Does the French Connection (Poincaré, Lautman) Provide Some Insights Facing the Thesis That Meta-mathematics Is an Exception to the Slogan That Mathematics Concerns Structures?

Gerhard Heinzmann

Abstract There are at least two versions of modern structuralism and each has its proper difficulties: if one adopts the *in re* version, the crucial feature is that the background ontology is not understood in structural terms; if one adopts the *ante rem* version, the crucial feature is that the talk about structures is exposed to a kind of third man objection.

The main thesis of this paper is that Poincaré's conventionalism and Lautman's structuralism must be ranked among these sources of structuralism that try to escape the mentioned difficulties.

Poincaré uses a psycho-physiological approach in order to justify his conventionalism in geometry, which is an improvement of an attenuated version of *ante rem* structuralism, and Lautman proposes a metaphysical dialectic in order to justify his anti-foundationalist position, which brings *ante rem* and *in re* structuralism together. Poincaré's approach fails for technical reasons whereas Lautman's approach fails for its aporetic conceptual vagueness.

My present concern is to incorporate the French historical inheritance in the systematic discussion of mathematical structuralism.

Introduction

I first give a short outline of standard results on structures from a philosophical point of view as they can be find in the works of Shapiro 1997, Resnik 1997 or Chihara 2004.

I shall call a domain of objects together with certain functions and relations on the domain satisfying certain given conditions a "system". A special group, e.g.

G. Heinzmann (✉)
Laboratoire d'Histoire des Sciences et de Philosophie—Archives Henri Poincaré (UMR 7117), Université de Lorraine/CNRS, 91, av. de la Libération, F-54001 Nancy, Cedex
e-mail: Gerhard.Heinzmann@univ-lorraine.fr

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Poincaré's group of displacements is a system. The abstract form of a system is called a structure. Under structuralism I understand the philosophical thesis that mathematics is not concerned with any particular ontology but with structures or with systems that share a common structure.

According to the structuralists' account, mathematical objects are places in structures. What is the difference between "place" and "object"? There are different possibilities to answer this question and, consequently to interpret the relation between structure and systems that exemplify them.

The first position says that places are offices and not officeholders. The "places are-offices perspective" presupposes a background ontology that supplies objects that fill the places of the structure. A structure is anything that can be exemplified by a type of systems, but there is no such thing as "the structure". The number structure is a pattern common to all number systems, which are not necessarily isomorphic. This position is a so-called structuralism without structures, and it is in this sense anti-realistic and eliminative. The eliminative structure program paraphrases places-are-objects statements in terms of a places-are-offices perspective with respect to systems different from the structure. For an anti-realistic structuralist place-are-objects statements are not to be taken literally: the apparent singular terms mask implicit bound variables. They are disguised definitions. Talk of numbers is convenient shorthand for talk about all systems that exemplify the structure. Talk of structures generally is convenient shorthand for talk about systems. The essence of a number being just its structural relation to other numbers, anything at all can be "4" when it occupies the place corresponding to the office "4" in a system exemplifying the natural number structure. But we cannot and need not answer the question whether $1 \in 4$ (von Neumann's notation) or not (Zermelo's notation). We have no objects with an internal composition so that the last question is a meta-mathematical one concerning the background-theory, which is normally the set hierarchy V. In other words, the back-ground theory cannot be a mathematical theory if mathematics is considered in an anti-realistic structural way. Otherwise, one should find a background-theory for the set theoretical structure etc.

The second interpretation of the token-type relation between systems and structure is a realistic one: tokens can, so to say, be destroyed. Contrary to the *in re* structuralist, the *ante rem* structuralist takes the pattern to exist independent of any systems that exemplify it. The structure is prior to the mathematical objects it contains. For the *ante rem* realist the distinction between position and object is a relative one. The idea is that the places of the natural number structure, considered from the "places-are-objects perspective", can be organized into a system, and this system exemplifies the natural number structure whose places are now viewed from the "places are offices-perspective". The "places-are–office perspective" refers here not to a system different from the structure. So, in a sense, each structure exemplifies itself. When we invoke the "places-are object-perspective", in "2 is $\{\{\emptyset\}\}$" the "is" is an identification, in contrast, when we invoke the "places are-offices perspective", the "is" constitutes a copula relative to a system exemplifying the structure.

