Macroscopic Locality

Miguel Navascués

1 Introduction

From the beginning of the 20th century, we have had at our disposal three alternative models of reality—newtonian, einstenian and quantum—each of which appears logically consistent. These theories, though, are not independent: there are inclusion relations between them. Indeed, note that the framework needed to describe physical systems in classical physics can be recovered as a low energy limit of the framework used in general relativity. Likewise, quantum theory allows recovering classical dynamics when measurements are sufficiently coarse-grained [1], or in the limit $\hbar \rightarrow 0$. Moreover, the fact that both quantum mechanics and general relativity provide accurate descriptions of reality in their respective domains implies that these two models emerge as limits of a third (yet unknown) physical theory.

This third theory is expected to reveal its nature in particle experiments dealing with energies of the order of the Planck mass $m_P \approx 1.2 \times 10^{19}$ GeV/c^2, inaccessible with current technology. In order to infer properties of this mysterious model that contains Quantum Physics and General Relativity as particular cases, we are thus bound to rely on logical reasoning and physical intuition rather than experimental feedback.

One approach to the problem, initiated by Hardy [2] and further developed in [3–7], is to find new formulations of Quantum Mechanics in terms of physically compelling axioms. The idea stems from the fact that while it is fairly easy to propose extensions of General Relativity, any small modification of Quantum Mechanics will most likely lead to inconsistencies. The hope here is that reducing Quantum Mechanics to a set of physical properties should point out new ways to generalize it.

M. Navascués (✉)
H.H. Wills Physics Laboratory, University of Bristol,
Tyndall Avenue, Bristol BS8 1TL, UK
e-mail: M.Navascues@bristol.ac.uk

G. Chiribella and R.W. Spekkens (eds.), *Quantum Theory: Informational Foundations and Foils*, Fundamental Theories of Physics 181,
DOI 10.1007/978-94-017-7303-4_13

Up to now, this line of research has proven very fruitful, allowing to recover finite dimensional Quantum Mechanics from first principles [3–7].

Another, more operational, approach was introduced in [8], where Rohrlich and Popescu proposed to classify physical theories regarding their ability to generate correlations between distant points in space.

For the sake of clarity, suppose, for instance, that two space-like separated parties (call them Alice and Bob) share a given bipartite physical system. Alice and Bob are allowed to measure certain observables X and Y (out of a finite set) on their subsystems, and these measurements will report them some outcomes a and b, respectively. Assuming a complete ignorance about the Physics involved in these processes, Alice and Bob could regard their system as a black box where they input a pair of symbols X, Y and obtain a pair of outputs a, b.

Popescu and Rorlich proposed that the choice of Alice's interaction should not affect Bob's statistics and viceversa. This principle, known as the *no-signaling condition*, translates at the level of probabilities as

$$\begin{aligned}\sum_a P(a,b|X,Y) &= \sum_a P(a,b|X',Y) \equiv P(b|Y)\\ \sum_b P(a,b|X,Y) &= \sum_b P(a,b|X,Y') \equiv P(a|X),\end{aligned} \tag{1}$$

for whatever interactions X, X' (Y, Y') available to Alice (Bob).

Despite its simplicity, the no-signaling condition imposes a strong constraint on the set of possible correlations present in a given physical theory. When represented in a real space, the set of all no-signaling distributions forms a polytope, i.e., a convex set defined by a finite number of linear inequalities.

Unfortunately, the no-signaling constraint is not strong enough. Indeed, the *no-signaling polytope* contains probability distributions that are so weird that soon people started to think that they could not be present in any reasonable physical theory. This motivated different works that ruled out some of the correlations of the no-signaling polytope on the grounds that they would make communication or computation trivial [9, 10], or violate the principle of information causality [11], see the corresponding chapter in this volume.

In [12], the authors proposed reduction to Classical Physics in the macroscopic limit as a fundamental axiom to be satisfied by any reasonable physical theory. Note that the theory that describes our universe has to recover Quantum Theory and General Relativity in some limits, and both these theories allow recovering the framework of Classical Physics. It thus seems inevitable that any reasonable physical theory must reduce to Classical Physics in suitable limits.

Notice also that a connection with Classical Physics may come together with a Correspondence Principle to derive the dynamics of the theory from classical models of the physical system at stake. A classical macroscopic limit is thus desirable from a practical point of view: since Classical Physics is the only known theory that relates Quantum Mechanics and General Relativity, it seems natural to resort to it in order to find a consistent dynamics for a deeper theory.

Finally, the notion that any theory has to recover Classical Physics is somehow implicit in Rorlich and Popescu's formalism. Expressions like $P(a, b|X, Y)$ assume that Alice is bound to apply only one out of the set of possible interactions, and not, for example, a linear combination of interactions X_1, X_2. The observer itself is thus regarded as classical in this scenario, and so the world in which it lives should also have some notion of classicality.

