpful to complete the data collection with interviews of the teacher as well as of some focus students.

The analysis of whole classroom discussions according to the RBC-model is delicate because, in such a context, the interventions of individual students may be sporadic. However, whole classroom discussions can complete analyses in a succession of activities. (See also comments in the concluding section.)

We prepare our analysis by reading the entire transcript and often viewing the videos in order to get an overview of the entire learning process; usually, we first prune the transcript to leave relevant data, then split the pruned transcript into episodes (Chi 1997) the end of an episode being determined by either cognition, content or external factors that influence content or cognition. Cognition means that a change occurs in students' orientation, in questions asked or approached by the students, or in the methods they use to attempt making progress. A change in content is often indicated by the transition to a new task or subtask. External factors, like the teacher calling attention to a whole class discussion, frequently have a similar effect of introducing a break or change or orientation. Among all episodes, we then identify those, in which it appears that new knowledge may have emerged for one or several focus students. Later stages of our analysis will begin from these episodes.

8.2.4 Need

Although the RBC-model focuses by definition on epistemological aspects of mathematical activity, abstraction stems from a need and overt expressions of this need may sometimes be identified in the preliminary analysis: enthusiasm, uncertainty, surprise, curiosity or bafflement, for example as the consequence of a cognitive conflict, are emotional states whose appearance are good threads to begin with to identify and explain the emergence of a constructing action. Similarly, learners may express the need to justify for themselves or for others a mathematical fact or an idea that they just discovered.

As researchers, we also have the task of identifying needs whose manifestation is subliminal, and often these can be detected. There are many possible indicators for a need as a precursor of constructing: Student's questions, manifestations of uncertainty, of surprise or even just a request for time of reflection that is related to the situation at hand, are all indicators of a possible need for a specific construct, even though the student is at this stage unlikely to be aware of this. Of course, this manifestation may remain undeveloped. Also, one cannot establish whether the above behaviors really express a need or a whimsical emotional expression lacking clear volition. A brief analysis of the episode generally clarifies this issue. Besides these emotional expressions as indicators of a need, natural places at which to look are self-regulatory or self-monitoring manifestations: the learner's explicit orientation in the problem space (e.g., by attempting to establish what has been achieved and what still needs to be achieved).

However, in some cases, constructing actions can occur without the researchers having been able to pinpoint a specific need for a new construct. Kidron et al. (2010) have shown that in some situations constructing actions can occur on the basis of a general epistemic need rather than on the basis of a specific need for a new construct, and that this general need is closely connected to Davydov's description of abstraction as a transition from the vague to the precise.

8.2.5 Analysis According to the RBC-Model

After the preliminary analysis and the possible identification of episodes where a need is detectable, one can turn to the heart of the method, the analysis of the episodes with R, B and C epistemic actions as building blocks of the process of abstraction. The goal in this RBC analysis is twofold: first, to unveil the processes by which the students' new constructs emerge as a vertical reorganization of previous constructs in the current context; second, to contribute to the refinement of AiC through the unfolding of the processes that occurred during the episodes (as will be shown further on).

The operational definitions generated in the a priori analysis are crucial for the RBC analysis. A construct's operational definition provides clear criteria for assessing whether a student's utterance or action provides evidence that a constructing action for the said construct has occurred. Sometimes, such an action or utterance by the students will occur during or immediately after the student's constructing action and sometimes only at a later stage. This may depend on how involved the student is in interaction with other students, with the teacher, or with the interviewer at the critical junctures. This does, of course, not mean that students less involved in the interaction do not construct; it only means that the researchers may not have immediate access to these students' constructs, and need to work on the basis of their later actions or utterances.

The analysis by the RBC+C model is a micro-analysis that proceeds utterance by utterance through the analyzed episodes. This often enables us to find the exact moment where a constructing action has been completed, sometimes by direct evidence and sometimes by concluding back from the evidence to earlier utterances. It also enables us to follow the constructing process and the way epistemic actions are nested in each other. This fits exactly our definition of abstraction: The three epistemic actions are the windows through which we can evaluate the main building blocks of the process of abstraction, and the micro-analysis enables us to trace these epistemic actions and the "mathematical glue" which ties them together.

In order to achieve this goal, researchers might begin by identifying a constructing action that has occurred, mark the relevant action/s or utterance/s in the transcript as the end of the constructing action of the said construct and begin to work backwards through the transcript to identify the recognizing and building-with actions that contribute to the said constructing action. Other researchers trace the epistemic actions in more flexible way: After reading and obtaining a general overview of the episode they go forward and backward, being led by their need to understand and interpret.

The researcher can usually identify recognizing actions quite easily on the basis of an action on or an explicit mention of a previous construct by the learner. The recognized previous construct may then become one of the building blocks being reorganized during a constructing action. Evidence for recognizing actions is usually rather obvious, and our experience shows that disagreements arise rarely with respect to the constructs being recognized by the learners and the evidence that shows that they are recognized. We stress explicitly that recognizing does not refer to a potential new construct that may emerge from the current activity but to previous constructs available to the learner.

Similarly, building-with actions refer to making use of previous constructs having been recognized by the learner as relevant to the problems situation at hand, and not a potential new construct that may emerge from the current activity; indeed, such a potential new construct, having not yet been expressed or used even once by the learner, cannot yet possibly be available to this learner. Building-with refers to explicit actions of computing, sketching, justifying, reasoning, etc. with previous constructs. For this very reason, building-with actions are similarly easy to identify as recognizing actions. Here also, our experience shows that disagreements are rare and novice researchers need only moderate training. Building-with actions also help the researcher to assess which recognizing actions are relevant, according to whether the recognized previous construct has been used in one of the building-with actions. A previous construct that is recognized but not used in any way does not play a relevant role in the constructing action.

If working backward through the transcript, the goal of the researcher is to identify and mark all actions and utterances that contribute to the emergence of the emerging construct. These utterances and actions need not be, and except in simple cases are not, contiguous and some may have occurred much earlier, long before the place where the end of the constructing action has been identified. Even so, this central part of the analysis frequently allows us to find the roots of the constructing action as well as its parts, and allows the researcher to produce a fine-grained description of how the construct emerged by vertically reorganizing previous constructs by means of recognizing and building-with actions. In some cases (e.g., Kidron et al. 2011), we have even been able to identify quite early seeds for a constructing action, for example the observation by students of phenomenological patterns that at the time of observation did not lead anywhere but later supported the initiation of a constructing action.

We have often found it convenient to represent a process of abstraction in a table such as Table 8.1. For purposes of demonstration, we assume that the activity under discussion relates to only two new constructs C_1 and C_2, and one previous construct C_a. (In concrete examples, we tend to give meaningful indices to the constructs – see the next section for examples.) The table is an extension of the transcript. Its leftmost column numbers the utterances; the second column shows the names of two students, A, B and the teacher T. The third column would contain the utterances (omitted here). A column with comments on gestures, timing etc. might be added. The fourth column in Table 8.1 exhibits the analysis of constructing action C_1 and the fifth one that of C_2. In these two last columns, we enter R- and B-actions with

Table 8.1 A hypothetical table of epistemic actions

No.	Name	Utterance	C_1	C_2
1	A	…		
2	B			
3	A			
4	B		R_a	
5	T			
6	B			
7	A			
8	B		B_a	
9	A		R_aB_a	
10	B			R_a
11	A		B_a	B_a
12	A			R_1
13	B			B_1
14	A			B_1
15	B			B_a B_1

previous constructs, denoting clearly which previous construct the action refers to. In the C_1–column, only R_a and B_a actions (with C_a) are possible. In the C_2–column, R_a, B_a, R_1 and B_1 are possible. We again stress that in column C_k, actions R_k and B_k cannot possibly occur since the constructing action of C_k has not ended and hence C_k is not available to the student. In rows that contribute to the constructing action C_k, the corresponding cells of column C_k are shaded. An utterance may contribute to more than one constructing action. The end of a constructing action is marked by a bold horizontal separation below the relevant cell. (For a detailed example of using such a table for the analysis of a constructing action, see Dreyfus et al. 2001.)

The preceding description shows that a construct has to be recognized as relevant before being used to build-with. Hence an R_k action is usually nested within any B_k action. It similarly follows that the R- and B-actions with previous constructs are nested in C-actions, for example, R_a, B_a, R_1 and B_1 are all nested in C_2. Constructing actions can go on in parallel, as in the table, where constructing C_2 begins before constructing C_1 has been completed. Parallel constructing actions can interact, branch and combine in rather complex ways (Dreyfus and Kidron 2006).

Colleagues and research students alike often ask us how to distinguish constructing actions from building-with actions. This question is eminently justified since there is no analytic way to distinguish them by mechanical analysis. The decision whether a task requires building-with or constructing from a learner depends not only on the task but also on the learner's personal history: If the sequence of learner actions and utterances when dealing with the task expresses vertical reorganization and hence the emergence of a construct that is new to the learner, then the action is a constructing action; otherwise, it is a building-with action. Someone might argue that this may introduce ambiguities into the RBC-analysis since different researchers may interpret a sequence of actions and utterances differently; we agree that there is a measure of interpretation involved, but our experience shows that differences of interpretation are rare and can be successfully resolved in interaction between researchers.

Students' constructs as revealed by the RBC-analysis can be expected ones that correspond closely to the knowledge elements identified in the a priori analysis, or they can be quite unexpected ones. This is fortunate, as good design for learning should not look like conveyor belt production in factories. Instead of considering unexpected knowledge elements as design failures, one should consider them as ways students handle opportunities for learning provided to them. An example for an unexpected construct will be presented below. In fact, students' constructs may, and often will correspond partially to the intended knowledge elements. Occasionally, this will lead to a modification of the a priori analysis, not in the sense of making it into an a posteriori analysis but rather in the sense of correcting what could or should have been expected. In other words, we as researchers have in quite a few cases used the data to discover that our analysis was not as fine-tuned or as reasonable as might be expected and to learn from the students' actions what might have been expected a priori. On other occasions, some students' constructs will be only partially correct from the mathematical point of view or sufficient to achieve only partially the tasks set by the activities. In this case, we speak of partially correct constructs – PaCCs (Ron et al. 2010). PaCCs are, by definition, partial with reference to an intended knowledge element and curriculum, and not in some absolute sense. PaCCs are examples of manifestations that were not considered in the first articulation of the RBC theory but that were incorporated in a refinement of the theory through further experiments.

8.2.6 Consolidation

Our theoretical approach to abstraction as based on activity theory lends itself naturally to the consideration of abstraction as including not only constructing processes but also the use of the resulting constructs further on. This implies that a student may be considered as having participated in a constructing action without fully being aware of it. Alternatively, the student may have fully taken responsibility for her constructing but may not have properly used the resulting construct further on. Examining student actions after the emergence of a construct is the focus of consolidation. In order to investigate consolidation, our analysis hence typically works forward from the moment the C-action has ended and identifies R- and B-actions with this construct. These are most frequently found either as B-actions carried out to deal with a task that does not lead to a new construct or, more interestingly, as R- or B-actions serving as part of a constructing action of a further construct as R_1 and B_1 have served in the constructing of C_2 in Table 8.1.

A first step in the analysis of consolidation of a construct is therefore to identify all R- and B-actions with this construct, a step that is necessary anyway for the purpose of analyzing the constructing of further constructs. The second step in the analysis of consolidation is to determine whether and in what way these R- and B-actions provide evidence of the student's confidence in using the construct, of the immediacy with which the student uses the construct, of the self-evidence the construct has for the student, of the flexibility the student demonstrates in using the construct and the student's awareness of the construct. Each of these separately (and a fortiori some or all of them together) provides evidence for the continuing consolidation of the construct.

8.2.7 Who Is Constructing?

Just as choices have to be made which data to collect, choices sometimes have to be made, which data to analyze. While no such decision needs to be taken when analyzing an interview with a single student, this situation is prone to excessive researcher intervention, for example when the student gets stuck, or simply in order to make the student express her thoughts. We therefore tend to work with small groups of students. However, even when a pair of students is collaborating in an interview situation, one student may be leading, or the pair may be interacting in a symmetric manner. In each case, the researcher has to decide whether to analyze constructing actions of the pair as a whole or of each student separately; in some cases, it may be impossible to separate the epistemic actions of the students or information about one of the students may be sparse. The situation clearly gets more complex when groups have three or four participants; it gets even more complex, when the group works in a classroom. This is a big challenge because of the many variables that play a role in a whole class situation and the potentially messy data. Hence we had to deal with the question of what constitutes knowledge shared by a group of individual students as they construct and consolidate it in the classroom (Hershkowitz et al. 2007). In this setting, we emphasize the interactive flow of knowledge from one student to the others in the group, until they reach a shared knowledge—a common basis of knowledge, which allows them to continue constructing further knowledge in the same topic together.

8.3 A Focus Group in a Classroom as an Example

In this section, we provide an example whose purpose it is to illustrate the use of the methodology set forth in the previous section. The example is taken from a research project (Hershkowitz et al. 2007) whose main goal was to investigate aspects of small groups' knowledge constructing in working classrooms. The topic chosen for instruction was elementary probability. Five grade 8 classes in different schools and their teachers participated in the project.

The probability unit deals with concepts and problem-solving aspects of empirical versus theoretical probability, and one- and two-dimensional sample spaces. This overall construct of probability includes three stages:

A. Sample spaces in one dimension (1D), for example the probability of obtaining 3 when rolling a die; theoretical probability as the ratio of the number of relevant outcomes to the number of all possible outcomes; experience with the fact that empirical probability values tend to the theoretical value as the number of trials increases.
B. Sample spaces in two dimensions (2D), where all simple events are equiprobable and can be counted and organized into a table, and the probabilities of the compound events can hence be found by inspection, for example, the probability of obtaining at least one 3 when two dice are rolled simultaneously.
C. Sample spaces in 2D, where each dimension has two possibilities whose probabilities are not necessarily equal.

8.3.1 Design for Abstraction

The project team designed a ten-lesson probability unit with the specific aim of giving students opportunities for constructing knowledge related to stages A, B, and C above, and to cooperate in constructing knowledge. The underlying design principles include a didactical hierarchy with occasions for hypothesizing, for facing conflicts, and for arguing (Hadas et al. 2008).

The unit is structured hierarchically in such a way that each stage (and often each task) introduces new elements, which evolve from the previous stages (globally) and tasks (locally). The global aspect of the didactic hierarchy principle can be demonstrated by the appearance of the calculation of probability values as a relevant ratio at each stage of the unit. It is reflected in the tools used for presenting the probability values: In stage A, the students present the probability values of 1D events on a chance bar (a [0,1] number segment). In Stage B, the students deal with situations involving the probability of 2D events such as two dice or a spinner and a die; a table model based on the equiprobability of the simple events represented by the table cells is introduced as a tool for finding such probabilities. In Stage C, the students deal with binomial but not necessarily equiprobable 2D events using an area model for probabilities in the resulting four-event sample space. Stage C is designed on the shoulders of Stages A and B, which are interwoven together with the two dimensions being represented by two orthogonal chance bars generating a square. The unit area of the square represents the probability of the sample space as a whole, while the area of each of the four rectangle-cells in the square represents the probabilities of the four events. This briefly illustrates the didactical hierarchy design principle.

Task 1

1a Joe and Ruth roll two white dice. They decide that Ruth wins if the numbers of dots on the two dice are equal and Joe wins if the numbers are different.

Do you think the game is fair? Explain!

1b The rule of the game is changed: Joe wins if the dice show consecutive numbers. Do you think the game is fair?

1ci How many possible outcomes are there when rolling two dice?

1cii Reconsider your answers to Tasks 1a and 1b: Are the games fair?

1d Suppose Joeand Ruth play with one red die and one white die. Does this change the answers to 1a, 1b and 1c?

Task 2

We again roll two white dice. This time, we observe the difference between the larger number of dots and the smaller number of dots on the two dice. (If the numbers of dots on the two dice are equal, the difference is 0.)

2a Make a hypothesis whether all differences have equal probability. Explain!

2b How many differences are there?

Fig. 8.2 Tasks 1 and 2 of Stage B

We demonstrate the other three design principles using the first two tasks of Stage B. These tasks are presented in Fig. 8.2. The hypothesizing principle is most explicit in Task 2a. Experience in teaching probability led the designers to expect that some students will hypothesize equal probabilities for all differences, and that many of the others are likely to hypothesize that the probability is decreasing and, in particular, that the probability of difference 0 is larger than the probability of difference 1. Making use of this expectation for creating a situation favorable for constructing one of the central knowledge elements in 2D probability, the design continued after Task 2b, in the following lesson, by means of a computer simulation that creates a histogram for the differences; this histogram is likely to generate a conflict between what students expect based on their hypotheses and what the computer simulation shows–hence realizing the conflict principle. This conflict caused by the refutation of their hypothesis by computer experiment is a highly fertile situation for constructing knowledge about the probabilities of events in context of the 2D sample space of two dice. This is a suitable moment to encourage students to argue their reasons for and against the hypothesis and to evaluate the cogency of different, sometimes contradictory arguments, thus engaging also the argumentation principle in support of constructing knowledge. It will be shown below that this specific case closely relates to the hierarchy of knowledge elements produced by the a priori analysis hence demonstrating also the local aspect of the hierarchy principle.

8.3.2 A Priori Analysis

The RBC+C analysis will be exemplified below using data from students' work on the tasks at the beginning of Stage B of the learning unit (see Fig. 8.2). A preliminary knowledge element from Stage A that may become relevant here is E_{PV1} (PV1 stands for *probability value in 1D situations*): When all possible outcomes are known, the theoretical probability value can be computed as the ratio of the number of relevant outcomes to the number of all possible outcomes. We note that this knowledge element is built on and includes knowledge elements such as *event* or *outcome*, *relevant outcome*, and *all possible outcomes*. Obviously, additional knowledge about dice having six faces numbered 1–6 and about differences and fractions are needed to deal with the activity but these are assumed to be generally known to students in grade 8 and we do not list them.

Four basic probability concepts are considered here as the main knowledge elements that underlie the learning design and may be expected to be constructed by the students when working on Tasks 1 and 2: *simple event, compound event, sample space* and *probability value*. The simple event element has two constituent elements, *pair* and *order*. Figure 8.3 presents the knowledge elements as well as the relationships between them. For example, the probability of a 2D compound event is calculated as the ratio between the number of simple events in the compound event and the number of simple events in the sample space. Hence, the probability value element contains the compound event element and the sample space element as constituents, while both the compound event element and the sample space element contain the simple event element as constituent. These knowledge elements will below be defined operationally in such a way that the researcher can tell from students' actions and utterances whether or not they are using these elements. We note that strictly speaking, students are never constructing a knowledge element E_X or building-with a knowledge element E_X; rather, in the constructing action, constructs emerge for the students that (hopefully) correspond at least partly to knowledge elements intended by the designer. This is what we will, for simplicity, refer to as "a

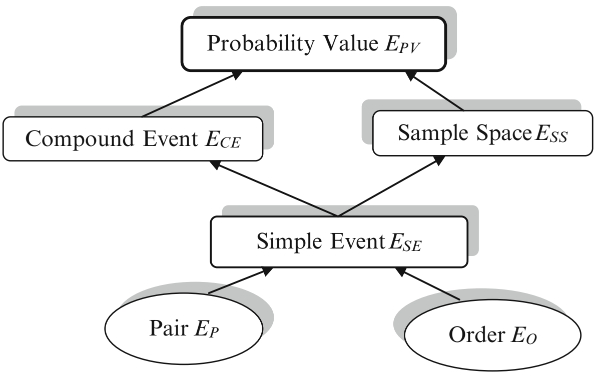

Fig. 8.3 The knowledge elements

student constructed or built-with E_X"; however, we always take this to mean that the student constructed or built-with his or her personal construct corresponding to E_X.

Simple event E_{SE} A 2D simple event E_{SE} is an ordered pair of 1D simple events, for example getting 3 on the first die and 4 on the second die. When tracing the process of constructing the simple event element we shall focus on two constituent elements, Pair E_P and Order E_O. We shall say that students have constructed the pair element when they, by their words or actions, relate to the occurrence of the two 1D simple events as to a single event. We shall say that students have constructed the order element if they relate to pairs like (□,Δ) and (Δ,□) as to two distinct events. The order element can be constructed only if the pair element has been constructed.

Compound event E_{CE} A compound event is a set of simple events, which share a common attribute, for example "the two dice show a difference of 3". We shall say that students have constructed the compound event element, if they organize or list all relevant simple events in a systematic way, with the aim of finding their number, depending on what they consider as a simple event. In particular, if students did not construct the order element, but count all the non-ordered relevant pairs, we shall say that they have constructed the compound event element. This is a reflection of our approach, which considers processes of constructing knowledge from the student's existing perspective rather than from an expert's perspective.

Sample space E_{SS} The sample space is the set of all possible simple events. We shall say that students have constructed the sample space element, if they carry out actions to organize or list all events of the sample space in a systematic way, with the aim of finding their number, depending as above on what they consider as a simple event.

Probability value E_{PV} The probability of a 2D compound event is calculated as the ratio between the number of simple events in the compound event and the number of simple events in the sample space. We shall say that students have constructed the probability value element if they calculate the probability of a compound event as the ratio between the number of simple events in the compound event and the number of simple events in the sample space.

Our list of knowledge elements is, of course, incomplete. On one hand, it is based on the didactical decisions for the design of the probability unit. On the other hand, the aim of the research to analyze knowledge constructing processes, dictates concentrating on knowledge elements that were observed in the students' work. Thus both, the design and research considerations have influenced the list of elements and its organization.

The designers' intention was that dealing with Tasks 1 and 2 may be expected to lead to a need for constructing of E_P and at least partially constructing E_{SS} (for example Task 1c) and E_{CE}. The intention of Task 1d is to motivate students to deal with the dilemma of E_O. Task 2 represents a more sophisticated situation since there are six different differences, each of which is a complex event with a different probability whose calculation requires E_{PV}.

8.3.3 Data Collection and Preliminary Analysis

In all lessons taught in the framework of the project, researchers were present and documented the lesson by means of two video cameras. One camera was directed at a focus group of students (the same group in each lesson) and the second camera at the teacher and the whole class activity. The researchers also took field notes and collected students' completed worksheets and written tests.

In the specific lesson serving as example here, the focus group consisted of three girls, Noam, Rachel and Yael. We consider the focus group while all students in the class were working in small groups as part of the regular class work. We present parts of the analysis of the work of the focus group on Tasks 1 and 2. The preliminary analysis of the data yielded a division into three episodes as used in the next subsection.

8.3.4 Analysis According to the RBC-Model, Including Need and Consolidation

The need for a new construct, its emergence and its consolidation are three interlinked stages that form part of a process of abstraction for this target knowledge. Although these three stages have in the previous section been treated separately for added clarity, they are in practice closely interwoven and are therefore presented here in combination.

8.3.4.1 Episode 1: Constructing E_P

After a brief introduction during which she reminds the students of the notions of frequency and relative frequency, the teacher distributes the worksheets with Tasks 1 and 2. Rachel reads 1a aloud. Noam and Yael immediately agree that the game is not fair, that Ruth's chances are lower, with Noam even saying 1 in 6 and Yael, counting $1+2+3+4+5+6=21$ and adding another 6 arriving at 6 out of 27. This provokes Noam to ask for clarification.

[21] Noam	What are you doing? Eh?
[22] Yael	I did: What are the chances? One one, two two, three three, four four, five five, six six. And he has one two, one three, one four, and two one, two two. No two one and one two are the same thing… one two, one three, one four, five plus 4 is nine, nine plus three is twelve
[23] Rachel	I don't understand what you are doing. What are you doing?
[24] Yael	Because I have to know what is our whole. It's like, what results can be, what results might occur? … It's either (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6) or … (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and then 2.
[25] Noam	Yael, I don't understand at all what you are doing.
[26] Rachel	Yes, me neither.

Yael clearly and repeatedly refers to pairs of numbers [22] as simple events, is already busy enumerating, more or less systematically, all the relevant events for calculating Ruth's and Joe's chances to win [22, 24], and asks explicitly what are all possible simple events [24].

Using the operational definition for E_P given above, we interpret that Yael has constructed a suitable construct for E_P in the context of two dice before [21], and without providing any explicit evidence for the constructing action to the researcher. She is already recognizing and building-with E_P in [22, 24] and beyond. This is the reason why we have freely used the notation (a, b) for pairs from [24] onward, although the parentheses can hardly be heard in the students' utterances.

Using the operational definitions for E_{CE} and E_{SS} we conclude that Yael is in the process of constructing these elements: She is attempting to organize lists of all relevant simple events, with the aim of finding their number, but does not yet do this quite systematically.

We also note that Yael gives a brief indication of being aware of the issue of order (of the elements of a pair) in [22] and, in fact, negates order; she may be in the process of constructing what might be called an "anti-order" construct, which, of course, does not form part of our list of knowledge elements.

There is no parallel evidence for Rachel and Noam; quite the contrary, their questions seem to indicate that they may not yet have constructed a view of events as pairs of numbers (E_P), and they do not appear to understand what, how, and why Yael is counting [21, 23, 25, 26].
Yael seems to realize this and repeats

[27] Yael	Listen, there are some possibilities that 1 will appear: (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and we finished with 1, now 2: (2, 3) (2, 4) (2, 5) (2, 6).
[28] Rachel	O.K., O.K., we understood that, but why are you adding? I don't understand.

We observe that Yael's [27] is more explanatory and more systematic than [24], and interpret this has showing that she has now constructed E_{SS}. Although there is no explicit evidence from Rachel's [28] that she has now constructed E_P, the fact that her question has changed from a vague expression of unease [23, 26] to a precise question about Yael's counting system [28] may indicate this. And indeed, as will be shown in Episode 2 below, Rachel acts as if building-with E_P, while she and Yael continue to discuss the numbers 6, 15 and 21 of simple events for Ruth, Joe and the total, respectively, in [29–57]. During this entire excerpt, events as pairs are mentioned only a single time, and even then by Yael and not by Rachel. Nevertheless, in [64] Rachel provides explicit evidence for having constructed E_P.

[58] Noam	See 'cause you didn't … you did like one side on the die is 3 and the other side is 4 and you did like 3 plus 4 and that's like …
[59] Yael	I didn't do 3 plus 4. I'll tell you exactly what I did.
[60] Noam	No wait a second, second. This is what I understood from what you did
[61] Yael	I'll tell you what I did.

[62] Noam One minute! No! You have to do 3 and 4; it is one possibility, and 4 and 5 is a second possibility, so it is two [possibilities].

[63] Yael What?

[64] Rachel That is what she did; (3, 4) is one possibility and (5, 4) is one possibility.

Rachel [64] explains that this is exactly what Yael did, by repeating Noam's explanation. Here Rachel uses E_P, which she had presumably constructed earlier, to build-with it her explanation. She thus provides evidence that she is already consolidating E_P. This excerpt also allows us to follow in some detail Noam's constructing of E_P. Her transition from "3 plus 4" in [58] to "3 and 4; it is one possibility" in [62], may be interpreted as the moment she begins to see the pair (3, 4) as a unit and hence has constructed E_P. She still expresses this fact rather clumsily, and this illustrates well "the first time the new construct is expressed or used by the learner", which forms part of our definition of constructing and indicates that her construct is still fragile. This fragility has disappeared by 109 where Noam builds-with and thus consolidates E_P when explaining what is the meaning of the difference in Task 2.

[109] Noam If we look on the difference of one die and the other die: (2, 2), then the difference is 0.

In summary, we conclude from the data that the three students constructed and began consolidating E_P. While Yael's process of constructing was quick to the point of being immediate (and hence not observable until after the effect) and Rachel's process of constructing was hidden because she did not express it, Noam's was explicit, even for this very simple construct.

The question arises, in what sense the three students shared the knowledge they have constructed. The knowledge flow started from Yael. The constructs of the two other students emerged first through interacting questions and explanations from Yael to the two others, and then through the explanations of Yael and Rachel to Noam. At this point in time, it seems that E_P has become a shared common basis of knowledge for the group, which they can use together for further constructing. From observations of later activities, we have evidence that E_P was consolidated and remained available as part of the group's shared knowledge. The article by Hershkowitz et al. (2007) presents a more detailed analysis of the sharing aspects of this focus group's constructing processes.

8.3.4.2 Episode 2: Partially Constructing E_{CE} and E_{SS}

The knowledge elements E_{SS}, E_{CE} and E_{PV} for the 2D case as needed in Stage B are all generalizations of the corresponding 1D knowledge elements, which the students met, and presumably constructed, in Stage A. While the generalizations might seem straightforward, they are not necessarily so for students, and may therefore require processes of abstraction. For the three students in our report, the generalization of E_{PV} was indeed automatic and never formed the topic of specific attention. However, the generalization of E_{SS} and E_{CE} was associated with considerable difficulty, at least for Rachel and Noam. As in Episode 1, Noam lagged behind, and we will focus on Rachel and use her process of constructing E_{SS} (she constructed E_{CE} concurrently) and use it to illustrate additional methodological aspects of AiC.

Chronologically, Episode 2 develops partially in parallel to Episode 1. We differentiate between the two episodes in order to clearly separate the constructing of E_P, which is a prerequisite for the constructing of E_{CE} and E_{SS}, and because the constructing process of E_P is quite different in nature from that of E_{CE} and E_{SS}.

As described in Episode 1, while working on Task 1 Yael is counting events [22, 24] and explains that she is trying to find all the possible simple events in the sample space (in her words, “the whole” [24]) in order to calculate the required probabilities. Rachel, who has already constructed E_P, still does not follow why and how Yael is counting [28], and Yael explains again:

[32] Yael What is my whole? This is my whole ‘cause… O.K., look! There are 6 out of… 5+4 is 9, and 3 is 12, and 2 and 1 is 15, and the 6 is…
[33] Rachel 21

Rachel adds the numbers; however, she still wonders about the meaning of the number 21, as we conclude from

[41] Yael I know that I am right. His frequency is 15 out of 21…
[42] Rachel What?

and later
[48] Rachel Why 21?

and

[51] Rachel What is 21? I don’t understand what 21 is! [laughs with embarrassment]
[52] Yael O.K., there are 6 possibilities that we will get the same number on the dice.
[53] Rachel True!
[54] Yael Then 15 plus 6 is 21.
[55] Rachel Ah, O.K.
…
[69] Noam Why 21?
[70] Yael Ufff !!
[71] Rachel Because there are 15 possibilities for him and for her 6 possibilities and 15 plus 6 is 21.
[72] Yael It is the whole, all the possibilities that can occur on the dice, two dice.
[73] Noam Ah, you roll…
[74] Yael Then, 6 divided by 21, it’s her frequency, and 15 divided by 21 it’s his frequency.
[75] Rachel [dictating the answer to Question 1a] No, because the chance of two equal numbers is 6 divided by 21 and the chance of different numbers is 15 divided by 21.

We omitted the segment [56–68] from here since most if it was discussed in Episode 1; in this segment, Rachel speaks only in [64] explaining to Yael what Noam meant – and this has to do with E_P, rather than with the constructs of interest presently.

In [69]–[75], Rachel and Yael, together and in similarly lucid language, explain the meaning of the number 21 to Noam. Based on this, we infer that Rachel's construct for E_{SS} emerged previously, and given what she asked in [51], this can only have happened in [52]–[55]. In other words, we interpret her "True!" in [53] and her "O.K." in [55] as expressions of her constructing action. This interpretation is strengthened by the fact that Rachel does not shy away from admitting her lack of understanding ([23], [26], [28], [42], [48], [51]). The evidence of Rachel's constructing E_{CE} is similar: It occurred no later than [52]–[55] for the same reasons as E_{SS}. However, it is more difficult to locate precisely in a specific segment of the transcript because Rachel asked no questions expressing its lack. We can therefore locate the end of Rachel's process of constructing E_{SS} in [55] but find it difficult to locate its beginning. There are three distinct possibilities, between which we cannot decide on the basis of the empirical evidence. In other research studies, we have found that all three of these possibilities actually do occur. One possibility is that constructing E_{CE} precedes constructing E_{SS}, i.e. that Rachel's process of constructing E_{CE} has ended before [51]. Another is that the two processes partially overlap, with constructing E_{SS} beginning after constructing E_{CE} begins but before it ends, and constructing E_{CE} ending before constructing E_{SS} ends. The third one, and the one that seems to us most likely, is that constructing E_{CE} occurs within constructing E_{SS}, in other words that one constructing action is nested in another one. In some cases, such nesting occurs by necessity because the construct nested inside the other one is unavailable to the students and necessary for constructing the other one (see e.g., Dreyfus et al. 2001). This is not the case here; it is conceivable and has happened in our research project that students construct E_{SS} before they construct E_{CE}, that they construct E_{CE} before they construct E_{SS}, or that the two constructing actions partially overlap.

When explaining E_{SS} [71, 75], Rachel refers to the compound events "Ruth wins" and "Joe wins", which have 6 and 15 elements respectively, i.e. she recognizes these compound events as relevant for the sample space; she also builds-with them the sample space, which consists of the union of these two compound events. She thus recognizes and builds-with her construct for compound event. Similarly but less explicitly, she recognizes and builds-with her construct for simple event, without which the compound events do not exist. We stress again that the R- and B-actions nested in C_{SS} are not R- and B-actions with the SS construct currently being constructed but rather R- and B-actions with the previous constructs: R_P, B_P, R_{CE}, and B_{CE}.

We are, of course, aware that these actions are all implicit and hence interpreted into [52]–[55]; however, given the lucid explicit explanation Rachel gives to Noam in [71, 75], we feel justified to make this interpretation. Moreover, since we consider Rachel's constructing actions of E_{CE} and E_{SS} as terminated in [55], her explanation to Noam is an opportunity for her to formulate the constructs in her own words, linguistically correct and elaborate, which we consider as her beginning to consolidate the constructs E_{CE} and E_{SS}. Searching forward through Rachel's activity on further tasks in the next lessons, for example a task with two spinners, we find further evidence for consolidation of these constructs. We are also able to observe how the constructs for E_P, E_{CE} and E_{SS} become increasingly self-evident for Rachel and how she uses them with increasing flexibility, for example in other physical contexts. A brief example will be given in Episode 3. However, giving more detailed examples for this would require lots of additional space without adding much more to the explanation of our methodology.

If, on the other hand, we search backward through the transcript for expressions by Rachel of a need for the SS construct, we find a clear need for it in Rachel's insistent and repeated questions in [42], [48] and [51]. We note that the questions become more explicit and clearer as her thinking proceeds. Even in its clearest form, however, Rachel's question asks what is the meaning of 21 rather than what is a sample space. This is only natural since only in very rare cases can learners be expected to ask for something they do not yet know about. This is why we interpret questions of the type Rachel asked as a clear expression of need for a new construct.

Finally, we observe that from an expert's point of view, Rachel's (and Yael's) results are wrong. In fact, there are 36 possible events in the sample space for two cubes, not only 21. They do not take into account that (1, 2) and (2, 1) are different events. This is, of course, due to the fact that Yael and Rachel have constructed and work with a construct for simple event that includes a construct for Pair (E_P) but does not include a construct for Order (E_O). Their construct for simple event is therefore only partially correct and since the simple event construct is a crucial constituent of the compound event construct and of the sample space construct, Rachel's (and Yael's) construct for sample space is also only partially correct. Ron et al. (2010) give a detailed treatment of partially correct constructs, together with a methodology for identifying them.

8.3.4.3 Episode 3: The differences' Task (Task 2)

The students in our focus group quickly agree that the game in Question 1b (Fig. 8.2) is not fair either, because Ruth has six winning possibilities and Joe has only 5, that the answer to Question 1ci is 21, and that playing with coloured dice does not make a difference. Yael and Rachel give these answers. Noam reads the questions and occasionally asks for clarification. This changes when they start thinking about Question 2. From the start, Noam contributes. She reads the question [107] and is the first to build-with E_P the difference between a pair of numbers on the two dice [109, see above]. Possibly, the fact that she is now sharing the E_P construct with her peers gives her some self-confidence and motivation.

When Yael points out that "difference 1" is a compound event [115] and uses this to reformulate the question more concretely [119], Noam immediately produces an answer [120–126].

[115] Yael	If you get (1, 2), the difference is 1 and if you get (2, 3) the difference is also 1.
[119] Yael	Are the probabilities of 1 and 4 equal?
[120] Noam	I understand. No!
[121] Yael	Are the probabilities of 1, 2, 3, 4 equal?
[122] Noam	We understand! NO!
[123] Yael	How do you know? Did you calculate it?
[124] Noam	No! Because they are not equal.
[125] Yael	How do you know?
[126] Noam	If we say difference of 4 … or we say difference of 5 there is only one.
[127] Yael	One moment! Let's take 1, then we have (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) and the same number (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6).
[128] Rachel	What's the connection?
[129] Yael	Well, I check the probability of 0 and the probability of 1.

This is a case where the designer intended no new constructs but R- and B-actions were required to achieve a local goal. While Yael and Rachel [127–129] behave accordingly, building-with their existing constructs (E_{CE}, E_{SS}) to count how many pairs fit each difference in order to compute probabilities (E_{PV}), Noam uses only her compound event construct in order to construct a new strategy to achieve the same goal, namely comparing the frequencies of the compound events. She concludes that different frequencies mean different probabilities, without any need for calculating the probability value itself. We consider this strategy to be a new construct that has emerged for Noam. It is an unexpected and somewhat non-conventional construct, which the researchers did not include in the a priori analysis. We have met other such unconventional unexpected student constructs in other research studies (Ron et al. 2011).

In other cases, there are good reasons for the researchers to modify or refine the a priori analysis in order to include unexpected constructs; one example for this has been implicitly mentioned above: Yael (and many other students') construct for simple event did not take the order element into account. We therefore modified our original operational definition for the compound event element E_{CE} by adding, "if students did not construct the order element, but count all the non-ordered relevant pairs, we shall say that they have constructed the compound event element".

We note that when the three students summarize and write down their conclusions, which are identical, the difference in the approaches taken by Rachel and Noam leads to an argument between them that reaches far beyond the constructs under consideration here into constructs related to what it takes to justify a claim: For Noam, examples are sufficient to conclude that different differences have different probabilities. Rachel does not agree; she thinks that examples are not sufficient:

[143] Noam	No, let's give examples.
[144] Rachel	It decreases, you understand?

[145] Noam	Come on, let's give simple examples, O.K.? Let's give examples!
[146] Rachel	No! Because the difference between 0 and 1…
[147] Noam	It IS an example, you dupe!
[148] Rachel	[writes] Because as the difference grows, the probability becomes smaller.

In summary, we note that each student constructed her individual knowledge in her own way and time. While Yael and Rachel reached a stage where they shared their knowledge with respect to all constructs under discussion here, we have no evidence that Noam has constructs corresponding to E_{SS} and E_{PV}, and the strategy she constructed in Episode 3 may even point to the fact that she was lacking a construct for E_{SS}. However, this strategy constitutes an unexpected construct for Noam that could potentially be useful to build-with in other situation. All three students shared their lack of knowledge concerning the element E_O and hence all their other constructs were partial.

8.3.5 Additional Methodological Comments

There are a number of additional aspects of our methodology, which we were not able to demonstrate in this chapter because their presentation would require information about additional research studies and additional data and therefore be quite onerous in terms of additional space. We will therefore only comment on these aspects, referring the reader to other publications for more detailed information.

8.3.5.1 Knowledge Construction and Social Interaction

The study used for illustrative purposes above exhibits important aspects of the role of social context in knowledge constructing processes, including the role of the flow of knowledge between individuals during the interaction, which have been discussed in more depth by Hershkowitz et al. (2007). However, our previous study dealing with the relationship between peer interaction and the construction of knowledge (Dreyfus et al. 2001) already proposed a methodology using two parallel analyses of the protocols of the work of student pairs, an analysis of the epistemic actions of abstraction as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction. We also found an excellent fit between episodes defined for the epistemic actions analysis and episodes defined for the peer interaction analysis.

8.3.5.2 Tools as Contextual Elements in Knowledge Construction

Artifacts such as manipulative or computerized learnware are another element of the context that has been shown to possibly have a crucial role in the construction of knowledge. For example, following the students' work on Task 2 above and a computer simulation of the situation in Task 2 that led to conflict, teachers were asked to present the sample space of two dice as a six by six table of all simple events. This helped many students construct E_O.

In a more substantial study on the influence of tools, Weiss (2011) has analyzed the role of an analogical model in knowledge construction during model-based tasks. The tasks were in the area of fraction comparison and the analogical model was the tower of bars. From the methodological point of view, the study by Weiss required not only an RBC analysis but also an analysis based on the Emergent Models approach of Gravemeijer (1999), which is rooted in the Realistic Mathematics Education philosophy of the Freudenthal tradition and makes use of vertical mathematization. This double analysis has led to the identification of deep theoretical relationships between AiC and RME.

In a different study, Kidron and Dreyfus (2010) describe how instrumentation led to cognitive constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine, giving the CAS a major influence on interactions between different parallel constructing actions.

8.3.5.3 Revision of the Instructional Design on the Basis of RBC Analysis

Finally, the micro-analytic nature of the research methodology described in this chapter allows the researcher to observe constructing actions very closely. One of the related practical considerations is that this allows educators to identify problems in the micro-design of an educational intervention. Kouropatov and Dreyfus (2014) briefly describe an example of a revision of the micro-design of an activity in the realm of the integral as accumulation.

8.4 The Relationship of Theory and Methodology in AiC

The flow of theoretical and methodological paradigms that determine the frames for research in science and mathematics learning recently became richer, more divergent and more sophisticated. In addition, it seems that more than in the past, researchers today do not feel obliged to and/or satisfied with sticking to one methodological paradigm. Research trends in our area are nowadays characterized by flexibility and creativity in combining research methods and methodological tools that fit the researchers' theoretical framework and meet their goals and needs to explain and answer some 'big questions' emerging from their explorations.

In this section we discuss issues concerning the contour lines between the theoretical framework of AiC and the methods and methodological tools within AiC research. Hershkowitz (2009) argues that "these boundaries (the above mentioned contour lines) are flexible and even a bit vague in the sense that the same scheme or model may serve as a theoretical framework in one piece of research, as a methodological tool in a second one, and as both of them in a third piece of research" (p. 273). There are many examples for this in AiC. One is the notion of nesting, which was originally meant for R-actions being nested in B-actions and B-actions in C-actions, but empirical results showed that C-actions may be nested in higher level C-actions and this led us to expand both, our theoretical conception of nesting and our methodology for describing it (Dreyfus et al. 2001). A second example is a refined classification of B-actions into subcategories, distinguishing, for example, actions that provide an orientation in the student's problem space from actions that aim at solving a problem; this again led to an expansion of both, our theoretical conception of B-actions and our methodology for identifying them (Dreyfus and Kidron 2006). A third example is provided by the notion of PaCCs (Ron et al. 2010); introducing this notion led us to systematically carry out a priori analyses (a methodological tool), which in turn enriched the theoretical framework, as well as required a different method of analyzing the epistemic actions in order to identify PaCCs. Instead of providing more brief pointers to such local examples, we will in the remainder of this section discuss in some detail the most substantial and deepest case of intertwined growth of theory and methodology, which is related to the role of the social context in the construction of knowledge. This will also give us an opportunity to briefly review the history of AiC and conclude the paper with an outlook.

At the beginning of the AiC research work, we came up with a first hypothesis for the model using both theoretical considerations and the analysis of considerable amounts of data in parallel. In this undertaking, we were led by the need to give theoretical expression to the specific characteristics of our data, which pointed to constructing of knowledge by means of mathematical thinking. In the process, we took into account and incorporated elements of existing theories. As we described in the theory section, abstracting was taken as human activity of mathematization, specifically vertical reorganization of previous mathematical constructs, interweaving them into one process of mathematical thinking with the purpose of constructing a new mathematical construct—Abstraction in Context has emerged.

At that stage, the researchers found themselves in a circular situation where theory stemmed from the analysis of data, and the analyzed data and its interpretation served as evidence for validating the theory. We were quite aware of this situation and explained: "This definition (of abstraction) is a result of the dialectical bottom-up approach . . . a product of our oscillations between theoretical perspective on abstraction and experimental observations of students' actions, actions we judged to be evidence of abstraction" (Hershkowitz et al. 2001, p. 202). It is clear that for analyzing the above actions, the researchers had to use some basic methodologies, which fit protocol analyses of an individual and the more complicated analysis of cognitive and interactive work within dyads and groups. The three epistemic actions, recognizing, building-with and constructing, and the dynamically nested relationships between them were hypothesized as the main building blocks of the model. These building blocks and the dynamic relationships among them express the vertical reorganization within the theory, and at the same time are validated by being used as the lens and compass to describe and interpret the data analyses themselves. Such a situation held for the first steps towards the validation of the model as a theoretical framework and as a methodological tool as well.

In these first steps, the research context and the theory itself were expanded. As an example let's follow the development of the socio-interactive dimension: In our first AiC research we interviewed individual grade 9 students. Along the interview the student has an interaction with the interviewer only (Hershkowitz et al. 2001). In our second AiC research (Dreyfus et al. 2001) we interviewed dyads of students; the interaction between students turned out to be a main factor in the process of knowledge construction and therefore took its role in the theory, specifically as part of the context. The methodology changed as well: The interviewer became an observer, and a graphic tool for demonstrating the RBC flow together with the interaction flow was invented.

In these as well as the following stages of development, the RBC + C model has been validated, both as a theoretical framework and as a methodological tool, in various social settings and learning environments. The settings considered include (teacher-led) classroom discussion, small-group problem-solving processes (see the examples in Sect. 8.3), tutoring situations and individual activities (e.g., introspective self-reports of single learners; see Kidron 2008). The age range of the learners extends from elementary school to adult experts and the longitudinal dimension varied. And, indeed, research made it clear that the RBC + C model is an appropriate tool/theory/theoretical tool/methodology to describe abstraction and provide insight into processes of abstraction in a wide range of situations of abstraction and consolidation on a medium-term timescale, where consolidation is a process by which the construct becomes progressively more self-evident, the student's awareness of the construct increases and the use of the construct becomes more flexible (Dreyfus et al. 2006; Hershkowitz et al. 2007). More than 10 years of research and more than 40 research publications, contributed by more than a few people, separate the 'birth' of the AiC framework from today's RBC + C model, as an empirically based theoretical framework.

We give a small number of illustrative examples for the linked expansion of theory and methodology from more recent research. Tabach et al. (2006) present an example of knowledge constructing within the context of peer learning in a working classroom. It shows how the design of the tasks and the computerized tools available to the students afford the constructing of conceptual knowledge (the phenomenon of exponential growth and variation, as it is expressed in its numerical and graphical representations). The researchers traced the constructing of knowledge through a series of dyadic sessions for several months in a classroom environment, and analyzed three sessions with intervals of a few months between them. The analysis showed that knowledge is constructed cumulatively, each activity allowing for consolidating previous constructs. This pattern indicates the nature of the processes involved in creating a new abstract entity: knowledge constructing and consolidating are dialectical processes, developing over time, even over time intervals that seemed to break the continuity. Hence, the main function of the RBC + C model in this research was to serve as a methodological tool to illustrate construction and consolidation processes. In addition, the data of this specific study served, in turn, to let a new methodological characteristic emerge: The RBC + C model is an appropriate tool for analyzing construction of knowledge over long intervals of time.

Wood et al. (2006) examined "the relationship between the patterns of interaction that exist in the classroom and children's expressed mathematical thinking" (p. 228) in classes from different cultures, and for this aim the RBC-model served as the "conceptual framework employed to examine the quality of students' expressed thinking" (p. 225). We think that the term conceptual framework, when applied to a certain model, expresses the flexibility with which this model may be applied as a framework for both theory and methodology. And indeed in Wood et al.'s research, it seems that on the one hand the authors believe theoretically that the model with its three epistemic actions expresses quite accurately the level of mathematical thinking that children have; on the other hand they used the model as a methodological tool for analyzing the protocols of the class members and identifying the levels of thinking expressed by class members. They accumulated these data for the purpose of quantitative analyses of the levels of thinking expressed by the class members in discussions in the different classrooms. This research shows some maturation of the model as a theoretical framework and as a methodology. The authors needed two conceptual frameworks in their study and the model is one of them. The model is not any more the focus of the study. It allows the researchers to determine levels of thinking that are available for inspection in the classroom.

Finally, Dooley's classroom research (2007) concerns what she calls collective abstraction processes, which emerged in one lesson, mostly during the last phase of the lesson, where a whole-class interaction took place. The researcher's aim was to show how the class community, as one entity, reaches 'sophisticated constructs'. She explained: "One pupil's 'recognizing' led to 'building with' by another and to 'constructing' of new ideas and strategies by others" (p. 1658). Again, this is a situation where the researcher uses the RBC-model as a methodological tool to explore the existence and nature of collective abstraction, and by doing this she confirms the RBC-model as a conceptual framework for this type of collective abstraction as well.

The RBC+C model may exemplify the flexible contour lines between a model as theory and a model as methodological tool: the model aims to serve as a framework for describing, analyzing and interpreting a human mental activity. The model is appropriate for exploring individual student mental activity as well as for exploring collective mental activity that is distributed in a group or a classroom among different individuals.

The model with its three epistemic actions, has a very general nature, general in the sense that it can be used in many and varied contexts. The nested relationships among the epistemic actions of the RBC+C model are global and the three actions of the model are observable and can be identified. Therefore the model lends itself easily to be adapted and to contribute to research in many different contexts of constructing abstract knowledge.

The research by Dooley raises questions that are at the heart of classroom research more generally. For example, in a study that focuses on "the travel of ideas" in the classroom, Saxe et al. (2009) claim that "developing methods to understand the travel of ideas is foundational to understanding learning in classroom communities" (p. 221), where the individuals in the classroom have a main role in the travel of ideas. Researchers who plan to observe and analyze in detail micro-processes of constructing knowledge, or argumentation, or any other learning processes which are related to the classroom activity in a given context, along a time segment that may range from minutes to weeks, and do not want to ignore the researched phenomena as it is expressed by the individuals, face great difficulties: The observation and documentation processes are complicated, data are messy and massive and there are no systematic clear-cut methodologies for analyzing them (Schoenfeld 1992).

It seems that there is no theoretical problem in conducting such classroom research, which analyses in parallel the knowledge-constructing processes in the different social settings, which are formed in a natural way in the classroom, including the paths of such processes within individuals and among them, and then interweaving the analyses together. This seems the optimal way for gaining insight into processes of constructing knowledge in the everyday classroom. But will the 'heavy battery' needed for documenting all of the above not affect the natural learning environment in the classroom? And what about creating the methodologies for analyzing the huge volume of the accumulated data needed for interweaving the findings together?

Looking back on our 'research journey' with AiC, we discern a trend from investigating an individual learner or dyad with an interviewer in a laboratory setting via investigating small focus groups in a working classroom, to investigating students' processes of constructing knowledge, and shifts of the constructed knowledge in a working mathematical classroom. The first phase served to develop the AiC theory and the RBC+C model, whereby we used the RBC+C model in two parallel roles: as a methodological tool for analyzing the data and for validating the theory as explained in the concluding section of this chapter. In the second phase, we applied the RBC+C model for analyzing students' abstraction processes as they worked in a group in a working classroom. In the third and current phase, we aim to develop a methodology to coordinate analyses of the individual, the group and the classroom collective in a working mathematical classroom (Tabach et al. 2014; Hershkowitz et al. 2014). For this purpose, we combine AiC with another theoretical framework, Documenting Collective Activity (Rasmussen and Stephan 2008), which serves to analyze whole class discussions. We hope that observing different kinds of interactions and learning settings in the classroom in parallel on one hand, and overcoming the methodological problems of analyzing, interpreting and interweaving the findings of the different research settings on the other, may give researchers a coherent and meaningful insight into natural processes of learning activities (in the widest sense) in the classroom.

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Part VI
Networking of Theories

Chapter 9
Advancing Research by Means of the Networking of Theories

Ivy Kidron and Angelika Bikner-Ahsbahs

Abstract Networking different theories is a rather new and promising way of doing research. This chapter presents the concept of the networking of different theories and its methodology, including networking strategies like research heuristics and cross-methodologies. The variety of networking is outlined by illustrating examples, and methodological reflections on the difficulties and benefits that accompany the networking are described.

Keywords Networking of theories • Networking strategies

9.1 Introduction

Recent research in mathematics education aims to understand how theories can be connected successfully while respecting their underlying conceptual and methodological assumptions, a process called "networking of theories". This process also demonstrates that taking into account the diversity of theories in mathematics education permits a better grasp of the complexity of learning and teaching processes (Bikner-Ahsbahs and Prediger 2010). Networking of theories not only aims at clarifying the notion of theory and working with different theories: it is essentially a methodological approach for theoretical and empirical research that connects different theories to broaden and deepen insight into problems. Research on networking informs methodological principles of how different theories can be used and what kind of benefit can be obtained by the use of different theories.

I. Kidron (✉)
Department of Applied Mathematics, Jerusalem College of Technology,
Havaad Haleumi Street, 21 Jerusalem 91160, Israel
e-mail: ivy@jct.ac.il

A. Bikner-Ahsbahs
Faculty 3 of Mathematics and Computer Science, University of Bremen,
Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de

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9.1.1 The Evolution of Networking

In their introduction paper to the proceedings of the theory group at CERME 4, Artigue et al. (2005) wrote that "the central term that emerged from the working group was *networking*" (p. 1242). They added that

> as a research community, we need to be aware that discussion between researchers from different research communities is insufficient to achieve networking. Collaboration between teams using different theories with different underlying assumptions is called for in order to identify the issues and the questions. (p. 1242)

Networking has become the topic of a working group at CERME5 (Arzarello et al. 2007), CERME6 (Prediger et al. 2009) and CERME7 (Kidron et al. 2011b) leading to several publications (see Radford 2008; Sriraman and English 2010; Prediger et al. 2008; Kidron 2008). Networking of theories was also discussed at a Research Forum at PME 34 (Bikner-Ahsbahs et al. 2010), at the colloquium in honor of Artigue (Kynigos 2012; Kidron 2012) as well as in ICMI 12 (Bikner-Ahsbahs et al. 2012; Kidron and Monaghan 2012).

Considering the evolution of our own experience in networking theories we point out that this chapter is the fruit of our personal involvement in the networking enterprise. In the next sections, this personal involvement and the way we collaborate between teams of the Networking Theories Group (cf. Bikner-Ahsbahs et al. 2014) will be explained as part of the methodology.

9.1.2 Why Networking?

We may well ask about the reasons for dealing with the complementary influence of different theoretical approaches on the research process in mathematics education. Teaching and learning processes and their environmental conditions and influences are at the center of interest in mathematics education. Due to their complexity, we might need different types of theories (Boero et al. 2002) and ways for approaching these processes in research. But different theoretical frameworks might provide different insights for instance into the description of processes that accompany the emergence of new mathematical knowledge structures. Hence, it is important to know how these different results can be linked. Sometimes data from empirical research are difficult to discuss and interpret within a single theoretical frame (Arzarello and Olivero 2006). In this case alien theories might offer a complementary way of analyses. The networking of theories broadens this view in regarding the diversity of theories as providing a potential to deepen our understanding of, for example, teaching and learning processes (Prediger et al. 2008). Radford (2008) also points to the rapid contemporary growth of forms of communication, increasing international scientific cooperation and that "such [networking of theories] efforts may reflect direct or indirect actions to cope with some of the needs that were brought to the fore by the new educational, political, and economical structures of the European Union and its institutionalizing forms of academic research" (comment by Radford in Bikner-Ahsbahs et al. 2010, p. 169).

9.2 Language for Networking

9.2.1 The Semiosphere

Radford (2008) elaborates a meta-language for networking. Referring to Lotman (1990) he describes the idea of *semiosphere* as "an uneven multi-cultural space of meaning-making and understanding generated by individuals as they come to know and interact with each other" (Radford 2008, p. 318). In the *semiosphere*, he considers a theory be a dynamic interrelated triplet (P, M, Q) formed by a system of theoretical principles (P), methodologies (M), and a set of paradigmatic research questions (Q). The fruitfulness of strategies for networking depends to an important extent on how "close" or "far" the networked theories are located in the semiosphere. Referring to Lotman (1990, p. 125), Radford points out that "one of the striking characteristics of the semiosphere is its heterogeneity" (Radford 2008, p. 318). The semiosphere is a multi-cultural space of theory cultures that is dynamically changing:

> Theory cultures constantly produce, re-produce and develop their identities, but at the same time they establish boundaries that separate their cultures from the others and immunize their cores. However, the boundary is also the place of exchange between the cultures. As Lotman (1990) stated, creative ideas normally are not born in the centre but in the periphery, at the boundaries of the cultures. Networking crosses these boundaries and therefore is a way of renewing theories in different ways. The semiosphere's main function is providing possibilities for dialogue, thus creating connections that are beneficial in different ways, such as deepening the identity of a theory, integrating different theories into a new one or just locally, or creating new kinds of research questions. According to this background, research about the networking of theories means investigating the theories within the semiosphere. In this way, sources and limits for the dialogue are uncovered through common research of different researchers representing their theory cultures. Dialogue in this sense links theories. (Bikner-Ahsbahs et al. 2010, p. 146).

These links can be made by building relationships between the principles, methodologies, and paradigmatic questions of the different theories involved into the specific networking study.

9.2.2 The Essence of Networking and Its Limits

In the paper by Radford (2008) we read that the conceptualization of theories in terms of triplets can also shed some light on the question of the limits of networking theories. Radford points out that although connections are always possible, there is nonetheless a limit to what can be connected. This limit is determined by the goal of the connection, but also by the specificities of the components (P, M, Q) of the theories that are being connected. This limit has to do with the boundary of each theory under consideration. We would like to use Artigue's word *decentration* for describing the effort that is needed when, working collaboratively (Kidron et al. 2008), we in the Networking Theories Group tried to understand our respective didactical cultures, to identify interesting similarities and complementarities between our perspectives, and boundary concepts that could support connections. As noted by Artigue, this effort requires from each of us a costly effort of decentration. She also added that the cost of this effort provides evidence for the strength of the coherences underlying our respective didactical cultures. Artigue's view of *decentration* while retaining the specificity of each theoretical framework with its basic assumptions is the essence of networking. This *decentration* consists of the ability to better understand the limits of our respective tools and what could be offered by networking them in ways that would not destroy their internal coherence.

9.3 Methodological Considerations

A methodology of networking consists of two main parts: the process of the networking of theories as it is, for example, implemented in empirical research, and reflections about the networking process concerning its benefit, limits, difficulties, methodological potential etc. In the first part, networking methods and techniques are invested to undertake this networking process depending on the research goal. This often is done by a cross-methodology that involves networking strategies.

9.3.1 Networking Strategies: The Spectrum of Networking Theories

Not every case of networking represents an effort towards integrating theories. Prediger et al. (2008, p. 170) conducted a case study based on empirical examples and thus gained a landscape of networking strategies that are linearly ordered according to their degree of integration (Fig. 9.1). Networking strategies are heuristics that aim at building relations between different theories and thus advancing

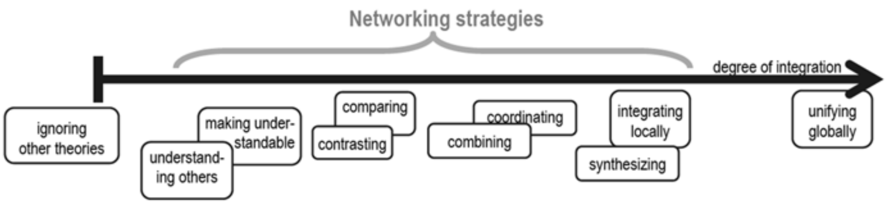

Fig. 9.1 A landscape for connecting theoretical approaches (Prediger et al. 2008, p. 170)

them in the direction of integration while at the same time respecting their identities. This is done by the four pairs of strategies in between the two ideal poles of ignoring all but one's own theory and the attempt to unify all theories (Fig. 9.1). The networking approach does not regard these two poles as useful; rather it acknowledges the diversity of theories within mathematics education as a rich resource for the development of theories in the community of mathematics education.

When working with the networking of theories, researchers must *understand* the alien theories involved and communicate their own ones to *help* colleagues *understand* their principles, methodologies and paradigmatic question (Fig. 9.1). On this basis, similarities and differences by *comparing* and *contrasting* can be identified. *Combining* is done when theories are juxtaposed leading for example to complementary views. By *coordinating*, the connection between theories becomes tighter while common frameworks and methodologies for research can be built. Finally *local integration* and *synthesizing* aim at connecting at least part of the theories on the level of methodologies or principles (for an overview see Prediger et al. 2008; and Bikner-Ahsbahs and Prediger 2010). Bikner-Ahsbahs et al. (2010) briefly describe these strategies in the following way:

> The first strategy pair in the landscape above describes that mutual understanding of theories is necessary when researchers start to practice networking; the second pair focuses on strategies of comparison; the third pair grasps the step that has to be gone towards other theories when linking them; and the fourth describes the balance of reducing theories by integrating at least parts of one theory into another one and building new theories that subsume others. Even if researchers want to integrate theoretical parts only locally into a new theoretical view they have to deeply understand the other theories before using the strategies of comparing, contrasting, coordinating and combining them in the course of integration. (p. 147)

Especially the notion of the fourth pair of strategies has developed over time. As a result of research on networking, the meaning of local integration is much more widely understood today than at the beginning. Local integrations may appear if new concepts such as the Epistemological Gap (Arzarello et al. 2009a) or the General Epistemic Need (Kidron et al. 2010, 2011a) at the boundary of theories are integrated into different theories.

9.3.2 Cross-Methodologies for Networking

We use the term cross-methodology for special techniques and methods employed to enhance and enable networking, such as cross-experimentation between research teams and cross-case analysis in which material gained in one theory team is experimented with or analyzed by another theoretical view. This methodology allows respecting the identity of each theory while at the same time links are explored. How far this methodology drives the networking process towards integration is deeply dependent on the goal and the theories involved.

A complete cross–methodology consists of five subsequent stages requiring the involvement of networking strategies towards local integration. At a first stage, the researchers decide for instance about the common data which will be analyzed or what kind of task is developed. Secondly, each team processes separately on the decision made for example by analyzing the data with its own theoretical lens. Thirdly, the results are exchanged and a discussion between the teams follows this process. Fourth, the teams rework their analyses, task design etc. and complementary, overlapping or contradicting insights gained in this process are considered. In the final stage the researchers' new results are exchanged again leading to a consensus about the outcomes or insights which have been gained.

9.4 Different Cases of Networking

We differentiate between different kinds of interest in the effort of networking theories. In some cases, the goal is to investigate the complementary insights that are offered when we analyze given data with different theories (Kidron 2008). In some other cases, the researchers start with an empirical phenomenon with the aim of developing their understanding of it better by means of connecting two or more different perspectives (Arzarello et al. 2009a). In some further cases, the aim of networking is to satisfy the need for an enlarged framework in relation to some new domain of research (Lagrange and Monaghan 2010). In such cases, each theoretical tool turns out to be insufficient to properly analyze the data. In other cases, the interest in the rich diversity of theories is to explore the insights offered by each theory to the others and at the same time also to explore the limits of such an effort (Kidron et al. 2008).

In the second chapter of this part, the readers will be offered a detailed description of a case of networking in which both authors have been intensively involved. In the next subsections we present several networking cases which show how networking as a methodology may vary depending on the respective aims in the effort of networking.

Case 1: Networking Two Theoretical Approaches Enriches Research
The goal of Arzarello et al. (2009b) paper is to show how networking different theories can help researchers in entering more deeply into their research questions. More precisely, the authors illustrate the limits of two theoretical approaches when used alone to analyze a classroom teaching situation, and the benefits of networking. As a result, data analysis and the understanding of learning processes are strongly enriched. The main question faced in the authors' research concerns how mathematical knowledge about the growth of the exponential function is achieved in a specific socially-supported learning process environment. The authors describe the starting point of their networking process as follows:

> The same teacher-student interaction is analysed from two theoretical perspectives that on the surface seem to be in conflict: the *interest-dense situation* [Bikner-Ahsbahs 2003, 2005, 2006] and the *semiotic bundle* analysis [Arzarello 2006]. (…) Using the former, it appears that the thought process of a student is disturbed by the social interaction with the teacher. However, no disturbances appear using the latter. (emphases as in the original, p. 1545)

The authors demonstrated that through "adding an epistemological perspective this conflict was cleared away" (Arzarello et al. 2009a, p. 1545) since the new concept of the epistemological gap provided a common view and thus deepened the insight from both theoretical perspectives. This case is an example of local integration, a networking strategy in the landscape offered in Fig. 9.1, which appeared, as a new concept could be integrated into the two theoretical lenses. (cf. Arzarello et al. 2009a, p. 1545)

Case 2: Three Theories and Their Complementary Role
Coordinating and combining as another strategy in the landscape in Fig. 9.1 is used by Kidron (2008), who investigates the contributions of three theoretical frameworks to a research process and the complementary role played by each. Her research process addresses the conceptualization of the notion of limit by means of the discrete continuous interplay. Kidron discerns that the different theoretical approaches intertwine. Moreover, she realizes that the research study demanded the contribution of more than one theoretical approach to the research process and that the differences between the frameworks could serve as a basis for complementarities. Indeed, concerns about students' cognition might be expressed in different ways when different frameworks are employed. In Kidron's paper, different ways to express concerns about cognition are presented: "cognitive difficulties that are inherent to the epistemological nature of the mathematical domain" (p. 198), inner details of the learner's epistemic actions in their cognitive processes, instrumental mediation and its influence on the learner's cognitive processes. Each way highlights a specific aspect that relates to cognitive processes. The different ways complement each other.

Case 3: Clarifying the Role of Concepts in Theories Through Networking
The aim of networking in this case is different than the aims in the two previous cases. In the study by Kidron et al. (2008), the goal was not to investigate the kind of complementarity that could result from studying the same data from different theoretical perspectives.

> The analysis presented in this paper constitutes a theoretical attempt at comparison of three theoretical frameworks: the theory of didactic situations (TDS) (Brousseau 1997), the nested epistemic actions (RBC+C) model for abstraction in context (AiC) (Schwarz et al. 2008 [the right year is 2009]), and the theory of interest-dense situations (Bikner-Ahsbahs 2003 [2005]). The aim of the present paper is to compare, combine and contrast these three theoretical approaches. We provide a concrete example in which we observe how networking permits to deepen the analysis of a given situation by a combined use of the three different theoretical frameworks. As an example to talk about networking we decide to exhibit, compare and contrast how social interactions, a phenomenon which is more and more considered as an essential dimension of mathematics learning processes, are taken into account by these different theoretical frameworks. (Kidron et al. 2008, p. 248)

The authors identified not only connections and contrasts between the three frameworks but also the additional insights which each of these frameworks can provide to each of the others. An interesting point is that the different views the three theories have in relation to social interactions force the authors to reconsider the theories in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks. This interesting point is well explained by Radford in his commentary (Bikner-Ahsbahs et al. 2010):

> The semantic value of a theoretical term (e.g. *social interaction*) in a theory results from its position in the main web of dynamic interconnections that characterize the theory as a whole. There is hence something positional or hierarchical about theoretical constructs that makes them impossible to be extracted from the whole, contrary to, e.g. the unproblematic way we extract a weed from the grass. (p. 169)

Case 4: The ReMath Example of a Multi-Design Project
Another case of networking is offered by the ReMath project (Artigue et al. 2009) starting from a given set of theories and aiming at designing a theoretically integrated development of digital artifacts. This is done by using concrete empirical research to develop conceptual and methodological tools for coordinating and combining theoretical approaches. The TELMA (Artigue 2009, p. 494) and the ReMath project (Artigue et al. 2009) illustrate how a productive "dialogue between theories" can be established through the development of appropriate methodologies.

> ReMath aimed at coordinating and combining theoretical approaches. The language used in the presentation of the project was indeed a language of theoretical integration, as it was planned to achieve it through a cyclic process combining the progressive elaboration of an integrated theoretical framework, the design of six dynamic digital artifacts (DDA), and their experimentation in realistic contexts. (Bikner-Ahsbahs et al. 2010, p. 149)

The method of cross-experimentation was used in this project according to the following aim: each research team was designing and carrying out research based on the use of a digital medium that had been designed and developed by another team and vice-versa. This method was developed in TELMA (Artigue and Cerulli 2008) and further developed and used in the ReMath project (see for example Artigue et al. 2009; Maracci et al. 2013; Artigue et al. 2010).

9.5 Methodological Reflections: Difficulties That Accompany the Networking

Different cases of networking according to the different aims of networking present us with different variations of the methodology of networking which aim to permit a fruitful dialogue between theories. Kidron and Monaghan (2012) consider the complexity of dialogue between theorists and the benefits and difficulties that are related to this complexity. One main difficulty relates to the relevance of data and its appropriateness to the different foci of attention that characterize the different theories. In all the cases of networking in which both of us were involved, we observed difficulties already in the first stage of designing common activities or selecting data towards the analysis which will permit the networking. As noted by Radford (2008) "It is through a methodological design that data is first produced; then the methodology helps the researcher to 'select' some data among the data that was produced but also helps the researcher to 'forget' or to leave some other data unattended" (p. 321). Different theories have different priorities with regard to the focus of analysis. Thus, there is an essential difference in the focus of analysis which influences the essence: if we focus for example on the social process, as a consequence, the analysis is in terms of and within the social process. The different priorities in the focus of the analysis might find their expression in the way the different frameworks have different cognitive and social strengths. Another main difficulty resides in the use of language, especially the plurality of meanings of a single word like milieu, institution, epistemic action. In the detailed example given in the next chapter of this part we will demonstrate how two theories might use the same words "epistemic actions" but with different meanings.

9.6 Benefits of Networking: Advancing Research by Means of the Networking of Theories

As Radford (2008) stresses one of the interesting aspects in the networking of theories is that it not only leads to a deeper acquaintance and understanding of the other theories with which our theory is dialoguing; it also leads to a better understanding of our own theory as well. According to Kidron and Monaghan (2012), the difficulties that accompany networking theories point to benefits like a better understanding of the potential of each frame but also its limits. Kidron and Monaghan consider the complexity of dialogue between theorists and the benefits and difficulties that are related to this complexity. Kidron et al. (2008) also realized the complexity and the benefits of defining the relevant data and its appropriateness to the different foci of attention. Defining the relevant data that is requested for each analysis,

> we certainly understand better today the functionalities each of us gives to the theoretical constructs she/he uses, how she/he uses them and what she/he is able to produce thanks to them; we also see better the limits of our respective tools and what could be offered by networking them in ways that would not destroy their internal coherence. (Kidron et al 2008, p. 263)

In the detailed example of a specific case of networking that will be described in Chap. 10, we illustrate difficulties but also benefits that accompanied the effort of designing common activities towards the analysis which will permit the networking between a cognitive theoretical approach and a social approach.

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Chapter 10
A Cross-Methodology for the Networking of Theories: The General Epistemic Need (GEN) as a New Concept at the Boundary of Two Theories

Angelika Bikner-Ahsbahs and Ivy Kidron

Abstract This example illustrates how research including the networking of two epistemic actions models from different theoretical perspectives is conducted and yields a new concept at the boundary of the two theoretical approaches. It illustrates a cross-methodology and the networking strategies described in the previous chapter of this part. The cross-methodology comprises five cross-over stages that systematically link the research process from the two perspectives in every methodical step and reveal an in-depth comprehension of the new concept from the two perspectives.

Keywords Networking of theories • Interest-dense situations • Abstraction in context • General epistemic need • Cross-methodology

10.1 Introduction

In this chapter we present an excerpt from the project "Effective knowledge construction in interest-dense situations"[1] to illustrate how a networking methodology led to enriching results (Kidron et al. 2010, 2011). From the beginning, the project aimed at the networking of two theories to gain insight into processes of

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A. Bikner-Ahsbahs (✉)
Faculty 3 of Mathematics and Computer Science, University of Bremen, Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de

I. Kidron
Department of Applied Mathematics, Jerusalem College of Technology, Havaad Haleumi Street 21, 91160 Jerusalem, Israel
e-mail: ivy@jct.ac.il

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constructing knowledge which merge individual and social views and which are characterized by the term *epistemic*. The two theories are Abstraction in Context (AiC) and the Theory of Interest-dense Situation (IDS), which both use similar epistemic actions models for analyzing processes of constructing knowledge, but which pursue different principles and methodologies. The project was conducted by a German research team, the IDS-team, and two Israeli research teams that we consider as the AiC-team. The networking of the two epistemic actions models provided a fruitful constellation to disclose mechanisms of how the individual and social construction of knowledge can be merged. Referring to Radford's (2008) theory concept (cf. chapter 9), this presentation will first dwell on describing the main *principles*, characteristics and some *methodological* aspects of the two theories specifically concerning the two epistemic models. Then the methodology that was used to network the two models will be outlined and illustrated by a concrete example of data analysis.

10.1.1 Abstraction in Context

The theory of Abstraction in Context (Hershkowitz et al. 2001; Schwarz et al. 2009; Dreyfus and Kidron 2014) describes abstraction processes from an individual point of view and in a given context as a reorganization of existing knowledge. Abstraction processes are initiated by a need for a new construct, e.g., a need to find a suitable function: this need directs an epistemic process. This epistemic process can be captured and investigated by an epistemic action model, which permits a micro analysis of the process of constructing a new construct. In the final phase of abstraction, this new construct is consolidated. These processes of abstraction deeply depend on the context in which they arise. *Context* is everything that does not belong to an epistemic action, for example the learner's biography, the task, the material available, the social context and also the social actions. New constructs are obtained through carrying out the three *nested epistemic actions* recognizing (R), building-with (B) and constructing (C) that shape the epistemic action model. These actions are mental actions which contribute to gaining new insight. They can be observed while solving tasks through producing text, practical and communicative actions. **R**ecognizing happens when a previous construct is activated as relevant for solving the problem at hand. Through **B**uilding-with, these previous constructs are used to reach relevant insight by connecting the previous constructs in a relevant direction for solving the task. **C**onstructing involves recognizing and building-with actions and yields new constructs. In the third phase, recognizing and building-with actions are also used to **C**onsolidate (+C) the new constructs and to prepare new constructing processes. These four actions together shape the epistemic actions model, to which the AiC-team briefly refers by the abbreviation RBC+C-model.

Methodologically, the AiC-team (cf. Dreyfus and Kidron 2014; Dreyfus, Hershkowitz, & Schwarz, Chap. 8 in this book) focuses mainly on cognitive aspects of epistemic processes while students solve mathematical tasks. A priori analyses reveal hypotheses about constructs that might be observed during the solving processes. A posteriori analyses investigate video data of these processes by the use of the epistemic action model as a methodical tool to disclose how the epistemic process within the specific context happens, how constructs appear or why they might not occur.

10.1.2 The Theory of Interest-Dense Situations

This approach (Bikner-Ahsbahs 2003, 2005; Bikner-Ahsbahs and Halverscheid 2014) is based on an interpretive view of people interacting with each other with a focus on solving a mathematical task. This view entails that people understand other people through interpreting the other people's actions. This interpretation is indicated by the subsequent reacting actions (Jungwirth 2003). Interpretation is the only way for researchers to access people's meaning-making in empirical research analyses. However, in contrast to people's interpretation in everyday-situations, empirical analyses in interpretive research are methodically controlled re-interpretation of the participants' interpretations (Jungwirth 2003, cf. Part II and Part III in this book).

The theory of interest-dense situations focuses on the construction of knowledge as an epistemic process in social interactions which emerges when different people together solve a mathematics task. Interest-dense situations can be characterized through three features: the participating students become deeply involved in the social interactions of solving the problem, they further develop and further deepen the epistemic process, and value highly the mathematics they work on. In all interest-dense situations the students follow their own train of thought, and if a teacher is present he/she focuses on following the students. The dynamics of the epistemic process of interest-dense situations can be described and investigated by an epistemic actions model (GCSt-model) which consists of three collective epistemic actions: *gathering* mathematical meanings that is animated by the task, *connecting* mathematical meanings by bringing these gathered meanings together to solve the task and *structure-seeing*, which means seeing patterns as a unity with a potential to concretize this pattern by many examples. Theoretically, a saturation of gathering and connecting meanings is necessary to make structure-seeing emerge. Theoretically this saturation can be observed in all interest-dense situations. Interest-dense situations are likely to initiate situational interest expressed by the students' deep involvement in solving the task and indicating meaningfulness to the work at hand (Mitchell 1993). Every interest-dense situation identified so far leads to structure-seeing; but these structures need not necessarily be new, they also can be recognized in unfamiliar contexts.

10.1.3 General Description of the Cross-Methodology of Networking the Two Theories

The two approaches share a certain view on knowing and therefore the networking of the two theories could fruitfully be done. Knowing emerges when students elaborate vague mathematical ideas within solving mathematical problems. This is done by *epistemic actions*, i.e. actions which lead to constructing mathematical knowledge. However, the two theories differ in what they call an epistemic action. AiC defines an epistemic action as a mental action that individuals pass through while solving a task: this action is a constituent for abstracting. Social interactions might be seen as a part of the context. In contrast, in IDS, epistemic actions are constituted within collective processes of social interactions while solving a task. Individuals contribute to these actions as they participate in these solving processes. Knowledge construction is constituted through negotiating mathematical meanings within social interaction.

This description of the different meanings of epistemic actions already shows the complementary nature of the two approaches in solving a task: AiC focuses on the individual's construction of knowledge: social actions may contribute to this process; IDS focuses on the social construction of knowledge in social interactions: the individual's actions may contribute to this process. This complementary nature first appeared as a difficulty in the analyses but as soon as we became aware of this complementarity the two epistemic actions models turned out to become the motive to undertake a deep networking process on which the project "effective knowledge construction in interest-dense situations" was built. In this project three mathematical tasks were designed for pairs of students in grade 10 from Germany and Israel to investigate their epistemic processes. Already in this designing process the teams met difficulties on how open the tasks should be. Besides other aims, more specific aims in this process were to find out how the need for a new construct arises, what drives the students to proceed with solving the task even when the process is long, and what obstacles or constraints are met and how this is related to acting.

The methodologies which were developed can be distinguished into two within-methodologies, which according to Radford's (2008) notion of theories are specific for each of the two teams and their theories, and one between-methodology through which the networking of the theories was conducted (cf. Mok and Clarke, Chap. 15). In Radford's words: the between-methodology establishes a "semiospheric methodology" of connecting the theories involved (cf. Radford's comments in Bikner-Ahsbahs et al. 2010, p. 170; Radford 2008). It is a cross-methodology based on five cross-over procedural stages which guide the networking strategy of *coordinating* (cf. Chap. 9) and may lead to the strategy of *local integration* (cf. Chap. 9). It will now be described, and later illustrated by an example.

The project pursued five methodical steps: task development, piloting tasks, data collection and data processing, data analysis, and reflection on the networking process.

Every step was conducted by a series of *five cross-over stages* guiding the research towards common results:

- *deciding cooperatively* about what will be done, for example deciding about the part of the transcript to be analyzed;
- *separate processing*, for example analyzing separately;
- *exchange of the results and working with alien results*, for example exchanging the analysis results and commenting them;
- *reworking home results*, for example re-analyzing the part of the transcript, and
- finally a collaborative meeting aimed at *building consensus* about the work done.

These cross-over stages are based on the networking assumption of respecting the theories' identities, but also allow to stepwise link the results leading to their possible local integration. Since qualitative research must respect the quality criteria of content sensitivity these five stages were adapted to the methodical steps of the project. Sometimes cyclic processes of exchange and separate re-working processes were repeated several times before the next stage could be reached. Not only the between-methodology was further developed; each research group also developed their methods and techniques as part of their within-methodology further.

In order to make the networking methodology, and its benefits and difficulties, explicit, every networking project is finalized by common reflections on the methodology and the results gained.

10.2 An Illustrative Example: Investigating the General Epistemic Need

On the basis of data from the pilot study, we investigated in more detail the need for a new construct connecting it to situational interest. In some cases, the need for a new construct could not exactly be identified as predicted. In the paper by Kidron et al. (2010) an epistemic need that is more general than the need for a new construct, the General Epistemic Need (GEN), is described as

> a need to progress, to reinforce a vague image into a more definite one. Hence, it [this action] is a constructing action according to Davydov's [1972/1990] view of abstraction: the transition from an initial vague to a more precise notion (of infinite process and of limit). (p. 175)

It seems to be initiated by the features of the situation, and can be described as the students' need to proceed in an epistemic process looking for ideas to solve a task. It is expressed by individuals but can also be shared as a driving force for coming to know in social interaction. It can become more specific, for example as a need to be more precise, shown by the students' actions to make things more precise, and, thus, can be observed.

The GEN, as it emerged in the study by (Kidron et al. 2010), was described as the driving force that makes students progress in learning processes according to the challenge they meet within a situation, individually and socially. Later, Kidron et al. (2011) offered a first discussion how a GEN leads to a need for a new construct. The analyses in the present paper portray a revised version of those in Kidron et al. (2011) with a focus on cross-methodology. We describe the subsequent cross-analysis on the concept of GEN as part of this methodology. The analyses aimed at further investigating the GEN and checking whether our hypotheses about its relevance for epistemic processes could be confirmed.

10.2.1 The Task and Its Setting

In an interview situation, the two grade-10 students, T and M, work on a continued fraction task (Fig. 10.1). Due to the criteria of IDS, the interviewer's role is not to guide and intervene but to support the students only when they get stuck, with weak

How can we interpret the continued fraction? $1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1+\dots}}}$

1. Construct a sequence of fractions representing the continued fraction, like this:

$$f(0)=1$$
$$f(1)=1+\frac{2}{1}=1+2=3$$
$$f(2)=1+\cfrac{2}{1+\cfrac{2}{1}}=1+\frac{2}{1+2}=1+\frac{2}{3}=\frac{5}{3}$$
$$f(3)=1+\cfrac{2}{1+\cfrac{2}{1+\cfrac{2}{1}}}=$$

2. Add 3 more terms: Calculate: $f(4)$, $f(5)$, $f(6)$.
3. Look at the seven terms you calculated and at the way you calculated them. Can you find a pattern when passing from one term to the next one?
4. Explain the pattern - why does it work?
5. Add more terms to the sequence, using the pattern you found, until you have 20 terms in the sequence. Fill in the following table. Use a calculator to represent the fractions of the sequence as decimal fractions. Copy all the digits from the calculator pane.
6. Look at the sequence in the table and write a conjecture.
7. Write your conjecture from 6. by means of x. Now justify your conjecture.
8. How does this justify the conjecture in 6?

Fig. 10.1 The continuous fraction task (cf. Kidron et al. 2011, p. 2452) hints just to help the students continue and find their own line of thought. In the task, the students were asked to calculate the first seven fraction; and find an expression $f(x)$ for the xth fraction, beginning with $f(0)=1$.

A table was prepared for the first 20 fractions to be written as simple fractions and decimal numbers (Table 10.1). *f(0), f(1), f(2)* already were included in the table (Table 10.1). Based on these preparation tasks, the students were asked to find patterns, make conjectures, and explain why these conjectures are true (Fig. 10.1).

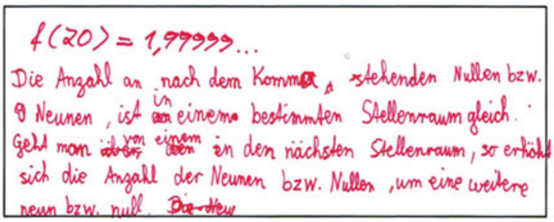
We now illustrate the cross-analysis as part of the cross-methodology.

10.2.2 Beginning a Cross-Analysis

Our cross-analysis used the video and the transcript of T's and M's epistemic process as they worked on the task described above. The first *crossing stage* referred to the *common decision* on a piece of data. The IDS-team normally analyzes the whole transcript from the beginning in a sequential way, whereas the AiC-team prefers to concentrate on specific parts of the process of constructing of knowledge, especially on actions that indicate the central stage of the emergence of a new construct. In the situation to be presented, the decision was made to focus on utterances 1333 up to 1512 of the transcript (available from the researchers).

