Chapter 3
Basic Electromagnetic Theory with Special Attention to Lightning Electromagnetics

The goal of this chapter is not to provide a reference for the theory of electricity and magnetism. The study of the physics and effects of lightning flashes entail certain elements of electricity and magnetism that are used to describe various interactions. Some of the equations of electromagnetic theory that will be used either directly or indirectly in the book are presented here. For a complete treatment of the subject the reader is referred to Refs. [1] and [2].

3.1 Electric Field Generated by a Point Charge

Consider a point charge located at point O. Let us denote by P the point of observation where we would like to study the effect of the electric charge (Fig. 3.1). The electric charge gives rise to an electric field in space, and the direction and magnitude of this electric field change from one point to another. According to Coulomb's law, the electric field (measured in volts per meter) produced at P by a point charge q (whose magnitude is measured in Coulombs) located at O is

$$E = \frac{q}{4\pi\varepsilon_0 r^2}\mathbf{a_r}, \tag{3.1}$$

where $\mathbf{a_r}$ is a unit vector directed toward OP. If the charge is positive, then this electric field is directed away from point O. In the preceding equation, the parameter ε_0 is called the *permittivity of free space*. Its value is 8.85×10^{-12} Farads/m (F/m).

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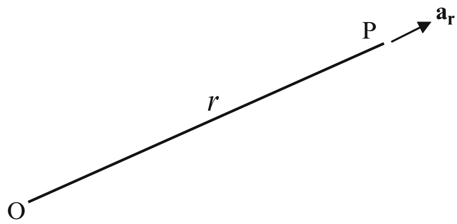

Fig. 3.1 A point charge is located at point O. The electric field produced by the point charge at point P is directed toward unit vector $\mathbf{a}_r$. (Figure created by author)

3.2 Electric Potential of Point Charge

Consider a point charge at point O. The potential of point P due to the presence of the point charge is

$$V = \frac{q}{4\pi\varepsilon_0 r}. \tag{3.2}$$

This is the amount of work that must be done (or the energy necessary) to bring a point charge of 1 C from infinity (where the potential is zero) to point P. The potential is measured in volts. Thus, the work necessary (measured in Joules) to bring a charge q_1 from infinity to point P is

$$W = \frac{qq_1}{4\pi\varepsilon_0 r}. \tag{3.3}$$

At the same time, if the charge q_1 is moved from point P to infinity, then the amount of energy released is also equal to the value given by Eq. 3.3. This shows that when we transfer a charge Q across a potential difference of V, the energy released (or the work that needs to be done, depending on the direction of movement of the charge) is equal to QV.

3.3 Gauss's Law

According to Gauss's law, the flux of an electric field coming from a closed volume is related to the total charge located inside that closed volume. Let **E** be the electric field at any point on a closed surface and **ds** a small area vector (the direction of the vector is the outward normal to the surface; Fig. 3.2). Then the flux of the electric field coming from the closed volume is given by $\oint_s \mathbf{E.ds}$. The 's' sign on the integral shows that it is a surface integral, and the circle shows that it is performed around the closed surface. Gauss's law states that this is equal to

$$\oint_s \mathbf{E.ds} = \frac{Q}{\varepsilon_0}, \tag{3.4}$$

where Q is the total charge located inside the closed volume. Now let us apply this equation to calculate the electric field in the vicinity of a long charged line or a channel. Let us assume that the charge per unit length of the channel is ρ. Since the channel is very long, the electric field lines by symmetry are radial to the channel. Now consider a closed surface in the form of a cylinder having radius r (Fig. 3.3). Applying Gauss's law to this cylinder one can write directly that

$$2\pi r l E = \frac{\rho l}{\varepsilon_0}. \tag{3.5}$$

Note that since the field lines are radial, the flux coming from the edges of the cylinder is zero (at the edges, $\mathbf{E.\,ds} = 0$). Thus, the radius r at which the electric field is equal to E is given by

$$r = \frac{\rho}{2\pi \varepsilon_0 E}. \tag{3.6}$$

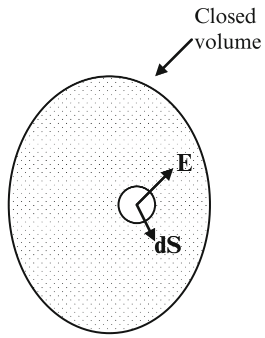

Fig. 3.2 Geometry pertinent to definition of Gauss's law. The *white patch* is a small surface area on the closed surface. The outward normal to the surface area is **dS**, and **E** is the electric field vector at the location of the elementary area (Figure created by author)

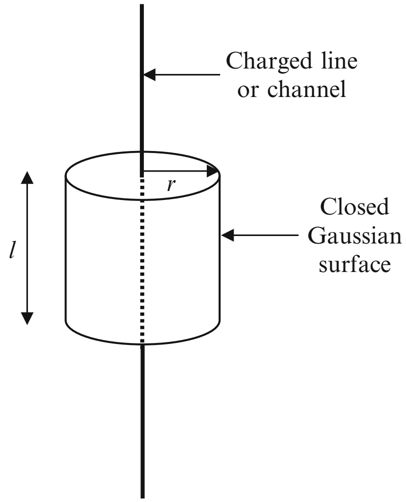

Fig. 3.3 Geometry pertinent to application of Gauss's law to a long charged line or channel. The closed volume is composed of the cylindrical surface together with the two circular surfaces covering the two ends. Since the electric field is radial, all the flux of the electric field passes through the cylindrical surface (Figure created by author)

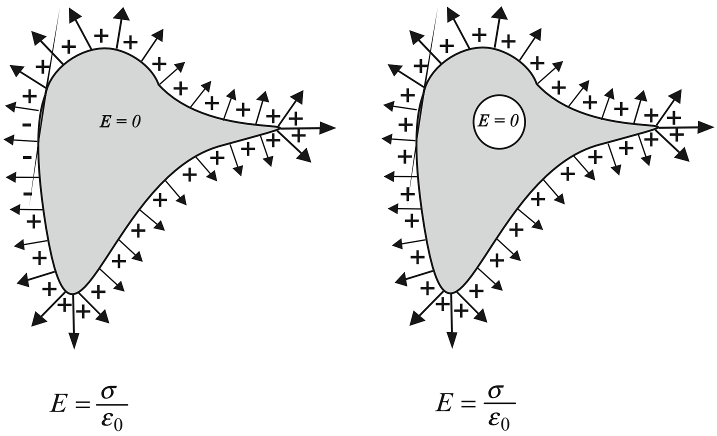

Fig. 3.4 The electric field on the surface of a perfect conductor is always perpendicular to the conductor. The local electric field is proportional to the local surface charge density, σ, and is given by σ/ε_0. The local charge density is proportional to the curvature of the surface. The surface charge and, hence, the electric field is large at sharp points on the conductor. The electric charge resides always on the surface of the conductor, and the electric field is zero inside a closed conductor. For the same reason the electric field inside a cavity of the conductor is also zero. This is the origin of the Faraday cage principle (Figure created by author)

3.4 Electric Field Inside and on Surface of Perfect Conductor

On the surface of a perfect conductor the electric field is always perpendicular to the surface. The component of the electric field parallel to the surface is zero (Sect. 3.16). This can be understood as follows. If there is an electric field parallel to the surface of a good conductor, free charges in the conductor will displace along the conductor and create an electric field opposite to that of the applied electric field. The displacement of the charge continues until the two electric fields cancel each other, making the electric field parallel to the surface zero. For the same reason the electric field inside a perfect conductor is also zero. That is, the potential at any point of the conductor is the same. This also explains why the electric charge of a charged conductor resides on the surface of the conductor. These facts can be used together with Gauss's law to show that the electric field inside the cavity of a perfect conductor is zero irrespective of the amount of charge that resides on the surface. This is the origin of the Faraday cage principle of lightning protection. Some of these points are illustrated in Fig. 3.4.