In this chapter we will show that the existence of a classical limit bounds the strength of the correlations measured by space-like separated observers in a non-trivial way. In a nutshell, the fact that there exists a classical limit implies that 'natural' macroscopic experiments involving distant parties measuring many microscopic systems in a coarse-grained way admit a local hidden variable model. This property, which we will call *macroscopic locality* [12], in turn imposes strong restrictions on the correlations generated by such microscopic systems. The main goal of this chapter is to characterize such restrictions.

The structure of this chapter is as follows: first, we will illustrate the meaning of macroscopic locality (ML) by means of a specific example. Later, in Sects. 2.2, 2.3, we will define and characterize Macroscopic Locality in a more general framework where experimentalists are allowed to apply sequential interactions. In Sect. 2.4, we will attain one of the main goals of this chapter, that is, to characterize the set Q^{ml} of all multipartite correlations *compatible* with ML. We will next prove that quantum correlations are contained in Q^{ml}, and, in Sect. 4, we will study the differences and similarities between the two sets. Finally, we will present our conclusions.

2 Macroscopic Locality

2.1 Some Preliminary Thoughts

Think of the following bipartite scenario (Fig. 1): a particle pair is produced and two experimentalists, call them Alice and Bob, receive one particle each. Within the given setup, Alice (Bob) can interact with her/his particle in two different ways $X = 0, 1$ ($Y = 0, 1$). As a result of each interaction, Alice's (Bob's) particle will follow one of two possible paths, the upper or the lower, and eventually will impinge on one of Alice's (Bob's) two detectors, as shown in Fig. 1. If Alice and Bob repeat the experiment many times, they will be able to estimate the probabilities $P(a, b|X, Y)$, i.e., the probability that Alice's and Bob's particles impinge on detectors $a, b = 0, 1$ when they apply the interactions X, Y.

This is the schema of a bipartite experiment of non-locality. We say that $P(a, b|X, Y)$ is *local* (sometimes called *classical*) if it can be expressed as

$$P(a, b|X, Y) = \sum_{\lambda} P(\lambda) P_A(a|X, \lambda) P_B(b|Y, \lambda), \tag{2}$$

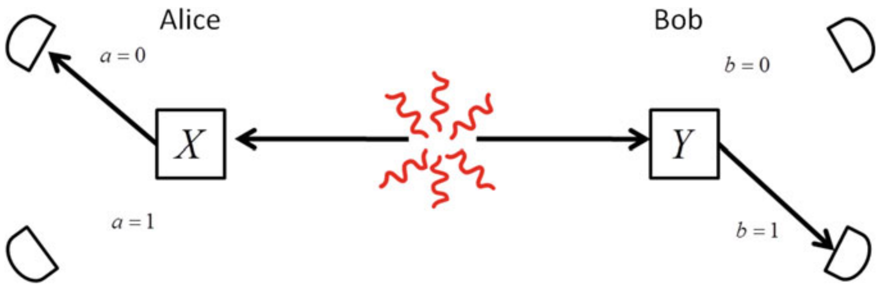


Fig. 1 A microscopic experiment of non-locality. A particle pair is produced. The interaction that Alice (Bob) subjects her (his) particle will make it take the path $a = 0$ or $a = 1$ ($b = 0$ or $b = 1$). Clicks on the detectors at the end of each path are associated to measurement outcomes. In the figure, the outcomes are $a = 0, b = 1$

for some probability distributions $P(\lambda)$, P_A, P_B. Equivalently, a distribution is local if there exists a global probability distribution for the variables $\{a_X, b_Y : X, Y\}$ with marginals $P(a_X, b_Y) = P(a, b|X, Y)$. Otherwise, we say that $P(a, b|X, Y)$ is *non-local*. There is plenty of evidence that our world is non-local (or non-classical) at the microscopic level, so it should not surprise us that an experiment like the one described above produces a non-local distribution.

Suppose now that $P(a, b|X, Y)$ is of the form:

$$P(a, b|X, Y) = \frac{\epsilon}{2}\delta_{a\oplus b, XY} + \frac{1-\epsilon}{4}, \tag{3}$$

where $0 \leq \epsilon \leq 1$. Such a probability distribution is known as an *isotropic Popescu-Rorlich (PR) box* [13]. It is local for $\epsilon \leq \frac{1}{2}$, and quantum for $\epsilon \leq \frac{1}{\sqrt{2}}$.

If there existed a microscopic event producing a particle pair following a distribution of the form (3), one would expect to find natural 'macroscopic' sources of $N \gg 1$ independent identical pairs. According to our guiding principle, namely, the existence of a classical limit, the observations performed by two classical experimentalists situated at a distance from this source should admit a classical description.