In order to share the same data base, the AiC team also looked at the whole transcript up to line 1333 in detail but still focused more on the chosen part. As a first step both teams undertook separate analyses of the lines before they exchanged and re-analyzed them, and finally discussed the results. In the utterances 719–1353 the students had investigated the decimal fractions and observed a mathematical pattern about the growth of the numbers of ninths/zeros after the decimal comma, which they called "space of place" as represented in Fig. 10.2.

What was important is the fact that this term resulted from the need to share interpretations, since it was very complicated to always describe what was meant when referring to this pattern. This need also seemed to result from a view on the pattern that was neither coherent nor consistent, varying between two interpreta-



Translation of the text:
The number of zeros/ninths after the comma are in a specific Space of Places the same. If one turns from one to the next Space of Places the number of ninths/zeros increases by one nine/zero.

Fig. 10.2 The students' definition of the space of places

Table 10.1 Representing *f*(x) as simple fractions and decimal fractions by the students

	Bruchzahl	Dezimalzahl	Abstand zu 2
$f(0)$	1	1	1
$f(1)$	3	3	
$f(2)$	$\frac{5}{3}$	1.666666667	$\frac{1}{3}$
$f(3)$	$\frac{11}{5}$	2,2	0,2 $\frac{1}{5}$
$f(4)$	$\frac{21}{11}$	1,909090909	$\frac{1}{11}$
$f(5)$	$\frac{43}{21}$	2,047619048	$\frac{1}{21}$
$f(6)$	$\frac{85}{43}$	1,976744186	$\frac{1}{43}$
$f(7)$	$\frac{171}{85}$	2,011764706	$\frac{1}{85}$
$f(8)$	$\frac{341}{171}$	1,994152047	$\frac{1}{171}$
$f(9)$	$\frac{683}{341}$	2,002932551	
$f(10)$	$\frac{1365}{683}$	1,998535871	
$f(11)$	$\frac{2731}{1365}$	2,000732601	$7{,}32601 \cdot 10^{-4}$
$f(12)$	$\frac{5461}{2731}$	1,999633834	$3{,}6616 \cdot 10^{-4}$
$f(13)$	$\frac{10923}{5461}$	2,000183117	$1{,}83117 \cdot 10^{-4}$
$f(14)$	$\frac{21845}{10923}$	1,99990845	
$f(15)$	$\frac{43691}{21845}$	2,000045777	
$f(16)$	$\frac{87381}{43691}$	1,999977112	
$f(17)$	$\frac{174763}{87381}$	2,000011444	
$f(18)$	$\frac{349525}{174763}$	1,999994278	
$f(19)$	$\frac{699051}{349525}$	2,000002861	

tions of the space of places "as an interval on the *x*-axis, which numbers the elements, specifically the interval in which the number of nines/zeros remains the same" (Kidron et al. 2011, p. 2453); and as a "part of the decimal expansion of *f*(*x*), specifically the part containing the nines (or zeros)" (ibid.). Despite its fuzzy meaning, this term helped the students to communicate. The AiC team indicated the space of places as a seed for a later constructing process and identified evidence for this interpretation in the students' double interpretation that "reinforces our interpretation of the SP [Space of Places] as a seed for later constructing, as something that is not precise and needs to be elaborated" (ibid., p. 2453).

10.2.3 Separate Analysis from the AiC-View

The cross-over stage of *separate processing* in this section is described by the use of the analysis of the AiC-team that has already been published (Kidron et al. 2011, see pp. 2453–2455).

As mentioned earlier, the AiC researchers observed some phenomenological identification of patterns. These patterns were regarded as seeds for constructing actions that take place later. By *phenomenological* the AiC researchers refer to the fact that the elements of the sequence are viewed as strings of digits rather than as numbers. In this first phase, the researchers noticed some indication for need in the efforts of the students to clarify the notion of Space of Places (SP) and to assign it a name. Expressing the same result, the same idea of SP in different ways was a clear indication for a GEN. The AiC researchers observed one specific aspect of the GEN: "the need to understand the present situation in terms of the previous knowledge or previous experience, to engage with the challenges offered by the task" (Kidron et al. 2011, p. 2453). Then, they noted how this need and the limitation of the previous knowledge (especially, when the previous knowledge was adequate to empirical computations while the strings of digits were explicitly written and observed) led to another specific aspect of the GEN: "The need to be more general" and "The need to clarify" (ibid.). The essential point was that these specific aspects of the GEN led to the emergence of the need for a new construct. This need appears in the next phase, in which the researchers observed a striking change in the students' way of thinking: they start giving reasons rather than only phenomenological descriptions. This change might be interpreted as a consequence of the interviewer's initiative of asking questions concerning the SP for f(1000) and f(1000000).

> The students express their need to understand the new situation in terms of their previous empirical experience, and express their thinking that they need to do all the 1000 computations [….] The limitation that results ('we cannot do all the 1,000 computations') leads to the '*need to be more general*' and to apply their patterns in a more general way. (ibid., p. 2453)

In response to the next initiative of the interviewer "How would it work and go on?" (line 1424) the students again experience a limitation in their previous experience which "leads them to the need to think in a more general way. This need directed the students towards the beginning of a constructing phase in which infinity

plays a role." (ibid., p. 2454). The AiC researchers paid attention to the important role of the interviewer, but at the same time to the fact that the notion of infinity was expressed on the students' initiative. "The students express a need to understand the meaning of 'infinite[ly many] zeros,' […] 'and infinite[ly many] nines'" (ibid., p. 2454) and at the same time the meaning of keeping "closer to zero," "to two" (line 1427). As a consequence, the researchers observe four constructing actions that relate to convergence.
A first constructing action, C_0: *"Convergence as very close to."* appears in

1418. T Yes ,so it is very close to two already

and is clearer in

1427. T and it keeps on leaning closer to zero- ,closer to two, both numbers

Then, we observe the construct C_1: *the Potential Infinite process view* in 1427 above but also later in

1473. T If one looks at it precisely ,it never reaches two ,even if there are infinite nines ,after it there always comes ,seven three two ,whatever. ,can be anything (.) ,the following numbers ,we have not even looked at them yet ,could be that they have a pattern too ,but ,I don't- ,personally I don't see anything there (M laughs).

Immediately follows C_2: *Infinite as "a façon de parler: very, very large but finite*" and C_3, *the strange infinite object is legitimate only in the mathematical world*, for example in

1437. T If- ,if you insert infinity, it theoretically equals two
1457. T Yes ,otherwise we just write f of infinity equals two

The students wrote "$f(\infty)=2$" on their worksheet. This led to another aspect of C_3: *Transition from infinite as potential infinite, as "a façon de parler" to infinite as a legitimate object in the mathematical world.*

Constructions C_1, C_2, C_3 as observed in the AiC a posteriori analysis are in accord with its a priori analysis of the knowledge elements intended by the design. The AiC a priori analysis in the present study also identified different stages, from the intuitive notion of limit as a process to a conception of limit as an object that could partly be observed in the data.

C_1, C_2, C_3 develop in parallel for a long time from 1425 to 1473. "At the same time when the potential infinite process view is expressed the student also identified an expression of the kind: let us manipulate the infinite as a legitimate object as we have done previously for large but finite numbers. This experience seemed too complex to have so different (and somewhat contradictory) constructions in parallel. Therefore, the research team inferred that this situation leads to a feeling of unease, of confusion which is expressed in

1473. T If one looks at it precisely ,it never reaches two ,even if there are infinite nines ,after it there always comes ,seven three two ,whatever. ,can be anything (.) ,the following numbers ,we have not even looked at them yet ,could be that they have a pattern too ,but ,I don't- ,personally I don't see anything there.

A need for a new construct is then expressed in

1478. T The best would be of course if we had a functional equation right' (.) ,thus if one could say exactly ,f of x exquals (...)

The AiC researchers consider this need as a consequence of the limitations of the students' previous experience—they are no longer able to use what they know from the finite case. Therefore, a need for a new view is expressed. This need leads to the construction C_4: *Transition from a numerical way of thinking (with empirical results calculated by the students) to a more general way of thinking (which does not depend on specific cases)*. "A new construct is needed to permit this transition. The need for this new construct is explicitly expressed in 1478." (ibid., p. 2455)

During this search, the students continue their "new" (since 1354) approach of giving reasons rather than only describing phenomena. This new approach, in which the students explain their way of thinking, demonstrates the passage from the seeds of constructing to the beginning of the construction process.

10.2.4 A Re-Analysis from the IDS-View and Its Results

The cross-over stage of *reworking home results* is now portrayed by summarizing the re-analyses of the IDS-team as has already partly been published by Kidron et al. (2011, see pp. 2455–2458). The IDS-team reconstructed what they call the *flow of mathematical ideas* that produces mathematical knowledge. This *flow of ideas* is regarded as a

> horizontal scanning of the mathematical [neighbouring, incl. by the authors] aspects of a problem expressed in the utterances towards oneself and the other in order to describe, concretize, understand, progress, …, but also to inform the other, to take up her idea and develop it further, negotiate, explain, … It is an evolution of ideas associated with a given mathematical problem, building on previous experiences. (Kidron et al. 2011, p. 2455; cf. notion of travel of ideas by Saxe et al. 2009)

It is built by the epistemic actions of *gathering* and *connecting mathematical meanings, and structure-seeing*. IDS analyses of the epistemic process in the episode resulted in the diagram of Fig. 10.3 with six idealized phases represented by pictographs (cf. Bikner-Ahsbahs 2006). Every phase is initiated by the interviewer (represented by the arrows). Phase I mainly involves gathering actions, in phase II, III and VI gathering and connecting actions are merged. In phase IV and V, structure-seeing occurred, followed by making the structures concrete and validating them (the meaning of the symbols are described in Bikner-Ahsbahs 2006, cf. Chapter 6).

The analyses below start with line 1397 in phase III (1397–1423) of the transcript. We illustrate three different kinds of flow of ideas which are concerned with approximating the number 2 by the continuous fractions, how this flow is driven by a GEN, and how the GEN is transformed into more specific epistemic needs, and finally into the need for a new construct (which is a concept coming from AiC).

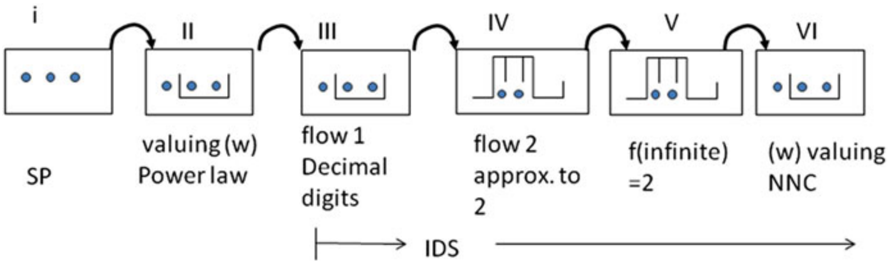


Fig. 10.3 Phase diagram of the analyzed scene (1333–1512). (Kidron et al. 2011, p. 2455). The diagram in Fig. 10.3 consists of pictographs, phase I (the points indicate gathering meanings): 1333–1353, phase II (the sign of the open rectangle indicate connecting meanings, the included points gathering meaning): 1354–1401, phase III (this pictograph begins with a kind of the letter S for seeing a structure which e.g. is made concrete or checked by gathering and worked out or justified by connecting actions): 1402–1423, phase IV: 1424–1454, phase V: 1455–1466, phase VI: 1468–1512 (cf. Kidron et al. 2011, p. 2455)

A flow of ideas may prepare structure seeing: This flow refers to what the digits of $f(x)$ for $x = 1000000$ may look like. Beforehand, the students had emphasized the power law for the length of the Space of Places (SP) was a conjecture which only may offer estimations. Now they include it as a means for further investigation.

1397.	I	And ‚f of ‚one million’
1398.	T	Ohm
1399.	M	(sighs) F of one million
1400.	/T	we would have to cal- calculate now ‚what’s the root of one million ‚and then round it down
1401.	/M	what kind of’
1402.	I	You- ‚you really don’t need to do it accurate now now
1403.	/M	no ‚now we are doing it (laughs)
1404.	/I	(spoken simultaneously) ok.
1405.	M	Thousand
1406.	/T	(spoken simultaneously) is thousand ‚so exact thousand the set ‚of the space of places
1407.	I	Hmmh
1408.	T	So th-
1409.	I	And how would f of one million and one look like’
1410.	T	Ohm that would still be a spa- ‚that is just the set of the space of places
1411.	/M	so one (looks at the calculator) ‚ah never mind
1412.	/T	we just can’t the- ‚still thousand ‚until ‚one thousand and one results
1413.	/M	But what we do know in any case is ‚that eeh there is a one before the decimal point ‚well not for one thousand and one ‚for thou- for one-
1414.	/T	No ‚for one thousand and one there is a one in front of the decimal point ‚well no wait yes ‚a two
1415.	/M	That’s an odd number ‚yes

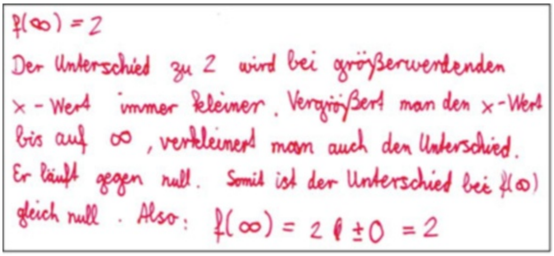
1416. T Two point, ,zero zero zero zero zero
1417. /M Yes because it's an odd eeh ,place
1418. /T Yes ,so it's very close to two already
1419. /M yes
1420. /T Those are then about a hundred zero or so (laughs) ,and then comes some ,different number

T expresses the need to apply the power law to *f*(1000000) (line 1400) but *f*(1000000) cannot be computed. The interviewer seems to fear the students are on the wrong track and allows them to be less accurate. However, M expresses his interest to go on (1403). We take his utterance as evidence that he does not want the flow of ideas to be disturbed, and therefore as an expression of the GEN to proceed. He calculates the root of one million (1405) as an application of the power law but still the string of *f*(1000000) remains vague. This vagueness causes a need for certitude as M assures what is certainly known, the "one before the decimal point" (1413). This need for certitude is shared by T since he immediately corrects the utterance of his peer by indicating the reason that this is an odd number (1414), which convinces M (1415). Concerning the flow of ideas, an interesting instance happens now. T illustrates the string of digits (1416) as *2.00000*, hence, connects the digit 2 with the idea of the Space of Places as a string of zeros. Comparing this with the number 2 it is only a tiny step to see a first structure of *2 as a limit number* (1418). This is expressed by "very close to two already" (1418, 1419) and strengthened by speculating that there are 100 zeros and then different digits (1420).

This *flow of ideas* can be described by scanning (that is gathering) ideas about how the decimal number *f*(1000000) might look including actions of connecting and structure seeing. It produced the idea of approximation to 2 that is then elaborated by another flow of ideas initiated by the interviewer by asking "and how would it work then" (1424). This flow leads to structure-seeing.

A flow of ideas may allow structure seeing: This flow is just listed as steps and interpreted according to the theoretical frame (cf. Kidron et al. 2011, p. 2456):

1. s*tructure seeing*: "it keeps on going" (1425) (seed for infinite as a process),
2. concretizing this structure: "an infinite number" (1426) (length of the decimal number),
3. structure-seeing: "leaning … closer to two, both numbers" (1427), (because of the leaning-key-idea of approximation, grasping the convergence to 2, and referring to both numbers that converge from both sides),
4. connecting two structures from 1. and 3.: "but no never becomes 2" (1429) (sequential process of potential infinity),
5. concretizing by connecting: "there are always infinite zeros" (1429) (the value of the digits connected to the process directs the view to the actual infinite),
6. inferring from 5. "it's infinite that's just it" (1430) (the length of the decimal number),
7. Connecting: "at the end there are infinite zeros, or infinite nines, and there is something" (1431) (the image of the infinite length here is rooted in the experience of the finite).



Translation of the text:
The distinction to 2 becomes smaller and smaller when y grows. If one increases the value of x until infinite one will minimize also the distinction. It goes to zero. Thus, the distinction at f(∞) equals zero.

Fig. 10.4 The students' justification that the continuous fraction converges to 2

A flow of ideas may result in dispersing views: Whereas before, the flow of ideas was concerned with the infinite length of the decimal numbers and the convergence to the number 2, the students now take infinite as an actual value of x writing $f(\infty)$. This cannot be computed, therefore, the students argue hypothetically: if "we insert infinity" (T, 1435), "will always be the same" (M, 1436), "if you insert infinity ,it theoretically equals two" (T, 1437), "then it would be two" (M, 1438). "one point nine period" (M, 1440), "equals two then" (M, 1442), "equals about two" T, 1443), "equals two" (M, 1444), "so close- ,ah ok" (T, 1445). (cf. Kidron et al. 2011, p. 2457)

In the hypothetical argumentation, the flow of ideas leads to dispersing individual views. T imagines potential infinity (1437, 1443, 1445) whereas M talks about actual infinity (1434, 1442, 1444). Since these views are incompatible, they cannot be merged. A need for certitude makes them refer to what the teacher has told them. M recognizes an argumentation for 1.9999… =2 given by their teacher as relevant, which they both recall by the manipulations as they have learnt it in school. The final structure about the convergence to 2 has been written as represented in Fig. 10.4.

The GEN can be transformed into more specific epistemic needs: The flow of ideas can lead to the experience of limitations, for example when the students reach a point where they do not have the tools to continue solving the problem or when different forms of understanding cannot be overcome. Limitations do not necessarily force the students to give up; they can be a source for transforming the GEN into a more specific epistemic need which often is expressed by fulfilling this need. This was shown in several ways (cf. Kidron et al. 2011, p. 2457 ff.), for example by *changing the conditions* as a reaction on the need to progress which cannot be fulfilled or by *going back to a clear situation* as a reaction to the need for certitude.

Although the IDS principles were different from the AiC principles the IDS team has gained results that validated those of the other team:

- *Taking a more general view* is a reaction initiated by a need to be more general. This need occurred after the students experienced limitations like "one can't say anything else" (in 1359) about 1000 and 1001. The need for being more general was expressed by a reference to their power law "wait we have our theory" (1360) which made them change their view.

- *Finally the students expressed the need for a new construct (NNC)* in 1478 as a need for a function, T: "It would be the best if we had a function equation right' (.) ,well if one could say exactly ,f of x equals (…) ,wait ,what's that' (points at the sheet) ,no ,that's not a sum (not understandable)" (1478); /M: "but we had- ,oh well a function equation (holds his head) ,we do have a func-" (1479); /T: "yes ,we only have a function equation ,depending ,on the previous number value" (1480). This need seems to be a reaction of a deep personal irritation when the GEN drives the students to proceed but they do not at all know how to go on: "personally I don't see anything there" (1473).

Situational interest may empower the students' to act epistemically: In 1468 T values the demand to justify their conjecture as being "more difficult". In 1469 the interviewer acknowledges "I find your last aspect just now most interesting". This kind of valuing influences the students to confirm: "Yes ,that's really is interesting how-" (1470). T adds what is interesting: "yes ,so theoretically it keeps on leaning closer *to* two" (1471). It is the kind of approximation that is most interesting. The attempt to justify this approximation results in the experience of difficulties for the students, which in turn provokes T to express a need for a new construct "it would be best if we had a function equation…" (1478) which he values highly contrasting it with the reason why the expression they have is insufficient. This need is socially shared because both students agree upon fulfilling it together: "right',shall we try to discover something like that ,cause that would be" (1485), "one that depends on x right'" (1486), "Yes ,so f of" (1487). This shows "deep involvement, accompanied by meaningfulness and ending up with valuing highly what the students do not yet have[:] expressing a *need for a new construct (NNC)* driven by situational interest" (Kidron et al. 2011, p. 2459).

10.3 Methodological Reflections About the Networking Process

The concept of GEN emerged when the AiC team tried to identify the need for a new construct but could not find it. This difficulty was overcome by learning from the IDS team that there might be a more general epistemic need that drives the epistemic process. At the same time, the IDS team had difficulty identifying situational interest and learned from the AiC team that it may be a driving force with roots in epistemic needs. Both teams agreed upon the relevance of the general epistemic need for epistemic processes, but sometimes this GEN remained implicit. This difficulty was solved by postulating its existence, and starting to investigate it; thus, both analyses disclosed evidence for the relevance of the concept. For the AiC team, with the theory based on Davydov's (1972) ideas it was clear that the constructing process starts from vague and undeveloped ideas towards more definite ones which might be elaborated as new constructs. But, with the notion of GEN, the AiC team was able to consider seeds as the starting points of constructing processes in which the GEN sometimes is transformed into the NNC. This may happen when the students meet limitations in solving a problem which are experienced as personally disturbing.

The IDS team also investigated the role of the GEN, but it was focused on the role it plays in the support of situational interest. It turned out that the GEN and the situational interest mutually inform each other and, hence, together are driving forces for epistemic processes. The results described above were considered as evidence that the GEN can turn into more specific needs which are fulfilled by epistemic actions, for example to make things more precise if the need to be more precise is experienced or to argue hypothetically if there is a need to proceed but the mathematical objects cannot be computed. A GEN becoming more specific supports overcoming the experience of a gap of knowing: given this overcoming is successful, the experience of competence may enhance situational interest, which in turn may support the GEN and finally lead to the need for a new construct.

The collective and the individual process of constructing knowledge is linked by the flow of ideas as a process of social construction of knowledge that provides a source for individual constructions of knowledge which in turn contributes to the flow of ideas. While solving a problem, a GEN may drive a flow of ideas providing material for an individual *recognizing* an idea as a relevant previous construct which may be further elaborated within social interaction; collectively *connecting* ideas may provide the bases for individual *building-with* actions and leading to *seeing a structure* within social interactions or individually *constructing* a new construct.

The described cross-methodology afforded to employ networking strategies (see Chap. 9), i.e. to *understand* the other theory and *make* one's own *understandable* through offering analyses of the same transcript. *Comparing* and *contrasting* also took place, but implicitly. It became more explicit when the groups met difficulties in the analyses of the transcripts. Through the five cross-over stages in the between-methodology, it was possible to *coordinate* the networking process, and led to a new kind of concept of which both theories could make sense. This concept of GEN at the theories' boundary was then *locally integrated* into both approaches. Hence, the networking of the two theories involved all the networking strategies except synthesizing. However, what is also important is the fact that the two teams learned from each other, and in this way deepened their within-analyses, hence, according to Radford (2008), they improved their theoretical approaches.

Transcription Key

S(s), T	student(s), teacher
EXECT	loud voice
<u>exect</u>	with stressed voice
e-x-a-c-t	prolonged
exact.	dropping the voice
exact´	raising the voice

(continued)

,exact	with a new onset
exact-	voice remains suspended
(.),(..)(…)	1, 2, 3 s pause
(....)	more than 3 s pause
(5 s)	5 s pause, if necessary
(gets up)	nonverbal activity, the duration of non verbal activity need not be fixed unless it is special, a pause of 2 s afterwards (..), interpreted (*slow*)
(exact??)	assumed utterance

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Part VII
Multi-Level-Analysis

Chapter 11
Understanding Learning Across Lessons in Classroom Communities: A Multi-leveled Analytic Approach

Geoffrey B. Saxe, Kenton de Kirby, Marie Le, Yasmin Sitabkhan, and Bona Kang

Abstract This chapter presents a methodology for studying classroom communities as microcultures, with a focus on processes of teaching and learning over significant spans of time. In Sect. 11.1, we present a conceptual framework that treats classroom activity at two levels of analysis, collective and individual. Both levels are geared for understanding the reproduction and alteration of a common ground of talk and action through time. Key concerns are the emergence of collective norms and individuals' use of representational forms to serve varied functions in classroom communicative and problem solving activity. In Sect. 11.2, we show how the conceptual framework was used to organize two related programs of empirical research. First, we present design research that led to a 19-lesson sequence on integers and fractions, which uses the number line as a central representational form. Second, we use the framework to organize an empirical analysis of a single classroom community over the 19-lesson sequence. We illustrate empirical techniques for capturing the reproduction and alteration of a common ground with shifting lesson topics. The chapter concludes with an analysis of the way the analytic approach illuminates core processes of teaching and learning and the utility of the approach for future work.

Keywords Multi-level-analysis • Classroom as a microculture • Mixed methods • Design research

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G.B. Saxe (✉) • K. de Kirby • M. Le • B. Kang
Graduate School of Education, University of California Berkeley,
Berkeley, CA 94720-1670, USA
e-mail: saxe@berkeley.edu; kenton@dekirby.net; marie.x.le@gmail.com; bona_kang@berkeley.edu

Y. Sitabkhan
RTI International, Washington, DC 20005, USA
e-mail: ysitabkhan@rti.org

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Our purpose in this chapter is to present a methodology that can advance our understanding of the dynamics of teaching and learning in classrooms. Our methodological goals are unusual but important—to develop productive ways of analyzing classroom communities as microcultures, and then to develop techniques to study processes of teaching and learning over significant spans of time (across lessons). We regard the cross-lesson cultural focus as under studied and under theorized, and, in this regard, our chapter is intended to support new conceptual and empirical efforts.[1]

In our methodological treatment, we distinguish between a conceptual framework and empirical techniques, both elements of a general methodology. We take a conceptual framework to be a working theory that elevates some phenomena as central while backgrounding others. Empirical techniques, on the other hand, provide ways of generating data relevant to a working theory, allowing researchers to explore conjectures, tease apart rival hypotheses, and refine theories.

The chapter is organized in two sections. In Sect. 11.1, we describe our conceptual framework for a cross-lesson treatment of classroom communities. We argue that a cross-lesson treatment requires two mutually constitutive levels of analysis: One level focuses on collective (or joint) activity, and the other on individual activity. We introduce these two levels in the analysis of excerpts from a fourth grade classroom that participated in our design research project, Learning Mathematics through Representations (LMR). We show how these two levels of analysis illuminate the dynamics entailed in teaching and learning as lessons progress.

In Sect. 11.2, we illustrate two sets of empirical techniques for the cross-lesson study of classrooms, each guided by our conceptual framework. The first set was used as a part of design research that led to a strong 19-lesson sequence particularly well suited for our cross-lesson analysis. We used a second set of empirical techniques to analyze a single classroom community and its implementation of the designed sequence across the 19 lessons.

11.1 A Conceptual Framework for Analyzing the Generation of Common Ground

We consider two mutually constitutive levels of analysis in the conceptual framework that we elaborate: collective and individual. At the collective activity level, our treatment builds on several core and related ideas concerning communication. A central idea is that in communications with one another people do not have direct access to one another's intended meanings. Hence individuals face coordination problems (Lewis 1969) in joint activity—the challenge of interpreting the

[1] There are notable exceptions that do incorporate cross-lesson perspectives. One example is Paul Cobb, Kay McClain, and Koeno Gravemeijer's on statistical reasoning about data in the middle school grades (Cobb 1999; McClain et al. 2000). The project is the focus of a special issue of *The Journal of the Learning Sciences.* Anna Sfard and Kay McClain served as Editors of the special issue (Sfard and McClain 2002), and convergent analyses produced from a range of methodological perspectives (Cobb 2002; Forman and Ansell 2002; Macbeth 2002; McClain 2002; Saxe 2002; Schliemann 2002; Sfard 2002).

communicative displays of others in talk, text, or action. To address the communicative challenge, interlocutors behave cooperatively: They tailor their displays to their audience and assume that others are doing the same, as they make inferences and presuppositions about the other's intents (Grice 1989). The result of such exchanges is often an appearance of shared meanings in collective activity, but in fact, these meanings can only be "taken as shared" (whether the presupposed meanings are accurate or not) because individuals can never know one another's minds in any direct sense (Bauersfeld 1994, 1995; Cobb et al. 1992; von Glasersfeld 1984, 1987).

In our exposition of the individual activity level, we build upon a second line of scholarly tradition that treats cognition as a dynamic process, requiring a genetic method for its study and analysis. The genetic method is well represented in Piaget's treatment of the equilibration of cognitive structures (Piaget 1970, 1977), Vygotsky's treatment of mediation and the development of higher cognitive functions (Vygotsky 1978, 1986), and Werner's treatment of symbol formation and his orthogenetic principle (Langer 1970; Werner 1948; Werner and Kaplan 1963). Though emphases and constructs differ across these scholars, all share a concern with understanding individuals' progressive differentiation and integration of knowledge and activity.

Most centrally, we use a recent framework that coordinates an analysis of collective activity and individual development over significant spans of time. The framework was developed through an analysis of a remote Papua New Guinea cultural group with a focus on a 60-year span of local history studying the cultural development of mathematical ideas (Saxe 2012). In the New Guinea work, the focus was on the emergence of forms of mathematical representations and their shifting functions through historical time as local people (the Oksapmin) engaged with shifting collective problems of daily life. In the present work, we draw on Saxe's framework on the cultural development of mathematical ideas, adapting it to an analysis of classrooms, which we treat as microcultural communities engaged with shifting collective problems.

11.1.1 Core Constructs

In our conceptual framework specialized for the study of classroom communities, we use the term common ground (a term borrowed from Clark 1996) to index our focus on a taken-as-shared public discourse (Sfard 2008). We take the production of common ground to occur at both collective and individual levels. Figure 11.1 shows the constructs we use at each level.

At the collective level, we take common ground to be the production of taken-as-shared norms (Yackel and Cobb 1996) that support the coordinated actions of participants in joint activity (Fig. 11.1, left branch).[2] These include norms for participation, like norms for turn-taking or contributing to a classroom discussion. They also include sociomathematical norms for mathematical argumentation and justification that may emerge in particular communities.

[2] We take for granted that norms have no stable existence beyond the participation of individual actors. However, as a heuristic, we consider norms as a collective level construct.

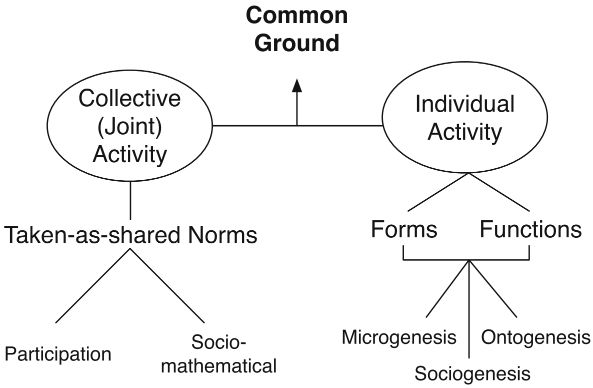


Fig. 11.1 Common ground at collective and individual levels of analysis

At the individual level, we focus on the cognitive work of individual actors. Here, we take common ground to be generated as individuals produce and interpret displays of mathematical thinking, making use of representational forms (linguistic, graphical, gestural) to serve communicative and problem solving functions (Fig. 11.1, right branch). As individuals solve problems in their public displays, their actions contribute to the common ground of the classroom community. To understand this cognitive work and its history in classroom communities requires analyses of the microgenesis, sociogenesis, and ontogenesis of relations between representational forms and the functions that those forms serve, a focus that we will elaborate in subsequent sections.

When analyzed as a "snapshot" of a classroom community, common ground may appear fixed at both collective and individual levels. But we find problematic an approach that objectifies common ground as a state rather than a process. Norms have no objective status. They are descriptions of community expectations that are exhibited through patterns of behavior. Far from static, these patterns and expectations are continually reproduced and altered in the flow of activity. Similarly, at the individual level the functions of representational forms may appear fixed and stable over time, but we note that forms have no inherent functions; representational forms take on functions only in activity. Furthermore, multiple forms can be used to serve the same function, and a single form can be used to serve multiple functions. So, at both collective and individual levels, common ground is a moving target, and we refer throughout the chapter to common ground as being reproduced and altered in activity, an expression that captures its dynamism and emergent properties. It also motivates a cross-lesson perspective on common ground—processes of reproduction and alteration as lesson topics shift.

11.1.2 The Reproduction and Alteration of a Common Ground: An Illustrative Exchange

To illustrate the framework, we draw on an observation in a fourth grade classroom engaged with an LMR lesson on integers. In this episode, we note that students are able to communicate with one another successfully so that learning is seemingly

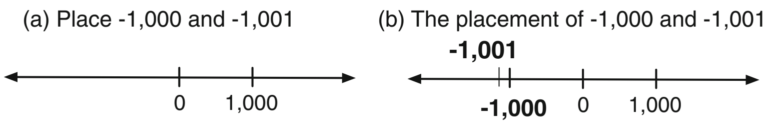


Fig. 11.2 (**a**) A problem of the day for the seventh lesson on integers: "Place –1,000 and –1,001 on the number line below" and (**b**) the class' solution

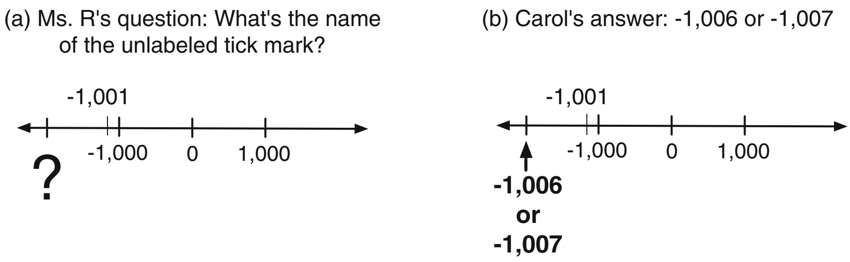


Fig. 11.3 (**a**) Ms. R's question and (**b**) Carol's answer

achieved, despite some threats to communication like the expression of conflicting ideas, elliptical references to mathematical definitions, and use of negative integers (numbers that have no material embodiments). The example provides a particularly clear illustration of the way individuals draw upon and contribute to a common ground of talk and action.

In this episode, the class is engaged with a challenging mathematical problem, depicted in Fig. 11.2a: Place –1,000 and –1,001 on a number line for which only 0 and 1,000 are marked. The class has solved the problem by placing both –1,000 and –1,001 on the white board as depicted in Fig. 11.2b.

Now, Ms. R elaborates the problem by placing an unlabeled tick mark at the position of –2,000, asking what the value is (see Fig. 11.3a). Carol, a student who is already at the board, reflects, subtly making a counting motion with her hand to the unlabeled tick mark (translating the –1,000 to –1,001 interval to the unlabeled tick mark); she then reports that the tick mark should be –1,006 or –1,007 as indicated in Fig. 11.3b. Although there are some elements of Carol's solution that appear to respect canons of the number line (e.g., respecting the order of numbers on the line), Carol's solution is a curious one. It is unclear whether Ms. R or the class has insight into Carol's rationale, threatening breakdown of communication between Carol and the class. Now there is conflict with chatter in the class and hands raised.[3]

Kail now comes to the board and voices her disagreement with Carol, indicating that the label for the unlabeled tick mark should be –2,000 (not –1,006 or –1,007). To support her judgment, Kail makes reference to a mathematical definition established

[3] An analysis of a video record revealed that Carol appears to take the distance between –1,000 and her mark of –1,001 as a unit interval (see Fig. 11.3). She translates that interval to the unlabeled tick mark about 6 or 7 times, yielding the label "–1,006 or –1,007."

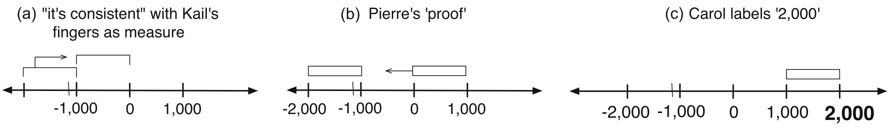


Fig. 11.4 Kail and Pierre's uptake on Carol's placement, and the shift in Carol's thinking

in a prior lesson, gesturing to the interval between the unmarked point (−2,000) and −1,000 and then to the other interval between −1,000 and 0, as depicted in Fig. 11.4a, saying that, "they are consistent," a term used repeatedly in prior lessons. Some students voice their agreement with Kail, and Kail proceeds to label −2,000 in its place.

Ms. R gives Carol another opportunity to justify her thinking in light of Kail's alternative. Carol struggles, appears unsure, but despite the class support for Kail's solution, Carol is unrelenting in her assertion that the unlabeled mark should be called −1,006 or −1,007. Although students (and teacher) do appear to have a host of compatible presuppositions about number lines that supports a level of communication, what is salient at the moment are two contradictory names for the unlabeled point: Carol's insistence on −1,006 or −1,007 as contrasted with Kail's insistence that the point should be called −2,000.

Ms. R invites Pierre to the board perhaps to help the class work towards a collective resolution. Pierre thinks, grabs a yellow wooden Cuisenaire™ rod (C-rod), and fits it between the interval from 0 and 1,000 and again at the interval between −1,000 and −2,000 as shown in Fig. 11.4b.[4] Corroborating Kail's solution but using a different approach, he explains, "this (the interval between 0 and 1,000) is the same thing as this (the interval between −1,000 and −2,000). {…} therefore, this {points to the tick mark} is −2,000." At Ms. R's urging, the class notes that the yellow rod is equivalent to a value of 1,000 units on the line.

After reflecting on Pierre's C-rod display, Carol now appears to have an epiphany. She announces that after listening to the discussion, she agrees that the tick mark should be labeled as −2,000. In a move that shows that she grasps his explanation, Carol grabs a yellow C-rod, places it at the interval between −1,000 and 0, then moves the rod to the interval between 0 and 1,000, saying that "this is the same as this." Carol then, responding to Ms. R's request, further illustrates her grasp of Kail and Pierre's solution, additionally placing 2,000 on the number line using the yellow C-rod (Fig. 11.4c).

In this episode, we found that students were able to communicate with one another successfully, despite numerous threats of breakdown (which does often occur). How was this successful communication achieved? In the next section, we draw upon our prior distinction between collective and individual levels of analysis to understand how the classroom community drew upon and contributed to a common ground of talk and action.

[4] Cuisenaire™ rods are manipulative wooden rods used in Ms. R's classroom.

11.1.3 Analyzing Common Ground at Collective and Individual Levels

To understand more about the process whereby a common ground of talk was generated in the integers discussion in Ms. R's classroom, we argue that two levels of analysis are needed, one at the collective and the other at the individual level. We take these levels to be mutually constitutive, with the collective providing form and social meaning for individual activity, and individuals' actions creating the collective.

11.1.3.1 The Collective Level

At the collective level, our focus is on norms, taken-as-shared expectations about the range of appropriate actions and communications in the classroom community. We take norms to constitute part of the common ground of talk and action that supports successful sense making of the intended meaning of others' displays. At the same time, because norms are inherently fluid, they are also part of the generation and re-generation of a common ground in collective life. In this chapter, we focus on participation norms and sociomathematical norms (Valentine et al. 2005; Yackel and Cobb 1996).

Consider first participation norms that capture expectations and obligations for how individuals participate in discussion, like norms for taking turns in conversational moves or displaying one's thinking clearly to an audience, akin to Gricean maxims (cf. Grice 1989; Keller 1994). Let's return to the short excerpt from Ms. R's classroom and an illustration of a norm for which we found considerable support: One's display should be clearly visible and audible to the class as a whole.

One illustration for the display norm comes when Pierre approaches the board to share his solution. As he begins talking and making marks, Ms. R repeatedly asks him to "tell us, not the board." Once Pierre has turned around to address the class and speaks louder, Ms. R allows him to continue his explanation. On other occasions we see students themselves enacting the public display norm. Consider the interchange between Carol and Kail. They were both standing at the board, having just offered different solutions. At Ms. R's request, Carol attempts to justify her original solution in response to Kail's objections. She begins talking and moves in closer to Kail to point to certain features of the inscription. She then gestures to Kail to move backwards (pictured in Fig. 11.5), presumably positioning herself so that she "has the floor" and the class can better see her display (see Davies, and Harré 1990; Harré et al. 2009).

Pierre and Ms. R's interaction and Carol and Kail's interaction both reproduce this display norm, in the sense that they are drawing upon expectations for behavior that undoubtedly have their origins in the history of this classroom community. At the same time, these exchanges are also contributing to the evolution of this norm, carrying it forward in ways that are inherently unique to present circumstances,

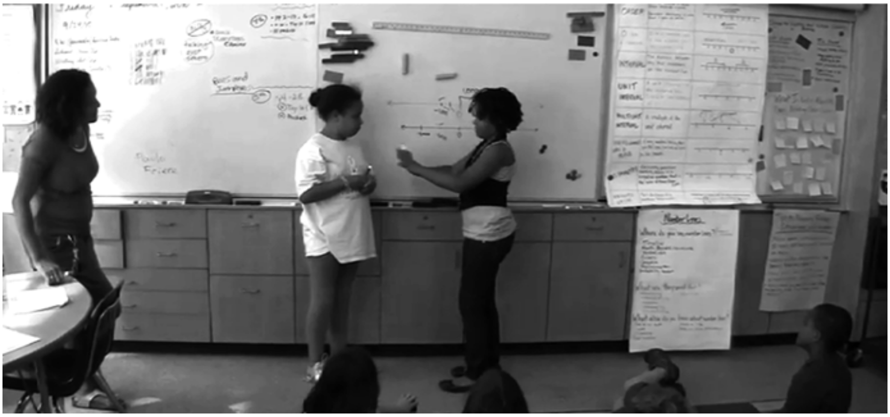


Fig. 11.5 Carol positioning Kail so that the class will be better able to see her (Carol's) display

and in ways that may reverberate in future lessons. Hence, Pierre and Ms. R, and Carol and Kail, are simultaneously drawing on and contributing to common ground at the collective level.

By its very nature, this specific display norm (dictating that one's display should be clearly visible and audible to the class as a whole) would make it more likely that individuals will grasp each other's intended meanings. Pierre's way of making his thinking accessible to the class afforded Carol's reflection on new ways to solve the problem. This apparent norm for participation is therefore part of the common ground that enables successful communication in this episode.

Now consider sociomathematical norms. Unlike participation norms, sociomathematical norms are specific to mathematical explanation and argumentation, capturing a set of collective expectations for how individuals should explain their thinking and justify their solutions (Yackel and Cobb 1996). In the case of Ms. R's classroom, we find that from early on in the lesson series, Ms. R worked with the class to formulate mathematical definitions of key ideas and displayed these definitions on a public poster (in Fig. 11.5 the poster is visible to the right of the students). Ms. R elevated terms like "consistent" as key to the use of unit intervals and multiunit intervals in both generating labels for unlabeled tick marks and creating points for numbers on the number line (like labels for the position of −2,000). It was the classroom community's taken-as-shared expectation that such terms would be used in mathematical explanation and argumentation.

We see evidence of the enactment of this norm in students' elliptical references to "consistent." For example, recall that Kail argued that the unlabeled tick mark would be −2,000 because "it's consistent," gesturing toward the interval between 0 and −1,000 and the adjacent interval between −1,000 and the unlabeled point. Responding to Carol's subsequent defense of her solution, Ms. R said to the class that "we're forgetting a very important principle" and called Pierre up to the board, repeatedly asking him to articulate this principle. Pierre's mention of "consistent" is emphatically repeated by Ms. R. Carol herself appealed to "consistent" in explaining why she changed her mind about the value of the unlabeled tick mark, and why she was persuaded by Kail and Pierre's explanations. Based on these observations, we surmise that use of the word 'consistent' and appeals to the idea it indexes are a valued form of explanation and argumentation in Ms. R's class.

Like the participation norm, this sociomathematical norm is part of the common ground that supported successful communication and learning in this episode. We observe that students appeal to "consistent" as a valued resource for explanation and justification, and in a way that assumes the import of the term is familiar to their audience. In enacting this norm, members of the class were able to convey intended meanings, negotiate different solutions and, ultimately, to persuade each other. This sociomathematical norm, which undoubtedly has a history in prior lessons, is reproduced in this episode. At the same time, students are extending the norm to a new and challenging problem, making it part of the common ground that the classroom community is working to construct.

As we have emphasized, the status of norms as an object of inquiry is not fixed. Their existence is fleeting, appearing only in the productive and interpretive actions of individuals. For this reason, we take norms to be reproduced and altered in activity. Thus, an analysis of norms, whether participation or sociomathematical, requires an analysis over time (or lessons). The norms that we have pointed to in this section no doubt have origins in prior lessons and have projections into future lessons. In Sect. 11.2, we take up empirical techniques geared for exploring origins of norms in Ms. R's classroom community.

11.1.3.2 The Individual Level

At the individual level, we focus on how individuals' public displays tailor representational forms to serve varied functions in classroom communities. At this level, common ground is reproduced and altered in individuals' construction of form-function relations through time.

Figure 11.6 contains exemplars of forms (left) and functions (right) that we have observed in LMR lessons. The forms depicted in Fig. 11.6 include geometric forms such as elements of the number line, including the line, arrowheads, tick marks as well as kinds of intervals (unit intervals, multiunit intervals)[5]; numeric forms like number words and written numerals; manipulative forms like C-rods of ten different color-coded lengths, like the yellow rod referred to the exchange in Ms. R's classroom; and they also include mathematical definitions introduced in the lessons, such as definitions for unit interval, multiunit interval, and order (recall reference to "consistent" linked to a definition introduced in Ms. R's classroom).

[5] We note that conventions vary in the use of forms. For example, in the United States in elementary mathematics classrooms, s contain arrows on the left and right ends, indicating that lines are extended in both directions. In contrast in other countries, number lines are sometimes depicted as rays, with arrowheads on the right end only.

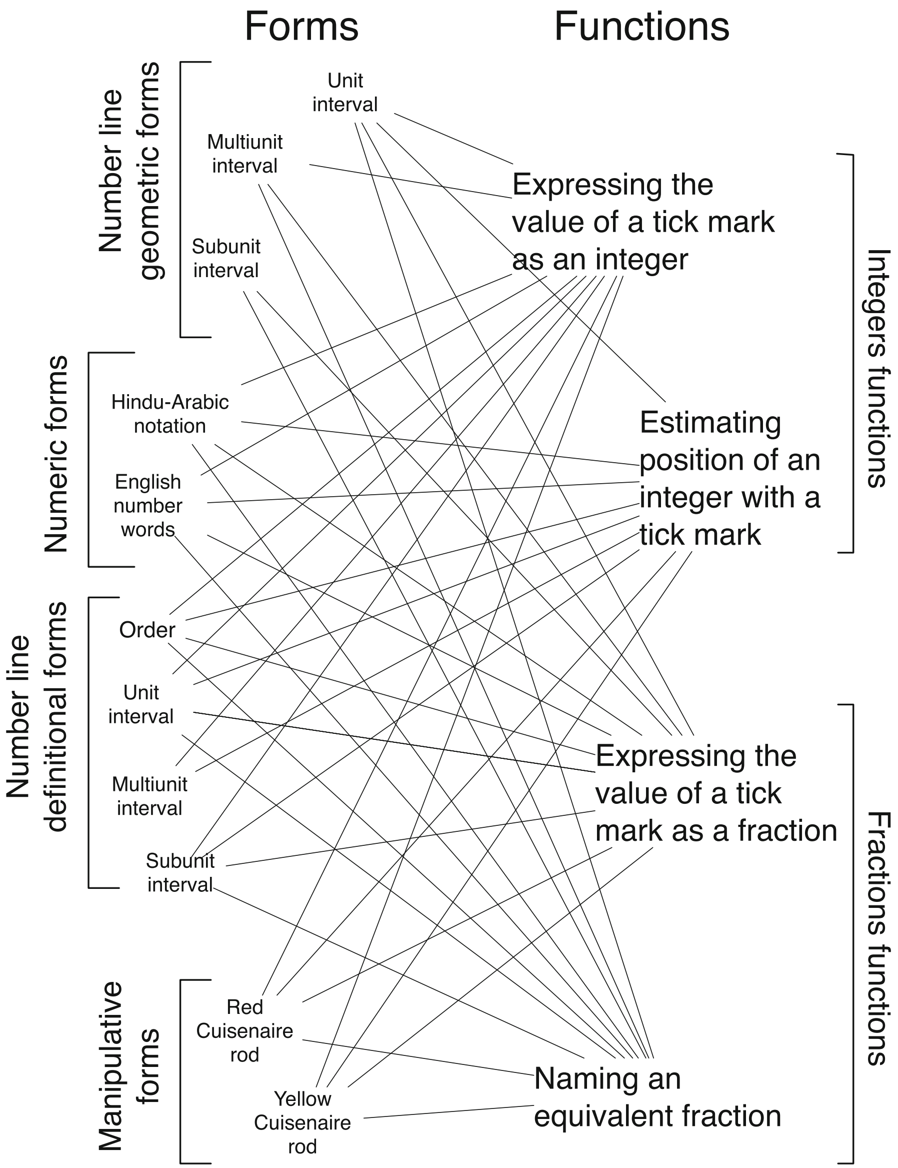


Fig. 11.6 Some exemplar forms, functions, and form-function relations

Like the forms depicted in Fig. 11.6, the possible functions that these forms afford are varied and many. Examples include (a) expressing the numerical value of a tick mark with an integer value, as in the interchange between Carol, Kail, and Pierre (–1,006 or –2,000), and (b) estimating a place for a number on the line, as when the class positioned –1,001 on the line (however imprecisely).

The lines connecting forms and functions in Fig. 11.6 show that forms may be tailored to serve multiple functions, and the listed functions may be served by any number of forms, often used in concert with each other. For example, recall students' use of different forms to serve the same function in the prior exchange: Carol used a unit interval form (the interval between −1,000 and −1,001) to express the value of the unlabeled tick mark as an integer (−1,006 or −1,007). By contrast, Kail and Pierre used a multiunit interval form to express the value of the same tick mark as −2,000. At the same time, we might imagine that a geometric form on the number line can be used to serve any number of functions. One can imagine, for example, how a unit interval might enter into naming an equivalent fraction through partitioning it into equivalent lengths (subunits), and at the same time enter into the estimation of where an integer is located on the same line as Carol did in her display.

Three genetic strands are needed to understand the generation of a common ground of forms and functions from the perspective of individual activity. A microgenetic strand allows for analysis of the construction of representations, as individuals tailor representational forms to serve functions. An ontogenetic strand allows for the analysis of continuities and discontinuities as individuals reproduce and alter form-function relations in their own learning trajectories. Finally, a sociogenetic strand allows for the analysis of distributions in individuals' use of forms to serve functions in a community, both at a single moment in time and as distributions shift. Although the genetic strands are grounded in the same activity, we consider each strand separately for purposes of analysis.

Microgenesis

The microgenesis of representations involves tailoring forms to serve functions as problems are conceptualized and accomplished, often in public displays. In this process, individuals draw upon a common ground of representational forms and functions, reproducing and altering them as they work to communicate and address recurring problems. Thus, a microgenetic analysis illuminates the way individuals contribute to the generation of a common ground. As an illustration, let's consider the classroom episode described previously and Carol's microgenetic construction (represented in Fig. 11.7) in which she called Ms. R's unlabeled point, "–1,006 or –1,007."

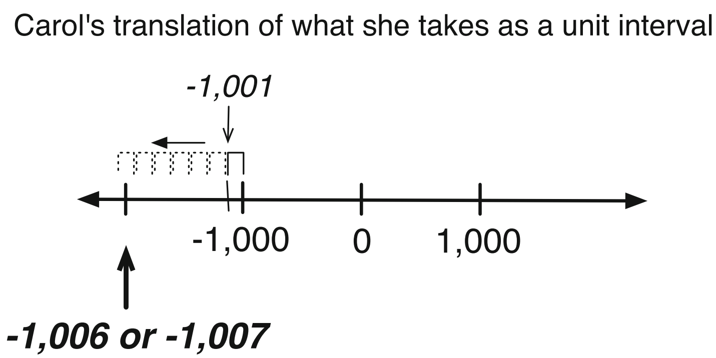


Fig. 11.7 Carol's microgenetic construction

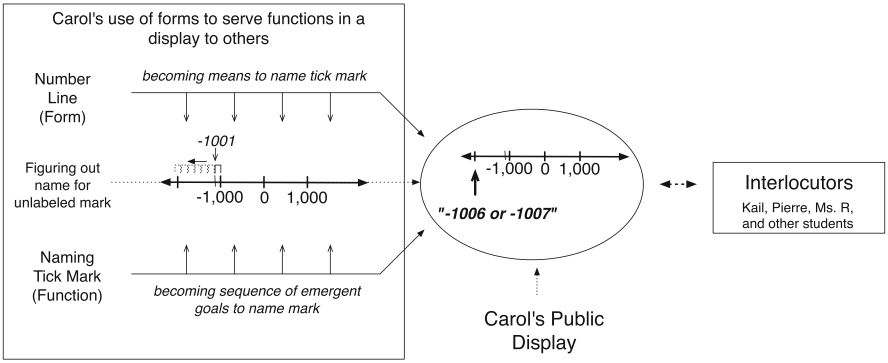


Fig. 11.8 Schematic of Carol's microgenetic process of naming the unlabeled point

Figure 11.8 contains a schematic of what we regard as key elements of Carol's microgenetic construction. In the middle section of the figure we show Carol's public display that named a point on the line. To the right we depict Carol's audience—those to whom Carol is making efforts to communicate, adjusting her talk and actions to what she conceives as the communicative constrains of the situation. To the left is the process whereby Carol draws upon what she takes to be shared forms of representation and orchestrates them to serve the function of naming the unlabeled point on the line in a way that communicates to others. In this process she turns the interval from −1,001 to −1,000 (a mathematical form) into a means by iterating it successively to the left as she accomplishes a sequence of emergent goals linked to her approach to naming the unlabeled tick mark. The process is one in which she both reproduces a host of aspects of the class' common ground of talk—number words and unit intervals—but also alters them to suit the function of naming the unlabeled mark.

Two features are noteworthy in Carol's construction, features that are generally true of microgenetic constructions. First, Carol's microgenetic construction is a process of recruiting forms with no inherent functions. The initial marking of the −1,001 point by the class served a function (estimating the position of a point) that was not directly related to the function that it served for Carol: to name an unlabeled tick mark. Rather, Carol herself turned the unit interval of −1,001 to −1,000 into a means to locate the unlabeled point.

Second, the representation itself is coherent. It emerges out of Carol's structuring the unit distance (−1,001 to −1,000) in what we take to be a successive iteration; each translation is produced such that the left end point of one imaginary translation is the right end point of the subsequent translation. Further, each translation is accumulated and coordinated with numerical units and number word forms, such that six translations is equivalent to the value of 6 on the number line.

As we noted earlier, individuals' microgenetic constructions in public displays, like Carol's, contribute to the reproduction and alteration of a common ground. In Carol's microgenetic construction of a form-function relation (a unit interval form to name a tick mark), she reproduces translation approaches that she has observed in prior classroom activities, though her particular display is also an alteration. Her display makes use of a particular unit interval on the line on the whiteboard, an interval quite imperfectly drawn, and produces a novel solution, naming the tick mark at the −2,000 position as "−1,006 or −1,007." Further, her display creates a context for subsequent displays of her classmates, which also become ingredient to the reproduction and alteration of a common ground of the classroom community.

Ontogenesis

The ontogenesis of form-function relations involves continuities and discontinuities in learning trajectories of individuals. Of particular interest are individuals' use of forms that are familiar to them (continuity) to serve new functions (discontinuity) in activity, and individuals' use of new forms (discontinuity) to serve developmentally prior functions (continuity). Such continuities and discontinuities in individuals' learning trajectories shape their microgenetic displays. For this reason, students' ontogenetic trajectories have implications for the common ground of the classroom community. Thus, ontogenetic analyses illuminate shifts in microgenetic processes, and by extension, the reproduction and alteration of a common ground.

To illustrate, at the point of the featured episode from Ms. R's class, students have participated in a number of prior lessons, and in the process have created their own learning trajectories, drawing upon what they had learned to fashion solutions to novel problems. From an ontogenetic perspective, the problem discussed by Kail, Carol, and Pierre represents a significant moment. After all, it was only in the prior lesson that negative numbers were introduced, and the problem they are solving is the first involving both large intervals (e.g. from 0 to −1,000) and small intervals (e.g. from −1,000 to −1,001) on the same representation. This problem takes students into new territory, calling upon them to substantially extend and coordinate their understandings of numerical order, unit intervals, and multiunit intervals. Even within the boundaries of a single episode of problem solving, we can gain some insight into construction of learning trajectories. Carol in particular provides a useful case in illustrating ontogenetic shifts in form-function relations.

Carol's apparent shift in thinking—albeit over a short frame of time and in the context of a single problem—nonetheless represents an important ontogenetic development. Her initial strategy was to use the unit interval between −1,000 and −1,001 as a key resource, which she iterated until she reached the unlabeled tick mark. After watching Kail and Pierre's displays and hearing their reasoning, she suddenly expressed agreement that the unlabeled tick mark should be labeled as −2,000 instead of −1,006. She gives voice to her new understanding, which clearly builds on her classmates' explanations. Recall that she grabs a yellow Cuisenaire™ rod, places it at the interval between −1,000 and 0, then moves the rod to the interval between 0 and 1,000, saying that "this is the same as this." In characterizing this ontogenetic development, we would say that Carol shifted in the representational form she uses to serve the function of labeling the unlabeled tick mark.

We can also begin to frame questions about the ontogenetic process that underlies the shift in her thinking. In particular, we can inquire about why Carol abandoned her original answer of −1,006. Certainly, the social forces at play—the pushback she received from her classmates—constitute part of the story. Still, Carol initially attempted to refute Kail's argument, demonstrating her willingness to resist being challenged. We take her initial resistance as an indication that a purely social explanation is inadequate. Indeed, what is missing from this explanation is an account of why she became dissatisfied with her original thinking and came to understand Kail's alternative as a better solution. Because our discussion here is limited to the data of her public communications in this episode, we could only speculate about this aspect of her ontogenetic process.

Inquiring into Carol's ontogenetic process also entails identifying the conceptual resources she may have for constructing a new solution that stands in contrast to her initial thinking. What understandings might she have developed in prior lessons that would support her in making sense of Kail and Pierre's displays? We know, for instance, that in prior lessons Carol has displayed knowledge of units, multiunits, and C-rods, and the ways in which they can be used as resources in problem solving. In all likelihood, she brought this knowledge to bear on understanding and evaluating Kail and Pierre's displays.

Sociogenesis

Sociogenetic processes also involve microgenetic constructions of forms and functions produced in individual activities. But sociogenetic analyses require attention to the way microgenetic constructions are distributed over individuals. That is, a sociogenetic analysis considers how individual displays are part of a wider distribution of form-function relations, as well as how such distributions shift over time. Thus, a sociogenetic analysis illuminates the reproduction and alteration of a common ground of form-function relations, taking multiple individuals into account in a classroom community. Let's return to the interchange in Ms. R's classroom to illustrate.

Consider that Carol's microgenetic construction of −1,006 or −1,007 was not the only way in which forms and functions were elaborated to name the unlabeled point. Indeed, Kail's microgenetic construction (Fig. 11.9a) provides an interesting contrast. Recall that moments after Carol's display, Kail came to the board to correct what she perceived to be an error in Carol's answer. She immediately asserts the product of her microgenetic construction, which is a label of −2,000 for the unlabeled tick mark. She then gives the class a window into the microgenetic process through which she reached this answer. She points out that the interval between the unmarked point and −1,000 is the same size as the interval between −1,000 and 0. Like Carol, Kail also used a given interval on the line as a resource in her problem solving. However, Kail's use of the interval from 0 and −1,000, which she conceptualizes as a value of 1,000, results in a different solution. In this move, her construction avoids the imprecise measurement of Carol's unit interval of −1,001 to −1,000.

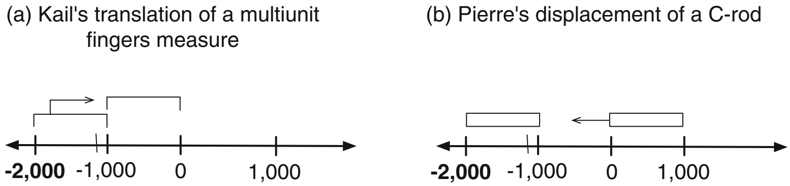


Fig. 11.9 The alternate microgenetic constructions of (**a**) Kail and (**b**) Pierre

Pierre also produced a microgenetic construction that corroborated Kail's analysis, but he used a different form. Recall that, after Kail made her display, Carol remained unconvinced of the solution of –2,000 and Ms. R called Pierre up to the board. Pierre grabbed a C-rod fitted to the interval from 0 to 1,000. He translated the rod, placing it over the interval from –1,000 to the unlabeled tick mark. He explained, "this (the interval between 0 and 1,000) is the same thing as this (the interval between –1,000 and –2,000) {…} therefore, this {points to the unlabeled tick mark} is –2,000."

Carol's, Kail's, and Pierre's displays were public constructions that made use of different forms to serve similar functions during the same interactional exchange. There may well be many unnamed students who also engaged in varied microgenetic constructions that were undocumented in students' displays. To understand a common ground of talk and action at the level of individual activity, we need to move beyond a level of analyzing the microgenetic processes of single constructions of form-function relations. Rather, what is needed is an analysis of the distribution of form-function constructions across individuals.

The distributional properties of form-function relations are not fixed in a classroom community but are themselves in motion, shifting through time. Indeed, if we were to examine an interchange during a subsequent lesson, we would likely observe shifts in the distribution of forms students used and the functions that they used them to serve. For example, we might find that a preponderance of (but not all) students are using a newly introduced form to serve a familiar function. We might also find that some students are tailoring familiar forms to serve new functions, adapting them to novel problems. To understand the shifting common ground of talk and action requires approaches to capturing both the character of such distributional shifts and the dynamics that lead to them.

There are multiple sources that could lead to distributional shifts in form-function relations over time. One source is the emergence of novel problems in the classroom community. As we mentioned, the problem of placing negative numbers and intervals of such discrepant sizes was relatively novel to students. To solve this problem and explain their reasoning, Carol, Kail, and Pierre each had to adapt familiar forms to suit this new mathematical context, tailoring them into problem solving and communicative resources in new ways. In doing so, they appeared at once attentive to valued forms in the history of the classroom community—with Kail's elliptical reference to a mathematical definition ("consistent") and Pierre's use of the yellow C-rod—but at the same time altered the use of these forms relative to their treatment of the problem.

Another source contributing to distributional shifts are the social positions that individuals take up in the classroom community. In their displays, individuals position themselves and are positioned by others. Let's consider Carol's epiphany at the end of the classroom episode. It may well be that Carol regards Kail and Pierre as mathematically competent and her reflections on their displays and eventual epiphany were influenced by her regard for them. In this shift, Carol draws upon the C-rod form and multiunit form to confirm the name for the unlabeled mark as −2,000. What is revealed here is a shift in the distribution of form-function relations in the classroom community, possibly linked to students' social positions. These kinds of shifts capture the reproduction and alteration of a common ground of form-function relations in a classroom community.

The Interplay Between Micro-, Onto-, and Sociogenetic Developments in Collective Activities

We have argued in the prior pages that microgenetic, ontogenetic, and sociogenetic processes do not occur in isolation from one another. To the contrary, they are intrinsically related. Each microgenetic construction is a point in an individual's ontogenetic development as the individual adapts forms to serve communicative and problem solving functions. Further, in a microgenetic act, the individual draws from and contributes to the sociogenesis of form-function relations. Indeed, over a period of time in any community, multiple individuals are reproducing and altering form-function relations as they produce displays and interpret displays of others. These many microgenetic acts, spread across a community, result in continuities and discontinuities in the sometimes stable and other times shifting distributions of form-function relations.

We close our discussion of the individual level of analysis with a schematization contained in Fig. 11.10. The schematization captures not only the intrinsic relations between micro-, socio-, and ontogenetic processes in activity, but also their interplay through time. The horizontal organization of the figure presents three individuals, I1, I2, and I3. The vertical organization depicts three time periods, past, present, and future. To understand the interplay between the genetic processes, consider first the microgenetic act of Student #2 (middle row) in the present time (middle column), depicted as microgenesis2b (in bold). In that microgenetic act, Student #2 is engaged in solving a problem in a public display by drawing on prior forms and functions, as when Carol draws upon and translates the unit interval form to label a point on the number line. Also in present time, students #1 (upper row) and #3 (lower row) are solving the same problem (microgenesis2a and microgenesis2c); these varied constructions are enabled and constrained by students' own prior ontogenetic constructions (microgenetic constructions 1a, 1b, and 1c) and their efforts to link their displays to the interpretive efforts of their interlocutors. The three microgenetic displays (microgenesis2a, microgenesis2b, and microgenesis2c) constitute the distribution of form-function relations in the community in present time, distributions that have roots in past time and have implications for the future

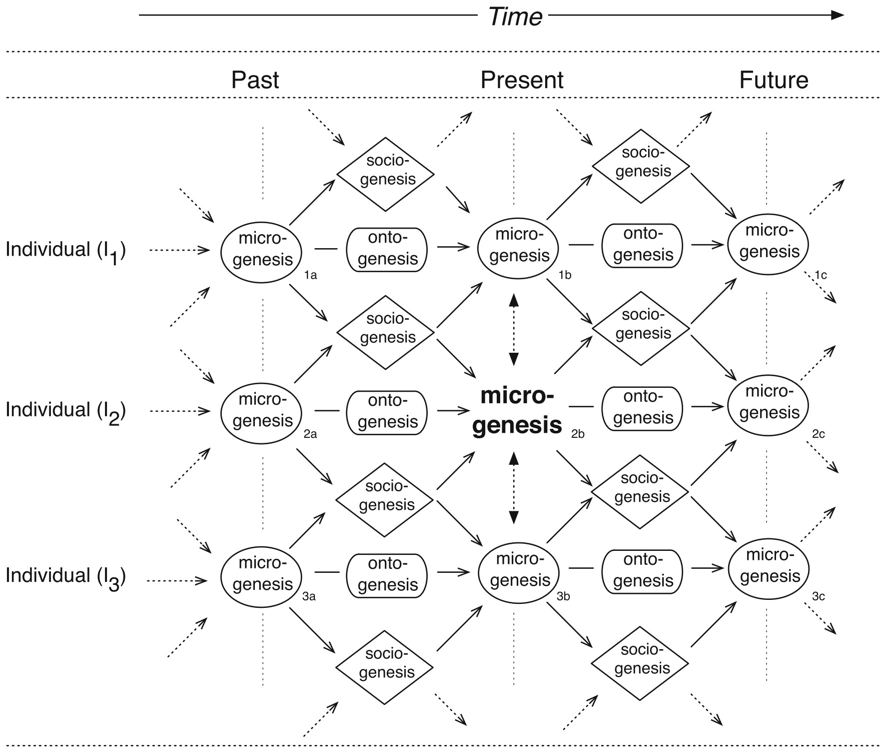


Fig. 11.10 The interplay between micro-, onto-, and sociogenetic processes through time in a classroom community (Figure adapted from Saxe 2012, p 326)

(sociogenesis). Together, they are a constitutive dynamic that results in the reproduction and alteration of a common ground through time in communities.

11.1.3.3 A Final Note on Collective and Individual Activity

In our analytic approach to the reproduction and alteration of a common ground, individual and collective activity become part of the same frame for analysis. On the one hand, collective activities are constituted by the concerted work of individuals at moments in the microgenesis, sociogenesis, and ontogenesis of form-function relations. On the other hand, individuals' activities take on social meaning in relation to social norms, conventions, artifacts and institutions. The reflexive process in which individual and collective activity take form in relation to one another leads to both continuities and discontinuities in the reproduction and alteration of a common ground. This shifting common ground is constituted by both the elaboration of form-function relations and the reproduction and alteration of collective norms.

11.2 An Illustration of Empirical Techniques: The Learning Mathematics Through Representations Project

We partition our description of empirical techniques into two sub-sections, both of which are grounded in the conceptual framework described in Sect. 11.1. First, we illustrate empirical techniques for our approach to design research (for a more general discussion of design research in educational research see Cobb et al. 2003; Lehrer and Schauble 2004). Our main goal was to make research-informed design choices in constructing a lesson sequence. Throughout, our intention was to support a common ground of talk and action. Second, we illustrate empirical techniques for analyzing how a common ground is reproduced and altered over time in classroom communities. To this end, we draw upon extended observations from Ms. R's classroom.

11.2.1 Empirical Techniques Used to Inform Design Choices for the LMR Lesson Sequence

Our first application of the framework is in design research that would culminate in the LMR lesson sequence. We detail how the conceptual framework informed key design choices through the use of a broad range of empirical techniques, and we review key aspects of the product of this work: a complete version of the lessons.

11.2.1.1 Preliminaries

At the outset of the design research, it was our intention to develop a lesson sequence for the upper elementary grades. The targeted domains were integers and fractions, hard-to-learn ideas central to the transition between elementary and secondary mathematics. We were well aware of critiques of curricular and instructional approaches to integers and fractions and our concern was to address them. For example, coverage of these domains in instruction is diffuse—many topical issues are covered, but the coverage often lacks depth (Schmidt et al. 1997). Further, integers and fractions are often treated as entirely separate topics, even though they are deeply related.

To address the problems of diffuse and superficial treatment of integers and fractions, we planned to develop a curricular approach that would engage a classroom community with a progressive elaboration of a common ground of talk and action. In accord with the discussion presented in Sect. 11.1, we would need to be attentive to collective and individual levels of activity to inform design choices, and empirical techniques would need to be geared accordingly. At the individual level, our focus would be on supporting the micro-, onto-, and sociogenesis of form-function relations in students' generation (and re-generation) of a rich common ground

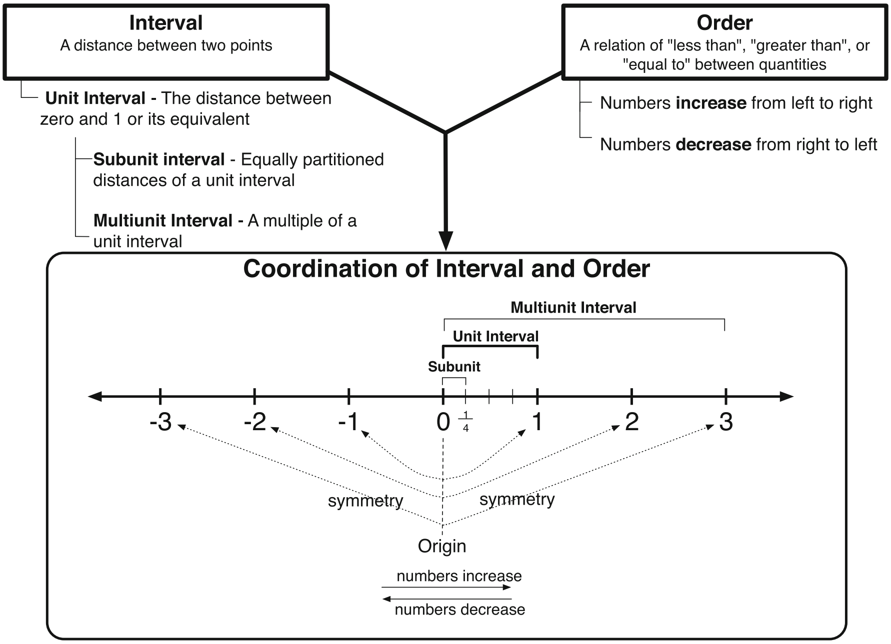


Fig. 11.11 Core ideas definitional to number line representations

through a central mathematical terrain. At the collective level, our focus would be on supporting productive interactions (participation norms) and conceptually oriented argumentation (sociomathematical norms) in the community (part and parcel to a rich common ground).

A foundational design choice that we considered early in the LMR development effort was whether to use the number line as a principal representational form through the lessons. There were strong reasons for using the number line. Across lessons, the number line could support continuity of representational forms (curricular coherence) in the context of discontinuity of topics across domains of integers and fractions[6] (Wu 2008, 2011). Consider, for example, Fig. 11.11, which contains the key number line elements that supported our curricular approach. As indicated in the figure, we planned to treat the number line as a coordination of two ideas at the crux of a linear measurement model of number: Order relations and metric relations, with order defined on the line as an increase in magnitude from left to right, and metric relations defined in terms of line segments or intervals on the line.

[6] Throughout the chapter we make use of the term "fractions" where "rational number" would often be more appropriate. We take "fractions" to be forms of representing rational numbers—a number expressed with a numerator and denominator (common fraction, mixed numbers) or a number expressed as a decimal (decimal fraction), and rational numbers to be the numbers expressed by these representational forms. For simplicity, we use the expression "fractions" to refer sometimes to rational numbers and other times to the representational form of common fractions (including proper and improper common fractions).

For integers, these topics would engage issues of order and different kinds of units (units and multiunits); they would also engage related properties of symmetry of positive and negative integers and absolute value and origin. Fractions would additionally entail a treatment of subunits and a related host of topics rooted in a treatment of subunits (e.g., equivalent fractions, benchmark fractions, fractions less than one, fractions greater than one). In this way, using the number line as a consistent representational context promised to support continuity across lessons as topics shifted, fostering a common ground of talk and action through some difficult mathematical terrain.

11.2.1.2 Empirical Techniques and Design Choices

We drew upon a host of empirical techniques to evaluate the utility of the number line as a central representational form across lessons and to inform various secondary but no less important design choices. At the highest level, our empirical techniques included various study designs; these consisted of interview, tutorial, classroom, and efficacy studies, each of which had a role in supporting particular design choices. Further, within each study type, we created data collection techniques (e.g., approaches to video recording) and data reduction techniques (e.g., approaches to the analysis of video records, qualitative and quantitative methods for analysis).

Table 11.1 provides an overview of our design research that led to our lessons. The first column describes our empirical techniques in the form of different study types. The second column describes the purpose of each study type and how the empirical techniques extended the conceptual framework. The third column summarizes major findings of each study type, and the final column lists key design choices that followed from the findings.

Interview Studies

The interview studies were designed to reveal patterns in students' microgenetic constructions as they made use of number line forms to serve problem solving functions in varied contexts. All studies were conducted with samples of fifth grade students from urban classrooms, and they used a clinical interview format in which a researcher presented number line problems to students. While several interview studies were conducted, here we illustrate findings from two studies that focused on integers (see Saxe et al. 2013).

In one study, we examined students' microgenetic constructions on non-routine number line problems ($n=24$). We had good reasons to expect that non-routine problems would reveal student understandings and confusions in ways that more routine problems would not (Ginsburg 1997; Piaget 1979). Such tasks would problematize students' well-used procedures for problem solving that bypass conceptual analysis. For the purposes of this study, we defined non-routine problems as featuring

Table 11.1 Illustrations of empirical techniques and their purposes in supporting design choices

Empirical techniques: study types	Studies' purposes in relation to constructs	Design-related findings and design decisions	
		Finding	Decision
Interview Studies on Integer Problems. Engaged students (n = 24) in solving non-routine number line problems in varied contexts	*Microgenesis of form-function relations*. To examine how students, in their microgenetic constructions across varied non-routine problem types and thematic contexts, made use of elemental forms of the number line (e.g., units, multiunits) to serve problem solving functions	*Number line.* Number line could be used to build upon student intuitions (e.g., order) to support inquiry into more advanced mathematical topics	*Number line*. The number line would be used as a representational context across lessons
		Types of number line problems. Non-routine number line problems revealed student understandings and confusions	*Types of number line problems.* Lessons would privilege non-routine over routine problem types
		Unit-multiunit coordinations. Non-routine number line problems revealed many students' difficulties in coordinating units and multiunits; a racecourse number line context supported students' differentiations and coordinations	*Unit-multiunit coordinations.* Problems would be geared for supporting the coordination of units with multiunits and include racecourse contexts for particularly challenging topics
Tutorial Studies on Integer Problems. Examined efficacy of a tutorial approach (n = 19 students) as contrasted with control (n = 19 students) to support learning	*Ontogenesis* (form-function relations). To explore useful ways to support students' use of forms (definitional forms, manipulative forms, number line forms) to serve new functions as problem types shifted	*A problem sequence* that supported the progressive elaboration of big ideas (e.g., order, unit, multiunit, symmetry, absolute value) proved effective	*Problem sequencing*. The lessons would sequence problems with the aim of supporting the progressive elaboration of big ideas
	Norms (socio-mathematical). To investigate how norms could be promoted in a way that supports a productive common ground	*C-rod forms*. Students were able to use C-rods to serve communication and problem solving functions related to big ideas (e.g., concatenating and translating rods on the number line)	*C-rod forms*. C-rods would be introduced and re-introduced as lesson topics shifted
		Definitional forms. Definitional forms became norms for communicating and explaining, and resources for solving number line problems	*Definitional forms*. Definitional forms would feature prominently across lessons to support big ideas (e.g., order, interval, unit, etc.)

(continued)

Table 11.1 (continued)

Empirical techniques: study types	Studies' purposes in relation to constructs	Design-related findings and design decisions	
		Finding	Decision
Classroom Studies on Integers and Fractions Lessons. Iteratively refined lessons over the course of 2 years with partner teachers (n=4)	*Microgenesis*. To support our understanding of student thinking as they solved lesson problems	*Phase structure*. Six-phase structure of lessons proved useful but in need of refinement	*Phase structure*. Lessons would feature a five-phase structure (opening problems, opening discussion, partner work, closing discussion and closing problems)
	Ontogenesis. (a) To corroborate and modify initial conjectures on sequencing of lessons (b) To assess student learning gains through participation in lessons	*Problems*. Opening and closing problems provided useful formative assessments, but some problems proved to be too challenging and others not challenging enough	*Problems*. Lessons would feature problems that were appropriately challenging
	Norms. To create supports for the emergence of sociomathematical and participation norms	*Lesson sequencing*. Lessons clearly built upon one another, but students needed additional support between some lessons	*Lesson sequencing*. Sequence would include additional lessons prior to introduction of topics that proved overly challenging
		Definitional forms. Definitional forms were taken up in classrooms but (a) their coherence could be improved and (b) additional definitions were needed to support difficult transitions across lessons	*Definitional forms*. (a) To maximize coherence, several definitional forms would reference the same core idea (interval). (b) To support difficult transitions, new definitions would be included

Efficacy Study on the Refined Lesson Sequence.	*Ontogenesis*. To evaluate ontogenetic developments in integers and fractions in LMR and Comparison classrooms. Data source: pre, interim, post, and end of year (final) assessments	*Learning gains in LMR classrooms and comparison classrooms*. LMR classrooms performed better than Comparison classrooms: Hierarchical Linear Model (HLM) analyses showed that students in LMR classrooms did better on both number line and non-number line tasks at post and final assessments than students in Comparison classrooms	–
Evaluated whether designed lessons in LMR classrooms (n = 11) provided greater support for students' growth than in Comparison classrooms (n = 9)	*Sociogenesis*. To evaluate whether there were differential student growth curves as a function of starting levels of understandings	*No differential gains in students who began instruction at different pretest levels*. Findings produced through HLMs indicated that there was, for the most part, consistent gain across all students in LMR classrooms with minimal to no differential gain as a function of starting knowledge	

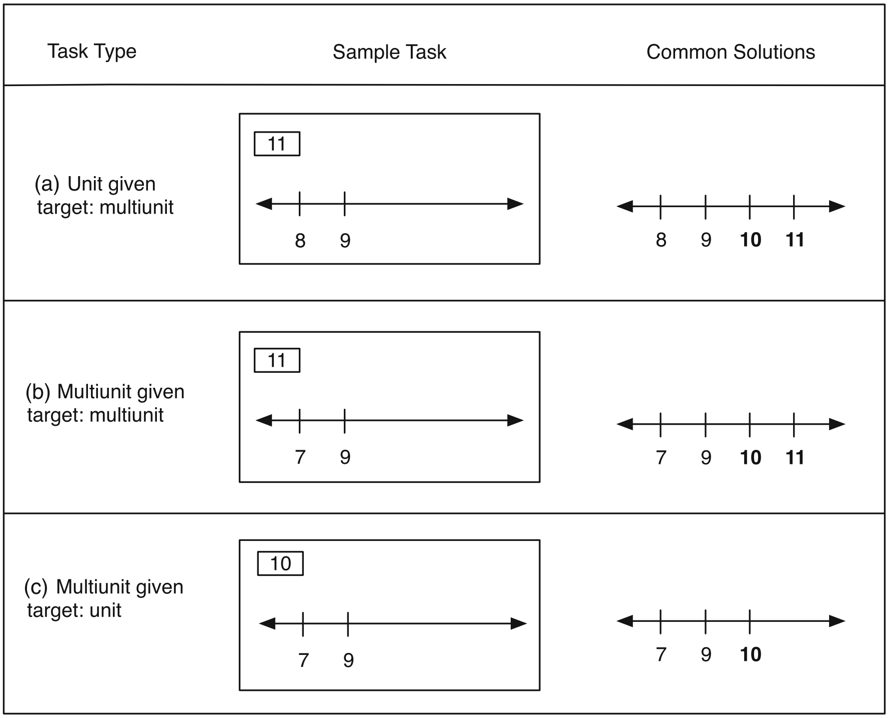


Fig. 11.12 Examples of tasks used in interview study 1

number line representations in which only two numbers are marked, and students were required to place a third number (see Fig. 11.12). Problems were of two types: One in which two consecutive numbers created a unit interval (Fig. 11.12a), and the other in which two nonconsecutive numbers created a multiunit interval (Fig. 11.12b, c). To solve the task in Fig. 11.12a (unit interval given), students needed to treat the 8,9 interval as a unit interval, placing 11 at the appropriate position. To solve the task in Fig. 11.12b (multiunit interval given), students needed to treat the 7,9 interval as a multiunit interval, placing 11 at the appropriate position. To solve the task in Fig. 11.12c, students once again needed to treat the 7,9 interval as a multiunit interval, but now must partition a multiunit interval into unit intervals.

A central finding of the study was that the non-routine problems illuminated strengths and difficulties in students' microgenetic constructions. On all versions of our tasks, all students placed numbers in the correct order, and most (90 %) completed the unit interval translation tasks correctly (tasks like Fig. 11.12a). However, when students were given a multiunit interval (tasks like Fig. 11.12b, c), their solutions were often incorrect, and the most common incorrect solution was to treat the given multiunit interval as a unit interval, not differentiating the two. Thus the common solution to Fig. 11.12b was to place 11 at the location for 13. The common solution to Fig. 11.12c was to place 10 at the location for 11. This study demonstrated that while the number line supported some student intuitions (e.g., order, unit translations), confusions surfaced in other areas, and non-routine number lines

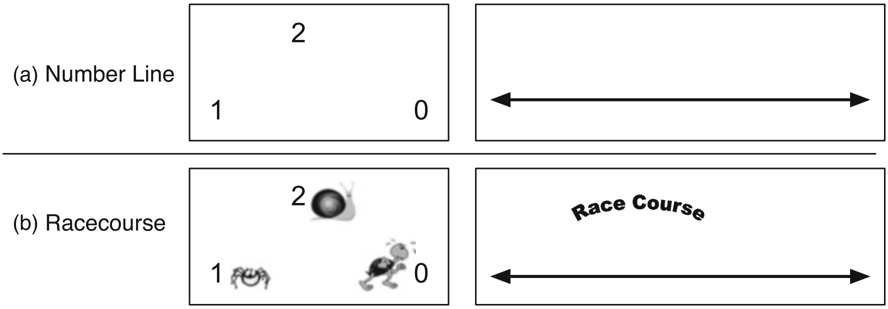


Fig. 11.13 Examples of the racecourse and number line tasks

were particularly well suited to revealing the difficulty that many students had in coordinating multiunit intervals with unit intervals.

In another interview study, we examined how a familiar measurement context, a narrative about a racecourse in the form of a number line, can support students' microgenetic constructions. Of particular interest was whether the racecourse might support students' differentiation and coordination of units and multiunits. To this end, we assigned one group of 5th grade students (n = 24) to a condition in which they placed numbers on conventional number lines, and the second group of fifth graders (n = 24) were assigned to a condition in which they placed cartoon characters on number lines represented as 'racecourses' (Fig. 11.13). Most students in both groups placed consecutive whole numbers (e.g., 5, 6, 7) at appropriate linear distances, but the racecourse group was more likely to place non-consecutive whole numbers at appropriate linear distances (e.g., partial consecutive sequences such as 9, 12, 13 or non-consecutive sequences such as 7, 11, 14). These results indicate that students' differentiation and coordination of units and multiunits was aided by the addition of the racecourse context.