3.5 Electric Field of a Point Charge Over a Perfect Conductor

Now consider a point charge located over a perfectly conducting plane of zero potential. In problems dealing with electrostatics or slowly changing fields, the Earth's surface can be treated as a perfect conductor. For all practical purposes we can also treat the potential of the ground as zero. Now, the presence of charge q (assumed to be positive) repels any positive charge from points directly below the ground while attracting negative charge into that region. This gives rise to a charge distribution (induced) on the surface of the ground (recall that charges reside on the surface of a conductor). The electric field at any point above the ground is the sum of the electric field produced by the electric charge q and this induced charge. Moreover, as discussed earlier, the electric field at the ground plane is perpendicular to it because it is a good conductor. The field configuration of a point charge over a perfectly conducting ground plane is shown in Fig. 3.5a. Note that without affecting the field distribution, one can replace the conducting plane by placing a negative charge of the same magnitude at the mirror point (Fig. 3.5b). That is, the effect of the induced negative charge on the ground can be represented by a negative charge, equal in magnitude to the positive charge, located at the mirror point of the charge. Now consider a point charge q located at height h above a perfectly conducting ground plane of potential zero (see Fig. 3.5c). The electric field at point P located above the ground plane consists of two components, one from the real charge (positive) and the other from the image charge (negative). The component from the real charge is

$$E_r = \frac{q}{4\pi\varepsilon_0 r_r^2}\mathbf{a_{rr}}, \tag{3.7}$$

where $\mathbf{a_{rr}}$ is a unit vector directed toward r_r (see Fig. 3.5c). Separating the electric field components into vertical and horizontal directions we obtain

$$E_{rx} = \frac{q}{4\pi\varepsilon_0 r_r^2}\sin\theta_r, \tag{3.8}$$

$$E_{rz} = -\frac{q}{4\pi\varepsilon_0 r_r^2}\cos\theta_r. \tag{3.9}$$

Similarly, the electric field due to the image charge is

$$E_i = \frac{q}{4\pi\varepsilon_0 r_i^2}\mathbf{a_{ri}}. \tag{3.10}$$

Separating this into z and x components we obtain

$$E_{ix} = -\frac{q}{4\pi\varepsilon_0 r_i^2}\sin\theta_i, \tag{3.11}$$

$$E_{iz} = -\frac{q}{4\pi\varepsilon_0 r_i^2}\cos\theta_i. \tag{3.12}$$

The total x and z components are the sum of these components. These are given by

$$E_x = \frac{q}{4\pi\varepsilon_0 r_r^2}\sin\theta_r - \frac{q}{4\pi\varepsilon_0 r_i^2}\sin\theta_i, \tag{3.13}$$

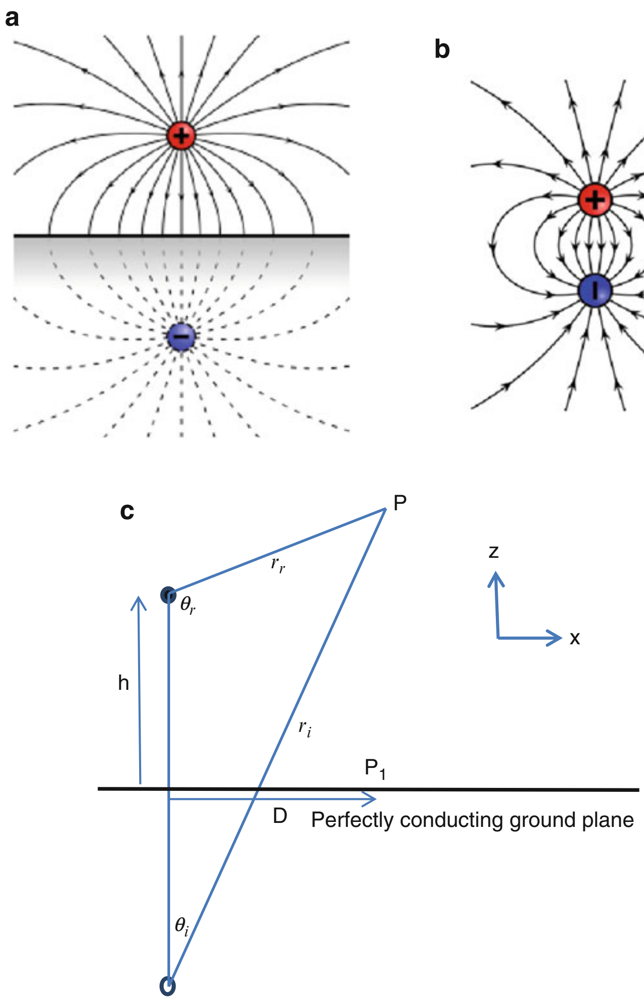

Fig. 3.5 (**a**) Field configuration of a point charge located over a perfectly conducting ground plane. (**b**) One can replace the perfect conductor with a charge of opposite polarity located at the image point without changing the field configuration above the conducting plane. (**c**) Geometry relevant to calculation of electric field from point charge located at height *h* over perfectly conducting ground plane. The perfectly conducting plane is replaced by a charge of equal magnitude but of opposite polarity located at the mirror image point (Panels **a** and **b** from Wikipedia (http://en.wikipedia.org/wiki/Electric_charge), Panel **c** created by author)

$$E_z = -\frac{q}{4\pi\varepsilon_0 r_r^2}\cos\theta_r - \frac{q}{4\pi\varepsilon_0 r_i^2}\cos\theta_i. \tag{3.14}$$

One can see directly at ground level, where $r_r = r_i$ and $\theta_r = \theta_i$, the x component, i.e., the component parallel to the surface, goes to zero. The vertical electric field on the surface of the ground at a horizontal distance D from the point charge is (see Fig. 3.5c for the geometry)

$$E_{z,\text{ground}} = -\frac{qh}{2\pi\varepsilon_0 \left(D^2 + h^2\right)^{3/2}}. \tag{3.15}$$

3.6 Ampere's Law and the Magnetic Field due to a Long Conductor

Ampere's law relates the line integral of the B-field or the H-field around a closed path to the electric current I passing through the closed path (Fig. 3.6):

$$\oint_l \mathbf{B} \bullet \mathbf{dl} = \mu_0 I. \tag{3.16}$$

In the above equation the letter 'l' on the integral sign denotes that it is a line integral and the circle indicates that it is performed around a closed path. Applying this to a long current-carrying conductor and using the facts that the magnetic field forms closed loops around the conductor and the magnetic field has the same magnitude along any of these loops, one obtains the B-field at a radial distance r from the conductor as

$$2\pi r B(t) = \mu_0 I(t), \tag{3.17}$$

$$B(t) = \frac{\mu_0 I(t)}{2\pi r}. \tag{3.18}$$

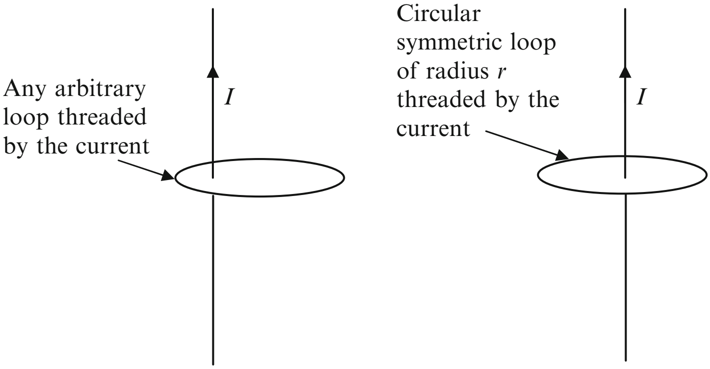

Fig. 3.6 Geometry relevant to definition of Ampere's law, which says that the line integral of the B-field around the loop is equal to $\mu_0 I$, where I is the current passing through the loop. The law is valid irrespective of the orientation of the loop or the point of intersection of the loop and the wire (Figure created by author)

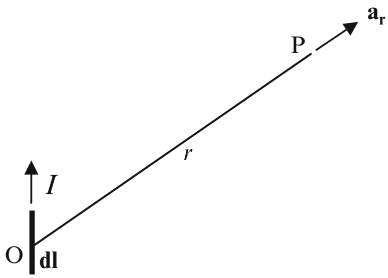

Fig. 3.7 Geometry relevant to definition of magnetic field produced by a current element according to Biot-Savart's law. In the diagram, **dl** is an elementary current element (or a small piece of the conductor carrying a current). The direction of the vector **dl** is the same as the direction of the positive current flow (Figure created by author)

3.7 Magnetic Field Produced by Current Element

Consider a current element *dl* through which a current *I* is flowing (Fig. 3.7). The current element can be represented by a vector **dl** with magnitude *dl* and direction specified by the direction of current flow. According to Biot-Savart's law, the B-field produced by this current element at point P is given by

$$dB = \mu_o I \cdot \frac{\mathbf{dl} \times \mathbf{a_r}}{4\pi |R|^2}, \tag{3.19}$$

where $\mathbf{a_r}$ is a unit vector in the direction of OP (Fig. 3.7). Note that the direction of the magnetic field is perpendicular to both the current element **dl** and the vector joining the current element and the point of observation (i.e., $\mathbf{a_r}$). The magnitude of the magnetic field is proportional to the current in the current element, and its strength decreases with $1/R^2$.

The magnetic field produced by a conductor of any shape can be calculated by dividing the conductor into elementary sections and summing up the contribution to the B-field from each element using the preceding equation.