And what would two classical observers see in the vicinity of such sources? That will depend on Alice and Bob's 'classical' measurement devices, which we will model through their microscopic counterparts. Correspondingly, we will assume that Alice's (Bob's) interactions over her (his) beam will affect each particle individually, and so the net effect of such interactions will be to split the incident beam into two sub-beams of lower intensity, see Fig. 2. Also, since $N \gg 1$, Alice and Bob's detectors will measure sums of clicks or intensities rather than individual clicks. That is, as opposed to noticing a click in detector a, Alice will measure the intensity $I^a_{A|X}$.

Now, it is very unlikely to recover Classical Physics in a macroscopic experiment if we allow Alice's and Bob's detectors to have an arbitrary (microscopic) precision. For instance, if Alice and Bob realized that all their intensities are multiples of a smaller quantity, they could postulate that their beams are composed of pairs of

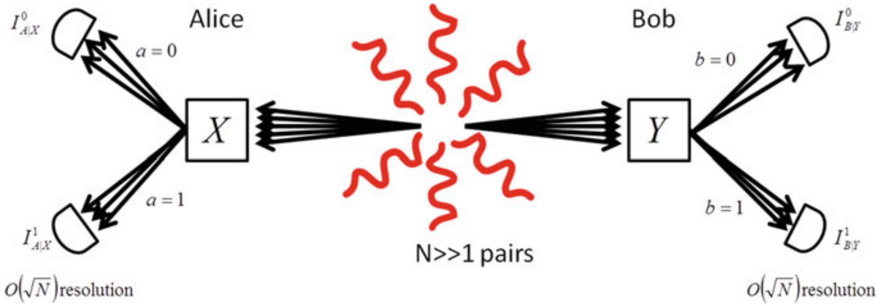


Fig. 2 A macroscopic experiment of non-locality. N independent particle pairs are produced. Alice's and Bob's interactions apply to all particles on each beam. This time, the intensities measured at each detector (with precision $O(\sqrt{N})$) are the outcomes of the experiment

correlated elementary particles, and derive the (in general, non-local) microscopic distribution $P(a, b|X, Y)$ from their classical data. This consideration leads to the extra assumption that the resolution of such detectors should just be able to detect intensity fluctuations of order $O(\sqrt{N})$. According to the Central Limit Theorem, such will be the expected size of the intensity fluctuations every time they repeat the experiment, so the assumption seems quite natural.[1]

We thus conclude that, under the above conditions (where two parties are conducting coarse-grained extensive measurements over a natural source of particle pairs), any macroscopic experiment should be describable in terms of a classical physical model. A necessary condition for this is the existence of a local hidden variable model (LHVM) for the distributions[2]

$$P(I_A^0, I_B^0|X, Y). \tag{4}$$

That is actually the original definition of Macroscopic locality [12]: namely, the requirement that coarse-grained ($O(\sqrt{N})$) extensive observations of macroscopic sources of N independent particle pairs admit a LHVM in the limit $N \to \infty$.

By the central limit theorem, when $N \to \infty$, $P(I_A^0, I_B^0|X, Y)$ become bivariate gaussian distributions with covariance matrix[3] proportional to

$$\gamma_{XY} = \begin{pmatrix} \frac{1}{4} & \frac{\epsilon}{2}\delta_{X\cdot Y,0} - \frac{\epsilon}{4} \\ \frac{\epsilon}{2}\delta_{X\cdot Y,0} - \frac{\epsilon}{4} & \frac{1}{4} \end{pmatrix}, \tag{5}$$

[1] A similar coarse-graining was required in [1] to prove the emergence of macroscopic realism from quantum mechanical systems. Note, however, that the resolution Δm considered there satisfies $\Delta m \gg O(\sqrt{N})$.

[2] Note that, more generally, we could have demanded the existence of a LHVM for $P(I_A^0, I_A^1, I_B^0, I_B^1|X, Y)$. However, in this class of experiments, it can be observed experimentally that $I_A^0 + I_A^1 = I_B^0 + I_B^1 = N$, and so the locality condition reduces to (4).

[3] The covariance matrix of a set of random variables $\xi_1, \xi_2, \ldots$ is defined as $\gamma_{ij} \equiv \langle\xi_i\xi_j\rangle - \langle\xi_i\rangle\langle\xi_j\rangle$. It can be verified that any covariance matrix must be positive semidefinite, see Appendix 1.

see Appendix 2.

Now, suppose that there exists a global measure $d\rho$ (or LVHM) for the intensities $\{I^0_{A|X}, I^0_{B|Y} : X, Y = 0, 1\}$. Then one could use such a measure to define a *global* covariance matrix of the form

$$\Gamma = \begin{pmatrix} \frac{1}{4} & \lambda_1 & \frac{\epsilon}{4} & \frac{\epsilon}{4} \\ \lambda_1 & \frac{1}{4} & \frac{\epsilon}{4} & -\frac{\epsilon}{4} \\ \frac{\epsilon}{4} & \frac{\epsilon}{4} & \frac{1}{4} & \lambda_2 \\ \frac{\epsilon}{4} & -\frac{\epsilon}{4} & \lambda_2 & \frac{1}{4} \end{pmatrix}. \tag{6}$$