Overall, interview study findings informed design choices in the development of the lesson sequence. Both studies corroborated our sense that the number line would be a useful representational context. Prompted by the first study, we decided to consistently feature non-routine number line problems in the lessons. These problems elicited student thinking and revealed students' difficulty with differentiating and coordinating units and multiunits. Further, because of the challenges that students had with coordinating units with multiunits, we would design problems geared for supporting the differentiation and coordination of these ideas, like those featuring a racecourse context.

Tutorial Studies

To explore ways of supporting ontogenetic progression towards a generative use of units and multiunits on the number line, we designed a tutorial study (see Saxe et al., 2010). Our overarching concern was to explore ways of supporting learning through the generation of a common ground of talk and action between tutor and student. At the individual level, our concern shifted from microgenesis to ontogenesis.

We wanted to know if student intuitions about the number line representation could be used as a productive foundation for the development of rich and generative understandings. Our tack was to introduce particular forms—definitions and C-rods—and attempt to support students in using these forms in a coherent and coordinated way to serve an expanding repertoire of functions, both in solving problems and justifying their reasoning. At the same time, the tutorial study was also motivated by our concern with the functioning of common ground at the collective level. We were interested in how ways of using these forms might become constitutive of the sociomathematical norms that emerged between tutor and student, and in turn how these norms might support learning.

The tutorial was organized as a sequence of 13 number line tasks partitioned in two sessions. To evaluate the efficacy of the tutorial design, we conducted an experimental study. We administered a pretest to fifth grade students and also asked teachers to rate students' overall performance in mathematics. Using these two indicators of student knowledge, we then created two matched groups through random assignment: a tutorial group ($n = 19$) and a control group ($n = 19$).

Recall from the interview studies that many fifth grade students did not coordinate unit and multi-unit intervals. This motivated us to provide students with two kinds of mathematical forms—Cuisenaire™ rods (C-rods) and mathematical definitions—to support their thinking and communication. We chose C-rods because they can serve as models of linear magnitudes off and on the number line. Each rod color is a unique length, and thus there are stable relationships between rods of different colors; for example, a single purple rod is the same length as two red rods. The tutorial sequence was organized such that tasks called for the use of rods to serve progressively differentiated functions. In the initial phase, student and tutor solved modeling problems using unit intervals (rods with the value of 1), like marking the length of four red rods on an open number line with only zero represented and no other tick marks. A subsequent problem engaged student and tutor with modeling problems of multi-unit intervals—for example, marking the length of six red rods using purple rods. As students used rods to record specified lengths, the rods served as the functional equivalent of units or multiunits, supporting students as they calibrated the line with tick marks. As the tutorial progressed, the tasks engaged students with units and multiunits defined on the line itself, with C-rods shifting in function to measurement tools; later tutorial problems did not make use of rods at all.

The second type of form built into the tutorial was definitional forms. The tutorial was structured as a communication game of sorts, in which tutor and student worked to resolve any discrepant solutions that emerged between them. As shown in Fig. 11.14, the tutor and the student played a game in which each was required to mark the same position on a number line but could not see one another's activities. After solving the problem independently, the tutor and student compared their solutions, constructing agreements on a public sheet. The agreement sheet served as a public record that tutor and tutee could refer to when resolving discrepancies and solving additional tasks. One agreement, for example, was Order: On the number line, numbers increase in value from left to right and decrease in value from right to left.

As shown in Fig. 11.15, pre-/posttest contrasts showed that the students who were tutored gained more than controls, with a large effect size. Most notably, tutorial

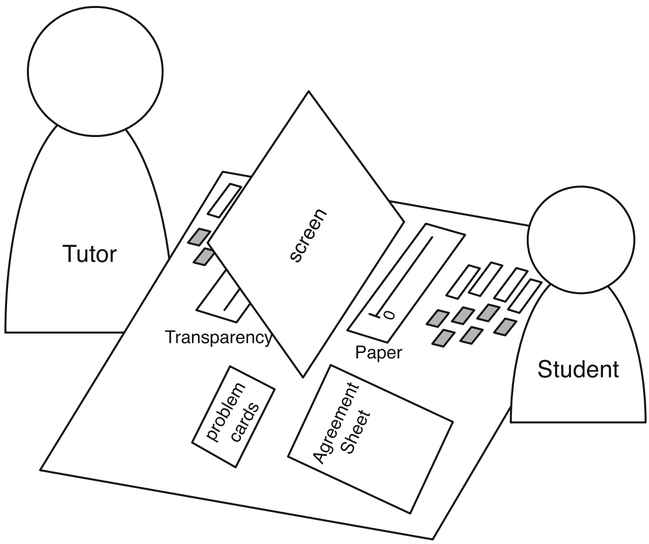


Fig. 11.14 Illustrations of written agreements

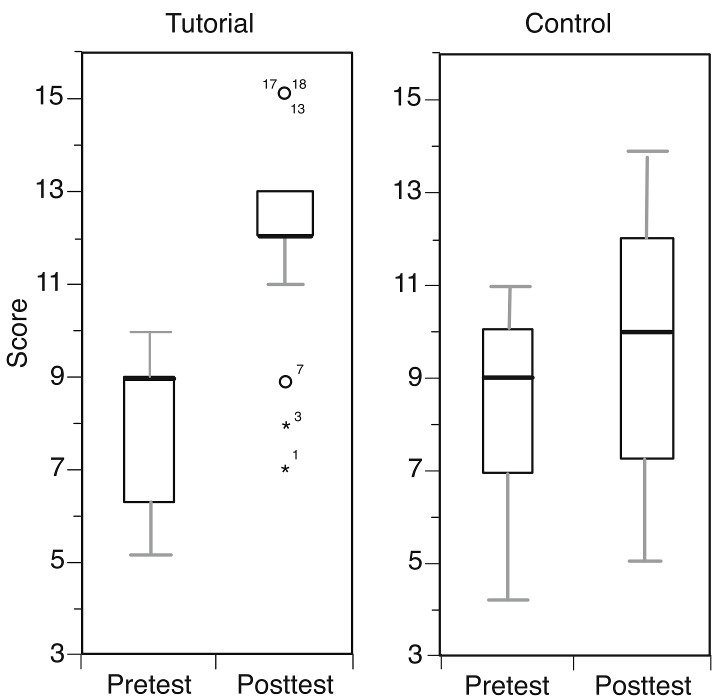


Fig. 11.15 Contrasts between pretest and posttest performances for tutorial and control groups

students' use of agreements predicted their learning gains. There was a strong correlation between students' uses of appropriate agreements when reasoning about their solution and their gains in scores from pretest to posttest.

From these results, we concluded that the problem sequence, the C-rods, and the definitional forms contributed to a supportive context for students' ontogenetic development. Over the course of the tutorial, these forms supported students in making the kinds of coordinations that proved difficult in the interview studies. We also concluded that particular uses of the definitional forms and C-rods functioned effectively as socio-mathematical norms, fostering not only clear communication between tutor and student, but also generative understandings that guided problem solving. Based on the overall effectiveness of the tutorial, and these conclusions in particular, we made a number of design choices that guided the development of our lesson sequence. For one, both C-rods and definitional forms would feature prominently in the lessons. C-rods would be introduced and re-introduced as lesson topics shifted, acting as resources to ease students through difficult transitions. Definitional forms would be a critical feature of the lessons, helping to create coherence and continuity and to support a generative understanding of big ideas. Additionally, the lessons would feature a sequence of problems that would support the progressive elaboration of these big ideas.

Classroom Studies

The purpose of the classroom studies was to engage in a process of iterative refinement of preliminary lessons with a concern to support teachers' and students' generation of a common ground of talk and action as lesson topics shifted. At the collective level, our intention was to support useful sociomathematical and participation norms. At the individual level, our intention was to support the micro-, onto-, and sociogenesis of form-function relations that reflected rich understandings. At the conclusion of the process, our plan was to have produced a full lesson set in which design choices were guided by these intentions and informed by systematic research. The classroom studies were completed in several phases.

Preliminary Classroom Studies

Even prior to the interview and tutorial studies summarized in Table 11.1, we worked with sixth grade teachers and their classrooms. In these early classroom studies, we designed pilot lessons and studied their implementation. We were sensitive to pedagogical and developmental issues at both collective and individual levels. At the collective level, we explored problem types and lesson structures that would support an entire classroom community in building norms for participating in a rich mathematical discourse. At the individual level, we were particularly interested in ways students were producing microgenetic constructions of form-function relations and ways in which distributions of form-function relations in the classroom were reproduced and altered in sociogenetic processes. Analyses that resulted from these early classroom studies are reported in prior publications (Saxe et al. 2009; Saxe et al. 2007). These early studies set the stage for the interview and tutorial studies previously described, and they also became valuable resources as we crafted blueprints of the LMR lesson sequence.

LMR Classroom Studies: Preliminary Lessons and Their Iterative Refinement

With the completion of the early classroom research and the subsequent interview and tutorial studies, we developed preliminary lesson plans and recruited two highly recommended elementary school teachers as collaborators (one fourth and the other fifth grade). We presented the teachers with drafts of lessons that were informed by our prior research efforts, and then began, in collaboration with them, a process of iterative refinement. Our collaboration also extended to outside consultants: Deborah Ball (mathematics educator) and Hyman Bass (mathematician), a consultation that originated through their Elementary Mathematics Laboratory at the Park City Mathematics Institute prior to LMR funding.

Building upon our prior research, we guided lesson construction with a concern to cover key mathematical content and at the same time support the generation of a common ground. At the collective level, a concern was to produce lesson structures and problem types that supported norms for participation and for mathematical argumentation informed by the prior research. At the individual level, an overarching concern was to produce lesson structures, problem types, and representational forms (mathematical definitions, C-rods) that would support public displays of student thinking. In turn, students' displays over the lessons could support teachers' insight into students' ways of construing and accomplishing problems (in microgenetic processes), students' developmental trajectories in using new mathematical forms and functions (in ontogenetic processes), and the way the classroom population as a whole was or was not shifting in approaches to conceptualizing and accomplishing problems as lesson topics progressed (sociogenetic processes).

As we cycled through one to three lessons per meeting with partner teachers, our process of iterative refinement proceeded as follows: (a) We engaged our partner teachers in discussions, refining our preliminary blueprints so that they became workable lessons; (b) we observed and videotaped teachers' implementation of lessons in classrooms, using observations and joint debriefings to further refine lessons; (c) we re-drafted lesson plans that would eventually become principal components of our Teacher's Guide to the Lessons.

Through our 2-year period of lesson refinement, we maintained the overarching intention to support the elaboration of a common ground through the lessons at both individual and collective levels. But many of our ideas about how to implement this intention evolved. Some of the refinements were major. For example, we developed new lessons to address key transitions from one topic to another. Additional major revisions cut through all lessons: We modified the lesson structure used across lessons to more effectively engage students, and we developed a poster on which the teacher would record definitions throughout the entire lesson sequence. Other changes were structurally minor but no less significant: (1) For some lessons, we created new definitions (or modified included definitions) to support the crystallization of key ideas. (2) We tailored language used in definitions so that it was more accessible to students. (3) We developed differentiated partner worksheets with problems at multiple levels of difficulty in order to support the participation of all students.

To evaluate the promise of the lessons in our two partner teachers' classrooms, we developed a preliminary assessment of students' understanding of integers and fractions. We administered pretests and posttests in each domain at our last iteration of the respective integers and fractions units. With the impressive gains we found in both of our partner teachers' classrooms for each unit, we sought to field test the lessons with teachers unfamiliar with the curriculum.

Support for Use of the Lesson Sequence with New Teachers

We recruited two additional teachers who had no familiarity with the lessons. We piloted an approach to professional development with these teachers, and through this process, we gained insight on what needed refinement in our approach to supporting teachers' use of the lessons. We also gained insight on how the lessons were occasionally transformed by teachers in ways that were at odds with our intentions. With this information, we returned to refine our approach to teacher training and further adjusted the lessons to support their use in classrooms.

Efficacy study

To evaluate the implementation and impact of the LMR curriculum, we conducted an efficacy study that made use of a quasi-experimental design (Saxe et al. 2013). We sampled classrooms from school districts that were all using the same elementary mathematics curriculum, Everyday Mathematics. We matched classrooms on a number of variables (demographics, language background, teacher professional experience) and then assigned 11 classrooms to an LMR implementation group and 10 classrooms to a non-LMR implementation group. We engaged the LMR teachers in several training sessions in the use of the 19 LMR lessons, and then studied their use in the Fall of 2010. Of principal concern in these analyses was to evaluate whether students in LMR classrooms showed greater growth in integers and fractions knowledge than students in comparison classrooms. Further, we also wanted to evaluate whether greater or lesser performing students differentially benefitted by the LMR lessons, and whether students in LMR classrooms, regardless of their entry-level understandings of integers and fractions, showed similar growth curves relative to comparison students.

Student Assessment Instrument

We built upon and refined the instrument that we had used in our classroom studies to assess students' understanding of integers and fractions. In this process, we generated three forms of an assessment instrument (about 30 items on each instrument with 18 common items). The instrument was composed of items drawn from our assessment used in the classroom studies, as well as released items from varied standardized assessments, including the National Assessment of Educational Progress (NAEP) and the California Standards Tests (CST). It also included items drawn from the Everyday Mathematics curriculum. In the design of the instrument, we selected integers and fractions items that included both number line representations and those that did not. We expected to find that students in LMR classrooms would not only show greater gains on problems that made use of number line representations, but that they would also extend that knowledge to problems that did not involve number lines.

Student Assessments and Growth

To assess students' growth through the intervention and carry over through the winter and spring, we administered the assessment to LMR students on four occasions and to Comparison students on three occasions:

- Prior to the start of the LMR lessons in September (LMR and Comparison classrooms)
- After the integers but before the fractions unit (LMR only)
- At the end of LMR fractions unit in December (LMR and Comparison classrooms)
- At the end of the school year in May (LMR and Comparison classrooms)

We used Item Response Theory to calibrate item difficulty over the assessments. The result yielded multi-leveled data: Students nested in classrooms (see Saxe et al. 2013). We used Hierarchical Linear Models (HLMs) to analyze growth curves of students to evaluate intervention effects. Our findings can be summarized as follows:

- HLM analyses documented greater achievement for LMR students than comparison students on both the end-of-unit and the end-of year assessments of integers and fractions knowledge, showing a moderate to large effect size.
- Gains for LMR students occurred on item types that included number line representations and those that did not.
- The growth rates of LMR students were similar regardless of entering ability level.

These findings confirmed our expectations and demonstrated the efficacy of the LMR sequence in supporting teaching and learning in the domains of integers and fractions.

11.2.2 The Complete Lesson Series

We now describe the complete LMR lesson sequence, beginning with a broad overview of the lessons. We then detail the ways that the lessons supported a common ground in classroom communities.

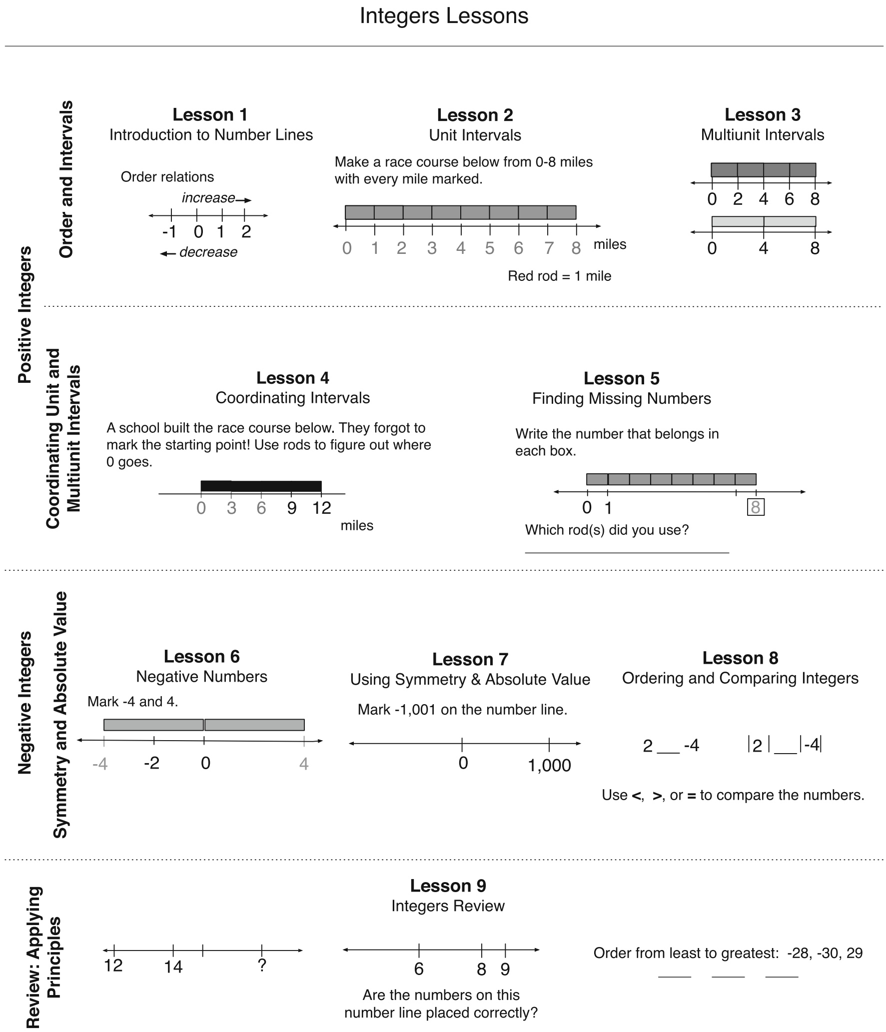


Fig. 11.16 Integers lessons in the LMR sequence

Figures 11.16 and 11.17 contain overviews of the lessons on integers and fractions. In each figure, we present a thumbnail sketch of each lesson; we indicate the lesson topic as well as how the topics are addressed in the number line context through depictions of lesson-specific problems. The leftmost column in the figures reveals the broader grouping of topics. For example, in Fig. 11.16, Lessons 1–5 cover positive integers and 6–8 cover negative integers, with an integers review provided in lesson 9. Some core ideas covered across the topics are also indicated in the figures. Thus, order and intervals are the core ideas for Integers Lessons 1–3, coordinating unit and multiunit intervals for Lessons 4 and 5, and symmetry and absolute value for Lessons 7 and 8. For fractions (Fig. 11.17), Lessons 1–5 cover part-whole relations and 6–9 cover multiplicative relations, with a fractions review given in Lesson 10.

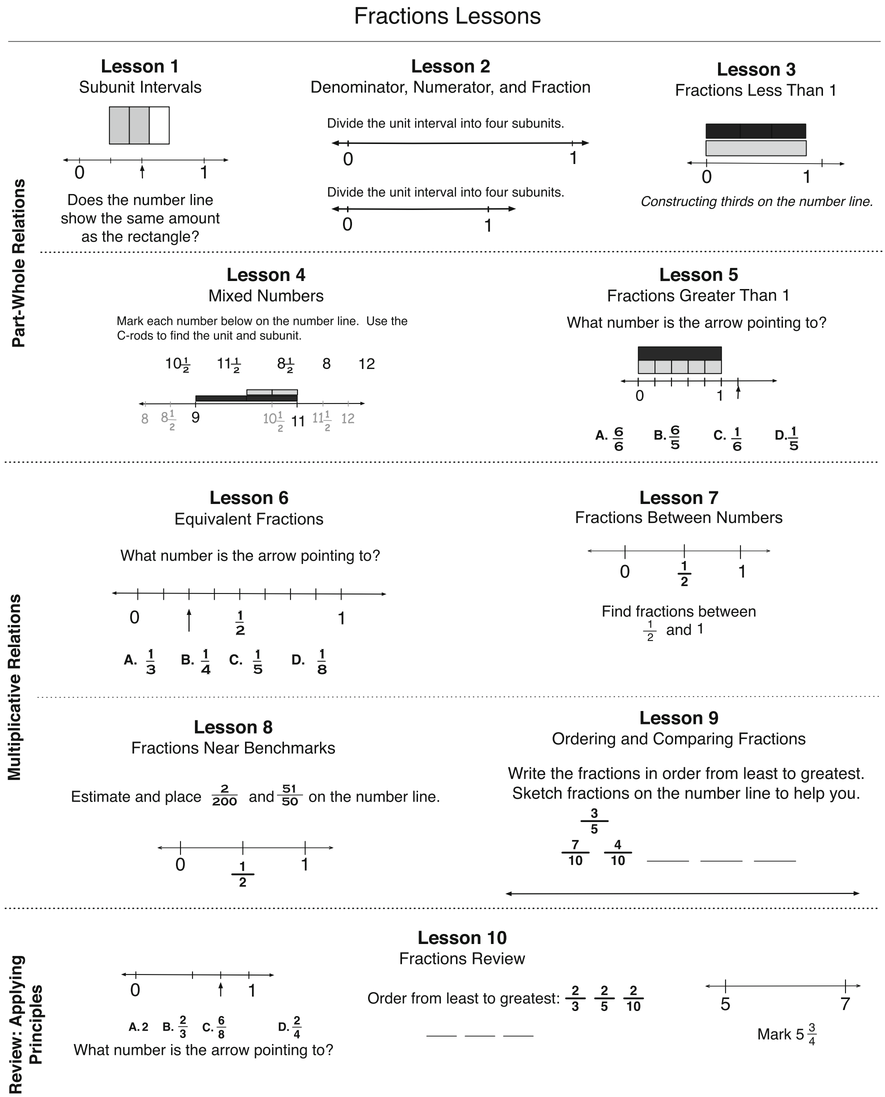


Fig. 11.17 Fractions lessons in the LMR sequence

11.2.2.1 Supports for a Common Ground of Talk and Action with Shifting Lesson Topics

Across the lessons, the problems we developed were intended to support a good deal of discussion. Recall the discussion recounted in Sect. 11.1 in which Ms. R's class engaged with the problem of marking −1,001 on a number line on which only the positions of 0 and 1,000 were given (Lesson 7). In such discussions, the problems were intended as occasions for students to surface their thinking in public displays, thereby provoking conflict between alternative solutions and encouraging resolution, a process that could support the generation of common ground. However, we realized this process alone would not ensure the construction of rich mathematical insights. Indeed, the design research led us to build in multiple supports to facilitate a common ground of rich mathematical talk. These supports were of four principal kinds, and we review each below.

Ordering of Lesson Topic

Faced with the challenge of how to support a classroom community's maintenance of a common ground with shifting lesson topics, we ordered lesson topics so that central ideas built upon one another. With each new lesson topic, our concern was that students and teachers would be able to draw upon ideas from prior lessons as resources to construct new ways of communicating and solving problems. To illustrate, consider Integers Lessons 2 through 4, which covered some of the core, generative ideas in positive integers (depicted in Fig. 11.16).

Lesson 2 introduced the idea of interval and later unit interval. In this progression, the idea of interval was used as a foundation to define a unit interval, the interval between 0 and 1. Later in the lesson, the idea was further extended to any distance of 1 on the number line, like the 4,5 or 5,6 distances. With an establishment of a common ground related to a unit interval in Lesson 2, Lesson 3 then further differentiated the idea of interval, building upon the idea of unit interval as foundational to the idea of multiunit interval. Multiunits are defined as multiples of unit intervals, like intervals of 2, 3, 4, etc. This idea then serves as a foundation for students' understanding of arithmetic series (e.g., 0, 2, 4, 6…; 0, 3, 6, 9…).

Definitions and Principles

Over the course of the lessons, key generative ideas were introduced in the form of explicitly formulated number line definitions and principles. The definitions and principles were progressively introduced and displayed on large posters in front of the class. The complete set of definitions used for integers lessons are contained in Fig. 11.18, and those for fractions are contained in Fig. 11.19. The definitions and principles were intended as resources to support the generation of a common ground as topics shifted. They supported a common lexicon that indexed key ideas that students used in communication and problem solving. They also supported the resolution of discrepant solutions.

The introduction of definitions and principles was coordinated with the sequencing of topics in the lesson sequence in order to maximize their utility in communication and problem solving. Like the lessons, the definitions and principles built upon one another; subsequent definitions often followed as entailments of prior definitions. For example, the definition for unit interval was presented in Integers Lesson 2: the distance of 1 on a number line. The definition of multiunit interval followed in Integers Lesson 3 as a logical extension: the multiple of a unit interval. In turn, these definitions led to the principle for positive integers, every (whole) number has

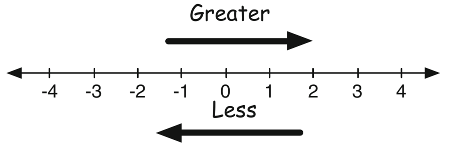
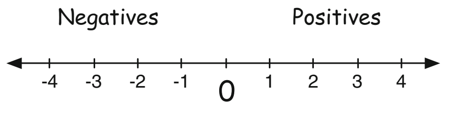
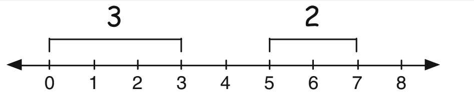
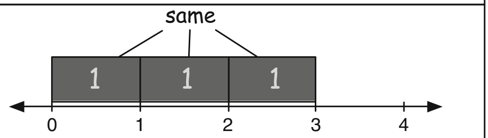
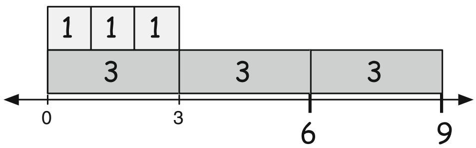
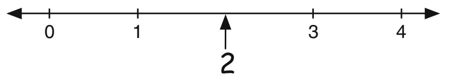
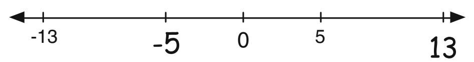
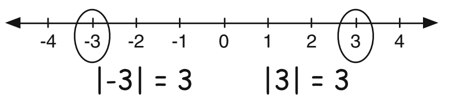
Name	Definition	Example
Order	Numbers increase in value from left to right. Numbers decrease from right to left.	
0 is a number	0 is a number, so it has a place on the number line.	
Interval	The distance between any two numbers on the number line.	
Unit Interval	A unit interval is the distance from 0 to 1 or any distance of 1.	
Multiunit Interval	A multiple of a unit interval.	
Every number has a place	Every number has a place on the line, but not all need to be shown.	
Symmetry	For every positive number, there is a negative number that is the same distance from 0.	
Absolute Value	The distance of a number from 0.	Circle two numbers with the same absolute value. 

Fig. 11.18 Integers definitions and principles used in the LMR lessons

a place. The subsequent definitions led to the introduction of negative integers (definitions of symmetry and absolute value). At the beginning of the fractions unit, the definition of subunit built upon the prior definitions of unit and multiunit.[7]

[7] From this point we refer to "definitions and principles" simply as definitions, even though some of the ideas to which "definitions" refer are actually in the form of principles (e.g., *every number has a place but doesn't need to be shown*).

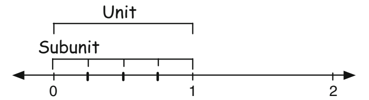
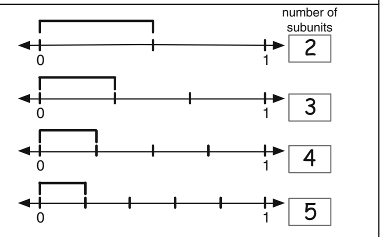
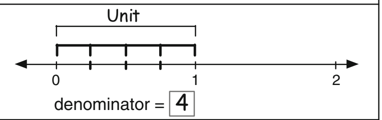
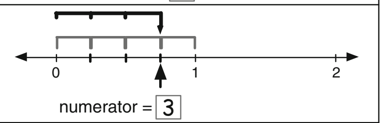
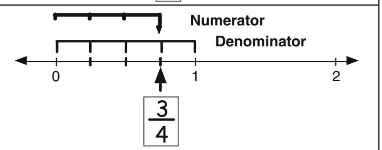
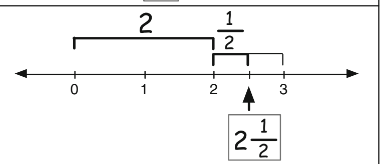
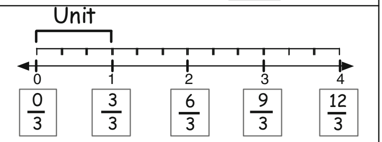
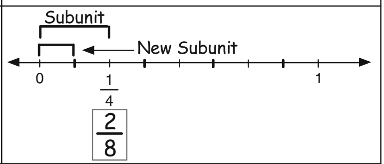
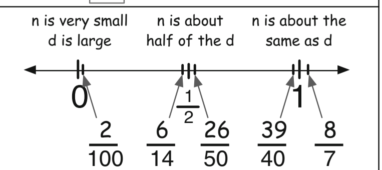
Name	Definition	Example
Subunit	Dividing a unit into equal distances creates subunits.	
Length of the subunit	The more subunits in a unit the shorter the subunits are.	
Denominator	The number of subunits in a unit.	
Numerator	The number of subunits.	
Fraction	numerator / denominator	
Mixed Number	A whole number and a fraction.	
Whole Numbers as Fractions	A whole number can be written as a fraction	
Equivalent Fractions	Fractions that are in the same place but with different subunits.	
Benchmarks	0, 1/2, and 1 are benchmarks. You can tell ABOUT how big a fraction is by comparing the numerator and denominator.	

Fig. 11.19 Fractions definitions and principles used in the LMR lessons

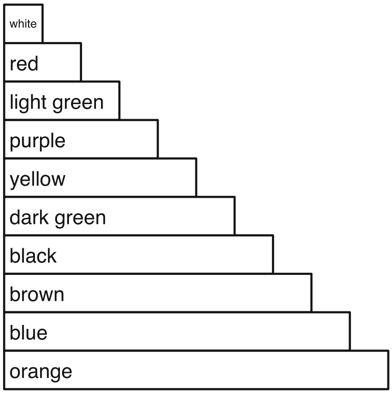


Fig. 11.20 C-rods used to support common ground

Cuisenaire™ Rods (C-Rods)

The productive use of C-rods in both the tutorial and classroom studies by students and teachers led us to incorporate them at challenging transitions in the lesson sequence. In the lessons, we used C-rods of different kinds, including large/magnetized rods, translucent rods, and small wooden rods. The different rod types allowed them to be used in different contexts in the classroom—on the whiteboard, on the overhead projector, and at students' desks. All C-rods are graded in size with relative sizes indexed by rod color (as depicted in Fig. 11.20); the white is the shortest and each consecutive rod length adds the length of another white rod. The size ratios afford the additive and multiplicative composition and decomposition of length.[8]

We expected rods to be helpful at key transitions because they could support student thinking (serving functions like defining and measuring units, multiunits, and subunits) and render it more visible to others, properties that would support the generation of a common ground related to complex ideas. Recall, for example, the heated discussion in Ms. R's classroom in Integers Lesson 7, when the class was moving from positive to negative integers and to labeling points for values with unit lengths impossible to discriminate by perception. To solve the problem, Pierre used the yellow C-rod on the white board to represent a multiunit length of 1,000 units, asserting that the unlabeled point on the number line is −2,000. This use of the C-rod supported Carol's epiphany that her initial answer of −1,006 or −1,007 was not correct.

Another example of the use of C-rods to support a difficult transition comes from Fractions Lesson 3. In this lesson, students are engaged with the transition from identifying unit fractions (fractions with a numerator of 1) to non-unit fractions,

[8] For example, for additive relations, the length of 1 red (2 units) plus 1 white (1 unit) equals 1 light green (3 units); inversely, the length of 1 light green (3 units) minus the length of 1 red (2 units) equals the length of 1 white (1 unit). Similarly, the rods can be used to express multiplicative relations: 1 light green is the length of three whites, and 1 white is 1/3 the length of the light green.

fractions in which the numerator is greater than 1. We used C-rods to support this transition, designing problems in which the use of C-rods would support the partitioning of units into subunits (e.g., a partition of a unit into three congruent parts [subunits] to create thirds) and then the use of multiple subunits to create a non-unit fraction (to locate 2/3 by using two 1/3 subunits).

Recurrent Lesson Structure

For students and teacher to generate a common ground in classrooms requires that they participate actively in attempting to get across their communicative intentions and in actively making sense of the communicative intentions of others. Indeed, the supports that we built into the lessons for generating a common ground—the ordering of lessons, the support for definitions, the use of C-rods—all require a back and forth of reflective talk.

To privilege a back and forth of reflective communications in classroom discourse, we settled on a recurrent five-phase structure for lessons (depicted in Fig. 11.21). Each lesson begins with independent work on opening problems (opening problem phase), which support student initial reflection on core ideas targeted in the lesson prior to whole class discussion. What follows is a whole class discussion that provides opportunities for students to share their reflections and engage with others' communicative displays (opening discussion phase). Subsequently, students communicate about and solve problems in dyads (partner work phase). Students then return to whole class communicative interactions in which problematic ideas are surfaced and resolved in talk guided by the teacher (closing discussion phase). In the final phase, students make use of the knowledge generated in the class to individually solve closing problems (closing problem phase); their solutions afford the teacher a window into student thinking that can inform further efforts to support common ground in subsequent lessons.

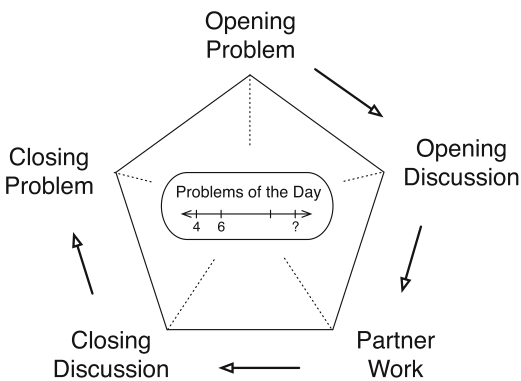


Fig. 11.21 Five phase lesson structure used in LMR lessons

11.2.3 *Empirical Techniques Used to Analyze the Reproduction and Alteration of a Common Ground of Talk and Action in a Classroom Community*

We now turn to the use of the LMR lesson sequence as a context to explore empirical techniques that can illuminate dynamics of the reproduction and alteration of a common ground within a classroom microculture over time. Guided by our conceptual framework, we sought to capture this process at both collective and individual levels. We selected a single classroom community that we treat as a laboratory, illustrating both data collection techniques and techniques for reduction and analysis.

11.2.3.1 Empirical Techniques: Data Collection

Figure 11.22 provides a bird's eye view of the data sources from Ms. R's classroom, the classroom that we selected to serve as our laboratory.[9] Some of these data were collected as a part of the LMR efficacy study reviewed previously, like the assessment of students' integers and fractions knowledge (pre, interim, post, and final assessment) as well as video of selected lessons. But while such data sources would help us evaluate the efficacy of the lessons relative to comparison classrooms, these data were too sparse in themselves. They could not inform an analysis of shifting common ground. For this reason, we collected more intensive data in our laboratory classroom. The complete set of data sources are detailed below.

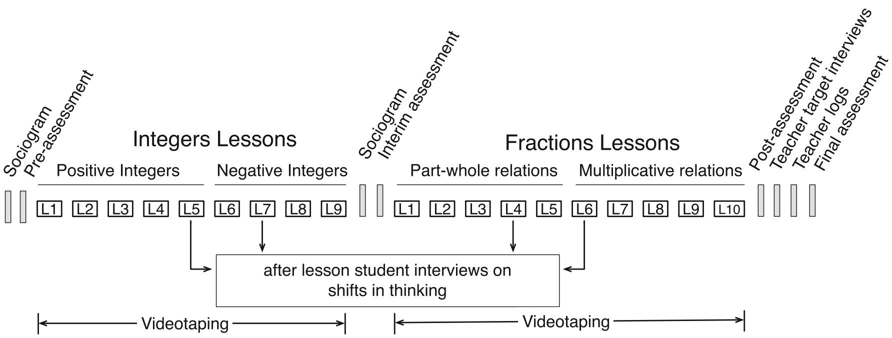


Fig. 11.22 Data collection techniques used in Ms. R's case study classroom

[9]There were two classrooms that we treated as laboratories, Ms. R. 's and another, both of whom were partner teachers who participated in the development of the LMR lessons (see classroom studies reviewed earlier). Our choice of Ms. R's classroom over the other teacher's classroom was arbitrary.

Video Records (of Lessons)

Video records were collected during all 19 lessons using at least one high-definition, digital video camera and tripod.[10] During whole class discussion, the camera was placed in the rear of the classroom and oriented toward the teacher and the white-board or overhead projector. While students were working at their desks, the camera operator followed the teacher as she circulated around the class observing students' work and offering assistance.

For four focal lessons (Integers Lesson 5 and 7; Fractions Lesson 4 and 6), an additional camera and tripod was used. During whole class discussion, this camera was oriented towards students, who were either sitting at their desks or at the foot of the whiteboard. As students were engaged in work at their desks, additional cameras captured small working groups of preselected students.

Assessment of Integers and Fractions Knowledge

Ms. R, like all of the LMR teachers, implemented the 19-lesson sequence during the Fall semester, 2010. We targeted four points for assessment, each point with a different function in mind (see Fig. 11.22). A pre-assessment was administered to assess students' baseline knowledge of integers and fractions prior to the use of the LMR unit. The interim assessment was administered after the 9 lessons on integers, but before the 10 lessons on fractions; the test was used to assess gains on integers but to also determine whether lessons supported an understanding of fractions, even without formal instruction on the topic. The post-assessment was administered just after completion of the fractions unit to assess immediate gains in both integers and fractions domains. The final assessment was administered at the end of the school year, about 5 months after the LMR unit was implemented to assess whether any gains documented on the post-assessment were evident several months after the LMR curriculum was used. Each assessment lasted about 30 min.[11]

We included in the assessments approximately the same number of items presented in number line and non-number line formats for both integers and fraction domains. In this regard, we anticipated the concern that since the focus of LMR lessons was using the number line format, students' progress may well be limited to the number line representation. To determine whether that was the case, we included non-number

[10] Depending upon the focus of videotaping, we used between one and five cameras in the classroom on a given day. When multiple cameras were used, the focus was on teacher and the class (two cameras) and teacher, class, and partner work (five cameras). One of our team members led the development of a video manual to organize positioning of cameras to maximize coverage (Katherine Lewis).

[11] Though we administered four assessments, we made use of three different forms, with the post-assessment and final assessment being the same form. Each of the three measures consisted of about 30 items. 18 problems were shared across measures (to enable scaling using an IRT model). On each of the assessments, we included number line and non-number line items for both fractions and integers. Of particular interest was whether learning gains for students in Ms. R's class and all LMR classrooms might be limited to number line items or whether student learning might be reflected on both number line and non-number line items.

line items. Because the curricular focus was on generative understandings, we expected that student gains would not be limited to number line representations.

After Class Interviews

Following the four focal lessons mentioned above, another source of data was obtained through after class interviews with select students. During the initial phase of the lesson, while students were engaged with solving the opening problems, three interviewers circulated around the class. Each interviewer attempted to identify two students who incorrectly solved one of the problems. Once a student was identified, the interviewer knelt down next to the student and asked him or her for an explanation, taking notes on the student's solution and how he or she justified it.

After the lesson concluded, each of the identified students was interviewed using a structured protocol. This protocol was constructed with the intention of understanding the role of principles and definitions in the students' thinking. First, the interviewer presented the student with a new copy of the opening problems and asked him or her to attempt one of them a second time (the problem the student had answered incorrectly). Whether or not the second solution was correct, the student was then asked a series of probes about his or her use of principles in solving the problem, or if principles could be used to justify the solution. Next, the student was presented with a counter suggestion. Students whose second attempt at the problem was correct were given an incorrect counter suggestion, and vice versa. A series of probes followed the student's reaction, once again targeting the student's ability to make connections between the different solutions and the number line principles.

Sociogram

Sociogram data were collected prior to the lessons to provide evidence of students' initial social positions. Students were asked to name classmates they would like to sit next to in math, as well as those from whom they would ask for help in math. These two questions were designed to gather information about different kinds of social standing: the general social status of individual students, and the social status of individual students specifically with regard to math. The different types of justifications children gave for their nominations helped to corroborate that students thought about these two questions differently. Sociogram data were collected again midway through the lessons in order to determine shifts in social positions upon the completion of the Integers Unit.

Teacher Interviews

After the lessons were taught, Ms. R was interviewed about two students in her class, Carol and Kail. These two focal students were chosen because they frequently participated in class discussions, and as a result we could more clearly observe the evolution of their mathematical ideas over time. The interview was meant to gather information on the teacher's perspective of shifts and changes in these students' learning over the course of the LMR lessons. Questions asked were related to performance in LMR, participation in class discussions, and ways in which the students managed their learning. A secondary goal of the interview was to get a sense of the teacher's goals for creating norms for participation.

11.2.3.2 Empirical Techniques: Data Reduction and Analytic Approach

The data sources provided an extraordinarily rich corpus of materials for analysis. But a host of challenges arose as we considered ways of reducing these so that they could leverage our efforts to understand the reproduction and alteration of a common ground over lessons. Preliminary challenges involved addressing the core questions of where to begin our analysis (should one data source be privileged over others?), and with a data source selected, how to begin to reduce that data source into productive units of analysis.

Where to begin? Our central research concern was to understand the reproduction and alteration of a common ground of mathematical talk over the lessons, and there seemed no better starting point than the video records of interaction over the 19 lessons, with other data sources to be drawn into our analyses to explore and constrain conjectures about collective norms and genetic processes. But to make progress in the analysis of the enormous video record (about 20 hours of video), we needed a "road map" of the continuities and discontinuities in the use of representational forms that could orient us to processes of reproduction and alteration of a common ground over the lessons.

How might we create a "road map"? We viewed the answer to this question as key to productive work in the analysis of a common ground over the lessons at both the individual and collective levels. We appreciated that video records of the sort that we collected from Ms. R's classroom are often mistakenly regarded as data. As many investigators have pointed out, they are data only in a very weak sense (Erickson 2006). They constitute only the pointing of a camera and pressing an on-switch. Thus a major challenge is how to turn video records into data that would allow for insight into the shifting common ground over these many hours of video. Such a transformation requires the construction of analytic units, their operationalization, and their application.

We considered varied possible alternatives for analytic units to track common ground in mathematical talk over lessons. We ultimately chose to focus on one set of representational forms that in retrospect seems quite natural: references to definitions by teacher and students. We chose to focus on definitional forms for several key reasons: (1) Definitional forms were a distinguishing feature of the lesson sequence design; (2) they were developed with the explicit goal of supporting communication across shifting lesson topics; (3) they captured central mathematical ideas of the lessons; (4) their appearance in public displays could be readily documented. A systematic focus on teacher's and students' references to definitions and principles over the lessons could provide just the sort of leverage needed to begin an analysis of the reproduction and alteration of common ground, but it only partly answered our "how to begin?" question.

Having decided to focus on definitional forms, we lacked clarity on how to implement a coding of the video records across lessons. As a solution, we turned to video software that would support this effort. The software that we used was StudioCode. The software enabled us to code each occasion for which an individual referred to a definition (by naming it) in public discourse. The coding produced with the software provided a way to visualize the distribution of definition use in each lesson and shift in definition references across lessons. Such a visual representation provided a bird's eye view of broad continuities and discontinuities in patterns of definition use and at the same time supported our efforts to dig deeper into the video record of the 19 lessons at key segments. We developed conjectures about why the contours of definition references took the form that they did, identified parts of the video for close analysis, and brought our additional data sources to bear as we worked through these conjectures.

The visual representation that we produced with the software is contained in Fig. 11.23. In the figure all 19 lessons are represented serially, from left to right. Each row corresponds to a single definition or principle, and the rows are organized from top to bottom by the order in which the definitions and principles were introduced in the lesson sequence. Each column corresponds to a lesson, and columns are blocked into integers and fractions lessons, each with dedicated subtopics. Each tick mark in the timeline corresponds to a single reference by an individual (student or Ms. R) to a definition or principle.[12]

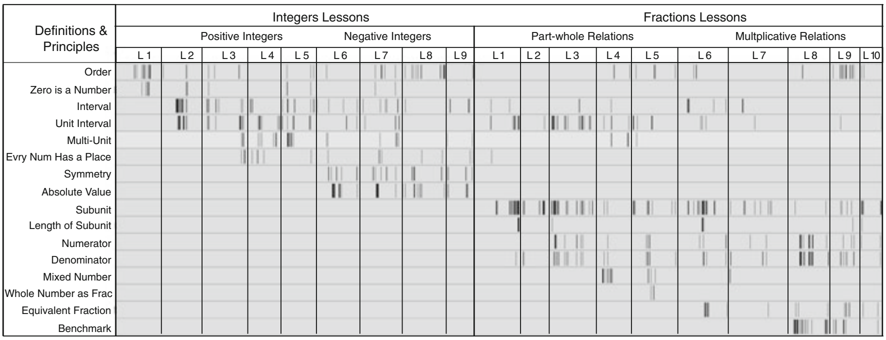


Fig. 11.23 Timeline for references to definitions and principles in Ms. R's classroom over the 19 LMR lessons (opening and closing discussions only)

[12] In our coding procedure, we noted that sometimes references to definitions varied from their canonical forms inscribed on the poster in front of the class. For example, the following variants were also coded as "unit interval": "unit," "interval of one," "distance of one." We used this scheme to code all references to definitions or principles across all 19 lessons for both partner teachers. We established the reliability of the references to the definitions in one partner teacher's classroom by having two coders independently re-code reference to definitions/principles for 20 % of the video data evenly distributed across all of the lessons. We computed percent agreement to be 86 %.

To illustrate the timeline's organization, consider the first row, references to order, the first definition introduced in the lesson series. The tick marks that populate the order row indicate each time that order was mentioned in each lesson's opening/closing discussion (represented by the columns, with varying column widths corresponding to varying durations of lessons). Thus, one can note that references to order were dense in the middle of Integers Lesson 1 and reference to order continued through every integers lesson with the exception of Lesson 4.

With the timeline generated, our first step was to consider some of the timeline's contours that capture frequency patterns in students' and teacher's references to definitions and principles. We note three contours below, and we use these contours to guide inquiry in subsequent sections.

Contour #1. High overall density of references. Over the entire timeline, we find many references to definitions. Indeed, for every lesson column, we find at least some references, and, across the set of integers lessons, we counted 378 references, and across the fractions lessons, we counted 536 references. Clearly, there was a great deal of definition use, especially given that the timeline only reflects opening and closing discussions.

Contour #2. Pockets of densities. The density of references to any specific definition was not equally distributed across lessons. Some, like interval, had pockets of greater reference when they were first introduced. For example, references to interval was exceptionally dense in Integers Lesson 2, the first lesson in which it appears. A similar trend appears for other definitions as well, like unit interval, absolute value, subunit, and benchmarks.

Contour #3. Continuity in use of definitions over lessons. Some definitions continued to be referenced over the course of many lessons. For example, unit interval received a great deal of use in Integers Lesson 2, and then was reproduced throughout subsequent lessons on integers and continued through many of the fractions lessons. Others did not.

Our second step was to make use of empirical techniques to explore how the contours emerged through the lessons. These contours became points of entry for our analyses of dynamic processes in the reproduction and alteration of a common ground. Employing the conceptual framework articulated in Sect. 11.1, we began a coordinated analysis at collective and individual levels to illuminate processes of reproduction and alteration in the public displays of form-function relations that resulted in these contours.

Collective Level: A Focus on Emergent Norms in the Classroom Community

Recall from Sect. 11.1 that a collective level analysis requires a focus on sociomathematical and participation norms that emerge in a classroom. Motivated by our concerns to understand patterns in references to mathematical definitions, our illustrative analysis focuses on sociomathematical norms. In order to posit and corroborate conjectures about sociomathematical norms, we conducted observational analyses of key segments of video that included reference to definitions and principles.

What emerged from our analysis were two sociomathematical norms: The first norm entails the use of mathematical definitions when explaining thinking or justifying reasoning. This norm we argue helps explain the first contour, why definition use is so dense over the lessons. The second norm engages the way definitions are used in public displays: Definitions should be either connected to particular problem contexts or follow entailments of other definitions. This norm illuminates the second contour, the periods of frequent references during which teachers and students are beginning to negotiate how to use particular definitions.

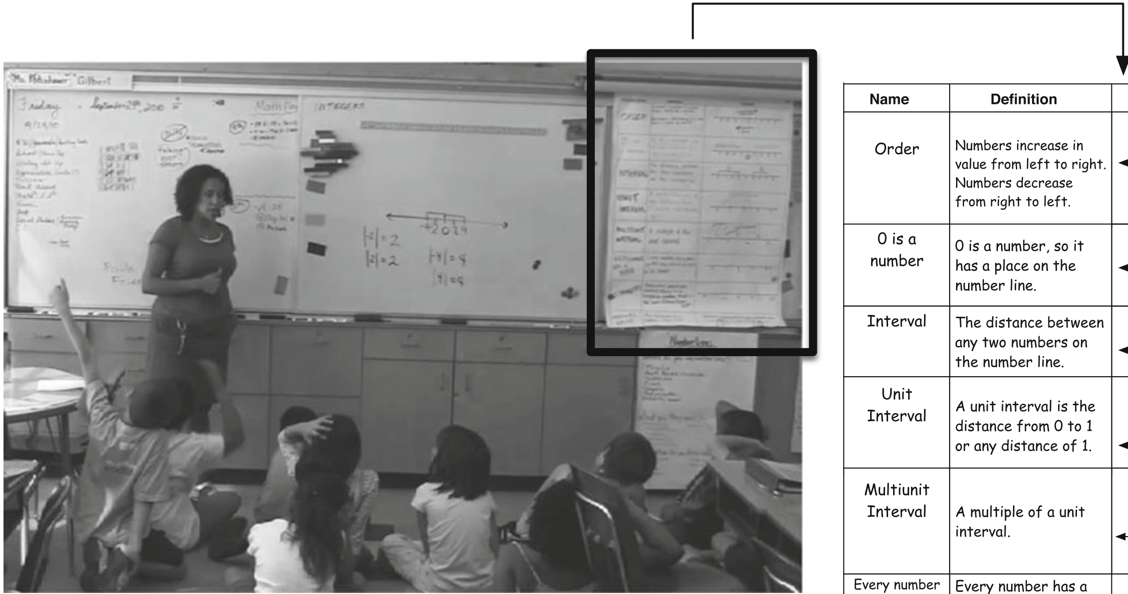
Sociomathematical Norm #1. Use Definitions to Support Your Ideas: Especially When Explaining Thinking or Justifying Reasoning

Guided by the timeline, we began by noting the enormity of references to definitions (Contour #1). To understand what was occurring in talk during these references, we examined coded instances. We discovered that the specific contexts of these references were varied. Some occurred as Ms. R and her class were engaged with discussions of the definitions poster (see Fig. 11.24). But others occurred in classroom discussions as students justified their claims, or as Ms. R pushed students to consider how a definition was relevant to problem solutions. In spite of the variation in contexts, we observed consistently that Ms. R was both actively using definitions and actively supporting their use by students in their public displays and argumentation. This led us to conjecture that the use of definitions to justify reasoning and argumentation constituted a sociomathematical norm in Ms. R's classroom. We also found that students appeared responsive to this norm of definition use. Such a sociomathematical norm would partially explain the overall density of definition use over the lessons.

Consider more specifically the ways that Ms. R supported this norm. Sometimes Ms. R made meta-discursive comments about the importance of definitions, like exclaiming "awesome word!" when referring to the definition of a subunit. Other times when students used a definition, she noted with pleasure students' references to definitions. Still other times, she would prompt students to use a definitional term in talk, with a marked intonation, starting a sentence for them to complete. For example, she attempted to elicit the name of the unit interval definition saying that, "This interval is a ___." (pointing to the interval between 3 and 4 on the board), or "an interval of one is called a ____". Finally, prior to posting a definition on the classroom poster, she spent preparatory time creating a context to motivate the need for a definition.

Yet another source of evidence that Ms. R supported students' references to definitions (as normative) comes from ways that she treated their absence in students' talk. Sometimes she re-voiced a student's talk to include references to definitions. Other times, Ms. R treated a student's lack of reference to a definition as a breach, offering the student an opportunity to reformulate their own thinking in accord with the sociomathematical norm.

Ms. R's valuing of definition use can help to explain the relative density of references to definitions across the lessons. But there was more to definition use than simply praising their appearance in student talk or providing occasions for invoking them. Ms. R also supported norms for how to use definitions.



Name	Definition	Example
Order	Numbers increase in value from left to right. Numbers decrease from right to left.	Greater; -4 -3 -2 -1 0 1 2 3 4; Less
0 is a number	0 is a number, so it has a place on the number line.	Negatives, Positives; -4 -3 -2 -1 0 1 2 3 4
Interval	The distance between any two numbers on the number line.	3, 2; 0 1 2 3 4 5 6 7 8
Unit Interval	A unit interval is the distance from 0 to 1 or any distance of 1.	same; 1 1 1; 0 1 2 3 4
Multiunit Interval	A multiple of a unit interval.	1 1 1; 3 3 3; 0 3 6 9
Every number has a place	Every number has a place on the line, but not all need to be shown.	0 1 3 4; 2
Symmetry	For every positive number, there is a negative number that is the same distance from 0.	-13 -5 0 5 13
Absolute Value	The distance of a number from 0.	Circle two numbers with the same absolute value. -4 -3 -2 -1 0 1 2 3 4; |-3| = 3 |3| = 3

Fig. 11.24 Definitions posted on the wall recorded by students in their workbooks at the beginning of Lesson 7, "Using symmetry and absolute value"

Sociomathematical Norm #2. When Definitions Are Used, Connect Them to a Particular Problem Context and/or to Other Definitions

Guided by the timeline, we began by exploring what was occurring in talk during the initial flurries of definition use that constitute the second contour. Our process was iterative. We first considered the initial references to one definition and developed a conjecture about what interactional work was occurring between teacher and students that might explain the density of definition use during these episodes. With this conjecture in mind, we then considered additional definitions and refined our conjecture. This process led us to posit a sociomathematical norm of "connectedness" that led to repeated reference to definitions: When public displays are made involving references to definitions, they should be either connected to particular problem contexts or to other definitions.

In the episodes of high density, we found that Ms. R was initiating the negotiation of connectedness as a sociomathematical norm. For example, to support connectedness, Ms. R pursued two approaches to elaborating definitions that build upon one another: formal definitions and ostensive definitions. Formal definitions provide articulations in speech and writing of the criteria for what is included and what is not included in a target category. Ostensive definitions present (or point to) particular instances of the idea. The flurry of references to definitions was in part accounted for by Ms. R's rapid back and forth between formal and ostensive definitions.

To illustrate, we consider her introduction of two definitions—interval and unit interval—that occurred towards the beginning of the lesson sequence. Early in Integers Lesson 2, Ms. R draws a number line on the board and labels 0 through 4 (Fig. 11.25a). She asks whether anyone has heard of the word "interval" before and

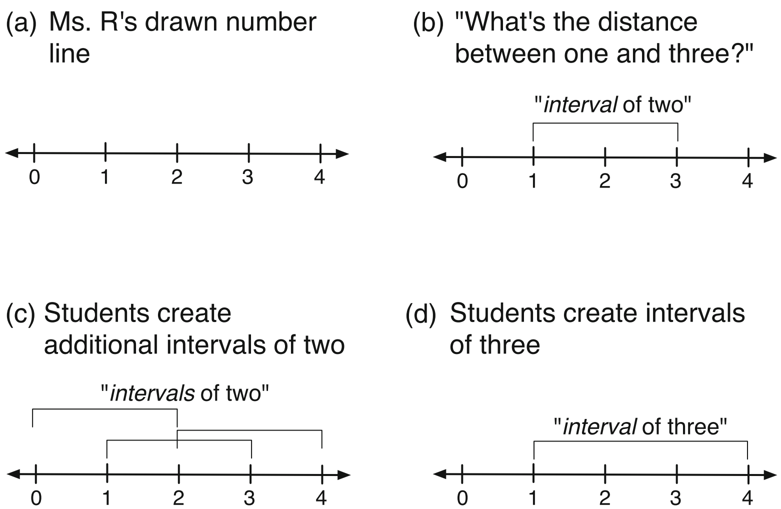


Fig. 11.25 Students' participation in movement between formal and ostensive definitions

tells the class that it is a word they will be learning about. Her focus returns to the number line on the white board and she asks what would be the distance from 1 to 3 (Fig. 11.25b). She reiterates a student's answer, calling it a "distance" of two. She then asks if anyone can find another "interval" of 2 (referring back to the prior ostensive definition of an interval of 2). Once a few more have been identified by students (Fig. 11.25c), she asks for someone to identify an interval of 3. When a student says "four to one" (Fig. 11.25d), Ms. R reinforces the answer and responds "he went four to one; we can also say one to four," pointing to the equivalence of two statements that are each ostensive definitions of an interval of 3. In her summary of the discussion to the class, she offers a formal definition, saying that "interval just means the distance between one number and another number."

Upon inspection, we find that Ms. R's introduction of the definitions for interval is rich. She orchestrates the discussion by supporting a dynamic back-and-forth movement between the word form, "interval," and formal and ostensive definitions of it. She begins by casting the word as an important learning goal but immediately moves away from it, asking what might seem like an unrelated question about the distance from one to three. Without making explicit the connection between this "distance" and the idea of an interval, she asks students for another exemplar (ostensive definition), this time phrasing the question to include the word form "interval." Only once a number of intervals have been identified does she make explicit the relation between "distance" and "interval," formulating a formal definition for the term. In addition to this back-and-forth movement from word form to exemplar, there is also movement from one exemplar to another. Even after repeated instances of an interval of 2 are identified, Ms. R doesn't yet offer a definition. She also asks for students to name intervals of 3 and implies that the same interval can be named in 2 different ways (as "four to one" or "one to four").

In this episode we also find that Ms. R promotes connections between definitions, in this case between interval and unit interval. After the discussion of interval, Ms. R introduces the idea of unit interval by stating, "there's a special type of interval that we have to know," and marks the interval from 0 to 1 and asks for its size, which she also does for the intervals from 2 to 3 and 3 to 4. An interval of one, she says, has "a special name." She points out that "an interval can be any number. I can do an interval of 50, I can do an interval of 63, I can do an interval of 1,000,402, I can do an interval of 4, but a unit interval can only be an interval of…" and the class responds in chorus: "One!"

Here, Ms. R is supporting students in making connections between a formal definition (interval) and another formal definition (unit interval), as well as what counts as an exemplar of each (ostensive definition). In this process, she follows an entailment of the class' constructed definition of interval to generate a definition of a unit interval. Unit interval is defined in terms of the established definition of interval. A key idea is that unit intervals are a subset of intervals and, accordingly, the definition of an interval is more inclusive than the definition of unit interval.

Our focus on the collective level of activity—the sociomathematical norms of definition use and connectedness of ideas, like the interplay between ostensive and formal definitions in collective activity—is central to understanding the reproduction and alteration of a common ground of talk and action in Ms. R's classroom. Further, this level captures an important explanation for the first two contours—the great number of references to definitions through the lessons (Contour #1), and the concentration of references early in the use of particular definitions or "pockets of density" (Contour #2). But we have yet to explore the functions that Ms. R and individual students were using definitional forms to serve. To explore the relations between definitional forms and their functions, we shift to the level of individual activity, turning our attention to the micro-, onto-, and sociogenesis of form-function relations with particular attention to Contour #3, the continuity in references to definitions over lessons.

Individual Level: A Focus on the Microgenesis, Ontogenesis, and Sociogenesis of Form-Function Relations

Recall from Sect. 11.1 that an individual level analysis of common ground entails a focus on how individuals tailor forms to serve mathematical functions in communication and problem solving. In turning our attention to empirical techniques that illuminate the individual level, we seek to explain the third contour—the recurring references to the same definition over the course of lessons. In our previous analysis of the timeline (Fig. 11.23), we noted that references to definitions like order, interval, and unit interval were introduced in earlier lessons but were also referenced repeatedly through later lessons, even as mathematical topics shifted. At first blush, the recurrence of principles may present a puzzle: Why do certain definitions continue to reappear long after they are introduced? This puzzle led us to consider whether these definitions are being repurposed, which would explain their continued relevance across shifting mathematical terrain. To explore this conjecture about recurring references to definitions introduced early in the sequence (Contour #3), we needed empirical techniques to illuminate the functions that forms serve in activity.

We began our inquiry by selecting a single definitional form, unit interval. The unit interval is a core, generative idea in the definition of a number line. It is also constitutive of additional core, generative definitions that lesson topics address, including definitions like multiunit, subunit, and equivalent fractions (see definitions of these terms in Figs. 11.18 and 11.19). Further, we noted that references to the unit interval reflect the third contour. Indeed, the dedicated timeline contained in Fig. 11.26 shows references throughout the entire lesson sequence, long after the definition was initially introduced. In some fractions lessons it was referenced with considerable frequency.

In the sections that follow, we illustrate a set of empirical techniques that are geared for explaining the presence of continued references to the unit interval definition. In this effort, we build on the treatment of the micro-, onto-, and sociogenesis of relations between definitional forms and the functions that they serve in activity.

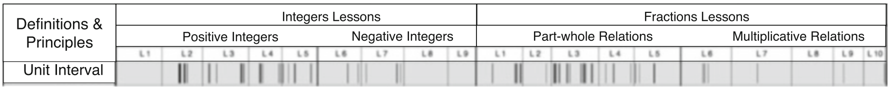


Fig. 11.26 References to the unit interval definition across integers and fractions lessons

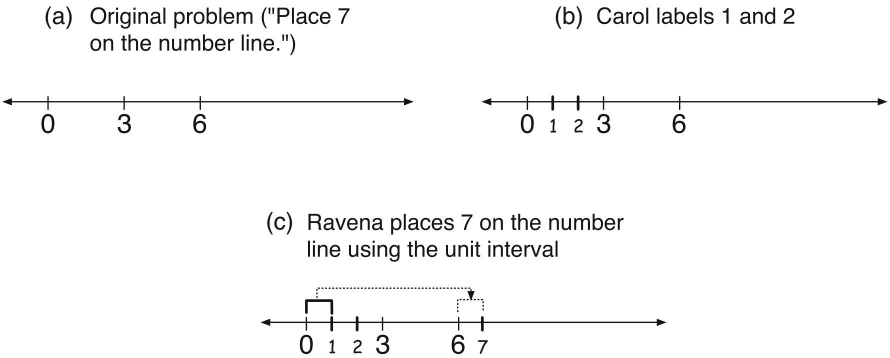


Fig. 11.27 Place 7 on the number line: Two students' microgenetic constructions involving the unit interval

Microgenesis: Empirical Techniques and the Unit Interval

In Sect 11.1 we noted that, in the course of problem solving, individuals produce microgenetic constructions by tailoring representational forms like definitions, number lines, and C-rods to accomplish emerging goals. Microgenetic constructions are pervasive in our video records and in other data sources. In this section, we analyze a single classroom episode. Our intention in this analysis is to reveal the intrinsic property of microgenesis that makes it possible for the unit interval to shift in function over time. Our analysis will illustrate that microgenetic constructions inherently reproduce aspects of prior forms or functions and, at the same time, generate alterations or variants. In what follows, we illustrate, during an integers lesson, how a variant form-function relation emerges in the process of microgenesis.

Our illustrative case occurs during Integers Lesson 4, not long after unit interval was defined in the class. By this point in the lesson sequence, we expected that individuals would use references to the definition in ways that were consistent with the original definition but that would also introduce productive alterations. In this episode, the class is collaboratively solving a problem. Students have been asked to place 7 on a number line where 0, 3, and 6 are marked (as depicted in Fig. 11.27a).

During the whole class discussion of the problem, a student, Carol, has come to the board and subdivided the interval between 0 and 3 into smaller, equally spaced segments using tick marks, which she labels "1" and "2" (Fig. 11.27b). Ms. R then calls on Ravena to help move the class towards a solution. In Ravena's display, she promptly places the numeral 7 at approximately the correct location, apparently making use of Carol's generated unit interval to do so, the 0 to 1 interval (see Fig. 11.27c). Interestingly, she does not use C-rods or other measurement tools, nor does she insert tick marks at 4 or 5. She appears to have made a mental translation of the unit interval distance created by Carol in order to approximate the location of 7 on the line.

What is significant about Ravena's microgenetic construction is the novel function that the unit interval serves. There is nothing explicit in the way that the unit interval was defined that indicated the unit interval could be translated to locate a number. What we see in Ravena's construction is perhaps a productive alteration that she tailors to the particular problem at hand.

Ravena's novel alteration of the unit interval is one of myriad alterations in form-function relations produced in students' and Ms. R's activities over the course of the lessons. Microgenetic constructions like Ravena's help maintain a common communicative ground with others, insofar as they reproduce aspects of the prior definition that are familiar to the classroom community. But such constructions also introduce novelty as an adaption to new collective problems that may become more frequent as lessons progress. Over time, such alterations would allow definitional forms like the unit interval to remain relevant across shifting problems and topics. This continued relevance would of course make it far more likely that earlier definitional forms would be referenced across lessons (Contour #3).

Ontogenesis: Empirical Techniques and the Unit Interval

We now shift our analytic lens to ontogenesis: the development of individuals' thinking and constructive activity over time. Microgenetic constructions do not occur in a vacuum. Rather, they are situated in a sequence of prior constructions that constitute an individual's ontogenetic trajectory. Thus, each microgenetic construction produced by an individual is both enabled and constrained by their prior constructions—enabled in the sense that prior constructions provide a frame for current activity, and constrained in that subsequent constructions operate within limits set by prior ones. Put another way, individuals reproduce and alter their own prior constructions. They both create novel functions for forms that they already know and recruit new forms to serve functions that they already understand. An analysis of the form-function shifts that are inherent in ontogenesis can further illuminate the third contour, explaining how a definition like the unit interval could remain relevant across shifting topics.

To illustrate empirical techniques to study the ontogenesis of form-function relations in ways that further illuminate the third contour, we make use of two data sources. We first consider our after lesson interviews, and the way that they illuminate shifts in function of the unit interval through a lesson for an individual student. We then turn to our pre-, interim, post-, and final assessments of students' progress; we focus on a single item that captures each student's use of the unit interval to name a point on the number line with an improper fraction over the course of the entire lesson sequence.

After Class Interviews: Ravena's After Class Interview and Shifts in Thinking Through the Lesson

As described in our section on data sources, the after class interview was a structured protocol that explored short-term ontogenetic changes in students' thinking over the course of the lesson. The procedure included two phases. The first phase occurred while the lesson was in progress, just as students were completing the solutions to the opening problems. As our research team members observed targeted students finishing, they quietly kneeled down and quickly queried students on their solution processes. The second phase occurred just after the math lesson concluded and consisted of a videotaped after class interview in which students were asked to solve the same problem again. Possible discrepancies between the student's initial and after class thinking about the problems were then discussed between the interviewer and student.

We selected Ravena's after class interview from Fractions Lesson 4 as an illustrative case (the same "Ravena" that we described in the prior section on microgenesis). In our analysis, we have two purposes: To illustrate the affordances of the after class interview technique in illuminating the form-function shifts in students' ontogenetic developments and to show how such data can be used to support an explanation of Contour #3 from an ontogenetic perspective.

Recall that in Integers Lesson 4, Ravena produced a microgenetic construction in which she repurposed the unit interval definition so that it served a translation function, providing a means to locate a positive integer on a number line. In turning to an ontogenetic analysis, we consider a problem from Fractions Lesson 4 where translating the unit interval could also be a viable strategy for Ravena. But now our focus is on how Ravena reproduced and altered her own prior constructions.

The opening problem in Fractions Lesson 4 asked students to locate the mixed number 1 1/3 on a number line in which only 0 and 1 are labeled (see Fig. 11.28). The problem is particularly challenging because there is no tick mark that defines the 1 to 2 interval. To produce a correct solution to the problem, students (and Ravena specifically) would need to understand a new set of functions for the unit interval, functions that entail coordinating translations and subdivisions.

Ravena's initial solution during opening problem phase of the lesson. In the opening problem phase, Ravena's solution of the problem (see Fig. 11.29a) was to mark 1/3 (not 1 1/3, as the problem calls for) by partitioning the unit interval into three equal subunits and labeling the first as 1/3 (see Fig. 11.29b). In doing so, she produced a microgenetic construction in which she used the unit interval to serve the function of subdivision, generating the location for a fraction on the number line. At the same time, Ravena did not make use of the translation function that she used in the integers lessons, reviewed in the prior section. A coordination of subdivision and translation would have allowed her to locate the correct position of 1 1/3.

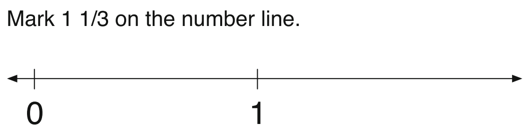


Fig. 11.28 An opening problem for Fractions Lesson 4: Locating a mixed number on the number line

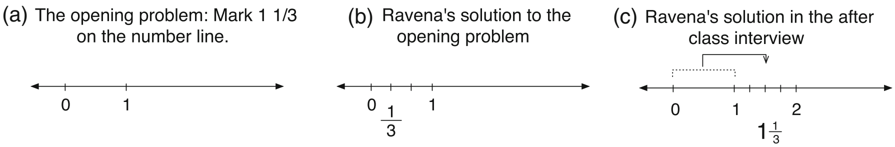


Fig. 11.29 The opening problem and Ravena's solution and her subsequent solution in the after class interview

Ravena's revised solution during the after class interview. In the after class interview, we found that Ravena was able to (partially) coordinate the functions of subdivision and translation in her effort to locate the mixed number, 1 1/3. She located 1 1/3 between 1 and 2, and she also used equal intervals in her partitioning. However, Ravena subdivided the interval between 1 and 2 into four subunits rather than three, and she had difficulty labeling the subunits (see Fig. 11.29c).

Our analysis of Ravena's after class interview reveals the way that she drew upon and extended her prior microgenetic constructions to support her solution to a novel problem. In her after class solution, she translated the unit interval to create the interval from 1 to 2, much as she had during the Integers lessons. She also subdivided the unit interval as she had done in the opening problem and earlier fractions lessons (see her solution to the Opening Problem). Her after class solution reflected an adaptation in which she coordinated these translation and subdivision functions, now used to locate a mixed number on the number line. This coordination, however, remains incomplete. She did not accomplish the subdivision of the 1,2 interval in a way that would allow her to identify the appropriate point. Thus, Ravena's prior ontogenetic constructions enabled new advances, but they also constrained her ability to fully coordinate functions in a way that would lead to a correct solution.

Such short-term case studies of individual students provide rich material to explore ontogenetic trajectories; indeed, they can contribute to our understanding of processes involved in repurposing forms to serve novel functions over a single lesson. But such case studies are limited to short time spans. We now turn to a complementary data set that allows us to explore shifts in each student's thinking over a greater span of time, an approach that makes use of a different set of techniques.

Student Assessment Instrument to Document Longer-Term Ontogenetic Shifts: Pre-, Interim-, Post-, and Final Assessments

For the longer-term ontogenetic analysis, we made use of the student assessment instrument previously referenced in our efficacy study. Rather than focus on a total score on the assessment, our concern now was to make use of these assessment data to document shifts in the function of the unit interval in students' ontogenetic trajectories over the lesson sequence. To this end, we focused on a single item. We looked for evidence of shifting functions of the unit interval in each student's responses across four points in time: prior to the LMR intervention (pre-assessment), after the integers lessons but before the fractions lessons (interim assessment), immediately following the fractions intervention (post-assessment), and at the end of the school year (final assessment).

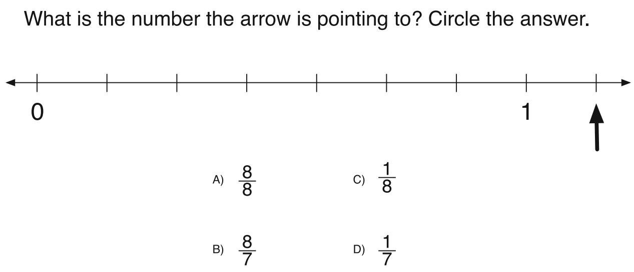


Fig. 11.30 A recurring number line item used on pre-assessment, interim assessment, post-assessment, and final assessment (correct answer: (b) 8/7)

Figure 11.30 contains the selected item. The item asked students to label a point as an improper fraction on the number line, with possible solutions presented in multiple-choice answer format. We deliberately selected each possible multiple-choice answer based upon observations from the classroom studies described earlier, when students were solving and justifying solutions to similar problems. Of particular interest for the present analysis is students' choice of the denominator, 7 (in 8/7 and 1/7) or 8 (in 8/8 and 1/8), and what we surmise the choice suggests about the function of the unit interval in students' solutions.

Note that for the number line contained in Fig. 11.30 there are eight equal-size intervals, one of which extends beyond the longer (0–1) unit interval. Based upon classroom observations, we anticipated the problem would present an intellectual challenge for students and that their answer choices would reflect an ontogenetic trajectory in the functions that they use the unit interval to serve. At a less sophisticated level, students would not differentiate between equal-size intervals within the unit interval and those beyond the unit interval. Instead, students would treat all equal-size intervals in the same way—as subunits—leading to a denominator of 8. Over the course of the fractions lessons, however, students would shift in the function that they used the unit interval to serve. This shift would be marked by students' use of the unit interval as a boundary in the construction of subunits and lead to choice of 7 as the denominator.

Figure 11.31 contains a representation of each student's solutions to the problem at pre, interim, post, and final assessments. To support qualitative analysis, we organized each student's trajectory as a single line. The vertical axis of the graph represents the four different solutions, where the top two solutions feature 7 as the denominator (8/7 and 1/7), and the bottom two solutions feature 8 as the denominator (1/8 and 8/8). The horizontal axis of the graph represents the four points of assessment: pre-assessment (prior to LMR instruction), interim assessment (after integers but before fractions lessons), post-assessment (just after the fractions unit) and final assessment (at the end of the school year). The numbers that populate the graph are student identification numbers and their associated lines represent the trajectories of the individual student.

The graph reveals three noteworthy features relevant to shifts in function of the unit interval. First, as expected, almost all students (10 of 13) begin by using 8 as the

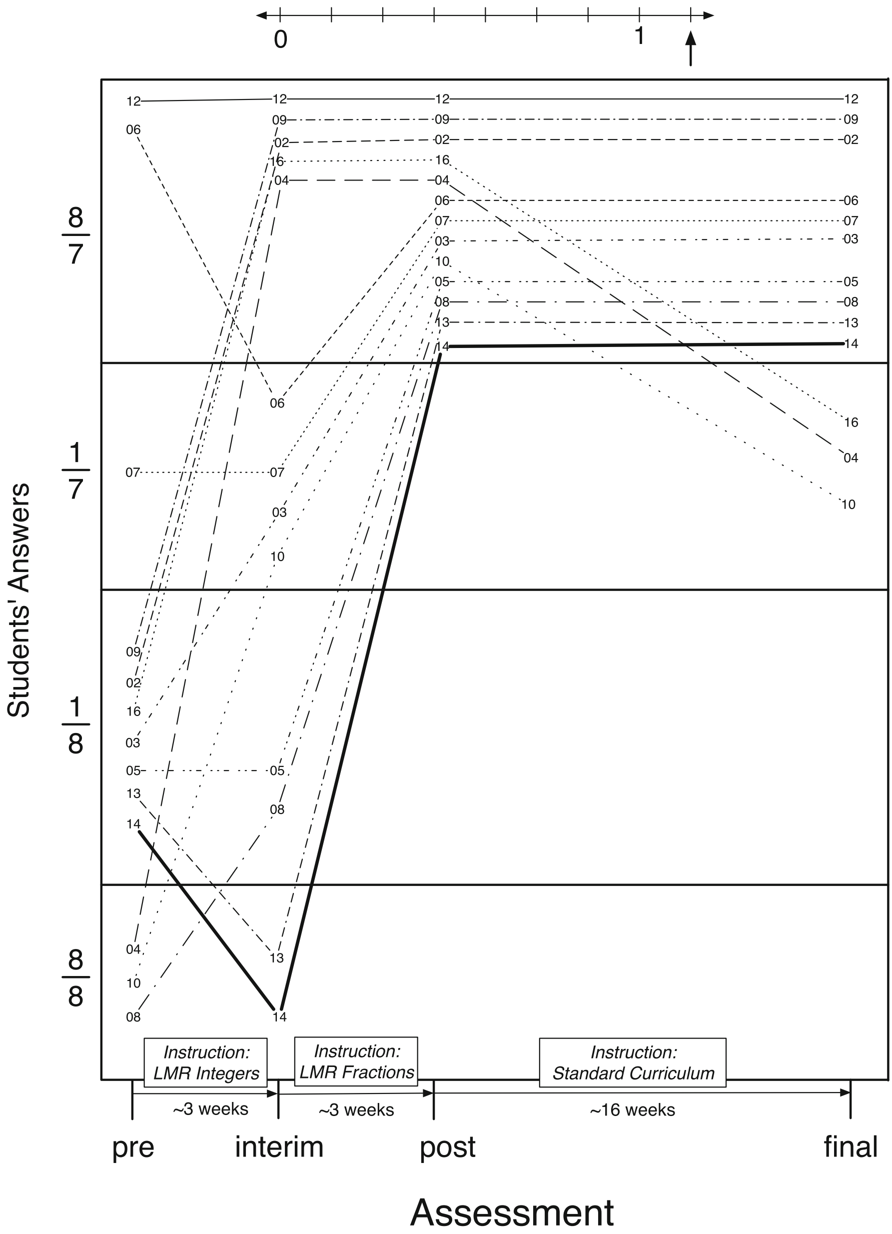


Fig. 11.31 Ontogenetic trajectories in Ms. R's class on a fractions assessment item: "What is the number the arrow is pointing to? Circle the answer: 8/8, 8/7, 1/8, 1/7." (Numbers in the graph represent student identification numbers.)

denominator; by post and final assessment all students shift to 7 as the denominator. We take such cross-lesson trajectories to indicate that students are constructing new functions for the unit interval as they repeatedly engage with labeling the denominator for an improper fraction.

Second, at the interim assessment, several students have already made the shift to 7 as the denominator. At first blush, this may be surprising. Recall that between the pre and interim assessments, LMR lessons covered integers, not fractions. Why might students have shifted, absent the introduction of the subunit definition in the lesson sequence? One plausible conjecture is that students extended their developing understanding of the unit interval in identifying an improper fraction, building upon whatever instruction they had on fractions prior to the LMR integers lessons.

Third, after the unit on fractions (post-assessment), all students had shifted to 8/7, the correct answer.[13] In this answer, the numerator reflects the total number of subunits (8) from 0 to the unlabeled tick mark. We take this shift in answer choices as indicating an ontogenetic trajectory, one in which the unit interval serves a changing function in students' constructions of numerator as well as the denominator.

Considered together, these three features capture a long-term shift in function of the unit interval. The shift sheds further light on the third contour—the repeated use of a definitional form through the lesson sequence. Students appeared to be building upon the definition of the unit interval in their construction of subunits and the identification of an improper fraction on the number line. This classroom-wide shift in function allowed the unit interval to remain relevant long after it was originally introduced in the integers lessons.

Sociogenesis: Empirical Techniques and the Unit Interval

Recall from Sect. 11.1 that our treatment of sociogenesis focuses on the population-wide distributional properties of form-function relations. At any one time in a community, numerous form-function relations are generated across the constructive activity of individual actors. Individuals may be using different forms to serve the same function, or the same form to serve different functions. Hence, certain form-function relations will be more widespread than others during a given interval of time. Furthermore, these distributions of form-function relations inevitably shift over time. For example, formerly widespread form-function relations may fall into abeyance; new functions for familiar forms may spread; and familiar functions may become increasingly accomplished by new forms. This focus on population-wide distributions marks a departure from processes of microgenesis and ontogenesis, which are limited in scope to the constructive activities of particular individuals.

[13] The graph also reflects students' performances on the final assessment, about 16 weeks after LMR instruction. We find that most students reproduced their post-assessment solutions to the problem, 8/7, while three shifted to 1/7 as the solution. No students reverted to a solution for which 8 was the denominator.

A sociogenetic approach to an analysis of Ms. R's classroom community leads us to inquire into the character of distributions of form-function relations and shifts in these distributions over time. Further, the approach would also lead to an analysis of the processes that support and constrain such distributional shifts. Attempts to make progress in this inquiry create an empirical challenge: How might we craft empirical techniques that provide information on shifting distributions of form-function relations as well as illuminate factors that support continuities and discontinuities in distributions through time?

Documenting Shifts in the Distributions of Form-Function Relations

In this section we focus on patterns in the co-occurrence of references to definitional forms across lessons. We conjectured that the co-occurrence of definitional forms might provide a window into shifting functions of the unit interval through the lessons. Our assumption was that, when the unit interval is used in the context of a definition like multiunit, it might be serving a different function than when it occurs in the context of a reference to subunit.

Figure 11.32 contains the timeline (presented previously) but now with several features made salient. Consider the different patterns of co-occurrence of selected definitional forms in Integers Lesson 4, Fractions Lesson 3, and Fractions Lesson 6. Across the three lessons (marked by bolded rectangles for emphasis), the figure captures the shifting contexts of the unit interval's co-occurrence with definitions of multiunit interval, subunit interval, numerator, denominator, and equivalent fractions. These shifting definitional contexts for the unit interval suggest that it is serving different functions in public discourse over time.

To focus more closely on these changing distributional contexts and their implications for shifts in function of the unit interval, we generated a bar chart contained in Fig. 11.33. To produce the chart, we calculated the proportions of references in

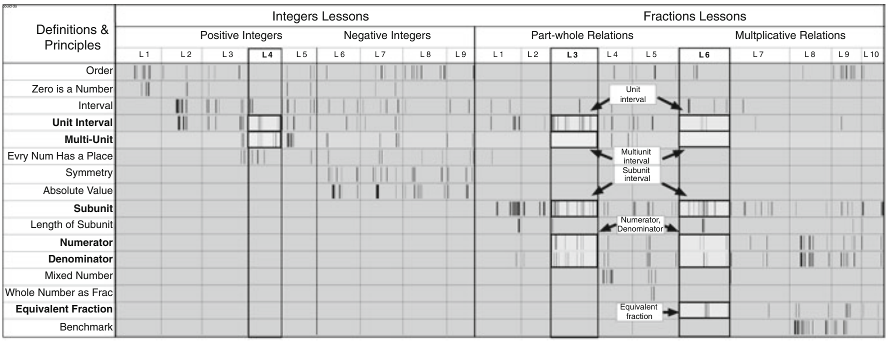


Fig. 11.32 Timeline of references to definitions with a focus on references to unit interval and references to related definitions

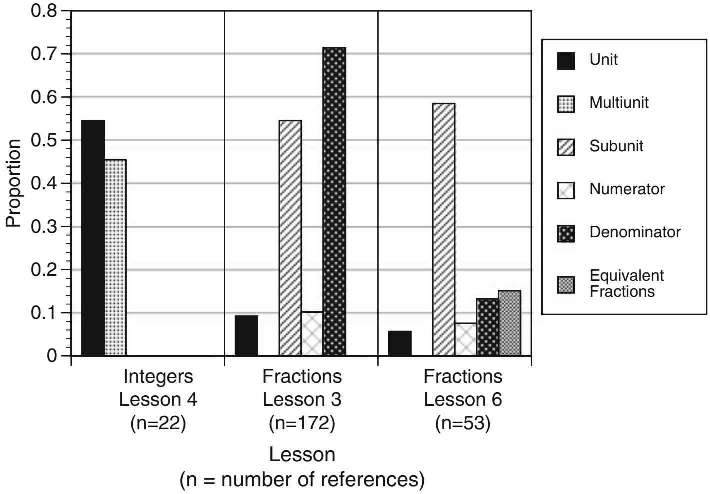


Fig. 11.33 Distribution of references to five selected definitions for Integers Lesson 4 and Fractions Lessons 3 and 6. References are represented as proportions

each targeted lesson using data contained in the time line.[14] The bar chart further reveals shifting patterns of co-occurrence of the unit interval with the other targeted definitions, including multiunit interval, subunit interval, numerator, denominator, and equivalent fractions. These shifting definitional contexts, based upon what we know about the lesson topics, imply shifts in the function the unit interval is serving: In Integers Lesson 4, we find that of the targeted definitional forms, the only referenced definition that accompanies the unit interval is the multiunit interval. In contrast, in Fractions Lesson 3, a lesson in which the focus is on identifying fractions greater than 1 on the number line, the distribution of references shifts. Now, the unit interval is still referenced, but there are many more references to the other definitional forms (and an absence of references to multiunit). Denominator appears with the greatest frequency, but there are also references to subunit and numerator. In Fractions Lesson 6, when equivalent fractions are introduced, the distribution again shifts markedly. Now, in this lesson, the focus is on the construction of equivalent fractions, using subunit as a means of generating new fraction names for the same point (and again, with no reference at all to multiunit).