3.8 Faraday's Law and the Voltage Induced in a Loop in the Vicinity of a Current-Carrying Conductor

The essence of Faraday's law is that it defines and quantifies the natural law that a changing magnetic field gives rise to an electric field. Consider a closed path in a region where there is a changing magnetic field. According to Faraday's law, the E-field, **E**, generated by this changing magnetic field is such that

$$\oint_l \mathbf{E} \bullet \mathbf{dl} = -\frac{d\psi}{dt}, \tag{3.20}$$

where the left-hand side is the line integral of the electric field taken along the closed path and the right-hand side is equal to the negative rate of change of magnetic flux ψ passing through the closed path. The magnetic flux passing through the closed path can be calculated as

$$\psi = \int_s \mathbf{B} \bullet \mathbf{ds}, \tag{3.21}$$

where the surface integral is carried out over a surface bounded by the closed path (Fig. 3.8a). The positive direction of **ds** can be decided as follows. Place a right handed screw inside the closed path and rotate it in a circular direction in which the line integral is performed. If the screw moves out of the surface then the positive direction of **ds** is the outward normal to the surface. If the direction of motion of the screw is into the surface the positive direction of **ds** is the inward normal to the surface. Since the electromotive force, *emf*, generated around the closed path under consideration is given by the line integral of the E-field along that path, we can write

$$emf = -\frac{d\psi}{dt}. \tag{3.22}$$

Let us consider a square loop located in the vicinity of a current-carrying conductor, as shown in Fig. 3.8b. The magnetic field produced by this conductor at a radial distance r from the conductor is given by

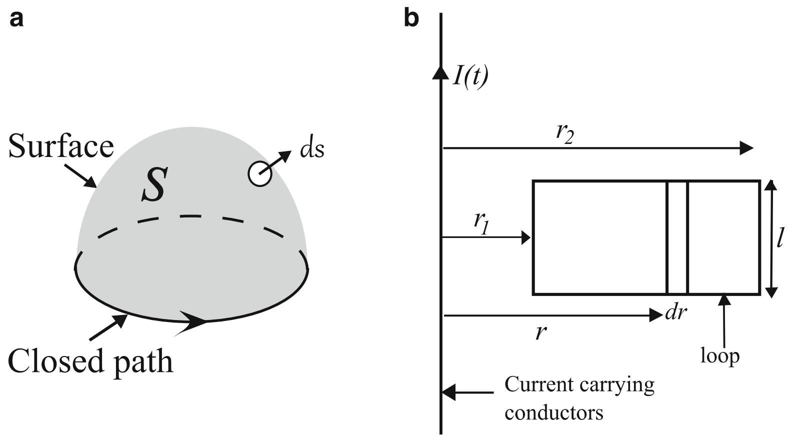

Fig. 3.8 (**a**) The definition of a surface bounded by a closed path. The white patch is a small area element on the surface. The magnitude of the vector **ds** is equal to the area of the element and the direction of the vector is decided by the right hand screw rule (see the text). (**b**) Geometry relevant to application of Faraday's law in case of a conducting loop located in vicinity of current-carrying conductor (Figure created by author)

$$B(t) = \frac{\mu_0 I(t)}{2\pi r}. \tag{3.23}$$

Now, let us divide the square loop into small elementary rectangular sections, as shown in Fig. 3.8b. The flux through the small rectangular element located at distance r is

$$d\psi = \frac{I(t)\mu_0}{2\pi r} l dr. \tag{3.24}$$

The total flux is obtained by integrating this result from distance r_1 to r_2. The result is

$$\psi = \int_{r_1}^{r_2} \frac{I(t)\mu_0 l}{2\pi} \frac{dr}{r}. \tag{3.25}$$

This reduces to

$$\psi = \frac{I(t)\mu_0 l}{2\pi} \ln \frac{r_2}{r_1}. \tag{3.26}$$

Thus, the rate of change of magnetic flux through the loop is

$$\frac{d\psi}{dt} = \frac{dI(t)}{dt} \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1}. \tag{3.27}$$

The induced *emf* in the loop according to Faraday's law is

$$emf = -\frac{dI(t)}{dt} \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1}. \tag{3.28}$$

Note that the *emf* is proportional to the rate of change of the current passing through the conductor.

3.9 Force Between Two Current-Carrying Conductors

Consider a conductor carrying a current I located in a magnetic field **B** (Fig. 3.9a). Consider a small element **dl** on this conductor. The element is represented by a vector, with a magnitude dl and the positive direction the same as that of the direction of positive current flow. The force acting on the element **dl** due to the magnetic field is given by

$$d\mathbf{F} = I\mathbf{dl} \times \mathbf{B}. \tag{3.29}$$

This can be expanded as

$$dF = BIdl \sin \theta. \tag{3.30}$$

Now consider a situation where two parallel current-carrying conductors are separated by a distance a (Fig. 3.9b). We represent the current in the two conductors as $I_1(t)$ and $I_2(t)$. Assuming that the radii of the conductors are much smaller than the separation a, the magnetic field generated by conductor 1 at the location of conductor 2 is

$$B(t) = \frac{\mu_0 I_1(t)}{2\pi a}. \tag{3.31}$$

Since this magnetic field is perpendicular to conductor 2 (i.e. $\sin\theta = 1$ in equation 3.30), the force per unit length (in N/m) on conductor 2 due to the current in conductor 1 is (using Eq. 3.30)

$$F(t) = \left(\frac{\mu_0 I_1(t)}{2\pi a}\right) I_2(t). \tag{3.32}$$

The same force also acts on conductor 1. If the two currents are equal [say $I(t)$], then the force between the two conductors is

$$F(t) = \left(\frac{\mu_0}{2\pi a}\right) I^2(t). \tag{3.33}$$

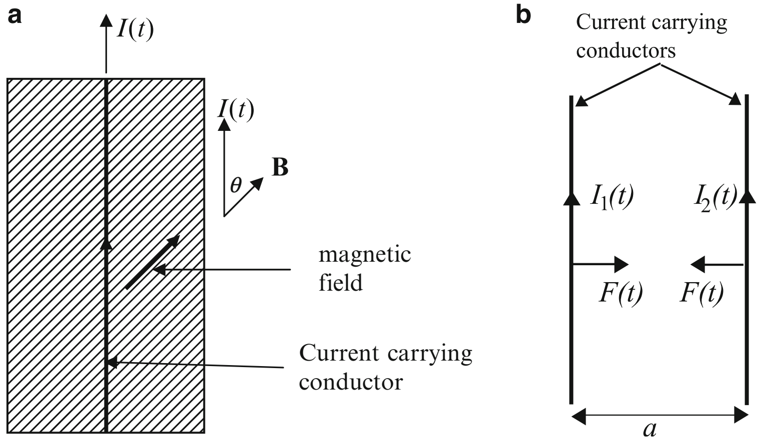

Fig. 3.9 (**a**) Current-carrying conductor located in magnetic field. (**b**) Two parallel current-carrying conductors separated by a distance a (Figure created by author)

If a transient current flows along the conductors, the change in momentum of the conductors (i.e., the relative movement due to the force) is given by the impulse, S, due to the force. This is given by

$$S = \left(\frac{\mu_0}{2\pi a}\right)\int_0^\infty I^2(t)dt. \tag{3.34}$$

In lightning research, the parameter $\int_0^\infty I^2(t)dt$ is called the *action integral*.

3.10 Electric Fields Generated by a Tripolar Thundercloud

Consider a tripolar cloud with three charge centers. The heights of the charge centers are h_1, h_2, and h_3. The amount of charge in the respective charge centers is q, Q, and $–Q$, respectively. In the literature, the charge center closest to the ground and located close to the base of the cloud is called the *positive charge pocket*. The charge center in the middle, which carries a negative charge, is called the *negative charge center*. The charge center at the top is called the *positive charge center*. The geometry is depicted in Fig. 3.10. The z-axis is directed out of the conducting plane (or the ground plane), as shown in the figure. In presenting the results of the calculations we treat the electric field directed along the positive z-direction (i.e., a vector directed away from the ground surface) as positive. This is called the *physics sign convention*. The opposite sign convention, where the electric field directed into the ground is assumed to be positive, is called the *atmospheric sign convention*. The vertical electric fields at ground level generated by individual

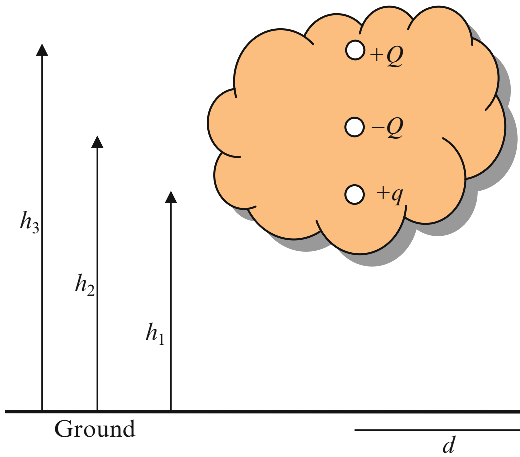

Fig. 3.10 Idealized tripolar cloud with three charge centers located over ground. The amount of charge in the respective charge centers are denoted by q, Q, and $-Q$. In the calculations the positive direction of the electric field is in the z-direction (Figure created by author)

charges at a point of observation located at a horizontal distance d from the axis of the charge centers are given by

$$E_1 = -\frac{q}{2\pi\varepsilon_0}\frac{h_1}{\left(h_1^2 + d^2\right)^{3/2}}, \tag{3.35}$$

$$E_2 = \frac{Q}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}}, \tag{3.36}$$

$$E_3 = -\frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.37}$$

The total vertical electric field E at distance d is then given by

$$E = E_1 + E_2 + E_3. \tag{3.38}$$

Figure 3.11 depicts the electric field at a point of observation as a function of the distance to the thundercloud. In the calculations, we assume $Q = 100$ C, $h_1 = 4$ km, $h_2 = 6$ km, and $h_3 = 9$ km. Results are shown for three values of q equal to 0, 5, and 10 C to show the effect of the positive charge pocket. This diagram also shows how the electric field at ground level varies as the thundercloud approaches a point