The problem raised by this interpretation is the status of a structure as type. Are there some identity-conditions? Is a structure an entity? The answer is surely: no! But what does it mean, then, to speak of a structure?

Poincaré

Now, concerning Poincaré, it is obvious that he was strongly influenced by and attuned to a philosophical movement consisting of a mixture of positivism and Neo-Kantianism, namely the so-called "Boutroux Circle" (Nye 1979).[1] The members of the circle criticized at the same time Comte's determinism and Kant's static view of the mind's structure. The existence of consistent non-Euclidean geometries, used by Poincaré in order to overcome difficulties of the theory of "real" geometry as a tool in function theory, leads him to study "the structural relations between Euclidean and non-Euclidean geometry" (Nye 1979, 111). Whereas the existence of different geometries prevents one from considering geometric propositions as necessary truths determined by a priori intuitions, mathematical exactness and the impossibility of defining distance empirically prevent one from considering geometric propositions as empirical descriptions. This is why Poincaré introduces "propositions" in the core of his theory of mathematical knowledge not as genuine bearers of general semantic information, but as hypotheses. The axioms of metric geometry are *apparent hypotheses,* i.e. conventions, neither true nor false. The introduction of this formal and decisional aspect in mathematics was held always as the modernist aspect of Poincaré's philosophy of science. On the contrary, it is little known that Poincaré should be ranked with Duhem among the forerunners of Quine who survive his criticism of logical empiricism:

(a) for his conception of geometrical conventions as a kind of bicephalous selection of analytical but non-logical propositions, "guided" at the same time by experience.[2]
(b) for his "relationalism" which in certain aspects comes close to Quine's doctrine: both attempt to account philosophically for the incompleteness of (scientific) objects. Poincaré argues that what science can attain

> is not things themselves, as the dogmatists in their simplicity imagine, but the relations between things; outside those relations there is no reality knowable. (Poincaré 1902, XXIV)

What are the links between Poincaré's relationalism and the geometrical conventions? They are nothing but two different aspects of Poincaré's structural approach. Concerning the relational aspect, David Stump remarks:

[1] Cf. for the following (Heinzmann/Stump 2013).

[2] Because Poincaré distinguished very well between analytic and synthetic sentences, the analogy to Quine must be restricted to conventions.

> Poincaré and Hilbert argue for a new conception of geometrical systems [. . .] Poincaré holds that outside of the context of an axiomatic system, geometrical primitives mean nothing [. . .] He argues that geometry concerns only the relations expressed in the axioms, and not some inherent features of the primitives [. . .] The set of relations that holds between the primitives constitute the form, not the matter, of geometric objects, and these are what is studied. (Stump 1996, 482–84)

Geometry is the study of the form of the group together with its properties. This form (=structure) of the group preexists in our minds, and certain relations of our experience are represented by *apparent hypotheses* or *conventions.* The empirical objects satisfying the relations as matter of the form are described by *indifferent hypotheses*. They concern the *ontological* but not the structural determination of elementary phenomena. Such indifferent hypotheses are mere metaphorical crutches, useful for thought but "unverifiable" and "useless" as such (Poincaré 1902, 156). They are conventional in the usual sense of the word; that is to say, they are "arbitrary," but compelled by rational agreement. In general, the ontological determination of singular objects is, from a scientific perspective, an over-determination: scientific objectivity is purely relational, while the *relata* remain inaccessible to human knowledge.

Thus, the form of the group constitutes the relational aspect of Poincaré's structuralism. But what does it mean that the relational form is described by apparent hypotheses or conventions?