To corroborate our conjecture that shifting patterns in the co-occurrence of the unit interval with other definitions reflect shifts in the function that the unit interval

[14] To determine proportions, we counted the number of total references to each of the five definitions for each lesson. In this computation, we divided the number of references by the total number of references for each lesson. Note that total number of references to the selected definitions (n) differs markedly by lesson. There are two reasons for these differences. First, other definitions were referenced in the lessons, and so the number of instances does not reflect the total number of instances referenced in a given lesson. Second, Ms. R decided to devote two math periods to F-3; as a result, more references were made to definitions for that lesson than the other two lessons.

serves, we turned to video records of lessons. What we found did indeed corroborate our conjecture. In Integers Lesson 4, for example, some students used the unit interval (in coordination with references to multiunits) to translate distances that had a value of 1 on a line to label unidentified points. In contrast, in Fractions Lesson 3, when trying to label a point greater than 1 with an improper fraction, some students used the unit interval as a means to compute both the denominator and the numerator. Then, in Fractions Lesson 6, the unit interval was repeatedly partitioned to identify equivalent fractions on the number line.[15] Thus, inspection of video records provided additional support for our interpretation of the shifting patterns of the co-occurrence of references to definitional forms—that these patterns reflected shifting functions of the unit interval in classroom discourse.

Explaining Shifts in the Distributions of Form-Function Relations

What might contribute to the shifts in function of the unit interval in Ms. R's classroom community? We have already remarked that the sociomathematical norms in Ms. R's classroom support definition use. We have also already demonstrated the important role of the shifting topics in the LMR curriculum. Beyond these factors, what other elements of collective life could influence shifting distributions of form-function relations?

In this section, we examine social positions of participants in the classroom community as a possible answer. To capture social positions in Ms. R's classroom, we employed a sociogram to capture how Ms. R's students view one another's mathematical competence. We conjectured that students' understanding of one another's competence may well be implicated in whether and how they take up form-function relations that appear in one another's public displays. This uptake would then contribute to the shifting distributions of form-function relations in the classroom community.

Figure 11.34 displays the results of the procedures that we used to generate the sociogram data. The left column contains the students' answers to the question, "who would you seek math help from?" The right column contains the names of the students they would target for help. Thus, the number of convergent lines on names in the right column provides an index of the "social position" of students with respect to perceived competence in math. We observed that several students are nominated numerous times by others.

As evidence of the potential utility of the sociogram in explaining the shifting distribution of form-function relations in a sociogenetic analysis, we return to the initial excerpt from Ms. R's classroom. Recall that Carol appeared to shift in her

[15] We note an issue that is important but that we have not taken up in our analysis of sociogenesis: In Ms. R's classroom during any particular lesson, students may well have been using the same form, like the unit interval, to serve different functions in classroom displays during the same period of time (like a whole class discussion). Our coding schemes did not allow us to document these form-function distributions. Variant functions for the same form are to be expected in sociogenetic processes, and we expect that these phenomena were very much a part of the nuance of classroom discourse in Ms. R's community.

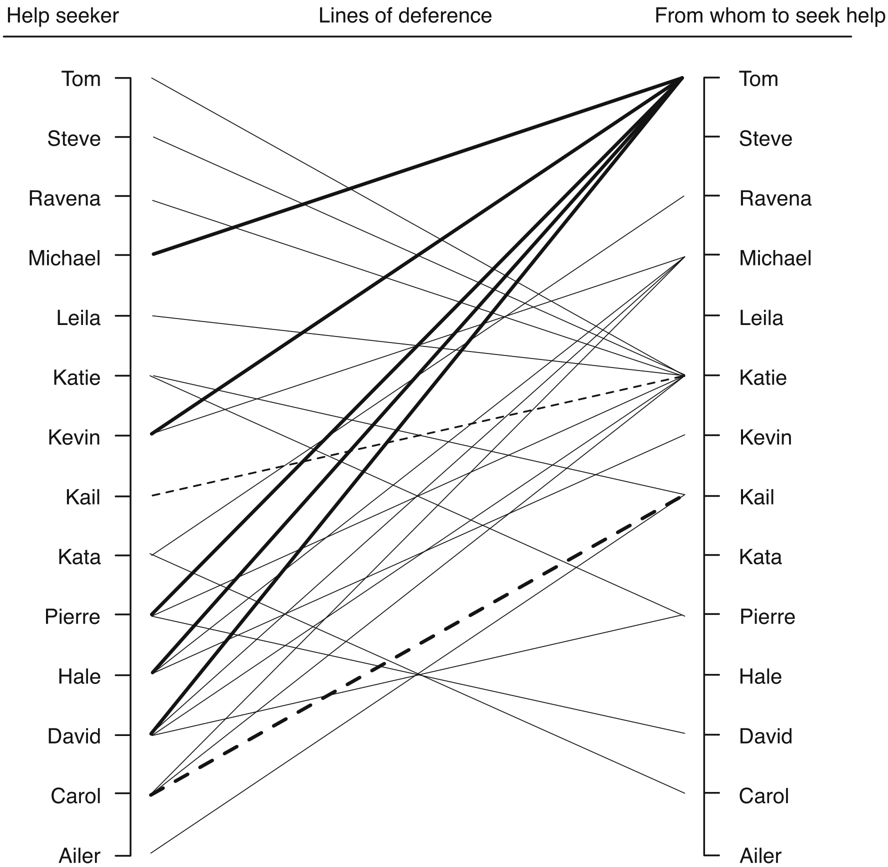


Fig. 11.34 Sociogram showing who students would prefer to seek mathematical help from on a hard problem (direction of preference, *left* to *right*)

thinking over the course of the episode. Although she initially used the unit interval to provide a value for an unlabeled tick mark on the number line (−1,006 or −1,007 for the position of −2,000), she ultimately shifted to using a multiunit interval (1,000) to serve the same function, achieving a more adequate solution. Recall also that the shift occurred after Carol observed public displays of Kail and Pierre, each of whom objected to Carol's answer (which was produced through the translation of a unit interval). In contrast to Carol's display, Kail and Pierre's displays made use of a multiunit interval (multiunit interval of 1,000).

We believe that it is more than coincidental that Carol nominated Kail as someone she should go to for help in math (see sociogram, Fig. 11.34). We suspect that Carol's view of Kail's mathematical competence may have shaped her reflections on Kail's solution and ultimately contributed to a reconsideration of her own solution approach. We are certainly aware that there are other factors that mediated Carol's shift including her interpretation of Kail's and Pierre's displays, an interpretation that initially privileged her own solution as the more appropriate. Indeed, we noted that Carol does not immediately defer to Kail, reiterating publically her own solution with a logic that makes sense to her. However, it may well be the social position she accords to Kail that pushes her to reconsider, and it is her reconsideration that leads to her epiphany.

11.3 Final Thoughts and Next Steps

In this chapter, we addressed two challenges. The first was to extend a framework for the study of the cultural development of mathematical ideas (Saxe 2012) to the analysis of classroom communities. We presented this extension in the first section of the chapter, directing it towards a central but under-conceptualized (and understudied) dimension of teaching-learning interactions: the reproduction and alteration of a common ground of talk and action over lessons in classroom communities. In the extension of the framework, we treated classroom communities as microcultures reproducing and altering a common ground of mathematical talk and action as topics shifted over lessons. To accomplish this, we specialized, refined, and elaborated constructs used in cultural treatments of cognition, adapting them to the world of classrooms.

The second challenge was to explore the utility of the framework through the development and deployment of a coordinated set of empirical techniques. To address this challenge, we engaged in two related strands of empirical inquiry that we described in the second section of the chapter. One strand was preparatory. It involved a coordinated set of interview, tutorial, and classroom studies that supported the development of LMR's 19-lesson sequence. The central interest of the LMR team was to generate a lesson series that would afford continuity over lessons through the use of the number line and related definitional forms. Throughout, our focus was on the framework-motivated construct of common ground, central to processes of teaching and learning in whole class discussions. The experimental study corroborated the utility of the design approach by demonstrating strong effects on student learning when compared to students in classrooms that were implementing a well-regarded (non-LMR) curriculum.

The other strand of the chapter on empirical techniques focused on an analysis of the lessons themselves as they came to life in a classroom community. Our goal was to understand the reproduction and alteration of a common ground as lesson topics shifted, drawing upon data sources generated in a single classroom. To leverage our empirical inquiry, we focused on the definitional forms that the lessons supported and the variant functions that the definitions were used to serve. Using our archive of video records along with video analysis software, we produced a timeline of references to definitional forms during whole class discussions. The timeline supported a two-pronged investigation. First, we pursued an analysis of collective activity: the emergence, reproduction, and alteration of norms that regulated joint activity in the classroom. Guided by trends in the timeline, we developed conjectures

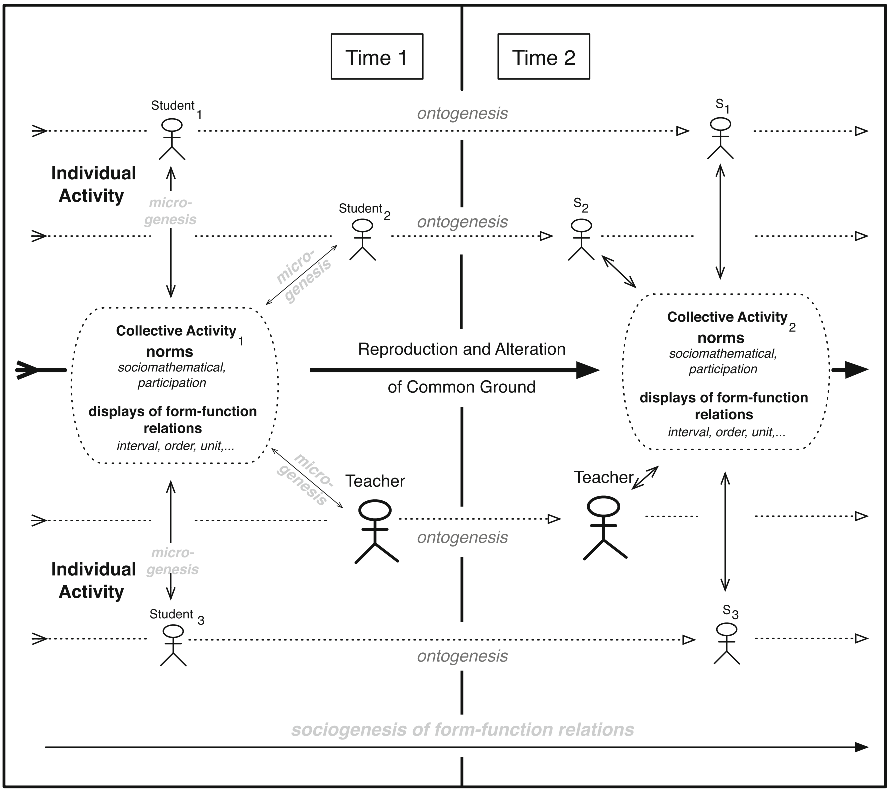


Fig. 11.35 A schematization of individual and collective activity over lessons

about the emergence of sociomathematical norms and supported them with close analysis of video records. Second, we pursued an analysis of individual activity. We coordinated (a) microgenetic analyses of references to definitions in our video records and after class interviews, (b) ontogenetic analyses that drew particularly upon our pre, interim, post, and final assessments, and (c) sociogenetic analyses that drew on the shifting distribution of form-function relations in the classroom community as well as analyses of social positions of students that may have supported the reproduction and alteration of particular form-function relations.

We now bring our inquiry to a close, revisiting our orchestration of key constructs of the framework. We represent those constructs in Fig. 11.35. Featured in the figure is a display of the reproduction and alteration of a common ground through time, including the intrinsic role of collective and individual activity, as well as associated constructs of sociomathematical norms and genetic processes.

The figure schematizes two sequential time periods in a classroom, Times 1 and 2, the temporal dimension being at the crux of our analytic approach. The two time periods can be conceptualized as two consecutive lessons, two consecutive phases within a single lesson, or two durations of time within a lesson phase. Regardless of the unit of time, the figure depicts a common ground emerging through time as a result of an interplay between collective and individual activity.

At Time 1, the figure portrays a nexus of relations constitutive of a common ground as collective and individual activity become reciprocally constituted. The collective activity of classroom life (e.g., the sociomathematical norms like the use of definitions in argumentation, social positions like who is "good" at math, public displays of form-function relations used in the community) affords social meaning for the constructive activity of individuals. At the same time, individual activity is constitutive of collective activity. In the figure, Student1, Student2, Student3, and the teacher are producing public displays of their microgenetic constructions, tailoring forms to serve communicative and problem solving functions (and at the same time contributing to the reproduction and alteration of collective norms).

At Time 2, the figure shows a reproduction and alteration of a common ground. Although the actors remain the same, the conditions of activity have changed with shifting topics or problems. To illustrate, consider the unit of time to be a lesson in the LMR sequence: A problem may have changed from labeling a whole number on the number line to labeling a negative integer on the line. In such changing conditions, we may find both continuity and discontinuity across time periods. At the collective level, we expect to find both the reproduction and alteration of norms, as students explain their thinking to one another and justify their claims with references to definitions. At the individual level, students at Time 2 may be introduced to new definitional forms to solve problems in their microgenetic activity. Individuals at Time 2 are also drawing on their prior constructions, creating continuity and discontinuity in their ontogenetic developments. Finally, the directional line at the bottom of the figure represents the sociogenesis of forms and functions. In their constructions, teacher and students are at once drawing upon form-function relations used previously while at the same time, they are contributing to the reproduction and alteration of form-function relations and their distribution in the community.

A lingering and important question in our chapter is whether the general treatment that we have offered has legs: Is the analytic approach and methodology that we have elaborated applicable to the study of other classrooms beyond those implementing the LMR curriculum? As we alluded to at the beginning of Sect. 11.2, in the construction of the lesson sequence we included features that afforded systematic inquiry about the interplay between processes of teaching and learning in classroom communities, like the definitional forms that leveraged our analysis of the reproduction and alteration of a common ground. Given the leverage afforded by the LMR lessons, we ask the question, is the conceptual and methodological progress that we have made inextricably linked to the LMR lesson sequence alone?

We are cautiously optimistic that the framework and techniques for cross-lesson analyses will be useful for other projects. Indeed, one can ask about any classroom community an important question at the crux of teaching-learning interactions: What is the character of the common ground that is being reproduced and altered across lessons? To what extent are students and teachers operating on one another's meanings in ways that both lead to a productive common ground and seed the construction of new generative ideas? At the same time, we expect that researchers studying classrooms will develop new focal points for analysis and will need to create new empirical techniques that differ from those we have developed. Indeed, we regard the methodological approach that we have developed as a useful starting point, opening up new conceptual and empirical territory in classroom research.

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Part VIII
Mixed Methods

Chapter 12
The Combination of Qualitative and Quantitative Research Methods in Mathematics Education: A "Mixed Methods" Study on the Development of the Professional Knowledge of Teachers

Udo Kelle and Nils Buchholtz

Abstract Research about education in mathematics is influenced by the ongoing dispute about qualitative and quantitative research methods. Especially in the domain of professional knowledge of teachers one can find a clear distinction between qualitative, interpretive studies on the one hand and large-scale quantitative assessment studies on the other hand. Thereby the question of how professional knowledge of teachers can be measured and whether the applied constructs are developed on a solid theoretical base is heavily debated. Most studies in this area limit themselves to the use of either qualitative or quantitative methods and data. In this chapter we discuss the limitations of such mono-method studies and we show how a combination of research methods within a "mixed methods design" can overcome these problems. Thereby we lay special emphasis on different possibilities a mixed methods approach offers for a mutual validation of both qualitative and quantitative findings. For this purpose, we draw on data and results coming from an empirical study about a teacher training program in mathematics, where quantitative data measuring the development of professional knowledge of student teachers were related to qualitative in-depth interviews about the training program.

Keywords Mixed methods • Teacher education

U. Kelle (✉)
Helmut-Schmidt-Universität Hamburg, Holstenhoffweg 85, 22043 Hamburg, Germany
e-mail: kelle@hsu-hh.de

N. Buchholtz
Arbeitsbereich Didaktik der Mathematik, Universität Hamburg,
Fak. 04, FB ErzWiss 05, Binderstraße 34, 20146 Hamburg, Germany
e-mail: Nils.Buchholtz@uni-hamburg.de

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12.1 Introduction

The dispute about qualitative and quantitative research methods and about possibilities to combine these methods is a crucial and up to date topic in the field of research in mathematics education. Current discussions do not only touch questions about the practical implementation of qualitative and quantitative research designs and about their empirical outcomes—there are also deeper epistemological questions at stake, as Philipp Mayring pointed out in a plenary lecture at the 44th annual meeting of the German Society of Didactics of Mathematics (GDM) in Munich (Mayring 2010), thereby drawing on the well-known expression of "Science Wars" (Bammé 2004). Although in empirical research in the social sciences qualitative and quantitative methods are often used simultaneously within the same research project, the differing methodological approaches underlying qualitative and quantitative research are still an important topic within methodological debates in the field of mathematics education.

With the well-known large-scale comparative studies such as TIMSS (Baumert et al. 2000a, b) or PISA (Baumert et al. 2001) in the 1990s and 2000s in Germany a vivid and ongoing discussion about methodological and epistemological fundamentals and tenets of different empirical research methods, which is taking place all over the social sciences, has also reached the field of mathematics education. Recent years saw a growing empirical research interest in the domains of teacher education and teacher training programs, a development also sparked by certain large-scale comparative studies like the international comparative IEA study TEDS-M (Blömeke et al. 2010a, b) or the national COACTIV study (Kunter et al. 2011), which follow the tradition of PISA. Meanwhile, there is a variety of quantitatively and qualitatively oriented approaches in research on the so-called professional knowledge of teachers and a methodological dispute takes place about these different approaches. By now, only few empirical studies in mathematics education try to combine qualitative and quantitative research methods in a common design. The aim of this chapter is to discuss the advantages as well as the challenges of such an approach.

In the first two sections of the chapter we describe some crucial aspects of the dispute over "qual" and "quan" and discuss the methodological background of the qualitative-quantitative divide first in the social sciences in general and secondly in research on mathematics education. In both areas much of the (often justified) methodological criticism of both sides is not used in a productive manner, that means: the criticism of the other side is not taken seriously as an inducement to deal with the weaknesses of one's own approach but answered in a "tit for tat" way by pointing to the shortcomings of the other side.

"Mixed methods" is a brand name for a methodological movement which offers ways out of the described dead end streets of endless methodological warfare and which now looks back on 20 years of practical research experiences and methodological debate. In section four we discuss terminological distinctions relevant for mixed methods research, in particular the distinction between the integration of different types of data and analysis techniques on the one hand and the integration of qualitative and quantitative sub-studies within one research design on the other hand. To conceptualize the latter, the notion of triangulation is of great importance. We describe two types of triangulation, namely triangulation for the sake of mutual validation and triangulation as a mutual supplementation of research results which provides a more complete picture of the research domain.

At the core of mixed methodology is the idea that both qualitative and quantitative research methods have specific limitations with respect to certain research questions and research domains. In the fifth section of the chapter we discuss the most important strengths and weaknesses of both qualitative and quantitative methods and how they can be balanced by drawing on different types of mixed methods designs. Thereafter we describe the implementation of a mixed methods approach in research on mathematics education with the help of a practical example. This example makes clear how qualitative methods are helpful for the interpretation of quantitative results and how they may provide extensive background information for the evaluation of educational processes in mathematics teacher education on the level of performance tests. It also shows that "mixed methods" does not mean to merely make concessions to both camps in the ongoing qual-quan-debate, but rather to develop a complex methodology bringing about a variety of new challenges.

12.2 "Mixed Methods": Challenging the Qualitative-Quantitative Divide in Social and Educational Research

Looking at current methodological writings one may think that the long-lasting "paradigm wars" (Gage 1989) between the quantitative and the qualitative methods tradition have lost much of their attraction in the past years, especially since the advent of a movement in educational research to end paradigm wars, which now has also influenced other social science disciplines, e.g. sociology and psychology. At present this movement is widely known under the label "mixed methods" which means the integration of qualitative and quantitative methods (Tashakkori and Teddlie 2003, 2010). The experience of research practitioners that both qualitative and quantitative methods can be necessary in empirical research can already be learnt from famous social studies from the 1930s onwards—among them the "Hawthorne Study" (Roethlisberger and Dickson 1939) or the "Marienthalstudie" (an investigation among Austrian unemployed workers shortly after the great depression; Jahoda et al. 1982/1933). However, until nowadays it remains still difficult to obtain methodological advice on how qualitative and quantitative findings can be related to each other in order to achieve valid research results. This may be due to the fact that "paradigm warriors" often restrict their arguments to general epistemological ideas on the nature of reality (emphasizing, for example, that there are "multiple realities") , whereas "pacifists" or "integrationists" (e.g. Bryman 1988; Brannen 1992; Cresswell 1994; Tashakkori and Teddlie 1998, 2003, 2010) have mainly developed methodological guidelines for method integration and regard theoretical and substantive aspects as a matter to be decided according to the requirements of each respective research project. However, the efforts to combine qualitative and quantitative methods often lack a solid methodological basis in research practice, as meta-analyses about mixed method studies have shown (Bryman 2005, 2008): researchers frequently combine quantitative and qualitative methods without providing a clear rationale for their choice of methods. And qualitative and quantitative findings are often not integrated in a coherent way when results from such research projects are presented. That makes clear that the mixing of methods in a certain research design is a challenging task: it is never sufficient simply to piece together different types of data and analysis methods. On the contrary, researchers must make sure that the different data and methods used are related to an overarching conceptual framework, so that the mixing of methods does not break the research question, research topics and theoretical base of the project into unrelated parts. Furthermore, in order to relate qualitative and quantitative data, methods and research results to each other in a meaningful way, the research logic as well as the specific methodological problems of weaknesses of either tradition must be taken into account.

Reasons for the unsatisfactory situation that in many mixed methods studies methods are only juxtaposed but not truly integrated do not only lie in a lack of competencies of empirical researchers but also in the state of the methodological discussions about mixed methods. In this debate it is often emphasized that the use of methods should be predominantly influenced by substantive research questions, and not by methodological and epistemological considerations alone. Moreover, it is often maintained that all methods have specific limitations as well as particular strengths, and that they should be combined in order to compensate for their mutual and overlapping weaknesses (Johnson and Turner 2003, p. 299). However, crucial questions regarding the relation between research domains, research questions and research methods have been still not addressed sufficiently in methodological discussions. For which types of research questions are qualitative and quantitative methods suited better? What are the typical weaknesses and strengths of qualitative and quantitative methods in relation to particular research domains? To develop mixed methods approaches it would be thus helpful not to simply dismiss the debate between qualitative and quantitative approaches as an outmoded "paradigm war" but to use criticisms coming from the both sides in this debate in a productive way. For this purpose we must look at the rational arguments behind the polemic attacks which relate to real shortcomings of either tradition (for details cf. Kelle 2006).

A crucial limitation of quantitative research, for instance, stems from the requirement to construct precise and clear cut hypotheses before data are collected. The reason for this requirement is clear. For serious statistical analyses standardized data are needed. To collect such data one has to have a clear cut idea about the entities one is observing and their attributes. Quantitative research is thus strongly tied to a hypothetico-deductive (HD-) approach: research is conceptualized as a process whereby elaborated hypotheses are first deduced from theories, then operationalized and subsequently tested with the help of empirical data. Qualitative critics of this approach point out correctly that in certain areas of investigation social researchers do not have enough theory and precise concepts at hand to construct instruments for the categorization of phenomena in the field. In such areas it would require the availability of culture-specific knowledge to operationalize theoretical concepts and to develop measurement instruments. A meticulously constructed questionnaire, for instance, may yield an invalid and highly misleading picture of the investigated domain if research subjects understand a question in a different way than the researchers, or if the topics treated are not relevant for the respondents. On the other hand, from the beginning of the qualitative tradition, statisticians blamed qualitative researchers for not providing a basis for sound generalizations because of the lack of representativeness of small N studies. In qualitative studies case numbers are often small and one may get the impression that they are somewhat arbitrarily selected—this raises doubts about the generalizability or (to use a more careful word: "transferability," cf. Guba 1981) of findings. Furthermore quantitatively minded critics have often noticed with unease that data in qualitative research seem to lack objectivity and are not collected following an explicit theoretical rationale.

There is an extended discussion about epistemological implications of such arguments and this methodological dispute as well as proposals how to overcome such differences (which may seem irreconcilable at least at first sight) can be found in almost all social science disciplines, and consequently also in research on mathematics education. In the following we want to trace this dispute and discuss strategies of how basic arguments from it can be used in a productive manner for the application of mixed methods designs.

12.3 The Dispute About "Quan" and "Qual" and Mixed Methods in Research on Mathematics Education

In the 1960s and 1970s a methodological shift took place in research on mathematics education. Up to this point, Piaget's developmental psychology (Piaget and Szeminska 1965) was dominant and empirical research focused on (qualitative) observations. But as early as in 1962 Bruner noticed a "revival" of interest in cognitive processes, which later led to the so-called cognitive revolution in psychology that stimulated standardization of research methods and set in motion a movement towards quantitative research:

> The past few years have witnessed a notable increase in interest in and investigation of the cognitive processes – the means whereby organisms achieve, retain and transform information. This increase in interest and effort should, we suppose, be counted as a "revival," since there was an earlier time (the years before the First World War), when the Higher Mental Processes constituted a core topic within psychology. (Bruner et al. 1962, p. vii)

In the 1980s and 1990s, due to the growing importance of cognitive processes for psychological research, theories from cognitive psychology (for example Anderson 1980) began to play a more important role in mathematics education. And with this a broader discussion started between more elaborated constructivist qualitative approaches in research on the one hand and adherents of critical rationalism on the other hand who normally favored quantitative methods (e.g. Brezinka 1968). However, the integration of qualitative and quantitative methods was not a topic at that time, although some American researchers already tried to combine methods in small N studies in research about teachers' professional knowledge (Carpenter et al. 1988, 1989; Carpenter and Fennema 1992; Fennema et al. 1996). In an overview about methodological research at that time, Wellenreuther (1997) distinguished between four different research approaches: (1) diagnostic/descriptive research about learning behavior, attitudes and strategies, (2) research about mathematics instruction (teaching and behavior of students), (3) experimental research as prospective research and (4) developmental research. The author related each of these approaches either to the constructivist or to the rationalist paradigm and argued for a methodological pluralism in research on mathematics education. In particular, he criticized the prevailing dominance of qualitative research in that field; especially since such qualitative studies often led to empirical results that were not empirically testable. However, Wellenreuther did not bring up the possibility to integrate qualitative and quantitative research methods to overcome such problems. Others mentioned this option only as a possible combination of methods in the context of the same research paradigm. Beck and Maier for example proposed a supplementary use of interviews in research on mathematics education in the following way:

> To get valid test results, many projects make the attempt to combine several methodological instruments. One method is supposed to complete or control the findings that were obtained with the help of the other method(s). However, no objection, as long as the various methods are based on a unified paradigm.[1] (Beck and Maier 1993, p. 174, translated by N. Buchholtz)

In the late 1990s the TIMSS study (Baumert et al. 2000a, b) had a great influence on the German debate about mathematics education. Thereby, the methodological discussion between proponents of quantitative approaches and followers of qualitative methods began to escalate. At that time, Kaiser maintained the possibilities and limitations of different research approaches in the field of empirical research, especially for international comparative studies (Kaiser 2000). In doing so, she identified problems and limitations of both quantitative and qualitative approaches: According to Kaiser, large-scale studies were criticized for the fact that (1) their results depend on the selection of the items used, (2) that curricular validity cannot be guaranteed and (3) that the applied tests claim to measure one single latent ability (i.e. mathematical literacy). This assumption of unidimensionality, however, could be merely a product of the research methodology applied. It is still a substantive question, whether certain heterogeneous mathematical skills can be subsumed under a single ability at all. On the other hand, qualitative studies were criticized since (1) they often do not comply adequately with the usual methodological standards of qualitative research and since (2) often data are not satisfactorily connected

[1] Beck and Maier distinguish slightly differently between the "normative" and the "interpretive paradigm" going back on Wilson (1970).

to theoretical assumptions. For this reason, Kaiser recommends the *integration* of the different methodological approaches:

> In the analysis of the capabilities of quantitative-statistical studies the interpretation of the data more than once referred to accompanying qualitative investigations. Without these investigations a substantial interpretation of the data would not have been possible. […] This makes clear that international comparative studies, which intend to produce more than just rankings and which intend to explain the background of their results require an integration of both types of methodological approaches. (Kaiser 2000, p. 188f, translated by N. Buchholtz)

Recent critiques of quantitative approaches in research on mathematics education were formulated by researchers who work in the field of qualitative research (Jahnke and Meyerhöfer 2006; Hopmann et al. 2007) shortly after the publication of results of the first PISA studies (Baumert et al. 2001) in the last decade. Putting aside a more general discomfort with a "culture industry" produced by the increased requirement for large-scale studies (Mayerhöfer 2006), substantial criticism is primarily aimed at technical and methodological mistakes of these studies. In particular, technical flaws in the formulation of the items used to describe the tasks the respondents had to perform are bemoaned (Hagemeister 1999; Bender 2004, 2005, 2006; Kießwetter 2002). Wuttke (2006) calls attention to the lack of transparency of the analytical methods, since the manuals describing the data analysis published by the PISA consortium are inadequately formulated and flawed in his opinion. Meyerhöfer (2004a, b) mentions the problem that systematic guessing may affect the results of quantitative analyses and criticizes the arbitrary nature of proficiency scaling. Another point of contention refers to questions of measurement. In particular, qualitatively oriented researchers criticize that the edumetric scaling pragmatism of quantitative approaches (see Baumert et al. 2000a, p. 67) leads to arbitrarily constructed instruments. Furthermore, critics complain about the lack of theoretical considerations about how to measure mathematical or didactical knowledge, so that only isolated aspects of the underlying constructs could be assessed. Drawing on theoretical considerations, Jablonka (2006) and Gellert (2006) maintain that studies like PISA are not able to measure mathematical skills in a valid manner, since the test construct "mathematical literacy" is insufficiently designed. Especially its connection to the didactical theory of the mathematician Hans Freudenthal (1983), which (according to the editors) represents the underlying theoretical approach (Deutsches PISA Konsortium 2000) is said to be deficient. According to critics, the lack of validity of the construct of mathematical competence which the studies intend to measure can be seen from the restrictions regarding test items. In order to provide a questionnaire which respondents are able to work on in a limited amount of time, the underlying theoretical constructs usually have to be transferred to sometimes rather simple tasks in the process of operationalization. As Bender (2006) states:

> The teaching and learning of mathematics and the terminology developed for this purpose by the didactics of mathematics are much more complex than comparable concepts in those empirical sciences for which statistical methods had been developed in the first place such as medicine, psychology, economics, etc. In these disciplines a statement such as

> "Intelligence is, what the intelligence test measures" may be considered as a good working basis. One may address PISA in a similar way, defining "mathematical competence as that what the PISA test measures" […], etc. One would then talk about "mathematical competence as defined by PISA," etc., and everyone would understand the meaning of such an expression which carries a remarkably reduced meaning compared to what ordinary mathematics educators see as "mathematical competence", […] or "mathematical literacy" […]. (Bender 2006, p. 235, translated by N. Buchholtz)

In response to such criticisms, researchers involved in large-scale studies have sometimes accused their critics of mere misunderstandings (Baumert et al. 2000c) but have also made attempts to newly interpret the measured psychological constructs and the potential of the used tasks (Neubrand 2004) as well as to explain its validity in relation to curricula (Baumert et al. 2000c). The discussion between adherents of different methodological approaches is still ongoing (Wuttke 2009; Jahnke 2009, 2010).

As far as the relation between qualitative and quantitative methods is concerned we thus find in the field of mathematics education the same dissatisfying situation as in other social science disciplines. Adherents of both camps in the debate denounce the respective other sides for their methodological shortcomings and thereby they often answer the other side's criticism in a tit-for-tat manner. Critical arguments are not utilized as a means to detect limitations and blind spots of the own approach and to improve and to develop one's own methods; the potential of a controversial debate is not harnessed. Especially the insight coming from the discussion about mixed methods that qualitative and quantitative methods have complementary strengths and weaknesses which could be balanced out in a design using both approaches is rarely referred to. At present most studies about mathematics education, especially in the area of education and professional competence of teachers, still utilize either exclusively qualitative or quantitative methods (for example Blömeke et al. 2010a, b; Eilerts 2009; Schwarz 2013). Only in recent years a few attempts have been made to combine the two different methodological approaches in order to achieve multi-perspectivity (e.g. Kuntze 2011; Klieme and Bos 2000; Schulz 2010; Kaiser and Buchholtz 2014). Schulz (2010), for example, uses a mixed methods research design in a study in Luxemburg to capture innovation processes in mathematics teaching and the implementation of competence orientation. In an evaluation study about the effects of innovative approaches in mathematics teacher education Kaiser and Buchholtz (2014) relate results of a quantitative longitudinal study to findings of a qualitative-based interview study. Nevertheless, the mixed methods methodology is particularly suitable and extremely fruitful when applied in research in mathematics education, because shortcomings, which also arise specifically in the area of mathematics educational research, can be overcome by combining qualitative and quantitative methods. Despite the discussion on the subject and theory of mathematics education we can identify the specific task of research in mathematics education according to Wittmann (1995) and Steinbring (1998) in an analytical research dimension next to a constructive research dimension, which aims to develop learning environments based on theoretical knowledge. The analytical dimension of research focuses on the study of social and mathematical processes of mediation and in relation to school practice on the observation and analysis of learning processes of students. The strengths of a mixed methods approach in research are particularly evident here: If, for example, a new learning environment is studied in school practice, there is usually little knowledge at hand about its effects at the beginning, and so it is very difficult to articulate good hypotheses for monitoring quantitative research before entering the empirical field. Exploratory qualitative studies in which the effects of learning environments are observed can therefore help to generate hypotheses that can be examined afterwards with the help of large quantitative studies. Furthermore, qualitative studies can be used to deepen the findings from quantitative monitoring studies because there are many theories in mathematics education that can be applied at the micro level. In the long term, qualitative studies may thus support the development of new learning environments that can also be empirically investigated using quantitative methods.

12.4 Basic Methodological Concepts of Method Integration

Discomfort with the problems of (qualitative and quantitative) "Mono-method-research" is widespread, not only in the field of mathematics education. The specific limitations and weaknesses of qualitative and quantitative approaches can often be identified easily. Researchers, however, who undertake assiduous efforts to compensate for these weaknesses by drawing on the strengths of the respective other tradition in a mixed methods study will often find that good advice about how qualitative and quantitative methods can be combined in research practice may be difficult to find. Mixed methods is a relatively new and quickly expanding field; researchers who work in it still struggle for a common terminology. Thus a newcomer drawing on the available collected readings or handbooks (e.g. Tashakkori and Teddlie 2010) may get confused from the variety of differing approaches, concepts and nomenclatures. Unfortunately, there is still a lack of consensus about the exact terminology and nomenclature of different "mixed methods," "multiple method" or "multimethod designs" which are used in research practice.

To gain a first overview over the field it is essential to be clear about whether one talks about a *combination of methods and techniques during data collection and analysis* or about the *integration of methodological approaches within one research design.*

12.4.1 Combination of Methods and Techniques During Data Collection and Analysis

Quantitative research usually means the statistical analysis of collected standardized data, for instance, with the help of questionnaires or by other highly structured techniques; in qualitative research non-standardized data are obtained through open interviews or by writing field notes which are analyzed with the help of non-numerical (interpretive, hermeneutic) methods. Since the early times of social research many attempts have been made to blend these different techniques of data collection and analysis: one may, for instance, analyze non-standardized data (e.g. newspaper articles) with statistical methods by counting words or occurrences of words etc. Or one may include open ended questions providing qualitative data into a standardized questionnaire. Such a mixture of qualitative and quantitative research techniques is not free from methodological risks, especially if one tries to analyze the same data with separate techniques. Different methods entail separate methodological standards and quality criteria—a good qualitative interview requires openness towards the (possibly idiosyncratic) perspectives and parlances of the actors in the field, whereas a main purpose of quantitative data collection is to obtain comparable and repeatable bits of information. Context-relatedness of data is a crucial issue for qualitative researchers and the idea to isolate words from their context to count them, for instance, may sound odd in the framework of an interpretive approach. Consequently, blended techniques of that kind frequently tend to become either methodological oxymora or lead to the development of distinct research methodologies with own quality criteria and methodological standards like, for instance, Quantitative Content Analysis (Krippendorff 2004).

12.4.2 Integration of Methodological Approaches Within one Research Design

For that reason the term "mixed methods" is mainly used to denote empirical studies which comprise different small sub-studies in order to answer specific questions which can be combined in order to answer the project's general research question. Normally in each of the sub-studies one type of data is analyzed with one (qualitative or quantitative) method. Qualitative and quantitative data are then collected and analyzed separately and results from that are related to each other, although the collection and analysis of the different data sets can have an impact on each other: researchers, for instance, may draw a small qualitative sample from a huge representative quantitative sample. Depending on the research purpose there are various possibilities to combine quantitative and qualitative sub-studies to a mixed methods design. Until now a variety of proposals have been made to classify types of combinations and designs (for an overview see Nastasi et al. 2010, p. 311 ff.); for a broad overview about the topic area it is helpful to draw on the straightforward notation system proposed by Morse (2003) which refers to the type of methods employed in the sub-studies ("qual" and "quan"), to the timing (whether the sub-studies are conducted sequentially, depicted by an arrow "→", or whether they take place at the same time, marked by a "+") and to the possible dominance of one approach (which is signified by the use of big letters). "Qual → QUAN" thus denotes a design, whereby a small qualitative study is followed by a great quantitative study (the latter more important for the research purpose). "QUAN + QUAL" would be used to describe a design comprising a qualitative and a quantitative sub-study carried out simultaneously, whereby both studies have comparable importance for the research question. We describe such a design with the help of an empirical example in Sect. 12.6.

A competent construction and use of such designs is dependent on the ability to draw adequate inferences from the results of the respective sub-studies and to integrate these inferences into a common framework. Thus the integration of research results forms a crucial part of method integration. To understand the problems connected with this task it is helpful to draw on the methodological concept of "triangulation." This term was initially borrowed from the field of trigonometry for use in quantitative psychological research. Later on it was used to denote the combination of different kinds of research methods and has become a popular term also in the methodological debate surrounding mixed methodology (cf. Denzin 1978; Fielding and Fielding 1986; Lamnek 1995; Flick 1991, 1992, 1998; Erzberger and Kelle 2003; Kelle and Erzberger 2004).

In its methodological debate in the social sciences the term triangulation has acquired two different meanings – both of them remote from its original trigonometrical understanding: triangulation as a mutual validation of research results and triangulation as an integration of complementary perspectives on the subject under investigation in order to achieve a more complete image of the research domain (see also Erzberger and Kelle 2003).

In the 1950s the term was used for the first time to describe a research strategy that employs different measurement operations or empirical results to answer a certain research question. In the context of a theory of psychological testing, Campbell and Fiske (1959) proposed to supplement or to further test empirical results by the use of different instruments. According to these authors, "multitrait-multimethod matrices" should be constructed using correlation coefficients between scores obtained through different tests. These matrices should then serve as a means to determine the degree of convergence as an indicator for the validity of research results: "Validation is typically convergent, a confirmation by independent measurement procedures" (Campbell and Fiske 1959, p. 81). In their book about unobtrusive measures Webb and his colleagues picked up Campbell's and Fiske's idea and transferred it to a broader methodological framework (cf. Webb et al. 1966) arguing that the collection of data from different sources and their analysis using different strategies would improve the validity of results: "Ideally, we should like to converge data from several different data classes, as well as converge with multiple variants from within a single class" (Webb et al. 1966, p. 35). This idea was picked up by a dedicated advocate of qualitative methods in social research: Norman Denzin used the argument of Webb and colleagues that a hypothesis which had survived a series of tests with different methods could be regarded as more valid than a hypothesis tested only with the help of a single method. Since different methods entail different weaknesses and strengths, Denzin opted for "*methodological triangulation*" which consists of a "complex process of playing each method off against the other so as to maximize the validity of field efforts" (Denzin 1978, p. 304) leading to a reduction of "threats to internal and external validity" (p. 308). The idea that the convergence of results of different measurement operations would enhance the validity of research findings led to the adoption of the term triangulation in the methodological debate in the social sciences. But the concept of triangulation as a means of mutual validation has also been criticized as being inadequate by many authors (see for instance Fielding and Fielding 1986; Flick 1991, 1992, 1998; Rossman and Wilson 1985, 1994; Lamnek 1995). Since in the 1980s qualitative methods became more accepted in the social sciences in general (however, as has been discussed above, not in the field of mathematics education) the idea gained acceptance that different methods of social research reflect different and even diverging epistemological standpoints and can relate to different empirical phenomena and that it thus may be difficult to simply compare research results acquired by means of different methods in order to check their validity.

This view stimulated an alternative understanding of triangulation: the use of different methods to investigate a certain domain of reality can be compared with the examination of a physical object from two different viewpoints or angles. Both viewpoints provide different pictures of this object which may not be useful to validate each other, but which may yield a fuller and more complete picture of the phenomenon concerned if brought together. To use another metaphor: empirical research results obtained with different methods are like the pieces of a jigsaw puzzle which provide a full image of a certain object if put together in the correct way.

These differing understandings of "triangulation" show the limitations of this notion as well as its systematic ambiguities. In navigation and land surveying it denotes a clearly defined technique of determining the spatial position of a point C by measuring angles to it from two points A and B with a known distance [AB] between them. The point C can be fixed as the third point of a triangle with the known side [AB] and two known angles α and β (Fig. 12.1).

In social and behavioural research, however, terms like "spatial position of a point" or "angle" are not defined and represent metaphors at best: the "determination of a point by measuring angles from different points" can be understood, for instance, in a way that

1. The s*ame phenomenon* is investigated with the help of different methods,
2. *Different aspects of the investigated phenomenon* or even *different phenomena* are examined with the help of different methods.

This differentiation is not a mere play of words: only methods which refer to the same phenomena can yield results which may be used for mutual validation

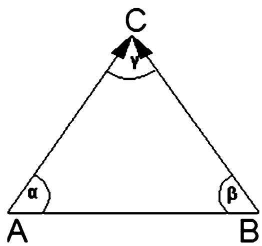


Fig. 12.1 Triangulation in trigonometry

of methods. Different results would indicate validity problems here; but if separate aspects of the investigated phenomenon or even separate phenomena were examined with different methods we would expect different (but certainly not contradictory) results.

To make sense of abstract methodological considerations and concepts we have to relate them to research practice – if we look at concrete research projects applying mixed methods designs we will find that each of the following four outcomes can arise (cf. Erzberger and Prein 1997; Erzberger 1998; Kelle 2001; Kelle and Erzberger 2004):

1. qualitative and quantitative results *converge*,
2. qualitative and quantitative results relate to different objects or phenomena, but are *complementary* to each other and thus can be used to *supplement* each other,
3. qualitative and quantitative results are *divergent* or *contradictory*,
4. qualitative and quantitative results refer to unrelated phenomena.

This makes clear that both types of triangulation are applicable and can make sense within a mixed methods design: *triangulation as validation* may lead to convergent qualitative and quantitative findings or it may result in divergent findings which point to validity problems; *triangulation as investigating different aspects of the research subject* may yield complementary results (if applied successful) or it may render unrelated results (if this triangulation strategy fails).

12.5 Capabilities and Functions of Mixed Methods Designs

Whether method integration is used for mutual validation of methods and results or whether it aims at the examination of different aspects of the research subject, the main rationale behind it is always the search for a compensation for the limitations of qualitative or quantitative methods by drawing on the strengths of the respective other tradition. But which specific strengths and weaknesses of qualitative and quantitative research can be balanced in a mixed methods design?

12.5.1 Strengths and Challenges of Quantitative Methods

The standardization inherent in quantitative methods aims at data which fulfill the classical requirements of reliability (replicability) and objectivity (observer-neutrality). Standardization thus is a necessary prerequisite for quantification: by ensuring that objects are counted which are equal with regard to certain attributes the risk of "comparing apples with pears" can be minimized. Furthermore, this allows for the examination of large numbers of cases in a straightforward manner. If the sample is drawn according to specific rules, error margins of statistical results can be determined by using well known concepts and formulas from sampling theory providing a theoretically grounded basis for generalizations. For many elements of quantitative research explicit rules can be formulated, and it is in principle possible to structure the research process according to an overarching schedule, starting with the definition of the problem and the search for explaining theories that may account for it (although by doing research in the real world one should always be prepared to experience nasty surprises when following a strict research agenda). Quantitative research is ideally suited for a theory driven, "hypothetico-deductive (HD-)" approach, whereby precise hypotheses are formulated at the onset, then operational definitions are formulated for the concepts these hypotheses comprise, measurement instruments are constructed and data are collected and analyzed subsequently. Quantitative methods can be appropriate tools to examine clear cut causal statements like "Does the teaching method X improve student's achievement, measured by their final grades of the course or with the help of a specific measurement instrument?"

But these obvious strengths of quantitative research can easily turn into drawbacks, if theories applicable to the field and clear cut hypotheses derived from them are not available. This situation does not only occur if the researchers' theoretical knowledge is too limited or if the corresponding discipline has not yet developed a satisfactory body of theories. Achieving certain goals in education requires social action and interaction, and the understanding of such social phenomena always requires knowledge about context-bound patterns, structures and rules which form an integral part of culturally specific stocks of everyday knowledge in particular life worlds. Such "local" knowledge can relate to a certain culture, to a specific organization and even to a very small group. In developing theories, in formulating hypotheses and in constructing research instruments like standardized questionnaires social and educational researchers employing an HD-approach often utilize their personal common sense knowledge (cf. Kelle and Lüdemann 1998). In many cases, these *heuristics of common sense knowledge* causes no harm, especially if research takes place within the researcher's own culture. But, since a great deal of common sense knowledge is self-evident or implicit, the application of this heuristic is usually not discussed explicitly– instead it serves as a "shadow methodology".

The use of this shadow methodology may become hazardous, if the socio-cultural background of researchers and respondents differs. If respondents belong to another social class, gender or ethnic group, researchers often have no access to culture-specific stocks of knowledge to formulate hypotheses, to define variables and to construct research instruments.[2]

[2] It should be clear from the preceding discussion that this is not so much a problem of quantitative research *per se*—it may occur if one strictly follows a hypothetico-deductive approach (which is for many reasons advisable if quantitative methods are applied) *and* if researchers lack empirically contentful hypotheses, workable theories and/or specific knowledge about the domain under study. The latter is often not so much the fault of uninformed researchers but a consequence of the fact that social action is often structured by culture-bound rules and "local knowledge".

Certain methodological problems of quantitative research may arise from that. First of all, such insufficient knowledge may result in *problems of theory building* and hypothesis construction, leading to *misspecification* of statistical models. That means that

1. important explaining variables are omitted with low levels of explained variance as a consequence,
2. intervening variables are neglected or functional relations between certain variables are not correctly specified so that causal processes underlying the investigated phenomena are not adequately understood.

Furthermore, limited socio-cultural and local knowledge also leads to another problem one often faces in quantitative surveys. There one is often confronted with statistical distributions and correlations which are difficult to understand and interpret on the basis of known theories (often in the social and educational sciences theories at hand are far too general to account for certain interesting and unexpected findings). In a study about the effects of a certain teaching method one may find, for instance, that the method has, contrary to expectations, only very limited or temporary effects. In classical HD-design data that may explain this unexpected finding are only available, if researchers were able to foresee such a possibility as a hypothesis before data collection took place.

Insufficient knowledge about investigated life worlds, organizations or groups may also lead to *problems of operationalization* and *measurement*: a carefully designed questionnaire may yield invalid data, if respondents do not understand questions in the same way as the researchers, or if the mentioned issues are not relevant for them. Problems of that kind do not only refer to the meaning of words or phrases or to the relevance of themes of a questionnaire, but to the whole process of data collection, which is not a mere process of information exchange, but a complex social interaction, which can be defined and framed in highly different ways by the persons involved. Mutual understanding in a research interview or the answering of questions in self-administered questionnaires depends (as any other form of communication does) on the ability of the participants to interpret the motives of their counterparts and to identify and understand their expectations. Respondents may have a variety of motives beside the purpose of merely helping researchers with information, they may hide their own motives, conceal facts, hold back information etc. Interviewing can be a difficult social interaction with a high risk for misunderstanding and conflict. Unintended misunderstandings, accidental mistakes, willful omissions or deceits or the consequences of social desirability will lead to invalid and biased data, which are sometimes impossible to detect. In some cases high item nonresponse rates or suspicious distributions of certain variables give clues to such processes which usually remain hidden in standardized data collection and can only be analyzed and described in detail if qualitative data material (verbatim transcripts of standardized interviews, cognitive interviews about the process of answering a questionnaire etc.) is available.

12.5.2 Strengths and Challenges of Qualitative Methods

Qualitative research allows for the exploration of previously unknown social life worlds. Thereby researchers can get access to knowledge from the investigated field and may discover social rules and structures about which they had no idea before (and about which they were not able to formulate hypotheses before entering their empirical field). Qualitative methods are powerful tools for exploration, detection and discovery which help to construct new theoretical concepts, categories and sometimes even whole theories about the domain under study. This strength has a price as well: if researchers aim at the discovery of issues they do not even have a dream of in the onset of a project, data collection becomes tedious and its success incalculable. Qualitative research relies on unstandardized data which have to be interpreted, paraphrased or categorized in a sometimes painstaking and always time consuming process. Contrary to quantitative surveys researchers cannot conduct hundreds or thousands of interviews, but have to restrict themselves to a carefully selected choice of cases. This immediately raises questions about the generalizability of findings and about the intersubjectivity of interpretations. The analysis of unstandardized data is a process highly dependent on the individual researcher and it is often doubtful whether someone else would draw the same conclusions from a certain body of unstandardized data. For many social and educational scientists this is a source of constant and serious trouble, which was already brought to a point by the sociologist Lundberg in a review of famous qualitative studies of the 1920s:

> The scientific value of all these (studies) depends, of course, upon the validity of the subjective interpretations of the authors as well as the extent to which the cases selected are typical. Neither the validity of the sample nor of the interpretations are objectively demonstrable on account of the informality of the method. (Lundberg, 1929/1942, p. 169)

First attempts of qualitative researchers to counter such a criticism in the 1930s and 40s were deeply influenced by sociological structuralism (nowadays, arguments of this kind are still put forward by adherents of structuralist qualitative approaches like conversation analysis or German "objective hermeneutics"). Florian Znaniecki, for instance, argued in 1934 that qualitative analysis rests on a kind of generalization superior to statistical inference which does not depend on the mere number of cases, "*but on the strength of the theoretical reasoning*" (Seale 1999, p. 109). The underlying idea is that some general social process or social structure is at work in every single case which can be revealed by a deep, penetrating and elaborated analysis which helps to identify essential structural characteristics and to exclude accidental aspects even if idiosyncratic cases are investigated. This idea, proposed by Znaniecki to justify small N-studies, can be even traced back to Emile Durkheim who claimed that all social facts and entities are brought about by evolutionary processes, such that simple forms of social life always contain essential structures shared also by more complex forms (cf. Seale 1999, p.112).

Znaniecki had proposed a strategy of theory building and theory testing whereby hypotheses are formulated on the basis of single cases and then tested by searching for "contradictory instances" (Znaniecki 1934, p. 279ff.). Although this is an attempt to employ a falsificationist approach in line with the HD model to qualitative research (see also Hammersley 1990, p. 604) this concept has provoked serious criticism, since it "presupposes that there are scientific laws of human social life (and deterministic rather than probabilistic ones at that)" (ibid.) This idea is not very common in the social sciences nowadays, especially not among qualitative oriented researchers. According to Znaniecki, extremely small numbers of cases are fully sufficient to construct and test hypotheses since it is the task of social research to identify the universal "static" laws (Znaniecki 1934, p. 279) which govern each single case. This method, however, must fail if one takes the (more realistic) stance that regularities in the social sciences are probabilistic in nature. Furthermore, micro-sociological approaches like symbolic interactionism which formulated basic theoretical tenets of qualitative research are hardly compatible with the idea of a stable social order reigned by ahistoric universal social laws—this tradition rather emphasizes the role of interpretation in social interaction and the resulting complexity, variability and uniqueness of social phenomena and the context-boundedness and flexibility of social rules. Social processes as well as human history as a whole are considered as contingent und unpredictable: "uncertainty, contingency, and transformation are part and parcel of the process of joint action." (Blumer 1928, p. 72).

In the 1980s qualitative researchers who follow post-structuralist or post-modern approaches like Norman Denzin, Egon Guba or Yvonna Lincoln have radicalized this conception by claiming that the principle of context-boundedness of social phenomena must lead to a general renouncement of all attempts of generalization in social research (Lincoln and Guba 1985). However, taking this claim serious would mean to exclude a great variety of phenomena from qualitative inquiry. Typical macro-social phenomena (cultural norms, for instance) can hardly be investigated without any reference to the idea of (at least limited and context-related) generalization. This is even more the case if one adheres to the interpretivist postulate that social order is highly flexible and evolves through processes of interpretation in micro-social contexts. The resulting pluralization and heterogeneity of social structures and patterns of action poses serious challenges for any methodological approach which relies on the investigation of a small number of cases.

Of course, the investigation of a single person, group and organization can be a research goal justified in itself, but at the same time poses the danger of focusing on marginal cases. A crucial question here would be: to which kinds of people or organizations does the knowledge derived from a certain study also apply? To address that question, the term "transferability" was proposed as an alternative to "generalizability" in the context of qualitative research. However, transferability and generalizability are notions closely related to each other, especially since the latter term does not necessarily imply generalization towards *universal laws* valid in all places at all times. Any research project aiming at the investigation of cultures, societies, organizations etc. as limited wholes situated in concrete spatiotemporal contexts has to address questions of limited generalization (or "transferability" if one prefers that term), for instance: *Do certain problems experienced by teachers at a particular school reflect deeper lying problems of the whole organization or of schools in a* *certain state or country or are these problems only an expression of the situation in that specific school*? *Are certain rules of behavior followed by members of a small gang of youths generally accepted habits in a certain youth culture*? One need not cling to the idea of universal social laws to largely benefit from quantitative methods in an inquiry of social life worlds, organizations or groups where heterogeneous norms and patterns of action play an important role.

This point was already made by one of the founders of symbolic interactionism, Herbert Blumer, who countered the critique of sociological structuralists that statistical methods are unable to disclose universal laws with the argument that statistical methods take into account the *"complexity, variability or uniqueness"* of social phenomena (Blumer 1928, p. 47ff.). The interpretive tradition in the social sciences can thus provide arguments in favor of *"the importance of statistical analysis"* (Hammersley 1989, p. 219): quantitative research can capture social heterogeneity by providing information about great numbers of persons or situations. Mixed methods designs offer different possibilities to deal with the sometimes limited transferability of qualitative findings, for instance by using large scale quantitative surveys to further examine qualitative findings based on small numbers of observations and interviews (cf. Barton and Lazarsfeld 1969). This strategy, already proposed by Paul Lazarsfeld and Allan Barton decades ago, is often wrongly accused of restricting qualitative research to unsystematic pilot studies. But qualitative sub-studies will only yield useful results if they are carried out in a structured and methodical way so that empirical descriptions and theoretical hypotheses can be really grounded in the (possibly small number of) investigated cases. This can hardly be accomplished within a casual pilot study but requires a considerable amount of resources in terms of time and person-power whereby the selection of cases is of utmost importance.

The qualitative tradition nowadays offers different forms of a methodologically controlled purposive selection of cases like "theoretical sampling" (Glaser and Strauss 1967). The idea underlying theoretical sampling, the maximization and minimization of differences, is also used with other forms of qualitative sampling, for instance in the search for extreme, deviant or typical cases (Patton 2002; cf. also Silverman 2000, p. 102ff. or Gobo 2004). Quantitative research may inform all types of purposeful qualitative sampling. A quantitative sub-study in a mixed methods design may, for instance, give an overview about the distribution of certain problems, structures or patterns of action relevant for the overall research question. Or it can provide a sampling frame which allows for the comfortable selection of typical, deviant or extreme cases.

12.5.3 Types of Mixed Methods Designs and Their Function in the Research Process

In the contemporary debate about mixed methods or multimethodology it is often maintained that mixed methods is "a strategy for overcoming each method's weaknesses and limitations by deliberately combining different types of methods"

(Brewer and Hunter 1989, p. 11; see also Hunter and Brewer 2003). However, this debate still provides only sparse information about which mixed methods design could overcome which weaknesses of mono-method research. A complete overview about all problems that may occur in mono-method studies and their possible solutions within a mixed methods design would clearly go beyond the scope of this chapter—but the following remarks may show how the most basic of the problems mentioned above can be addressed within a simultaneous or sequential mixed methods design.

In a *sequential qualitative-quantitative design* (QUAL → QUAN) a qualitative study helps to identify core issues and to develop theoretical concepts and hypotheses, which can be further examined in a subsequent quantitative study. This design helps to overcome two shortcomings of qualitative or quantitative mono-method research: in regard to qualitative research the limited transferability of findings from small N studies, and in respect to quantitative research the mentioned problems of the heuristics of common sense knowledge (a lack of context-related local or culture-specific knowledge). By starting the research process with a qualitative sub-study, researchers may get access to local knowledge of the field which helps to develop theoretical concepts and hypotheses suited for the domain and to construct standardized research instruments later on which grasp relevant phenomena by meaningful and relevant items.

It can also be sensible to reverse the order of qualitative and quantitative methods in a sequential design. In a *sequential qualitative-quantitative design* (QUAN → QUAL) a quantitative study is performed to identify problem areas and research questions which can be further investigated with the help of qualitative data and methods. Typical problems of quantitative research which can be treated in this way are the incomprehensibility of statistical findings (which often can be only adequately understood if additional context-related knowledge from a qualitative study is available). Furthermore, the quantitative sub-study in such a design can guide systematic case comparison in the subsequent qualitative study by helping to identify criteria for the selection of cases and by providing a "qualitative sampling frame." In this way an important threat for validity of qualitative research can be addressed—a focus on remote and marginal cases. And a further problem often experienced in qualitative research can be treated within this design: since a large scale quantitative study can capture heterogeneity in the field by describing the distribution of predefined phenomena, such quantitative data may help to avoid a qualitative study with an "oversized scope," that means a study with a research domain too heterogeneous to be captured by a small qualitative sample. To take an easy example: a qualitative study about the influence of family forms on the academic achievement of students nowadays must take into account more different forms of families than a similar study in a traditional rural community in the beginning of the twentieth century. By drawing on statistical information about the distribution of different family forms researchers can learn about minimal requirements for qualitative sampling and may downsize the research question and research domain (to a limited number of family forms with a certain social background) such that it can be covered by the planned investigation.

A *parallel qualitative-quantitative design* (QUAL + QUAN) can fulfill similar functions than a sequential design yet in a sometimes restricted manner: the qualitative sub-study can provide information which supports the interpretation of statistical findings, the development of explanations and the identification of additional variables which help to explain variance in the quantitative data. However, a disadvantage of a parallel design is that qualitative sampling and data collection as well as the construction of the standardized research instrument in the quantitative sub-study cannot be supported by information coming from the respective other sub-study—data from the qualitative sub-study may thus often provide only limited answers to questions coming from the quantitative study since they were not collected for that purpose. A great benefit of a parallel qualitative-quantitative design, however, is that it can help to identify measurement problems and methodological artifacts of both qualitative and quantitative data, since data from the same persons can be obtained with the help of different (qualitative and quantitative) techniques.

12.6 An Example of a Mixed Methods Research Design in Mathematics Education

To illustrate possibilities for the integration of qualitative and quantitative methods in research on mathematics education we now present an example from a mixed methods study in this field. In the following we first describe the research design and the methodological approaches applied in the "Teacher Education and Development Study TEDS-Telekom" (Buchholtz and Blömeke 2012; Kaiser and Buchholtz 2014; Buchholtz and Kaiser 2013), an evaluation study subsidized by the German Telekom Foundation. Thereafter we present some selected results from the study to demonstrate how qualitative and quantitative findings can be related to each other and thus help to overcome the limits of the respective other method.

12.6.1 Research Purpose and Mixed Methods Design of the TEDS-Telekom Study

Two teams of researchers at the universities of Giessen and Siegen developed a research and development program called "*Mathematik Neu Denken*" ("*Thinking mathematics in a new way*") to restructure the teacher training program in mathematics (Beutelspacher et al. 2011). The project aimed at a long-term quality improvement of the training of future mathematics teachers for the higher track schools ("Gymnasium"). Courses normally attended both by teacher students and by students aiming for a general degree in mathematics were split into teacher students' courses on the one hand and courses for other students on the other hand. This project aimed at the integration of university mathematics and "elementary mathematics from an advanced standpoint" (Klein 1932), the latter seeking for a comprehension-oriented teaching of basic mathematical concepts in academic courses in order to enable student teachers to grasp elementary mathematical ideas for teaching (see also Kirsch 1977). A further important goal of the project was the early integration of mathematics didactics into teacher education. In particular the introductory courses of analysis and linear algebra were restructured by enriching their curricula with content about teaching and elementary mathematics from an advanced standpoint. Each university thereby implemented the program in a slightly different way (Siegen focused on analysis, Giessen on linear algebra/ analytic geometry).

The main purpose of the TEDS-Telekom study is to evaluate this innovative approach from an external point of view. By now one can find only very few reliable empirical research results upon the effects of university teaching programs in general. However, when analyzing complex learning environments like university teacher training programs under the perspective of an evaluation, issues on the macro as well as on the micro-level are of particular interest. Thus, an evaluation aims on the macro-level initially on a standardized comparison between universities with and without a specific treatment. The research focuses in the first place on the question of how the student teachers' professional competencies develop in a longitudinal observation, i.e. the impact of the intervention regarding the development of mathematical, didactical and pedagogical knowledge of the prospective teachers together with the development of the teacher students' corresponding beliefs. To answer this research question in TEDS-Telekom a standardized questionnaire was developed in a quantitative sub-study to measure the prospective teachers' professional knowledge in their first, second and fourth semester. However, since it is not certain whether with a comparison of the development of knowledge the impact of the intervention is sufficiently analyzed in detail, the study also focuses on the micro-level by attempting to identify the concrete elements of the program which actually influence the individual development of students' competencies. Yet, there exist no sufficiently reliable instruments to also carry out standardized surveys in this area. On the contrary: Since the quantitative approach of the study is restricted to a mere description of individual competence development and cannot provide results about the impact of specific aspects of the teacher training program on the individual acquisition of competence, the methodological approach of the study had to be enriched.

In a qualitative sub-study of TEDS-Telekom, problem-centered interviews (Witzel 2000) with prospective teachers of the universities involved were carried out to gain an in-depth understanding of the effects of didactical efforts at the different universities taking part in the study. In addition the interviews should help to investigate the influence the separate teaching and learning conditions had on the individual internal perception of the prospective teachers and on their acceptance of particular components of teacher education. These interviews were conducted shortly after the last survey with voluntary student teachers who participated in all three surveys of the questionnaire. The aim was to be able to relate results of both the quantitative and qualitative sub-studies to each other at a later stage of the research process (for example by assigning the results of students who have participated in both sub-studies by an individually generated personal code in the two sub-studies). This mixed methods design can be described in terms of Morse (2003) with (QUAN + QUAL) since quantitative and qualitative sub-studies were carried out independently and simultaneously. Through this “blind spots” of (qualitative and quantitative) mono-methods should be detected and a broader range of effects of the reconstructed teacher training program should be captured (“triangulation as the investigation of different aspects of the research subject”).

12.6.2 The Quantitative Sub-Study

The quantitative sub-study of the TEDS-Telekom evaluation study is theoretically based on the conceptualization of professional competence of prospective mathematics teachers as a multi-dimensional construct, as it has been developed in general by Weinert (1999) and by Bromme for teaching (1992, 1997). Thereby we drew on approaches from the international comparative study “Teacher Education and Development Study – Learning to Teach Mathematics” (TEDS-M; Blömeke et al. 2010a, b) as an external reference framework. According to this approach, professional competence includes subject-related and interdisciplinary cognitive dispositions of performance, as well as affective-motivational beliefs as part of a teacher’s personality. Due to the feasibility of the study, TEDS-Telekom had to be restricted to the analysis of cognitive components of professional competence (professional knowledge of teachers as outlined by Shulman (1986) and Bromme (1992, 1997)). In the realm of personality features the study focuses on beliefs concerning the subject and its teaching and learning.

The evaluation study concentrated on the following dimensions of professional competence (see Fig. 12.2):

- academic mathematical content knowledge (MCK) in the area analysis and linear algebra/analytic geometry;
- mathematical content knowledge in the domain of elementary mathematics from an advanced standpoint (cf. Klein 1932 and Kirsch 1977); pedagogical content knowledge in mathematics or didactics of mathematics referring to upper secondary level (MPCK);
- general pedagogical knowledge focusing on action-related aspects, such as the structuring of teaching, motivation, classroom management, assessment and dealing with heterogeneity;
- beliefs about mathematics as a science and about learning and teaching of mathematics according to Grigutsch et al. (1998);

In order to capture the achievements of first-year student cohorts at the respective universities as well as the development of these achievements, the quantitative sub-study of the TEDS-Telekom study was designed as a real longitudinal study. The assessment of students by means of a 90-min paper-and-pencil test took place at the

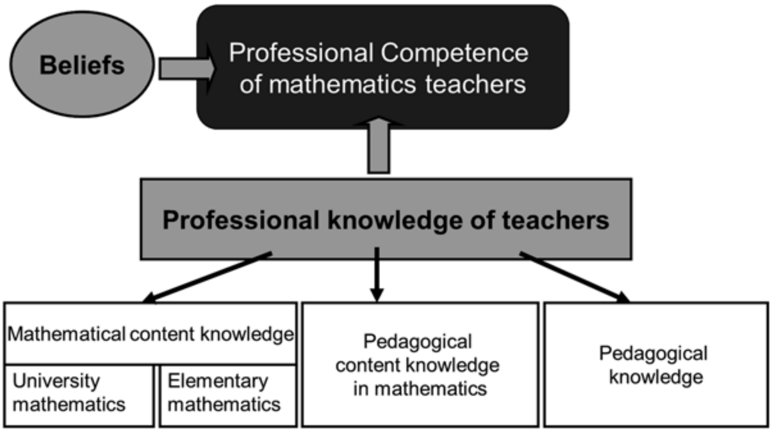


Fig. 12.2 Model of professional competence in the "TEDS-Telekom-study"

beginning and the end of the first year (December 2008 and July 2009) and at the end of the second year (July 2010). In the quantitative sub-study the main hypotheses regarding the evaluation of test results predicted a reasonable improvement of the achievement from the beginning of the first half term (semester) until the end of the fourth semester. In addition, it was assumed that the degree of achievement increase would be different depending on the level of achievement at the beginning, on the students' learning preconditions and on the learning opportunities provided by the universities (which represented the innovative potential of the study programs). Comparative reference groups at other universities which also agreed to evaluate their teacher training program set an external benchmark. All in all, first-year cohorts at five universities (Giessen and Siegen as the innovative programs, Bielefeld, Essen and Paderborn as external reference universities without a specific treatment) were analyzed, including also students aiming at a general degree in mathematics. In total more than 400 students participated in the study. Since the implementation of the TEDS Telekom study was depending on situational conditions such as size of classes and access to the prospective teachers, the study has an unbalanced, non-representative sample. Some students could not be reached again and participated only in one or two measurements, other students participated at a later measurement point. To account for the panel attrition occurring in the longitudinal design the first statistical analyses were restricted to those 115 first-year students who did participate in all three measurements. However, it is planned to extend these analyses to those participants who did not participate in each wave. Table 12.1 gives an overview about the sub-samples.

As an indicator of the school-related precognition, data about the kind of mathematics courses attended during the last 2 years of high school were collected (cf. Briedis et al. 2008). The options "Basic course" (courses on a basic high school mathematical level), "Advanced course" (courses on an advanced high school mathematical level that exceeded the basic class) and "neither basic nor advanced course"

Table 12.1 Comparison of sub-samples considering gender, age and *Abitur*-grade

Reference group	N	Female students (%)	Average age (SD)	Average *Abitur* grade[a] (SD)
Gießen	32	59.4	20.4 (1.78)	2.16 (0.54)
Siegen	14	64.3	19.9 (2.20)	2.19 (0.50)
Teaching	39	48.7	20.4 (1.68)	2.23 (0.54)
Non-teaching	30	13.3	20.2 (1.76)	2.01 (0.49)

[a]The grades can differ from 1.0 (best grade) to 4.0 (worst grade)

Table 12.2 Comparison of sub-samples considering school-related precognition

Reference group	N	Advanced course (%)	Basic course (%)	Neither Basic nor Advanced (%)
Gießen	32	71.9	28.1	0.0
Siegen	14	57.1	35.7	7.1
Teaching	39	76.9	20.5	2.6
Non-teaching	30	96.7	0.0	3.3

(in some federal states of Germany the mathematics courses are named differently and thus cannot be distinguished) were given. Basic and Advanced courses differ in the depth of the content and also in the amount of attended lessons per week (e.g. the Advanced course treats the quotient rule of derivation while the Basic course only treats product- and chain rule). Table 12.2 shows the proportions of students having attended the specific courses in high-school.

The comparison shows that the group of non-teaching students is better prepared for mathematics. In the teaching groups there are substantial proportions of students who have only attended a Basic course in mathematics at school and therefore have less good preconditions (see also Köller et al. 1999), especially in Siegen. This percentage is even larger in the Gießen and Siegen groups than in the reference group.[3]

The items for the measurement of mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK) were developed by the research team at the University of Hamburg (see also Buchholtz et al. 2012). All items were revised in workshops by experts in mathematics didactics from universities participating in the study. The items relating to pedagogy had been developed by the working group at Humboldt University of Berlin (see also König and Blömeke 2010). The test also contained items from the TEDS-M study so that results of the evaluation study could be evaluated and interpreted with reference to an external

[3]A methodological adjustment of the treatment groups by measures of treatment evaluation (e.g. propensity score matching) has been omitted so far as the use of elaborate statistical methods to determine treatment effects appeared disproportionate due to the small group sizes. Furthermore, the group differences in *Abitur* grades are not significant and the relationship of school-related pre-cognitions considering the attendance at Advanced or Basic course merely reflects the pre-cognitions of local convenience samples.

US25)	We know that there is only one point on the number line that satisfies the equation 3x = 6, namely x = 2.
Let us now transfer the equation to a plane with coordinates x and y, and then to space, with coordinates x, y and z. What is the set of points that satisfy the equation there?	
Tick one box per row.	

		A point	A straight line	A plane	Else
A)	The solution of 3x = 6 in the plane	☐	☐	☐	☐
B)	The solution of 3x = 6 in space	☐	☐	☐	☐

Fig. 12.3 TEDS-M 2008-item

standard later on. To give an example one of the TEDS-M 2008 items used in the TEDS-Telekom study is presented in Fig. 12.3 together with the respective solution frequencies[4]: the task US25 refers to the content area of academic mathematical knowledge about linear algebra and analytic geometry and requires basic knowledge about the geometry of the plane and the space. The number of points that satisfy the equation $3x=6$ in the plane is a straight line; in space it would be a plane.

In TEDS-M 2008 72 % of the German prospective teachers were able to solve item A correctly and item B was correctly answered by 68 %. The student teachers of the University of Giessen solved both items with 75 % at approximately the same height, as well as the comparative teacher training group (71.4 % for item A and 61.9 % for item B). From the student teachers of the University of Siegen only 50 % could provide the correct answer, which may indicate the different focus of the implementation of the teacher training program at both universities.

To evaluate the results of the tests in comparison with large-scale studies like PISA or TEDS-M 2008, models of multidimensional-IRT (Rost 2004; Hartig and Kühnbach 2006) were used. At present, longitudinal data from each survey are available. For first results of the three evaluations and for curricular validity of the instrument see Kaiser and Buchholtz (2014) or Buchholtz and Kaiser (2013).

12.6.3 The Qualitative Sub-Study

In order to investigate the influence of institutional conditions and of various aspects of didactics of higher education on the individual acquisition of competence from a different, more qualitatively-oriented point of view, problem-centered guided

[4] It needs to be noted that performance on the level of individual items can vary due to chance and thus should not be over-interpreted.

interviews (Witzel 2000) were carried out with 19 prospective teachers from all participating universities simultaneously with the third wave of the quantitative survey. At each university, they were chosen randomly from the group of prospective teachers that indicated their willingness to participate in the interviews voluntarily. These interviews were conducted by using a guideline containing several aspects of perception and estimation of university teaching experienced by the students during the introductory stage of their training. The different aspects covered in the interviews related to the different sub-dimensions of professional competence of mathematics teachers that were also used in the quantitative sub-study:

- Integration of visualization, examples and example-bound argumentations and real-world applications in mathematical lectures in the area of MCK;
- integration of elementary mathematics from an advanced standpoint in mathematical lectures;
- interweaving of mathematical and mathematics didactical content in university courses in the area of MPCK;
- beliefs about teaching and learning of mathematics and their influence on the acquisition of competence.

The interviews are analyzed by means of “qualitative content analysis” (Mayring 2008; for first results from this analyses see Buchholtz and Blömeke 2012).

The focus of the qualitative content analysis lies on the empirical, methodologically controlled analysis of texts within their context of communication, following content analytical rules and step by step models in a process of interpretation (Mayring 2000). The method comprises a clear, theory-guided procedure that allows for a systematic empirical examination of emerging categories and hypotheses. Figure 12.4 shows a model for qualitative content analysis according to Mayring (2008) modified for the purpose of our study.

The process starts with the research question—in our case, the research question referred to the individual acquisition of professional competence in the restructured teacher training program. The objects of the analysis were problem-centered interviews with students from the participating universities. Based on the research question, structural selection criteria were formulated which corresponded to the interview guideline questions, referring, for example, to the initial study phase or to the integration of visualization, applications, and elementary mathematics in mathematical lectures. In the deductive step of category development, several different categories for evaluation were formulated based on these selection criteria, like for example “motivation for studying” or “experiences with visualizations.” Subsequently, a selection of cases was coded with the help of these categories. Coding here means assigning text passages and statements from the interviews to individual categories. In this process the existing categories were modified and revised. At the same time, new categories were formulated from the material in an inductive or abductive manner. The combination of inductive, abductive and deductive category building led to an explicit description of the categories in coding manuals. These were used to analyze and subsequently interpret the whole data material from the interviews, using the software MaxQDA, a qualitative data analysis

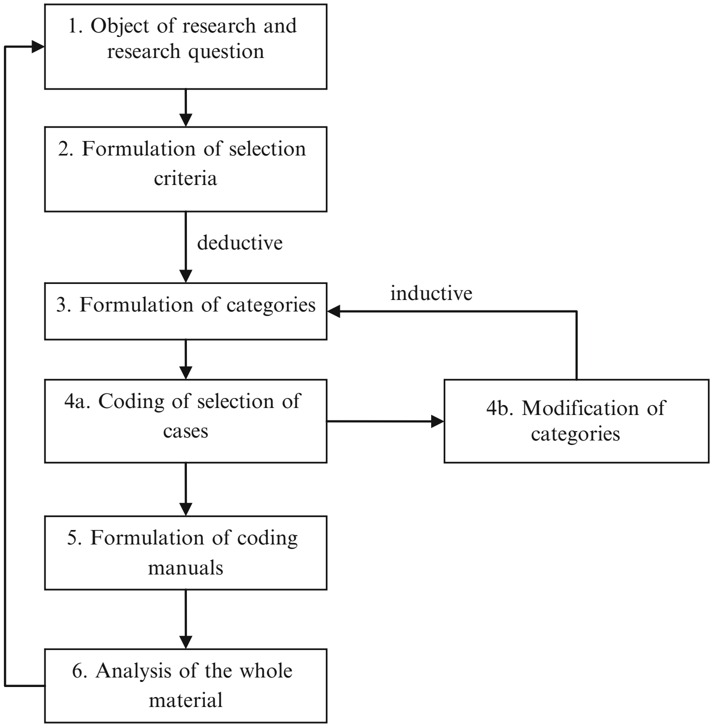


Fig. 12.4 Modified model of the procedure of qualitative content analysis (cf. Mayring 2008)

software for the systematical evaluation and interpretation of textual data. Through the whole process of analysis and category building the emerging categories were related to the initial research questions.

12.6.4 Triangulation in the Mixed Methods Design: Relating Quantitative and Qualitative Findings to Each Other

How can the results of the quantitative study be related to the findings from the qualitative interviews? Concerning the quantitative sub-study of the TEDS-Telekom study, statements about the achievement of the different groups at the different points of measurement can be made via description and through the comparison of group means. These results provide only partial answers to the research question of the impact of the innovative programs, insofar as they enable comparison of the development of the prospective teachers' achievement at different universities in a descriptive manner. With the help of the qualitative sub-study one can identify institutional influences on the individual acquisition of skills, as well as describe individual experiences of the student teachers in the acquisition of competence and perceptions of their study conditions during their first four semesters. Obviously, both sub-studies used their respective methods to focus on different aspects of the outcomes of the teacher training program and therefore yielded different (but complementary) results.

For relating the qualitative and quantitative findings to each other, issues had to be identified which could be related to both qualitative and quantitative data, for example different achievements of student teachers at different universities. Which elements of the program contributed to the success or failure in terms of an increase of the teacher students' competencies? In respect to such questions the quantitative data provided results which could be interpreted in different ways and helped to develop new hypotheses about potential influences of the teacher training program on the acquisition of professional knowledge. These hypotheses could be examined empirically with the help of the qualitative interviews. In this way information about certain aspects of professional competence development could be obtained, which allow for a fuller understanding of quantitative results. Thereby, qualitative data helped to make quantitative findings more comprehensible. Furthermore, qualitative data also provided information about additional factors which influenced the acquisition of professional knowledge and which could not be identified exclusively by standardized paper-and-pencil tests. Given the current state of research and debate, the qualitative findings can also be used to derive new hypotheses – for example hypotheses about different learning strategies pursued by student teachers who may benefit in various ways from the institutional learning conditions provided by their universities. Such hypotheses can be subsequently examined with the help of the quantitative data. The design of our study thus helped to relate qualitative and quantitative results to each other via triangulation (cf. Fig. 12.5).

The integration of quantitative and qualitative findings in the TEDS Telekom study can be further clarified with the help of some examples from the data which concern students' mathematical content knowledge (MCK). We thereby focus on the possibilities offered by the mixed-method design for obtaining *complementary* research results which can be used to *supplement* each other.

In Fig. 12.6, the ability parameters estimated through IRT scaling are presented graphically. The WLEs (*weighted likelihood estimates,* Warm 1989) of all measurement points were transformed to an average value of M = 100 and a standard deviation of SD = 20. The quantitative data showed that the reference group of the students aiming not at the teaching profession shows the best achievements in the area of MCK. This indicates that the knowledge measured by the corresponding test items is highly dependent on the expertise in mathematics. The mathematical content knowledge of all first-year student teacher groups increased significantly during their first 2 years although site-specific differences could be identified. Students at the University of Giessen showed a significant increase in MCK between the second and third point of measurement. In Giessen, the courses on analysis have to be attended in the second year, while at the other universities these courses are usually taken by students in the first year. Although only the first year courses on linear algebra were restructured, the achievement increase between the second and the

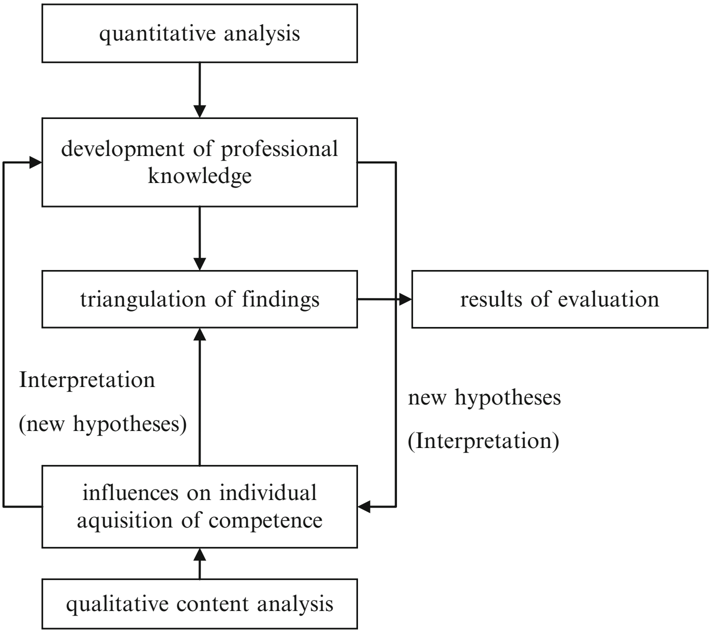
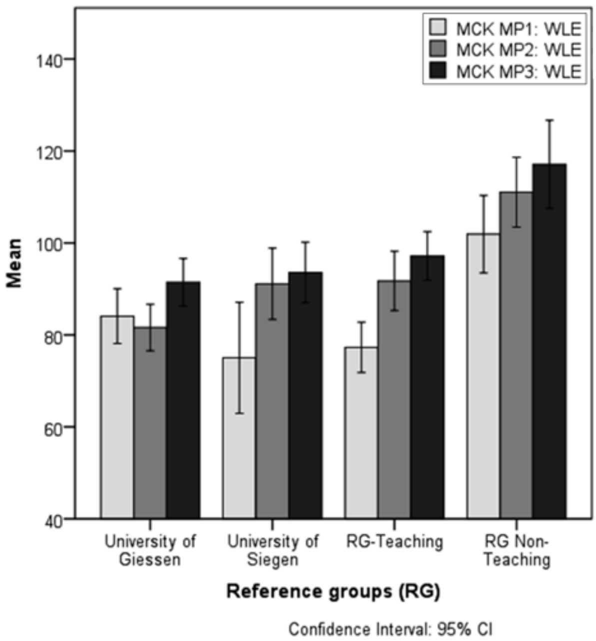


Fig. 12.5 Model of triangulation in the parallel mixed-method design

Fig. 12.6 Ability parameters (WLE) in the area Mathematical Content Knowledge



third point of measurement was considerably high. These results suggest that the used test items are strongly influenced by the knowledge that is gained in the courses on analysis. Compared to the students in Giessen, the first-year student teachers in Siegen as well as the student teachers from the reference universities showed significant performance improvements in the field of MCK already between the first and second point of measurement. In total the student teachers from Giessen and Siegen show almost the same achievement in MCK at the third point of measurement as the student teachers from the reference group.