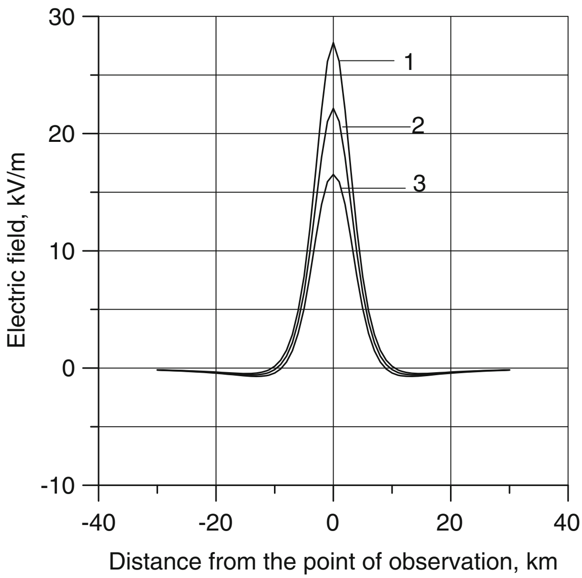

Fig. 3.11 Electric field at point P (Fig. 3.10) located at surface of ground as function of distance d to tripolar thundercloud (distance to vertical axis where charges are located). In the calculation, $Q = 100$ C. Results are shown for three values of q: (1) 0, (2) 5 C, and (3) 10 C. In the calculation, $h_1 = 4$ km, $h_2 = 6$ km, and $h_3 = 9$ km. In the presentation the electric field produced by a negative charge in the cloud is assumed to be positive. If the charge q is significantly larger than the preceding values, the electric field at 0 km could also become negative (Figure created by author)

of observation, passes over it, and then recedes from the point of observation. Note also how the polarity of the electric field changes as a thundercloud approaches the point of observation.

3.11 Electric Field Change Due to Cloud Flash

Let us consider a tripolar cloud. The heights of the charge centers are h_1, h_2, and h_3. The amount of charge in the respective charge centers before a cloud lightning flash is q, Q, and $-Q$, respectively. The cloud lightning flash leads to the neutralization of the ΔQ charge from the negative and positive charge centers (Fig. 3.12). Let $Q' = Q - \Delta Q$. The electric field at ground level at a horizontal distance d from the cloud just before the lightning flash is

$$E_i = -\frac{q}{2\pi\varepsilon_0}\frac{h_1}{\left(h_1^2+d^2\right)^{3/2}} + \frac{Q}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2+d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2+d^2\right)^{3/2}}. \tag{3.39}$$

After the lightning flash the field reduces to

$$E_f = -\frac{q}{2\pi\varepsilon_0}\frac{h_1}{\left(h_1^2+d^2\right)^{3/2}} + \frac{Q'}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2+d^2\right)^{3/2}} - \frac{Q'}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2+d^2\right)^{3/2}}. \tag{3.40}$$

The field change ΔE caused by the cloud lightning flash is

$$\Delta E = E_f - E_i. \tag{3.41}$$

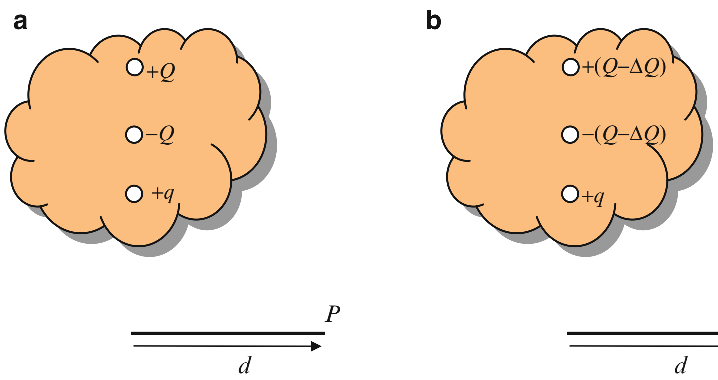

Fig. 3.12 Charges in a thundercloud (**a**) before and (**b**) after a cloud lightning flash. Note that the cloud flash neutralizes a charge of magnitude ΔQ. The parameters used to define the heights of the charge centers are given in Fig. 3.10 (Figure created by author)

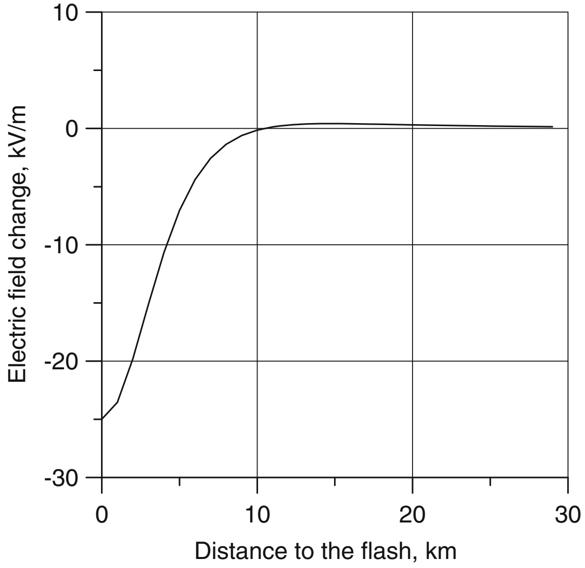

Fig. 3.13 Electric field change at ground level produced by cloud flash depicted as function of distance d to thundercloud (Fig. 3.12). In the calculation, $Q = 100$ C and the charge neutralized by the cloud flash $\Delta Q = 10$ C; $h_2 = 6$ km and $h_3 = 9$ km (Figure created by author)

After substitution from the preceding expressions we obtain

$$\Delta E = -\frac{(Q - Q')}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} + \frac{(Q - Q')}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.42}$$

Figure 3.13 depicts this field change as a function of the horizontal distance from the thundercloud. In the calculation, it is assumed that $\Delta Q = 10$ C, $h_1 = 4$ km, $h_2 = 6$ km, and $h_3 = 9$ km. Note the change in polarity of the field changes with distance.

3.12 Electric Field Change Due to Ground Flash

Let us consider a tripolar cloud. As before, the heights of the charge centers are h_1, h_2, and h_3. The amount of charge in the respective charge centers before a ground lightning flash is taken to be q, Q, and $-Q$, respectively. In the calculation, we assume that the ground lightning flash leads to the complete neutralization of the positive charge pocket and a transfer of $-\Delta q$ to ground from the negative charge center (Fig. 3.14). The total charge, $-\Delta Q$, removed from the negative charge center is equal to $-\Delta q - q$. Let $Q' = Q - \Delta Q$. Thus, the charge in the negative charge center after the ground flash is equal to $-Q'$. The electric field at ground level at a horizontal distance d from the cloud just before the lightning flash is

$$E_i = -\frac{q}{2\pi\varepsilon_0}\frac{h_1}{\left(h_1^2 + d^2\right)^{3/2}} + \frac{Q}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.43}$$

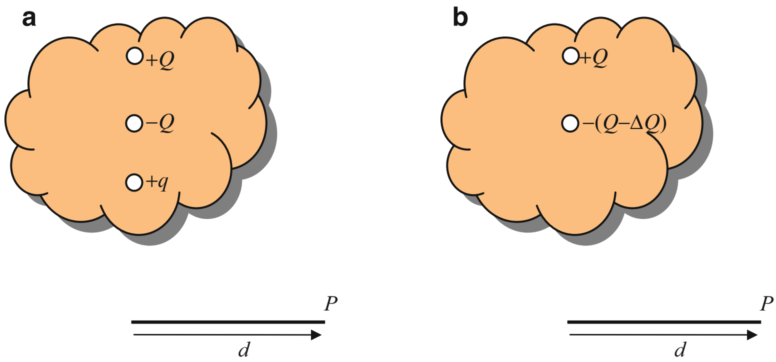

Fig. 3.14 Charges in thundercloud (**a**) before and (**b**) after ground lightning flash. Note that a charge of $-\Delta Q$ is removed from the negative charge center and part of this negative charge is used to neutralize the charge q of positive charge pocket. The rest of the charge $(-\Delta Q + q)$ is transferred to the ground. The parameters used to define the heights of the charge centers are given in Fig. 3.10 (Figure created by author)

After the lightning flash the electric field at the same point is

$$E_f = +\frac{Q'}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.44}$$

Thus, the field change at the point of observation caused by the ground flash is

$$\Delta E = E_f - E_i. \tag{3.45}$$

After substitution of the respective parameters we obtain

$$\Delta E = \frac{q}{2\pi\varepsilon_0}\frac{h_1}{\left(h_1^2 + d^2\right)^{3/2}} - \frac{(Q - Q')}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}}. \tag{3.46}$$

Figure 3.15 depicts the electric field change generated by a ground flash. In the calculation, it is assumed that $\Delta Q = 10$ C, $h_1 = 4$ km, $h_2 = 6$, and $h_3 = 9$ km. Note that, unlike the field change due to a cloud flash, the polarity of the field change remains the same with distance.