Here the rows and columns of the matrix correspond to the intensities $I^0_{A|X=0}$, $I^0_{A|X=1}$, $I^0_{B|Y=0}$, $I^0_{B|Y=1}$, and $\lambda_1, \lambda_2 \in \mathbb{R}$ resp. represent the values $\langle I^0_{A|X=0} I^0_{A|X=1}\rangle - \langle I^0_{A|X=0}\rangle\langle I^0_{A|X=1}\rangle$, $\langle I^0_{B|Y=0} I^0_{B|Y=1}\rangle - \langle I^0_{B|Y=0}\rangle\langle I^0_{B|Y=1}\rangle$ as calculated via the measure $d\rho$. Note that λ_1, λ_2 are not observable, and thus can only be computed with the extra knowledge $d\rho$.

At this moment there comes a crucial observation: in order for $\Gamma(\lambda_1, \lambda_2)$ to be a covariance matrix, it must be positive semidefinite. This implies that there must exist a choice of λ_1, λ_2 such that $\Gamma(\lambda_1, \lambda_2) \geq 0$.

By symmetry under the exchange of Alice and Bob, it is easy to see that we can take $\lambda_1 = \lambda_2 = \lambda$. Since the minimum eigenvalue of $\Gamma(\lambda, \lambda)$ is $1/4 - (1/4)\sqrt{2\epsilon^2 + 8\epsilon|\lambda| + 16\lambda^2}$, the condition for positivity is thus equivalent to the existence of $\lambda \in \mathbb{R}$ such that

$$2\epsilon^2 + 8\epsilon|\lambda| + 16\lambda^2 \leq 1. \tag{7}$$

It is easy to see that the above equation can only hold if $\epsilon \leq \frac{1}{\sqrt{2}}$, i.e., if the isotropic box belongs to the quantum region.

We have just shown that post-quantum isotropic PR boxes are incompatible with the principle of Macroscopic Locality (ML).

2.2 The Macroscopic Scenario

In this section, we will consider the transition from microscopic to macroscopic experiments in complex multipartite scenarios where each party is allowed to interact sequentially with its particle beam. Before starting, though, some comments about basic notation are in order. Despite its popularity in nonlocality research, denoting probabilities by $P(a, b|X, Y)$ and intensities by $I^a_{A|X}$ soon becomes messy when we have to deal with macroscopically local models. For this reason, along this article we will adopt the representation introduced by Tsirelson [14]. In this notation, any possible outcome we may measure after the application of an interaction X is to be denoted by a symbol a that allows identifying X. That way, interactions X can be regarded as disjoint sets of possible outcomes a. For any pair of interactions

X, Y, available at Alice's and Bob's lab, respectively, and any pair of outcomes $a \in X, b \in Y$, the expression $P(a, b|X, Y)$ thus becomes redundant, and can be substituted by $P(a, b)$.

In a generic multipartite microscopic experiment, each of the space-like separated sites has access to a set of local interactions $X, Y, Z, \ldots$. Given a possible outcome a, the mappings $X(a) = X$, $O(a) = i$ will return, respectively, the measurement setting and site i where such a measurement is performed.

An *experimental setting* S is any arrangement of interactions that an experimentalist at lab i can prepare in order to measure intensities in a macroscopic experiment. For example, in Fig. 3, the experimental setting on site 1 consists on applying interaction X over the main beam and then interactions Z and Z' to the particles following a trajectory a or a', respectively. Given an experimental setting at site i, we will call *arc* to the trajectory followed by a particle since its arrival at lab i until it impinges on a detector. Any arc s can thus be completely specified by an ordered sequence of outcomes $s \equiv a_1 \to a_2 \to a_3 \to \cdots \to a_m$, with $O(a_j) = i$, and therefore any experimental setting can be identified with the set of all arcs it generates. Interactions applied in different arcs will be regarded as different, i.e., the specification of each interaction must make reference to all prior interactions. Coming back to Alice's experimental setting S in Fig. 3, we hence have that $S = \{a \to c, a \to c', a' \to d, a' \to d'\}$.

We say that two different arcs s, s' are *locally orthogonal* if they both appear in the same experimental setting, that is, if $s = s_1 \to a \to s_2$ and $s' = s_1 \to a' \to s_2'$, with $X(a) = X(a')$, $a \neq a'$. Also, two arcs s, s' are *space-like separated* if they correspond to experimental settings on different sites, i.e., iff $O(s) \neq O(s')$. Given two space-like separated arcs $s \equiv a_1 \to \cdots \to a_m$, $t \equiv b_1 \to \cdots \to b_{m'}$, $P(s, t)$ will represent the probability that the first particle of a pair returns the sequence of outcomes $(a_1, \ldots, a_m)$ when interactions $X(a_1), \ldots, X(a_m)$ are sequentially applied, and the second particle outputs $(b_1, \ldots, b_{m'})$ when $X(b_1), \ldots, X(b_{m'})$ are effected.