These results are surprising, looking both at the student teachers' pre-conditions in the different universities and the curriculum covered in their courses. On the one hand, the students' cognitive dispositions at the beginning of their study (less the degrees of the entrance qualification for higher education, in Germany the so-called "*Abitur*" but the attended mathematics courses in high school) indicated that the groups of the student teachers in Giessen and Siegen had less good pre-conditions than the students at the reference universities. On the other hand, the courses for the student teachers covered less academic mathematics, because they were enriched with teaching content. The students therefore had fewer opportunities to learn the academic mathematics than in the reference groups. Nevertheless, these student teachers reached comparable high achievements than students at the other universities which would be a success of the innovative program and which also indicates that the process of selection in the first year at university did not take place in the usual way (see for details Kaiser and Buchholtz 2014). Thus the hypothesis from these findings can be derived that student teachers benefit in a specific way from a modified treatment of mathematics in the mathematical courses by improving their competencies. But what did the program do to establish mathematical courses that facilitate the acquisition of knowledge in the field of MCK, especially in the courses on analysis?

Following the idea of triangulation as the investigation of different aspects of the research subject, qualitative and quantitative data was combined to gain a deeper understanding of the development of professional competence and the different results regarding achievements. In order to obtain a fuller picture of the effects of a modified treatment of mathematical content within courses about analysis, questions about the acquisition of mathematical content knowledge were focused during the analysis of the qualitative interviews. The student teachers were asked in particular to describe those learning opportunities in academic mathematics courses from which they had benefitted most. Furthermore the students were asked which issues need to be addressed in order to ensure a more comprehension-oriented university teaching.

Due to the limited space we restrict the discussion to a single topic covered by the guided interviews, namely *"integration of visualization in mathematical lectures*." Thereby we concentrate on student teachers from the University of Giessen, where most of the respondents´ statements refer to the course of analysis which is offered only in the second year and was not included into the restructuring measures of mathematics teacher education: That means the analysis course is taught in the traditional abstract way and is attended by the prospective teachers together with the mathematics students not aiming at the teaching profession (for details cf. Kaiser and Buchholtz 2014).

> If now I'm thinking back on school, [in analysis] we had a bit the evaluation of functions, a bit derivations. And actually in analysis—the time at school, now I do not remember it so well—but I think there were hardly any proofs. That is precisely the opposite at the university, there are definitions, proofs … And analysis in the university context actually consists only of proofs and definitions. […] No, there were clearly dropouts. Due to analysis there were clearly dropouts, but I can imagine, if we have had that in the first semester, the dropout rate would have been even higher. (Prospective teacher, female, 26, grade 1.3)

> The terminology of analysis remained abstract, 75 % of it I would even not know what to do with, what does it mean, for what I am actually doing that … That was just stupefying learned by heart and simply written down, that what the professor wanted to hear. (Prospective teacher, male, 21, grade 2.2)

> It extends, the learning process extends. At home I sit down and work upon it by exemplification, so that I can understand it by myself. And so it takes much longer. If that would have been contributed by the lecture, one would not need to work on exemplification afterwards on one's own. (Prospective teacher, female, 26, grade 1.3)

The statements clearly demonstrate that a high grade of abstraction in mathematics lectures may cause obstacles for understanding and in the worst case even cause some students to cancel their studies. Considering knowledge gain, it shows that if the content remains abstract, what has been learned will be forgotten immediately. The prospective teachers obviously need a more illustrative way of teaching mathematical content, which often cannot be realized in university courses attended both by prospective teachers and mathematics students not aiming at becoming a teacher:

> At the end actually, because there are so many prospective teachers, I personally think it would be wonderful if it would really be possible to separate Bachelor [(i.e. non-teacher) students] and prospective teachers, completely, and not only for selected courses. And the Bachelor [students] do not need these references of reality, the exemplification as strongly as prospective teachers need it. I think, because, the Bachelor [students] do not teach that later. (Prospective teacher, female, 26, grade 1.3)

Obviously, the lecturers at the University of Giessen were successful with their first-year course on linear algebra for teacher students and succeeded at least partly in the overcoming of comprehension problems by embedding exemplification into the mathematical content courses. Many of the interviewees made clear that this exemplification had a key role for their own understanding. Respondents even recognized the significance of visualizing and illustrating mathematical concepts via analogies, metaphors and examples for teaching mathematics at school.

> In [linear] algebra, concerning vector spaces, it was beautifully made clear, that a vector is not just an arrow which is just drawn, but that it has a direction and which properties it has. Because, one has quasi developed an imagination of it, how it looks like. And therefore later it is good for the students, one can better explain it. (Prospective teacher, male, 21, grade 2.2)

> I now also try to apply the exemplification in the private tutoring center, where I work. I try to put this also into the foreground. Because the experience, the short experience, that I could make now, has shown that the more exemplifying the beginning is, the more the pupils are willing to get to work on theory. (Prospective teacher, male, 21, grade 2.2)

The teaching of mathematical content in an understanding-oriented way may foster learning on the one hand, but is on the other hand also very time-consuming because the pace of learning may be reduced. But students do not consider that as impairing.

> Yes, exemplification I think is quite important, in order to have reference, so that one knows what one is doing there. If you have an image right in front of your eyes, then the theory remains more rooted in your head, later it is like this at school. And yes, then it is okay for me, if then in only one week lecture can be worked on only the half, but one knows: the students do understand it now. (Prospective teacher, female, 20, grade 1.8)

> Yes, I just say, I personally think it makes more sense to work on less content, but to understand it, instead of working on more content of which one does not know anything at the end after having struggled through. (Prospective teacher, male, 20, grade 1.8)

It can be assumed, that the particular courses offered at the universities of Giessen and Siegen have a strong influence on the knowledge development of prospective teachers. One reason for that might be a slower pacing and the empathic, exemplifying way of teaching applied in the courses. Such a style of teaching in mathematics courses has obviously a motivating effect—this could be confirmed by statements from the qualitative interviews which represented the perspective of students. These students did not experience the sometimes slower pace of learning as an obstacle, but to the contrary, as strengthening their learning efforts. The student teachers from the University of Giessen also distinguish between the course on linear algebra and analysis; they assess teaching in the latter as being less helpful, while in the former they experienced comprehension-oriented teaching. The pure transmission of factual knowledge in academic teacher education may of cause lead to high achievement scores in performance tests as the quantitative findings suggest (which show a significant increase between the second and third survey), but not to finally sustainable results and even may cause students with comprehension problems to give up their studies. Nevertheless, the teacher students from Giessen with comparable low previous competencies and knowledge in mathematics who took part in the new course do not show significant performance deficits in the area of mathematical content knowledge compared to the reference universities. Student teachers obviously did well with the modified treatment of academic mathematical knowledge and increased their achievement independently from the way the subject matter was taught. However, the restructuring of the introductory courses may have stronger impacts on endurance, motivation, beliefs, and sustainable learning, which have to be further investigated, especially for students with restrained pre-conditions. In order to relate the findings of the qualitative and quantitative sub-studies more intensively to each other and to further examine the hypotheses derived from the quantitative findings, the interviews have to be further analyzed focusing on experiences of student teachers with comprehension-problems. At the current state of research it can be assumed that these students can deal better with the traditional way of teaching in the second year, because their experiences with comprehension-oriented teaching in the first year prepared them for that what follows.

12.7 Different Functions of Mixed Methods Designs: An Overview

Mixed methods designs provide important tools to overcome limitations of both qualitative and quantitative mono-method research:

- a quantitative study can help to corroborate findings from a qualitative study and to transfer these findings to other domains,
- results from the qualitative part of a mixed methods design can help to understand previously incomprehensible statistical findings,
- qualitative research can help to discover a lack of validity of quantitative measurement operations and instruments,
- results from qualitative interviews can help to identify unobserved heterogeneity in quantitative data as well as previously unknown explaining variables and misspecified models,
- in a sequential quantitative-qualitative design quantitative research can help to guide the selection of cases in qualitative small-N studies,

Thus quantitative and qualitative methods can fulfill different yet complementary purposes within mixed method designs:

Quantitative methods can give an overview about the domain under study and can describe its heterogeneity, whereas qualitative methods can be used to gain access to specific knowledge in the field in order to develop theoretical concepts and explanations which cover phenomena which are relevant for the research domain. Quantitative and qualitative methods thus cannot substitute for each other but help to illuminate different aspects of the investigated phenomena: quantitative methods, for instance, may describe the actions of large numbers of different actors, whereas qualitative methods provide information about possible reasons for these actions. In such cases qualitative and quantitative methods help to answer different questions: the results of statistical analyses show *what kinds of actions* social actors typically perform (use specific techniques of calculation, methods of teaching, etc.), while the analysis of qualitative data helps to answer *why*-questions (e.g. *for what purposes* do teachers use specific methods of teaching, how do they perceive and define their situation, which norms do they acknowledge? etc.). Here qualitative and quantitative results are not interchangeable. It is not possible to analyze the aggregated results of actions (e.g. the results of performance tests of large numbers of pupils) with the help of qualitative interview data, whereas local knowledge typical for a certain culture or life world often cannot be investigated using standardized questionnaires, since the researchers do not have sufficient knowledge to construct such research instruments.

Since the application of qualitative methods to a yet unknown field carries the danger that researchers focus on remote phenomena and marginal cases, an important function of quantitative methods in mixed methods research is to guide the selection of cases in the qualitative sub-study. Using a metaphor from geography and geology one could say that quantitative methods provide us with a general picture of the surface of the research field, while qualitative research can be used to drill deep holes into the field yielding the information necessary for in-depth explanations. The problem of hazardous generalizations from small N studies can be further mitigated if quantitative methods are used for the corroboration of results coming from a qualitative study. Best practice in mixed method research thus comprises a chain of alternating steps of qualitative and quantitative research. Quantitative methods can be used to describe the investigated phenomena and explananda on an aggregated level and to guide qualitative sampling. Qualitative research provides information necessary for elaborated explanatory arguments which can be further examined by subsequent quantitative research. Thereby further quantitative studies may lead to new questions which require additional qualitative research and so forth.

In our study presented as an example the re-orientation of the mathematics teacher training at various German universities on the basis of a mixed methods research approach is being investigated. Of particular interest hereby is the comparative development of the student teachers performance. The development of diagnostic tools for performance measurement in the context of quantitative research methods here makes an important contribution. The results of the quantitative-oriented sub-study showed a different performance development at the different universities, including the level of initial pre-cognitions of the student teachers. But the question, which factors of the re-orientation of the mathematics teacher training may be responsible for the different performance development, could rather be examined by an analysis of interviews with student teachers of the respective universities about the institutional framework of their studies, evaluated by qualitative research methods. Within the qualitative sub-study, which focuses on the perceptions of student teachers of their studies, by this on the one hand the quantitative results can be partially explained; on the other hand the qualitative sub-study provides independent findings about individual cases, which afterwards can be checked for the whole group on the basis of the quantitative data. In this way, within the research design a multi-step process is initiated, in which qualitative and quantitative research methods complement each other.

Methodological rules for the integration of qualitative and quantitative methods can certainly not serve as recipes to be exactly followed in a step-by-step manner. At most, they are general guidelines whose significance varies according to the research question at hand and the empirical domain under investigation, and depending on the specific methods employed.

1. The selection of adequate methods should not be made mainly on the basis of sympathies towards a certain methodological camp or school. Methods are tools for the answering of research questions and not *vice versa*. Consequently, decisions about the applied methods should not be made before the research questions are formulated
2. Each method is well suited to specific empirical domains, while there are other empirical fields of interest where the same method will not yield meaningful results.

3. There is not one single methodological model of method integration available (that claims, for instance, that qualitative and quantitative empirical investigation must always lead to convergent or complementary results). The model of triangulation for mutual validation as well as the complementarity model both have strengths and weaknesses depending on the research questions posed and the empirical domains under investigation. Consequently, the aim of method integration, be it the mutual validation of data and methods or the complementarity of research results, has to be determined on the basis of theoretical and substantive considerations for each research project.
4. If method integration is carried out with the purpose of mutual validation, convergence of research results may provide good arguments for their validity, but can never fully prove this validity, for it is always possible that all the convergent results are biased for the same reason and in the same direction.
5. The crucial function of method integration performed for the purpose of complementary results is to provide additional data material in an empirical research domain where one single method is insufficient for the investigation of the complete empirical basis of a theoretical assumption.
6. If the qualitative and quantitative methods applied simultaneously lead to divergent results, in principle two explanations are possible: either the divergence is the result of mistakes made when applying one (or both) methods and thus represents a methodological artifact, or the initial theoretical assumptions have to be modified and revised.

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Part IX
Qualitative Content Analysis

Chapter 13
Qualitative Content Analysis: Theoretical Background and Procedures

Philipp Mayring

Abstract Qualitative Content Analysis designates a bundle of text analysis procedures integrating qualitative and quantitative steps of analysis, which makes it an approach of mixed methods. This contribution defines it with a background of quantitative content analysis and compares it with other social science text analysis approaches (e.g. Grounded Theory). The basic theoretical and methodological assumptions are elaborated: reference to a communication model, rule orientation of analysis, theoretical background of those content analytical rules, categories in the center of the procedure, necessity of pilot testing of categories and rules, necessity of intra- and inter-coder reliability checks. Then the two main procedures, inductive category formation and deductive category assignment, are described by step models. Finally the procedures are compared with similar techniques (e.g. codebook analysis) and strengths and weaknesses are discussed.

Keywords Qualitative content analysis

13.1 Methodological Background of Qualitative Content Analysis

The techniques of Qualitative Content Analysis have become a standard procedure of text analysis within the social sciences. In their bibliometrical analysis of the Social Sciences Citation Index (SSCI, 1991–1998), Titscher et al. (2000) found Qualitative Content Analysis in seventh place (after Grounded Theory, Ethnography, Standardized Content Analysis, Critical Discourse Analysis, Conversation Analysis and Membership Categorization Device). On a predominantly German language database (Psyndex, Sociofile, WISO-Social Science and MLA International

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P. Mayring (✉)
Institut für Psychologie der Alpen, Alpen-Adria-Universität Klagenfurt,
Universitätsstr. 65-67, Klagenfurt, Austria
e-mail: Philipp.Mayring@aau.at

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Bibliography) they found qualitative, open content analysis in first place. A reason for this could be that it can be located between open hermeneutic approaches and quantitative measurement. Thus Hussy et al. (2010) discuss it as hybrid qualitative–quantitative approach within the mixed methods approach.

But how could qualitative and quantitative methodologies come together? In social sciences a "science war" is diagnosed (Ross 1996). On the one hand stands a rigid positivist conception of research with quantitative, experimental methodology; on the other hand an open, explorative, descriptive, interpretive conception working with qualitative methods. Norman Denzin has subtitled his *Qualitative Manifesto* (Denzin 2010) as "a call to arms", so it seems for him impossible to overcome the contradiction.

If we are looking at approaches to text analysis, we can differentiate the two positions as coming from different epistemological backgrounds (cf. Guba and Lincoln 2005):

– The hermeneutical position, embedded within a constructivistic theory, tries to understand the meaning of the text as interaction between the preconceptions of the reader and the intentions of the text producer. Within the hermeneutical circle the preconceptions are refined and further developed in confrontation with the text. The result of the analysis remains relative to the reading situation and the reader.
– The positivistic position tries to measure, to record and quantify obvious aspects of the text. Those aspects of the text can be detected automatically, and their frequencies can be analyzed statistically. The results of the analysis claim objectivity.

A strict adherence to one of these positions overlooks the possible convergences: the social constructivist theory formulates the possibility of agreement between different individual meaning constructions and allows by that the concept of a socially shared quasi-objective reality. Modern hermeneutical approaches try to formulate rules of interpretation. By this the analysis gains objectivity. On the other hand, positivistic positions had been refined to post-positivism, or critical rationalism (Popper). Here only an approximation of reality, by critical efforts of researchers to refute hypotheses, is held to be possible; again there is the notion of an agreement process in talking about reality rather than a naive copy of reality.

If there are possibilities of bringing together opposing positions in the qualitative–quantitative debate, the floor is open for models of combination and integration, now discussed under the label of mixed methods (cf. Mayring et al. 2007; Creswell and Clark 2010). Qualitative Content Analysis tries to establish such a mixed methods approach in text analysis. We combine two fundamental steps of analysis: the first is a qualitative-interpretative step following a hermeneutical logic in assigning categories to text passages; the second is a quantitative analysis of frequencies of those assignments (if the same categories are coded in several text passages) (cf. Mayring 2002).

13.2 Development and Definition of Content Analysis

Following this background to the procedures of Qualitative Content Analysis, we first define and characterize the basic ideas of (quantitative) Content Analysis. There is general agreement that the aim of content analysis is to analyze material derived from any kind of communication, but content analysis has not concerned itself solely with analyzing the content. On this point even the definition by the author of the first textbook on content analysis, Bernhard Berelson, is not precise: "Content analysis is a research technique for the objective, systematic, and quantitative description of the manifest content of communication" (Berelson 1952, p. 18). Not only description of content, but also formal aspects of communication and underlying meaning structures have become the object of analysis. Thus scripts of dialogues with psychotherapy patients are scrutinized for formal characteristics such as sentence corrections, incomplete sentences, word repetitions, "ers" and "erms", etc., in order to register indications of a patient's anxiety level (Pool 1959). Even American propaganda research during the Second World War, which was directed by Harold D. Lasswell and contributed significantly to the development of content analysis, does not restrict itself to the actual contents of communication. In fact, many analysts are altogether suspicious of the concept "content", as they are more interested in the latent meanings than in the overt communicative content. Thus Budd et al. (1967) define as follows: "Content analysis is a systematic technique for analyzing message content and message handling" (p. 2), while George (1959) points in a different direction when he calls it "a diagnostic tool for making specific inferences about some aspects of the speaker's purposive behaviour" (p. 7); or, again, in a more generalized form, Krippendorff (1969): "Content analysis may therefore be redefined as the use of replicable and valid methods for making specific inferences from texts to other states or properties of its source" (p. 11). As can be seen from this, content analysis has long ceased to concern itself solely with content. Pool (1959), in summary, identifies three objectives:

– describing texts;
– drawing inferences from texts to their antecedents;
– drawing inferences from texts to their effects.

With this background two main techniques of quantitative content analysis have been developed. First, and primarily, **frequency analyses** and techniques derived from them. The simplest method of content-analytical procedure is to count certain elements in the material and compare them in their frequency with the occurrence of other elements. Of special importance here is the use of comprehensive category systems (so-called "dictionaries"), which are supposed to include all aspects of a text and form the basis for a computer count of language material. The General Inquirer (Stone et al. 1966) seems to have been the first attempt in this direction. Dictionaries now exist, for instance, for psychologically relevant issues (e.g. *Harvard Psychological Dictionary*), the latest editions of which can be conveniently used on a PC (cf. Weber 1990). On this basis frequencies are computed and analyzed statistically. The dictionary must also of course be able to recognize different grammatical forms of a word within the context of a sentence. This, however, can cause problems:

- multiplicity of meaning (e.g. “madly” in the colloquial meaning, say, of “very”; or “madly” as pertaining to psychological disturbance);
- the nuances and connotations conferred on terms by the context;
- contextual modification of meaning (for instance in the case of “no anxiety”, “little anxiety” and “a lot of anxiety”, “anxiety” will be counted once in each case);
- the contextual relationship of the term counted (e.g. with “I am afraid of X” or “X is afraid of me”, “afraid” is counted once in each case);
- the problem of pro-forms (e.g. with “I didn’t notice any of that” the computer does not know what “of that” refers to);
- dialect expressions (which occur in interview scripts regularly) need a great deal of re-working.

Several more problems could be added to the list. Attempts have in fact been made to check and control contextual influences of this kind (e.g. KWIC – Key Word In Context program, cf. Weber 1990). For this a list of the text passages within which a category was found, that is, the category in its different contexts, is drawn up for each concept or term counted. This, however, only makes it possible to recognize the problem, not to solve it. In any case, lists such as this are difficult to process with large quantities of text. One example of a more complex frequency analysis is the Gottschalk–Gleser Speech Content Analysis for the measurement of affective states (anxiety, aggressivity) (Gottschalk and Gleser 1969), which has also been adapted for the German language (Schoefer 1980).

This brings us to the second group of tested techniques of content analysis: **contingency analyses**. The development of such techniques goes back above all to Charles Osgood (Osgood 1959). The objective here is to establish whether particular text elements (e.g. central concepts) occur with particular frequency in the same context and are connected with one another in any way in the text, that is, are contingent. The intention is that by discovering many such contingencies one may extract from the material a structure of text elements associated with one another. Examples of this are the classical contingency analysis of Osgood (1959) or semantic field analysis (Weymann 1973).

However, there are fundamental criticisms of quantitative content analysis to the extent that, today, one can say that the methodology discussion has reached a point of stagnation. An increasing number of critical voices describe the technique as inadequate and unable to fulfill requirements. The joke about “discontent analysis” can be heard with increasing frequency. Koch et al. (1974), for example, tested six fairly recent journalistic content analyses from German-speaking countries according to customary standards of quality. From them, content analysis gets a bad report: “If conclusions are drawn on the basis of the work reviewed here, then it must be stated that up to now no one has succeeded in developing a handy instrument for describing and analysing news publications with the help of content analysis” (Koch et al. 1974, p. 83). Manfred Ruehl also denies that content analysis has a chance of achieving “social-scientific status capable of gaining general acceptability”

(Ruehl 1976, p. 377). It achieves only superficial polish through quantitative techniques, and has pushed the problem of sense and meaning to one side, he argues. "The results of content analysis remain highly pseudo- and parascientific … as long as content analysts do not know how to equip their scientific criteria better for methodological testing" (Ruehl 1976, pp. 376–377). The fact that the quantification approach and orientation to manifest content tend to sidestep the problem of what language symbols actually mean is reason enough also for Ingunde Fuehlau to declare that content analysis is a failure: "This is why content analysis, if pursued strictly according to its own tenets, must inevitably lead to distorted results. If the method was stringently applied which actually is almost never really the case — it must either produce irrelevant descriptions of the subject — albeit in a very "objective manner" — or on the other hand meaningful descriptions of communication content, to which, however, if judged according to its own criteria, it can only assign a highly subject value. In either case, therefore, it fails as a method" (Fuehlau 1978, pp. 15–16; cf. also Fuehlau 1982).

13.3 Basics of Qualitative Content Analysis

Qualitative Content Analysis tries to retain the strengths of quantitative analysis and against this background to develop techniques of systematic qualitatively oriented text analysis. The following points are central:

13.3.1 Embedding of the Material Within the Communicative Context

A particular advantage of the content-analytical procedure as compared with other approaches to text analysis is the fact that it has a firm basis in the communicative sciences. The material is always understood as relating to a particular context of communication. The interpreter must specify which part of the communication process he wishes his conclusions from the material analysis to relate to. This content-analytical particularity should be retained at all costs for qualitative content analysis because many quantitative content analyses have neglected this point. The text is thus always interpreted within its context, i.e. the material is examined with regard to its origin and effect.

13.3.2 Systematic, Rule-Bound Procedure

Preserving the systematic procedure of content analysis is one of the main concerns of the methods suggested here. Systematic procedure in this connection means, first and foremost, orientation towards rules of text analysis laid down in advance.

Several points need to be made in this regard. The establishing of a concrete procedural model of analysis is of central importance. Content analysis is not a standardized instrument that always remains the same; it must be fitted to suit the particular object or material in question and constructed especially for the issue at hand. This is laid down in advance in a procedural model (examples of such models will be found below) which define the individual steps of analysis and stipulate their order. But it is also continually necessary to establish additional rules. It is an axiom precisely of content analysis, in contrast to “free analysis”, that every analytical step and every decision in the evaluation process should be based on a systematic and tested rule. Finally, the systematic quality of content analysis is reflected also in its method of “dissection” or line-by-line analysis rather than a more holistic interpretation.

The definition of content-analytical units (recording units, context units, recording unit) should in principle be retained also in qualitative analysis. Concretely this entails deciding in advance how the material is to be approached, which parts are to be analyzed in what sequence, what conditions must obtain in order for an encoding to be carried out. In the process of inductive category formation it can be useful to keep such content-analytical units very open-ended. Despite this, however, the process here also is characterized by dissection of the material carried out progressively from one passage to the next. Certainly, it is precisely this last point which has frequently been criticized by some proponents of the qualitative approach. Latent structures of meaning cannot be revealed in this way, they say. One answer to this, in the case of such an analytical objective, is to define the units more broadly. Nevertheless, it is important that such units be theoretically well founded, in order to allow other analysts access to the logic and method of the analysis. The system should be so described that another interpreter may carry out the analysis in a similar way.

13.3.3 Categories as the Focus of Analysis

The category system is the central point in quantitative content analysis. Even with qualitative analysis, however, an attempt should be made to concretize the objectives of the analysis in category form. The category system constitutes the central instrument of analysis. It also contributes to the inter-subjectivity of the procedure, helping to make it possible for others to reconstruct or repeat the analysis. In this connection qualitative content analysis will have to pay particular attention to category construction and substantiation. However, precious little help is given in this respect by standard works on content analysis. Krippendorff thus writes: “How categories are defined … is an art. Little is written about it” (Krippendorff 2004, p. 76). That of course is unsatisfactory. It is precisely the methods described in this work which may be of further assistance in this regard.

On this point also, some proponents of qualitative analysis make the objection that orientation to categories entails an analytically dissecting procedure which impedes more holistic comprehension of the material. In answer to this it can be said that qualitative content analysis also provides methods which accord prominence to synthetic category construction, that is, where the category system actually constitutes the findings of the analysis. This is the case for inductive category formation procedures and summarizing content analysis (see below). On the other hand, working with a category system is an important contribution to the comparability of findings and the evaluation of analysis reliability.

13.3.4 Object Reference in Place of Formal Techniques

On the other hand the methods of qualitative content analysis should not simply be techniques to be employed anywhere and everywhere. The alliance with the individual object of analysis is an especially important concern. This is seen in the fact that the procedures discussed here are oriented to the way language material is ordinarily experienced and dealt with in everyday life. The three basic techniques of summary, explication and structuring (see Sect. 4) are based on it. This clearly demonstrates that it is the object of analysis which is paramount. The methods are not intended to be conceived of as techniques which can be blindly and automatically transferred from one object to the other. The appropriateness of the method must be demonstrated with regard to the particular material in each individual case. This is why the methods suggested here must themselves always be adapted to suit the individual study.

13.3.5 Pilot Testing of the System of Categories and the Content Analytical Rules

Qualitatively oriented content analysis does not use fully standardized instruments. The category system and the related content analytical rules usually are developed for the specific material in respect to the specific research question. Initially that means a disadvantage compared with quantitative research and is why methods should be tested in a pilot study. After working through a substantial part of the material the coder is requested to stop coding and revise the category system and the coding rules. Are they adequate to the material and the research question? If a revision is done as consequence, the coding process has to start from the beginning. In the procedural models (see below), these steps are included through the presence of reverse loops. What is important in this is that the trial runs are also documented in the research report.

13.3.6 Theory-Guided Character of the Analysis

It must by now have become clear that qualitative content analysis is not a rigidly delineated technique, but a process in which new decisions regarding basic procedure and individual stages of analysis constantly have to be made. What are such decisions based upon? In qualitatively oriented research it is repeatedly stressed that here theoretical arguments must be used. Technical fuzziness is compensated for by theoretical stringency. This applies above all to the explication of the particular issue, but it also concerns detailed analyses. Theory-guidedness means that in all procedural decisions systematic reference is made to the latest research on the particular subject and on comparable subject fields. In qualitative content analysis content-related arguments should always be given preference over procedural arguments.

13.3.7 Integrating Quantitative Steps of Analysis

As has already been emphasized above, efforts are made to combine qualitative and quantitative methods. Putting it more exactly, the chief task is to determine those points in the analytical process at which quantitative measures can be sensibly brought in. Reasons for their use should then be carefully explained and the results should be analyzed in detail. Quantitative steps of analysis will always gain particular importance when generalization of the results is required. In case study procedures it is important to show that a certain case recurs in similar form with particular frequency. But within content-analytical category systems, too, registration of how often a category occurs may give added weight to its meaning and importance. Of course, this must be given adequate justification in the respective case. A precisely based qualitative assignment of categories to a certain material (e.g. through the structuring method, cf. below) can also be supplemented by more complex statistical evaluation techniques, as far as these are appropriate to the purpose of analysis and suited to the object involved.

13.3.8 Quality Criteria

It is precisely because here the harsh methodological standards of quantitative content analysis have been softened and applied more flexibly in some respects that the assessment of results according to quality criteria such as objectivity, reliability and validity is especially important even in qualitative content analysis. For quantitative content analysis it is inter-coder agreement which is of particular significance. Several content analysts work on the same material independently of one another and their findings are compared. In general this should also be attempted with qualitative content analysis, although negative findings do not necessarily have to lead to the immediate abandoning of the analysis. Here the main point, again, is to understand and interpret unreliabilities. Such a search for sources of error is especially important during the pilot phase, as it can lead to the instruments of analysis being modified. That is to say, it can lead to inquiry into arguments for reliability and validity while the process of analysis is actually going on, instead of leaving this exclusively to a single assessment at the close of the analysis.

13.4 Basic Procedures or Techniques of Qualitative Content Analysis

From an analysis of common qualitative oriented text analysis techniques (cf. Mayring 2010a, b) we can show that they can be reduced to three fundamental forms of interpreting: summary (text reduction), explication and structuring:

- **Reducing procedures:** The object of the analysis is to reduce the material such that the essential contents remain, in order to create through abstraction a comprehensive overview of the base material which is nevertheless still an image of it.
- **Explicating procedures:** The object of the analysis is to provide additional material on individual doubtful text components (terms, sentences, …) with a view to increasing understanding, explaining, interpreting the particular passage of text.
- **Structuring procedures:** The object of the analysis is to filter out particular aspects of the material, to give a cross-section through the material according to pre-determined ordering criteria, or to assess the material according to certain criteria. In those procedures the categories are formulated in advance in the sense of a deductive category assignment.

These basic forms, however, must be further differentiated before an exact description of procedures is possible. In addition to the usual summaries, the same ongoing process is useful for inductive category formation; a criterion for the categories is defined and aspects of this criterion are stepwise gathered in the material. Forms of explication are possible which use the textual context for the elucidation of a particular text passage (narrow context analysis); however, the most common method of hermeneutical interpretation is to use further material beyond the textual context for explication (broad context analysis). With structuring, too, subgroups must be distinguished. The categories which are brought deductively to the material can consist of a list of aspects (nominal scale). Or the categories form an ordinal scale (e.g. more – less) and serve as a rating procedure for the text. In addition, some mixed procedures have been described (Mayring 2010a, 2013). One such is that in content structuring or theme analysis the material is deductively ordered to categories and within each category material an inductive process of category formation is performed. Type analysis is a similar procedure where categories in the first step have to meet a typologizing criterion (typical types, extreme types, frequent types, theoretical types). In category refinement a deductive category system is modified and supplemented with new categories in an inductive way. Parallel forms execute several procedures in one passage through the material.

Through this differentiation we arrive at ten distinct forms of analysis:

Reductive:	(1) Summary	(2) Inductive category formation
Explicating:	(3) Narrow context analysis	(4) Broad context analysis
Structuring/Deductive:	(5) Nominal categories	(6) Ordinal categories
Mixed:	(7) Content structuring	(8) Type analysis
	(9) Category refinement	(10) Parallel forms

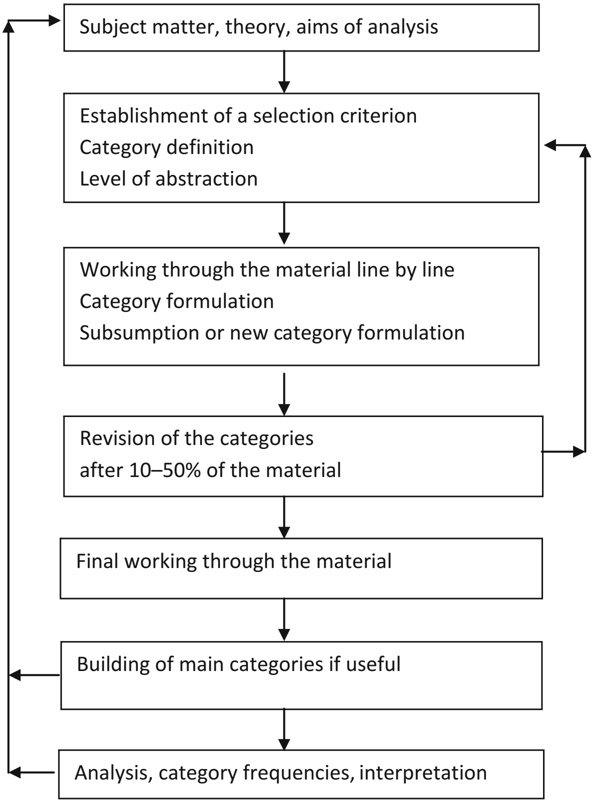
These procedures have been described extensively elsewhere (Mayring 2010a, 2013). We now demonstrate two central techniques of Qualitative Content Analysis: inductive category formation and deductive category assignment (structuring).

13.4.1 Inductive Category Formation

On the basis of summarizing qualitative content analysis a technique for inductive category formation can be developed. We have heard that category definition is a central step in content analysis, a very sensitive process, "an art" (Krippendorff 2004). There are two possible procedures: deductive category definition tries to develop categories out of theoretical considerations, with theories or theoretical concepts used in a process of operationalization in direction of the material; inductive category formation develops categories directly out of the material. For qualitative content analysis the second is very fruitful. The ongoing inductive process has great importance within qualitative research. It aims at a true description without bias due to the preconceptions of the researcher, an understanding of the material in terms of the material. Inductive category formation is a central process within the approach of Grounded Theory (Strauss 1987; Strauss and Corbin 1990), which they call "open coding". They developed many rules of thumb for open coding, and they recommended a systematic, line-by-line procedure. For content analysis, nevertheless, inductive category formation has to be more systematic. And it can use the same logic, the same reductive procedures, as in summarizing content analysis. The following process model (Fig. 13.1) will now be explained.

Within the logic of content analysis, the level or theme of categories to be developed must be defined previously. There has to be a criterion for the selection process in category formation. This is a deductive element and is established within theoretical considerations about the subject matter and the aims of analysis. The second basic content analytical rule for inductive category formation is the establishment of the abstraction level. This comes from summarizing content analysis which reduces the material from one abstraction level to the next. If this level is not defined the

Fig. 13.1 Process model of inductive category formation



categories remain chaotic, out of order. After those two rules are determined, the material is worked through line by line. The first time material fitting the category definition is found, a category has to be constructed. A term or short sentence which stands as near as possible to the material serves as category label. The next time a passage fitting the category definition is found it has to be checked whether if it falls under the previous category, in which case it can be subsumed under this category (a reductive process); if not, a new category has to be formulated.

After working through a good deal of material (c. 10–50 %) no new categories are to be found. This is the moment for a revision of the whole category system. It has to be checked whether the logic of categories is clear (e.g. no overlaps) and whether the level of abstraction is adequate to the subject matter and aims of analysis. Perhaps the category definition has to be changed. If there are any changes in the category system, of course the complete material has to be worked through once again. After this analysis we have a set of categories to a specific topic, connected with specific passages in the material. The further analysis can go different ways: the whole system of categories can be interpreted in terms of the aims of the analysis and used theories; or the links between categories and passages in the material can be analyzed quantitatively (e.g. we can look at those categories occurring most frequently in the material).

For example, in a study on learning emotions (Glaeser-Zikuda and Mayring 2003) we analyzed open-ended interviews and daily diaries on concrete learning experiences of 24 students of eighth grade. With inductive category formation we built up categories concerning positive learning experiences. We generalized those categories on a medium level of abstraction. Here are the most frequent categories:

C1: Happy about the interesting learning activities today	(21 occurrences)
C2: Happy to master the subject and having understood everything	(21 occurrences)
C3: Amazing subjects in the lesson (literature, poems)	(16 occurrences)
C4: Enjoyed the positive feedback by the teacher	(14 occurrences)
C5: Nice group work or partner work	(11 occurrences)
C6: Interesting problems (electricity) in the lesson	(3 occurrences)

Those inductive categories give an impression about positive emotions in learning processes. In a second step we found two main categories within this list: positive emotions about the learning processes (C1, C2, C4, C5) and positive emotions about the learning content (C3, C6). We then compared the occurrences of those main categories between the two groups of high and low achievers and found a correlation between positive emotions about learning processes and high classroom achievement. That means that it seems to be more important for good teaching to associate positive emotionality with successful learning processes than with learning content.

13.4.2 Deductive Category Assignment (Structuring)

This is the content-analytical method which is probably most often used. It has the goal of extracting a certain structure from the material. This structure is brought to bear on the material in the form of a category system. All text components addressed by the categories are then extracted from the material systematically. If one wishes to describe the structuring procedure quite generally, a few points are especially important: the fundamental structuring dimensions must be exactly determined; they must derive from the research question and must be theoretically based; these structuring dimensions can be further subdivided, split up into individual features or values; the dimensions and values are then brought together to form a category system.

The particular categorization of a given material component is something that must be determined precisely. A procedure for this, based on everyday life processes of categorization, has proven useful (cf. Ulich et al. 1985). Within developmental psychology (learning of categories in speech development) and within general psychology (categorization theories, cf. Murphy 2002) it has been shown that categories are imagined in form of explicit definitions, prototypes and demarcation rules. So a category can be defined best if all three determining approaches are used:

- **Definition of the categories**
  It is precisely determined which text components belong in a given category.
- **Anchor samples**
  Concrete passages belonging in particular categories are cited as typical examples to illustrate the character of those categories.
- **Coding rules**
  Where there are problems of delineation between categories, rules are formulated for the purpose of unambiguous assignment to a particular category.

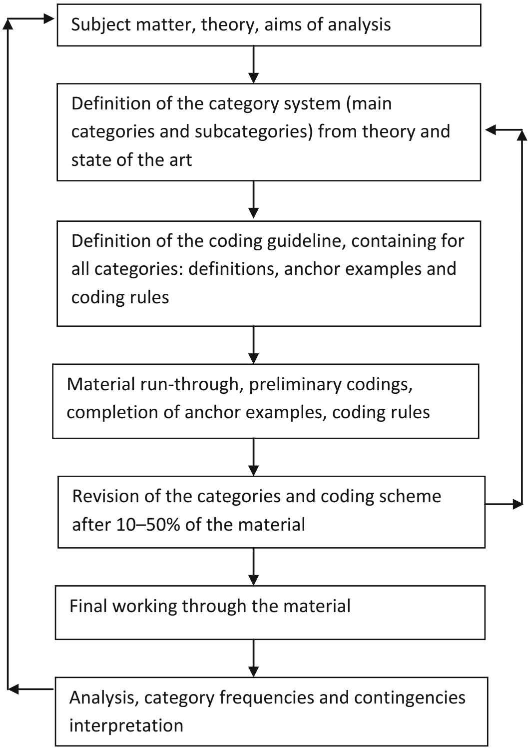
Test extracts are taken from the material to check whether the categories are at all applicable and whether the definitions, anchor samples and coding rules make category assignment possible. This trial run-through, like the main run-through proper, is subdivided into two steps of operation. First of all the text passages in the material are marked in which the category concerned is addressed. These "points of discovery" (cf. Hausser et al. 1982) can be marked by noting the category number in the margin of the text or through differently colored underlinings in the text itself. In the second step the material thus marked is processed in accordance with the structuring intention (see below) and copied out of the text. As a rule this trial run-through results in a revision and partial reformulation of the category system and its definitions. Now the main material run-through can finally begin, again split up into the two stages of marking the points of discovery and extracting and processing them. This general description of a structuring content analysis can be shown in a procedural model (Fig. 13.2).

To further explain the procedure for all techniques of Qualitative Content Analysis, rules of interpretation have been formulated. Those step-models and content analytical rules are explained in detail in Mayring (2010a, 2013). Here we just demonstrate the idea of rule-orientated text analysis.

In the above-mentioned study on learning emotions (Glaeser-Zikuda and Mayring 2003), we developed a category system with 3-point ordinal scales (much – some – no) for four central learning emotions: joy, interest, anxiety and boredom. We established a coding guideline containing definitions, anchor examples and coding rules for those 12 categories. Every student was coded one value (much – some – no) for the four emotions. We divided the sample into high and low achievers and compared the emotion results. In the material (interview, learning diary) of high achievers, significantly more joy was coded ($p<0.05$; Mann–Whitney-U $=42.00$), more interest ($p<0.05$, Mann–Whitney-U $=12.50$), but no significant difference in boredom was found.

To conduct a qualitative content analysis (inductive or deductive) it would be very helpful to use computer software, because most of the texts today are already data files and because normally we are handling huge amounts of texts. There are several programs available (Computer Assisted Qualitative Data Analysis, CAQDAS) and it is possible to use them for Qualitative Content Analysis, even if they are more orientated on Grounded Theory. But it is not easy to implement all content analytical steps and procedures adequately. So in the meantime we have developed special open access software which supports exactly the steps of qualitative content analysis (www.qcamap.org).

Fig. 13.2 Process model of deductive category application (structuring)



13.5 Final Appraisal of the Qualitative Content Analysis

First, compare the procedures of Qualitative Content Analysis with similar approaches of qualitative oriented social science text analysis (cf. Mayring 2010b).

Within media analysis, David Altheide (1996) has developed a procedure ("ethnographic content analysis") working with deductive categories (codes), which were refined in the process of analysis. Then he summarizes the results for each category. This has similarities with our approach but is not so rule-oriented as Qualitative Content Analysis. In the USA there exists an approach coming from quantitative content analysis which is called Codebook Analysis (Neuendorf 2002). It is a deductive category application procedure, which defines all categories in the codebook and gives examples from the text. But this definition is not so systematic as the coding scheme (definitions, anchor examples and coding rules) in our procedure. In some ways similar is Thematic Text Analysis (Stone 1997), which looks in the text for central themes, using theoretical preconceptions or empirical word frequencies and word contingencies. In both cases Qualitative Content Analysis defines the procedure more precisely. The related concept of Theme Analysis covers more free, phenomenological procedures (Meier et al. 2008). Some similarities can be found between Qualitative Content Analysis and text analysis following Berg

(2004). He describes deductive ("analytic") and inductive ("grounded") categories which have to be defined explicitly, but it remains unclear how this has to be done.

In comparison with those text analytical approaches, Qualitative Content Analysis seems to be most broad (describing a wide set of different procedures) and most exact (prescribing clear step models and analytical rules). So Steigleder (2008), after a praxis test of Qualitative Content Analysis, comes to the conclusion that "it has proven its worth in many studies. With its different techniques of analysis and its methodological concept it is excellently adapted to analyse qualitatively collected material" (Steigleder 2008, p. 197). But it should not be argued that Qualitative Content Analysis is the only legitimate text analysis procedure. It depends on the concrete research question and the quality of the material which procedure should be chosen. If use of the strict category relatedness and rule orientation of Qualitative Content Analysis neglects important deeper aspects of the material (e.g. repressions in the sense of psychoanalysis), then other procedures (e.g. psychoanalytical text interpretation) would be more adequate.

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Chapter 14
A Study on Professional Competence of Future Teacher Students as an Example of a Study Using Qualitative Content Analysis

Björn Schwarz

Abstract Subsequent to the general description of Qualitative Content Analysis as described in the preceding chapter of Philipp Mayring this chapter aims to give a concrete example of a study based, from a methodical point of view, on Qualitative Content Analysis. The study described for this purpose focuses on structures of professional competence of future mathematics teachers. Based on the concept of competence by Weinert ((2001). Concept of competence: A conceptual clarification. In D. Simone Rychen & L. Hersh Salganik (Eds.), *Defining and selecting key competencies* (pp. 45–65). Seattle et al.: Hogrefe & Huber.) and common distinctions of teachers' professional knowledge (e.g. Shulman, *Educational Researcher, 15*(2), 4–14, 1986) a questionnaire was developed and evaluated by means of Qualitative Content Analysis. This chapter emphasises the methodical aspects of the study and only subordinately considers its results.

Keywords Qualitative content analysis • Professional competence

14.1 Introduction

This chapter describes a qualitative study which uses the methodical approach of Qualitative Content Analysis in order to illustrate the general presentation of Qualitative Content Analysis by Philipp Mayring in the preceding chapter (Chap. 13). The study thereby focuses on the professional competence of future mathematics teachers and Qualitative Content Analysis was used for evaluating the future teachers' written answers to open questions. This chapter addresses the methodical approach of the study, rather than a detailed description of the study as a whole which can be found in Schwarz (2013). Hence in the next section only a brief sketch

B. Schwarz (✉)
Fakultät für Erziehungswissenschaft, Arbeitsbereich Mathematikdidaktik,
Universität Hamburg, Von-Melle-Park 8, 20146 Hamburg, Germany
e-mail: bjoern.schwarz@uni-hamburg.de

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of the theoretical framework is given. The main part of the chapter focuses on the data-evaluation within the framework of Qualitative Content Analysis. In this context then first the question why and second the question how Qualitative Content Analysis was applied in this study are discussed before this chapter closes with a summary.

14.2 Theoretical Framework and Research Question of the Study

The problem of how to describe teachers' professional competence is neither a new nor an already solved issue. Instead many different approaches have been developed with different emphases or from different perspectives (for an overview see for example Baumert and Kunter 2006). Furthermore several recent large scale comparative studies have been carried out in order to evaluate the efficiency of teacher education (e.g. the international comparative studies about the education of future mathematics teachers TEDS-M (for future primary teachers, Blömeke et al. 2010a; for future secondary teachers, Blömeke et al. 2010b) and MT21 (Blömeke et al. 2008), for further research with regard to the efficiency of teacher education see Blömeke (2004)) or to evaluate the professional competence of practicing teachers (e.g. with regard to German mathematics teachers COACTIV (Kunter et al. 2011)). Despite their individual differences with regard to their conceptualisations all these studies more or less share an underlying understanding of the concept of competence based on the approach of competence formulated by Weinert (2001). His understanding is the following:

> The theoretical construct of action competence comprehensively combines those intellectual abilities, content-specific knowledge, cognitive skills, domain-specific strategies, routines and subroutines, motivational tendencies, volitional control systems, personal value orientations, and social behaviors into a complex system. Together, this system specifies the prerequisites required to fulfill the demands of a particular professional position, of a social role, or a personal project. (p. 51)

One immediately identifies the fundamental distinction between a cognitive part and a more affective part in this understanding of competence, both of which again can be conceptualized in many different ways. Prominent examples of the cognitive part, with regard to teachers, are the distinctions of different areas of teachers' knowledge formulated by Shulman (1986) and their subsequent serial works and concretisations (e.g. Bromme 1997, 1994).

The study described in this chapter arises from the research context of MT21 ("Mathematics Teaching in the 21st Century") and was conceptualized as a national supplementary study to MT21. MT21 in general was an international comparative study about the efficiency of teacher education which was conducted in six countries. It thereby approached the research topic by a multi-dimensional approach including the individual perspective of teacher education as one dimension. With regard to this dimension in MT21 all phases of teacher education were taken into account and respectively future teachers in the beginning, in the middle and at the end of their education were asked to participate in the study (Blömeke et al. 2008). While on one hand, in line with MT21, the study described here also focuses on future teachers of mathematics, on the other hand in contrast to the wider defined group of participants in MT21 the study focuses only on future teacher students who are still in their first phase of teacher education taking place at a university.

With regard to its theoretical conceptions the study, like MT21, starts from the concept of competence by Weinert (2001) as described above with its distinction between cognitive and affective parts. To provide a more detailed concretisation and operationalization of this concept with regard to future mathematics teachers the study exclusively focuses on content-related parts of the future mathematics teachers' professional competence. More specifically, this means taking into consideration, for the cognitive side of competence, the areas of content knowledge and didactical knowledge (in a sense of a combination of pedagogical content knowledge and curricular knowledge) related to the work of Shulman (1986) and Bromme (1997, 1994), and for the affective side of competence a focus on mathematical beliefs (Grigutsch et al. 1998; Pehkonen and Törner 1996). These selected parts of professional competence are then further divided into subareas. So for example didactical knowledge is divided into the subareas of teaching-related didactical knowledge on one side and learning-process-related didactical knowledge on the other side. Beliefs are divided into beliefs focusing on mathematics itself on the one hand and beliefs focusing on the teaching and learning of mathematics on the other hand.

Taking this theoretical framework as a basis, the research question then is: What kind of structures can be reconstructed between the selected parts of professional competence for future mathematics teachers? Structures here more precisely can be understood as frequent common appearances of manifestations of the different areas of professional competence. In this context the methodical approach of the study is described in order to illustrate the use of Qualitative Content Analysis.

14.3 Why Was Qualitative Content Analysis Chosen?

Some fundamental methodical decisions with regard to the design of the study are described below. As these characteristics of the study are not exclusively related to Qualitative Content Analysis but are of a more general nature they are only shortly sketched in order to set the focus of this paper on the use of a Qualitative Content Analysis.

So starting with basal methodical characteristics of the study, the first of these fundamental decisions was whether the study should be qualitative or quantitative in nature. Here the aim of the study dictated the decision for qualitative research. The study was not intended to identify relations within the professional competence of the future mathematics teachers in the sense of statistical correlations between dimensions of competence. Instead the research goal was to be able to find and describe on a more detailed level characteristics of the structures that existed between components of professional competence. It was accepted that this goal of more detailed descriptions would yield only hypotheses (for example the hypothesis that there is an influence of future teachers' individual representations of areas of content knowledge on the application of this content knowledge in the context of pedagogical content knowledge, see below) rather than representative results, which, however, could be used as a starting point for a subsequent quantitative study (Mayring 2008; Flick 2006). Against this background of the choice of qualitative research, the decision was made to use a questionnaire in order to be able to survey more future mathematics teachers than would have been possible using an interview approach. Thus, a questionnaire with five[1] tasks was developed with each task assigned to a mathematical and school related background, for example focussing on a task for pupils or a mathematical theorem discussed in school. Furthermore, each task consisted of several subtasks, each of which was assigned to one of the parts of professional competence as described above.[2] With regard to the underlying areas of mathematics, all tasks were assigned to either the area of mathematical modelling (Kaiser et al. 2011) or to the area of argumentation and proof (Hanna 2000) in order to try to find areas which represent both a more dynamic and a more static side of mathematics (cf. Grigutsch et al. 1998). Following the qualitative fundamental structure of the study then all subtasks were open. The questionnaire was answered by 79 future teachers who were students in either an earlier or later phase of their university studies, and who intended to teach in either primary or secondary schools.

Given these fundamental methodical characteristics of the study it was decided to analyse the data with a Qualitative Content Analysis according to Mayring (see Chap. 13, also Mayring 2000). A first and central reason for the decision was that fundamental possibilities for analysing the data offered by Qualitative Content Analysis, in comparison to other methods, fit very well with the aims of the study. The study was intended to search for structures within the professional competence of future teachers, which precisely equates to structuring as the third of the "Basic procedures or techniques of Qualitative Content Analysis" (Chap. 13, Sect. 13.4) named by Mayring. More precisely the approach of scaling structuring (cf. ibid., Mayring 2008) was used as it was meaningful to introduce the idea of scales for describing the formations of the future teachers' areas of professional competence. The latter results from the study's focus on different areas of knowledge and beliefs, that is on areas of professional competence that can be held in different degrees (namely one could know more or less or has a stronger or less strong belief or stronger or less strong affirmation or disaffirmation towards something). The possibility

[1] Only four of these five tasks formed the basis for the final data evaluation. For details see Schwarz (2013).

[2] Often within one task more than one subtask was assigned to a respective part of professional competence. In summary 20 subtasks were evaluated of which nine focused on didactical knowledge, five on content knowledge and six on beliefs.

of distinguishing between different degrees implies ordinal scales which therefore were often used in the study.

This using of ordinal scales thereby especially offered advantages when answers were to be distinguished in a hierarchic way as ordinal scales allowed coding beyond a dichotomous distinction of the answers. Instead finer grades between different answers could be distinguished. An example for this is the coding of a subtask focussing on mathematical content knowledge. With regard to future teachers' professional competence one might agree that there is a difference between answers which contain a substantial mathematical mistake and answers which contain a minor mathematical mistake such as a calculation error. A coding using ordinal scales here allowed a distinction between such answers, in contrast to a coding which only distinguishes between right and wrong answers and therefore would cover both kinds of answers as wrong and thereby treat them equally. Another example can be found when two answers are both correct but one is more elaborated or covers more relevant aspects than the other answer. Then again a dichotomous distinction between the answers would both equally cover them as right while the use of ordinal scales instead offers the possibility of evaluating one answer as of a higher adequateness than the other. Nevertheless in addition to the use of ordinal scales also nominal scales were used when a distinction between different answers without grading them was more appropriate.

A second also central reason for the decision to choose Qualitative Content Analysis was the already existing theory about the different areas of professional competence, for example with regard to didactical or mathematical knowledge or with regard to different beliefs on mathematics and the teaching and learning of mathematics. Therefore analysing the future teachers' answers on the open questions required two aspects. On the one hand all qualitative analysis of data in general contains an act of individual interpretation of the material in the sense that different persons can have different connotations or a different understanding of the same material. In this regard it was necessary to interpret the future teachers' answers as they were formulated in an open way. But, on the other hand, there is already an existing theory about what the future teachers are writing about. So despite the need of interpreting the future teachers' answers it was also an appropriate methodological decision to follow precise rules when evaluating them in order to take this theory into account. This for example covers the evaluation of how adequate an answer is. Thus in the case of the study, necessary consideration of existing theories while analysing the qualitative data was also offered by Qualitative Content Analysis. More specifically, this method is even centrally characterized by aspects which allow a controlled data evaluation based on and considering existing theory, namely aspects of "systematic, rule-bound procedure" (Chap. 13, Sect. 13.3), "categories as the focus of analysis" (ibid.), and the "theory-guided character of the analysis" (ibid.).

The third, and final, reason for choosing Qualitative Content Analysis was related to the practical requirement of evaluating quite a large amount of data. This amount directly resulted from the aim of enriching the qualitative study's results by considering—for a qualitative study—a quite big sample of 79 future teachers. Thus, a methodical approach was necessary which on the one hand in line with the previous reason allows a rule-bound categorization of the material, but furthermore also is "economical" on the other hand in order to, at the same time, facilitate the evaluation of all questionnaires. Both aspects are fulfilled by Qualitative Content Analysis as they were even the starting point for developing this method (cf. ibid.).

One of the fundamental "Basics of Qualitative Content Analysis" (ibid.), even the first Mayring names in the homonymous section of his chapter, is the "embedding of the material within the communicative context" (ibid.). That means that "the material is always understood as relating to a particular context of communication" (ibid.) and "examined with regard to its origin and effect" (ibid.). Often this means that the researcher analyses communication that took place between one or more senders and one or more recipients. Mayring (2008) thereby describes a communication model basing on the model of Lagerberg (1975). Therein "source of information", "sender", "product", "receivers" and "target group" (ibid., p. 275) are distinguished. Mayring (2008, p. 50 ff.) extends this model and especially emphasizes background variables of the participants involved in the communication, the role of the researcher as an analyser of the product, information about the context in which the communication takes place, and an analysis of the communicated text. This communication model generally enables the interpreter to "specify which part of the communication process he wishes his conclusions from the material analysis to relate to" (Chap. 13, Sect 13.3). Transferred to this study the act of communication based on the fulfilling of the questionnaires by the future teachers[3] and the following reading of the questionnaires by the researchers. This means that the text was not produced by the senders for a target group and in addition is analysed by the researchers. Rather here the target group, the actual receiver, and the researcher accord in the sense that the questionnaires were originally written by the future teachers as senders in order to be analysed by the researchers. Hence furthermore the reader of the study in addition can be seen as part of the target group and as a receiver. Regarding the background of the senders, the future teachers knew that the questionnaire was part of a study, so here the senders explicitly took part in the act of communication under the perspective of participating in a study. Therefore one can assume that the future teachers as senders did not connect any further intentions with the communication going beyond participating in the study. All senders furthermore have a common background insofar as they all are mathematics teacher students and in contrast differ in their background for example with regard to the phase of their study or the school level they later want to teach in. The context of the communication then is the framework of university teacher education for future mathematics teachers. Regarding the transmitted product, in this case it was fixed already by the senders as the future teachers in the act of communication wrote down their answers. This is also important as it can be expected that the necessity of writing down an answer instead of verbally formulating it influences the characteristics of this answers, for example with regard to the elaboration or length of an answer. In summary using this communication model the study is an attempt to evaluate a product in order to formulate hypotheses about the professional competence of the group of senders of the product.

[3]This also includes considerations of the future teachers about how to solve the task which under the perspective of describing the act of communication can be seen as internal argumentations.

A further aspect which combines more practical and more theoretical aspects of analysing data is the question of a possible segmentation of the data. Having any kind of material to be analysed there is always the necessity of "deciding in advance how the material is to be approached, which parts are to be analyzed in what sequence" (ibid.). This leads to the "definition of content-analytical units" (ibid.) which in general also represents the "systematic quality of content analysis" (ibid.). In this study the segmentation of the material in units could be directly derived from the research question mediated by the structure of the questionnaire. The theoretical-based distinction between different areas of professional competence in the research question corresponds with the concept of separate subtasks in the questionnaire. Each subtask thereby is related to one of the areas of professional competence. The segmentation into different units of interpretation then follows the structure of the questionnaire with these different subtasks. Thus each answer to a subtask is taken as one unit of analysis. Also the sequence of interpreting the different answers could be adopted from the structure of the questionnaire. In this regard the answers were evaluated following their position in the questionnaire. This way, the coder knew what the future teacher had answered in the preceding subtasks referring to the same content. Therefore the coder could consider possible implicit references to previous answers when interpreting the actual answer.

14.4 How Was Qualitative Content Analysis Used in This Study?

This section specifies the particular methods used in this study to evaluate the collected data. The choice of methods reflect the position that there is not *the* method of Qualitative Content Analysis but a compilation of possible forms of analysis covered by Qualitative Content Analysis. Furthermore "the methods […] must themselves always be adapted to suit the individual study" (ibid.). Thus the following concrete example of using Qualitative Content Analysis like all forms of using this form of data evaluation needs to be considered within the context of its particular theoretical framework and research question.

The first part of data evaluation was carried out using the scheme of "deductive category application" (ibid., Sect. 13.4) described by Mayring (ibid.). First, before the actual coding of the data began, manuals were formulated that contained all relevant information for the coders. According to Mayring (ibid.) these manuals consist of a definition of the possible values of each category, as precise as possible formulated coding-rules for each value and finally so-called anchor examples for each value taken from the data that illustrate either very typical or border examples for a respective value. Data evaluation was then carried out by *deductive* coding[4] of all subtasks by two independent coders.

[4] In this chapter the following terminology is used: The *process* of allocating one of the possible values of a category to a certain future teachers' answer is called coding. If this coding is done by using a deductively developed coding manual, it is called deductive coding, while it is called inductive coding whenever an inductively developed coding manual is used. The *result* of coding all future teachers' answers to one subtask according to a certain coding manual is called a code of

With regard to deductive coding, one of the “especially important” (ibid.) aspects for Mayring is that “the fundamental structuring dimensions must be exactly determined; they must derive from the research question and must be theoretically based.” (ibid.). In this study these dimensions directly arose from the research question and the related theoretical framework. As described above, the study is theoretically based on the distinction between different areas of future teachers’ professional competence, more precisely, the areas of content-knowledge and didactical knowledge and mathematical beliefs. Therefore it was meaningful to conceptualize these areas of professional competence—or more precisely, the corresponding subareas as described above—as the fundamental dimensions for structuring. The detailed differentiation of these dimensions into particular categories for coding then was derived from the questionnaire. As every subtask was coded separately the precise definition of the different categories followed from the foci of the subtasks. For every subtask its theoretical based reference point served as category for the coding of the subtask. Thus in summary the theoretical basis of the categories followed from the theoretically-based development of the questionnaire.

Also the various values of a category and its belonging definitions and coding rules were developed theory-based. As every category is related to a certain subtask, the conceptualisation of the values of a category can be derived from the theory about the topic the subtask focuses on. For example the values of a category belonging to a question on mathematical content knowledge could be developed by considering which characteristics form a more or less adequate answer with regard to the mathematical theory about the subtask’s topic.

It is important to be aware that although Qualitative Content Analysis is strongly rule-guided and follows documented procedures for data evaluation it still is a qualitative method which involves interpreting the data. Therefore as Mayring (ibid., Sect. 13.3) points out an important quality criterion for the evaluation of data within the framework of Qualitative Content Analysis is intercoder-reliability as a measure of how strongly different coders agree in their estimation of the same data. It can be expected that this intercoder-reliability is connected to the precision of the coding manuals in a sense that more precise coding manuals allow a more distinct coding and thus a higher accordance of the codes of different coders. Nevertheless as all coding contains an act of interpretation one can not expect a full accordance of different coders. In this context often the intercoder-reliability is summarized by a measure like Cohen’s Kappa (Mayring 2000; for Cohen’s Kappa, Bortz 2005, Bortz et al. 2000) and as full accordance can not be expected Mayring (2000) advises 0.7 as a sufficient value.

In this study, in order to raise the inter-coding agreement, the two independent coders were well familiarized with the theoretical background of the study and hence Cohen’s Kappa of 0.7 was reached in all but one subtask. In order to come to one conclusive code which can be used for the following analyses at the end of the first part of the data evaluation deviant codes made during the deductive coding were discussed between the coders and one code was consensually decided (cf. Schmidt 1997). As expected, the necessity of interpretation or in other words the

the subtask. A code resulting from deductive coding is called a deductively defined code while a code resulting from inductive coding is called an inductively defined code.

Within an investigation the following problem was given to pupils in class 8:

> *Violet collects different seashells from a beach and puts three of them on a scale at home. They weigh 27g all together. After that she puts two more seashells on the scale. Which weight does the scale indicate now? Explain your answer.*

Fig. 14.1 Starting point for the tasks for the future teachers in the questionnaire described in the following focusing on a reality-related task for pupils (Basically see Cukrowicz et al. 2002, p. 29, see also Kaiser et al. 2003, p. 98 and Maaß 2004, p. 337) (translated)

possibility of formulating a non-ambiguous coding manual that allows for a high agreement between different coders strongly depends on the underlying content of the respectively coded subtask. For example, higher coder agreement will occur the more precisely one can estimate the correctness of a mathematical answer or the adequateness of a didactical reflection. On the other hand, lower agreement between coders occurs whenever more interpretation is required in estimating the future teachers' answers to questions concerning beliefs or knowledge.

To further illustrate this issue of inter-coder agreement as well as to clarify the study's process of data evaluation, the following example presents a particular subtask and its deductively developed coding manual. Starting point is a reality-related mathematical task (Fig. 14.1) which originally was designed as an exercise for pupils (basically see Cukrowicz et al. 2002, p. 29). It was used in and therefore taken from both the SINUS-project in Hamburg (Kaiser et al. 2003, p. 98) and the study of Maaß (2004, p. 337) and again served in these studies as an exercise which was given to pupils in order to analyse their solution approaches.

The reality-related task for the pupils thereby apparently has no distinct solution. Anyhow furthermore it is not impossible to formulate any answer to the question based on the given information either (in contrast to so-called "captain-exercises", cf. Stern 1992). Rather it is possible to meaningfully apply the information given in the task in order to answer the question. In the questionnaire, to probe for mathematical content knowledge, the future teachers were asked, "How would you answer this question?" (subtask 2a): Figure 14.2 describes the deductively developed coding manual for subtask 2a[5].

With a value of Cohens's Kappa of .962 this subtask was one of the subtasks in which the coders mostly agreed when independently coding which most likely was based on the possibility of a quite clear distinction between different future teachers' answers. As expected, this accordance between the coders goes down somewhat when the future teachers' answers can be interpreted more differently, as with the

[5] It should be kept in mind that all coding manuals were formulated in order to be most applicatory for the coders and not in order to be most suitable for publication.

Coding manual for subtask 2a

2 a) "How would you answer this question?"

Deductive coding of the subject-related adequateness of the solution and reflection with regard to the reality-related mathematical task

Value	Definition	Coding rules
+2	Very high competencies when dealing with reality-related mathematical tasks are evident from a subject-related adequate solution approach, which furthermore features a meaningful usage of the given numbers.	The solving strategies in this value are characterized by: 1. a context-related reflection about the ambiguity of the solvability of the task which can also be indicated by key words, estimations etc. 2. a meaningful usage of the numerical data going beyond the context-related reflection (see 1.) and rule of three. This can for example be done by giving an estimation of the interval of solutions. Explanation: The meaningful usage of the numerical data required in 2. is very meaningful for tasks without a unique solution. This indicates that the student reflected more upon the transition between reality and mathematics.
+1	High competencies when dealing with reality-related mathematical tasks are evident from a subject-related adequate solution approach, but which does not feature a meaningful usage of the given numbers.	The solving strategies in this value are characterized by: 1. a context-related reflection about the ambiguity of the solvability of the task which can also be indicated by key words, estimations etc. 2. a meaningful usage of the numerical data going beyond the context-related reflection (see 1.) which does not go beyond the rule of three.
0	Average competencies when dealing with reality-related mathematical tasks are evident from a reflection of a solution approach without using the numerical material at all or with the wrong usage of the given numbers.	The solving strategies in this value are characterized by: 1. a context-related reflection about the ambiguity of the solvability of the task which can also be indicated by key words, estimations etc. 2. The context-related reflection about the ambiguity of the solvability of the task entails that the numerical data from the task either is not used with regard to the solution or is used for incorrect calculations.
-1	Low competencies when dealing with reality-related mathematical tasks are indicated by solving-strategies without further reflections however without mistakes.	There is no context-related reflection about the ambiguity of the solvability of the task; instead, the simple rule of three is used to solve the task mathematically.
-2	Very low competencies when dealing with reality-related mathematical tasks are indicated by solving-strategies without further reflections and with mistakes or by an inadequate mathematical proceeding.	There is no context-related reflection about the ambiguity of the solvability of the task; instead, the simple rule of three is used to solve the task mathematically and the calculation contains errors. Or There is no context-related reflection about the ambiguity of the solvability of the task and the solving strategy is inadequate, or does not lead to a result.

Value	Anchorexamples
+2	"A weight between 30 and 70g. Tiny light shells can be added as well as bigger heavier ones. Thinking about shells from the Baltic and the North Sea, the weight will not exceed 70g. In countries that are further in the south this would probably be different." "Hard to answer: Without further knowledge about the size of the five shells, no result referring to reality can be given. Otherwise: Exemplary assumption: All shells weigh the same. Then, the result is about 45g. It is also possible to give a range of results: e.g. result ∈ [28g; 77g]"
+1	"There is no definite answer because we do not know how much the different shells weigh. If we did, the answer would be 45 g." "Hard to answer when you do not see the shells (size, shape). Otherwise, if forced, I would calculate as follows: 27gr. : 3 = 9gr. 9g × 2 = 18gr. 18gr. + 27gr. = 45gr." "I would assume that the first three shells are representative for the other big/heavy shells. Then calculate the weight of an average shell. Calculate the weight of the two new shells like this and add it to the other."
0	"This cannot exactly be stated. Firstly, I do not know the size and weight of each of the first three shells. And secondly I do not know this about the two which I add to the first either." "More than 27g. A more precise answer is not possible as there is no information about the size of the shell. Probably the first three shells are very big (small) and the other two shells are small (big)?" "Not at all because I cannot infer the weight from three shells to others." "The question cannot be answered precisely because the shells can be of very different size and weight (also apart from the fact that we do not know how well Violet cleaned them from sand). It could be possible to get a book about shells and look for data that state the lower and the upper limit."
-1	"rule of three: 3 shells = 27g 1 shell = 27/3 g 5 shells = 27g/3 × 5 = 9 × 5=45g"
-2	"As I have to assume that the shells don't always weigh the same, the weight of the two added shells can only be inferred by using a set of linear equations with two variables."

Comment: Solutions which contain a division by 1, e.g. 27g/3=9g/1 are handled as solutions with a rule of three.

Fig. 14.2 Deductively developed coding manual for subtask 2a case of evaluating how adequate a future teachers' answer is with regard to didactical knowledge or in the case of evaluating an answer related to beliefs. However also in these cases a theory-led distinction between different answers could be formulated upfront with regard to the respective theory. In the case of didactical knowledge, for example, content-related didactical analyses about respective topics or didactical principles formed the basis for formulating deductively developed coding manuals which distinguish between more and less adequate answers. With regard to a distinction between different answers related to beliefs, for example, the relevant theory about different categorizations of mathematical beliefs could serve as a starting point for developing respective deductively developed coding manuals.

The above is an example of how the process of deductive coding was applied to task responses. For this type of evaluation, coding manuals were formulated a priori and at first independently of the existing data. Especially in the case of subtasks focusing on areas of knowledge this deductive coding mainly focused on the adequacy of the answer.

Deductive coding was followed by a second part of data evaluation that focused on inductive coding. It was not to be expected that all aspects of the collected material which are relevant with regard to the research question could be evaluated by the a priori formulation of coding manuals. Rather, the future teachers' answers were found to differ in characteristics that were not covered by the a priori rules, for example when answers, despite their differences, were equal in their adequateness while the a priori rules covered the degree of adequacy of the answers. Examples for these different characteristics included different ways of solving a problem, different ways of formulating an answer or different focal points included in the answer. These different characteristics were noted by the coders during the first coding process that used the deductively developed coding manuals. Afterwards the impressions of additional possibilities for distinguishing answers were discussed by all coders and led to new manuals each focussing on one of these characteristics according to which the answers also differ. As these manuals and their development are based on the impression of the material they are inductively developed (Chap. 13, Sect. 13.4). Often during this phase more than one additional inductively developed coding manual was developed for a respective subtask as the coders often noted more than one characteristic according to which the answers to the respective subtask differ. In this case for each characteristic taken into consideration a separate inductively developed coding manual was developed and therefore more than one inductively developed coding manual could be allocated to one subtask. Then all subtasks were again coded according to the newly formulated inductively developed manuals. In contrast to the previous part of the data evaluation, in this part, all subtasks were coded by only one coder and only answers difficult to code were discussed with other coders. This restriction to one coder was, on the one hand, a practical choice and, on the other hand, related to the fact that, due to the inductive development of the coding manuals, a good and quite distinct applicability of the determined rules in order to code the concrete material was to be expected, as the determined rules were derived from this material.

So the basic principle of a “process model of inductive category formation” (ibid.) was used but in a way that was strongly adapted to this particular study. Thus for example the criteria according to which the material was later distinguished also arose from the material, in contrast to the theory-based criteria used for coding in the preceding first phase of the data evaluation. This example illustrates how the general concept of the different approaches offered by Qualitative Content Analysis can and have to be adopted in every case of data evaluation to the respective study (ibid., Sect. 13.3).

The inductive phase of data-evaluation is illustrated by the following example. The respective subtask is again focused on the reality-related task about the shells. In subtask 2b the future teachers were given three different pupil solutions:

- Solution I is based only on the rule-of-three
- Solution II is based only on reflections about the real world context
- Solution III includes both a mathematical attempt using the rule-of-three and reflections about the real world context.

To probe their didactical knowledge, the future teacher students were asked the following: “What would you tell each of the three students if you were their teacher? Please explain your decision.” The future teachers’ answers were first evaluated using deductively developed coding manuals under the perspective of whether their answers were adequate or not with regard to didactical knowledge. During this process the coders, among others, observed that different future teacher’s responses to a pupil’s answer were sometimes, on one hand, quite similar with regard to the estimation of the pupil’s answer but, on the other hand, differed with regard to the usage or not usage of commendations. This observation then led to the development of an inductively developed coding manual which focuses on whether or not the future teachers formulate a commendation for the respective pupil’s solution independent from how adequate the respective future teacher’s answer is. This inductively developed coding manual is described in Fig. 14.3. Thereby as well as in the deductively developed coding manuals also in the inductively developed coding manuals all anchor items directly arose from the material.

Following these phases of the data evaluation, all subtasks were coded according to the deductively and inductively developed coding manuals. However, this was not sufficient to identify structures between the areas of competence as was needed. Instead these structures can be observed when codes are related to each other. Thus, in the third and final part of the data evaluation different codes were set into relation in pairs. Here different combinations of types of codes are possible and occur. More precisely this means that deductively defined codes can be set into relation with other deductively defined codes and inductively defined codes can be set into relation with other inductively defined codes as well as deductively defined codes can be set into relation with inductively defined codes.

From the observations on these relations of codes finally hypotheses were derived in order to answer the research question. A part of such an analysis is given below in order to briefly illustrate the procedure. In general codes resulting from the previously described codings again were taken into consideration and the common distribution of two codes at a time was analysed. Thereby the search for structures

Coding manual for subtask 2b

2 b) "What would you tell each of the three students if you were their teacher? Please explain your decision."
Inductive coding of the usage of commendation for the different pupil's solutions

In general: Each future teacher's answer is coded with regard to every pupil's answer separately, that means that usually one future teacher's answer is allocated to three codes.

Value	Definition	Coding rules
Commendation of the pupil's answer	The respective pupil's solution is brought out by a commendation in the future teacher's answer.	This code is given if a commendation is explicitly related to the respective pupil's solution. Thereby in a wider sense all positively orientated verbal emphases of the respective pupil's solution are considered a commendation.
No commendation of the pupil's answer	The respective pupil's solution is not brought out by a commendation in the future teacher's answer.	This code is given if no commendation is related to the respective pupil's solution in the sense described above.

Value	Anchorexamples
Commendation of the pupil's answer	"Solution I: His solution is quite good, but is it that all shells are precisely equal? Solution II: Good, but how about equal shells? Solution III: Very good! So I would try to watch that the pupils realize that there could be different solutions."
No commendation of the pupil's answer	"Solution I: I would ask him whether he has seen shells before. If so, I would ask him whether it is realistic, that all shells weigh the same. Afterward I would ask him in which sense his calculation is not wrong (estimation). Solution II: I would ask her what she estimates how heavy the shells are. Solution III: Why he has done it in this way."

Fig. 14.3 Inductively developed coding manual for subtask 2b

within the future teachers' professional competence in the sense of the research question was especially focused on identifying structures which could be found often in the data. This focus on more frequent structures was laid because it was not the aim of the study to reconstruct a preferably broad variety of possible structures independent of their distribution in the data. Rather especially frequent and in this sense typical structures were to be reconstructed so as to derive hypotheses therefrom which might have the potential to be confirmed in more general samples. Thus quantitative methods were used for the analysis in the sense that frequencies are compared. More precisely, in particular the frequencies of combinations of values were considered and interpreted by the meaning of the values in order to therefrom derive descriptions of structures.

For the following example of the analysis, especially those future teachers' answers were taken into consideration which in general contain commendations but in which commendations of only one or two pupils' solutions are included. That means that all future teachers' answers in which all three pupil's solutions received commendation were not regarded. This was done because a future teachers' use of only one or two commendations was interpreted as an indication that the future teacher used commendations to specially emphasize certain pupil's solutions. This emphasis normally is expected to result from a special appreciation of the respective pupil's solution. This selective use of commendations was to be distinguished from the use of commendations for all three pupils' solutions which was interpreted as a use of commendations by the future teachers to generally motivate the pupils. Table 14.1 then describes the common distribution of the respective inductively defined code of 2b and the deductively defined code of 2a. (Because of the possibility of using the commendation once or twice, future teachers' answers in which two pupil's answers are allocated to a commendation are counted twice)[6]:

Table 14.1 Common distribution of the deductively defined code of subtask 2a and the inductively defined code of subtask 2b

		Inductively defined code of subtask 2b with regard to the usage of commendation for the different pupil's solutions Commendation of …		
		Solution I	Solution II	Solution III
Deductively defined code of subtask 2a with regard to the subject-related adequate solution and reflection of a reality-related mathematical task	**–2**	0	1	1
	–1	6	2	5
	0	2	12	9
	1	2	13	12
	2	2	3	3

[6] It may be noted that because of the restriction to only take those future teachers' answers into consideration that allocate a commendation to only one or two pupils' solutions, quite a lot of future teachers' answers that contain commendations were not taken into consideration in this table, as the majority of future teachers who use commendations then allocate a commendation to each of the three pupils' solutions.

In the exemplary analysis discussed below, especially the answers coded as −1, 0, +1 in subtask 2a are to be discussed. For this, these answers are divided into two groups for further consideration under the perspective of the code of subtask 2a and then the answers are discussed relating both codes to each other, that is in view of the codes of 2a and 2b. The division of the answers into two groups thereby was done according to whether or not the answers contain context-related reflections about the impossibility of formulating a distinct number as solution. The first group consists of those answers coded in subtask 2a with 0 or +1. According to the coding rules described above (Fig. 14.2) these answers contain only, or among others, a context-related reflection about the ambiguity of the solvability of the task. Table 14.1 shows that future teachers formulating such answers often allocated a commendation to the solutions II and III (see above in this section), that means solutions that contain only, or among others, a remark about the impossibility of formulating a distinct solution because of reflections about the real-world-context. The second group consists of answers coded with a −1 in subtask 2a. These are answers which according to the coding rules only contain a usage of the number material in the task to the rule-of-three with no context-related reflection about the ambiguity of the solvability of the task. Again looking at Table 14.1 one can see that future teachers whose answers are coded with −1 in 2a often allocated a commendation to the solutions I and III, solutions that also contain the use of the rule-of-three (see above in this section). In summary, one can see that the future teachers who selectively allocate commendation tend to allocate commendation to those pupil's solutions which are completely or in parts close to their own solution. These observations suggest a hypothesis concerning the future teachers' content knowledge and its influence on their didactical knowledge. More precisely here with regard to the content knowledge individual representations of the future teachers' content knowledge are considered. Thereby the future teachers' preferences in solving mathematical tasks served as an indicator for these individual representations. The application of the didactical knowledge in turn here is represented by the future teachers' formulations of responses to pupil's answers. Under this perspective the described observations suggest the hypothesis that individual representations of the future teachers' content knowledge have an influence on the application of their didactical knowledge related to the respective content knowledge.

This part of the chapter mainly focuses on the methodical aspects of the study. Thus, this exemplary result remains the only one described here. Further results and a more detailed description are contained in the original study (Schwarz 2013) that describe not only the structures between different areas of professional competence of future mathematics teachers but also relations between these areas and the practical teaching experience the future teachers have already gained, for example, in their previous teacher education.

14.5 Summary

In summary the paper tried to give an overview about the methodical approach of a study on the professional competence of future mathematics teachers in order to give a concrete example of a study using Qualitative Content Analysis. The described study, or more precisely in particular the related data-evaluation, thereby is presented in order to give an example of some central characteristics of Qualitative Content Analysis (see Chap. 13).

This chapter illustrates that Qualitative Content Analysis does not consist of one fixed method for evaluating qualitative data but, rather, is made up of a plurality of approaches. The study shows one possibility of how methods from the assemblage of approaches covered by Qualitative Content Analysis can be combined and adapted to a particular study. This accommodation of the methodical approach within the concept of Qualitative Content Analysis to the theoretical framework of the study thereby is one central aspect of Qualitative Content Analysis. In particular, the data-evaluation excerpts described above exemplify how future mathematics teachers' answers to open questions are coded according to both deductively and inductively developed coding manuals, to address the research question related to structures between areas of the future teachers' professional competence. The combination of different coding approaches offers the opportunity to consider both predefined expectations about what the future teachers could answer, on the one hand, and, on the other hand, their personal ways of answering the questions, that is, of expressing their professional competence. Analysing the data according to theory-led developed criteria as well as allowing for attributes to arise from the material offers the possibility to more broadly evaluate the data with regard to the research question.

Furthermore, the study demonstrates that Qualitative Content Analysis is a qualitative approach that is strongly based on rule-guided procedures. This chapter illustrates how categories and related coding manuals serve as focal points for applying Qualitative Content Analysis. While this methodical approach still is clearly a qualitative one involving interpretations of fixed communications, such as written answers, this chapter also illustrates the important role of the intercoder-reliability and the use of coding manuals in the rule-guided data-evaluation within Qualitative Content Analysis.

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Part X
Triangulation and Cultural Studies

Chapter 15
The Contemporary Importance of Triangulation in a Post-Positivist World: Examples from the Learner's Perspective Study

Ida Ah Chee Mok and David J. Clarke

Abstract Triangulation has become a reference construct when issues of methodological rigour are discussed. In this chapter, we argue that conceptions of triangulation must be broadened if it is to be relevant to a community increasingly committed to interpretivist and critical methodologies. We suggest that the metaphoric entailments of triangulation can usefully inform contemporary research efforts and the development of new methodologies, particularly those required by cross-cultural comparative research. Our argument is illustrated by examples taken from the Learner's Perspective Study (LPS). This study examined the patterns of participation in competently-taught eighth grade mathematics classrooms in 18 countries in an integrated and comprehensive fashion, using different theoretical frameworks to address a variety of significant research questions. The complementary accounts generated by the application of the different theories are at the heart of the methodological shift that has required the progressive reconception of triangulation, where the ultimate goal is not a unique finding (proposition or relationship) warranted by a process of cross-validation leading to the convergence of multiple data points on a single truth, but rather the multi-faceted portrayal of a complex social situation (e.g. dyadic collaboration or teacher-led discussion). Acts of cross-cultural comparison are poorly served by the use of triangulation as a mechanism of convergence and benefit from the triangulation of accounts interpreted as complementary. In the case of the LPS, these complementarities are enacted at the level of the participants' social, organisational and cultural affiliations and at the level of the researcher's

I.A.C. Mok (✉)
Faculty of Education, The University of Hong Kong, Room 405,
Runme Shaw Building, Pokfulam Road, Hong Kong
e-mail: iacmok@hku.hk

D.J. Clarke
International Centre for Classroom Research, The University of Melbourne,
109 Barry Street, 3053 Carlton, VIC, Australia
e-mail: d.clarke@unimelb.edu.au

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theoretical affiliations. The paradigmatic shift in the nature and utility of triangulation is captured precisely in this movement from convergence to complementarity.

Keywords Triangulation • Cross-cultural comparison

15.1 Introduction

The Learner's Perspective Study (LPS) (Clarke et al. 2006b) was designed to examine the practices of eighth grade mathematics classrooms in Australia, Germany, Japan and the USA in an integrated, comprehensive way and the project was originally designed to complement other international studies that reported national norms of student achievement and teaching practices. Since its inception, the LPS community has expanded and the research teams now participating in LPS are based in universities in Australia, China (Beijing, Hong Kong, Macau and Shanghai), the Czech Republic, Finland, Germany, Israel, Japan, Korea, New Zealand, Norway, The Philippines, Portugal, Singapore, Slovakia, South Africa, Sweden, the United Kingdom, and the USA. Each research team shares the same research design for the collection of data in their own research site (see http://www.lps.iccr.edu.au/ for details). The resultant shared database includes a rich documentation of lesson videos, lesson materials, post-lesson video-stimulated interviews with focus students and teachers. This complex, interconnected dataset exemplifies "mixed methods" approaches to research (Johnson and Onwuegbuzie 2004), in which a variety of data types relating to the same situation or phenomenon are strategically generated for both qualitative and quantitative analysis. The contemporary recognition of the viability and value of mixed methods research designs (in the paper by Johnson & Onwuegbuzie, in Chap. 12 of this book by Kelle & Buchholtz, and in a very large body of research literature) provides a belated, retrospective acknowledgement of the co-existence of qualitative and quantitative aspects in all studies. Triangulation is not to be either confused or equated with mixed methods research designs. While mixed methods approaches may afford triangulation of data types, the significance of "triangulation" lies in its function as a particular form of design logic: generating evidence through the strategic juxtaposition of design elements. The effects of triangulation of informants, triangulation of research techniques, triangulation of cultural settings, plus triangulation of researchers' theoretical frameworks lie in the inevitable generation of interpretive accounts that must logically be viewed as complementary.