3.13 Electric Field Change Caused by Stepped Leader

For simplicity let us consider a bipolar cloud with negative and positive charge centers at heights h_2 and h_3, with charges $-Q$ and $+Q$, respectively. At time $t = 0$ a stepped leader starts moving down from the negative charge center. We do not specify here how the negative charges are removed from the charge center and

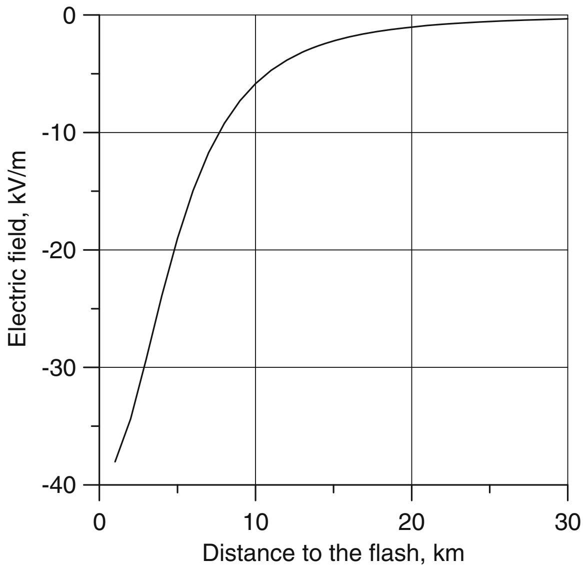

Fig. 3.15 Electric field change at ground level produced by ground flash depicted as function of distance d to thundercloud. In the calculation, it is assumed that $Q = 100$ C and the total charge removed from the negative charge center is -10 C. Part of this charge is used to neutralize the positive charge pocket ($q = 5$ C) and the rest is transported to the ground. In the calculation, $h_1 = 4$ km, $h_2 = 6$ km, and $h_3 = 9$ km (Figure created by author)

transferred to the leader channel (this point will be discussed in a subsequent chapter). Let us assume that the charge on the leader channel is uniform. Let us denote the magnitude of this uniform linear charge density by λ. Other charge distributions could also be assumed, but the procedure for calculation is the same. We also assume that the leader channel moves down at a uniform speed equal to v. Just before the creation of the leader the electric field at a horizontal distance d from the thundercloud is

$$E_i = +\frac{Q}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.47}$$

At time t after the generation of the leader, the physical situation is depicted in Fig. 3.16a. During this time the leader travels a distance of vt. To calculate the electric field caused by the leader, let us divide the leader channel into elementary sections. Consider an element dz located at height z from ground level (Fig. 3.16b). The charge on this element is λdz. The electric field produced by this charge element at a point located at ground level at a horizontal distance d is (note that the charge on the leader channel is negative)

$$dE = \frac{\lambda dz}{2\pi\varepsilon_0}\frac{z}{\left(z^2 + d^2\right)^{3/2}}. \tag{3.48}$$

The total field produced by the leader can be obtained by integrating this from $h_2 - vt$ to h_2. That is,

$$E_l = \int_{h_2 - vt}^{h_2} \frac{\lambda dz}{2\pi\varepsilon_0\left(z^2 + d^2\right)^{3/2}}\,z. \tag{3.49}$$

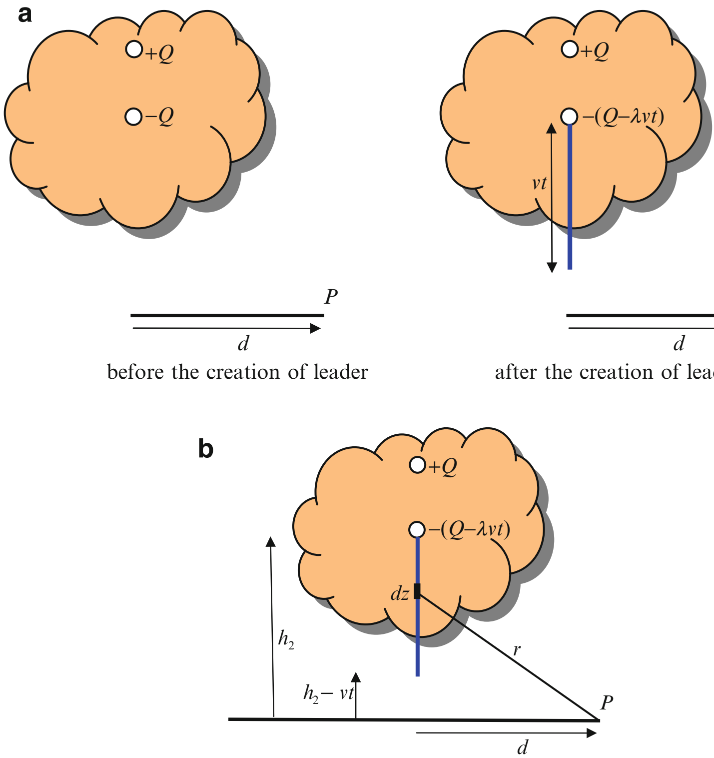

Fig. 3.16 (**a**) Idealized physical situation at time t after creation of leader. The leader initiates from the negative charge center and travels straight to ground at speed v. At time t it had traveled a distance vt. For ease of analysis only the main charge centers are considered in the calculation. (**b**) Geometry relevant to calculation of electric field produced by downward moving stepped leader. In the figure dz is an element of the stepped leader channel located at a distance r from the point of observation P. The length of the leader channel at time t is vt. The tip of the leader channel is located at a height of $h_2 - vt$ (Figure created by author)

This reduces to

$$E_l = \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{(h_2 - vt)^2 + d^2}} - \frac{1}{\sqrt{h_2^2 + d^2}}\right]. \tag{3.50}$$

The total vertical electric field at the point of observation at time t is

$$E_t = +\frac{(Q - \lambda vt)}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0}\frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}} + E_l. \tag{3.51}$$

Thus, the field changes caused by the leader and other charges at time t are

$$\Delta E = E_t - E_i. \tag{3.52}$$

Substituting for different components we obtain

$$\Delta E = -\frac{\lambda vt}{2\pi\varepsilon_0}\frac{h_2}{\left(h_2^2+d^2\right)^{3/2}} + \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{(h_2-vt)^2+d^2}} - \frac{1}{\sqrt{h_2^2+d^2}}\right]. \tag{3.53}$$

This can also be written as

$$\Delta E = -\frac{\lambda vt}{2\pi\varepsilon_0 d^3}\frac{h_2}{\left((h_2/d)^2+1\right)^{3/2}} + \frac{\lambda}{2\pi\varepsilon_0 d}\left[\frac{1}{\sqrt{((h_2/d)-(vt/d))^2+1}} - \frac{1}{\sqrt{(h_2/d)^2+1}}\right]. \tag{3.54}$$

Figure 3.17 depicts the field change caused by the stepped leader as a function of time at different horizontal distances. In the calculation it is assumed that $\lambda = 0.001$ C/m, $v = 10^6$ m/s, and $h_2 = 6$ km. Note how the polarity of the field change varies with distance.

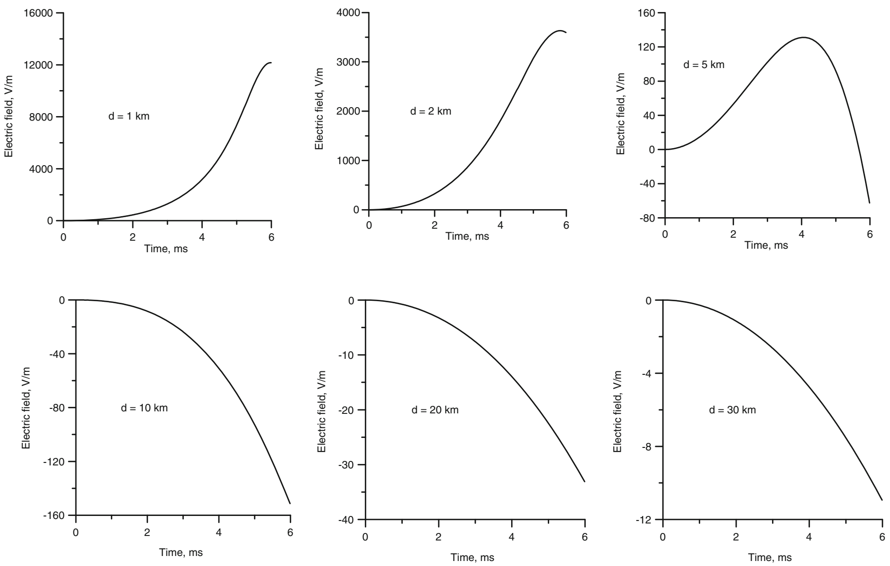

Fig. 3.17 Electric field change at ground level caused by stepped leader as function of time at different horizontal distances (Fig. 3.16b). In the calculation, the speed of propagation of the leader is $v = 10^6$ m/s, and the linear charge density on the leader channel is 0.001 C/m. The height of origin of the leader is 6 km (Figure created by author)

3.14 Electric Field Change Caused by Leader Return Stroke Combination

The electric field at the point of observation at any time during the progress of the leader, i.e., $t < h_2/v$, is given by Eq. 3.51. The electric field at the point of observation when the leader reaches ground, i.e., $t = h_2/v$, is

$$E_{t=h_2/v} = \frac{(Q - \lambda h_2)}{2\pi\varepsilon_0} \frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0} \frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}} + \frac{\lambda}{2\pi\varepsilon_0 d}\left[1 - \frac{1}{\sqrt{(h_2/d)^2 + 1}}\right]. \tag{3.55}$$