Poincaré's approach is not identical with Hilbert's axiomatic approach. His often quoted structural *Credo,* saying that in mathematics the word "existent" means "exempt from contradiction" (Poincaré 1902, 44; 1905a, 819), must be seen under a non-Hilbertian light. For reasons concerning above all the involvements of impredicative procedures, Poincaré excludes proving mathematical reliability by a consistency proof in the Hilbertian way. He takes a structural position without completely disengaging meaning and knowledge from ostension. Nevertheless, Poincaré begins his alternative reliability construction only *apparently* with sensations as ostensive contacts with the given. In reality, he introduces, similarly to Helmholtz's conception of intuition as imagined sensible impressions, a representation of a two-places sensation relation, based on the *imagination* of single sensations.[3] They are the office-holders of the categories (forms) of sensible space and of groups. Poincaré's conventions in geometry are the tools to close the gap between the *exactness* of a structure and the *objectivity* of sensation-relations based on an imagined ostensive contact (*reflecting on sensations*). If this interpretation is right, then Poincaré's concept of structure is not the new Hilbertian one deriving from his axiomatization of Euclidean geometry, but constitutes a development of the

[3]Representation of an object in the sensible space means nothing else than the deliberate and conscious reproduction of muscular sensations *thought* necessary to reach the object: "When it is said, [. . .] that we "localise" an object in a point of space, what does it mean? It simply means that we *represent to ourselves* these movements that must take place to reach that object [. . .] When I say that we represent to ourselves there movements, I only mean that we *represent* to ourselves the muscular sensations which accompany them" (Poincaré 1902, 57; *our emphasis*).

traditional algebraic one and concerns continuous groups. His epistemological project has a strong affinity with Schlick's *General Theory of Knowledge*.

By recognizing Poincaré's legacy for the structural point of view, Schlick insisted on the conventional aspect of his structuralism. He saw in Poincaré's conventions a third type of definitions between the axiomatic or implicit and concrete or ostensive ones. These conventions are, as (Friedman 2007, 100) observes, "crucial for an understanding of how we achieve a coordination between concepts and empirical reality in the mathematical exact sciences". I quote Schlick:

> To define a concept implicitly is to determine it by means of its relations to other concepts. But to apply such a concept to reality is to choose, out of the infinite wealth of relations in the world, a certain group or complex and to embrace as a unit by designating it with a name. By suitable choice it is always possible under certain circumstances to obtain an unambiguous designation of the real by means of the concept. Conceptual definitions and coordinations that come into being in this fashion we call conventions (using this term in the narrower sense, because in the broader sense, of course, all definitions are agreements). It was Henri Poincaré who introduced the term convention in this narrower sense into natural philosophy; and one of the most important tasks of that discipline is to investigate the nature and meaning of the various conventions found in natural science. (Schlick 1979, 91f)

So, according to Schlick, Poincaré's conventions combine conceptual definitions and coordinations.

How should one understand this affirmation?

An answer can be found by examining four steps in Poincaré's psycho-physiological reconstruction of the genesis of geometrical space. It gives an instantialist view of geometrical relations. In his early articles, Poincaré argues that geometry concerns only the relations expressed in the axioms, and not some inherent features of the primitives: "What we call geometry is nothing but the study of formal properties of a certain continuous group; so we may say, space is a group" (Poincaré 1898, 41).

The first step of Poincaré's construction of geometrical groups proceeds from the *observable* fact that a set of impressions can be modified in two distinct ways: on the one hand without our feeling muscular sensations, and on the other, by a voluntary motor action accompanied by muscular sensations. So, similarly to Carnap's Aufbau, the starting point is here the definition (guided by experience) of two two-place relations: an *external chance* α (with '$x\ \alpha\ y$' for '*x changes in y without muscular sensation*') and an *internal change S* (with '$x\ S\ y$' for '*x changes in y accompanied by muscular sensations*').