To illustrate this in relation to the LPS: Embedded in the design of the study is the anticipation of multiple entry points for analysis of classroom practice using different theoretical frameworks to address a variety of significant research questions. The researchers' reflection and construction of meaning in the analysis of the various mathematics classrooms is enlightened and enriched by the spirit of collaboration in the LPS community. This collaboration took as its goal the construction of complementary (rather than consensual) accounts of classroom practice. As the founders of the project, Clarke et al. (2006b) stated:

> Complementarity is fundamental to the approach adopted in the Learner's Perspective Study. This applies to complementarity of participants' accounts, where both the students and the teacher are offered the opportunity to provide retrospective reconstructive accounts of classroom events, through video-stimulated post-lesson interviews. It also applies to the complementarity of the accounts provided by members of the research team, where different researchers analyse a common body of data using different theoretical frameworks. (pp. 4–5)

The results of the Learner's Perspective Study are reported in a Book Series, journal papers and conferences. The reports (and the underlying analyses) take many different forms including comparison between teachers within the same school system or culture, comparison of lesson structures and lesson event types between different cultural settings, and case study reports of the practices of a single classroom by juxtaposition of complementary accounts of the lessons from the perspectives of multiple classroom participants. It is these acts of implicit comparison, central to the LPS research design, that make the various LPS analyses such rich sources of examples of triangulation.

The multiple analyses undertaken within the LPS project have been reported in four books. In the first book, *Mathematics Classrooms in Twelve Countries: The Insider's Perspective* (Clarke et al. 2006b), the authors of each chapter are insiders in their own cultures and school systems and carry out their analyses from that position; also, the voices that constitute the data of this research are the voices of the insiders in the classrooms studied—the students and their teachers.

In the second book, *Making Connections: Comparing Mathematics Classrooms Around the World* (Clarke et al. 2006a), the authors addressed a variety of macro and micro level concerns in mathematics education through several comparative analyses of mathematics classrooms in different countries. The various reports demonstrate how the data base generated by the LPS research design affords different levels and units of analysis, from consideration of lesson structure, to the characterisation of the iconic classroom activities we call "lesson events," to the characteristics of mathematical tasks and the fine-grained analysis of language use in different mathematics classrooms. In the same way that comparison is central to LPS analyses, so is collaboration, particularly when collaboration involves the connection of analyses grounded in different theoretical perspectives. It is an irony of design that the commitment to comparison and collaboration should demand a research design in which key elements are kept invariant and the need for consensus definitions is critical.

For example, to facilitate collaboration in the analysis of data, a definition for "lesson event" based on its form and function was developed:

> A 'lesson event,' as we conceived it, was characterised by a combination of form and function, both of which were subject to local variation, but with an underlying familiarity and frequency of use that suggested both cross-cultural relevance and utility. Each individual lesson event had a fundamentally emergent character, suggested by the classroom data as having a form (visual features and social participants) sufficiently common to be identifiable within the classroom data from each of the countries studied. … Each lesson event required separate and distinct identification and definition from within the international data set. (Clarke et al. 2007, p. 287)

The third LPS book, *Mathematical Tasks in Classrooms around the World* (Shimizu et al. 2010), is devoted entirely to research into the role of mathematical tasks. In combination, the various analyses in this book enhance our understanding of the nature of tasks and provide alternative analytical frameworks for investigation of tasks and their role in mathematics lessons; and culturally-specific differences in the nature of the tasks demonstrating the situatedness of the instructional use of mathematical tasks and the need to consider an instructional task in the curricular, organizational and social contexts. The focus of the fourth book, *Student Voice in Mathematics Classrooms around the World* (Kaur et al. 2013), is the occurrence and function of "student voice" in the mathematics classrooms analysed, the identification of patterns of classroom discourse and the relative contribution of student and teacher voice to those discourse patterns.

The aim of this chapter is to utilise selected examples from among the LPS analyses to raise issues associated with the methodological function and significance of triangulation as an essential consideration in contemporary interpretivist research design, particularly with regard to international comparative research. The selected reports illustrate how triangulation was employed to address particular research questions, while also providing an illustration of the capacity of various mixed methods approaches to facilitate several forms of triangulation.

It has been argued elsewhere (Clarke 2003) that we need to attend to the cultural authorship of any research report. Inevitably, a researcher's participation in a research project will reflect values, beliefs and priorities that have their origins in that individual's personal pedagogical, curricular, organisational and socio-cultural situation. The vast majority of reports of educational research reflect a Western and, frequently, American perspective. Culturally familiar perspectives can conceal assumptions and ignore alternative (unstated) possibilities. In an attempt to foreground these considerations, we take the experiences, challenges and findings of the Hong Kong LPS team as our particular example.

15.2 Triangulation

Over the past decades, researchers have taken triangulation and redefined it to meet perceived needs (Denzin 2010). Triangulation was the emerging fad in social science during the formative period in the history of mixed methods research (MMR) (Denzin 2010). This period has been outlined in terms of three moments, namely, paradigm debates in 1980s, procedural development in 1990s, and the advocacy and separate design period since 2000 (Creswell and Clark 2007; Denzin 2010). The paradigm debates were the quantitative-qualitative debates (Johnson and Onwuegbuzie 2004; Kelle & Buchholtz, this book) where the quantitative purists and qualitative purists constructed themselves as mutually exclusive (Johnson and Onwuegbuzie 2004; Denzin 2010). The assumptions of quantitative purists are consistent with what is commonly called a positivist philosophy in contrast to the constructivist and interpretivist positions advocated by qualitative purists. The term triangulation became problematic when the two warring paradigm camps, the quants and quals, became exclusively aligned with positivist and interpretivist paradigms respectively. The debates end with the mixed method research paradigm that sees both quantitative and qualitative methods are important, useful and can be used in combination.

Triangulation has been broadly defined as using different research methods in the study of the same phenomenon (Denzin 1970; Jick 1979). The triangulation metaphor is taken from navigation and surveying, and refers to the use of multiple reference points to locate the position of an object (Jack and Raturi 2006). In a simplistic way, triangulation is often used as a validation technique for research in different disciplines such as nursing and social science, where identification of a single "best method" may be essential to the work of the particular discipline (Denzin 1970, 1989; Flick 2004; Guion et al. 2011; Jack and Raturi 2006; Mathison 1988). Different types of triangulation have been discussed. Denzin discussed four different forms of triangulation, namely triangulation of data, investigator triangulation, triangulation of theories and methodological triangulation (Denzin 1970, 1989; Flick 2004; Jack and Raturi 2006). Triangulation of data combines data drawn from different sources including verbal data such as interviews and group discussions; and visual data such as videos and photos. Investigator triangulation refers to interpretations and analysis of collected data to be carried out in groups, evaluation teams or by several investigators to check for subjective views and to balance out individual views. Triangulation of theories means approaching the data from multiple perspectives and hypothetical assumptions, typically by bringing together people from different disciplines or people in different status positions within disciplines (Guion, et al. 2011). Methodological triangulation can be further differentiated into within-method and between-method (Casey and Murphy 2009). Within-method triangulation, that may be carried out by inviting differently positioned narratives and focusing on differently positioned individuals' accounts of experiences of concrete episodes. This within-method approach intends to create complementary perspectives on the research issues as well as to clarify the different facets of the researcher's inevitably subjective position. Between-method triangulation, also known as across method (Bekhet and Zauszniewski 2012), is the combination of different methods, often qualitative and quantitative methods. This difference is central to an understanding of triangulation: for example, (i) the comparison of two interviews with two participants within the same activity and setting; and (ii) the comparison of interview and survey data relating to the same issue. In each case, triangulation is employed to enrich or strengthen the products of the analytical process.

Why triangulate? The most discussed type of triangulation refers the use of multiple methods in the examination of a social phenomenon (Denzin 1970; Jonsen and Jehn 2009). The objective of using triangulation as a validation strategy is the most discussed (Denzin 1970; Jack and Raturi 2006; Mathison 1988). The use of triangulation as a research strategy is based on some assumptions:

> First is the assumption that the bias inherent in any particular data source, investigator, and particularly method will be cancelled out when used in conjuction with other data sources, investigators, and methods. The second, and related, assumption is that when triangulation is used as a research strategy the result will be a convergence upon the truth

> about some social phenomenon…This assumption suggests that when a triangulation is used the result will be convergence on a single perspective of some social phenomenon. (Mathison 1988, p. 14)

The identification of these basic assumptions in the early years of the development of mixed-method evaluation designs placed triangulation in a positivist paradigm with the intent to seek convergence in the classic sense of the quantitative purist, precluding complementarity as a legitimate research goal (Greene et al. 1989; Mathison 1988). However, the role of triangulation changed with the evolution of research methodologies in education and the social sciences, and triangulation became no longer so simplistic in its conception nor so positivist in its application. As suggested by Denzin (1970, 1989), there are other conceptions of triangulation that are equally valid in different types of studies. As already noted, triangulation can have a different function in the qualitative paradigm, where a research question is addressed from multiple perspectives. Importantly, the introduction of multiple perspectives brings with it the likelihood of arriving at inconsistencies leading to deeper understanding of the issues. Mathison (1988) suggests that triangulation may result in convergent, inconsistent and contradictory results, any of which may be useful to researchers. It is essential that triangulation is seen as not only a validation strategy for the purpose of obtaining convergence or confirmation of findings, but also as an approach to the generalization of discoveries; or elucidating divergent findings as a route to additional knowledge (Flick 2004; Denzin 2012). Jack and Raturi (2006) suggest three rationales for using methodological triangulation, namely, completeness, contingency and confirmation. Triangulation has the advantages of "increasing confidence in research data, creating innovative ways of understanding a phenomenon, revealing unique findings, challenging or integrating theories, and providing a clearer understanding of the problem" (Thurmond 2001, p. 254). Restricting triangulation to only a validation strategy is to deny a great part of its research value. Ellingson (2008) moves the conversation of triangulation to another level and endorses crystallization as a postmodern form of triangulation that embodies multi-genre strengths in the analysis (Denzin 2012). Over the past decades, the very term triangulation has been redefined to meet different needs but the complexities continue to pose challenges (Denzin 2010).

We support the importance attached to triangulation. In fact, we argue in this chapter that the function and value of triangulation in contemporary research methodology is sometimes too narrowly conceived. The convergent research paradigm within which triangulation originally assumed significance was a positivist one, where the goal of research was the identification of a singular truth and the validity of any conclusion was subject to the required convergence of corroborative data. Nonetheless, inconsistencies can be seen as opportunities for deeper understanding of the issues (Guion et al. 2011). To return to the surveying analogy from which the metaphor arose, three observers differently situated around a mountain may provide bearings that enable the geographic location of the mountain's peak to be identified with a high level of precision. This example mirrors the trigonometric example in the Chap. 12 by Kelle and Buchholtz. But the same three differently positioned observers might also describe the mountain as it appears from their distinct perspectives.

In this case, their descriptions (e.g. steep or gradual slope, barren and rocky or heavily wooded) may or may not have some features in common, depending on the nature of the mountain face confronting each observer. What can be said is that a description of the mountain that attempts to combine the separate descriptions of all three observers is likely to provide more information about the mountain and thereby be more useful. The description of a social situation may be undertaken from a variety of perspectives as well without incurring any obligation to converge on a single consensus interpretation. Instead, both similarity and difference in the complementary accounts provide insight and stimulus for further investigation.

Contemporary enthusiasm for "data mining" of large scale (typically quantitative) data bases, highlights the growing recognition that data can be re-purposed or constructed anew from a common data source to meet the needs of different analyses addressing different questions. The gatekeepers to large data bases, such as those generated by the TIMSS and PISA studies (IEA and ACER), have invited researchers to undertake secondary analyses in order to optimise the societal value of these highly expensive data bases. Implicit in the accepted legitimacy of these multiple analyses of a common data set is the recognition that any situation is open to interpretation from different perspectives and that each resultant interpretive account can be held accountable for its consistency with the originating data source, without the combined accounts being required to be laterally consistent with each other. In such an interpretivist paradigm, triangulation becomes the aspiration to more thorough portrayal, rather than the aspiration to more precise location.

15.3 Research as the Mobilization of Bias[1]

Our engagement with research reflects our existing knowledge and beliefs and these, in turn, are a product of our personal history and culture. These attributes of the researcher have the effect of focusing her efforts on particular issues and particular situations. And not on others. By way of example, the decision to establish a Hong Kong research team within the LPS project was variously motivated:

15.3.1 Mathematics Teaching

The Hong Kong team joined the LPS project in 2000 with the primary aim of identifying strengths and weaknesses in mathematics teaching in China (Hong Kong and Shanghai) through an in-depth comparison with mathematics teaching in some other countries (Australia, Germany, Japan, Sweden, and the USA).

[1] The phrase "research is the mobilization of bias" was used by Barry McDonald (1985) as part of a public lecture on research methodology at Monash University. It usefully highlights the inevitable process whereby the researcher translates their own history and priorities into a research agenda.

15.3.2 Chinese Learners' Paradox

In international comparisons of achievement in mathematics, students from "Confucian Heritage Countries" such as Singapore, Japan, Korea and Hong Kong had outperformed students from most other countries. On the other hand, there was a widely accepted view that Chinese classrooms (including those in Hong Kong) are characterized by rote learning and passive students, which in combination constitute a learning environment potentially detrimental to the development of creative, flexible, and widely useful modes of thinking. These observations sat in uneasy juxtaposition and led to the identification of the phenomenon known as the "Chinese Learners' Paradox" (Biggs and Watkins 1996; Watkins and Biggs 2001). The Hong Kong LPS team sought insight regarding the "Chinese Learners' Paradox."

15.3.3 National Pedagogies

Classroom practices should be studied as part of culturally-mediated belief systems and traditions. The assumptions about learning, knowledge and teaching underlying classroom practices are often parts of coherent and rational systems of thinking about learning and of acts aimed at bringing learning about. Such systems of values and beliefs might be called "national pedagogies". Earlier research by members of the Hong Kong team (e.g. Leung 1992, 1995) identified differences between the instructional styles in Beijing, Hong Kong and London. Hong Kong involvement in the LPS project was also motivated by interest in whether a "Chinese" pedagogy of mathematics education could be identified and whether comparison with other culturally specific pedagogies of mathematics education would reveal the strengths and weaknesses of those practices, providing a basis for further development.

15.3.4 Research Design Characteristics

The TIMSS Video Study (Stigler and Hiebert 1999) utilized a random sub-sample of the full TIMSS sample of classrooms to compare activity in mathematics classrooms in Japan, Germany, and the USA. This study reported different "scripts" used by teachers in the three countries. In attempting to characterize national norms of teaching practice, the TIMSS Video Study accepted certain limitations. Only one camera was used and the primary focus of data collection and analysis was the teacher, and only one lesson was videotaped for each classroom sampled. Compared with the TIMSS Video Study, the key characteristics of the LPS research design are the recording of a sequence of at least 10 lessons by the same teacher, the use of three cameras (teacher camera, student camera and whole class camera), and the use of video-stimulated recall in interviews conducted with the teacher and selected students immediately after each lesson to obtain the teacher's and students' reconstructions of the lesson and the meanings that particular classroom events held for them personally (see Clarke 2006a, for details).

The LPS research design, as implemented by the Hong Kong research team, had very specific characteristics:

- It involved two different cities in China, Shanghai and Hong Kong, thus facilitating the identification of a distinct Chinese pedagogy (if there was any), as well as the variations within any such pedagogy.
- It complemented studies reporting national norms of student achievement and teaching practices with an in-depth analysis of classroom learning from the perspective of the learner.
- It utilized multiple frameworks in the analysis of the data in order to get a deeper understanding of classroom teaching from the learner's perspective.
- It examined the Shanghai-Hong Kong differences in teaching in light of what the students actually learned from the lessons through the use of post-lesson video-stimulated interviews.

In combination, the above reasons provided strong motivation and a coherent (and culturally-situated) rationale for the Hong Kong research team to participate in the Learner's Perspective Study. Each participating research team did so with their own unique set of motivations and purposes. Each team brought to the LPS research community a perspective grounded in their own culture and national priorities and each implemented the LPS research design in a form aligned with those priorities.

Triangulation was enacted at the level of data type (video, interview, questionnaire and classroom artefacts), informant (teacher, students and observing researcher), analytical perspective (e.g. discourse analysis, variation theory, socio-cultural theory) and cultural perspective. Positivist aspirations to identify a 'correct view' were discarded for the richness of the aggregated 'complementary accounts' available at each of the levels just outlined. Consistent with contemporary conceptions of triangulation, the accounts generated from different data types or analytical or cultural perspectives "may not be useful to validate each other, but … may yield a fuller and more complete picture of the phenomenon concerned if brought together" (Kelle and Buchholtz, Chap. 12 in this book). In this discussion of triangulation, it is useful to examine data examples and analyses at each of these levels.

15.4 Characteristic Features of the Hong Kong LPS Research Implementation

In implementing the LPS research design in Hong Kong and Shanghai, a very specific modification was made regarding the sampling of classrooms. Whereas the standard LPS research design took the grade level as its primary sampling unit and matched that grade level to the level at which TIMSS achievement testing and the TIMSS Video Study were carried out, the Hong Kong team chose "mathematical content" as the primary sampling unit: specifically, simultaneous equations. The major consequence of prioritizing mathematical content as the point of analytical comparison was that data were collected in eighth grade mathematics classes in Hong Kong and seventh grade mathematics classes in Shanghai. In other respects, the LPS research design was implemented consistently.

City Selection: Hong Kong and Shanghai, representing two urban/metropolitan cities with different historical backgrounds, were selected.

Teacher Selection: Three teachers were selected in each city. These teachers were recognized as competent by their respective local professional communities (their school principals, colleagues in schools, local teacher educators or researchers). Each teacher had at least 5 years of experience as a qualified teacher.

Video-recording Procedures: Before the recording of a continuous sequence of at least 10 lessons for each class, two to three lessons were recorded to allow both the teacher and the students to become accustomed to the presence of the video cameras in the classroom. This strategy was intended to maximize the validity of the video data as representing something approximating normal classroom practice.

Student Selection: A particular group of four students were videotaped continuously for each two of the consecutive lessons. The groups were selected by rotation. Using this protocol, 20 students in each class were studied in some detail. If possible, it was intended that there should be a range of competence across the 20 students, although not necessarily within any given group of four students. Two students were each interviewed individually after each lesson, providing a minimum of 20 post-lesson student interviews per classroom.

Camera Configuration and Integrated Video: The research design employed three cameras—a "Teacher Camera", a "Student Camera" and a "Whole Class Camera". An audio-video mixer was used to combine the Teacher Camera and Student Camera images in a split-screen arrangement, to form the "Learner Practice Composite Image". This integrated image was used to prompt student and teacher reconstructive accounts in interviews.

Fieldnotes: The research assistant/videographer recorded the time and type of all *changes* in instructional activity. The purpose of the fieldnotes was to augment the video record. The completion of fieldnotes was a lower priority than the maintenance of a continuous video record of teacher and students and was never allowed to distract the videographer(s) from their primary purpose.

Student Written Work: All written work produced by the four students "in camera" during any lesson was photocopied, together with any text materials or handouts used during the lesson. Each student was asked to bring their text and all written material produced in class with them to the post-lesson interview. The materials (text pages, worksheets, and students' written work) were photocopied immediately after the interview and the original returned to the student.

Student Interviews: At least two of the four students on camera were interviewed for about 30 min maximum as soon as possible subsequent to the lesson. The interview followed the protocol developed for the Learners' Perspective Study, although additional questions might be added in relation to the specific content of the lesson. For consistency of data collection across all classrooms and all countries, the minimum requirement was two student interviews for every lesson videotaped. During the post-lesson interview, the integrated lesson video was played back and the student was invited to pause the video at any instance that they saw as important, explaining why they considered each chosen event to be important and what they were doing and thinking during the selected instances.

Teacher Questionnaires: Three types of teacher questionnaires were administered: (i) a preliminary teacher questionnaire about each teacher's *goals* in the teaching of mathematics; (ii) a very brief *post-lesson* questionnaire administered after each lesson; and (iii) an adaptation of the TIMSS-video Teacher Questionnaire to generate teacher reflection on the *lesson sequence* recorded on videotape after the whole sequence of 10 lessons.

Teacher Interviews: Three teacher interviews were carried out during the period of the recording of the lessons. The video-stimulated recall technique was used. The teacher was invited in each interview to choose the video of one lesson to play back during the interview to provide a catalyst for the discussion of the critical issues found during the lesson.

The resultant data set afforded a wide variety of potential points of comparison between the practices in different lessons in the same classroom, between different classrooms in the same city, and between classroom practices in different cities. In addition to the observable practices of the classrooms, the post-lesson interviews with teacher and students provided participant accounts of both classroom events and the learning associated with those events.

In considering issues of triangulation, the comparison of the teacher's account of a classroom event with student accounts of the same event was not undertaken with the intention of identifying "what really happened" but with the goal of understanding the intentions, actions and interpretations that each participant identified with the particular situation or event. Only by understanding classroom situations and events from the perspectives of the participants could the researcher hope to identify those aspects that were conducive to or restrictive of learning and those aspects that might prove amenable to improvement or more widespread promotion.

The variety of data types available for analysis afforded selective triangulation in relation to a given activity, event or situation, through the juxtaposition of researcher analyses of different subsets of that extensive database.

15.5 Triangulation and Acts of Cross-Cultural Comparison

The LPS research data set was sufficiently complex to support the investigation of socio-mathematical regularities such as lesson patterns. The data set was intended to complement the approach taken in the TIMSS Video Study by (1) documenting sequences of lessons, rather than single lessons; and (2) by recording students' private conversations as well as public communications during the lessons. Another key feature was that the mathematics lessons were taught by three teachers designated as competent in the local context, and that the classrooms were intended to represent a variety of demographic contexts to the extent that this was possible, depending on the geographical location of the research site. However, it is important to reassert here that there was no intention to characterize the teaching of a country or a culture on the basis of the very selective sample. The overall intention of the LPS research design was to generate a large international data set recording the practices of mathematics classrooms from a wide diversity of geographical locations and cultural settings. In this way, any commonalities of practice across such diverse settings would assume particular significance, while the design offered the opportunity to investigate just how different the practices of eighth grade mathematics classrooms might be around the world. In the same way, for the Chinese sample, any commonalities between the Hong Kong and Shanghai classrooms were of interest, while the diversity of practice across the classrooms of six competent teachers indicated the variation possible within recognizably Chinese cultural settings.

This anticipation of insight arising from both similarity and difference is characteristic of triangulation as it was employed in this study. Denzin (1989) distinguished triangulation of data, investigators, theories and methodologies. As noted in the preceding section, the LPS research design engaged in triangulation of data types, informants, analytical perspectives and cultural perspectives. However, in each instance, the goal was not convergence on a single "true" measurement or account. In its original positivist form, triangulation provided enhanced confidence in a measure by subjecting it to the requirement of convergence of multiple measures or the corroboration of multiple accounts. One might see "positivist triangulation" as privileging reliability of measure through the goal of mutual validation of account. But, as has been argued in Sect. 15.2 of this chapter and extensively by other authors, including Kelle and Buchholtz in this book, triangulation offers much more than simple validation.

Triangulation in the LPS and other contemporary research designs is directed towards the achievement of a different form of research confidence; one much more closely associated with validity rather than reliability. The multiple, complementary accounts generated at each level of the LPS research provided 'thick' description, complex and comprehensive portrayal, and the opportunity for pattern recognition and hypothesis formulation. This greater detail of portrayal met higher expectations regarding the legitimate characterization of social interactions and the capacity of research to move beyond description to explanation. Two types of examples are provided to illustrate the role of triangulation in an interpretivist research design such as the one employed in the Learner's Perspective Study: Lesson patterns and Lesson events. Each presumes a different goal for the research process and each makes use of triangulation in a different way.

15.5.1 Lesson Patterns or Lesson Structure

One of the widely reported bases of comparison between classrooms and countries was lesson patterns or lesson structures, the discussion of which originated with the TIMSS studies. Based on the analysis of 81 single, randomly sampled, eighth grade mathematics lessons, Stigler and Hiebert (1999) reported that US lessons could be generally characterised by the recurrence of four distinct classroom activities: Reviewing previous material; Demonstrating how to solve problems for the day; Practicing; and Correcting seatwork and assigning homework; and it was suggested that these activities, when placed in a particular sequence, formed the basis of a national lesson pattern (Stigler and Hiebert 1999). Similarly, from the analysis of 100 German eighth grade mathematics lessons, Stigler and Hiebert (1999) reported the following lesson pattern for German mathematics lessons: Reviewing previous material; Presenting the topic and the problems for the day; Developing the procedures to solve the problem; and, Practicing. Finally, an analysis of 50 Japanese lessons produced the pattern: Reviewing the previous lesson; Presenting the problem for the day; Students working individually or in groups; Discussing solution methods; and, Highlighting and summarizing the major points. Each reported lesson pattern appeared entirely plausible to educators in each country and elsewhere. Certainly, the component elements of each lesson were familiar and each posited sequence seemed plausible. Despite the apparent plausibility of the three lesson patterns reported, it remained to be demonstrated that teachers would actually employ the same lesson structure for the introductory lesson in a topic sequence as they employed in the middle and final lessons of the topic.

Since the LPS data documented sequences of at least ten lessons for each classroom, one of the initial analyses of the LPS data was to determine whether the sequenced activity categories reported in the TIMSS Video study (Stigler and Hiebert 1999) could be identified in an analysis of the corresponding LPS data in American, German, and Japanese classrooms. A minute-by-minute analysis was carried out for all lessons of the nine classrooms (90 lessons) in Germany, Japan and USA in the LPS data set to determine which of the various activities identified in the TIMSS Video Study best described the classroom behaviour for each minute of every lesson. Two researchers carried out the analysis independently and the results were compared and discussed and a consensus coding constructed. The posited lesson structures did not appear in any lesson anywhere in the LPS data for any of the three countries: the USA, Germany or Japan. Taking the US Classroom 1 as an example, Lessons 1–5 appeared radically different in structure from Lessons 7–10. Overall, for the coding of the full US data set, Clarke et al. (2007) found that “the differences in lesson structure and topic structure between teachers suggested that each teacher combined and sequenced the various activities in ways that were not only a reflection of the mathematical topic being taught, and of the location of the lesson in the topic sequence, but also of the pedagogical style of the individual teacher” (p. 286). The full report can be found in the publication by Clarke et al. (2006c). This analysis suggested that the lesson components (events) identified in the TIMSS report were separately evident in the data from each of the three countries, but that they did not occur in the predicted *sequence*. This suggested that the pedagogy in each mathematics classroom was better represented by a separate, more detailed analysis of each of the characteristic lesson events.

In this comparison, we see the paradigmatic shift from convergence on a single typification (the lesson pattern) to the attempt to capture the diversity and complexity of social interactions over time. The aspiration to typify practice by aggregating over topics, teachers and time runs the risk of concealing precisely those differences (topics, teachers and chronological location) that might be most influential in determining lesson structure and the most important in assisting our understanding of teacher instructional decision-making in different culturally-situated classrooms.

Clarke and his co-authors (2007) suggested that the real contribution of the Stigler-Hiebert analysis was in the identification of characteristic lesson components that did seem to recur from lesson to lesson, but which required much more detailed individual analysis to determine similarity and variation in purpose and function. It was suggested that analysis of the teacher's deployment of these "lesson events" would provide greater insight into culturally-based differences in teaching practice. The Hong Kong data can be used to illustrate the difference between normative characterisations of lesson *pattern* and the detailed description of those lesson *events* from which teachers constructed their lessons.

15.5.2 The Hong Kong Investigation of Lesson Structure

The Hong Kong study addressed the question of whether a pattern of lesson structure existed in the data set generated in the six classrooms (over 60 lessons) in Hong Kong and Shanghai. In addition to patterns of lesson structure, the Hong Kong team attempted to investigate the variation of teaching approaches and methods between the lessons of the same teacher. Lopez-Real et al. (2004) analyzed a sequence of lessons by one teacher in meticulous detail and demonstrated that variation occurred between lessons at the level of lesson structure (see Fig. 15.1), characterizing a particular teacher's approach.

Rather than applying the TIMSS categories of activities, derived from their analysis of German, Japanese and US lessons, a classificatory scheme was developed specifically for the Hong Kong and Shanghai classrooms (Table 15.1). Six categories of lesson activity were identified, characterized by different levels of descriptive detail, indicative of the extent to which variation was possible in the performance of each lesson element. Developing a schematic visual representation for the sequence of activities for a lesson, it was possible to compare the structure of different lessons effectively (Fig. 15.1). For example, the teacher-directive approach was very obvious in Lessons L04 to L08, whereas Lesson L01 obviously demonstrated a different approach with greater emphasis on pedagogy of an exploratory nature.

In combination, Table 15.1 and Fig. 15.1 illustrate the dilemma faced by any researcher seeking to typify national or cultural practice for the purposes of comparison. This has been discussed at greater length elsewhere (Clarke 2006b; Clarke et al. 2012). If, however, the aspiration is to portray rather than to typify, then categorization schemes such as that shown in Fig. 15.1 can serve dual purposes: (i) As a matter of definition, each designated category will accommodate the variety of actions or activities held to be sufficiently similar to be associated with that category, which thereby provides a classificatory basis by which to compare one

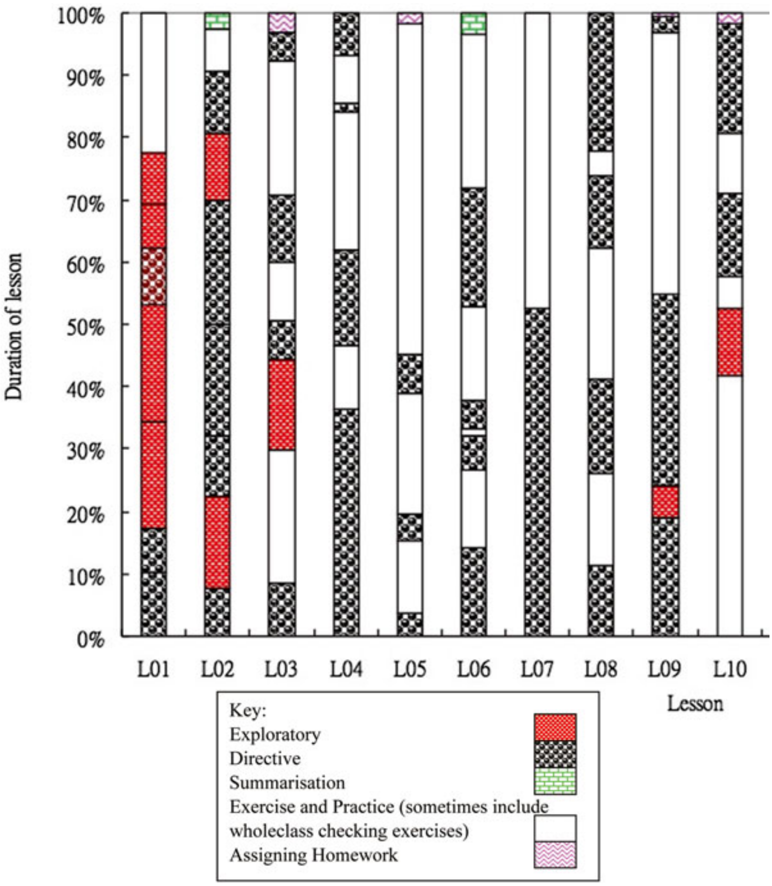

Fig. 15.1 Visual representation of the lesson patterns showing the variation between the lessons of one of the Hong Kong teachers

classroom or lesson with another; and (ii) the variations possible within each category delineate how the teachers may engage in pedagogical decision-making and select from their instructional repertoire the particular version or adaptation of the lesson component that will meet their needs in the particular lesson.

In elaborating the various categories of events of each of the lesson activities identified in Table 15.1, the researcher benefits from (i) data triangulation, where the video record of a lesson event is supplemented by retrospective explanatory interviews or copies of the written student products arising from that event; and (ii) informant triangulation, where the significance of each lesson event must be understood from the perspective of the various classroom participants and from the perspective of the researcher. Triangulation with respect to theoretical perspectives has been discussed elsewhere (Clarke 2011). The next section addresses the question of cultural triangulation, a research strategy that allows the research team to compare

Table 15.1 Criteria for categorizing activities demonstrating different teacher's approaches (Lopez-Real et al. 2004, pp. 411–412)

(B01) Exploratory
The focus is on a relative open or difficult problem which has more than one possible answers
The teacher gave a signal for pair discussion. (Sometimes this was skipped. The exploration was facilitated directly by the teacher in a whole class discussion.)
A whole class discussion with the following features: inviting more than one student to give answers, inviting for explanations, inviting for peer comments
(B02) Directive (Foundation; Consolidation)
No comment on the student's answer, no attempt to discuss the answer with the other students, simply stating what should be done, the conventional notation
Emphasis is purely on following a convention
No explicit discussion on the implicit identification (link) in the critical shift from a discrete context to abstract context
Insistent on precise language
Repetition of what had been learnt in an earlier lesson or in the earlier part of the lesson in a fast pace, using this as a foundation for establishing further knowledge
Insistent on articulation of procedures
Clear and directive summary (usually a definition of a concept or method) after an illustrative example or discussion
Teacher plays the role in directing students to work on problems
Probing
Directive explanation by teacher
(B03) Summarization
Teacher does summarization during the lesson, to summarize or to conclude the topics or problems discussed
(B04) Exercise and Practice (sometimes include whole class checking exercise)
In the situation of doing textbook exercise, there can be teacher talking about/ explaining the question, and students having seatwork
Teacher checks exercise with students
(B05) Assigning Homework
Teacher assigns homework or questions for students to do at home

and contrast the enactment of "lesson events" based on a shared definition (form) accommodating variation across different cultural systems (function) for a deeper understanding of the social and pedagogical situations in the classrooms.

15.5.3 Contrasting the Enactment of "Lesson Events" Across Different Cultural Systems

Analyses of Hong Kong and Shanghai classrooms, consistent with analyses of lesson structure in the classrooms of other participant LPS countries, suggested that teachers did not employ the same lesson structure for each lesson in a topic. Rather, teachers employed the various “lesson events” according to the location of the lesson in the topic sequence. Another analysis was therefore undertaken of the LPS lessons focusing on the form and function of the activities that made up the various parts in the lesson. The original conception of lesson events as providing a legitimate unit for comparison across different cultural systems was based on the observation that some recognizable components of classroom practice might happen in classrooms of different countries or systems, but that these would have differences and similarities in implementation in the various classrooms, particularly with respect to the function of the identified activity and its value in the local pedagogy. For example, the proportion of time that the Australian teachers spent in moving around the classroom while the students were working independently in class was much greater that that found in the Shanghai classrooms.

Crucial to the analytic process is the distinction between form and function. The LPS definition of a “lesson event” states that the lesson event is recognizable by “a form (visual features and social participants) sufficiently common to be identifiable within the classroom data from each of the countries studied” (Clarke et al. 2007, p. 287). Having identified by collaborative consensus between LPS team members that particular lesson events were recognizable by form in the classrooms of several countries, the focus of collaborative analysis was the function served by the identified lesson event in the various classrooms in which it appeared. Some examples of lesson events are: “beginning the lesson” (Mesiti and Clarke 2006), “Kikan Shido or between desks instruction” (O’Keefe et al. 2006), and “learning task” (Mok and Kaur 2006).

In making comparison of the enactment of a particular learning task in several different cultural settings, we see most clearly the contemporary use of triangulation. Consider, for example, the lesson event titled, “the learning task” (Mok and Kaur 2006). It was not the intention of the Hong Kong team to use the cultural triangulation of learning task performance in several cultural settings to take “cross-readings” on “learning task” as a category of activity and identify unambiguously those distinctive actions and functions that might constitute the set of criteria by which the event “learning task” might be uniquely defined. Rather, the goal was to document all possible forms of variation in the implementation of “learning task” across the various classrooms in order that the particular lesson event might be better understood through the differences in function and consequence that were evident in each setting. The combination of similar features, identified as holding constant across cultural settings, together with those functions that varied between settings, provide in their situated interconnection a more complete cross-cultural portrayal of the lesson event than would be provided by any positivist convergence on a unique set of indicators holding across all settings.

The remainder of this section makes use of the analysis of “learning task lesson events” undertaken by Mok and Kaur (2006) to show how the comparison of a particular type of lesson event between different cultural systems can help to understand culture-specific components of mathematics classrooms and identify explanations for any differences between different systems in terms of cultural values and traditions of teaching and learning. Mok and Kaur (2006) compared 18 identified “learning tasks” from Australia, Germany, Hong Kong, Japan, Shanghai, Singapore and the United States. The data analysed included the video record and lesson transcripts.

The emphasis of this discussion is not on the findings with respect to the lesson event "learning task" but on the methodological aspects of the work by Mok and Kaur (2006); that is, the identification of instances of the various units of analysis and the analytical approaches employed.

15.5.3.1 The Learning Task: One Example of a Lesson Event

The selection of examples of each of the various lesson events was undertaken as a collaborative effort by the team members of different countries. The first step was to develop a definition in terms of form with examples and non-examples, so the researchers of each country could select examples from the data set of their own research sites for inclusion in the combined set of examples of that lesson event. This combined set selected on the basis of similar form could then be analysed with respect to function.

For the purpose of defining the learning task lesson event, Mok and Kaur (2006) referred to the object of learning in the lesson:

> Every lesson will have an object of learning, that is, either a mathematical concept or skill which the teacher wants the students to learn in the lesson. This is often explained in the teacher's goals for the lesson in our data (the lesson tables). Some learning tasks are usually used in the lesson to illustrate or explain the concepts or skills. Students may be engaged in the learning tasks in whole class discussion led by the teacher, individually or in groups depending on the teacher's class arrangement. (Mok and Kaur 2006, p. 148)

As a result, the form of a learning task lesson event was represented by the segment of the lesson that contained the enactment of the learning task by the teacher and the students during the lesson (Mok and Kaur 2006; Mok 2010). With respect to the differentiation between examples and non-examples for the event, a special effort was made to differentiate a "learning task" as defined in this study from a practice item or repetitive exercise.

> A learning task aims to teach the students something new and the sequence of learning tasks show a coherent development of the object of learning. On the other hand, a practice item is mostly repetition of a taught skill. A common occurrence in Western classrooms, and also in Hong Kong, is for the teacher to do a worked example on the blackboard. This worked example is definitely a learning task. The students then are asked to do a set of problems that strongly resemble the worked example. According to the definition, it means that where the similarity is very high between the worked example and the problems subsequently attempted by the students, then the worked example should be seen as a learning task, but the subsequent problems are not. (Mok and Kaur 2006, p. 148)

This involved addressing the question of function in a general sense, without precluding locally-specific variations in the way the learning task event was carried out. For example, Mok and Kaur (2006) found, the various functions of the learning task lesson event carried out by the LPS Shanghai teachers included:

- For setting a background for the topic to be learned;
- For demonstration or explanation, often with visual display and interactive question-and-answers between the teacher and the students;

- For an in-depth investigation/discussion of a specific aspect of the object of learning. (Mok and Kaur 2006, p. 148)

These functions were consistent with the general introductory character of the learning task and also provided initial criteria for other researchers to identify learning tasks within the data set of their own countries. The criteria made possible the comparison between samples of the data selected by researchers from their home data (Australia, Germany, Hong Kong, Japan, Shanghai, Singapore and United States) (Mok and Kaur 2006). Given the nature of the LPS research design, focusing on the detailed documentation of practices in the classrooms of only three teachers in each community, the project was not intended to characterize the practices of a particular culture. The value of cross-cultural analysis lies in the observation that people tend to disregard some features of an event, despite its importance, because they are too familiar with what happens, whereas, these features may become prominent through the process of comparing examples from different backgrounds.

Each lesson event constituted a class of activities recognizable across different cultural settings, but diverse in the details of their enactment. Having identified video excerpts involving "learning tasks," different categories were generated to classify the particular instructional function served in each instance, resonating with the existing research literature and associated with the effective learning of mathematics. Three particular themes stood out, both as distinctive characteristics and as relevant to contemporary curricular priorities: differentiation of the mathematical process, realistic contexts and building connections. It was then possible for Learning Task lesson events that appeared to share one or more of these features, such as their utilization of realistic contexts, to be compared.

For example, under the dimension of "differentiation of the mathematical process," four examples were identified in algebra lessons from Germany, Singapore, Japan and Shanghai. These four examples all demonstrated how Learning Tasks, similar in form, might produce very different activity with respect to the development of an understanding of the mathematical procedure and making the procedure visible so that it could be shared by learners. The contrast between the four examples demonstrated a *spectrum of possibilities* in the emerging dimension, namely, students' sharing of non-public talk, teacher-guided class discourse with different levels of freedom and convergence in the process of constructing the solution, students' looking back at their own work. A summary is quoted here:

> Whilst the German example only has the students' own sharing of non-public talk, the Singapore teacher emphasises a correct demonstration of the process led by the teacher herself. There is hardly any possibility of any sidetrack and the students' answers were mostly to fill in what the teacher asked for. The Shanghai example falls in between the German and Singapore cases … The Japanese example is of a different kind compared with other three examples. The focus of the Japanese example is not on the process of how the students produce their work, but on looking back on what they have produced. Consequently, the invitation by the teacher to comment on the two students' work initiates discussion, the content of which includes ideas such as the meaning of a solution for equations and the presentation of checking. This kind of teaching strategy obviously demands another level of understanding of the procedure. (Mok and Kaur 2006, pp. 151–152)

The methodological point that is being made here concerns the capacity of cross-cultural comparison to reveal diversity and the utility of the documented differences in offering alternatives for practice, where a more convergent research paradigm would conceal difference through a process of aggregation and typification. In line with Kelle and Buchholtz (Chap. 12 in this book), we contrast triangulation employed for mutual validation with that employed for elaborated portrayal and emphasise the important role of triangulation in generating more complex, multi-perspectival descriptions with greater explanatory potential. Following Denzin (2010), we distinguish different types of triangulation functioning at different levels within the research design. But the key assertion of this chapter is that contemporary research efforts to understand classroom practice demand research designs that exploit rather than conceal difference, and that reveal diversity in ways that enhance the possibility of the recognition of pattern, connection and situatedness.

15.6 Conclusion

In constructing this chapter, we have used the LPS research design as our example of a contemporary methodological approach with particular features. Each of these features invokes the notion of triangulation in a different way. The multiplicity of data types generated by the mixed methods design contributes to a complex but interconnected account of any given social situation. The use of multiple informants both as participants in the situations of interest, but also as retrospective commentators on those same situations, foregrounds the separate legitimacy of each participant's interpretation of events, rather than their capacity to be mutually validating, and highlights the need to document the intentions, actions and interpretations of each, if we are to understand and thereby optimise the situations, actions and activities we find in classrooms internationally. The use of different theories to provide alternative analytical perspectives is not addressed in detail here, but provides a form of triangulation that is attracting increasing attention. Again, the essential methodological point is that such theoretically inclusive research designs are not intended to identify some consensus interpretation of events, but rather, through the juxtaposition of parallel, complementary, interpretive accounts, to offer a less partial portrayal of the situation of interest (Clarke 2011), and simultaneously to afford opportunities for the interrogation not only of the situation, but of the alternative theories themselves. Lastly, cultural triangulation is perhaps the most visible strategy employed in the LPS research design. Again, the question is "Triangulation to what end?" Certainly not the comparative evaluation of the effectiveness of different culturally-situated instructional approaches. By virtue of the differences in their cultural situation, the legitimacy of any such comparative evaluation is highly problematic (see Clarke et al. 2012). Instead, consistent with the other examples provided, cultural triangulation offers the opportunity for the rich portrayal of activities or phenomena having familiar form but varied function, in order to enrich our understanding of the *possibilities* of action, of the *interconnectedness* of the webs of activity found in classrooms, and of the *situatedness* of those actions and of the process of teacher decision-making by which each action was initiated.

By undertaking analyses at very different levels of granularity (Schoenfeld 1999), by examining and re-examining selected subsets of the data set, the initial research interest of the Hong Kong team, the question of a "Chinese" pedagogy for mathematics, has been investigated by analysis of both insiders' and outsiders' perspectives, by various choices of research foci and issues, and by the methods and frameworks developed by the various members of the LPS community. Better understanding of what might be meant by Chinese pedagogy has been obtained by the comparison between the two cities in China, by comparison of the Chinese examples with other places in the Asian context and with non-Asian contexts. These analyses have been primarily afforded by the particular forms of triangulation integral to the LPS research design. The examples discussed in this chapter are only selected snapshots, attempting to illustrate the multiple strategies and methodological concerns occurring during the conduct of the many analyses that have formed part of the Hong Kong team's participation in the LPS. These included: decisions as to what to compare across a significant number of classrooms situated in a broad range of cultures; how best to utilize a data set, simultaneously extensive and limited, to provide an understanding of generic pedagogical issues; and how to accommodate and exploit the combination of multiple theoretical approaches and the associated development of various coding schemes for seeking answers to very different research questions. In the context of cross-cultural classroom research, at a time when mixed methods approaches are recognised as the rule rather than the exception in educational research, we suggest that triangulation can serve the aspiration to accommodate and characterise complexity rather than conceal or minimise it.

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Part XI
Design Research as a Research Methodology

Chapter 16
An Introduction to Design-Based Research with an Example From Statistics Education

Arthur Bakker and Dolly van Eerde

Abstract This chapter arose from the need to introduce researchers, including Master and PhD students, to design-based research (DBR). In Sect. 16.1 we address key features of DBR and differences from other research approaches. We also describe the meaning of validity and reliability in DBR and discuss how they can be improved. Section 16.2 illustrates DBR with an example from statistics education.

Keywords Design based research • Statistics education

16.1 Theory of Design-Based Research

16.1.1 Purpose of the Chapter

The purpose of this chapter is to introduce researchers, including Master and PhD students, to design-based research. In our research methods courses for this audience and in our supervision of PhD students, we noticed that students considered key publications in this field unsuitable as introductions. These publications have mostly been written to inform or convince established researchers who already have considerable experience with educational research. We therefore see the need to write for an audience that does not have that level of experience, but may want to know about design-based research. We do assume a basic knowledge of the main research approaches (e.g., survey, experiment, case study) and methods (e.g., interview, questionnaire, observation).

Compared to other research approaches, educational design-based research (DBR) is relatively new (Anderson and Shattuck 2012). This is probably the reason that it is not discussed in most books on qualitative research approaches. For example, Creswell (2007) distinguishes five qualitative approaches, but these do not include DBR (see also Denscombe 2007). Yet DBR is worth knowing about, espe-

A. Bakker (✉) • D. van Eerde
Freudenthal Institute for Science and Mathematics Education, Utrecht University, Princetonplein 5, 3584 CC, Utrecht, The Netherlands
e-mail: a.bakker4@uu.nl; h.a.a.vaneerde@uu.nl

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cially for students who will become teachers or researchers in education: Design-based research is claimed to have the potential to bridge the gap between educational practice and theory, because it aims both at developing theories about domain-specific learning and the means that are designed to support that learning. DBR thus produces both useful products (e.g., educational materials) and accompanying scientific insights into how these products can be used in education (McKenney and Reeves 2012; Van den Akker et al. 2006). It is also said to be suitable for addressing complex educational problems that should be dealt with in a holistic way (Plomp and Nieveen 2007).

In line with the other chapters in this book, Sect. 16.1 provides a general theory of the research approach under discussion and Sect. 16.2 gives an example from statistics education on how the approach can be used.

16.1.2 Characterizing Design-Based Research

In this section we outline some characteristics of DBR, compare it with other research approaches, go over terminology and history, and finally summarize DBR's key characteristics.

16.1.2.1 Integration of Design and Research

Educational design-based research (DBR) can be characterized as research in which the design of educational materials (e.g., computer tools, learning activities, or a professional development program) is a crucial part of the research. That is, the design of learning environments is interwoven with the testing or developing of theory. The theoretical yield distinguishes DBR from studies that aim solely at designing educational materials through iterative cycles of testing and improving prototypes.

A key characteristic of DBR is that educational ideas for student or teacher learning are formulated in the design, but can be adjusted during the empirical testing of these ideas, for example if a design idea does not quite work as anticipated. In most other interventionist research approaches design and testing are cleanly separated. See further the comparison with a randomized controlled trial in Sect. 16.1.2.5.

16.1.2.2 Predictive and Advisory Nature of DBR

To further characterize DBR it is helpful to classify research aims in general (cf. Plomp and Nieveen 2007):

- To describe (e.g., What conceptions of sampling do seventh-grade students have?)
- To compare (e.g., Does instructional strategy A lead to better test scores than instructional strategy B?)

- To evaluate (e.g., How well do students develop an understanding of distribution in an instructional sequence?)
- To explain or to predict (e.g., Why do so few students choose a bachelor in mathematics or science? What will students do when using a particular software package?)
- To advise (e.g., How can secondary school students be supported to learn about correlation and regression?)

Many research approaches such as surveys, correlational studies, and case studies, typically have descriptive aims. Experiments often have a comparative aim, even though they should in Cook's (2002) view "be designed to *explain* the consequences of interventions and not just to describe them" (p. 181, emphasis original). DBR typically has an explanatory and advisory aim, namely to give theoretical insights into how particular ways of teaching and learning can be promoted. The type of theory developed can also be of a predictive nature: Under conditions X using educational approach Y, students are likely to learn Z (Van den Akker et al. 2006).

Research projects usually have one overall aim, but several stages of the project can have other aims. For example, if the main aim of a research project is to advise how a particular topic (e.g., sampling) should be taught, the project most likely has parts in which phenomena are described or evaluated (e.g., students' prior knowledge, current teaching practices). It will also have a part in which an innovative learning environment has to be designed and evaluated before empirically grounded advice can be given. This implies that research projects are layered. Design-based research (DBR) has an overall predictive or advisory aim but often includes research stages with a descriptive, comparative, or evaluative aim.

16.1.2.3 The Role of Hypotheses and the Engineering Nature of DBR

In characterizing DBR as different from other research approaches, we also need to address the role of hypotheses in theory development. Put simply, a scientific theory can explain particular phenomena and predict what will happen under particular conditions. When developing or testing a theory, scientists typically use hypotheses—conjectures that follow from some emergent theory that still needs to be tested empirically. This means that hypotheses should be formulated in a form in which they can be verified or falsified. The testing of hypotheses is typically done in an experiment: Reality is manipulated according to a theory-driven plan. If hypotheses are confirmed, this is support for the theory under construction.

Just as in the natural sciences, it is not always possible to test hypotheses empirically within a short period of time. As a starting point design researchers, just like many scientists in other disciplines, use thought experiments—thinking through the consequences of particular ideas. When preparing an empirical teaching experiment, design researchers typically do a thought experiment on how teachers or students will respond to particular tools or tasks based on their practical and theoretical knowledge of the domain (Freudenthal 1991).

In empirical experiments, a hypothesis is formulated beforehand. A theoretical idea is operationalized by designing a particular setting in which only this particular feature is isolated and manipulated. To stay objective experimental researchers are often not present during the interventions. In typical cases, they collect only pre- and posttest scores. In design-based research, however, researchers continuously take their best bets (Lehrer and Schauble 2001), even if this means that some aspect of the learning environment during or after a lesson has to be changed. In many examples, researchers are involved in the teaching or work closely with teachers or trainers to optimize the learning environment (McClain and Cobb 2001; Smit and Van Eerde 2011; Hoyles et al. 2010). In the process of designing and improving educational materials (which we take as a prototypical case in this chapter), it does not make sense to wait until the end of the teaching experiment before changes can be made. This would be inefficient.

DBR is therefore sometimes characterized as a form of didactical engineering (Artigue, 1988): didactical engineering: Something has to be made with whatever theories and resources are available. The products of DBR are judged on innovativeness and usefulness, not just on the rigor of the research process that is more prominent in evaluating true experiments (Plomp 2007).

In many research approaches, changing and understanding a situation are separated. However, in design-based research these are intertwined in line with the following adage that is also common in sociocultural traditions: If you want to understand something you have to change it, and if you want to change something you have to understand it (Bakker 2004a, p. 37).

16.1.2.4 Open and Interventionist Nature of DBR

Another way to characterize DBR is to contrast it with other approaches on the following two dimensions: naturalistic vs. interventionist and open vs. closed. Naturalistic studies analyze how learning takes place without interference by a researcher. Examples of naturalistic research approaches are ethnography and surveys. As the term suggests, interventionist studies intervene in what naturally happens: Researchers deliberately manipulate a condition or teach according to particular theoretical ideas (e.g., inquiry-based or problem-based learning). Such studies are necessary if the type of learning that researchers want to investigate is not present in naturalistic settings. Examples of interventionist approaches are experimental research, action research, and design-based research.

Research approaches can also be more open or closed. The term *open* here refers to little control of the situation or data whereas *closed* refers to a high degree of control or a limited number of options (e.g., multiple choice questions). For example, surveys by means of questionnaires with closed questions or responses on a Likert scale are more closed than surveys by means of semi-structured interviews. Likewise, an experiment comparing two conditions is more closed than a DBR project in which the educational materials or ways of teaching are emergent and adjustable. Different research approaches can thus be positioned in a two-by-two table as in Table 16.1. DBR thus shares an interventionist nature with experiments and action research. We therefore continue by comparing DBR with experiments (16.1.2.5) and with action research (16.1.2.6).

Table 16.1 Naturalistic vs. interventionist and open vs. closed research approaches

	Naturalistic	Interventionist
Closed	Survey: questionnaires with closed questions	Experiment (randomized controlled trial)
Open	Survey: interviews with open questions	Action research
	Ethnography	Design-based research

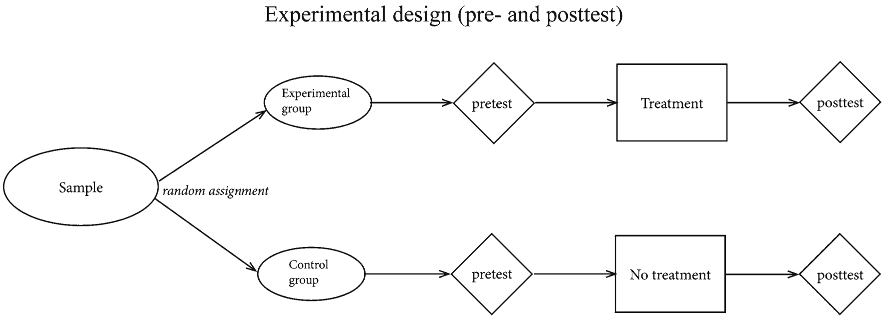

Fig. 16.1 A pre-posttest experimental design (randomized controlled trial)

16.1.2.5 Comparison of DBR with Randomized Controlled Trials (RCT)

A randomized controlled trial (RCT) is sometimes referred to as "true" experiment. Assume we want to know whether a new teaching strategy for a particular topic in a particular grade is better than the traditionally used one. To investigate this question one could randomly assign students to the experimental (new teaching strategy) or control condition (traditional strategy), measure performances on pre- and posttests, and use statistical methods to test the null hypothesis that there is no significant difference between the two conditions. The researchers' hope is that this hypothesis can be rejected so that the new type of intervention (informed by a particular theory) proves to be better. The underlying rationale is: If we know "what works" we can implement this method and have better learning results (see Fig. 16.1).

This so-called experimental approach of randomized controlled trials (Creswell 2005) is sometimes considered the highest standard of research (Slavin 2002). It has a clear logic and is a convincing way to make causal and general claims about what works. It is based on a research approach that has proven extremely helpful in the natural sciences.

However, its limitations for education are discussed extensively in the literature (Engeström 2011; Olsen 2004). Here we mention two related arguments. First, if we know what works, we still do not know why and when it works. Even if the new strategy is implemented, it might not work as expected because teachers use it in less than optimal ways.

An example can clarify this. When doing research in an American school, we heard teachers complain about their managers' decision that every teacher had to start every lesson with a warm-up activity (e.g., a puzzle). Apparently it had been proven by means of an RCT that student scores were significantly higher in the experimental condition in which lessons started with a warm-up activity. The negative effect in teaching practice, however, was that teachers ran out of good ideas for warm-up activities, and that these often had nothing to do with the topic of the lesson. Effectively, teachers therefore lost five minutes of every lesson. Better insight into how and why warm-up activities work under particular conditions could have improved the situation, but the comparative nature of RCT had not provided this information because only the variable of starting the lesson with or without warm-up activity had been manipulated.

A second argument why RCT has its limitations is that a new strategy has to be designed before it can be tested, just like a Boeing airplane cannot be compared with an Airbus without a long tradition of engineering and producing such airplanes. In many cases, considerable research is needed to design innovative approaches. Design-based research emerged as a way to address this need of developing new strategies that could solve long-standing or complex problems in education.

Two discussion points in the comparison of DBR and RCT are the issues of generalization and causality. The use of random samples in RCT allows generalization to populations, but in most educational research random samples cannot be used. In response to this point, researchers have argued that theory development is not just about populations, but rather about propensities and processes (Frick 1998). Hence rather than generalizing from a random sample to a population (statistical generalization), many (mainly qualitative) research approaches aim for generalization to a theory, model or concept (theoretical or analytic generalization) by presenting findings as particular cases of a more general model or concept (Yin 2009).

Where the use of RCTs can indicate the intervention or treatment being the cause of better learning, DBR cannot claim causality with the same convincing rigor. This is not unique to DBR: All qualitative research approaches face this challenge of drawing causal claims. In this regard it is helpful to distinguish two views on causality: a regularity, variance-oriented understanding of causality versus a realist, process-oriented understanding of causality (Maxwell 2004). People adopting the first view think that causality can only be proven on the basis of regularities in larger data sets. People adopting the second view make it plausible on the basis of circumstantial evidence of observed processes that what happened is most likely caused by the intervention (e.g., Nathan and Kim 2009). The first view is underlying the logic of RCT: If we randomly assign subjects to an experimental and control condition, treat only the experimental group and find a significant difference between the two groups, then it can only be attributed to the difference in condition (the treatment). However, if we were to adopt the same regularity view on causality we would never be able to identify the cause of singular events, for example why a driver hit a tree. From the second, process-oriented view, if a drunk driver hits a tree we can judge the circumstances and judge it plausible that his drunkenness was an important explanation because we know that alcohol can cause less control, slower reaction time et cetera. Similarly, explanations for what happens in classrooms should be possible according to a process-oriented position based on what happens in response to particular interventions. For example, particular student utterances are very unlikely if not deliberately fostered by a teacher (Nathan and Kim 2009). Table 16.2 summarizes the main points of the comparison of RCT and DBR.

16.1.2.6 Comparison of DBR with Action Research

Like action research, DBR typically is interventionist and open, involves a reflective and often cyclic process, and aims to bridge theory and practice (Opie 2004). In both approaches the teacher can be also researcher. In action research, the researcher is not an observer (Anderson and Shattuck 2012), whereas in DBR s/he can be observer. Furthermore, in DBR design is a crucial part of the research, whereas in action research the focus is on action and change, which can but need not involve the design of a new learning environment. DBR also more explicitly aims for instructional theories than does action research. These points are summarized in Table 16.3.

Table 16.2 Comparison of experimental versus design-based research

Experiment (RCT)	Design-based research (DBR)
Testing theory	Developing and testing theory simultaneously
Comparison of existing teaching methods by means of experimental and control groups	Design of an innovative learning environment long
Proof of what works	Insight into how and why something works
Research interest is isolated by manipulating variables separately	Holistic approach long white word
Statistical generalization	Analytic or theoretical generalization, transferability to other situations
Causal claims based on a regularity view on causality are possible	Causality should be handled with great care and be based on a realist, process-oriented view on causality

Table 16.3 Commonalities and differences between DBR and action research

	DBR	Action research
Commonalities	Open, interventionist, researcher can be participant, reflective cyclic process	
Differences	Researcher can be observer	Researcher can only be participant
	Design is necessary	Design is possible
	Focus on instructional theory	Focus on action and improvement of a situation

16.1.2.7 Names and History of DBR

In its relatively brief history, DBR has been presented under different names. *Design-based research* is the name used by the Design-Based Research Collective (see special issues in Educational Researcher, 2003; Educational Psychologist 2004; Journal of the Learning Sciences 2004). Other terms for similar approaches are:

- Developmental or development research (Freudenthal 1988; Gravemeijer 1994; Lijnse 1995; Romberg 1973; Van den Akker 1999)
- Design experiments or design experimentation (Brown 1992; Cobb et al. 2003a; Collins 1992)
- Educational design research (Van den Akker et al. 2006)

The reasons for these different terms are mainly historical and rhetorical. In the 1970s Romberg (1973) used the term *development research* for research accompanying the development of curriculum. Discussions on the relation between research and design in mathematics education, especially on didactics, mainly took place in Western Europe in the 1980s and the 1990s, particularly in the Netherlands (e.g., Freudenthal 1988; Goffree 1979), France (e.g., Artigue 1988, cf. Artigue Chap. 17) and Germany (e.g., Wittmann 1992). The term *developmental research* is a translation of the Dutch *ontwikkelingsonderzoek*, which Freudenthal introduced in the 1970s to justify the development of curricular materials as belonging to a university institute (what is now called the Freudenthal Institute) because it was informed by and leading to research on students' learning processes (Freudenthal 1978; Gravemeijer and Koster 1988; De Jong and Wijers 1993). The core idea was that development of learning environments and the development of theory were intertwined. As Goffree (1979, p. 347) put it: "Developmental research in education as presented here, shows the characteristics of both developmental and fundamental research, which means aiming at new knowledge that can be put into service in continued development." At another Dutch university (Twente University), the term *ontwerpgericht* (design-oriented) research was more common, but there the focus was more on the curriculum than on theory development (Van den Akker 1999). One disadvantage of the terms 'development' and 'developmental' is their connotations to developmental psychology and research on children's development of concepts. This might be one reason that this term is hardly used anymore.

In the United States, the terms *design experiment* and *design research* were more common (Brown 1992; Cobb et al. 2003a; Collins 1992; Edelson 2002). One advantage of these terms is that design is more specific than development. One possible disadvantage of the term *design experiment* can be explained by reference to a critical paper by Paas (2005) titled *Design experiment: Neither a design nor an experiment*. The confusion that his pun refers to is two-fold. First, in many educational research communities the term *design* is reserved for research design (e.g., comparing an experimental with a control group), whereas the term in design research refers to the design of learning environments (Sandoval and Bell 2004). Second, for many researchers, also outside the learning sciences, the term *experiment* is reserved for "true" experiments or RCTs. In design experiments, hypotheses certainly play an important role, but they are not fixed and tested once. Instead they may be emergent, multiple, and temporary. In line with the Design-Based Research Collective, we use the term *design-based research* because this suggests that it is predominantly research (hence leading to a knowledge claim) that is based on a design process.

16.1.2.8 Theory Development in Design-Based Research

We have already stated that theory typically has a more central role in DBR than in action research. To address the role of theory in DBR, it is helpful to summarize diSessa and Cobb's (2004) categorization of different types of theories involved in educational research. They distinguish:

- Grand theories (e.g., Piaget's phases of intellectual development; Skinner's behaviorism)
- Orienting frameworks (e.g., constructivism, semiotics, sociocultural theories)
- Frameworks for action (e.g., designing for learning, Realistic Mathematics Education)
- Domain-specific theories (e.g., how to teach density or sampling)
- Hypothetical Learning Trajectories (Simon 1995) or didactical scenarios (Lijnse 1995; Lijnse and Klaassen 2004) formulated for specific teaching experiments (explained in Sect. 16.1.3).

As can be seen from this categorization, there is a hierarchy in the generality of theories. Because theories developed in DBR are typically tied to specific learning environments and learning goals, they are humble and hard to generalize. Similarly, it is very rare that a theoretical contribution to aerodynamics will be made in the design of an airplane; yet innovations in airplane design occur regularly. The use of grand theoretical frameworks and frameworks for action is recommended, but researchers should be careful to manage the gap between the different types of theory on the one hand and design on the other (diSessa and Cobb 2004). If handled with care, DBR can then provide the basis for refining or developing theoretical concepts such as meta-representational competence, sociomathematical norms (diSessa and Cobb), or whole-class scaffolding (Smit et al. 2013).

16.1.2.9 Summary of Key Characteristics of Design-Based Research

So far we have characterized DBR in terms of its predictive and advisory aim, particular way of handling hypotheses, its engineering nature and differences from other research methods. Here we summarize five key characteristics of DBR as identified by Cobb et al. (2003a):

1. The first characteristic is that its purpose is *to develop theories about learning and the means that are designed to support that learning*. In the example provided in Sect. 16.2 of in this chapter, Bakker (2004a) developed an instruction theory for early statistics education and instructional means (e.g. computer tools

and accompanying learning activities) that support the learning of a multifaceted notion of statistical distribution.

2. The second characteristic of DBR is its *interventionist* nature. One difference with RCTs is that interventions in the DBR tradition often have better ecological validity—meaning that learning already takes place in learning ecologies as they occur in schools and thus methods measure better what researchers want to measure, that is learning in natural situations. Findings from experiments do not have to be translated as much from controlled laboratory situations to the less controlled ecology of schools or courses. In technical terms, theoretical products of DBR "have the potential for rapid pay-off because they are filtered in advance for instrumental effect" (Cobb et al. 2003a, p. 11).
3. The third characteristic is that DBR has *prospective and reflective components* that need not be separated by a teaching experiment. In implementing hypothesized learning (the prospective part) the researchers confront conjectures with actual learning that they observe (reflective part). Reflection can be done after each lesson, even if the teaching experiment is longer than one lesson. Such reflective analysis can lead to changes to the original plan for the next lesson. Kanselaar (1993) argued that any good educational research has prospective and reflective components. As explained before, however, what distinguishes DBR from other experimental approaches is that in DBR these components are not separated into the formulation of hypotheses before and after a teaching experiment.
4. The fourth characteristic is the *cyclic* nature of DBR: Invention and revision form an iterative process. Multiple conjectures on learning are sometimes refuted and alternative conjectures can be generated and tested. The cycles typically consist of the following phases: preparation and design phase, teaching experiment, and retrospective analysis. These phases are worked out in more detail later in this chapter. The results of such a retrospective analysis mostly feed a new design phase. Other types of educational research ideally also build upon prior experiments and researchers iteratively improve materials and theoretical ideas in between experiments but in DBR changes can take place during a teaching experiment or series of teaching experiments.
5. The fifth characteristic of DBR is that the *theory* under development *has to do real work*. As Lewin (1951, p. 169) wrote: "There is nothing so practical as a good theory." Theory generated from DBR is typically humble in the sense that it is developed for a specific domain, for instance statistics education. Yet it must be general enough to be applicable in different contexts such as classrooms in other schools in other countries. In such cases we can speak of transferability.

16.1.3 Hypothetical Learning Trajectory (HLT)

DBR typically consists of cycles of three phases each: preparation and design, teaching experiment, and retrospective analysis. One might argue that the term 'retrospective analysis' is pleonastic: All analysis is in retrospect, after a teaching experiment. However, we use it here to distinguish it from analysis on the fly, which takes place during a teaching experiment, often between lessons.

A design and research instrument that proves useful during all phases of DBR is the *hypothetical learning trajectory* (HLT), which we regard as an elaboration of Freudenthal's thought experiment. Simon (1995) defined the HLT as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p. 136)

Simon used the HLT for one or two lessons. Series of HLTs can be used for longer sequences of instruction (also see the literature on didactical scenarios in Lijnse 1995). The HLT is a useful research instrument to manage the gap between an instruction theory and a concrete teaching experiment. It is informed by general domain-specific and conjectured instruction theories (Gravemeijer 1994), and it informs researchers and teachers how to carry out a particular teaching experiment. After the teaching experiment, it guides the retrospective analysis, and the interplay between the HLT and empirical results forms the basis for theory development. This means that an HLT, after it has been mapped out, has different functions depending on the phase of the DBR and continually develops through the different phases. It can even change during a teaching experiment.

16.1.3.1 HLT in the Design Phase

The development of an HLT starts with an analysis of how the mathematical topic of the design study is elaborated in the curriculum and the mathematical textbooks, an analysis of the difficulties students encounter with this topic, and a reflection on what they should learn about it. These analyses result in the formulation of provisional mathematical learning goals that form the orientation point for the design and redesign of activities in several rounds. While designing mathematical activities the learning goals may become better defined. During these design processes the researcher also starts formulating hypotheses about students' potential learning and about how the teacher would support students' learning processes. The confrontation of a general rationale with concrete tasks often leads to a more specific HLT, which means that the HLT gradually develops during the design phase (Drijvers 2003).

An elaborated HLT thus includes mathematical learning goals, students' starting points with information on relevant pre-knowledge, mathematical problems and assumptions about students' potential learning processes and about how the teacher could support these processes.

16.1.3.2 HLT in Teaching Experiment

During the teaching experiment, the HLT functions as a guideline for the teacher and researcher for what to focus on in teaching, interviewing, and observing. It may happen that the teacher or researcher feels the need to adjust the HLT or instructional activity for the next lesson. As Freudenthal wrote (1991, p. 159), the cyclic alternation of research and development can be more efficient the shorter the cycle is. Minor changes in the HLT are usually made because of incidents in the classroom such as student strategies that were not foreseen, activities that were too difficult, and so on. Such adjustments are generally not accepted in comparative experimental research, but in DBR, changes in the HLT are made to create optimal conditions and are regarded as elements of the data corpus. This means that these changes have to be reported well and the information is stronger when changes are supported by theoretical considerations. The HLT can thus also change during the teaching experiment phase.

16.1.3.3 HLT in the Retrospective Analysis

During the retrospective analysis, the HLT functions as a guideline determining what the researcher should focus on in the analysis. Because predictions are made about students' learning, the researcher can contrast those conjectures with the observations made during the teaching experiment. Such an analysis of the interplay between the evolving HLT and empirical observations forms the basis for developing an instruction theory. After the retrospective analysis, the HLT can be reformulated, often more drastically than during the teaching experiment, and the new HLT can guide a subsequent design phase.

An HLT can be seen as a concretization of an evolving domain-specific instruction theory. Conversely, the instruction theory is informed by evolving HLTs. For example, if patterns of an HLT stabilize after a few cycles, these generalized patterns in learning or instruction and the insights of how these patterns are supported by instructional means can become part of the emerging instruction theory.

Overall, the idea behind developing an HLT is not to design the perfect instructional sequence, which in our view does not exist, but to provide empirically grounded results that others can adjust to their local circumstances. The HLT remains hypothetical because each situation, each teacher, and each class is different. Yet patterns can be found in students' learning that are similar across different teaching experiments. Those patterns and the insights of how particular educational activities support students in particular kinds of reasoning can be the basis for a more general instructional theory of how a particular domain can be taught. Bakker (2004a), for example, noted that when estimating the number of elephants in a picture, students typically used one of four strategies, and these four strategies reoccurred in all of the five classrooms in which he used the same task. Having observed such a pattern in strategy use, the design researcher can assume the pattern to be an element of the instruction theory.

For some readers, the term 'trajectory' might have a linear connotation. Although we aim for a certain direction, like the course of a ship, Bakker's (2004a) HLTs were non-linear in the sense that he did not make a linear sequence of activities in advance that he strictly adhered to (cf. Fosnot and Dolk 2001). Moreover, two subtrajectories came together later on in the sequence. In the following sections we give a more detailed description of the three phases of a DBR cycle and discuss relevant methodological issues. Further details about hypothetical learning trajectories can be found in a special issue of *Mathematical Thinking and Learning* (Mathematical Thinking and Learning 2004, volume 6, issue 2) devoted to HLTs.

The term HLT stems from research in which the teacher was a researcher or a member of the research team (Simon 1995). However, if the teacher is not so familiar with the research team's intentions it may be necessary to pay extra attention to what the teacher can or should do to realize the potential of the learning activities. In such cases, the terms *hypothetical teaching and learning trajectory* (HTLT) or *teaching and learning strategy* (Dierdorp et al. 2011) may be more appropriate.

16.1.4 Phases in DBR

16.1.4.1 Phase 1: Preparation and Design

It is evident that the relevant present knowledge about a topic should be studied first. Gravemeijer (1994) characterizes the design researcher as a tinkerer or, in French, a *bricoleur*, who uses all the material that is at hand, including theoretical insights and practical experience with teaching and designing.

In the first design phase, it is recommended to collect and invent a set of tasks that could be useful and discuss these with colleagues who are experienced in designing for mathematics education. An important criterion for selecting a task is its potential role in the HLT towards the mathematical end goal. Could it possibly lead to types of reasoning that students could build upon towards that end goal? Would it be challenging? Would it be a meaningful context for students?

There are several design heuristics, principles, and guidelines. In Sect. 16.2 we explain heuristics from the theory of Realistic Mathematics Education.

16.1.4.2 Phase 2: Teaching Experiment

The notion of a teaching experiment arose in the 1970s. Its primary purpose was to experience students' learning and reasoning first-hand, and it thus served the purpose of eliminating the separation between the practice of research and the practice of teaching (Steffe and Thompson 2000). Over time, teaching experiments proved useful for a broader purpose, namely as part of DBR. During a teaching experiment, researchers and teachers use activities and types of instruction that according to the HLT seem most appropriate at that moment. Observations in one lesson and theoretical arguments from multiple sources can influence what is done in the next lesson. Observations may include student or teacher deviations from the HLT.

Hence, this type of research is different from experimental research designs in which a limited number of variables are manipulated and effects on other variables are measured. The situation investigated here, the learning of students in a new context with new tools and new end goals, is too complicated for such a set-up.

Besides that, a different type of knowledge is looked for, as pointed out earlier in this chapter: We do not want to assess innovative material or a theory, but we need prototypical educational materials that could be tested and revised by teachers and researchers, and a domain-specific instruction theory that can be used by others to formulate their own HLTs suiting local contingencies.

During a teaching experiment, data collection typically includes student work, tests before and after instruction, field notes, audio recordings of whole-class discussions, and video recordings of every lesson and of the final interviews with students and teachers. We further find ‘mini-interviews’ with students, lasting from about twenty seconds to four minutes, very useful provided that they are carried out systematically (Bakker 2004a).

16.1.4.3 Retrospective Analysis

We describe two types of analysis useful in DBR, a task oriented analysis and a more overall, longitudinal, cyclic approach. The first is to compare data on students’ actual learning during the different tasks with the HLT. To this end we find the data analysis matrix (Table 16.4) described in Dierdorp et al. (2011) useful. The left part of the matrix summarizes the HLT and the right part is filled with excerpts from relevant transcripts, clarifying notes from the researcher as well as a quantitative impression of how well the match was between the assumed leaning as formulated in the HLT and the observed learning. With such analysis it is possible to give an overview, as in Table 16.5, which can help to identify problematic sections in the educational materials. Insights into why particular learning takes place or does not

Table 16.4 Data analysis matrix for comparing HLT and actual learning trajectory (ALT)

Hypothetical learning trajectory			Actual learning trajectory		
Task number	Formulation of the task	Conjecture of how students would respond	Transcript excerpt	Clarification	Match between HLT and ALT: Quantitative impression of how well the conjecture and actual learning matched (e.g., –, 0, +)

Table 16.5 ALT result compared with HLT conjectures for the tasks involving a particular type of reasoning

+				x	x			x	x	x	x	x		x	x	x	x	x	x
±	x		x										x						
–		x				x	x												
Task:	5d	5f	6a	6c	7	8	9c	9e	10b	11c	15	17	23b	23c	24a	24c	25d	34a	42

Note: an x means how well the conjecture accompanying that task matched the observed learning (– refers to confirmation for up to 1/3 of the students, and + to at least 2/3 of the students) take place help to improve the HLTs in subsequent cycles of DBR. This iterative process allows the researcher to improve the predictive power of HLTs across subsequent teaching experiments.

An elaborated HLT would include assumptions about students' potential learning and about how the teacher would support students' learning processes. In this task-oriented analysis above no information is included about the role of the teacher. If there are crucial differences between students' assumed and observed learning processes or if the teaching has been observed to diverge radically from what the researcher had intended, the role of the teacher should be included into the analysis in search of explanations for these discrepancies.

A comparison of HLTs and observed learning is very useful in the redesign process, and allows answers to research questions that ask how particular learning goals could be reached. However, in our experience additional analyses are often needed to gain more theoretical insights into the learning process. An example of such additional analysis is a method inspired by the *constant comparative method* (Glaser and Strauss 1967; Strauss and Corbin 1998) and Cobb and Whitenack's (1996) method of longitudinal analyses. Bakker (2004a) used this type of analysis in his study in the following way. First, all transcripts were read and the videotapes were watched chronologically episode-by-episode. With the HLT and research questions as guidelines, conjectures about students' learning and views were generated and documented, and then tested against the other episodes and other data material (student work, field notes, tests). This testing meant looking for confirmation and counter-examples. The process of conjecture generating and testing was repeated. Seemingly crucial episodes were discussed with colleagues to test whether they agreed with our interpretation or perhaps could think of alternative interpretations. This process is called *peer examination*.

For the analysis of transcripts or videos it is worth considering computer software such as Atlas.ti (Van Nes and Doorman 2010) for coding the transcripts and other data sources. As in all qualitative research, data triangulation (Denscombe 2007) is commonly used in design-based research.

16.1.5 Validity and Reliability

Researchers want to analyze data in a reliable way and draw conclusions that are valid. Therefore, validity and reliability are important concerns. In brief, validity concerns whether we really measure what we intend to measure. Reliability is about independence of the researcher. A brief example may clarify the distinction. Assume a researcher wants to measure students' mathematical ability. He gives everyone 7 out of 10. Is this a valid way of measuring? Is this a reliable way?

It is a very reliable way because the instruction "give all students a 7" can be reliably carried out, independently of the researcher. However, it is not valid, because there is most likely variation between students' mathematical ability, which is not taken into account with this way of measuring.

We should emphasize that validity and reliability are complex concepts with multiple meanings in different types of research. In qualitative research the meanings of validity and reliability are slightly different than in quantitative research. Moreover, there are so many types of validity and reliability that we cannot address them all. In this chapter we have focused on those types that seemed most relevant to us in the context of DBR. The issues discussed in this section are inspired by guidelines of Maso and Smaling (1998) and Miles and Huberman (1994), who distinguish between internal and external validity and reliability.

16.1.5.1 Internal Validity

Internal validity refers to the quality of the data and the soundness of the reasoning that has led to the conclusions. In qualitative research, this soundness is also labeled as *credibility* (Guba 1981). In DBR, several techniques can be used to improve the internal validity of a study.

- During the retrospective analysis conjectures generated and tested for specific episodes are tested for other episodes or by data triangulation with other data material, such as field notes, tests, and other student work. During this testing stage there is a search for counterexamples to the conjectures.
- The succession of different teaching experiments makes it possible to test the conjectures developed in earlier experiments in later experiments.

Theoretical claims are substantiated where possible with transcripts to provide a rich and meaningful context. Reports about DBR tend to be long due to the *thick descriptions* (Geertz 1973) required. For example, the paper by Cobb et al. (2003b) is 78 pages long!

16.1.5.2 External Validity

External validity is mostly interpreted as the generalizability of the results. The question is how we can generalize the results from these specific contexts to be useful for other contexts. An important way to do so is by framing issues as instances of something more general (Cobb et al. 2003a; Gravemeijer and Cobb 2006). The challenge is to present the results (instruction theory, HLT, educational activities) in such a way that others can adjust them to their local contingencies.

In addition to generalizability as a criterion for external validity we mention *transferability* (Maso and Smaling 1998). If lessons learned in one experiment are successfully applied in other experiments, this is a sign of successful generalization. At the end of Sect. 16.2 we give an example of how a new type of learning activity was successfully enacted in a new research project in another country.

16.1.5.3 Internal Reliability

Internal reliability refers to the degree of how independently of the researcher the data are collected and analyzed. It can be improved with several methods. Data collection by objective devices such as audio- and video registrations contribute to the internal reliability. During his retrospective analysis Bakker (2004a) ensured reliability by discussing the critical episodes, including those discussed in Sect. 16.2, with colleagues for peer examination. For measuring interrater reliability, the agreement among independent researchers, it is advised to calculate not only the percentage of agreement but also use Cohen's kappa or another measure that takes into account the probability of agreement by chance (e.g., Krippendorff's alpha). It is not necessary for a second coder to code all episodes, but ensure that a random sample should be of sufficient size: The larger the number of possible codes, the larger the sample required (Bakkenes et al. 2010; Cicchetti 1976). Note that the term internal reliability can also refer to the consistency of responses on a questionnaire or test, often measured with help of Cronbach's alpha.

16.1.5.4 External Reliability

External reliability usually denotes replicability, meaning that the conclusions of the study should depend on the subjects and conditions, and not on the researcher. In qualitative research, replicability is mostly interpreted as virtual replicability. The research must be documented in such a way that it is clear how the research has been carried out and how conclusions have been drawn from the data. A criterion for virtual replicability is 'trackability' (Gravemeijer and Cobb 2006), 'traceability' (Maso and Smaling 1998), or transparency (Akkerman et al. 2008). This means that the reader must be able to track or trace the learning process of the researchers and to reconstruct their study: failures and successes, procedures followed, the conceptual framework used, and the reasons for certain choices must all be reported. In Freudenthal's words:

> Developmental research means: experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience. (1991, p. 161)

We illustrate the general characterization and description of DBR of Sect. 16.1 by an example of a design study on statistics education in Sect. 16.2.

16.2 Example of Design-Based Research

In this second section we illustrate the theory of design-based research (DBR) as outlined in Sect. 16.1 with an example from Bakker's (2004a, b) PhD thesis on DBR in statistics education. We briefly describe the aim and theoretical background of this DBR project and then focus on one design idea, that of growing samples, to illustrate how it is related to different layers of theory and how it was analyzed. Finally we discuss the issue of generalizability. In the appendix we provide a structure of a DBR project with examples from this Sect. 16.2.

16.2.1 Relevance and Aim

The background problem addressed in Bakker's (2004a) research on statistics education was that many stakeholders were dissatisfied with what and how students learned about statistics. For example, in many curricula there was a focus on computing arithmetic means and making bar charts (Friel et al. 2001). Moreover, there was very little knowledge about how to use innovative educational statistics software (cf. Biehler et al. 2013, for an historical overview).

To solve these practical problems, Bakker's (2004a) aim was to contribute to an empirically grounded instruction theory for early statistics education with new computer tools for the age group from 11 to 14. Such a theory should specify patterns in students' learning as well as the means supporting that learning in the domain of statistics education. Like Cobb et al. (2003b), Bakker (2004a) focused his research on the concept of distribution as a key concept in statistics. One problem is that students tend to see isolated data points instead of a data set as a whole (Bakker and Gravemeijer 2004; Konold and Higgins 2003). Yet statistics is about features of data sets, in particular distributions of samples. The selected learning goal was therefore that distribution had to become an object-like entity with which students could see data sets as an entity with characteristics.

16.2.2 Research Question

Bakker's initial research question was: How can students with little statistical background develop a notion of distribution? In trying to answer this question in grade 7, however, Bakker came to include a focus on other statistical key concepts such as data, center, and sampling because these are so intricately connected to that of distribution (Bakker and Derry 2011). The concept of distribution also proved hard for seventh-grade students. The initial research question was therefore reformulated for grade 8 as follows: How can coherent reasoning about distribution be promoted in relation to data, variability, and sampling in a way that is meaningful for students with little statistical background?

Our point here is that research questions can change during a research project. Indeed, the better and sharper your research question is in the beginning of the project, the better and more focused your data collection will be. However, our experience is that most DBR researchers, due to progressive insight, end up with slightly different research questions than they started with.