In this scenario, the no-signaling condition translates as

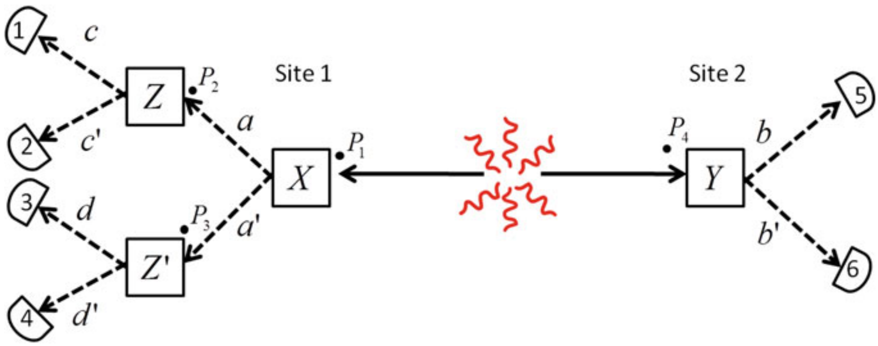


Fig. 3 A microscopic experiment with sequential measurements

Definition 1 No-signaling condition
Let $\{s_j\}_{j=1}^K$ be a collection of space-like separated arcs. Then, for any $k \in \{1, 2, \ldots, K\}$ and any interaction X, with $O(X) = k$,

$$\sum_{a \in X} P(s_1, \ldots, s_k \to a, \ldots s_K) = P(s_1, \ldots s_k, \ldots s_K). \tag{8}$$

When we bring a microscopic experiment of non-locality to the macroscopic scale, we end up in the scenario depicted in Fig. 4. Here the experimental outcomes are coarse-grained intensity measurements conducted at the end of each arc. The intensity measured at site i at the end of the arc s will be denoted by I^s, and by $\bar{I}^s$ we will refer to the measured intensity fluctuation, i.e., $\bar{I}^s = I^s - \langle I^s \rangle$. Notice that intensities corresponding to the same arc s, but belonging to different experimental settings, are identified with this notation. The reason is that the different interactions effected on each measurement setting can be space-like separated from arc s. For example, in the scenario depicted in Fig. 4 the regions where interactions Z, Z' are applied can be arbitrarily far away from each other, and so the intensities $I^{a\to c}$, $I^{a\to c'}$ ($I^{a'\to d}$, $I^{a'\to d'}$) would be regarded as independent of Z' (Z) by classical observers.

In a realistic situation, we cannot expect the K parties to be able to realize any possible experimental setting. Any experimentalist at site i will have to work under space and budget constraints. Consequently, the length of the available arcs will have to be limited, and so will be the (finite) set of accessible experimental settings $\mathcal{S}_{\text{acc}}$.

For any particular choice of experimental settings $\{S_i\}_{i=1}^K \subset \mathcal{S}_{\text{acc}}$, the K parties performing a macroscopic experiment can estimate the marginal probability distributions

$$P\left(\{I^s : s \in \cup_{i=1}^K S_i\}\right), \tag{9}$$

corresponding to all the intensities measured in a single experiment with space-like separated measurement settings $S_1, S_2, \ldots$ with precision $O(\sqrt{N})$.

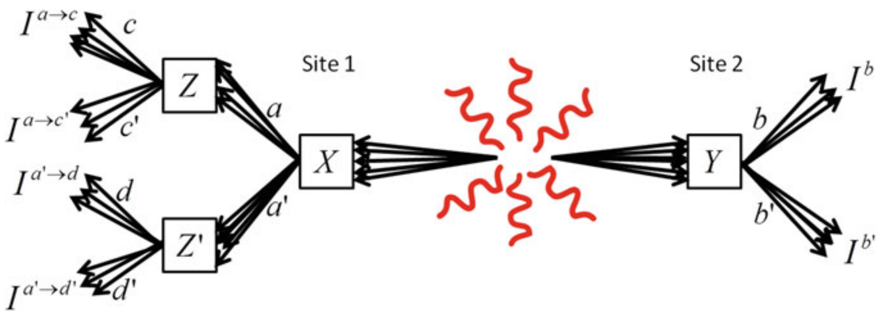


Fig. 4 A macroscopic experiment with sequential measurements

Following Sect. 2.1, the experiments performed by the K parties will satisfy ML iff there exists a joint measure $P(\{I^s : s \in S \in \mathcal{S}_{\text{acc}}\})$ for all the intensities I^s accessible in the family of experiments denoted by $\mathcal{S}_{\text{acc}}$, compatible with the marginal distributions (9) in the limit $N \to \infty$.