In the second step, he introduces a classification of external changes, some of which can be compensated by an internal change, while others cannot. The first are called **changes of position**, the second **changes of state**. One presupposes that the compensation is by convention exact and not approximate. In the third step, Poincaré defines, *modulo* an identity condition with respect to the compensation by internal changes, the equivalence class of changes of position, and calls it a *displacement* (see Poincaré 1905b, Chap. IV):

1. Two internal changes have to be considered identical if they have induced the same muscular sensations.

2.
$$\left.\frac{\alpha, \beta \text{ external}}{S \text{ internal}}\right\} \text{changes}$$

$$\alpha \sim \beta \quad \textit{iff} \quad \exists S \quad \left(S\alpha \doteq S\beta \doteq I\right)$$

that means: two external changes are equivalent if they possess a common character (i.e. they can be canceled by S).

3.
$$S \approx S' \quad \textit{iff} \quad \exists \alpha \quad \left(\alpha S \doteq \beta S' \doteq I\right)$$

4. If $\sim$, $\approx$ is an equivalent relation, then the equivalence class of the changes of position is a *displacement*. So we can recognize that two displacements are identical.

The fourth step and Poincaré's main result is that each set of displacement classes (external and internal) forms a group in the mathematical sense.

The group of displacement is in fact an adjustment to the general group preexisting in our mind as a form of our understanding, which the specific displacement-structure (=transformation group) exemplifies. In other words, the form in the mind leads to a special kind of platonic universals or *ante rem* structure. Thus, the genesis of geometry is based on an epistemological process founded on previous classifications, carried out as a relationship between a structure as norm of invariance and conventional adjusted systems as instantiations or exemplifications of these norms (the sensation compensation are only approximate). The exemplification of the group-structure by a variety of systems (=element of harmony) is not a logical but an esthetical operation without an explicit identity (harmony) criterion (=mathematician as artist). This is why the exemplification of structures and the esthetic perspective are "solidary" [VS, Chap. V].

Now, the special variant of convention, where there exists a choice between different possibilities, only becomes involved at a further step of the sensory-mathematical construction where the properties of the transformation group are studied and decisions are taken concerning the distance. It follows that the axiom of Euclidean distance is a conventional definition influenced by simplicity and commodity and guided as a whole by experience: it is a *disguised definition* or an *apparent hypothesis*. Poincaré uses the term *disguised definition* up to 1899, the year of Hilbert's famous *Foundations of Geometry*, to express the fact that language apparently used descriptively is not so in actuality. Certain axioms appear descriptive, but instead constitute the only way to *define* certain entities (see Poincaré 1899, 274). Such entities are found to be defined only up to structural equivalence: they reflect well the truth of certain relations between relata whose qualities remain—as according with Helmholtz and others—unknowable.

Contrary to *in re* structuralists, Poincaré's structure of a general group is not ontologically, but rather epistemically dependent on its instances. Contrary to *ante rem* structuralists, Poincaré doesn't speak of the structure as such but uses it as a metamathematical tool for his psycho-physiological genesis of *real* actions with *imagined* sensations: the structure is not itself a position in a meta-structure but the psycho-physiological procedure is the *ratio cognoscendi of* its existence in our mind. In this sense geometry is as such a whole system, understood by a *pragmatic* procedure, which is irreducible to a combination of clearly distinguished parts of conceptual analysis and aesthetic exemplifications.

As Philippe Nabonnand remarked, Poincaré's presentation of geometrical space is as a whole circular:

> in his 1898 paper, [he] put forward a (mathematical) explanation of the three dimensions of space. He observed that the Euclidean group, selected after many conventions, can be seen as acting on a space of three, four or five dimensions. The choice of a three-dimensional space is justified by considerations of commodity. Unfortunately, Poincaré's argument is vicious because the choice of the Euclidean group was grounded on Lie's classification of transformation-groups operating on R^3. (Heinzmann and Nabonnand 2008, 171)

Nevertheless, Poincaré noted his mistake and introduced in 1905 (VS) a three-dimensional physical continuum in order to justify his utilization of Lie's classification. The consequence is that Geometry is no longer independent of any mathematical space (Nabonnand). The structure of space must be presupposed as a primitive notion, contrary to the pragmatically suggested group notion existing in our mind!