As pointed out in Sect. 16.1, DBR typically draws on several types of theories. Given the importance of graphical representations in statistics education, it made sense for Bakker to draw on semiotics as an orienting framework. He came to focus on semiotics, in particular Peirce's ideas on diagrammatic reasoning. The domain-specific theory of Realistic Mathematics Education proved a useful framework for action in the design process even though it had hardly been applied in statistics education.

16.2.3 Orienting Framework: Diagrammatic Reasoning

The learning goal was that distribution would become an object-like entity. Theories on reification of concepts (Sfard and Linchevski 1992) and the relation between process and concept (cf. Tall et al. 2000, on *procept*) were drawn upon. One theoretical question unanswered in the literature was what the process nature of a distribution could be. It is impossible to make sense of graphs without having appropriate conceptual structures, and it is impossible to communicate about concepts without any representations. Thus, to develop an instruction theory it is necessary to investigate the relation between the development of the meaning of graphs and concepts. After studying several theories in this area, Bakker deployed Peirce's semiotic theory on diagrammatic reasoning (Bakker 2007; Bakker and Hoffmann 2005). For Peirce, a diagram is a sign that is meant to represent relations. Diagrammatic reasoning involves three steps:

1. The first step is to *construct* a diagram (or diagrams) by means of a representational system such as Euclidean geometry, but we can also think of diagrams in computer software or of an informal student sketch of statistical distribution. Such a construction of diagrams is supported by the need to represent the relations that students consider significant in a problem. This first step may be called *diagrammatization*.
2. The second step of diagrammatic reasoning is to *experiment* with the diagram (or diagrams). Any experimenting with a diagram is executed within a not necessarily perfect representational system and is a rule or habit-driven activity. Contemporary researchers would stress that this activity is situated within a practice. What makes experimenting with diagrams important is the rationality immanent in them (Hoffmann 2002). The rules define the possible transformations and actions, but also the constraints of operations on diagrams. Statistical diagrams such as dot plots are also bound by certain rules: a dot has to be put above its value on the *x* axis and this remains true even if for instance the scale is changed. Peirce stresses the importance of doing something when thinking or reasoning with diagrams:

   > Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra, permissible transformations are made. Thereupon the faculty of observation is called into play. (CP 4.233—CP refers to Peirce's collected papers, volume 4, section 233)

In the software used in this research, students can do something with the data points such as organizing them into equal intervals or four equal groups.

3. The third step is to observe the results of experimenting. We refer to this as the *reflection* step. As Peirce wrote, the mathematician observing a diagram "puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction" (Peirce 1976 III, p. 749). In this way he can "discover unnoticed and hidden relations among the parts" (Peirce CP 3.363; see also CP 1.383). The power of diagrammatic reasoning is that "we are continually bumping up against hard fact. We expected one thing, or passively took it for granted, and had the image of it in our minds, but experience forces that idea into the background, and compels us to think quite differently" (Peirce CP 1.324).

Diagrammatic reasoning, in particular the reflection step, is what can introduce the 'new'. New implications within a given representational system can be found, but possibly the need is felt to construct a new diagram that better serves its purpose.

16.2.4 Domain-Specific Framework for Action: Realistic Mathematics Education (RME)

As pointed out by diSessa and Cobb (2004), grand theories and orienting frameworks do not tell the design researcher how to design learning environments. For this purpose, frameworks for action can be useful. Here we discuss Realistic Mathematics Education (RME).

Our research took place in the tradition of RME as developed over the last 40 years at the Freudenthal Institute (Freudenthal 1991; Gravemeijer 1994; Treffers 1987; van den Heuvel-Panhuizen 1996). RME is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing educational materials for mathematics education. RME emerged from research and development in mathematics education in the Netherlands in the 1970s and it has since been used and extended, also in other countries.

The central principle of RME is that mathematics should always be meaningful to students. For Freudenthal, mathematics was an extension of common sense, a system of concepts and techniques that human beings had developed in response to phenomena they encountered. For this reason, he advised a so-called *historical phenomenology* of concepts to be taught, a study of how concepts had been developed in relation to particular phenomena. The insights from such a study can be input for the design process (Bakker and Gravemeijer 2006).

The term 'realistic' stresses that problem situations should be 'experientially real' for students (Cobb et al. 1992). This does not necessarily mean that the problem situations are always encountered in daily life. Students can experience an abstract mathematical problem as real when the mathematics of that problem is meaningful to them. Freudenthal's (1991) ideal was that mathematical learning should be an enhancement of common sense. Students should be allowed and encouraged to invent their own strategies and ideas, and they should learn mathematics on their own authority. At the same time, this process should lead to particular end goals. This process is called *guided reinvention*—one of the design heuristics of RME. This heuristic points to the question that underlies much of the RME-based research, namely that of how to support this process of engaging students in meaningful mathematical and statistical problem solving, and using students' contributions to reach certain end goals.

The theory of RME is especially tailored to mathematics education, because it includes specific tenets on and design heuristics for mathematics education. For a description of these tenets we refer to Treffers (1987) and for the design heuristics to Gravemeijer (1994) or Bakker and Gravemeijer (2006).

16.2.5 Methods

The absence of the type of learning aimed for is a common reason to carry out design research. For Bakker's study in statistics education, descriptive, comparative, or evaluative research did not make sense because the type of learning aimed for could not be readily observed in classrooms. Considerable design and research effort first had to be taken to foster specific innovative types of learning. Bakker therefore had to design HLTs with accompanying educational materials that supported the desired type of learning about distribution. Design-based research offers a systematic approach to doing that while simultaneously developing domain-specific theories about how to support such learning for example here on the domain of statistics. In general, DBR researchers first need to create the conditions in which they can develop and test an instruction theory, but to create those conditions they also need research.

Teaching experiment. Bakker designed educational materials with accompanying HLTs in several cycles. Here we focus on the last cycle, involving a teaching experiment in grade 8. Half of the lessons were carried out in a computer lab and as part of them students used two minitools (Cobb et al. 1997), simple Java applets with which they analyzed data sets on, for instance, battery life span, car colours, and salaries (Fig. 16.3). The researcher was responsible for the educational materials and the teacher was responsible for the teaching, though we discussed in advance on a weekly basis both the materials and appropriate teaching style. Three preservice teachers served as assistants and helped with videotaping and interviewing students and with analyzing the data.

In the example that we elaborate we focus on the fourth of a series of ten lessons, each 50 min long. In this specific lesson, students reasoned about larger and larger samples and about the shape of distributions.

Subjects. The teaching experiment was carried out in an eighth-grade class with 30 students in a state school in the center of a Dutch city. The students in this study were being prepared for pre-university (*vwo*) or higher professional education (*havo*). The students in the class reported on here were not used to whole-class discussions, but rather to be "taken by the hand" as the teacher called it; they were characterized by the three research assistants as "passive but willing to cooperate." These students had no prior instruction in statistics; they were acquainted with bar and line graphs, but not with dot plots, histograms, or box plots. Students already knew the mean from calculating their report grades, but mode and median were not introduced until the second half of the educational sequence after variability, data, sampling, and shape had been topics of discussion.

Data collection. The collected data on which the results presented in this chapter are based include student work, field notes, and the audio and video recordings of class activities that the three assistants and researcher made in the classroom. An essential part of the data corpus was the set of mini-interviews we held during the lessons; they varied from about twenty seconds to four minutes, and were meant to find out what concepts and graphs meant for students, or how the minitools were used. These mini-interviews influenced students' learning because they often stimulated reflection. However, we think that the validity of the research was not put in danger by this, since the aim was to find out how students learned to reason with shape or distribution, not whether teaching the sequence in other eighth-grade classes would lead to the same results in the same number of lessons. Furthermore, the interview questions were planned in advance as part of the HLT, and discussed with the assistants.

Retrospective analysis. In this example we do not illustrate how HLTs can be compared with observed learning (see Dierdorp et al. 2011). Here we highlight one type of analysis that in Bakker's case yielded more theoretical insights: a method resembling Glaser and Strauss's constant comparative method (Glaser and Strauss 1967). For the analysis, Bakker watched the videotapes, read the transcripts, and formulated conjectures on students' learning on the basis of transcript episodes. Numbering the conjectures served as useful codes to work with during the analysis. Examples of such codes and conjectures were:

C1. Students divide imaginary data sets into three groups of low, 'average', and high values.

C2. Students either characterize spread as range or look very locally at spread

C3. Students are inclined to think of small samples when first asked about how one could test something (batteries, weight).

C5. What-if questions work well for letting students think of aggregate features of a graph or a situation. What would a weight graph of older students look like? What would the graph look like if a larger sample was taken? What would a larger sample of a good battery brand look like?

C7. Students' notions of spread, distribution, and density are not yet distinguished. When explaining how data are spread out, they often describe the distribution or the density in some area.

C9. Even when students see a large sample of a particular distribution, they often do not see the shape we see in it.

The generated conjectures were tested against other episodes and the rest of the collected data (student work, field observations, and tests) in the next round of anal-

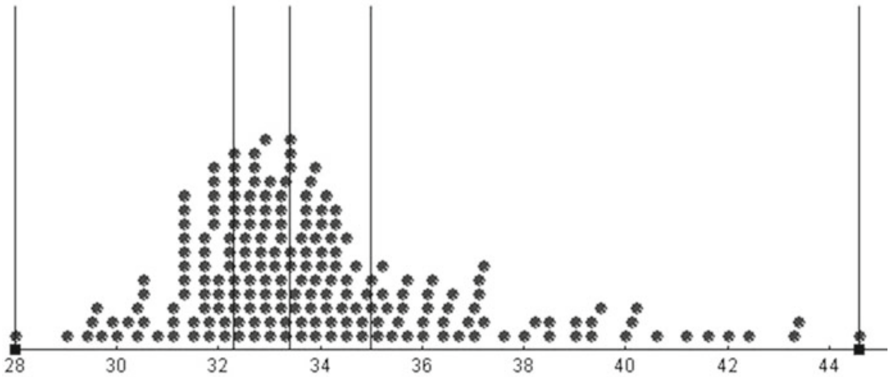

Fig. 16.2 Jeans data with four equal groups option in Minitool 2

ysis by data triangulation. Conjectures that were confirmed remained in the list; conjectures that were refuted were removed from the list. Then the whole generating and testing process was repeated. The aforementioned examples were all confirmed throughout this analysis.

To get a sense of the interrater reliability of the analysis, about one quarter of the episodes including those discussed in this chapter and the conjectures belonging to these episodes were judged by the three assistants who attended the teaching experiment. The amount of agreement among judges was very high: all four judges agreed about 33 out of 35 codes. A code was only accepted if all judges agreed after discussion. We give an example of a code that was finally rejected and one that was accepted. This example stems from the seventh lesson in which two students used the four equal groups option in Minitool 2 for a revised version of the jeans activity. Their task was to advise a jeans factory about frequencies of jeans sizes to be produced (Fig. 16.2).

Sofie Because then you can best see the spread, how it is distributed.
Int. How it is distributed. And how do you see that here [in this graph]? What do you look at then? (…)
Sofie Well, you can see that, for example, if you put a [vertical] line here, here a line, and here a line. Then you see here [two lines at the right] that there is a very large spread in that part, so to speak.

In the first line, Sofie seems to use the terms spread and distributed as almost synonymous. This line was therefore coded with C7, which states that "students' notions of spread, distribution, and density are not yet distinguished. When explaining how data are spread out, they often describe the distribution or the density in some area." In the second line, Sofie appears to look at spread very locally, hence it was coded with C2, which states that "students either characterize spread as range or look very locally at spread."

We also give an example of a code assignment that was dismissed in relation to the same diagram.

Int. What does this tell you? Four equal groups?
Melle Well, I think that most jeans are between 32 and 34 [inches].

We had originally assigned the code C1 to the this episode (students talk about data sets as consisting of three groups of low, 'average', and high values), because "most jeans are between 32 and 34" implies that below 32 and above 34 the frequencies are relatively low. In the episode, however, this student did not talk about three groups of low, average, and high values or anything equivalent. We therefore removed the code from this episode.

16.2.6 HLT and Retrospective Analysis

To illustrate relationships between theory, method, and results, this section presents the analysis of students' reasoning during one educational activity which was carried out in the fourth lesson. Its goal was to stimulate students to reason about larger and larger samples. We summarize the HLT of that lesson: the learning goal, the activity of growing a sample and the assumptions about students' potential learning processes and about how the teacher could support these processes. We then present the retrospective analysis of three successive phases in growing a sample.

The overall *goal* of the growing samples activity as formulated in the hypothetical learning trajectory for this fourth lesson was to stimulate students' diagrammatic reasononing about shape in relation to sampling and distribution aspects in the context of weight. This implied that students should first make diagrams, then experiment with them and reflect on them. The idea was to start with ideas invented by the students and guide them toward more conventional notions and representations. This process of guiding students toward these culturally accepted concepts and graphs while building on their own inventions is called guided reinvention. We had noted in previous teaching experiments that students were inclined to choose very small samples initially. It proved necessary to stimulate reflection on the disadvantages of such small samples and have them predict what larger samples would look like. Such insights from the analyses of previous teaching experiments helped to better formulate the HLT of a new teaching experiment. More particularly, Bakker assumed that starting with students' initial ideas about small samples and asking for predictions about larger samples would make students aware of various features of distributions.

The *activity* of growing a sample consisted of three phases of making sketches of a hypothetical situation and comparing those sketches with graphs displaying real data sets. In the first phase students had to make a graph of their own choice of a predicted weight data set with sample size 10. The results were discussed by the teacher to challenge this small sample size, and in the subsequent phases students had to predict larger data sets, one class and three classes in the second phase, and all students in the province in the third phase. Thus, three such phases took place as

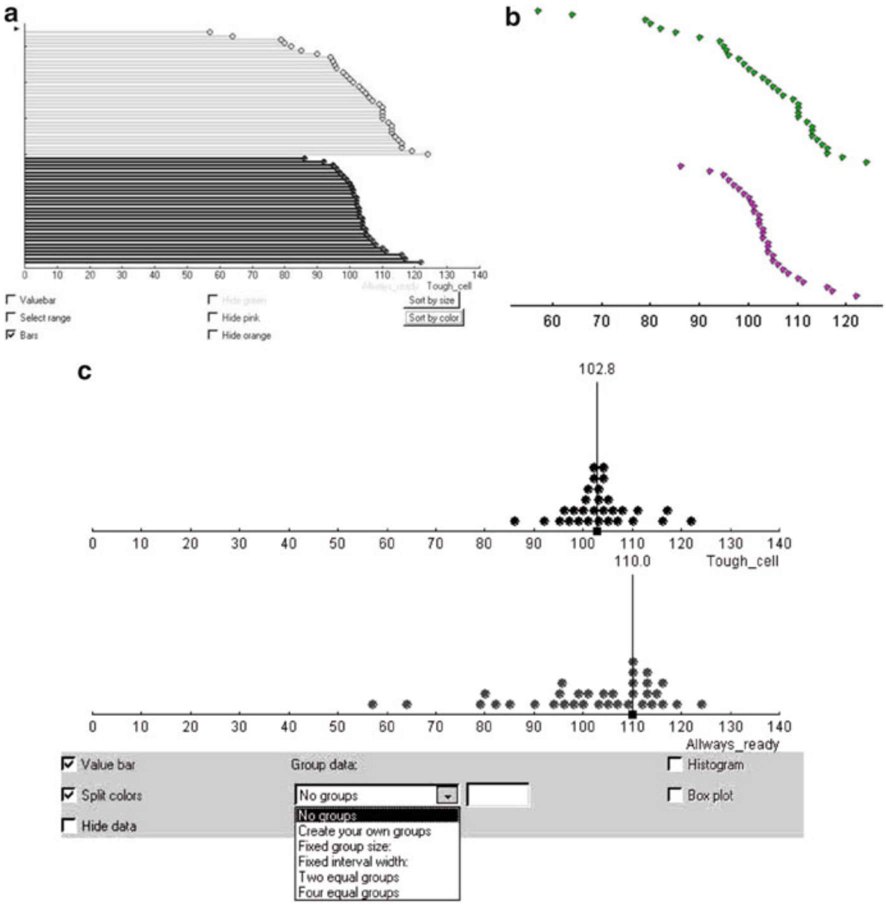

Fig. 16.3 (**a**) Minitool 1 showing a value-bar graph of battery life spans in hours of two brands. (**b**) Minitool 1, but with bars hidden. (**c**) Minitool 2 showing a dot plot of the same data sets

described and analyzed below. Aiming for guided reinvention, the teacher and researcher tried to strike a balance between engaging students in statistical reasoning and allowing their own terminology on the one hand, and guiding them in using conventional and more precise notions and graphical representations on the other. Figure 16.3b is the result of focusing only on the endpoints of the value bars in Fig. 16.3a. Figure 16.3c is the result of these endpoints falling down vertically on the x-axis. In this way, students can learn to understand the relationship between value-bar graphs and dot plots, and what distribution features in different representations look like (Bakker and Hoffmann 2005).

16.2.6.1 Analysis of the First Phase of Growing a Sample

The text of the student activity sheet for the fourth lesson contained a number of tasks that we cite in the following subsections. The sheet started as follows:

> Last week you made graphs of predicted data for a balloon pilot. During this lesson you will get to see real weight data of students from another school. We are going to investigate the influence of the sample size on the shape of the graph.
>
> Task a. Predict a graph of ten data values, for example with the dots of minitool 2.

The sample size of ten was chosen because the students had found that size reasonable after the first lesson in the context of testing the life span of batteries. Figure 16.4 shows examples for three different types of diagrams the students made to show their predictions: there were three value-bar graphs (such as in minitool 1—e.g., Ruud's diagram), eight with only the endpoints (such as with the option of minitool 1 to "hide bars"—e.g., Chris's diagram) and the remaining nineteen plots were dot plots (such as in minitool 2—e.g., Sandra's diagram). For the remainder of this section, the figures and written explanations of these three students are demonstrated, because their work gives an impression of the variety of the whole class. Those three students were chosen because their diagrams represent all types of diagrams made in this class, also for other phases of growing a sample.

To stimulate the reflection on the graphs, the teacher showed three samples of ten data points on the blackboard and students had to compare their own graphs (Fig. 16.4) with the graphs of the real data sets (Fig. 16.5).

> Task b. You get to see three different samples of size 10. Are they different from your own prediction? Describe the differences.

The reason for showing three small samples was to show the variation among these samples. There were no clear indications, though, that students conceived this variation as a sign that the sample size was too small for drawing conclusions, but they generally agreed that larger samples were more reliable. The point relevant to the analysis is that students started using predicates to describe aggregate features of the graphs. The written answers of the three students were the following:

Ruud	Mine looks very much like what is on the blackboard.
Chris	The middle-most [diagram on the blackboard] best resembles mine because the weights are close together and that is also the case in my graph. It lies between 35 and 75 [kg].
Sandra	The other [real data] are more weights together and mine are further apart.

Ruud's answer is not very specific, like most of the written answers in the first phase of growing samples. Chris used the predicate "close together" and added numbers to indicate the range, probably as an indication of spread. Sandra used such terms as "together" and "further apart," which address spread. The students in the class used common predicates such as "together," "spread out" and "further apart" to describe features of the data set or the graph. For the analysis it is important to

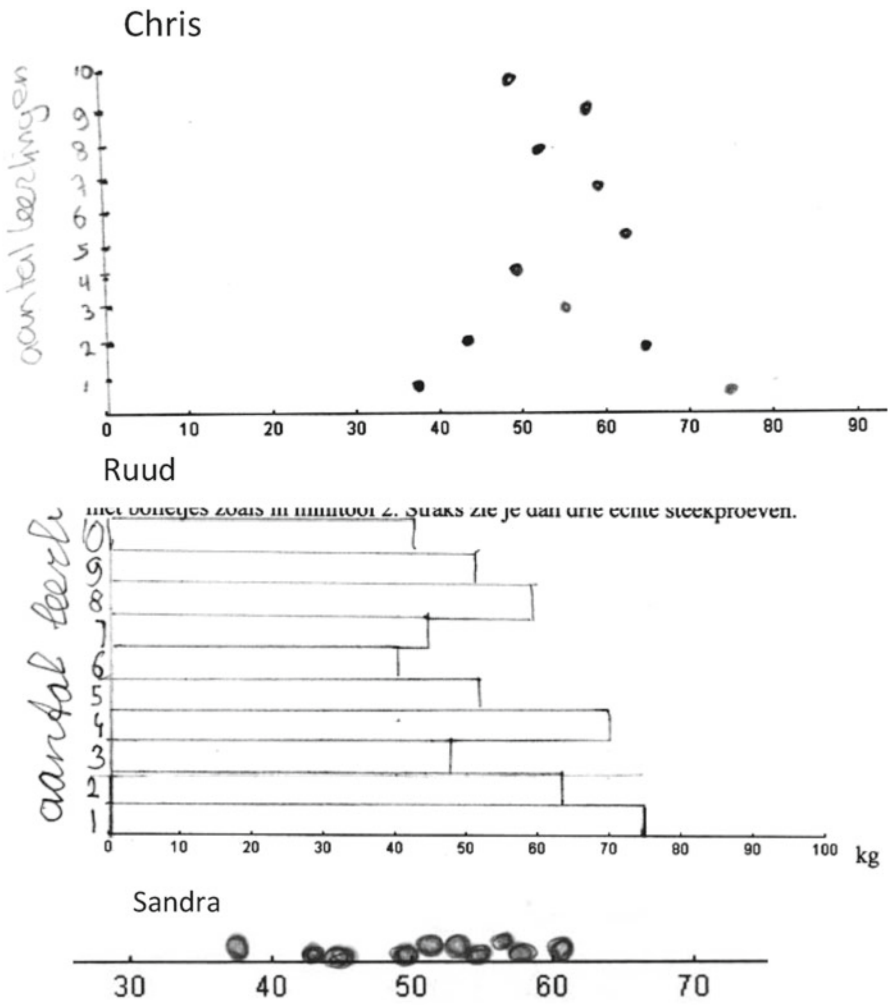

Fig. 16.4 Student predictions (Ruud, Chris, and Sandra) for ten data points (weight in kg) (Bakker 2004a, p. 219)

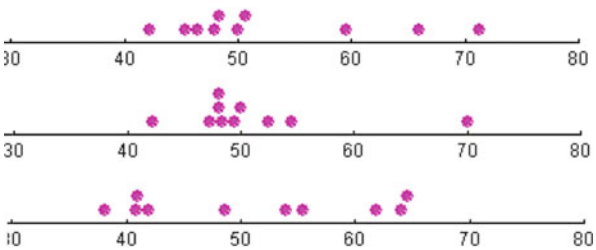

Fig. 16.5 Three real data sets in minitool 2 (Bakker 2004a, p. 219)

note that the students used predicates (together, apart) and no nouns (spread, average) in this first phase of growing samples. Spread can only become an object-like concept, something that can be talked about and reasoned with, if it is a noun. In the semiotic theory of Peirce, such transitions from the predicate "the dots are spread out" to "the spread is large" are important steps in the formation of concepts (see Bakker and Derry 2011, for our view on concept formation).

16.2.6.2 Analysis of the Second Phase of Growing a Sample

The students generally understood that larger samples would be more reliable. With the feedback students had received after discussing the samples of ten data points in dot plots, students had to predict the weight graph of a whole class of 27 students and of three classes with 67 students (27 and 67 were the sample sizes of the real data sets of eighth graders of another school).

> Task c. We will now have a look how the graph changes with larger samples. Predict a sample of 27 students (one class) and of 67 students (three classes).
>
> Task d. You now get to see real samples of those sizes. Describe the differences. You can use words such as majority, outliers, spread, average.

During this second phase, all of the students made dot plots, probably because the teacher had shown dot plots on the blackboard, and because dot plots are less laborious to draw than value bars (only one student started with a value-bar graph for the sample of 27, but switched to a dot plot for the sample of 67). The hint on statistical terms was added to make sure that students' answers would not be too superficial as (often happened before) and to stimulate them to use such notions in their reasoning. It was also important for the research to know what these terms meant to them. When the teacher showed the two graphs with real data, once again there was a short class discussion in which the teacher capitalized on the question of why most student predictions now looked pretty much like what was on the blackboard, whereas with the earlier predictions there was much more variation. No student had a reasonable explanation, which indicates that this was an advanced question. The figures of the same three students are presented in Figs. 16.6 and 16.7 and their written explanations were:

Ruud	My spread is different.
Chris	Mine resembles the sample, but I have more people around a certain weight and I do not really have outliers, because I have 10 about the 70 and 80 and the real sample has only 6 around the 70 and 80.
Sandra	With the 27 there are outliers and there is spread; with the 67 there are more together and more around the average.

Here, Ruud addressed the issue of spread ("my spread is different"). Chris was more explicit about a particular area in her graph, the category of high values. She also correctly used the term "sample," which was newly introduced in the second lesson. Sandra used the term "outliers" at this stage, by which students meant "extreme values," which did not necessarily mean exceptional or suspect values.

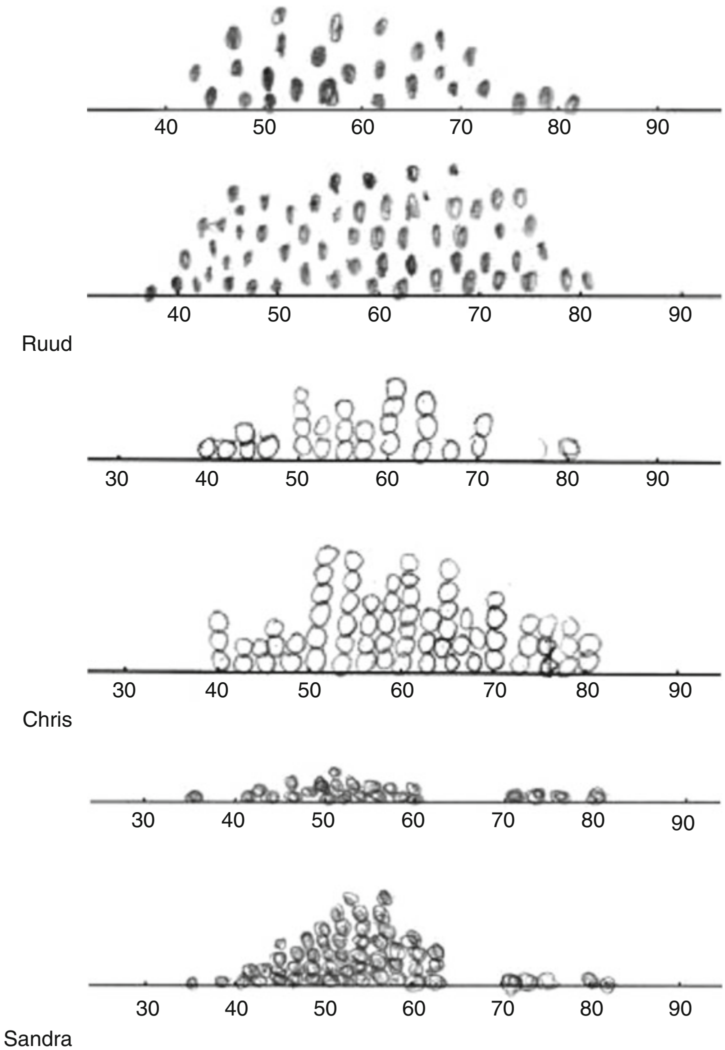

Fig. 16.6 Predicted graphs for one class (n = 27, top plot) and three classes (n = 67, bottom plot) by Ruud, Chris, and Sandra (Bakker 2004a, p. 222)

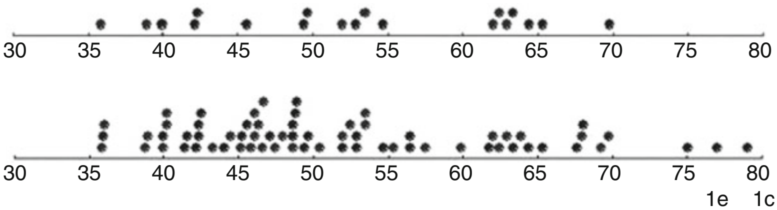

Fig. 16.7 Real data sets of size 27 and 67 of students from another school (Bakker 2004a, p. 222)

She also seemed to locate the average somewhere and to understand that many students are about average. These examples illustrate that students used statistical notions for describing properties of the data and diagrams.

In contrast to the first phase of growing a sample, students used nouns instead of just predicates for comparing the diagrams. Like others Ruud used the noun "spread" ("my spread is different") whereas students earlier used only predicates such as "spread out" or "further apart" (e.g., Sandra). Of course, this does not always imply that if students use these nouns that they are thinking of the right concept. Statistically, however, it makes a difference whether we say, "the dots are spread out" or "the spread is large." In the latter case, spread is an object-like entity that can have particular aggregate characteristics that can be measured, for instance by the range, the interquartile range, or the standard deviation. Other notions such as outliers, sample, and average, are now used as nouns, that is as conceptual objects that can be talked about and reasoned with.

16.2.6.3 Analysis of the Third Phase of Growing a Sample

The aim of the hypothetical learning trajectory was that students would come to draw continuous shapes and reason about them using statistical terms. During teaching experiments in the seventh-grade experiments (Bakker and Gravemeijer 2004), reasoning with continuous shapes turned out to be difficult to accomplish, even if it was asked for. It often seemed impossible to nudge students toward drawing the general, continuous shape of data sets represented in dot plots. At best, students drew spiky lines just above the dots. This underlines that students have to construct something new (a notion of signal, shape, or distribution) with which they can look differently at the data or the variable phenomenon.

In this last phase of growing the sample, the task was to make a graph showing data of all students in the city, not necessarily with dots. The intention of asking this was to stimulate students to use continuous shapes and dynamically relate samples to populations, without making this distinction between sample and population explicit yet. The conjecture was that this transition from a discrete plurality of data values to a continuous entity of a distribution is important to foster a notion of distribution as an object-like entity with which students could model data and describe aggregate properties of data sets. The task proceeded as follows:

> Task e. Make a weight graph of a sample of all eighth graders in the city. You need not draw dots. It is the shape of the graph that is important.
>
> Task f. Describe the shape of your graph and explain why you have drawn that shape.

The figures of the same three students are presented in Fig. 16.8 and their written explanations were:

Ruud	Because the average [values are] roughly between 50 and 60 kg.
Chris	I think it is a pyramid shape. I have drawn my graph like that because I found it easy to make and easy to read.
Sandra	Because most are around the average and there are outliers at 30 and 80 [kg].

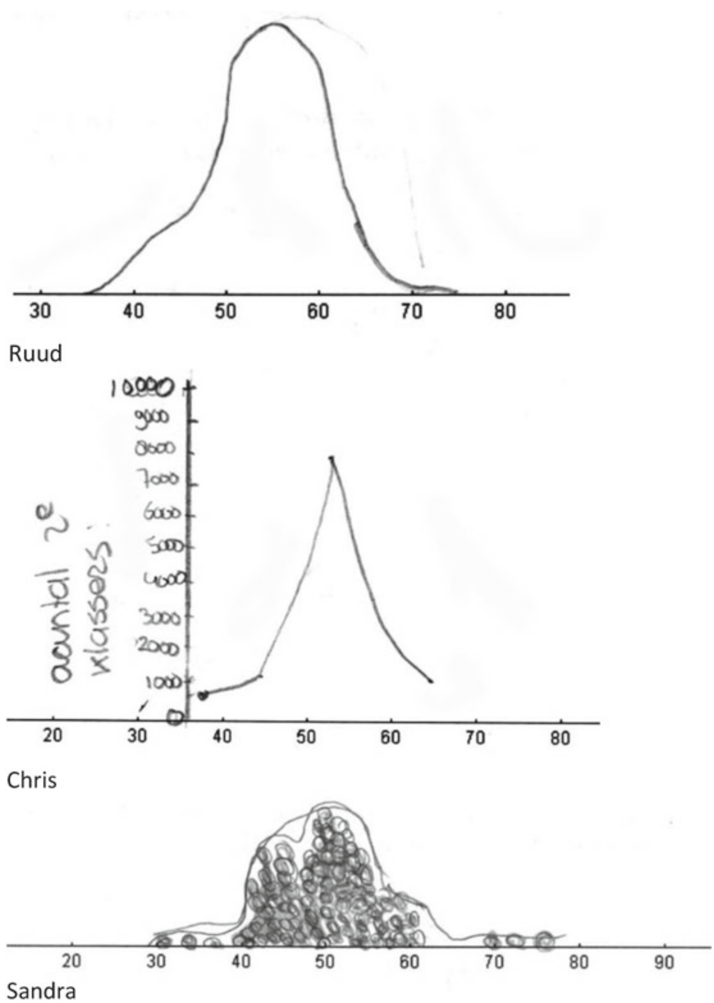

Fig. 16.8 Predicted graphs for all students in the city by Ruud, Chris, and Sandra (Bakker 2004a, p. 224)

Ruud's answer focused on the average group. During an interview after the fourth lesson, Ruud like three other students literally called his graph a "bell shape," though he had probably not encountered that term in a school situation before. This is probably a case of *reinvention*. Chris's graph was probably inspired by line graphs that the students made during mathematics lessons. She introduced the vertical axis with frequency, though such graphs had not been used before in the statistics course. Sandra may have started with the dots and then drawn the continuous shape.

In this third phase of growing a sample, 23 students drew a bump shape. The words they used for the shapes were pyramid (three students), semicircle (one), and bell shape (four). Many students drew continuous shapes but these were all symmetrical. Since weight distributions are not symmetrical and because skewness is an important concept, a subsequent lesson addressed asymmetrical shapes in relation to the weight data (see Bakker 2004b).

16.2.7 Reflection on the Example

The research question we addressed in the example is: How can coherent reasoning about distribution be promoted in relation to data, variability, and sampling in a way that is meaningful for students with little statistical background? We now discuss those key elements for the educational activity and speculate about what can be learned from the analysis presented here.

The activity of growing a sample involved short phases of constructing diagrams of new hypothetical situations, and comparing these with other diagrams of a real sample of the same size. The activity has a broader empirical basis than just the teaching experiment reported in this chapter, because it emerged from a previous teaching experiment (Bakker and Gravemeijer 2004) as a way to address shape as a pattern in variability.

To theoretically generalize the results, Bakker analyzed students' reasoning as an instance of diagrammatic reasoning, which typically involves constructing diagrams, experimenting with them, and reflecting on the results of the previous two steps. In this growing samples activity, the quick alternation between prediction and reflection during diagrammatic reasoning appears to create ample opportunities for concept formation, for instance of spread.

In the first phase involving the prediction of a small data set, students noted that the data were more spread out, but in subsequent phases, students wrote or said that the spread was large. From the terms used in this fourth lesson, we conclude that many statistical concepts such as center (average, majority), spread (range and range of subsets of data), and shape had become topics of discussion (object-like entities) during the growing samples activity. Some of these words were used in a rather unconventional way, which implies that students needed more guidance at this point. Shape became a topic of discussion as students predicted that the shape of the graph would be a semicircle, a pyramid, or a bell shape, and this was exactly what the HLT targeted. Given the students' minimal background in statistics and the fact that this was only the fourth lesson of the sequence, the results were promising. Note, however, that such activities cannot simply be repeated in other contexts; they need to be adjusted to local circumstances if they are to be applied in other situations.

The instructional activity of growing samples later became a connecting thread in Ben-Zvi's research in Israel, where it also worked to help students develop statistical concepts in relation to each other (Ben-Zvi et al. 2012). This implies that this instructional idea was transferable to other contexts. The transferability of instructional ideas from the USA to the Netherlands to Israel, even to higher levels of education, illustrates that generalization in DBR can take place across contexts, cultures and age group.

16.2.8 Final Remarks

The example presented in Sect. 16.2 was intended to substantiate the issues discussed in Sect. 16.1, and we hope that readers will have a sense of what DBR could look like and feel invited to read more about it. It should be noted that there are many variants of DBR. Some are more focused on theory, some more on empirically grounded products. Some start with predetermined learning outcomes, others have more open-ended goals (cf. Engeström 2011). DBR may be a challenging research approach but it is in our experience also a very rewarding one given the products and insights that can be gained.

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Appendix: Structure of a DBR Project with Illustrations

In line with Oost and Markenhof (2010), we formulate the following general criteria for any research project:

1. The research should be **anchored** in the literature.
2. The research aim should be **relevant**, both in theoretical and practical terms.
3. The formulation of aim and questions should be **precise**, i.e. using concepts and definitions in the correct way.
4. The method used should be **functional** in answering the research question(s).
5. The overall structure of the research project should be **consistent**, i.e. title, aim, theory, question, method and results should form a coherent chain of reasoning.

In this appendix we present a structure of general points of attention during DBR and specifications for our statistics education example, including references to relevant sections in the chapter. In this structure these criteria are bolded. This structure could function as the blueprint of a book or article on a DBR project.

	General points	Examples
Introduction:	1. Choose a topic	1. Statistics education at the middle school level
	2. Identify common problems	2. Statistics as a set of unrelated concepts and techniques
	3. Identify knowledge gap and relevance	3. How middle school students can be supported to develop a concept of distribution and related statistical concepts
	4. Choose mathematical learning goals	4. Understanding of distribution (2.1)

(continued)

	General points	Examples
Literature review forms the basis for formulating the research aim (the research has to be **anchored** and **relevant**)		
Research aim:	It has to be clear whether an aim is descriptive, explanatory, evaluative, advisory etc. (1.2.2)	Contribute to an empirically and theoretically grounded instruction theory for statistics education at the middle school level (advisory aim) (2.1)
Research aim has to be narrowed down to a research question and possibly subquestions with the help of different theories		
Literature review (theoretical background):	Orienting frameworks	Semiotics (2.3)
	Frameworks for action	Theories on learning with computer tools
	Domain-specific learning theories (1.2.8)	Realistic Mathematics Education (2.4)
With the help of theoretical constructs the research question(s) can be formulated (the formulation has to be **precise**)		
Research question:	Zoom in what knowledge is required to achieve the research aim	How can students with little statistical background develop a notion of distribution?
It should be underpinned why this research question requires DBR (the method should be **functional**)		
Research approach:	The lack of the type of learning aimed for is a common reason to carry out DBR: It has to be enacted so it can be studied	Dutch statistics education was atomistic: Textbooks addressed mean, median, mode, and different graphical representations one by one. Software was hardly used. Hence the type of learning aimed for had to be enacted.
Using a research method involves several research instruments and techniques		
Research instruments and techniques	Research instrument that connects different theories and concrete experiences in the form of testable hypotheses.	Series of hypothetical learning trajectories (HLTs)
	1. Identify students' prior knowledge	1. Prior interviews and pretest
	2. Professional development of teacher	2. Preparatory meetings with teacher
	3. Interview schemes and planning	3. Mini-interviews, observation scheme
	4. Intermediate feedback and reflection with teacher	4. Debrief sessions with teacher
	5. Determine learning yield (1.4.2)	5. Posttest
Design	Design guidelines	Guided reinvention; Historical and didactical phenomenology (2.4)
Data analysis	Hypotheses have to be tested by comparison of hypothetical and observed learning. Additional analyses may be necessary (1.4.3)	Comparison of hypothetical and observed learning
		Constant comparative method of generating conjectures and testing them on the remaining data sources (2.6)

(continued)

	General points	Examples
Results	Insights into patterns in learning and means of supporting such learning	Series of HLTs as progressive diagrammatic reasoning about growing samples (2.6)
Discussion	Theoretical and practical yield	Concrete example of an historical and didactical phenomenology in statistics education
		Application of semiotics in an educational domain
		Insights into computer use in the mathematics classroom
		Series of learning activities
		Improved computer tools
The aim, theory, question, method and results should be aligned (the research has to be **consistent**)		

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Chapter 17
Perspectives on Design Research: The Case of Didactical Engineering

Michèle Artigue

Abstract In what is often called the "French didactical culture," design has always played an essential role in research. This is attested by the introduction and institutionalization of a specific concept, that of *didactical engineering*, already in the early 1980s and by the way didactical engineering has accompanied the development of didactical research, both in its fundamental and applied dimensions. In this chapter, I present this vision of design and its characteristics as a research methodology, coming back to its historical origin in close connection with the development of the theory of didactical situations, tracing its evolution along the last three decades, and illustrating this methodology by some particular examples. I also consider current developments within this design culture, especially those linked to the integration of a design dimension into the anthropological theory of didactics and also to the idea of didactical engineering of second generation introduced for addressing more efficiently the development dimension of didactical engineering.

Keywords Didactical engineering • Theory of didactical situations

17.1 Introduction

Design has always played a substantial role in mathematics education up to the point that some researchers consider this field as a design science (see, for instance, Wittmann 1998; Cobb 2007). But the conception of design and the exact role it is given in research strongly depend on educational cultures. In this chapter we consider the case of what is often called the "French didactical culture" in which design has always played a fundamental role. This importance of design is attested by the introduction and institutionalization of a specific concept, that of *Didactical*

M. Artigue (✉)
Laboratoire de Didactique, LDAR, Université Paris Diderot – Paris 7,
Case 7018, 75205, Paris Cedex 13, France
e-mail: michele.artigue@univ-paris-diderot.fr

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Engineering (DE in the following) already in the early eighties. Since that time DE, which developed in close connection with the theory of didactical situations initiated by Brousseau (cf. (Warfield 2006) for an introduction and (Brousseau 1997) for a more detailed vision), has accompanied the development of didactical research, both in its fundamental and applied dimensions. This chapter is structured into four main sections. In the first section I briefly review the development of DE from its emergence in the early eighties until now, and clarify its links with the theory of didactical situations (see also (Bessot 2011)). In the second section I present its characteristics as a research methodology. In the third section I illustrate this methodology with examples taken at different levels of schooling. In the fourth section I consider two recent evolutions of DE. The first one is conveyed by the anthropological theory of didactics in terms of course of study and research that considers very open forms of design; the second one is "didactical engineering of second generation" introduced by Perrin-Glorian for addressing dissemination and up-scaling issues (Perrin-Glorian 2011). Beyond the many examples of realizations, the writing of this chapter has been especially inspired by some foundational texts such as (Chevallard 1982; Artigue 1990, 2002, 2009), and by the extensive reflection on didactical engineering carried out at the XV^e Summer School of Didactics of Mathematics in 2009 (Margolinas et al. 2011).

17.2 Didactical Engineering: An Historical Review

The emergence, consolidation and evolution of didactical engineering can be traced through the successive summer schools of didactics of mathematics organized every 2 years in France since 1980. In this brief historical review, I focus on three of these (1982, 1989, 2009) for which DE was a specific theme of study. Already, at the second summer school in 1982, DE was one of the themes addressed. Chevallard prepared a specific manuscript note for supporting the work of the summer school collective (Chevallard 1982); Brousseau gave a course, and practical sessions were organized around this theme. Accessible documents regarding this summer school show the shared conviction that didactical research should give a more central role to the construction and study of classroom realizations. French researchers expressed concerns about the observed tendency to privilege methodologies borrowed from established fields such as psychology (clinical interviews, questionnaires, pre-test/post-test comparisons…) for ensuring the scientific legitimacy of research in mathematics education. They pointed out that the didactics of mathematics is a genuine scientific field whose methodologies should be in line with its specific purpose: the study of intentional dissemination of mathematical knowledge through didactical systems, and the associated interaction between teaching and learning processes. As explained in Chevallard's note, the need for developing specific methodologies based on classroom realizations was justified by both theoretical and practical reasons. On the theoretical side, such methodologies were judged necessary for this essential part of scientific activity which is the production of phenomena (in this case, didactical phenomena), what Bachelard (1937) called *phénoménotechnique*. On the practical side, such methodologies were judged necessary for establishing productive relationships between research and practice, because they permit researchers to consider didactical systems in their concrete functioning, and to pay the necessary attention to the different constraints and forces acting on these, which could be neglected otherwise. Didactical engineering thus emerged as a research and development methodology based on classroom realizations in form of sequences of lessons, informed by theory and putting to the test theoretical ideas. At that time, what was predominant in the French didactical community was the theory of didactical situations that had emerged in the late 1960s. This theory became thus the natural support of DE. Its systemic perspective, constructions and values shaped DE, which progressively became the research methodology privileged within this community. In fact, it would be more adequate to say that theoretical constructions and DE jointly developed along the 1980s.

In 1989, for the second time, didactical engineering was a specific theme of the summer school and I was asked to give a course on this methodology. This course (Artigue 1990) contributed to the institutionalization of DE as a research methodology, making explicit its characteristics and its foundational links with the theory of didactical situations. It also pointed out that its privileged links with the theory of didactical situations did not prevent researchers using this methodology from relying on other theoretical approaches. For instance, several examples mentioned in the course or worked out in the practical sessions associated to it relied on the tool-object dialectics due to Douady (1986). Many contributions to the summer school indeed combined its specific constructs (through the attention paid in design to the dialectics to be organized between the tool and object dimensions of mathematics concepts and to the learning potential offered by moves between mathematical settings, numbers and geometry for instance) with those offered by the theory of didactical situations. In this course too, I pointed out that if DE had consolidated as a research methodology, the problem of establishing productive links between research and practice had not been solved. DE produced by research was disseminating through articles, educational resources and teacher education, but there was some evidence that along this dissemination process, it tended to lose its essence and value.

In fact, in coherence with the theory of didactical situations, in DE design, particular efforts had been made to create situations in which:

- the mathematical knowledge aimed at is an optimal solution to the problem to solve (which is captured in the theory by the idea of *fundamental situation*);
- students as a collective are able to reach this optimal solution through their interaction with the *milieu*[1] of the situation, without significant help from their teacher (which is captured in the theory by the idea of *adidactical situation*).

[1] In the theory, the milieu of a situation is defined as the system with which the student interacts, and which provides objective feedback to her. The milieu may comprise material and symbolic elements: artifacts, informative texts, data, results already obtained…, and also other students who collaborate or compete with the learner.

The teacher's role, for its part, had been mainly approached in terms of the dual processes of *devolution* and *institutionalization*, coherently with the vision of learning as a combination of *adaptation* and *acculturation* processes underlying the theory. Through the devolution process, the teacher tries to make her students accept the mathematical responsibility of solving the problem at stake. She tries to make thus possible the adidactic interaction with the milieu required for learning through adaptation. If the devolution process is successful, the students agree to forget for a while the didactical intention of the teacher; to concentrate on the search for mathematical solutions instead of trying to decipher the teacher's expectations. Through the process of institutionalization, the teacher connects the knowledge built by students through adidactic interaction with the milieu to the scholarly and decontextualized forms of knowledge aimed at by the institution, making the acculturation dimension of learning possible.

In 1989, even if the DEs produced by researchers had been able to approach in many cases this ideal-type, their functioning out of the control of research seemed difficult. Moreover, high attention was paid to the innovative situations designed for introducing new mathematical ideas or overcoming *epistemological obstacles,*[2] and much less to the more standard situations used for consolidating mathematical knowledge and techniques. This situation created a distorted vision of DE products that certainly had negative impact on the quality of their dissemination.

In 2009, 20 years later, DE was once again a theme for the summer school, in fact its unique theme. Since 1989, the didactic field had substantially evolved. The anthropological theory of didactics that was just emerging in the late 1980s had matured and gained in influence. Moreover, in the last decade, it had created its own design approach in terms of activities of research and study and then programs of study and research (Chevallard 2006, in press). A new theoretical framework had also emerged from the theory of didactical situations and the anthropological theory of didactics: the theory of joint action between teachers and students, proposing a renewed vision of the role of the teacher and of students-teacher interactions (Sensevy 2011, 2012). More generally, teachers' practices and professional development had become a focus of research, and this research had developed its own methodologies involving naturalistic and participative observations of classrooms. DE was still an important research methodology, especially each time the didactical systems one wanted to study could not be observed in natural conditions (as is for instance often the case in research about technology), but was no longer the privileged methodology (Artigue 2002, 2009). Didactical engineering had also migrated outside its original habitat. It has been extended to teacher education and to the study

[2] The notion of epistemological obstacle, introduced by the philosopher Gaston Bachelard, was imported in the educational field by Guy Brousseau (1983) for expressing the fact that the development of mathematical knowledge necessarily faces obstacles, due to prior forms of knowledge that were relevant and successful in specific contexts. Epistemological obstacles are those attested in the historical development of knowledge, and having played a constitutive role in this development. Their identification may help understand students' resistant errors and difficulties. Schneider (2014) provides a synthetic presentation and discussion of the notion, its development and use in mathematics education research.

of innovative pedagogical practices, including informal education; didacticians from other disciplines, for instance physical sciences or sports, had used it (Terrisse 2002); researchers educated in other countries and cultures, and having different theoretical backgrounds, had used it, for instance researchers referring to the socio-epistemological framework in mathematics education (Farfán 1997; Cantoral and Farfán 2003) or to semiotic approaches (Maschietto 2002; Falcade 2006). Moreover, design-based research perspectives had emerged and grown in other contexts, independently of it (Burkhardt and Schoenfeld 2003; Design-Based Research Collaborative 2003). These conditions created the need for a thorough reflection about the concept of DE and this was the exact purpose of the 2009 summer school. I have integrated some of the results of this reflection in the next section describing the characteristics of DE as a research methodology, and some others will be dealt with in the fourth section. Nevertheless the size of this chapter does not allow to pay full justice to the work carried out at this summer school and those interested are invited to read the report by Margolinas et al. (2011).

17.3 Didactical Engineering as a Research Methodology

In this section, I present the characteristics of DE as a research methodology, using for that purpose its most standard form: the conception, realization, observation, analysis and evaluation of classroom realizations aiming at the learning of a specific content. However, it should be clear that, while obeying fixed principles, this research methodology might take a diversity of forms in practice, according to the nature of the questions addressed by the researchers, and to the contexts involved. I will end this section by pointing out some similarities and differences with design-based research perspectives more and more influential in mathematics education.

One essential characteristic of DE as a research methodology is that, contrary to the traditional use of classroom realizations in educational research, it does not obey the validation paradigm based on the comparison of control and experimental groups. Its validation is internal and based on the comparison between the *a priori* and *a posteriori analyses* of the didactic situations involved. This methodological choice can be easily understood considering the educational culture in which DE has emerged. In this culture, as explained above, research in mathematics education (didactics of mathematics) is seen as a scientific field of its own whose ambition is the study of the intentional dissemination of mathematical knowledge through didactical systems. What is to be understood is the functioning of such didactical systems, and associated didactical phenomena, which requires entering into the intimacy of their functioning. Validating the hypotheses engaged in the conception phase of a DE cannot be thus a matter of comparison between experimental and control groups.

As a research methodology, DE is structured into different phases. These are the following: preliminary analyses, conception and *a priori* analysis, realization, observation and data collection, *a posteriori* analysis and validation.

17.3.1 Preliminary Analyses

Preliminary analyses set the background for the conception phase of the process. They cover different dimensions, and especially the three following:

- An epistemological analysis of the content at stake, often including an historical part. This analysis helps researchers to fix the precise goals of the DE and to identify possible epistemological obstacles to be faced. It also supports the search for mathematical situations representative of the knowledge aimed at, what the theory of didactical situations calls *fundamental situations*. These are problematic situations for the solving of which this knowledge is necessary or in some sense optimal. The epistemological analysis helps the researchers to take the necessary reflective position and distance with respect to the educational world they are embedded in, and to build a reference point.
- An institutional analysis whose aim is to identify the characteristics of the context in which the DE takes place, the conditions and constraints it faces. These conditions and constraints may be situated at different levels of what is called the *hierarchy of levels of co-determination* (Chevallard 2002) in the anthropological theory of didactics. They may be attached to curricular choices regarding the content at stake and associated teaching practices, to more general curricular characteristics regarding the teaching of the discipline, the (technological) resources accessible, the evaluation practices and the school organization. They can also be linked to the characteristics of the students and teachers involved, to the way the school is connected with its environment… Depending on the precise goals and context of the research, the importance attached to these different levels may of course vary.
- A didactical analysis whose aim is to survey what research has to offer regarding the teaching and learning of the content at stake, and is likely to guide the design.

The three dimensions organizing the phase of preliminary analyses reflect the systemic perspective underlying DE as a research methodology. Each of them has its methodological specificities and needs. The epistemological analysis often involves the use of historical sources and not just secondary sources; the institutional analysis also generally includes an historical dimension. As made clear by the theory of didactical transposition (Bosch and Gascón 2006), curricular organizations and choices are the result of a long-term historical process; they cannot be understood just by analyzing current curricula, official documents and textbooks. Such understanding is needed for making clear the strength of the constraints faced and the way some of these can be moved in the design. The didactical analysis has generally a substantial cognitive dimension, but this cognitive dimension is only one part of the global picture even if what is aimed at is the development of a didactical strategy allowing students to learn better some part of mathematics.

It must also be pointed out that, according to the precise goals of the research, what is exactly investigated in these dimensions, and the respective importance attached to each of them may vary substantially.

17.3.2 Conception and a Priori Analysis

Conception and *a priori* analysis is a crucial phase of the methodology. It relies on the preliminary analyses carried out, and is the place where research hypotheses are made explicit and engaged in the conception of didactical situations, where theoretical constructs are put to the test. Conception requires a number of choices and these situate at different levels. Some choices pilot the global project and in that case it is usual to speak of *macro-choices;* some are situated at the level of a particular situation, and in that case it is usual to speak of *micro-choices*. These choices determine *didactical variables,*[3] so we have both *macro-didactic and micro-didactic variables*. These variables condition the milieu, thus the interactions between students and knowledge, the interactions between students and between students and teacher, thus the exact opportunities that students have to learn, how and what they can learn. In line with the theoretical foundations of DE, in these choices particular attention must be paid to the epistemological pertinence of the problems posed and to the mathematical responsibility given to the students.

The *a priori* analysis makes clear the different choices and the way they relate to the research hypotheses and preliminary analyses. For each situation, it identifies the main didactical variables, that is to say those that affect the efficiency and cost of the possible strategies developed by students, and their possible dynamics. These variables can be attached to the characteristics of the tasks proposed to students, but they can also be linked to the resources provided to the students for solving these tasks (which in the theory corresponds to the *material milieu* of the situation) and to the way the students' interaction with the *milieu* is socially organized. From these characteristics, conjectures are made regarding the possible development of the situation, students' interaction with the *milieu*, students' strategies and their evolution, and the possible sharing of mathematical responsibilities between teacher and students. It is important to stress that such conjectures do not regard individuals but a *generic and epistemic student* who enters the situation with some supposed knowledge background and is ready to play the role that the situation proposes her to play. Of course, the realization will involve students with their personal specificities and history, but the goal of the *a priori* analysis is not to anticipate how each particular student will behave and benefit from the situation, but what the situation *a priori* can offer in terms of learning in the context at stake. It creates a reference with which classroom realizations will be contrasted.

[3]Among the many variables influencing the possible dynamics of a situation and its learning outcomes, didactical variables are those under the control of the teacher. In a situation of enlargement such as the well-known "Puzzle situation" by Brousseau, the number of pieces of the puzzle, their shapes and dimensions, the ratio of enlargement are didactical variables; the fact that students work in group, each student being asked to enlarge one piece of the puzzle is also a didactical variable.

17.3.3 Realization, Observation and Data Collection

During the realization phase, data are collected for the analysis *a posteriori*. The nature of the collected data depends on the precise goals of the DE, on the hypotheses put to the test in it and on the conjectures made in the *a priori* analysis. However, particular attention is paid to the collection of data allowing the researcher to understand students' interaction with the milieu, and up to what point this interaction supports their autonomous move from initial strategies to the strategies aimed at, and to analyze devolution and institutionalization processes. Generally collected data include the students' productions including computer files when technology is used, field notes from observers, audios and, more and more, videos covering group work and collective phases. The data, collected during the realization are generally complemented by additional data (questionnaires, interviews with students and teacher, tests) allowing a better evaluation of the outcomes of the DE. During the realization, researchers are in the position of observers. It is important to point out that the realization often leads to make some adaptation of the design during the realization, especially when the DE is of substantial size, or from one realization to the next one when several realizations are planned in the research project. In that case, adaptations are of course documented together with the rationale for them and taken into account when the *a posteriori* analysis is carried out.

17.3.4 A Posteriori Analysis and Validation

A posteriori analysis is organized in terms of contrast with the *a priori* analysis. Up to what point do the data collected during the realization phase support the *a priori* analysis? What are the significant convergences and divergences and how can they be interpreted? What happened that was not anticipated and how can it be interpreted? Through this connection between *a priori* and *a posteriori* analyses, the hypotheses underlying the design are put to the test. It is important to be aware that there are always differences between the reference provided by the *a priori* analysis and the contingence analyzed in the *a posteriori* analysis. As observed above, the *a priori* analysis deals with generic and epistemic students, which is not the case for the contingence of the realization. Thus, the validation of the hypotheses underlying the design does not impose perfect match between the two analyses.

The analyses carried out are qualitative in nature and local, even when the researchers use statistical tools such as for instance implicative analysis for identifying dependences. In accordance with the theoretical foundation of DE, what the researcher looks at is the dynamic of a complex system, and he does so through the comparison of the observed dynamics with the reference provided by the *a priori* analysis, trying to make sense of similarities and differences. The precise tools used for that purpose depend on the research questions at stake and the data collected. There is no doubt however that these tools have evolved along the years, influenced by the global evolution of the field and also by the technological evolution. In general, researchers combine and triangulate different scales of analyses. They more and more include microscopic analyses taking into account the multimodality of the semiotic resources used by students and teachers that technology makes accessible today. To this should be added that, as mentioned above, the validation of the research hypotheses generally combines the analysis of data collected during the classroom sessions themselves and of complementary data.

17.3.5 The Nature of the Results

It must be stressed that the results obtained through this methodology are mainly local, contextualized, and generally in form of existence theorems in their positive forms. For instance, in the research I developed about the teaching of differential equations in the mid-1980s (Artigue 1992, 1993), I used DE methodology to investigate the possibility of combining qualitative, algebraic and numerical approaches to the solving of ordinary differential equations in a university mathematics course for first year students. The research showed the possibility of organizing such a course in the French context, at that time, with the support of technological tools; it made clear what could be expected from such a course in terms of learning outcomes in this particular context and why. Beyond that, one important result was that a condition for the viability of the course was the acceptance by the didactical system of proofs based on specific graphical arguments, which violated the usual didactical contract[4] regarding proofs in Analysis at university. The difficulty of ensuring this acceptance out of experimental contexts and research control at that time hindered a large-scale dissemination of the developed didactical strategy, despite the fact that its robustness had been attested by realizations carried out with different categories of students. These results were certainly interesting but could not be generalized without precaution to another educational context. However, it would be unreasonable to consider that the results of this engineering work were limited to what we have summarized above.

As evidenced by the further use of this work by different researchers, the preliminary analyses carried out had a more general value, as well as the understanding gained on:

- the students' cognitive development in this area;
- the role played in it by the interaction between the quantitative and the qualitative, between algebraic and graphical representations;
- the affordances of technological tools for approaching the qualitative study of differential equations;

[4] The notion of didactical contract is a fundamental notion in the theory of didactical situations (Brousseau 1997). It expresses the mutual expectations, partly explicit but mainly implicit, of students and teacher regarding the mathematical knowledge at stake in a given situation. The rules of the didactic contract often become visible when they are transgressed by one actor or another.

- the characteristics of usual didactical contract regarding graphical representations and their didactical effects, especially the fact that proofs based on graphical arguments were not accepted.

Looking back at decades of DE research, what is evident indeed is that the results of DE research are far from being limited to the production and validation of didactical designs. DE research has also been a highly productive tool for understanding the functioning of didactical systems, and for identifying didactical phenomena. For decades, DE research has been an essential tool for the development of theoretical constructs paying justice to the complexity of the systems involved in the teaching and learning of mathematics.

What I have described here are the characteristics of the main form of DE: a research methodology based on the conception, experimentation and evaluation of a succession of classroom sessions having a precise mathematical aim. As already mentioned, this methodology has been extended to other contexts such as teacher education, to more open activities such as project work or modeling, and even to mathematical activities carried out in informal settings such as summer camps which obey a different form of contract, which Pelay (2011) defines as the *didactical and ludic contract.*[5] These extensions influence the content of preliminary analyses, but also what the design aims to control in terms of learning trajectories. The reference provided by the *a priori* analysis cannot exactly have the same nature, and this impacts the ways *a priori* and *a posteriori* analyses are contrasted.

17.3.6 Didactical Engineering and Design-Based Research

I will finish this section by situating didactical engineering with respect to design-based research, using the definition of it provided in the Encyclopedia of mathematics education (Swan 2014, p. 148):

> Design-based research is a formative approach to research, in which a product or process (or 'tool') is envisaged, designed, developed and refined through cycles of enactment, observation, analysis and redesign, with systematic feedback from end users. In education, such tools might, for example, include innovative teaching methods, materials, professional development programs, and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools, and is itself refined during the research process. Its goals are to create innovative tools for others to use, describe and explain how these tools function, account for the range of implementations that occur, and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children and other endusers.

[5] The didactical and ludic contract is defined as the set of rules that, implicitly or explicitly, fixes the respective expectations and regulate the behaviour of one educator and one or several participants, in a project combining ludic and learning aims.

This definition makes clear that design-based research and DE have some common methodological characteristics. Both methodologies are organized around the design of some educational tool; this design is informed by educational theory, but also contributes to its development. Moreover, both methodologies reject standardized validation processes based on the comparison of experimental and control groups through a pre-test/ post-test system. However, differences are visible. The global vision underlying design-based research is that of mathematics education as a design science whose aim is the controlled production of educational tools (Wittmann 1998; Collins 1992); the global vision underlying DE is of didactics of mathematics as a fundamental science, whose aim is the understanding of didactical systems and didactical phenomena, and which has also of course an applied dimension. This fundamental difference reflects in methodological characteristics. Design-based research is interventionist and iterative in nature, and the cyclic nature of its process is essential. Along the successive cycles, the design is refined but also experimented in wider contexts for studying how it functions with different categories of users, not involved in the design process, and what adaptations may be necessary for its large-scale use. Didactical engineering as a research methodology does not obey the same pattern. It is more a "phénoménotechnique" with the meaning given to this term by Bachelard (1937), a tool for answering didactical questions, for identifying, analyzing and producing didactical phenomena through the controlled organization of teaching experiments. This is the reason why the preliminary analyses with their different dimensions and the *a priori* analysis are a central part of the research work, and are given so much importance in the articles referring to this methodology. Of course, this does not mean that a DE used in research is built from scratch, but previous constructions when they exist are used to inform the *a priori* analysis; the process is not theorized as a cyclic process. Moreover, what concerns robustness and up-scaling is considered as a matter of development. I come back to this point in the last section of this chapter, but first illustrate the ideas developed up to now with two examples.

17.4 Two Particular Examples

17.4.1 A Paradigmatic Example: The Extension of the Field of Numbers by G. and N. Brousseau

The first example I will consider is the paradigmatic example of the didactical engineering developed by N. and G. Brousseau, more than three decades ago, for extending the field of whole numbers towards rational and decimals (Brousseau and Brousseau 1987, English version: Brousseau et al. 2014). This engineering which ranges over 65 classroom sessions is a very big object when compared with usual constructions whose size is much more limited. I cannot enter into its very details but would like to show how this construction is characteristic of a DE piloted by the theory of didactical situations.

17.4.1.1 Preliminary Analyses

This construction evidences first the importance attached to the preliminary analyses, and especially to their epistemological and didactical dimensions, the initial realizations having taken place in the COREM[6] where the institutional pressure was reduced. These analyses led Brousseau to question the usual educational strategy for extending the field of whole numbers. Usually indeed, the first step was the introduction of decimal numbers in connection with changes in units in the metric system, and fractions played a more marginal role. Emphasis was put on the continuity between the two systems of numbers (whole numbers and decimals), especially regarding the techniques for arithmetic operations, and the resistant cognitive difficulties that these strategies generated or reinforced were more and more evidenced by research. Brousseau made the hypothesis that, in their last years at elementary school, students were able to learn much more about rational and decimal numbers, for instance to differentiate the dense order of rational and decimal numbers from the discrete order of whole numbers, to appreciate the computational interest of decimal numbers and the possibility that this system offers for approaching rational numbers with arbitrary levels of precision. The didactical engineering developed aimed at testing the validity of this hypothesis with ordinary students.

17.4.1.2 Conception and Analysis a Priori

The epistemological analysis carried out inspired the first macro-choice, in clear rupture with established practices: to extend first the field of numbers towards rational numbers, and then to particularize decimal numbers among these for the facilities they offer in terms of comparison, estimation and calculation. Regarding the introduction of rational numbers, another macro-choice was made linked to the identification of two different conceptions for rational numbers: a conception in terms of partition of the unit (1/n is then associated with the partition of one unit into n equal parts and the rational m/n represents m such pieces of the unit) and a conception in terms of commensurability, which corresponds to the search for a common multiple to two different magnitudes for instance two lengths (the ratio of two magnitudes is expressed by the rational m/n if m times the second one equals n times the first one). Generally didactical strategies privilege the first conception in the context of pizza parts or other equivalent contexts. This constitutes an easy entrance in the world of fractions but Brousseau hypothesized that it could contribute to the observed cognitive difficulties. This led him to explore the potential offered

[6]COREM was the Center for observation and research in mathematics education created by Brousseau in Bordeaux in 1973. An experimental elementary school was attached to this center, with very advanced means for systematic data collection and storage. The data collected there during more than 20 years are still studied by researchers, for instance, in the frame of the national project VISA (http://visa.ens.lyon.fr). Detailed information is accessible at the following url: http://guy-brousseau.com/le-corem/acces-aux-documents-issus-des-observations-du-corem-1973-1999/

by an entry in terms of commensurability, and to search for a fundamental situation attached to this conception: a situation that would oblige to consider multiples of magnitudes to compare them.

The problem posed to the grade 4 students was the following: how to compare the thickness of different sheets of paper? There is no doubt that this problem answers the condition just mentioned. The thickness of a sheet of paper cannot directly be measured with usual instruments but taking a sufficient number of such sheets one obtains something measurable. This problem being fixed, different choices must be done for defining a situation. Evident didactic variables are the number of types of paper to compare and their respective thickness. Anticipating that a basic strategy for students is to use their senses (sight and touch) for ordering the different types, it is important to have papers of close thickness invalidating perceptive strategies. Other choices concern, as mentioned above, the organization of the material milieu and the students' interaction with this milieu, the social organization of the classroom. In the organization adopted in this DE, the material milieu was made of piles of sheets of different thickness which often were very close and students worked in groups. First, they had to find a way of comparing the thickness of the sheets provided to their group, then in a second phase, after selecting one type of paper, to write a message allowing another group of students having the same types of paper to find the paper they had selected. These messages became then themselves an object of study: did the messages produced by the different groups solve the particular problem each group had to address, and, beyond that, did they provide a technique for solving the problem of comparison in a general way? We can see here a construction which takes into account the distinction made in the theory of didactical situations between three different functionalities of mathematical knowledge: for acting, for formulating, for proving. Their development obeys different dialectics and thus supposes different types of situations: *situations of action* in the first phase, *situations of formulation* in the second phase (in which the key for success is the quality of the specific language developed) and *situations of validation* in the third phase (in which what is at stake is the validity of assertions).

In an implicit way, the winning strategy in this situation uses the fact that the thickness of a pile is proportional to the number of sheets, which constitutes a reasonable model under certain limits, of course. In fact, the different couples of whole numbers attached to the same paper obtained through manipulations are not exactly proportional, which shows the distance that separates the real world from mathematical models. In observed realizations, this strategy systematically emerged through a-didactic interaction with the milieu. This emergence is certainly fostered by the presence of piles of paper in the material milieu. In the *a priori* analysis, it was expected that each type of paper would be eventually characterized by one or several couples of whole numbers that are nearly proportional, in reference to the manipulations carried out by the students. For instance, it could be 1 mm for 27 sheets in one case, 2 mm for 40 sheets in another case. Once such couples are obtained, as they do not necessarily correspond to the same number of millimeters or to the same number of sheets, if students are not allowed more manipulations, the success of the comparison relies on proportional reasoning. For a good functioning of the interaction with the milieu, it is thus necessary that some knowledge about proportional reasoning be part of the mathematical knowledge shared by students. In the *a priori* analysis, this knowledge is supposed from the generic student. For instance, if the task is to compare the types of paper corresponding to the two couples mentioned above, one can develop the following reasoning: for the first paper, 2 mm should correspond to 54 sheets, and 54 is more than 40, thus the second paper is thicker. For close thicknesses, comparison may be more delicate for the reasons mentioned above, and several exchanges of messages might be needed.

What is mathematically at stake in the solving of this problem is the ordered structure of rational numbers seen as couples of whole numbers or more appropriately families of such couples, and the conception attached is clearly the commensurability conception. As shown by the many realizations carried out, substantial work can be developed in this context about equality and order of rational numbers, students can progressively discover a good number of properties in a-didactic interaction with the successive milieus organized for them, validate them pragmatically using piles of paper, and then use piles of paper more metaphorically for supporting computations and reasoning. However, the mathematical knowledge built still remains attached to this specific context. There is no reason that the notations introduced by students and progressively refined for reasons of economy and efficiency are the conventional notations. This is the responsibility of the teacher to decide when to connect these classroom notations to the usual ones expected by the institution, and also to organize the decontextualization of knowledge through appropriate situations. Of course, in the DE, these steps are also carefully designed.

In this DE, the same context is then used for extending addition to these new numbers. However it does not allow to extend multiplication to rational numbers in a similar way. For this extension, the choice is made of privileging a conception of multiplication as an external operation in terms of linear application for which the well-known situation of the puzzle is the associated fundamental situation. With this new situation, it is also expected to make students face the epistemological obstacle of the additive model.

17.4.1.3 Realization, Data Collection, *a Posteriori* Analysis, Validation and Further Outcomes

I cannot enter into more details in this DE structured in four main phases and invite the interested reader to consult the references mentioned above or the retrospective analysis provided by Brousseau and Brousseau (2007). In the description above, I have focused on the essential phases of design and *a priori* analysis of the methodology, trying to show how they were informed by the preliminary analyses and guided by the theory of didactical situations. The experimentations took place in the experimental school attached to the COREM, the sessions were observed by researchers according to specific guidelines and systematically video-recorded. The comparison of the *a priori* and *a posteriori* analyses, the complementary tests taken by the students, validated the hypotheses underlying the DE.

This DE was used year after year in the experimental school attached to the COREM. More than 750 students were exposed to it and its robustness was confirmed. However, as often stressed by Brousseau himself, it was never considered that it could be easily implemented in ordinary schools and become a standard teaching strategy. Moreover, the comparison of the successive dynamics attracted Brousseau's attention to the fact that the reproduction of the same situations, year after year, by a teacher generated what he called a phenomenon of obsolescence affecting the internal reproducibility of the DE. This phenomenon more globally raised the issue of the reproducibility of didactical situations that was theorized in further work (Artigue 1986).

It must also be stressed that this DE was in fact used for approaching a diversity of research questions, and for instance for investigating dependences between conceptions (Ratsimba-Rajohn 1982). In his doctoral thesis, indeed, Ratsimba-Rajohn, starting from the two strategies for associating a rational measure to a magnitude mentioned above (commensurability and partition of the unit), precisely differentiated these in terms of situations of effectiveness and mathematical knowledge engaged. This analysis led to the identification of a set of nine variables conditioning the effectiveness and cost of each strategy, depending on the type of task (game in the terminology used by the author, in line with the use of game theory in the theory of didactical situations). The author used this tool for investigating how students introduced to rational measures through the commensurability strategy, as was the case in the DE, could enrich their strategies by incorporating the partitioning strategy, *a priori* more intuitive and socially used. For that purpose, a sequence of three situations was designed as part of the DE. In the first situation, the commensurability strategy was extended to other magnitudes (length, weight, capacity); in the second situation, the tasks proposed were out of the domain of effectiveness of the commensuration strategy but could be solved using the partition strategy.[7] The goal of the third situation was to initiate the validation of equivalence of the two models when both strategies are effective. The corresponding lessons were implemented in two consecutive years. Students' strategies and their evolution were carefully documented. Different dynamics were identified. The most striking result was the difficulty that these students had at moving from commensuration strategies to partition strategies, even when commensuration was ineffective. These difficulties were confirmed by the evolution of students' answers at a test taken by the students before and after the teaching sequence in the first year of experimentation. All students significantly progressed in their answers to questions that favored the commensuration strategy or were neutral, only one student progressed on questions blocking the commensuration strategy. Difficulties met in using commensuration and efforts made for overcoming these difficulties in fact tended to reinforce this strategy and the associated conception of rational numbers; more was needed for

[7] This is the case for instance when pupils are asked to find a rational measure for a stick, a unit stick being provided, but the limitation of the physical space and material provided does not allow them to implement the strategy of commensuration.

integrating an alternative conception in terms of partition, despite the fact that it seemed *a priori* much more accessible than the commensuration conception.

17.4.2 An Example of Didactical Engineering Combining the Theory of Didactical Situations with Semiotic Perspectives

The second example I consider is substantially different. It corresponds to a didactical engineering developed by Maschietto in her doctoral thesis (Maschietto 2002) on the transition between Algebra and Analysis. The goal of this DE was to explore the possibility of introducing students very early to the local/global game on functional objects fundamental in Calculus and Analysis, through the introduction of the derivative in terms of local linear approximation. The main hypothesis was that, through an appropriate use of the potential offered by symbolic and graphical calculators, this local/global game could be initiated already in high school, and that the idea of derivative could be built by the students as mathematization of a perceptive phenomenon. Another aim of this DE whose theoretical framework combined the theory of didactical situations and the theory of semiotic mediations (Bartolini Bussi and Mariotti 2008) was to analyze how gestures and metaphors (Arzarello and Edwards 2005; Lakoff and Nuñez 2000) contributed to the mathematization process and the cognitive development of students, as summarized by Maschietto (2008, p. 208):

> The research hypothesis is that the transformations of the graphical representation of a function through the use of zoom-controls and the experience of the perceptive phenomena of "micro-straightness" that these transformations provoke, can give rise to the formulation of some specific language, the construction of metaphors and the production of gestures and specific signs by the students. Our hypothesis is also that adequately exploited by thke teacher, these germs can lead to an entrance in the local/global game, fundamental in Calculus and Analysis hardly observed at high-school level.

> We find in this DE interesting variations from the standard case; they illustrate how, while maintaining the foundational values of this methodology, researchers can adapt it to their theoretical culture and needs. In this presentation, I will try to make clear how the theoretical combination at stake affects the methodological work.

17.4.2.1 Preliminary Analyses

In this DE, we observe still the same attention paid to preliminary analyses. Maschietto developed a detailed analysis of the different perspectives that can be attached to a function: punctual, local, global, of the idea of local straightness, and of thinking modes in Analysis. Her epistemological analyses also aimed at understanding how, before the official introduction of the concept of limit, the language of infinitesimals could support the transition from Algebra towards Calculus, fostering the identification of rules for computations taking into consideration the respective order of magnitudes of the quantities involved.[8] From an institutional perspective, the DE was strongly constrained. Realizations could only be organized at the end of the school year in grade 11 in the Italian context, and in usual practices very few sessions were devoted to the topic. Moreover the use of calculators was usually limited in ordinary classrooms and that of symbolic calculators nearly non-existent. What was proposed was thus far apart from usual practices and would have been impossible to observe in naturalistic conditions. In fact, Maschietto worked with a teacher used to collaborate with researchers, but the institutional constraints limited the realization to a few sessions. Six sessions of 90 min were initially planned, but the thesis only analysed the three first sessions implemented in each of the three experimentations carried out.

Didactical analysis classically reviewed research carried out in that area which is substantial from the seminal work by Tall (1989). What this review showed nevertheless was that, even when the property of local straightness was put to the fore and the visualization potential of technology used for making students aware of it, the responsibility of the mathematization process was hardly devolved to them. Moreover, with few exceptions (see, for instance, Defouad 2000), the distance between what was seen on the screen of calculators or computers, or the equations provided by the calculator for tangent lines and the ideal mathematical objects was not necessarily questioned; thus the mathematization process was not fully developed. Research has also shown that when students enter Calculus, the idea of tangent is not new to them; they have coherent conceptions, geometric and algebraic ones, coming from the experience gained when working with circles. These conceptions lead to characterize the tangent to a curve as a line having a unique intersection point with the curve and staying on the same side of it, but not in terms of proximity (Castela 1995). This conception has to be questioned and as research also shows, usual teaching does not pay much attention to the reconstruction needed. Maschietto pointed out that, in Italy, these conceptions could be reinforced through the teaching of conics in grade 10. Her preliminary analyses also reviewed research developed on gestures and embodiment, as well as the metaphorical vision of mathematics developed by Lakoff and Nuñez (2000).

17.4.2.2 Conception and Analysis a Priori

The conception phase of the DE relied on these preliminary analyses. In the first situation, students were asked to consider six different functions and after entering them in the calculator and getting their graphical representation in the standard window, to make successive zooms around particular points and to explore what happened.

They were also asked to sketch the initial representation and those obtained after two zooms and at the end of the exploration (when they had the feeling that the graphical representation was more or less stable), before moving to another function.

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[8] For instance, taking into account the fact that, in the neighbourhood of 0, the order of magnitude of $x^2 + x$ is the order of magnitude of x.

The number and characteristics of the proposed functions and the selected points are evidently micro-didactical variables for this task. In the DE, the value of these was chosen so that students first met differentiable functions, then faced a function not differentiable at a point but having left and right derivatives (the function defined by $f(x) = -x^3 - 2|x| + 4$), a linear function and a function with a more complex behavior (the function defined by $f(x) = 4 + x.\sin(1/x)$ for $x \neq 0$ and $f(0) = 4$). It was hypothesized that the first examples would lead students to perceptively identify the local straightness phenomenon and to expect its emergence for further examples. The examples of non-differentiable functions would then oblige them to realize that there exist exceptions to this apparently common behavior and that these exceptions might present different characteristics. It was also expected that the dynamic process of zooming would make emerge discourses and metaphors able to support the further mathematization of the perceptive phenomena of local straightness. The drawings asked of the students were expected to be a useful support for this emergence, and for the substantial collective discussion at the end of the session. These drawings were also data to be used for the *a posteriori* analysis. Moreover, for each function two different points were selected for insisting on the local nature of the observed phenomenon. Students worked in pairs with one calculator for each pair and one common graphical production to deliver. This is a classical organization in DE for fostering verbal exchanges and making these accessible to researchers.

The aim of the second situation was the mathematization of this perceptive phenomenon. A differentiable function was selected, different from those already envisaged, and a particular point of its graphical representation. Students were asked to check its local behavior around this point and to find the equation of the line they had got on the screen. It was hypothesized that the different groups would manage the zooming process in different ways and stop it at different times, obtaining thus close but different lines. Using the Trace command or numerical values from the Table application of the calculator for getting coordinates of a second point of their line, they would thus get different equations. At this stage, it was planned that the teacher would collect and write on the blackboard all these equations and would launch a collective discussion. It was hypothesized that the view of the equations, close but different would lead students to consider all these lines as approximations of one ideal object: the tangent to the curve, whose equation they could conjecture from the equations written on the blackboard. The validation of this conjecture was not supposed to result from mere a-didactical interaction with the milieu. In the scenario for this session, it was planned that the teacher would ask students to find a common way of expressing the different computations and that, if this was not spontaneously proposed by them, she would introduce the idea of giving account of the commonalities between these different calculations through the use of a letter h representing the different small increments chosen by the students. From this point a collective computation was expected to lead to an equation for the line depending on h, but becoming the ideal equation when h was made equal to 0 (in some sense when infinite zooming was performed). This should allow the teacher both to institutionalize the definition of the tangent to a curve at a given point in terms of linear approximation, and the specific type of computation that allowed finding its equation. For this second situation, the characteristics of the function and of the point were the main micro-didactical variables of the task. In the DE, two different choices were successively made: a polynomial function of degree 2 and then one of degree 3, with simple coefficients and of a point whose coordinates were such that the ideal equation could be easily conjectured. Choosing a polynomial function and using the letter h in the symbolic computation resulted in the equation of the line described by a polynomial in h (after simplification by h), which made the reasoning easier. Choosing a polynomial of degree 3 made that the algebraic strategy known from these students for finding tangents to conics did no longer work. Once again students worked in pairs. In the third situation, it was planned to begin to consolidate the form of computation that had been introduced and also to connect this conception of the tangent in terms of approximation with those conceptions, geometric and algebraic, mentioned above, reinforced in grade 10, through the work with conics.

As mentioned above, it was hypothesized that during the three sessions, the students would combine gestures with the use of language and different semiotic representations for making sense of the situations and exchange with other students and the teacher. However, the exact forms these combinations would take, and the language that students were likely to introduce for qualifying local straightness was not anticipated. From that point of view, the DE had more an exploratory purpose.

Each session lasted 90 min and combined a phase of autonomous work by the students and a phase of collective discussion. Its *a priori* analysis was structured in the thesis around the following dimensions:

- the preparation of students' worksheets and analysis of them in terms of mathematical content, pre-requisites, didactical variables;
- the analysis of the role to be played by graphic and symbolic calculators in each phase of the session;
- the analysis of the work expected from the students, the anticipation of possible strategies and difficulties;
- the analysis of the work expected from the teacher in each phase of the session, and of the distribution of responsibility expected between students and teacher.

17.4.2.3 Data Collection, *a Posteriori* Analysis and Validation

The collected data consisted of students' worksheets and productions, videos of one particular group and of collective phases, observation notes for different groups (two or three depending on the experimentation) according to guidelines defined in the analysis *a priori*. A test taken by students 2 weeks after the teaching experiment and a questionnaire filled by them regarding their participation in this experience were added. The semiotic perspective impacted the collection of data (those in charge of video-recording for instance tried to capture students' and teacher's gestures as much as possible) and the *a posteriori* analysis of the sessions.

The *a posteriori* analysis of each session combined two levels. The first level presented a global analysis of the session in its relation to the *a priori* analysis (regarding the scenario of the session, the distribution between group work and collective discussions, the strategies developed by the students and the main characteristics of their work, the difficulties observed, the teacher's role…). The second level was a fine-grained analysis of the data collected during the session elucidating the conceptualization processes at stake and their characteristics, through the role of the calculator, of metaphors, of discourse and gestures, of interactions between students during group work and between students and teacher.

We illustrate this methodological work by a few examples taken from the *a posteriori* analysis of the first session. For this session, the global analysis was structured around four dimensions: the scenario, the localization of the perspective, the emergence of the invariant and the role of the teacher. Regarding the localization of the perspective for instance, the main elements taken into account in this global approach were the characteristics of the graphical representations drawn by the different groups. A specific list of codes had been developed in the *a posteriori* analysis of the first experimentation, starting from students' productions. It was used again in the *a posteriori* analysis of the second and third experimentation. These codes showed the expected evolution of representations along the zoom process, but they also made evident the strength of the usual didactic contract regarding graphical representations of functions and the difficulty most students thus faced when the zooming process makes the axes disappear.

The analysis of data for the observed groups and for the collective discussion then combined different semiotic elements for clarifying the conceptualization processes at stake and the characteristics of the situation that fostered these conceptualizations (characteristics of the task, of the milieu and of social interactions). In particular, discourse, inscriptions and gestures were tightly connected in the analysis.

In the *a posteriori* analysis, the different levels of analysis for one particular session were then combined for testing the conjectures made in the *a priori* analysis regarding this particular session. The same type of *a posteriori* analysis was made for the three sessions, then the different results were synthesized and triangulated with those resulting from the analysis of the final test and questionnaire.

The following two quotations by Maschietto (2008) in which the author gives a synthetic vision of her research work, illustrate the form that these analyses have taken. The first quotation (pp. 215–216) regards the emergence of the linear invariant and an interesting phenomenon accompanying this emergence. This phenomenon was not anticipated in the *a priori* analysis but it had a positive effect on the dynamics of the situation.