2.3 Characterization of Macroscopic Locality

By Appendices 1, 2, a necessary and sufficient condition for ML in a given set of accessible experimental settings S_{acc} is the existence of a positive semidefinite matrix γ, with rows and columns labeled by arcs, and satisfying

$$\gamma_{ss} = P(s) - P(s)^2, \tag{10}$$

$$\gamma_{ss'} = P(s, s') - P(s)P(s') \text{ for } s, s' \text{ space-like separated.} \tag{11}$$

$$\gamma_{ss'} = -P(s)P(s') \text{ for } s, s' \text{ locally orthogonal.} \tag{12}$$

for all $s \in S$, $s' \in S'$, with $S, S' \in \mathcal{S}_{\text{acc}}$.

Of course, if there is a LHVM behind all possible experiments that the K parties can perform without any limitation on the size of the settings, then there will exist a LHVM for all experiments involving a finite set of experimental settings $\mathcal{S}_{\text{acc}}$. However, in principle there could be weird K-partite microscopic correlations such that, for any finite set of available experimental settings $\mathcal{S}_{\text{acc}}$ there is a $\mathcal{S}_{\text{acc}}$-dependent LHVM that describes the observed intensity fluctuations, but nevertheless there is not a single classical model independent of $\mathcal{S}_{\text{acc}}$ that is compatible with all experimental data!

This possibility is ruled out by the next result.

Lemma 2 *Let P be a set of K-partite microscopic correlations. If for any finite set of available experimental settings $\mathcal{S}$ there exists a LHVM $P_{\mathcal{S}}(\{\bar{I}^s, s \in S \in \mathcal{S}\})$, then there exists a setting-independent LHVM compatible with all possible macroscopic experimental observations. That is, for any finite set of arcs $\vec{o}$ there exists a measure $P_{\vec{o}}$ for the intensity fluctuations $\{\bar{I}^s : s \in \vec{o}\}$ in agreement with experimental data, such that, for any two finite sets $\vec{o}, \vec{o}'$,*

$$\mathcal{P}_{\vec{o}}\left(\{\bar{I}^s : s \in \vec{o} \cap \vec{o}'\}\right) = \mathcal{P}_{\vec{o}'}\left(\{\bar{I}^s : s \in \vec{o} \cap \vec{o}'\}\right). \tag{13}$$

Proof Let $(\mathcal{S}_n)$ be a sequence of finite sets of experimental settings such that, for all n, $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ and such that, for any possible setting S, there exists an n such that $S \in \mathcal{S}_n$. According to the previous remarks, this implies that there exists a sequence of positive semidefinite matrices (γ^n) of increasing size that satisfy conditions (10), (11) and (12) for all $s, s' \in \mathcal{S}_n$. Note as well that all the entries of γ^n are bounded by $1/4$. If we extend to the infinity all the rows and columns of these matrices by adding zeros, we will end up with a sequence of vectors γ^n in the ball l_∞. It is not difficult to see that there exists an entry-wise convergent subsequence $(\gamma^{n_k})_{n_k}$, call $\bar{\gamma}$ its limit. The infinite dimensional matrix $\bar{\gamma}$ hence satisfies

1. Any finite submatrix of $\bar{\gamma}$ is positive semidefinite.
2. $\bar{\gamma}$ satisfies conditions (10), (11) and (12) for all $s, s' \in \vec{O}$.

Let $\mathcal{P}_{\vec{o}}$ be the gaussian probability distribution for the variables $\{\bar{I}^s : s \in \vec{o}\}$ with covariance matrix $\{\langle \bar{I}^s \bar{I}^{s'} \rangle = \bar{\gamma}_{s,s'}\}$ and zero displacement vector. Clearly, $P_{\vec{o}}$ satisfies the conditions of the lemma. □

Remark 3 The proof of the previous lemma shows that a microscopic distribution is macroscopically local iff there exists a matrix $\bar{\gamma}$ satisfying points 1 and 2. Now, consider the (infinite) matrix Γ, whose rows are columns are numbered by the symbol $\mathbb{I}$[4] and any arc $s \in \vec{O}$, and is defined by the following relation

$$\Gamma = \begin{pmatrix} 1 & \vec{p}^T \\ \vec{p} & \tilde{\gamma} \end{pmatrix}. \tag{14}$$

Here $\vec{p}$ is a vector whose components are numbered by arcs s and such that $p_s = P(s)$, and $\tilde{\gamma}_{ss'} = \bar{\gamma}_{ss'} + p_s p_{s'}$. Given $P(s)$, we can easily switch from one matrix to the other. Note also that, for whatever finite set of arcs $\vec{o}$, the submatrix $\{\Gamma_{\alpha\beta} : \alpha, \beta \in \vec{o} \cup \{\mathbb{I}\}\}$ is positive semidefinite iff $\{\bar{\gamma}_{\alpha\beta} : \alpha, \beta \in \vec{o}\}$ is positive semidefinite. Indeed, by Schur's theorem [15], $\Gamma_{\vec{o}\cup\{\mathbb{I}\}}$ is positive semidefinite iff $\tilde{\gamma}_{\vec{o}} - \vec{p}_{\vec{o}}\vec{p}_{\vec{o}}^T = \bar{\gamma}_{\vec{o}}$ is positive semidefinite.