Thus, the total field change caused by the leader is

$$(\Delta E)_{\text{leader}} = -\frac{\lambda h_2}{2\pi\varepsilon_0 d^3} \frac{h_2}{\left((h_2/d)^2 + 1\right)^{3/2}} + \frac{\lambda}{2\pi\varepsilon_0 d}\left[1 - \frac{1}{\sqrt{(h_2/d)^2 + 1}}\right]. \tag{3.56}$$

The return stroke removes the charge on the leader channel, and therefore, after the return stroke the electric field at the point of observation is

$$E_{t>h_2/v} = \frac{(Q - \lambda h_2)}{2\pi\varepsilon_0} \frac{h_2}{\left(h_2^2 + d^2\right)^{3/2}} - \frac{Q}{2\pi\varepsilon_0} \frac{h_3}{\left(h_3^2 + d^2\right)^{3/2}}. \tag{3.57}$$

Thus the field change caused by the return stroke (i.e., $E_{t>h_2/v} - E_{t=h_2/v}$) is

$$(\Delta E)_{\text{return}} = -\frac{\lambda}{2\pi\varepsilon_0 d}\left[1 - \frac{1}{\sqrt{(h_2/d)^2 + 1}}\right]. \tag{3.58}$$

The total field change caused by the leader return stroke combination is

$$(\Delta E)_{\text{leader}} + (\Delta E)_{\text{return}} = -\frac{\lambda h_2}{2\pi\varepsilon_0 d^3} \frac{h_2}{\left((h_2/d)^2 + 1\right)^{3/2}}. \tag{3.59}$$

This is simply the electric field caused by the removal of charge of magnitude $-\lambda h_2$ from the negative charge center.

Figure 3.18 depicts both the leader field change and the return stroke field change as a function of time at various distances. In the calculation, it is assumed that $\lambda = 0.001$ C/m and $h_2 = 6$ km. Note that in the case of the return stroke only the field change is depicted, assuming that it takes place instantaneously in the time scale of the leader.

3.15 Time-Varying Electromagnetic Fields

In the presence of time-varying currents, electrical charges, electric fields, and magnetic fields, the laws of electricity can be summarized by Maxwell's equations. They are as follows.

Integral form		Point form	
$\oint_l \mathbf{E} \bullet \mathbf{dl} = -\int_s \frac{\partial \mathbf{B}}{\partial t}\mathbf{ds}$	(3.60.1a)	$\mathrm{Curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(3.60.1b)
$\oint_l \mathbf{H} \bullet \mathbf{dl} = \mathbf{I} + \int_s \frac{\partial \mathbf{D}}{\partial t}\mathbf{ds}$	(3.60.2a)	$\mathrm{Curl}\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	(3.60.2b)
$\oint_s \mathbf{D} \bullet \mathbf{ds} = \int_v \rho_v$	(3.60.3a)	$\mathrm{Div}\,\mathbf{D} = \rho_v$	(3.60.3b)
$\oint_s \mathbf{B} \bullet \mathbf{ds} = 0$	(3.60.4a)	$\mathrm{Div}\,\mathbf{B} = 0$	(3.60.4b)

The other two equations of importance are

$\mathbf{J} = \sigma \mathbf{E}$	(3.60.5)	$\mathrm{div}\mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	(3.60.6)

Equation 3.60.5 defines the relationship between the current density and the electric field through the conductivity σ of the medium. Equation 3.60.6 is the continuity equation based on the fact that electric charges are conserved.

In the preceding equations, $\int_l$ indicates a line integral and $\int_s$ indicates a surface integral. The closed loop around the integral sign, i.e., $\oint$, indicates that the integral is performed around a closed path or over a closed surface. Note also that in isotropic media $E = D/\varepsilon_0$ and $B = \mu_0 H$. The electric and magnetic fields (E and B) can be calculated using time-varying scalar and vector potentials as defined by (Fig. 3.19):

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_v\left(r, t - \frac{r}{c}\right)}{r} dv, \tag{3.61}$$

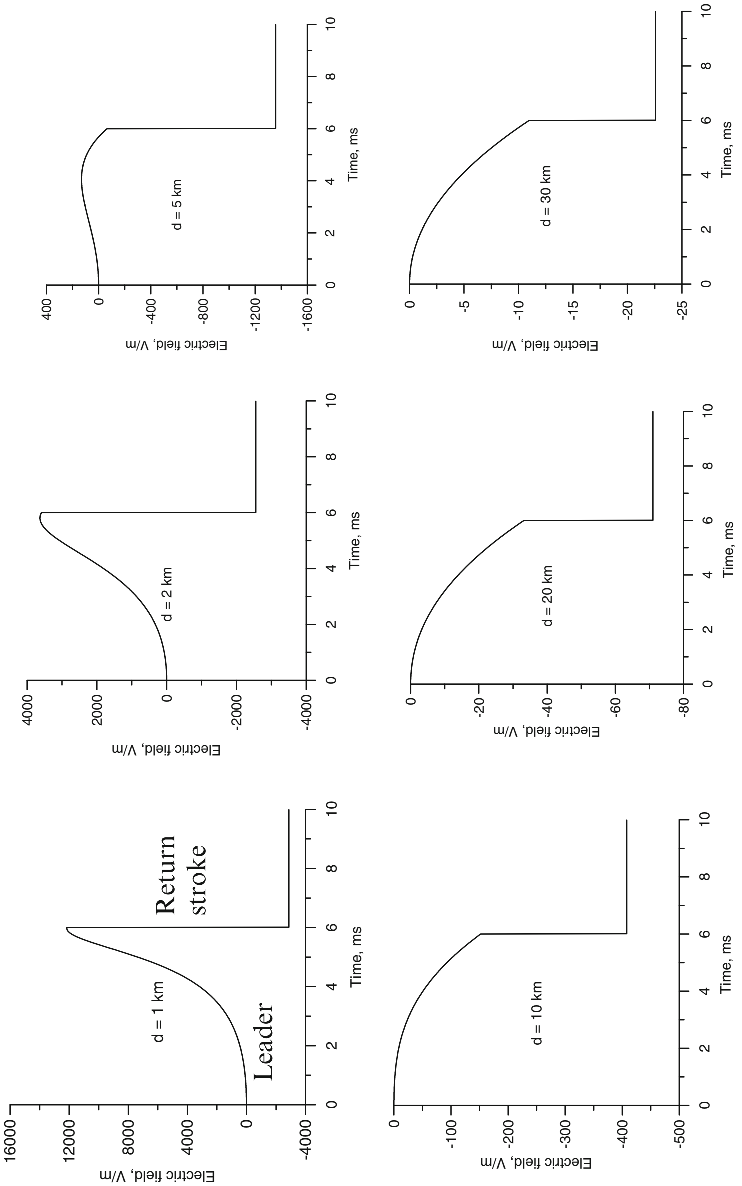

Fig. 3.18 Electric field change caused by leader (slowly varying portion of field change) and return stroke (step in field change) at different distances as function of time. The parameters of the leader are the same as in Fig. 3.17 (Figure created by author)

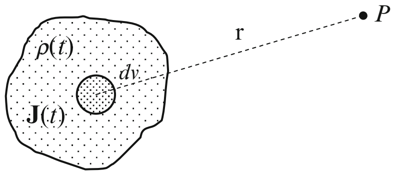

Fig. 3.19 Geometry relevant to definition of scalar and vector potential of time-varying charge and current distributions. In the diagram, $\rho(t)$ and $\mathbf{J}(t)$ are the time-varying charge and current densities; P is the point of observation where the potentials are needed (Figure created by author)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}\left(r, t - \frac{r}{c}\right)}{r} dv. \tag{3.62}$$

In the preceding equations, ϕ and $\mathbf{A}$ are the time-varying scalar and vector potentials and $\rho(t)$ and $\mathbf{J}(t)$ are the time-varying charge and current densities. The quantity $t-(r/c)$ is called the *retarded time*. The electric and magnetic fields can be calculated from these potentials using the relationships

$$\mathbf{E} = -\mathrm{grad}\phi - \frac{\partial \mathbf{A}(t)}{\partial t}, \tag{3.63}$$

$$\mathbf{B} = \mathrm{curl}(\mathbf{A}). \tag{3.64}$$

These laws must be complemented by the Lorentz force law, which specifies the force on a charge particle q in the presence of electric $\mathbf{E}$ and magnetic fields $\mathbf{B}$ as

$$\mathbf{F} = q\mathbf{E} - q\mathbf{v} \times \mathbf{B}. \tag{3.65}$$

In this equation the first term gives the force on a charged particle due to an electric field, while the second term gives the force of the charged particle caused by a magnetic field. In this equation, $\mathbf{v}$ is the velocity of the charged particle.