Excerpt 1: DAL-DF-MA group (Exp_A)

15. DF "Forward zoom" (*he carries out the 3rd ZoomIn*)
16. DF "Again" (*he carries out the 4th ZoomIn*)
17. DF "It becomes straighter and straighter"
18. DF "The drawing is the same as before. Even if the result is the same, we'll write it down".
After getting the representation in the standard window, DF does 2 ZoomIns

DF "I want the other piece of function. It's still a line! Draw at least one axis" (*addressed to MA. DF carries out the 3rd ZoomIn*)
DF "We'll stop here because it stays the same".

In the pencil-and-paper environment (Fig. 17.1a), the linearity is emphasised by the use of a ruler to draw the graphical representation that appears on the calculator display on the third sheet (end of the exploration).
In other protocols (Exp_B and Exp_C), the students try to explain the end-point of their exploration, for example: "REASON WHY WE STOPPED CARRYING OUT THE ZOOMS → *The more we used the ZoomIn, the more the curve sector considered tended to become a line"*. We observe here a dynamic language, that draws on the infinite approximation process.

In the protocols, there are two distinct phenomena, linked to the local point of view. The first regards the strength of the "straight" nature at a perceptive level. The second regards the interference of the global point of view with the local one. As far as the first phenomenon is concerned, the comments (for example, Excerpt 2) on the exploration of the corner (function y3[9]) highlight that at this stage the students have, in general, clearly identified the graphic phenomenon "it becomes straight using the zoom".

Excerpt 2: DAL-DF-MA group (Exp_A)
In all these cases the functions, even with the second zoom, are similar to a line with a gradient ≥0 but:

- y4[10] is similar to a line only after the 4th zoom [Note: at x = 1/pi]
- y3 is similar to two lines (one with m > 0 and the other with m < 0)

However, this recognition does not allow them to distinguish the situation of the function that is differentiable at the given point and that of the function having two different half derivatives and leading to a corner. In fact, these situations, mathematically different, are unified by their common "straightness" recognized at a perceptive level (Excerpt 2). The second function does not therefore represent a counter-example, unlike what is hypothesized in the a-priori analysis. Their distinction will only occur during the mathematization

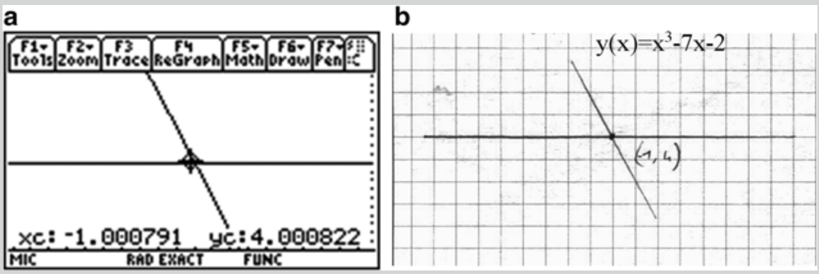

Fig. 17.1 Window at the end of the exploration process (Exp_A)

[9] $y3(x) = -x^3 - 2|x| + 4$ at $x = 0$.
[10] $y4(x) = 4 + \sin(1/x)$ at $x \neq 0, = 4$ at $x = 0$

process of the linear invariant. The real counter-example is provided by the y4 function, the graphical representation of which, after subsequent zooms, is perceptively different. In this case there is no move from the "curve" category to the "straight" category, as happens for all the other functions.

The second quotation (pp. 217–218) shows the importance attached to gestures in the *a posteriori* analysis:

In accordance with the a-priori analysis, the activity presented to the students shows its potential for the production of gestures and metaphors. These appeared both during the communication inside the groups and during the collective discussions. The analysis of the students' protocols and the discussions show that the conceptualisation of the zoom- controls, that supports the localisation of the view, appears through gestures that accompany the explanation of the exploration strategies and linguistic expressions that can be analysed in terms of metaphors.

A particularly representative example is the analysis of the gestures of one student, PM (Exp_A), while he is explaining the exploration of a graphical representation. The ZoomIn control is used in order to see some of the characteristics of the curve in a detailed way and is associated with a downward movement meaning an "entrance into the curve," that corresponds with moving into the curve (ZoomIn gesture, Fig. 17.2a). The ZoomOut control, which is used to obtain a bigger curve and to study its characteristics better, is associated with an upward movement meaning an "exit from the curve" (ZoomOut gesture, Fig. 17.2b), which also corresponds with moving away from the curve. PM's gestures lead the details of the curve to be interpreted as downwards and the overall curve as upwards. PM also creates a space in front of him for controlling these processes (the standard window of the calculator becomes a little rectangle that is constructed by his fingers, Fig. 17.2c).

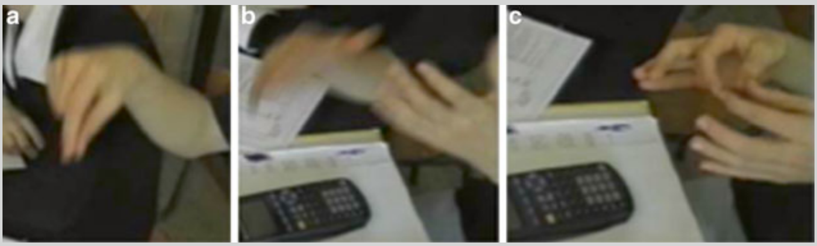

Fig. 17.2 PA's gestures (Exp_A): ZoomIn, ZoomOut, standard window

The reference to the ZoomOut control identifies the space under his eyes, while the palm of one hand is associated with the flat part that is obtained from the ZoomIn. In this way, PM has created his own space, which is suggested by the activity with the calculator, where the two different transformations of the curve can co-exist and be controlled.

The realization took place in three different classes as mentioned above, with some minor adjustments and evident regularities were observed. Globally the hypotheses mentioned above were confirmed despite the fact that it was not possible to cover all that had been planned and that, due to their previous experience with conics, some groups conjectured very early that the line was the tangent and privileged an algebraic strategy for finding its equation, persisting in that strategy with the polynomial of degree 3 in the second and third experiments. Some interesting and non-anticipated phenomena also occurred but they did not necessarily invalidate the *a priori* analysis. For instance, as shown in the first quotation above, it appeared that most students considered that straight lines and curves were objects belonging to different categories. This conception in fact helped them to consider that the linear representations they obtained by zooming were not exactly linear but just very close to a linear object, and that linearity could only be reached through an infinite succession of zooms. This helped them to make sense of the notion of tangent as an ideal object and of the computations carried out for finding its equation. This conception nevertheless also led them to think that the function admitting only left and right derivatives at a given point was not very different from the regular ones. This question was considered again later on once the derivative was properly defined. As expected also, gestures accompanied students' verbalizations and work, and the language and metaphors used by students showed evident embodiment. They introduced their own expressions for qualifying the phenomenon of local straightness, saying for instance that the functions were "zoomata lineare" at a particular point and these were accepted and used by the teacher. Validation of the DE did not just use the comparison of the *a priori* and *a posteriori* analysis of the sessions, but also the data from the questionnaire and interviews taken by the students after the completion of the process as mentioned above.

I cannot enter into more details here. The interested reader can find these in the references mentioned above. But I would like to stress a few points. According to the author, this methodological construction is a DE and I fully agree with this position, recognizing in it the fundamental features of DE presented above. This is nevertheless a construction sensibly different from that described in the first example. For instance, it is difficult to model the first situation as a game that students enter with basic strategies that they must make evolve towards winning strategies. Students are asked to stop their exploration when they have got the feeling that the graphical representations will no longer substantially evolve, which is a rather fuzzy condition. Moreover, if the situations are designed in order to ensure productive adidactical interaction with the milieu, in the construction of the situations an important role is given to collective discussions piloted by the teacher and to her mediations. These collective discussions are not just institutionalization phases. As evidenced by the *a posteriori* analysis, they play an essential role in the progression of knowledge beyond what has been achieved by each pair of students in the phase of autonomous work. In some sense, they play the role given in the theory of didactical situations to situations of formulation and of validation but they do not obey a similar organization; they are not supported by the same theoretical constructs. We can see here the effect of a combination of the theory of didactical situations and the theory of semiotic mediation. It shows us that, as a research methodology, DE can productively combine several theoretical approaches. Another close example is provided by the thesis by Falcade (2006) also combining the theory of didactical situations and the theory of semiotic mediation in an approach to functions using Cabri-Géomètre (see also Falcade et al. 2007).

17.5 Some Recent Developments of Didactical Engineering

17.5.1 Didactical Engineering and the Anthropological Theory of Didactics

After considering these two examples, in the last part of the paper, we enter into some recent developments of didactical engineering, referring more precisely to the work carried out at the 2009 summer school.

As mentioned earlier, the anthropological theory of didactics has developed in the last decade a design perspective based on the idea of Programme of Study and Research (PSR in the following). At the 2009 summer school, Chevallard proposed to refund didactical engineering around this idea (Chevallard 2011). I will not follow him up to this point but would like to situate Chevallard's perspective with respect to the vision of DE that has been presented in the first sections of this chapter, and briefly explore some possible complementarities between these.

Through PSR, Chevallard wants to build a new epistemology opposing what he calls the "monumentalistic" doctrine pervading contemporary school epistemology (Chevallard 2006, in press). As explained by Chevallard (2006):

> For every praxeology[11] or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its raisons d'être are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve. (p. 26)

[11] The notion of praxeology is central in the anthropological theory of didactics that considers that knowledge emerges from human practices and is shaped by the institutions where these practices develop. Praxeologies, which model human practices, at the most elemental level (punctual praxeologies), are defined as 4-uplets made of a type of task, a technique for solving this type of task, a discourse explaining and justifying the technique (technology), and a theory legitimating the technology itself.

In coherence with this vision, a PSR starts from the will to bring an answer to some generating question. In fact, at the 2009 summer school, Chevallard distinguished between different forms of PSR, and especially between finalized and open PSR. In finalized PSR, the main praxeologies aimed at are known. They correspond for instance to praxeologies aimed at by a given curriculum. The designer must found a question or a succession of questions which are able to generate the encounter of the corresponding types of tasks and the development of techniques and technological discourse constituting these praxeologies. This is done by a combination of study of existing works and inquiry processes. In open PSR, the situation is quite different. There is a generating question but the praxeological equipment needed for answering it is not *a priori* known; neither it is necessarily limited to mathematical praxeologies. This is for instance often the case in project work, and modeling activities.

Even in the case of finalized PSR, the proposed vision however is at some distance from the forms of DE mentioned above, especially in what concerns the milieu and its evolution. This is notably due to the place given to cultural answers to the question at stake in PSR. In the didactical schema that Chevallard proposes (Chevallard in press), a role is given to cultural answers or pieces of information accessible to the learners in the media and especially on the Internet. It is supposed that such cultural answers or pieces of information can enter the milieu on the initiative of teacher or students and that, duly studied and criticized, they should contribute to the elaboration of the expected answer to the question at stake. In the anthropological theory of didactics, this is encapsulated in the idea of *media-milieu dialectics*.

Differences with the classical vision of DE also concern more globally what the researcher ambitions to optimize and control in the design phase and consequently they affect the *a priori* analysis. This is especially the case for open PSR. For that case Chevallard denies the possibility of an *a priori* analysis. He thus introduces the idea of *analysis in vivo*, fully integrated into the inquiry work. This position can be questioned all the more as the publications of researchers working within this perspective show that they develop some form of *a priori* analysis to select questions having a strong generating power under the institutional conditions and constraints at stake. What is clear, however, is that, for such open PSR, in the *a priori* analysis researchers are more interested in investigating the didactical potential of the selected question, trying to make clear how its study can develop and generate new and interesting questions, motivate the study and progressive structuring of important praxeologies, than in the optimization of students' learning trajectories. In fact, the *a priori* analysis becomes an on-going process that develops and adjusts along the implementation phase of the DE. The doctoral thesis by Barquero (2009), (see also Barquero et al. 2008) analyzing the design and implementation of a PSR devoted to the modeling of population dynamics with undergraduate students provides a good example of such functioning.

There is no doubt that, from a DE perspective, the notion of open PSR makes it possible to address research issues attached to the functioning and viability of didactical forms more open than those usually addressed by existing DE such as project work and modeling activities. These didactical forms still have a marginal position in educational systems but they are also more and more encouraged as evidenced for instance by the number of European projects currently funded around inquiry-based education in mathematics and science.[12] As a research methodology, DE certainly needs some accommodation in order to cope efficiently with the research issues that emerge from this evolution, and also for taking into account the dramatic changes in access to information of the digital era. From this point of view, the design perspective offered by the anthropological theory of didactics seems promising.

17.5.2 Research and Development: Didactical Engineering of Second Generation

The second evolution I would like to mention is that introduced by Perrin-Glorian (2011) who distinguishes between DE of first and second generation. In this chapter, we have considered DE from a research perspective focusing on its characteristics as a research methodology. We cannot forget nevertheless that from its emergence DE had the ambition to contribute both to research and development. In the historical review we mentioned the difficulties met at converting DE developed for research aims into useful educational resources. This problem is still not solved but the increase in our knowledge of teachers' representations and practices, and of possible dynamics for their evolution makes us better understand the difficulty of the enterprise. The distinction introduced by Perrin-Glorian directly addresses this issue and we consider it because it can also affect the vision of DE as a research methodology. Contrasting RDE and DDE (research didactical engineering and development didactical engineering), she compares the levels of theoretical controls in which these two forms of DE engage. She thus points out that even if in both cases the analysis of the mathematical knowledge at stake and of the students' knowledge, the definition of the situations and associated milieus are under theoretical control, for DDE much more flexibility is needed for preparing the adaptation to a diversity of contexts. The loss of control is even greater with regard to the role of the teacher while institutional constraints cannot be partly removed as is often the case in RDE. These considerations lead her to postulate that before trying to implement a DE product coming from research in ordinary classes, it is necessary to plan at least two different levels of DE, each one having specific aims: This is the whole process that she names DE of second generation.

At the first level, the goal is the theoretical validation of the situations of the DE (i.e. their capacity in producing the knowledge aimed at) and the identification of the fundamental choices of the DE, separating what is essential from what is linked to the particular context and could be changed, and adapted. The associated realization takes place in a rather protected environment and under the control of researchers as is the case for RDE.

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[12] See the portal www.scientix.eu for information about these projects.

At the second level, the goal is the study of the adaptability of such validated situations to ordinary classrooms and teachers through the negotiation of the DE with teachers who have not been involved in the first phase. These negotiations and the transformations introduced by the teachers involved in this second phase are taken as objects of study together with their impact on the DE itself and its outcomes. It is expected that the results allow researchers to determine what concessions can be made in such negotiations, what should be preserved and why, and to identify what forms of control can be maintained.

As Perrin-Glorian points out, envisaging this second level modifies in fact the first level because it obliges researchers to move from a top-down conception of transmission of research results to an idea of adaptation much more dialectical. As she adds:

> The problem is no longer to control and disseminate engineering products coming from research but to determine the key variables, in terms of knowledge involved, piloting the didactical engineering that one wants to make a resource for ordinary teaching, and to study the conditions of their dissemination. (p. 69, our translation)

She then illustrates this vision by an example regarding the teaching of axial symmetry at the transition between elementary school and junior high school.

This reflection in fact points out that the transition from research to development needs specific forms of research, extending our view of the ways didactical engineering and educational research can be connected.

17.6 Conclusion

In this chapter, I have tried to present didactical engineering, focusing on its dimension of research methodology. To help readers make sense of this methodology, I have reviewed its history from its emergence in the early 1980s until now. I have tried to clarify its main characteristics and to show that this methodology, even if it has been shaped by the values and constructs of the theory of didactical situations, is a methodology that can be productively used beyond the frontiers of this theory, and is enriched by the different uses made of it. I have also tried to show that, as for many other constructs in educational research, didactical engineering is a living and dynamic concept which adapts to the evolution of the field, to the advances of educational knowledge, and to the evolution of the social and cultural contexts of mathematics education. I also hope to have made clear that this methodology, although flexible, imposes a systemic view of the field, a view of the classroom as a social organization, of learning as a combination of adaptation and acculturation processes and a particular sensitivity to the discipline and its epistemology.

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Chapter 18
Educational Design Research to Support System-Wide Instructional Improvement

Erin Henrick, Paul Cobb, and Kara Jackson

Abstract In this chapter, we describe a methodology for conducting educational design research to support system-wide instructional improvement in mathematics and draw on one of the few design studies that does this as an illustrative case. Design studies conducted at the level of an educational system are interventionist in nature, and can address both the complexity of educational settings and the problems that educational system leaders, school leaders, and teachers encounter as they work to improve the quality of classroom instruction, school instructional leadership, and ultimately, students' mathematics learning. This chapter describes the theoretical background for this approach, in which the issue of what it takes to support instructional improvement on a large scale is framed as an explicit focus of empirical investigation.

Keyword Educational design research on a large scale

Our purpose in this chapter is to describe a methodology for conducting educational design research to support large scale instructional improvement in mathematics. In countries with centralized education systems, large scale might mean instructional improvement at the national level. In countries with decentralized education systems, the appropriate organizational unit is the largest administrative jurisdiction that can support coordinated improvement efforts. In this chapter, this largest unit will be referred to as the educational system or system.

For the purpose of this chapter, we define design research as a family of methodological approaches in which research and the design of supports for learning are interdependent. On the one hand, the design of supports serves as the context for research and, on the other hand, ongoing and retrospective analyses are conducted

E. Henrick (✉) • P. Cobb
Department of Teaching and Learning, Vanderbilt University, Peabody College, PMB 230, GPC; 230 Appleton Place, Nashville, TN 37203, USA
e-mail: erin.henrick@vanderbilt.edu; paul.cobb@vanderbilt.edu

K. Jackson
College of Education, University of Washington, Box 353600, Seattle, WA 98195-3600, USA
e-mail: karajack@uw.edu

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in order to inform the improvement of the design (Gravemeijer 1994; Schoenfeld 2006). Design research methodology has become increasingly prominent in the learning sciences and in several related fields of educational research including mathematics education in recent years. Most design research studies focus on students' mathematical learning either as they interact one-on-one with a researcher (e.g., Cobb and Steffe 1983; Lobato 2003; Steffe and Thompson 2000) or as they participate in classroom processes (e.g., Cobb et al. 2003a; Design-Based Research Collaborative 2003). In comparison, design studies conducted to support and investigate teachers' learning are far less common, and design studies conducted to study the process of supporting improvements in the quality of mathematics teaching on a large scale have, until recently, been extremely rare. As a consequence, there are currently few examples of design research studies that have been conducted at the level of an educational system, or that focus simultaneously on teachers' development of instructional practices and the school and system settings in which they develop and refine those practices (e.g., Cobb et al. 2003b, 2009; Fishman et al. 2004). However, design research at scale (Cobb and Jackson 2012) and closely related approaches such as design based implementation research (Penuel et al. 2011) and improvement science research (Bryk 2009) are gaining momentum.

As Stein (2004) observed, research in mathematics education has not, to this point, investigated how the school and system settings in which mathematics teachers work can be organized to support their ongoing learning. Key aspects of these settings include the materials and associated resources that teachers use as a basis for their instruction, the formal and informal sources of assistance on which they can draw, as well as to whom and for what they are accountable. Design studies conducted at the level of an educational system is interventionist in nature, and can address both the complexity of these educational settings and the problems that educational system leaders, school leaders, and teachers encounter as they work to support improvements in the quality of classroom instruction, school instructional leadership, and ultimately, students' mathematics learning.

We illustrate the methodology for investigating and supporting system-wide instructional improvement by framing one of the few design studies of this type as a sample case. The study, Designing Learning Organizations for Instructional Improvement in Mathematics (known as MIST), was conducted in the United States and investigated how school- and system-level supports and accountability relations impacted the quality of mathematics instruction in middle-grades schools that served students aged 12–14. The study was a 4-year collaboration with district leaders, school leaders, and mathematics teachers in four city-wide school systems that served a total of 360,000 students. Before describing and illustrating the methodology, we first provide background information on the United States educational context. We also detail the vision of high-quality mathematics instruction that oriented our research agenda and recruitment of participating school systems.

18.1 The United States Context

The United States educational system is decentralized, and there is a long history of the local control of schooling. Each U.S. state is divided into a number of independent school districts. In rural areas, many districts serve less than 1,000 students whereas a number of urban districts serve more than 100,000 students. In the context of the U.S. educational system, urban districts are the largest jurisdictions in which it is feasible to design for improvement in the quality of instruction (Supovitz 2006).

The federal government's role in the educational system in the U.S. has increased significantly in recent years following the passing of the No Child Left Behind Act (NCLB) in 2001. States receive incentives to set standards for students' mathematics achievement, develop standardized assessments aligned with the standards, and implement accountability measures to promote increases in achievement for all students and for specific sub-groups (e.g., racial and ethnic categories, socio-economic status, students who receive special education services). Districts and schools are sanctioned if they fail to meet goals for "adequate yearly progress" (AYP) on state assessments.

As a result, school districts are under great pressure to improve student achievement in mathematics. In addition to responding to accountability pressures, urban school districts in the United States face a number of other challenges that impact improvement initiatives. These challenges include limited financial resources, under-prepared teachers, and high teacher turnover (Darling-Hammond 2007).

Unfortunately, most U.S. school districts do not have the capacity to respond to these accountability demands in a productive manner (Elmore 2006). Many districts are implementing short-term interventions aimed at "teaching to the test," and some are attempting to game the assessment system (Heilig and Darling-Hammond 2008). In addition, districts frequently expend considerable resources on different (and even conflicting) improvement policies, abandoning each for the next when student achievement does not improve quickly, without understanding the challenges of implementing particular policies. This policy churn (Hess 1999) can cause frustration for teachers and does not help the larger educational community understand how improvement in student achievement can be supported at the scale of a large school district.

A minority of districts is responding to accountability demands by attempting to improve the quality of classroom instruction. These districts are attempting to support teachers' development of high quality instructional practices that will ultimately lead to improvement in student achievement (Elmore 2004). Concurrently, the role of the principal is shifting from school manager to instructional leader, with an increased responsibility to support instructional reforms in each content area (Nelson and Sassi 2005; Fink and Resnick 2001). To date, efforts to support fundamental improvements in teachers' instructional practices on a large-scale have rarely been successful, and there are no proven models regarding how this can be accomplished (Elmore 2004; Gamoran et al. 2003). Furthermore, although research on mathematics teaching and learning has made significant advances in recent years, these advances have had limited impact on the quality of instruction in most U.S. classrooms. In addition, research in both mathematics education and in educational policy and leadership can provide only limited guidance to districts attempting to respond to high stakes accountability pressures by improving the quality of mathematics instruction.

18.2 An Orienting Vision of High-Quality Mathematics Instruction

The four urban school systems, or districts, that we recruited for the MIST study were all pursuing similar agendas for instructional improvement in mathematics. These agendas were oriented by goals for students' mathematics learning that are relatively ambitious in the U.S. context. These system-level goals emphasized students' development of conceptual understanding as well as procedural fluency in a range of mathematical domains, students' use of multiple representations, students' engagement in mathematical argumentation to communicate mathematical ideas effectively, and students' development of productive dispositions towards mathematics (U.S. Department of Education 2008; Kilpatrick et al. 2001; National Council of Teachers of Mathematics 2000). These student learning goals in turn oriented leaders of the four collaborating districts as they specified high-quality mathematics instructional practices that could be justified in terms of student learning opportunities (Kazemi et al. 2009). The resulting view of high-quality instruction has been referred to in the U.S. as ambitious teaching (Lampert and Graziani 2009; Lampert et al. 2010).

Ambitious teaching requires teachers to build on students' solutions to challenging tasks while holding students accountable to learning goals (Kazemi et al. 2009). Recent research in mathematics education has begun to delineate a set of high-leverage instructional practices that support students' achievement of ambitious learning goals (Franke et al. 2007; NCTM 2000). These practices include launching challenging tasks so that all students can engage substantially without reducing the cognitive demand of tasks (Jackson et al. 2013), monitoring the range of solutions that students are producing as they work on tasks individually or in small groups (Horn 2012), and building on these solutions during a concluding whole-class discussion by pressing students to justify their reasoning and to make connections between their own and others' solutions (Staples 2007; Stein et al. 2008). These practices differ significantly from the current practices of most U.S. teachers, and their development involves reorganizing rather then merely adjusting and elaborating current practices. The learning demands for teachers include developing a deep understanding both of the mathematics on which instruction focuses and of students' learning in particular mathematical domains. In addition, it involves developing the new high-leverage instructional practices outlined above (e.g., launching cognitively-demanding tasks effectively; orchestrating whole class discussions of students' solutions that focus on central mathematical ideas).

The agenda for instructional improvement that the four collaborating school systems were pursuing is specific to the U.S. context and was influenced by the recommendations of several professional organizations including the National Council of Teachers of Mathematics (1989, 2000), and it is compatible with the more recent Common Core State Standards Initiative (2010). Improvement efforts in other countries might be oriented by a different vision of high-quality mathematics instruction. The methodology that we describe will nonetheless be relevant to all cases where instructional improvement goals involve significant teacher learning and require teachers to reorganize rather than merely elaborate their current classroom practices.

In the remainder of this chapter, we describe the key aspects of design studies conducted to investigate and support system-wide improvement in mathematics instruction. Although we draw on the MIST study to clarify the rationale for certain tools and processes, our intent is to describe the methodology in broad terms.

18.3 Design Studies to Investigate and Support System-Wide Improvement in Mathematics Instruction

The overall goal of design research at the level of an education system is to investigate what it takes to support instructional improvement at scale (Bryk and Gomez 2008; Coburn and Stein 2010; Roderick et al. 2009) by testing and revising conjectures about school- and system-level supports and accountability relations. Design studies of this type aim to both support and investigate the process of instructional improvement at scale by documenting (1) the trajectories of (interrelated) changes in the school- and system-level settings in which mathematics teachers work, their instructional practices, and their students' learning, and (2) the specific means by which these changes are supported and organized across the system (Cobb and Smith 2008).

Design studies of this type have two primary objectives. The first objective is pragmatic, and is to provide leaders of the collaborating educational systems with timely feedback about how their improvement strategies or policies are actually playing out that can inform the ongoing revision of instructional improvement efforts. The second objective is theoretical, and is to contribute to the development of a generalizable theory of action (Argyris and Schön 1974) for system-wide instructional improvement in mathematics by synthesizing findings across multiple educational systems.

Design studies conducted at any level involve iterative cycles of designing to support learning and of conducting analyses that inform the revision of the current design. In contrast to studies conducted to investigate students' learning, design studies at the system level necessarily entail a partnership with system leaders. As a consequence, cycles at this level also include a feedback phase in which researchers share findings with system leaders who have the ultimate authority for making decisions about improvement strategies. The length of the cycles is much longer that in other types of design research. For example, in the MIST study, each cycle spanned an entire school year.

In the sections below, we describe the following aspects of the methodology:

1. developing an initial set of conjectures that comprise an initial theory of action about school- and system-level supports and accountability relations;
2. recruiting collaborating educational systems;
3. employing an interpretative framework for assessing an educational system's designed and implemented instructional improvement strategies;
4. conducting successive design, analysis and feedback cycles by: (a) documenting each collaborating system's current improvement strategies, (b) collecting and analyzing data on how those strategies are actually playing out, (c) sharing findings and recommendations with system leaders in time to inform their revision of improvement plans, and (d) assessing the influence of recommendations on the collaborating system's instructional improvement strategies;
5. testing and revising conjectures that comprise a theory of action for system-wide instructional improvement based on ongoing feedback analyses, the current research literature, and retrospective analyses of data collected in successive cycles.

18.4 Developing Initial Conjectures

The basic goal of a design study conducted at any level is to improve an initial design for supporting learning by testing and revising conjectures inherent in the design about the course of participants' learning and the means of supporting their learning (Cobb et al. 2003a). A key concern when preparing for a system-level design study is therefore to develop an initial set of conjectures for what it would take to support improvement in the quality of mathematics teaching across an entire system.

In the MIST study, we found it valuable to follow the basic tenets of design as articulated by Wiggins and McTighe (1998) and develop initial conjectures by mapping out from the classroom (cf. Elmore 1979–80). The first step in the process is to specify explicit goals for students' mathematical learning and an associated research-based vision of high-quality mathematics instruction. The learning demands for teachers can then be identified by comparing the vision of high-quality mathematics instruction that constitutes the goal for teachers' learning with their current instructional practices.

The second step is to develop an initial, tentative, and eminently revisable theory of action by formulating conjectures about both supports for teachers' learning and accountability relations that press them to improve their practices. These conjectures should clearly attend to teacher professional development and to instructional materials and associated tools designed for teachers to use. However, it also proved important in the MIST study to broaden our purview by considering other types of possible support such as mathematics teacher collaborative meetings scheduled during the school day, the colleagues to whom teachers turned for instructional advice during the school day, and mathematics teacher leaders or coaches who were charged with supporting teachers in their classrooms and during collaborative meetings. In addition, research on school instructional leadership oriented us to consider the role of principals and other school leaders in pressing and holding teachers accountable for improving the quality of instruction.

It is important to note that conjectures about supports and accountability relations for teachers' learning typically have implications for the practices of members of other role groups. For example, conjectures about the role of coaches in supporting teachers' learning have implications for the practices of system leaders responsible for hiring coaches and for supporting their development of effective coaching practices. Similarly, conjectures about school leaders' role in communicating appropriate instructional expectations to teachers have implications for the practices of others in the system who are charged with supporting them in deepening their understanding of high-quality mathematics instruction.

In following this process of mapping out from the classroom in the MIST study, it proved critical to balance the ideal with the feasible by taking account of each collaborating system's current capacity to support members of different role groups in improving their practices. As we worked through this process of formulating initial conjectures, we also found that the challenge of improving classroom instruction had implications for the practices of personnel at the highest levels of the four collaborating systems. As a consequence, it proved essential to formulate testable conjectures about the means of supporting the learning of mathematics teachers, mathematics coaches, school leaders, and system leaders in a coordinated manner. It also became apparent as we worked through this process that issues of mathematical content really matter. The mathematical learning goals for students have direct implications for the vision of high-quality instruction and thus for the learning demands on the teachers. These learning demands in turn have implications for conjectures about supports and accountability relations for teachers' learning, and thus for the practices of personnel at all levels of the system.

Research on instructional improvement at the level of an educational system is thin, and gets thinner the further one moves away from the classroom. In order to formulate MIST conjectures about potentially productive school- and system-level supports, we drew on the limited number of relevant empirical studies and conceptual analyses available in the mathematics education literature on mathematics teaching, professional development, and teacher collaboration (Kilpatrick et al. 2003; Cobb and McClain 2001; Franke and Kazemi 2001; Gamoran et al. 2000; Kazemi and Franke 2004; Little 2002; Stein et al. 1998; Coburn and Russell 2008) and the literature on education policy and leadership that viewed policy implementation as involving learning (Blumenfeld et al. 2000; Coburn 2003; McLaughlin and Mitra 2004; Stein 2004; Tyack and Tobin 1995).

The resulting conjectures specified school and district structures, social relationships, and material resources that we anticipated might support mathematics teachers' and instructional leaders' ongoing learning. These conjectures assumed that the district has adopted research-based, inquiry-oriented mathematics textbooks and would provide sustained teacher professional development.

One of our conjectures drew on research in educational policy and leadership that indicated the importance of teachers and school leaders having a common improvement agenda. We therefore conjectured that a shared vision of high-quality mathematics instruction in schools would be associated with instructional improvement. A second conjecture specified that instructional improvement will be supported both if school instructional leaders support and hold mathematics teachers accountable for developing high-quality instructional practices, and if district leaders hold school leaders accountable for assisting mathematics teachers in improving their instructional practices. We also conjectured that greater improvements in overall student mathematics achievement would occur if school mathematics programs were de-tracked so that classes were heterogeneous rather than organized by current student achievement. This conjecture drew on research that indicated that "tracking does not substantially benefit high achievers and tends to put low achievers at a serious disadvantage" (Darling-Hammond 2007, p. 324). A fourth conjecture indicated the importance that we attributed to the alignment of goals and strategies for instructional improvement across district central office units, particularly Leadership, the department responsible for supporting and holding school leaders accountable, and Curriculum and Instruction, the department responsible for selecting instructional materials and for providing professional development for teacher and coaches. Cobb and Smith (2008) provide more detail on these initial conjectures.

18.5 Recruiting Collaborating Educational Systems

When preparing for a system-level design study, it is essential to formulate explicit criteria for selecting educational systems to target for participation in the study. In doing so, it is important to remember that it is system leaders rather than researchers who have the ultimate authority to determine both goals for students' mathematics learning and what counts as high-quality mathematics instruction. One important selection criterion for any system-level design study is therefore that system leaders' views of high-quality mathematics instruction are similar to those of the researchers.

In the case of MIST, a second important criterion was that the district be typical of large urban districts in the U.S. in terms of persistent patterns of low student achievement (including disparities in achievement between historically disadvantaged and advantaged groups of students); high teacher turnover; and relatively large numbers of novice teachers.

A third selection criterion related to how a district was responding to accountability demands. Given our focus on instructional improvement at scale, we sought to recruit districts that were among the minority that were responding by focusing on the quality of classroom mathematics instruction and students' mathematical learning. In this regard, all of the districts with which we collaborated were atypical of urban districts in that they sought to improve student achievement in middle grades mathematics by supporting teachers' development of ambitious mathematics teaching practices of the type described earlier in the paper. The collaborating districts' goals for students' mathematical learning extended beyond improving achievement on state tests, and included a concern for students gaining admission to college and succeeding when they got there.

A fourth selection criterion was that the districts framed the problem of instructional improvement in terms of teacher learning and were attempting to implement a reasonably well worked out set of improvement strategies (Elmore 2006). Each of the four district's pre-existing improvement strategies for high-quality mathematics instruction aligned with current research on mathematical learning, and encompassed curriculum, teacher professional development, and school instructional leadership. Examples of such strategies include adopting an inquiry-oriented textbook series for middle-grades mathematics, providing high quality teacher professional development, scheduling time during the school day for mathematics teachers to collaborate, recruiting and supporting a cadre of school- or district-based mathematics coaches, and supporting instructional leaders' development of instructional leadership practices through professional development.

A fifth criterion was that middle-grades mathematics was a priority area for the districts. The four districts were committed to providing time and resources to further their instructional improvement efforts in middle-grades mathematics, and considered that participating in our study could contribute to these efforts.

Not surprisingly, the number of urban districts that met our five criteria was limited. We identified three of the districts with the assistance of the Institute for Learning, a national organization that partners with districts to guide their development and implementation of improvement policies.

In conducting a design study of instructional improvement at scale, it is typically not feasible to collect data in all schools in the systems that have been recruited. Given the intent of a study of this type, we recommend purposefully selecting schools that reflect the overall variation in student performance and capacity for instructional improvement across all schools in each system. Teachers might then be recruited randomly within schools, or they might be selected purposefully to reflect variation in quality of current instructional practices.

In MIST, we recruited 30 middle-grades mathematics teachers from between six and ten schools in each of the four districts, together with 20 school and district leaders in each district. We found that teachers often agreed to participate in the MIST study because they saw it as an opportunity to have their perspective taken into account when system leaders formulated district improvement policies for middle-grades mathematics.

18.6 Using an Interpretive Framework to Assess Designed and Implemented Improvement Strategies

Based on our experience in the MIST study, two types of conceptual tools are important when conducting investigations of this type. The first tool is a theory of action for large-scale instructional improvement in mathematics that consists of testable conjectures about supports and accountability relations. As we described above, we developed an initial theory of action in the MIST study by drawing on then current research in mathematics education and the learning sciences, educational leadership, and educational policy before we began working with the four collaborating systems. We tested and revised the conjectures that comprised this initial theory of action as we conducted successive cycles of design and analysis in the course of the study. The theory of action is central to the design phase of the iterative design and analysis cycles that characterize design research at the system level.

The second type of conceptual tool is an interpretive framework that can be used to assess the potential of the collaborating systems' designed or intended strategies to contribute to instructional improvement. This tool is central to the analysis phase of each cycle. During the first 2 years of the MIST study, we developed an interpretive framework that distinguishes between four general types of supports: new positions, learning events (including professional development), organizational routines, and tools. These types of supports capture all the improvement strategies that the four collaborating systems attempted to implement across the 4 years. In developing the framework, we drew on research in the learning sciences, teacher learning, and related fields to assess the potential of each general type of support to scaffold teachers,' coaches,' and school leaders' reorganization of their practices.

We clarify the nature of each type of support and its potential to support practitioners' learning in the following paragraphs. As will become apparent, the framework reflects the view that co-participation with others who have already developed relatively accomplished practices is essential when the learning demands of an improvement strategy require the reorganization rather than the extension or elaboration of current practices (Lave and Wenger 1991; Rogoff 1997; Sfard 2008).

18.7 New Positions

School- and system-level strategies for instructional improvement typically include changes in the responsibilities of existing positions, such as principals becoming effective instructional leaders in mathematics. In addition, improvement efforts often include the creation of new positions whose responsibilities include supporting others' learning by providing expert guidance. For example, an educational system might create the position of a school-based mathematics coach in each school, whose responsibilities include supporting their principals in becoming instructional leaders in mathematics. This improvement strategy assumes that the coaches have developed greater expertise as instructional leaders in mathematics and can therefore guide principals as they attempt to support mathematics teachers' improvement of their classroom practices (Bryk 2009; Spillane and Thompson 1997).

The importance that we attributed to the expertise or knowledge-in-practice of the holder of the new position follows directly from Vygotskian accounts of human development (Kozulin 1990; van der Veer and Valsiner 1991; Vygotsky 1978) and is supported by studies of apprenticeship and coaching (Brown et al. 1989).

We therefore view the provision of expert guidance by creating new positions (or changing the responsibilities of existing positions) as a primary support for learning. The extent to which the investment in the new position will pay off is likely to be influenced by a variety of factors in addition to the expertise of the appointee. These additional factors include the overall coherence of instructional improvement strategies and the extent to which the expert and target of policy co-participate in activities that are close to the intended forms of practice.

18.8 Learning Events

Large-scale instructional improvement efforts typically include professional development for teachers and, on occasion, for members of other role groups including principals. We view professional development sessions as instances of learning events, which we define as scheduled meetings that can give rise to opportunities for targets of policy to improve their practices in ways that further policy goals. We take account of both learning events that are intentionally designed to support targets' learning and those that might give rise to incidental learning.

18.8.1 Intentional Learning Events

A distinction that proved useful in the MIST study when analyzing the strengths and weaknesses of improvement strategies is that between intentional learning events that are ongoing and those that are discrete. The two key characteristics of ongoing intentional learning events are that they are designed as a series of meetings that build on one another, and that they involve a relatively small number of participants. As an example, a mathematics specialist might work regularly with middle-school principals as a group in order to support them in recognizing high-quality mathematics instruction when they make classroom observations. Because a small number of participants is involved, the group might evolve into a genuine community of practice that works together for the explicit purpose of improving their practices.

It is important to note that although communities of practice can be productive contexts for professional learning (Horn 2005; Kazemi and Hubbard 2008), the emergence of a community of practice does not guarantee the occurrence of learning opportunities that further policy goals (Bryk 2009). Recent research in both teacher education and educational leadership indicates the importance of interactions among community members that focus consistently on issues central to practice (Marks and Louis 1997) and that penetrate beneath surface aspects of practice to address core suppositions, assumptions, and principles (Coburn and Russell 2008). This in turn suggests the value of one or more members of the community having already developed relatively accomplished practices so that they can both push interactions to greater depth (Coburn and Russell 2008) and provide concrete illustrations that ground exchanges (Penuel et al. 2006). The critical role of expertise in a community of practice whose mission is to support participants' learning is consistent with the importance attributed to "more knowledgeable others" in sociocultural accounts of learning (Bruner 1987; Cole 1996; Forman 2003).

The key aspects of ongoing intentional learning events that we have highlighted are consistent with the qualities of effective teacher professional development identified in both qualitative and quantitative studies. These qualities include extended duration, collective participation, active learning opportunities, a focus on problems and issues that are close to practice, and attention to the use of tools that are integral to practice (Borko 2004; Cohen and Hill 2000; Desimone et al. 2002; Garet et al. 2001). We view ongoing intentional learning events that have these qualities as a primary means of supporting consequential professional learning that involves the reorganization of practice.

Discrete intentional learning events include one-off professional development sessions as well as a series of meetings that are not designed to build on each other. For example, system leaders might organize monthly meetings for principals. These meetings would be discrete rather than ongoing intentional learning events if principals engage in activities that focused on instructional leadership in mathematics only occasionally, and these activities do not build on each other. Discrete intentional learning events can be valuable in supporting the development of specific capabilities that elaborate or extend current practices (e.g., introducing a classroom observation tool that fits with principals' current practices and is designed to make their observations more systematic). However, they are by themselves unlikely to be sufficient in supporting the significant reorganization of practice called for in systems that are pursuing ambitious instructional agendas.

18.8.2 Incidental Learning Events

Learning opportunities are not limited to those that are intentionally designed, but can also arise incidentally for targets of policy as they collaborate with others to carry out functions of the school or educational system. For example, if principals meet regularly with mathematics coaches to discuss the quality of mathematics teaching in the school, these meetings could provide learning opportunities for the principal even though these meetings were not designed to support the principals' learning. In general, the extent to which regularly scheduled meetings with a more knowledgeable other involve significant learning opportunities depends on both the focus of interactions (e.g., the nature of teachers' classroom practices and student learning opportunities) and on whether the expert has in fact developed relatively accomplished practices and the novice recognizes and defers to that expertise (Elmore 2006; Mangin 2007). However, the strategy of relying primarily on incidental learning events to support professional learning appears to be extremely risky.

18.9 New Organizational Routines

In addition to creating new positions and planning learning events, instructional improvement policies sometimes include the specification of new organizational routines. Feldman and Pentland (2003) define organizational routines as “repetitive, recognizable patterns of interdependent actions, carried out by multiple actors” (p. 94). Investigations of organizational routines in school settings demonstrate that they can play a critical role in ensuring continuity and thus school stability in the face of high staff turnover (Spillane et al. 2007). In addition, these studies clarify that organizational routines often evolve incrementally in the course of repeated enactments and can therefore also be a source of organizational flexibility (Feldman 2000, 2004). Furthermore, as Sherer and Spillane (2011) illustrate, the introduction of carefully designed organizational routines can be an important means of supporting learning.

An illustration of an organizational routine would be system leaders expecting principals to conduct Learning Walks™ with the mathematics coach at their schools on a regular basis. A Learning Walk™ is a repetitive, recognizable pattern of actions that involves determining the focus of classroom observations (e.g., the extent to which teachers maintain the cognitive challenge of tasks throughout the lesson), selecting classrooms to visit, observing a classroom, and then conferring to discuss observations before moving on to the next classroom. In addition, a Learning Walk™ is carried out by multiple actors, namely the principal, mathematics coach, and the observed teachers. This organizational routine provides opportunities for the mathematics coach to support the principal in coming to recognize key aspects of high-quality mathematics instruction.

In this example, the organizational routine is conducted independently of any formally scheduled meetings. Other organizational routines might be enacted during either intentional or incidental learning events. For example, a mathematics specialist working with a group of principals might introduce an organizational routine that first involves having principals collect student work on the same instructional task from one or more classrooms in their schools, next having the principals analyze the quality of the student work in small groups, and finally pressing the principals to delineate the characteristics of high-quality work during a subsequent whole group discussion. We consider organizational routines in which a more knowledgeable other scaffolds relative novices’ learning as they co-participate in a sequence of activities that are close to practice to be a potentially productive means of supporting professional learning (Grossman and McDonald 2008; Lampert and Graziani 2009).

18.10 New Tools

In speaking of tools, we refer to material entities that are used instrumentally to achieve a goal or purpose. Work in the learning sciences and in teacher professional development indicates that introducing carefully designed tools is a primary means of supporting learning (Borko 2004; Cobb et al. 2009; Lehrer and Lesh 2003; Meira 1998). In the context of large-scale instructional improvement efforts, designed tools can also play a second important role by supporting members of a particular role group in developing compatible practices, and by supporting the alignment of the practices developed by members of different role groups (e.g., teachers, principals, coaches). Examples include textbooks, curriculum guides, state mathematics objectives, classroom observation protocols, reports of test scores, student written work, and written statements of school and educational system policies.

Large-scale instructional improvement efforts almost invariably involve the introduction of a range of new tools designed to be used in practice, including newly adopted instructional materials and revised curriculum frameworks for teachers, and new classroom observation protocols and data management systems for principals. The findings of a number of studies conducted in the learning sciences substantiate Pea's (1993) claim that the incorporation of a new tool into current practices can support the reorganization of those practices (Lehrer and Schauble 2004; Meira 1998; Stephan et al. 2003). However, it is also apparent that people frequently use new tools in ways that fit with current practices rather than reorganizing those practices as the designers of the tool intended (Wenger 1998). For example, the findings of a number of studies of policy implementation and of teaching indicate that teachers often assimilate new instructional materials to their current instructional practices rather than reorganize how they teach as envisioned by the developers of the materials (Cohen and Hill 2000; Remillard 2005; Spillane 1999). These findings suggest that the design of tools for professional learning should be coordinated with the development of supports for their increasingly accomplished use.

As a first design heuristic, it is important that users see a need for the tool when it is introduced (Cobb 2002; Lehrer et al. 2000). This implies that either the tool should be designed to address a problem of current practice or it should be feasible to cultivate the need for the tool during intentional learning events. As an illustration, consider a classroom observation protocol that has been designed to support principals in focusing not merely on whether students are engaged but also on whether significant learning opportunities arise for them. Most principals are unlikely to see a need for the new observation form unless it is introduced during a series of intentional learning events that might, for example, focus on the relation between classroom learning opportunities and student achievement.

Second, it is also important that the tool be designed so that intended users can begin to use it shortly after it has been introduced in relatively elementary ways that are nonetheless compatible with the designers' intentions and do not involve what A. Brown (1992) termed lethal mutations. In the case of our example, it would seem advisable to minimize the complexity of the observation protocol given the significant reorganization of practice that most principals would have to make to use it in a way compatible with the designers' intentions (Nelson and Sassi 2005).

Third, in using the tool in rudimentary but intended ways, users begin to reorganize their practices as they incorporate the tool. The challenge is then to support their continued reorganization of practice by scaffolding their increasingly proficient use of the tool either during intentional learning events or as they co-participate in organizational routines with an accomplished user (J. S. Brown and Duguid 1991; Lave 1993; Rogoff 1990). In the case of the observation protocol, for example, mathematics coaches might support principals' use of the tool as they conduct Learning Walks™ together. Just as the failure to provide sustained teacher professional development around a new curriculum can lead to difficulties (Crockett 2007), failure to scaffold principals,' coaches,' and others' use of new tools is also likely be problematic.

18.11 Summary

Our analysis of the four types of support for learning indicates that improvement strategies that are likely to be effective in supporting consequential professional learning involve some combination of new positions that provide expert guidance, ongoing intentional learning events in which tools are used to bridge to practice, carefully designed organizational routines carried out with a more knowledgeable other, and the use of new tools whose incorporation into practice is supported. We do not discount the support that discrete intentional learning events and incidental learning events might provide and recommend taking them into account when assessing systems' improvement strategies. However, research on professional learning and on students' learning in particular content domains indicates that they are, by themselves, rarely sufficient to support significant reorganizations of practice (Garet et al. 2001). The analysis we conducted during the MIST study of the four districts' instructional improvement efforts over a 4-year period is consistent with this conclusion.

18.12 Conducting Design, Analysis and Feedback Cycles

Thus far, we have discussed the key issues that need to be addressed when preparing for a system-level design study. We now focus on the process of conducting a study by enacting successive design, analysis, and feedback cycles. Each of the four cycles we conducted in the MIST study spanned an entire school year, which is much longer than in other types of design experiments (a day in the case of a classroom design study and a few weeks or less for a professional development study).

In planning cycles, it is important to take account of patterns in system leaders' work across the school year. In the U.S. educational systems, the school year runs from August until May or the beginning of June. In the MIST study, we delayed interviewing district leaders to learn about their current instructional improvement plans until October of each year after they had finalized their plans for that school year. We then determined that January-March would be the best time to collect data because it would give us enough time to conduct the feedback analyses, and would not interfere with standardized testing, which typically occurs near the end of the school year. We shared our feedback and recommendations with district leaders in May of each year so they could take account of our findings when they revised district instructional improvement strategies over the summer.

18.12.1 Documenting Current Instructional Improvement Strategies

The first phase of a cycle involves documenting the vision of high-quality mathematics instruction that orients each collaborating system's instructional improvement initiative and the strategies that each system is implementing in an attempt to achieve its vision. In the MIST study, it proved feasible to document the four collaborating systems' improvement strategies by interviewing six to ten key system leaders in each system and by collecting system-level planning and implementation documents in October of each year. The leaders were from a number of system units that had a stake in mathematics teaching and learning. They included Curriculum and Instruction that is responsible for selecting instructional materials and for providing professional development for teacher and coaches, Leadership that is responsible for providing professional development for school leaders and for holding school leaders accountable, ELL that is responsible for supporting the learning of English Language Learners, Special Education that is responsible for supporting the learning of students who receive special education services, and Research and Evaluation that is responsible for generating and analyzing data on students, teachers, schools, and the district.

In addition to asking about current initiatives in middle-grades mathematics, it proved useful to include interview questions that focused on student demographics, the impact of regional and national policies, and the historical context of the system including prior reform initiatives and previous mathematics instructional materials and assessments. (Interview protocols are downloadable at http://vanderbi.lt/mist).

The transcribed interviews and the artifacts can be analyzed through an inductive coding process in order to discern broad consistencies across participants in each system. The goal in conducting these analyses is to clarify the intended or envisioned practices of members of particular role groups (e.g., teachers, coaches, principals), the intended means of supporting the learning of members of those groups, and system leaders' rationales for why the supports might enable members of each role group to develop the envisioned forms of practice.

In the MIST study, we reported our findings for each collaborating system in a five-page document. This System Design Document named each district strategy and described the intended supports and accountability relations for members of each role group. We shared this document with system leaders to determine whether it accurately represented their plan for instructional improvement. We made revisions until the district leaders agreed that the document accurately represented their intended plan.

System Design Documents serve four useful purposes. First, they are useful in preparing for the next phase of a cycle that involves collecting data to learn how each system's intended strategies are being implemented in schools. Second, the major strategies identified in each document provide a framework for organizing the feedback given to the system leaders about how their improvement strategies are playing out. Third, system leaders who participated in the MIST study reported that they found these documents useful in clarifying their improvement strategies with others across the system. Finally, the System Design Documents produced in successive cycles provide a record of changes in a system's improvement policies over time, thus enabling the system leaders to monitor progress and researchers to document the influence of their recommendations on the improvement strategies that system leaders attempted to implement in the next cycle.

To illustrate, we refer to the System Design Document we created for District B, one of the four participating districts, during our first year of working with the district. (Table 18.1 provides a summary of District B's System Design Document, 2007–2008). The overall goal of the instructional improvement effort in District B was to ensure that all students had opportunities to learn through engagement with a rigorous mathematics curriculum, that teachers and school leaders had high expectations for students' learning, and that achievement disparities between White students and traditionally underserved groups of students were eliminated. District B was in its first year of implementing an inquiry-oriented mathematics curriculum. To support this implementation, the district had assigned a mathematics coach to each middle school the previous year and had provided them with a significant amount of professional development that focused on both teaching the new curriculum

Table 18.1 Summary of a System Design Document for District B, 2007–2008 school year

District B instructional improvement goal	
Ensure that all students have opportunities to learn through engagement with a rigorous curriculum, that teachers and school leaders have high expectations for students' learning, and that achievement gaps between White students and traditionally underserved groups of students are eliminated	
Improvement strategies	*Supports for role groups to develop the intended forms of practice*
1. Develop principals and coaches who work together to improve instruction	Professional development for principals on observing classroom and providing feedback to teachers
	Principal and the math coach are required to meet weekly to discuss classroom instruction and supports for teachers
	Professional development for math coaches
2. Support teachers in teaching a rigorous mathematics curriculum effectively	Professional development for teachers on the inquiry-oriented curriculum
	A comprehensive curriculum framework to support the implementation of the rigorous curriculum

effectively and coaching other mathematics teachers at their schools. Each coach taught for half of the school day and served as a coach for the remainder of the day.

The first improvement strategy that we identified was to support principals' and mathematics coaches' development as instructional leaders who worked together to improve the quality of mathematics instruction. Principals were expected to observe classroom instruction regularly to assess the quality of teachers' instructional practices and determine their needs based on these observations. Principals received professional development on observing and assessing the quality of mathematics instruction, and were expected to meet with the mathematics coach at their school every week to discuss the quality of classroom instruction and assess teachers' needs.

The second strategy was to support teachers in teaching the inquiry-oriented curriculum effectively. Supports for teachers' learning included teacher professional development provided by the mathematics coaches and a district Curriculum Framework that aligned the curriculum with the state standards and provided guidance on differentiating instruction for particular groups of students, especially English Language Learners and special education students.

We used the Interpretive Framework described above to assess the strengths and limitations of these two improvement strategies. District leaders clearly and consistently articulated the forms of practice they intended teachers, coaches, and principals would develop (e.g., principals were to observe classrooms and provide feedback to improve instructional practices). In addition, these intended forms of practice were compatible with the district's overall goal of supporting teachers' development of ambitious instructional practices. However, we considered it unlikely that the supports for various role groups' learning would be adequate.

With regard to the first strategy, principals would have to distinguish between weak and strong enactments of ambitious instructional practices if they were to give teachers effective feedback. The supports for principals' learning included professional development on observing classroom instruction. We questioned whether these ongoing intentional learning events would be effective because they focused on characteristics of high quality instruction that were independent of subject matter area, and because these characteristics were relatively global. Principals were also expected to meet regularly with the mathematics coach to discuss the quality of classroom instruction. Although these discussions might focus on content-specific instructional practices, we doubted whether the resulting incidental learning opportunities would be adequate. In addition, the coaches were new to the role and it was not clear that they had developed sufficient expertise to support principals in assessing the quality of instruction.

With regard to the second strategy, the effective implementation of the inquiry-oriented curriculum that the district had adopted required that most teachers significantly reorganize their instructional practices. Teachers participated in ongoing intentional learning events- 4 days of district professional development led by the math coaches. However, it was not clear that mathematics coaches had developed the expertise to lead this professional development effectively given that they were also teaching the new curriculum for the first time.

18.12.2 Documenting How Instructional Improvement Strategies Are Implemented

The next phase of the design cycle involves collecting data to document how each system's strategies are playing out in schools and classrooms. In the MIST study, we collected multiple types of data to document the four systems' instructional improvement efforts: audio-recorded interviews conducted with the 200 participants; on-line surveys for teachers, coaches, and school leaders; video-recordings of two consecutive lessons in the 120 participating teachers' classrooms, coded using the Instructional Quality Assessment (IQA) (Boston 2012; Matsumura et al. 2008); teachers' and coaches' scores on the Mathematics Knowledge for Teaching (MKT) instrument (Hill et al. 2004); video-recordings of select district professional development; audio-recordings of teacher collaborative planning meetings; and an on-line assessment of teacher networks completed by all mathematics teachers in the participating schools. In addition, the districts provided us with access to mathematics achievement data for students in the participating 120 teachers' classrooms. The interviews and online surveys focused on the school and district settings in which the participating teachers and school leaders worked and gave particular attention to the formal and informal supports on which they could draw to improve their practices, as well as to whom they were accountable and for what they were accountable.

As we had only 3 months to analyze data before district leaders began planning strategies for the following school year, we limited the data we analyzed to provide feedback about how districts' strategies were being implemented to the audio-recorded interviews conducted with the 50 participants in each district. (As our collaboration with each district continued over 4 years, we were able to share additional findings from other data sources, for example video-recordings of classroom instruction, in subsequent reports as they became available.)

One of the challenges when conducting a system-level design study is to analyze a large amount of data in a relatively short period of time while ensuring that the findings shared with system leaders are reliable. In this context, an important criterion for reliability is that claims about how improvement strategies are being implemented can be justified by backtracking through successive steps of the analysis to the raw data. This method involves using a series of structured tools to first summarize transcriptions of each participant interview, and then to triangulate and synthesize the responses both across participants in each school and across teachers, coaches, and school leaders in each collaborating system.

In MIST, a team member completed an Interview Summary Form (ISF) for each interview (teacher, coach, school leader, system leader). The ISF summarized each participant's response to interview questions that were central to understanding how improvement strategies were playing out in schools. This information was then synthesized across all participants in a school using the School Summary Form (SSF). This required the triangulation of participant responses at each school, citing evidence from the ISFs. Additional forms included a Principal Summary Form (PSF), a Coach Summary Form (CSF) and a Teacher Summary Form (TSF) that were used to synthesize information across members of a role group in a system (i.e., the TSF synthesized the interview summary forms for all the participating teachers in a system).

Once this initial analysis was complete, we returned to the System Design Document, and we identified gaps between each system's intended strategies and the strategies as they were being implemented in schools. We then examined why strategies were playing out as we had documented rather than as intended by focusing on the actual learning opportunities and press for improvement for members of each role group. In developing these explanations, we used one of our conceptual tools, the interpretive framework which differentiates between four general types of supports.

The final step in the analysis involved developing recommendations for how system leaders might revise their improvement strategies to make them more effective. In doing so, we drew on the conjectures about supports and accountability relations that comprised the current iteration of our theory of action for instructional improvement. It also proved essential to take account of each collaborating system's current capacity to support the learning of members of particular role groups. The resulting recommendations proposed feasible strategies for supporting teachers,' coaches,' and school leaders' improvement of their practices. Table 18.2 provides an illustration of this process.

As another illustration, our analysis of data measuring coaches' mathematical content knowledge (as measured by the MKT assessment), as well as the quality of coaches' instructional practices (as measured by the IQA) collected during our first year of collaborating with District B indicated that the knowledge and instructional practices of the school-based mathematics coaches were only slightly more advanced than those of the teachers they were expected to support even though they had received extensive professional development. In addition, few if any mathematics teachers had developed relatively accomplished instructional practices. Further, the district had only three district-level mathematics specialists (members of Curriculum and Instruction) who were expected to fulfill several different roles and responsibilities while serving 32 middle-grades schools. This lack of instructional expertise was a major constraint that we had to take account of when making recommendations about supports for teachers' and for coaches' learning. One of our recommendations therefore included leveraging the expertise of the three mathematics specialists by making their work in supporting the coaches' learning a priority.

18.12.3 Sharing Findings and Recommendations with System Leaders

We have emphasized that a system-level design study involves a genuine partnership with system leaders in which the leaders have the ultimate authority for making decisions about improvement strategies. It is therefore important for the researchers to develop a method for sharing findings and recommendations that is both feasible and relevant to system leaders' current concerns. In the MIST study, a two-step

Table 18.2 Example of an analysis for the District B year 4 System Feedback and Recommendations Report

Example District B coach interview transcripts (from January interview; edited slightly for readability)	Coach descriptions about district and principal expectations about the role of the math coach:	Principal expectations about the role of the math coach:	Excerpt from System Feedback and Recommendations Report
	Excerpt from Coach Summary Form (CSF) (includes 7 coaches)	Excerpt from Principal Summary Form (PSF) (includes 11 school administrators)	
I: So what are you being held accountable for as a math coach?	Four out of the seven coaches interviewed were a part of a state grant and received additional support from the state department of education. These coaches reported that the state expected them to model instruction, co-teach, provide training to develop teachers' instructional practices, conference, observe teachers, conduct a coaching cycle, monitor student learning, provide instructional materials for teachers, assist teachers with the curriculum, and help write lesson plans. In addition, one coach reported the expectation to work with teachers both one-on-one and groups. The coaches supported by this grant indicated they received a letter from the state that listed these expectations	Almost all of the administrators talked about the role of the mathematics coach involving working with teachers and not being an evaluative presence. Almost all principals expect the math coaches to work with teachers individually and in groups. Commonly cited specific activities included modeling, co-teaching, providing PD, and observing/intervening in classrooms. Multiple administrators said that the math coach was responsible for helping teachers with the curriculum	Our recommendation is that the district needs to clarify the role of the coach with both school leaders and the coaches by making it explicit that coaches should spend the majority of their time assisting either groups of teachers or individual teachers with instruction in their classrooms
C: Well I guess my principal definitely has a lot of expectations. He knows that no matter what he asks me to do I'm going to do it well, so even outside of what I should be doing. When we started the school year he asked me to help with scheduling, and we had a major issue. Maybe it is my job and maybe it's not my job, but it's important to me that every student started in the right math class			

(continued)

Table 18.2 (continued)

I have to do data analysis, so when students take assessments, I review and compile that data and then have discussions with the teachers regarding that data	The other math coaches did not describe as many concrete practices when asked about expectations for their role; three reported being expected to support teachers who are not strong with questioning and pacing and help struggling teachers with certain concepts	Some expectations emerged from only a few administrators. In at least two schools, the principal expected the math coach to lead collaborative time, and in one school, the math coach was expected to maintain a model classroom. Administrators in two schools also mentioned that the math coach should analyze data	
I have to hold our weekly grade level planning meetings and make sure that those are as productive as possible, that teachers are you know sharing information and are working together. Those are the things I would say I'm most accountable for, you know and then of course there's going out and coaching, but there's not much accountability with that			
You know obviously if a teacher's not doing a good job there's a discussion, but for the most part I really have a pretty solid department. Everybody can be coached and everybody can get better, so it's not like oh, you guys are solid, you don't need any coaching, but there's not a lot of accountability that goes with it	While most of the coaches say their principals' expectations are similar to the district's expectations, five coaches describe expectations above and beyond the district's expectations (e.g., tutoring, data analysis). Four coaches mentioned that in addition to working with teachers, they are expected to analyze data and run tutoring programs	Some themes emerged that were more operational in nature such as ordering supplies and obtaining resources. Some administrators also said that the math coach should be working with students individually	

Note that the table does not represent all of the data that were used to reach the findings and recommendations listed here. The data that were analyzed included the coach, teacher, and principal interviews (which were triangulated), previous years' video-recordings of classroom instruction and assessments of teachers' and coaches' mathematical knowledge for teaching, and previous years' feedback and recommendation reports. As an illustration, this table includes one coach's interview response related to expectations around their role as a math coach, along with the subsequent related syntheses from the CSF and the PSF process for communicating findings and negotiating future improvement strategies proved to be relatively effective.

The first step involved preparing a System Feedback and Recommendations Report of approximately 15 single-spaced pages for the leaders of each collaborating system. These reports built directly on the System Design Documents and were intentionally structured around the district's major strategies so that they related directly to the work district leaders were attempting to accomplish. For each strategy reported in the System Design Document, we reiterated the envisioned forms of practice that constituted the goal of the strategy and described the intended supports and accountability relations for the development of the envisioned practices. We then reported our findings about how that strategy was playing out in schools, explained why this was the case, and made our recommendations for adjusting the strategy. Based on our experience in the MIST study, we believe that this way of organizing reports for system leaders provides a useful model for others conducting system-level design studies. Redacted versions of reports produced in the MIST study are available at the MIST website (http://vanderbi.lt/mist).

The second step in sharing findings and recommendations with system leaders was a 2-h meeting with the leaders of each system scheduled approximately 1 week after we sent them the System Feedback and Recommendations Report. The intent of these meetings was to clarify the findings and to have a genuine conversation about their implications for the system's improvement strategies. We therefore recommend that researchers explicitly negotiate norms for these meetings with system leaders, and that they speak from notes rather than PowerPoint in order to encourage an open discussion.

In the case of MIST, these meetings usually included the head of Curriculum and Instruction responsible for all content areas, the head of the Mathematics Department, the district mathematics specialists (who work with the mathematics coaches and support schools), the head of Leadership, and leadership specialists who support and assess school leaders. In one district the superintendent attended the feedback sessions. These meetings were typically very productive. In every instance, the conversation was an open dialogue about the current status of the district's improvement efforts and about possible adjustments to those efforts.

18.12.4 Assessing the Influence of Recommendations on Collaborating System's Instructional Improvement Strategies

The first phase of the next data collection, analysis, and feedback cycle involves interviewing system leaders again to document their revised instructional improvement strategies. The influence of recommendations made to system leaders can be assessed by comparing their revised and prior improvement strategies. As we have noted, assessing the influence of the recommendations is important both because the pragmatic goal of a system-level design is to contribute to the collaborating systems' instructional improvement efforts, and because researchers have opportunities to test the conjectures that comprise their theory of action when system leaders act on their recommendations. A priori, sharing a written report and conducting a single meeting with system leaders to discuss its implications might appear to be a relatively weak mechanism for influencing system-wide improvement strategies. We were therefore gratified to find that the leaders in all four systems that participated in the MIST study revised their improvement strategies based on many of our recommendations. The data we collected the following year documented how the policy revisions we recommended were actually playing out in school and classrooms. We could therefore use these data to test, revise and thus improve our conjectures about the learning of members of different role groups and the means of supporting their learning. Thus, the collaborative partnerships in which we became co-designers of district improvement policies with district leaders enabled us to enact iterative cycles of design and analysis that are characteristic of the design research methodology.

There are several reasons why we believe this limited collaboration proved to be sufficient. First, we selected districts whose goals for students' mathematical learning and for teachers' improvement of their instructional practices were broadly compatible with those that we intended to investigate. Second, we prioritized the development and maintenance of relationships of trust with district leaders and school personnel. Thus, during the first year of the collaboration, we strove to produce feedback reports that district and schools leaders would view as extremely relevant and useful to their work. It was because districts leaders found this and subsequent reports useful that they were willing to continue to spend time with us three times a year (fall interview, January interview, and May feedback session) and to allocate resources to assist our data collection efforts. This in turn enabled us to achieve almost 100 % success in all aspect of our data collection each year in all four districts.

This approach of developing, testing, and refining theory by conducting tightly integrated cycles of analysis and (policy) design is at the heart of the design research methodology (Cobb et al. 2003a). On the one hand, we revised and elaborated the conjectures that comprised our evolving theory of action in the course of the analysis and feedback process. On the other hand, our evolving conjectures informed the formulation of the specific feedback recommendations we made to the districts. In a very real sense, the design for system-wide instructional improvement that is implemented was co-constructed by system leaders and the researchers. It is useful to distinguish between the co-constructed designs that are specific to a particular system and researchers' conjectures about the process of supporting system-wide instructional improvement more generally. On the one hand, these latter conjectures comprise a theory of action that can be used to make recommendations to system leaders. On the other hand, occasions when system leaders act on these recommendations constitute opportunities for the researchers to test and revise their conjectures and thus contribute to the development of a generalizable theory of action for system-wide instructional improvement in mathematics.

18.13 Testing and Revising Conjectures that Comprise a Theory of Action for System-Wide Instructional Improvement

To this point, we have focused on the pragmatic objective of providing leaders of the collaborating systems with timely feedback about how their improvement strategies are actually playing out that can inform the revision of their instructional improvement efforts. We now consider the theoretical objective of contributing to a generalizable theory of action for system-wide instructional improvement in mathematics. In doing so, we draw on our experience in the MIST study by discussing three types of evidence that can inform the revision of conjectures that comprise the theory of action: findings from feedback analyses about how the collaborating systems' instructional improvement strategies are being implemented, the current research literature, and the findings of retrospective analyses conducted by drawing on the multiple sources of data collected in each cycle.

18.14 Findings About the Districts' Instructional Improvement Strategies

When researchers formulate recommendations to collaborating educational systems, they necessarily have to address concrete organizational design challenges by proposing how the systems might support and hold members of particular role groups accountable for improving their practices. Addressing these challenges is a primary context for researchers' learning. Furthermore, researchers can step back after completing each data collection, analysis, and feedback cycle and frame the findings and recommendations for the collaborating systems as cases of attempting to support instructional improvement at scale. In doing so, it is important to determine whether any of the recommendations to a particular system represent refinements or elaborations of current conjectures, and if they do whether they might have more general implications and under what conditions. For example, the constraint of limited instructional expertise that we identified in District B proved to be a constraint in two of the other three collaborating districts. The recommendations we made to these districts for supporting teachers' and coaches' learning could therefore inform the revision of our initial conjectures for instructional improvement in districts that are constrained by limited instructional expertise.

18.15 Research Literature

As we have noted, relevant research that can inform the design of instructional improvement strategies becomes increasingly thin the further one moves away from the classroom (Cobb et al. 2013; Honig 2012). Nonetheless, findings reported in the literature can, on occasion, provide evidence for the revision of current conjectures.

This possibility is becoming increasingly likely as system-level design studies and closely related approaches become more common.

18.16 Retrospective Analyses

The retrospective analysis of data collected during successive design and analysis cycles is a key aspect of design studies conducted at any level. In the case of system-level design studies, a primary goal of retrospective analyses is to investigate key conjectures of the theory of action for instructional improvement. Based on our work in the MIST study, we recommend that mutually informing lines of retrospective analyses be established that focus on the major types of supports conjectured to be important for instructional improvement (e.g., teacher collaborative time, teacher networks, mathematics coaching, school instructional leadership).

As we have indicated, the types of data that can be analyzed to give collaborating systems feedback about how their improvement strategies are playing out is constrained by the need to ensure that the feedback is timely and can inform system leaders' revision of their strategies. Retrospective analyses that can inform the revision of the theory of action draw on a range of additional types of data that are collected during each data collection, analysis, and feedback cycle. The primary concern when making decisions about the types of data to collect is that the key constructs of each conjecture are assessed including the relevant aspects of teachers' knowledge and instructional practices. For example, if the vision of high-quality mathematics instruction that constitutes the goal for teachers' learning requires that teachers deepen their mathematical knowledge, then it is important to include an appropriate measure of this knowledge. Similarly, if teachers' informal professional networks are conjectured to be an important support for their learning, then it is important to develop instruments for assessing the relevant aspects of their networks (e.g., who teachers turn to for instructional advice, frequency of their interactions with those people, and content of their interactions).

The MIST team is currently conducting five interrelated lines of analysis that focus on district-level and school-level teacher professional development (including mathematics teacher collaborative meetings), teacher networks, mathematics coaching, school instructional leadership, and district instructional leadership. We discuss the current version of our theory of action for instructional improvement in mathematics in the next section of this chapter.

18.17 MIST's Current Theory of Action for Instructional Improvement in Middle-Grades Mathematics

Presenting the current iteration of our theory of action in any detail is beyond the scope of this chapter, and we refer the reader to Cobb and Jackson (2011). To illustrate our current conjectures, we focus on one component of the theory of action, school instructional leadership.

Our initial conjectures about school instructional leadership were relatively global and did not differentiate between the practices of mathematics coaches and school leaders. These conjectures indicated the importance of school leaders developing relatively sophisticated visions of high-quality mathematics instruction and both supporting and holding mathematics teachers accountable for developing high-quality instructional practices. Our revised conjectures indicate the potential value of a distributed model of school instructional leadership in which coaches and district mathematics specialists are primarily responsible for supporting teachers' learning, and school leaders are responsible for pressing and holding teachers accountable for developing the intended instructional practices (Elmore 2006; Spillane et al. 2004). In addition, our current conjectures specify three leadership practices that might be feasible goals for school leaders' learning. Two of these practices—observing mathematics instruction and providing feedback, and participating in mathematics teacher collaborative meetings—aim at pressing teachers to develop the intended forms of practice and providing teachers with adequate support. The third practice concerns the development of productive relationships with coaches.

We conjecture that by observing instruction and providing teachers with informed feedback, school leaders can both communicate expectations and hold teachers accountable for improving classroom instruction. We also conjecture that it is important that the feedback be specific to the instructional practices on which school and district teacher professional development focuses. However, the extent to which school leaders' feedback accomplishes these goals depends crucially on the professional development they receive.

We conjecture that school leaders' participation in mathematics teacher collaborative meetings signals the importance of teacher collaboration, enables school leaders to hold teachers accountable for using collaborative time productively, and constitutes a context for school leaders' learning, thus better positioning them to give productive feedback after observing instruction and to procure appropriate resources for teachers. In this regard, a meta-analysis conducted by Robinson et al. (2008) found that school leaders' participation in teacher professional development is strongly associated with improvements in student achievement.

Findings of a retrospective analysis indicate that coaches' effectiveness in supporting teachers' learning depends on school leaders assuming shared responsibility for instructional improvement with them (Gibbons et al. 2010). We therefore conjecture that it is important that school leaders understand the district-wide goals for students' mathematical learning and the guiding vision of high-quality instruction, and that they appreciate the critical role of coaches in supporting teachers' learning. In the course of our collaboration with the districts, we have documented several cases in which principals have assigned additional duties to coaches that took them away from their work with teachers (e.g., analyzing data to identify struggling students, tutoring struggling students). Our observations also indicate that principals protect coaches' time when they understand the coaches' role in the improvement effort. We conjecture that the development of shared responsibility for instructional improvement is facilitated if school leaders and coaches meet regularly to share their observations about the quality of teachers' instructional practices, discuss how the coach's work with teachers is progressing, jointly select teachers with whom the coach should work, and plan for future work with groups of teachers.

The ongoing analyses we have conducted while developing feedback for the collaborating districts indicate that it is challenging for school leaders, most of whom are not mathematics specialists, to develop the three instructional leadership practices that we have described. As a consequence, we have also developed conjectures about the nature of professional development that might support their development of these practices.

First, we conjecture that if school leaders are to effectively and realistically press teachers to improve the quality of instruction, professional development for school leaders should enable them to recognize the instructional practices that are the focus of teacher professional development, and to distinguish between low- and high-quality enactments of those practices. We also conjecture that a consistent emphasis on the same instructional practices across teacher, coach, and school leader professional development will contribute to the development of compatible visions of high-quality instruction and to the alignment of supports for teachers' learning.

Second, we conjecture that professional development should attend explicitly to how to provide feedback to teachers that communicates expectations for ambitious instruction. This might involve school leaders and district mathematics specialists observing instruction or watching video-recordings of specific phases of lessons and discussing the feedback they would provide.

Third, we conjecture that professional development should clarify the role of coaches and mathematics teacher collaborative meetings in supporting teachers' development of ambitious instructional practices. We have documented several cases in which a school leader has taken over the agenda of mathematics teacher meetings to the detriment of the participating teachers' learning. We therefore conjecture that it is important to give particular attention to how the distribution of instructional leadership between coaches and school leaders should reflect their complementary areas of expertise (Elmore 2006).

The contrast between our initial and current conjectures for school leadership is representative of the changes we have made as we have revised and elaborated our initial conjectures. The level of specificity of our current conjectures is essential if we are to provide district leaders with actionable guidance on how they might support instructional improvement in mathematics on a large scale. We regard the current iteration of our theory of action as a work in progress and are further testing and revising our conjectures as we continue to collaborate with two of the four districts for a further 4 years.

18.18 Conclusion

Our purpose in this chapter has been to describe a design research approach for studying and supporting improvements in the quality of mathematics teaching on a large scale. The aim of this methodology is to both provide the leaders of educational systems, such as urban school districts in the U.S., with feedback that can inform their instructional improvement efforts, and to contribute to the development of a generalizable theory of action for large-scale instructional improvement in mathematics. The successful use of the methodology depends crucially on researchers establishing a genuine collaborative partnership with educational leaders such that researchers become co-designers of instructional improvement policies. Only then is it possible for researchers to test and revise their conjectures about supports for instructional improvement by conducting successive data collection, analysis, and feedback cycles.

We noted early in this chapter that research in mathematics education has made considerable progress in recent years, but that the findings of this work have had little impact on the quality of mathematics instruction and thus student learning in most classrooms. Design studies of the type that we have described and illustrated are clearly non-trivial undertakings. The value of this methodology derives from the way in which it enables us to test, revise, and thus improve our understanding of what it takes to support large-scale instructional improvement in mathematics.

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Part XII
Final Considerations

Chapter 19
Looking Back

Angelika Bikner-Ahsbahs, Christine Knipping, and Norma Presmeg

Abstract In this final summary we reflect on the interconnection between methodology and research practice. This brings us to consider basic principles and paradigmatic questions that link methodologies with each other and with specific objects and goals of research. Methodologies are part of the theoretical frameworks used in research, and therefore deeply connected with the theory's principles and paradigmatic questions. However, the link between specific research objects and goals, methodology and theoretical principles may be stronger or weaker. Looking back over the parts of this book, this is reflected in their structures, with some having two distinct chapters focused on theory and research practice respectively, while others consist of a single chapter, and others have two chapters in which theory and research practice are integrated but differently emphasized. The connections between theory and research practice reflected in the book's structure is the main topic of this final summary.

Keywords Methodology • Research practice • Connection between theory and research practice

Initially, all the parts of this book were supposed to consist of two separate chapters, which would allow the reader to use the book as an actual guide for the selection of an appropriate methodology, based on both theoretical depth and practical implications. However, in the course of the emergence of the book we realized that not all methodologies could be described in two such separate chapters, i.e., one describing the methodology in a more general form including basic considerations and the other illustrating this general description with a specific research example. Some methodologies seemed to be much more tightly linked to research practice than we

A. Bikner-Ahsbahs (✉) • C. Knipping
Faculty 3 of Mathematics and Computer Science, University of Bremen,
Bibliothekstraße 1, 28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de; knipping@math.uni-bremen.de

N. Presmeg
Mathematics Department, Illinois State University, Normal, IL, USA
e-mail: npresmeg@msn.com

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had assumed beforehand. Therefore this strict separation was modified, to allow presentations of interesting new strands of methodologies to be more connected to their respective research practices, their research objects, and specific theoretical frameworks. In looking back, this seems so particularly interesting to us that we want to reflect on the interconnection between methodology and its research practice in our final summary.

Following the methodology concept introduced by Radford (2008, 2012), methodologies, encompassing methods and techniques, are parts of the theory involved in research, and therefore deeply connected with the theory's principles and paradigmatic questions. The link between methodology and the theory's principles may be of varying degrees. For instance, ideal type construction (Part III) and Grounded Theory methodology (Part I) both have their roots in interpretive sociology but they are not as deeply intertwined with the theoretical principles as in the case for Abstraction in Context (Part V).

Abstraction in Context is not only a theory, rather it also provides the tools for analyses leading to results that in turn allow a deepening of the understanding of the theory's principles and core concepts. This deep interrelatedness of principles, methodology and results gained in research reflects the way the authors have illustrated their methodology while drawing on examples of their "research journey". A separation into two separate chapters was just not suitable.

The chapter on semiotic research (Part IV) shows how methodology can be elaborated by research practice: Hence, these authors, too, delineate the methodology by an intense use of research examples, but also for other reasons. The authors' cultural historical view has spread out to their research process that naturally is regarded as a cultural historical activity that can only be thought of as being intertwined with research practice. As described in the chapter, principles, methodologies and research questions are brought about through research practice and reveal results, which in turn broaden the researcher's theoretical view and approach to the field (see also Radford 2012). Thus, the authors' methodological perspective on generalization was widened and changed as they realized the relevance of rhythm as a semiotic resource.

The two chapters on argumentation processes (Part II) at first glance seem to refer to the same research object since both papers use Toulmin's scheme for their analyses of argumentation structures. However, the argumentation processes described in the two chapters differ with respect to the students' age, the level of mathematics, their complexity and duration. Moreover, the foci of the papers are different. Because of the complexity of the investigated processes the authors of Chap. 4 use additional diagrams that allow the capturing of long lasting mathematical argumentation processes and their specificities. In Chap. 3, two theories are merged, a participation theory and an argumentation theory, resulting in a conjunction of two different methodologies that offer additional insights into both, argumentation and participation of the students. Hence, this chapter illustrates how methodological tools, theories and research objects mutually inform each other.

Similar to the chapter on semiotic research (Part IV), the authors of the chapters on the networking of theories (Part VI) regard themselves and their experience in the Networking Theories Group as parts of the methodology. At the beginning of this research strand, multi-theoretical empirical research was an attempt at deepening the understanding of the role of theory in research in mathematics education. In the course of doing research with a group of scholars using different theories, methodologies for the networking of theories were developed. As described in this part, these methodologies revealed new kinds of concepts at the boundary of the theories involved. Hence, the chapters on the networking of theories are an example that shows how new methodologies involving a variety of theories provide new ways of engaging in research practice, and new kinds of results.

The authors of Part VII undertake a multilevel analytical approach to investigate individual and social learning processes over time in classrooms. To capture these they address and connect microgenesis, sociogenesis, and individual ontogenesis strands of learning consisting of different mutually influencing sub-objects, which belong to the learning process. Their specific methodical approaches deeply reflect the intertwined influence of different kinds of research objects in processes of learning over significant spans of time in class. As the authors admit, their methodology might not be transferable to another project the way it is used in their work. However, their approach might serve as heuristics that may be converted and adapted by other researchers to investigate learning processes over time in another class or another environment to understand the learning of another topic.

The approaches of Mixed Methods (Part VIII), Qualitative Content Analysis (Part IX) and Triangulation (Part X) in the next three parts are methodological approaches of a more general character and are therefore better transferable to other research projects. Mixed Methods and Qualitative Content Analysis address the kind of data used. Mixed methods mean the combined use of qualitative and quantitative data and methods in the very same research project. This may be pursued by relating results of qualitative and quantitative analyses to each other in order to compensate for specific weaknesses of both types of research. Such a combination may lead to enhancing the validation of qualitative or quantitative findings being extensively discussed in Part VIII. Qualitative and quantitative research can also be combined by integrating quantitative methods into the analysis of textual data in qualitative content analysis—an approach presented and discussed in Part IX. The third aforementioned methodology (Triangulation in Part X) not only addresses the connection of different data and methods in research, but also the common use of different theoretical perspectives, informants, environments and specifically cultural settings. Even if these three approaches (Parts VIII, IX, X) are not so tightly connected to the specific research objects, they assist in pursuing specific research aims. For instance a research aim might require mixed methods either to deepen insight into a quantitative data set by qualitative data or to broaden or validate the view suggested by qualitative data by adding a quantitative approach. Another aim could be enhanced insight into and an overview of the complementary variation of classroom activities. This aim is pursued by the Learner's Perspective Study (LPS) with its implementation of different kinds of triangulation, focusing specifically on the triangulation of different cultural settings (Part X).

In the final part (Part XI) on design research, the methodologies described in two of the three chapters share a cyclic characteristic, although the methods used are different. This cyclic character is often at the core of design research methodologies that links design and theory, although each can play a different role in the research practice. The design can be the goal of research, informed by theory. Design can also be the object to be researched in order to gain theoretical insight into the design itself. Or design can be a way to understand the structure of specific mathematical content to be learned. The methodology described in the third chapter of this part, on didactical engineering, lacks a cyclic character, and design is used as a tool for research and theoretical insight informed by epistemological considerations. While all three chapters of this part share an emphasis on design, there are analytical distinctions between them. They all show that theorizing and designing inform each other, but either theory or design or both can be in the center of the specific project. Not only the objects of research but also the purpose for which the design is developed and the kind and the role of theories involved all determine the methods used.

While specificities of the methodologies described differ in many ways, the parts of this book have pointed out the connectivity between doing research and the (qualitative) methodologies involved. This connectivity has been brought to life by including illustrative and paradigmatic examples, and, looking back, it has been reflected on in the previous sections according to the methodologies' degree of tightness to the theoretical principles on the one hand and the role of research practice on the other. We may conclude that qualitative methodologies (and beyond) do not always serve as instruments for research that are completely determined beforehand; they rather also serve as heuristics and evolve through research practice, its focus, aims and objects and its results over time. As Radford (2012) describes it, results may retroact to the development of theory and this encompasses the development of methodology.

Overall, we believe the purpose of this book—as a contribution to a methodological debate and as an offer for scholars interested in qualitative research and beyond—has been fulfilled. We thank the authors for their scholarship and careful work, especially in providing the examples of research projects that illustrate the use of their various methodologies, intertwined as these are with the respective theoretical principles, and for illustrating the reflexive relationships among theory, methodology, and methods of data collection which allow each of these to develop further.

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