Conditions (10), (11) and (12), together with the definition of Γ, translate into the following rules

1. $\Gamma_{\mathbb{I}\mathbb{I}} = 1$.
2. $\Gamma_{\mathbb{I}s} = P(s)$.
3. $\Gamma_{ss} = P(s)$.
4. $\Gamma_{ss'} = P(s, s')$, for s, s' space-like separated.
5. $\Gamma_{ss'} = 0$, for s, s' locally orthogonal.

The remark above shows that the existence of a LHVM for all possible macroscopic experiments is equivalent to the existence of an object Γ satisfying conditions 1–5 and such that any finite submatrix of it is positive semidefinite. The advantage with respect to the previous formulation is that Γ only depends *linearly* on the original microscopic probabilities. In Sect. 4, this will allow us to compute maximal violations of linear Bell inequalities in general macroscopically local theories via semidefinite programming [16].

Remark 4 Remark 3, in combination with Lemma 2, suggests an operational hierarchy of constraints to be satisfied by any K-partite distribution $P(s_1, s_2, \ldots, s_K)$ in order to be macroscopically local (in the line of [17]). Given an increasing sequence of sets of experimental settings $(\mathcal{S}_i)$ such that no setting is left out, it is thus enough

[4] Intuitively, if Γ were a quantum moment matrix, "$\mathbb{I}$" would correspond to the identity operator.

to check that, for each i, there exists a positive semidefinite matrix Γ satisfying conditions 1–5 in Remark 3 for any $s, s' \in S \in \mathcal{S}_i$. From Lemma 2, it follows that such a hierarchy is complete.

2.4 The Set of Correlations Compatible with ML

Let P_1, P_2 be two independent distributions held by K parties. In principle, an experimentalist at site i conducting a microscopic experiment on the *composed distribution* $P_1 \otimes P_2$ could measure X_1 on P_1 and, depending on the outcome, measure Y or Y' on P_2. In this sort of experiments, a generic arc s_{12} at site i is thus decomposed as the interlacing of two arcs s_1, s_2 associated to measurements on the boxes $1, 2$, respectively.[5] Since systems P and P' are independent, the probability that the K particles follow the arcs $\{s_{12}^i\}_{i=1}^K$ is therefore given by $P(s_{12}^1, \ldots, s_{12}^K) = P(s_1^1, \ldots, s_1^K)P(s_2^1, \ldots, s_2^K)$.

The joint use of two or more independent distributions to generate new statistics is known as "wiring" [18], and in some circumstances it can be used to distill Bell inequality violations [13]. This observation raises the possibility of the existence of macroscopically local distribution P with the property that $P^{\otimes n}$ or some other allocation of many copies of the distribution P allows generating non-local macroscopic intensities. Such a distribution P, though macroscopically local, would not be *compatible* with the principle of macroscopic locality.

If such were the case, we would be in a conundrum. On one hand, it may be that distributions like P' never appear naturally at the macroscopic scale, and so such non-local intensities are never observed. This would not contradict the fact that Nature seems to be local at big scales, but would lead to a restriction on the dynamics of this Universe, that allows for the existence of macroscopic sources of P, but not of P'. On the other hand, we could simply postulate that *any* physical system can be brought to the macroscopic scale, and so we could ban the existence of P on the grounds that it allows to engineer P', which, in turn, generates non-local macroscopic correlations. This is the approach we will stick to along this chapter.

Once this point has been clarified, the next question to address is how to determine if a given probability distribution P is compatible with ML. In general, one would expect the answer to depend on the rest of available correlations $\{P'\}$, since it could well be that P and P' alone only lead to macroscopically local experiments, but nonetheless allow to distill macroscopic non-locality when they belong to the same space of physical states.

In this section we will show that such is not the case: any set of physical systems unable to produce non-local intensities by themselves cannot be wired into a macro-

[5] For instance, if a_1, a_2 are outcomes corresponding to P_1; and b, to P_2, the arc $s_{12} = a_1 \rightarrow b \rightarrow a_2$ corresponds to the event of measuring $X(a_1)$ on the first box, then $X(b)$ on the second and then $X(a_2)$ on the first, and obtaining the sequence of outcomes (a_1, b, a_2). In this case, the arc s_{12} corresponds to the interlacing of $s_1 = a_1 \rightarrow a_2$ and $s_2 = b$.

scopically non-local system when they are brought together. Ergo, there exists a maximal set of correlations Q^{ml} compatible with macroscopic locality that is closed under wirings. We will characterize this set at the end of the section.