Lautman

Between 1930 and 1940, three PhD students and friends at the *Ecole Normale Supérieur* at Paris, having a common interest in logic and philosophy of mathematics, were a driving force leading to the work of Bourbaki, and they had the common fate to disappear prematurely. Jacques Herbrand was a mathematician, Albert Lautman a mathematically well-trained philosopher, and Jean Cavaillès a philosopher and historian of set theory. Lautman is less well known than Cavaillès as a scientist (Jean Petitot wrote in 1987 one of the first articles on Lautman) and as a resistance fighter: nevertheless, like Cavaillès, he was killed in 1944 by the German occupying power.

Albert Lautman defended his PhD in 1937 with a principal and a complementary thesis, entitled respectively *Essai sur les notions de structure et d'existence en mathématiques* and *Essai sur l'unité des sciences mathématiques dans leur développement actuel.*[4] He shared with Poincaré the opinion that formalism and intuitionism fail together as reliable positions on the foundations of science, that is, as philosophical views of the nature of scientific objects and of scientific understanding (Lautman 2006, 181). His purpose was to solve the Hilbertian problem of the conflicts echoed in mathematical practice through the structural method used in Algebra and the constructive method, conceiving the real numbers

[4]Reprinted in (Lautman 2006).

and the operations of Analysis as generalizations from number theory. The tool he imagined is an *adequate* interpretation of the structural method so that the conflict in fact disappears in favor of the algebraic method (Lautman 2006, 87).

Lautman's intuitionistic opponents were Pierre Boutroux and Maximilien Winter, who formulated their theses in the books *l'Idéal scientifique des mathématiciens* (1920) and *La méthode dans la philosophie des mathématiques* (1911). Boutroux considered "independent mathematical entities with respect to the theories where they are defined." Speaking of "algebraic or logical clothes by which we seek to represent such a being," he presupposed, according to Lautman, a kind of neutrality of the formalism with respect to that which is formalized.

There was also a formalist opponent to Lautman: naturally, this was not Hilbert, but Carnap and the Vienna Circle around 1937. As did Cavaillès and Herbrand, Lautman went in the late twenties to Germany (Berlin, 1929). The French neo-Kantian tradition, enriched with the German experience of the fertility of structural relations, led him to oppose the reductionist and "static" character of Logical Empiricism. Theories, rather than isolated concepts or primitive notions linked by primitive logical propositions, have to be objects of the scientific philosophy. Mathematical reality should not be conceived as "being static" but as the result of the possibility of determining certain beings from one other, i.e. the result of a set of links (Lautman 2006, 226).

Lautman distinguished two points of view of the concept of structure: the *syntactic* or genuine structural perspective, and the *semantic* or extensive perspective. This distinction is identical to or at least very close to our modern *ante rem—in re* distinction (Lautman 2006, 66). Both perspectives belong, according to Lautman, to metamathematics: the first concerns the construction "of certain perfect structures, [...] and this regardless of whether there are concrete ("effective") theories having the properties in question" (Lautman 2006, 131). Lautman associated with this *syntactic* structural (*ante rem*) perspective such proof theoretic properties as "provable", "refutable", "irrefutable" or "non-contradictory".

The semantic perspective, concerning the existence of interpretations, uses the extensive processes of set theory by considering the fields of individuals that can serve as values to arguments in a formula of the theory. The semantic properties associated with this perspective are validity, satisfiability etc. (Lautman 2006, 182). According to this *in re* interpretation, the properties of mathematical beings are of a structural kind, exemplified by different systems. Take, for example, the property of divisibility of the number 21. If the domain is the field K of rational numbers, the result is: 3, 7; if it is the field K $\sqrt{-5}$, the result is: 3, 7, $(1+2\sqrt{-5})$, $(1-2\sqrt{-5})$.