3.16 Relaxation Time of a Conducting Medium

Consider a conducting medium or a conductor in an electric field. A conductor contains free electrons and under the influence of the electric field these electrons start moving in the conductor and accumulate at the edges of the conductor. This accumulation of electrons at the edges of the conductor give rise to an electric field that is opposite to that of the applied field. The flow and accumulation of electrons at the edges continue until the two electric fields cancel each other and the electric field inside the conductor is zero. This process of relaxation (or removal) of the electric field takes some time, and this time depends on the conductivity and the dielectric constant of the conducting medium. The same happens if electrical charges are placed inside a conducting medium. These charges create an electric field in the conducting medium and the free electrons move in the conducting medium so as to remove this electric field. The effect is the displacement of the electric charge placed inside the conducting medium to the outer surface of the conductor. Again, this removal of the charge from inside to the outer surface of the conducting medium takes some time, and this time again depends on the conductivity and the dielectric constant of the conducting medium.

Consider an isotropic and homogeneous conductor with a relative dielectric constant ε_r and conductivity σ. Assume that at time equal to zero an excess charge is placed inside the conductor with charge density $\rho_v(r, 0)$. This charge generates an electric field inside the conductor, generating a current that redistributes this charge and displaces it to the surface of the conductor (recall that electric charges may accumulate on the surface of conductors) and causing the electric field inside the conductor to go to zero. Let us evaluate how fast this process takes place.

From the equation of charge conservation we have

$$\mathrm{div}\mathbf{J}(r,t) = -\frac{\partial \rho_v(r,t)}{\partial t}, \tag{3.66}$$

where $\rho_v(r, t)$ is the charge density at any time inside the conductor. Substituting for $\mathbf{J}$ in the preceding equation from $\mathbf{J} = \sigma\mathbf{E}(r, t)$ we obtain

$$\sigma\mathrm{div}\mathbf{E}(r,t) = -\frac{\partial \rho_v(r,t)}{\partial t}. \tag{3.67}$$

We also know from Gauss's law that

$$\mathrm{div}\mathbf{E}(r,t) = \frac{\rho_v(r,t)}{\varepsilon_o\varepsilon_r}. \tag{3.68}$$

Substituting this into the previous equation we obtain

$$\rho_v(r,t) = -\frac{\varepsilon_o\varepsilon_r}{\sigma}\frac{\partial \rho_v(r,t)}{\partial t}. \tag{3.69}$$

The solution of Eq. 3.69 is

$$\rho_v(r,t) = \rho_v(r,0)\mathrm{e}^{-\sigma t/\varepsilon_o\varepsilon_r}. \tag{3.70}$$

The preceding expression for the variation in charge density inside a conductor shows that the charge inside the conductor decreases exponentially in time. The quantity $\varepsilon_o\varepsilon_r/\sigma$ is called the relaxation time of the conductor. In the same way, if we create an electric field inside a conductor, it decreases to zero exponentially with a time constant equal to the relaxation time. Recall that in Sect. 3.4 it was stated that the electric field on the surface of a conductor is perpendicular to the surface of the conductor. If a conductor with finite conductivity is placed in an electric field, it takes the relaxation time for the charges on the conductor to redistribute on the surface in such a way that the electric field becomes perpendicular at every point on the surface. In a perfect conductor the conductivity is infinite, and thus the relaxation time is zero. In this case the redistribution of the charges takes place instantaneously, and all the field components parallel to the conductor vanish instantaneously. This process of relaxation is of importance in understanding, among other processes, the response of the upper atmosphere, which is a conducting medium, to the electric fields generated by lightning flashes.

3.17 Electromagnetic Fields of a Dipole

The electromagnetic fields of a short electric dipole are used frequently in calculating the electromagnetic fields of different processes in a lightning flash. Here we present the electromagnetic fields of a short electric dipole in the frequency domain; the corresponding time domain fields are given in the next section.

Let the dipole length be l, and let it be directed in the positive z-direction with its center at the origin (Fig. 3.20). The current in the dipole is given by

$$I = I_0 \mathrm{e}^{j\omega t}. \tag{3.71}$$

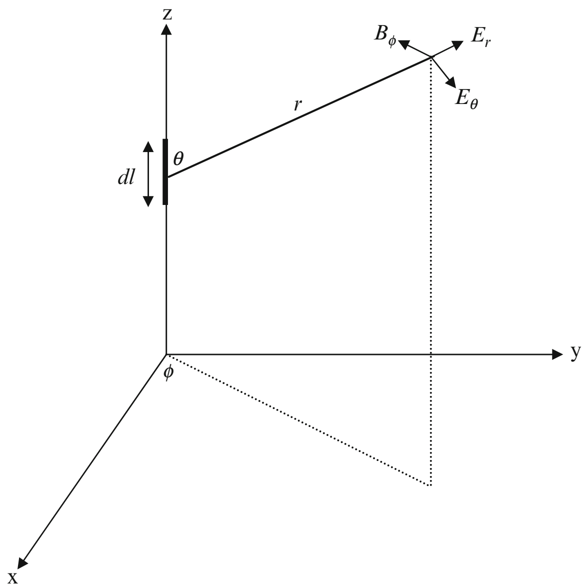

Fig. 3.20 Geometry relevant to calculation of electric and magnetic fields from electric dipole of length dl located at height h from ground plane (Figure created by author)

The electric and magnetic fields at any point in space generated by the short dipole can be calculated using scalar and vector potentials. When $r \gg l$ and $\lambda \gg l$, where r is the distance to the point of observation and λ is the wavelength, the electric and magnetic fields are given by

$$E_r = \frac{I_0 l \mathrm{e}^{j\omega t} \mathrm{e}^{-j\beta r}}{2\pi\varepsilon_0} \cos\theta \left[\frac{1}{cr^2} + \frac{1}{j\omega r^3}\right], \tag{3.72}$$

$$E_\theta = \frac{I_0 l \mathrm{e}^{j\omega t} \mathrm{e}^{-j\beta r}}{4\pi\varepsilon_0} \sin\theta \left[\frac{j\omega}{c^2 r} + \frac{1}{cr^2} + \frac{1}{j\omega r^3}\right], \tag{3.73}$$

$$B_\varphi = \frac{\mu_0 I_0 l \mathrm{e}^{j\omega t} \mathrm{e}^{-j\beta r}}{4\pi} \sin\theta \left[\frac{j\omega}{cr} + \frac{1}{r^2}\right], \tag{3.74}$$

$$\beta = \omega / c. \tag{3.75}$$

The directions of the electric fields in spherical coordinates are indicated in Fig. 3.20. Field components that vary inversely with distance are called *radiation fields*. When the distance to the point of observation is very large, only the radiation fields contribute to the total fields. The other field components attenuate rapidly with distance because they change as $1/r^2$ and $1/r^3$. Thus, the electric and magnetic fields at a point located far from the dipole are given by

$$E_{\theta,\mathrm{rad}} = \frac{I_0 l \mathrm{e}^{j\omega t} \mathrm{e}^{-j\beta r}}{4\pi\varepsilon_0} \sin\theta \left[\frac{j\omega}{c^2 r}\right], \tag{3.76}$$

$$B_{\varphi,\mathrm{rad}} = \frac{\mu_0 I_0 l \mathrm{e}^{j\omega t} \mathrm{e}^{-j\beta r}}{4\pi} \sin\theta \left[\frac{j\omega}{cr}\right]. \tag{3.77}$$

Note that the ratio $E_{\theta,\mathrm{rad}}/B_{\varphi,\mathrm{rad}}$ is equal to c, the speed of light in free space (observe that $c^2 = 1/\mu_0\varepsilon_0$).

3.18 Electromagnetic Fields of a Dipole Over a Perfectly Conducting Ground Plane

Let us now consider a dipole located over a perfectly conducting ground plane. The geometry is shown in Fig. 3.21. The electric field at any point over the conducting plane can be calculated by replacing the dipole with an image dipole. The vertical electric field at ground level (note that the horizontal electric field is zero over the surface of a perfectly conducting ground) at a horizontal distance d from the axis of the dipole is then given by

$$E_v = \frac{I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{\pi\varepsilon_0} \sin^2\varphi \left[\frac{1}{cr^2} + \frac{1}{j\omega r^3}\right] - \frac{I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{2\pi\varepsilon_0} \cos^2\varphi \left[\frac{j\omega}{c^2 r} + \frac{1}{cr^2} + \frac{1}{j\omega r^3}\right]. \tag{3.78}$$

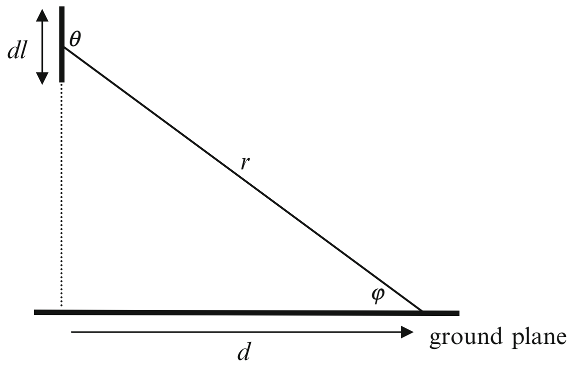

Fig. 3.21 Geometry relevant to calculation of electric fields from electric dipole of length *dl* located over perfectly conducting ground plane (Figure created by author)

This can be written as

$$E_v = -\frac{I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{2\pi\varepsilon_0}\left[\frac{\cos^2\varphi j\omega}{c^2 r} + \left(1 - 3\sin^2\varphi\right)\frac{1}{cr^2} + \frac{1}{j\omega r^3}\left(1 - 3\sin^2\varphi\right)\right]. \quad (3.79)$$