The question then arises, how did Lautman conceive the relation between the two perspectives on structure? The answer can be found by analyzing the split between the old *genetic* method and the new structural method in mathematics with respect to the relation between essence and existence (Lautman 2006, 65). According to the classical point of view, the question concerning this relation is still asking about the same being. According to the structural point of view, by contrast, when the transition from essence to existence is possible, it always concerns the passage of one kind of being to another kind of being. For example, according to the classical point of view in Analysis, the relationship between "discontinuous" and "continuous" or between "finite" and "infinite" is conceived as an expansion of the finite (discontinuous) or by the narrowing of the infinite (continuous), where the finite (discontinuous) is still considered in extension as a part of infinity (continuity). On the contrary, the structural point of view sees in the finite and the infinite not two extremes of a move to make, but two distinct kinds of being, each with its own endowed structure, supporting relations of similarity between them. But how should one compare them? At first glance, Lautman's answer sounds very vague: "By focusing on the frame of beings (armatures des êtres), which are compared, one indeed discovers between the finite and the infinite an analogy of structures" (Lautman 2006, 122/123). How can we speak about structures?

Lautman's answer is based on a more liberal concept of "structural content": to conceive "a structure whose elements are neither *entirely arbitrary* nor *built up really* but conceived as a *mixed form* that derives its fruitfulness of its dual nature" (Lautman 2006, 46).

In perspicacious way, Lautman identified the completeness theorem of the predicate calculus as a trivial technical realization of the intended dialectic between essence and existence, which is inadequate to be extended to more complex theories (Lautman 2006, 183/184). Naturally, when the system is not complete, there is no equivalence between the non-contradiction of the system and the existence of an interpretation of this system. The existence of a model "is a stronger requirement than non-contradiction, so that there will be a dissociation between the [genuine] structural view and the extensive point of view" (Lautman 2006, 184).

Now, in order to understand the internal unity of the *ante rem* and *in re* perspective on a structure, Lautman uses a biological metaphor: He notes: "It is obvious that the mathematical entity as we understand it is not unlike a dynamic living thing" (Lautman 2006, 140). However, the structural conception and the dynamic conception of mathematics seem at first opposed: "one tends, he says, to consider a mathematical theory as a whole, [. . .] independent of time, the other on the contrary does not separate the temporal stages of its development" (Lautman 2006, 130).

Hence Lautman's vision considered structures from a distinctive perspective, nearer to mathematical practice: the mathematical solutions of the problems they pose should contain an infinite number of degrees. Partial results and comparisons, stopped halfway when organized under the unit of the same theme, could perhaps, in their movement, manifest emerging links between abstract ideas. These links Lautman proposed to call "dialectical" (Lautman 2006, 131). He tried to develop a conception of mathematical reality that would combine the two kinds of structuralism with the life-metaphor attributed to theories. The understanding of a mathematical entity must involve two reciprocal aspects: "the essence of a form being realized in a matter created by the form, and the essence of a matter giving rise to the forms drawn by the structure of the matter" (Lautman 2006, 186).

Like Poincaré, Lautman viewed the ontological commitment as concerning relations and not objects, and the mathematical activity or experience as the *ratio cognoscendi* of a structure, determining within this process new elements. Contrary to Poincaré, however, he viewed the relations in question as only regulative, and saw no fixed structure preexisting in itself or in our minds. Like Neurath and Quine, Lautman was not seeking for an *ab ovo* sub-basement of mathematics, but staying afloat in the boat, he presupposed a preliminary background structure. Nevertheless,

> the reality of mathematics is not made of the act of the intellect that creates or understands, but it appears to us in this act and cannot be fully characterized independently of those mathematics that are their indispensable support. [. . .] The reality inherent in mathematical theories is that they are participating in an ideal reality that is dominant with respect to mathematics, and which is knowable only through it. [. . .] We see in mathematics a way of structuring a basic domain [structure] interpretable in terms of existence for some new things [. . .] that the structure of the domain seems to preform. (Lautman 2006, 66–68)

Now, I think that what interests us today in Lautman is not his platonistic solution itself, i.e. the proposal that the intrinsic reality of mathematical entities, facts or theories lies in their dialectical participation in ideas which dominate them (Lautman 2006, 237) and which are themselves realities. What is subtle is his insight in the essential difference between the nature of mathematics and the nature of the Dialectic. This insight leads to an alternative interpretation of Poincaré's and Quine's thesis of the incompleteness of mathematical objects and the ideas to which they belong: this incompleteness is neither an epistemic deficiency possessed finally by all objects according to Poincaré, nor purely a verbal accommodation with respect to a set theoretic progression possessing itself ontological commitment (Quine 1986, 401), but an ontological peculiarity: "Ideas are not models whose mathematical entities are merely copies. The Genesis is no longer seen as the creation of the concrete material from the idea, but the advent of concepts related to the concrete in an *analysis* of the Idea." (Lautman 2006, 238; *my emphasis*). In fact he distinguishes "notions" and "ideas" in order to underline the different status of philosophy and mathematics. While mathematical notions "describe existing relations between mathematical entities", the ideas describing dialectical relations do not assert any existing relation between *notions. Ideas* concern possible relationships between such *notions,* as, for example, between "formal systems" and their models or the relationship between the infinite and the finite. The analogies between structures cannot be expressed on the level of structures. The identity of a structure is not a mathematical subject, and the concept of structure as such not a mathematical object. The analogies between structures are as ideas "incarnated" or, as Poincaré would say, suggested "in the very movement of the mathematical theories" (Lautman 2006, 12). The systematical point of this parallel to Poincaré was seen by Lautman himself when he remarked that "the process of linking theory and experience symbolizes the relation between ideas and mathematical theories" (Petitot 1987, 105) The ideas have, according to Lautman, no ontological commitments and no anteriority with respect to their instantiations (Petitot 1987, 87), but rather raise questions and "are only the problematic issue relating to any of the existing situations." (Lautman 2006, 242/243). In short, as Petitot expressed it, the dialectic between ideas and notions is historic and ideas are by no means irreducible essences of an intellectual world (Petitot 1987, 95).

If we try treating structures as individuals and describing their relations, we are treating structures themselves as positions in a structure of structures. Lautman avoids such a circularity in the following way: "Metamathematics embodied in the generation of ideas [. . .] cannot give rise in turn to a meta-metamathematics; the regression stops when the mind has reached the patterns by which the dialectic is constituted. We see our reference to Platonism is well justified" (Lautman 2006, 232). In other terms, the classical view of structures cannot be substituted, without precautions, by a structural view of structures. Structures are neither mathematical objects nor properties of such objects, because they depend also, as we have seen, on a system of representation. Structures, as Lautmanian ideas (minus Platonism), are *patterns*. What I mean by "pattern" is a schema whose general and singular aspects are in a perpetual interplay or in a dialectical link. The concept of "pattern" makes it possible to avoid *ante rem* and *in re* structuralism (cf. Oliveri 2007, 163).

In this sense, Resnik is right, and Lautman would agree, that the structural approach to mathematics "would be no worse off than set theory, which cannot recognize its own universe of discourse as a set". Indeed, this limitation has only a negative bearing on structuralism, "if structuralism [is] purported to be a mathematical theory rather than a philosophical account of mathematics". But, we have seen that Lautman was not pursuing a foundational program, but rather hoping to achieve a deeper understanding of mathematical practice. His solution: philosophically, the concept of structure is dominated by a dialectical idea of a "pattern" that brings two perspectives together: (a) the structure as an essence of a form, realized in a matter, created by the form, and (b) the essence of a matter giving rise to the forms. This dialectical *ante rem—in re* solution seems no worse off than the *ante-rem* dialectic that considers structures in an alternating perspective, either from a "place-to-be-filled" or from a "places-are-objects" point of view, i.e. the doctrine that each structure can exemplify itself (Shapiro 1997, 89). This is formulated in a Lautmanian way by Chihara: "A structure is the abstract form of a system, and insofar as it exemplifies itself, it must be a system which has as its form the very form that it itself is" (Chihara 2004, 67).

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