The magnetic field, which is in the azimuthal direction, is given by

$$B_\phi = \frac{\mu_0 I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{2\pi}\cos\varphi\left[\frac{j\omega}{cr} + \frac{1}{r^2}\right]. \quad (3.80)$$

When the distance to the point of observation is large, only the radiation fields will contribute to the total field. Thus, the total fields at distances far from the dipole are pure radiation and are given by

$$E_{v,\mathrm{rad}} = -\frac{I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{2\pi\varepsilon_0}\left[\frac{\cos^2\varphi j\omega}{c^2 r}\right], \quad (3.81)$$

$$B_{\phi,\mathrm{rad}} = \frac{\mu_0 I_0 l \mathrm{e}^{j(\omega t - \beta r)}}{2\pi}\cos\varphi\left[\frac{j\omega}{cr}\right]. \quad (3.82)$$

If $d \gg h$, then $r \approx d$ and $\cos\varphi \approx 1$, and when $d/\lambda \approx r/\lambda$, the radiation fields reduce to

$$E_{v,\mathrm{rad}} = -\frac{I_0 l \mathrm{e}^{j(\omega t - \beta d)}}{2\pi\varepsilon_0}\left[\frac{j\omega}{c^2 d}\right], \quad (3.83)$$

$$B_{\phi,\mathrm{rad}} = \frac{\mu_0 I_0 l \mathrm{e}^{j(\omega t - \beta d)}}{2\pi}\left[\frac{j\omega}{cd}\right]. \quad (3.84)$$

3.19 Electromagnetic Fields of a Current Element in Time Domain Over a Perfectly Conducting Ground Plane

Consider a short current element located over a perfectly conducting ground plane. The current flowing in the current element is given by $I(t)$. The current transports charge from one end of the channel element to the other. The fields generated by the current element are identical to that produced by a short dipole in the time domain. Using Fourier transformation of the equations given earlier, the field components can be written directly as

$$E_v(t) = -\frac{l}{2\pi\varepsilon_0}\left[\frac{\cos^2\varphi}{c^2 r}\frac{dI(t-r/c)}{dt} + \left(1-3\sin^2\varphi\right)\frac{I(t-r/c)}{cr^2} + \frac{1}{r^3}\left(1-3\sin^2\varphi\right)\int_0^t I(z-r/c)dz\right], \tag{3.85}$$

$$B_\varphi(t) = \frac{\mu_0 dz}{2\pi}\left[\frac{\cos\varphi}{cr}\frac{dI(t-r/c)}{dt} + \frac{\cos\varphi}{r^2}I(t-r/c)\right]. \tag{3.86}$$

When the distance to the point of observation is large, only the radiation fields contribute to the total field. Thus, the total fields at distances far from the dipole are pure radiation fields. If $d \gg h$ then $r \approx d$, $\cos\varphi \approx 1$ and the field components reduce to

$$E_{v,\mathrm{rad}}(t) = -\frac{dz}{2\pi\varepsilon_0}\left[\frac{1}{c^2 d}\frac{dI(t-d/c)}{dt}\right]. \tag{3.87}$$

$$B_{\varphi,\mathrm{rad}}(t) = \frac{\mu_0 dz}{2\pi}\left[\frac{1}{cd}\frac{dI(t-d/c)}{dt}\right]. \tag{3.88}$$

Figure 3.22 depicts field components generated by a small current element at different distances assuming that the current in the short channel is a ramp function given by $I_p t$. In the calculation, the length of the dipole is assumed to be 1 m, $I_p = 30$ kA/s, and for simplicity the dipole is assumed to be located at ground level, i.e., $\cos\varphi = 0$. Note that as the distance to the point of observation from the source increases, the field becomes increasingly more similar to a step function, which is the radiation field associated with the source.

3.20 Electromagnetic Field of a Return Stroke

The electric and magnetic fields generated by a return stroke can be calculated easily by dividing the return stroke channel into a large number of elementary channel sections and treating each section as a short dipole. The geometry relevant to the derivation is given in Fig. 3.23. The total electric field can be calculated by

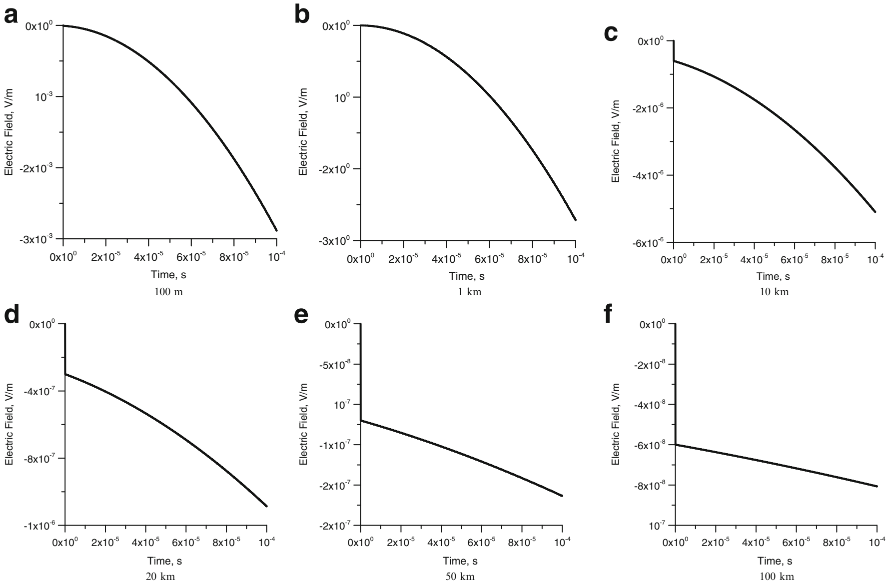

Fig. 3.22 Electric field at surface of perfectly conducting ground generated by small current element located at different distances from point of observation. In the calculation, a current element 1 m in length is assumed to be located at ground level. The current in the current element is a ramp function $I_p t$, where t is the time and $I_p = 30$ kA/s (Figure created by author). Note that in each diagram the electric field starts at zero

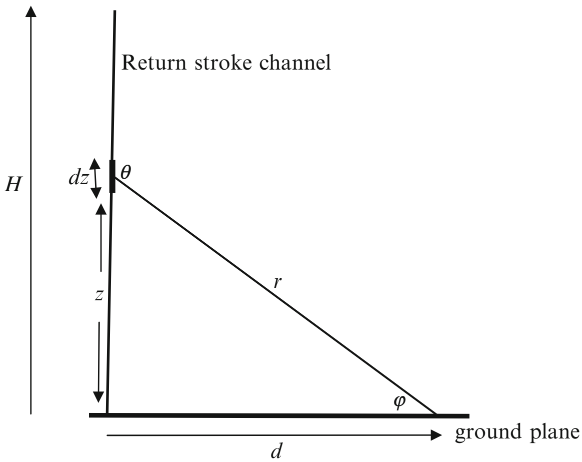

Fig. 3.23 Geometry relevant to derivation of equations pertinent to electric and magnetic fields produced by return stroke. In the diagram, H is the height of the return stroke channel and dz is a channel element located at height z on this channel (Figure created by author)

summing the contribution from each dipole. Let us represent the current at height z in the return stroke channel by $I(z, t)$. Then the vertical electric field at a horizontal distance d from the lightning channel is given by

$$E_v(t) = -\int_0^H \frac{dz}{2\pi\varepsilon_0}\left[\frac{\cos^2\varphi}{c^2 r}\frac{dI(t-r/c)}{dt} + \frac{(1-3\sin^2\varphi)}{cr^2}I(t-r/c) + \frac{1}{r^3}(1-3\sin^2\varphi)\int_0^t I(z-r/c)dz\right], \tag{3.89}$$

$$B_\varphi(t) = \int_0^H \frac{\mu_0 dz}{2\pi}\left[\frac{\cos\varphi}{cr}\frac{dI(t-r/c)}{dt} + \frac{\cos\varphi}{r^2}I(t-r/c)\right]. \tag{3.90}$$

In the preceding equations, H is the height of the return stroke channel. When the distance to the point of observation is large, only the radiation fields (i.e., terms varying inversely with distance) remains. When $d \gg H$, it is justified to assume that $\cos\varphi \approx 1$ and $r \approx d$. Under these conditions, the radiation field terms become

$$E_{v,\mathrm{rad}}(t) = -\frac{1}{2\pi\varepsilon_0 c^2 d}\int_0^H \frac{dI(t-r/c)\,dz}{dt}, \tag{3.91}$$

$$B_{\varphi,\mathrm{rad}}(t) = \frac{1}{2\pi\varepsilon_0 c^3 d}\int_0^H \frac{dI(t-r/c)\,dz}{dt}. \tag{3.92}$$

Note that the electric field is directed into the ground (the negative sign). The two components of the radiation field satisfy the condition

$$E_{v,\mathrm{rad}}(t) = cB_{\varphi,\mathrm{rad}}(t). \tag{3.93}$$

This shows that at large distances, where the radiation field is dominant, both the electric and magnetic fields have the same temporal variation, and the ratio of their amplitudes is equal to the speed of light.

References

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