2.4.1 A Closure Result

We will begin by showing that K-partite macroscopically local distributions are closed under local wirings. That is, once a number of such correlations has been distributed between the parties and those are not allowed to communicate classically, any wiring they may perform on their systems will not allow them to generate macroscopically non-local correlations. That $\otimes_i P_i$ cannot be used to distill macroscopic non-locality when each P_i is macroscopically local follows by induction from the next theorem.

Theorem 5 *Let P_1, P_2 be K-partite macroscopically local distributions. Then, $P_1 \otimes P_2$ is also macroscopically local.*

Proof Call $\vec{O}^1$ ($\vec{O}^2$) the set of all arcs s (s') pertaining to system P_1 (P_2); $\vec{O}^{12}$ will denote the set of all arcs generated by interlacing arcs from boxes P_1 and P_2. If P_1 and P_2 are ML, then from the last remark, there must exist two infinite matrices Γ_1, Γ_2 that satisfy conditions 1–5 for all $s, s' \in \vec{O}^1$ and $\vec{O}^2$, respectively, and such that any finite submatrix of them is positive semidefinite. Following the lines of [17], we have that there must exist two sets of vectors $V_1 = \{|s\rangle_1 : s \in \vec{O}^1 \cup \{\mathbb{I}_1\}\}$, $V_2 = \{|s\rangle_2 : s \in \vec{O}^2 \cup \{\mathbb{I}_2\}\}$, with $\langle s|s'\rangle_1 = \Gamma^1_{ss'}$ ($\langle s|s'\rangle_2 = \Gamma^2_{ss'}$) for all $s, s' \in \vec{O}^1 \cup \{\mathbb{I}_1\}(\vec{O}^2 \cup \{\mathbb{I}_2\})$.

Now, define the vectors[6]

$$
\begin{aligned}
|\mathbb{I}\rangle_{12} &\equiv |\mathbb{I}\rangle_1 \otimes |\mathbb{I}\rangle_2, \\
|s_{12}\rangle_{12} &\equiv |s_1\rangle_1 \otimes |s_2\rangle_2,
\end{aligned} \tag{15}
$$

where the arc $s_{12} \in \vec{O}^{12}$ is understood to arise by interlacing the arcs $s_1 \in \vec{O}^1$, $s_2 \in \vec{O}^2$ (without altering the order in which the outcomes of $\vec{O}^1$ and $\vec{O}^2$ appear).

Then we can construct the matrix Γ^{12} as

$$
\Gamma^{12}_{ss'} = \langle s|s'\rangle, \tag{16}
$$

for $s, s' \in \vec{O}^{12} \cup \{\mathbb{I}\}$. Clearly, any finite submatrix of Γ will be positive semidefinite.

We will now see that Γ^{12} satisfies conditions 1–5 of Remark 3 for the new set of correlations $P_1 \otimes P_2$.

[6] In case and $s_1 = \emptyset$ ($s_2 = \emptyset$), take $|s_1\rangle_1 = |\mathbb{I}\rangle_1$ ($|s_2\rangle_2 = |\mathbb{I}\rangle_2$).

1. $\Gamma^{12}_{\mathbb{I}\mathbb{I}} = \langle \mathbb{I}|\mathbb{I}\rangle_{12} = \langle \mathbb{I}|\mathbb{I}\rangle_1 \langle \mathbb{I}|\mathbb{I}\rangle_2 = 1$.
2. $\Gamma^{12}_{\mathbb{I}s} = \langle \mathbb{I}|s\rangle_{12} = \langle \mathbb{I}|s_1\rangle_1 \langle \mathbb{I}|s_2\rangle_2 = P(s_1)P(s_2) = P(s)$.
3. $\Gamma^{12}_{ss} = \langle s|s\rangle_{12} = \langle s_1|s_1\rangle_1 \langle s_2|s_2\rangle_2 = P(s_1)P(s_2) = P(s)$.
4. If $s, s' \in \vec{O}^{12}$ are space-like separated, then s_1 and s'_1 (s_2 and s'_2) are space-like separated. It follows that $\Gamma^{12}_{s,s'} = \langle s|s'\rangle_{12} = \langle s_1|s'_1\rangle_1 \langle s_2|s'_2\rangle_2 = P(s_1, s'_1)P(s_2, s'_2) = P(s, s')$.
5. Let s, s' be locally orthogonal. Then, $s = t_{12} \rightarrow a \rightarrow t'_{12}$, $s' = t_{12} \rightarrow a' \rightarrow t''_{12}$, where $a \neq a'$ but $X(a) = X(a')$. Suppose w.l.o.g. that $a \in O^1$. Then, $|s\rangle = |s_1\rangle \otimes |s_2\rangle$, and $|s'\rangle = |s'_1\rangle \otimes |s'_2\rangle$, with $s_1 = t_1 \rightarrow a \rightarrow t'_1$ and $s'_1 = t_1 \rightarrow a' \rightarrow t''_1$. That is, s_1 and s'_1 are locally orthogonal. This implies that $\langle s_1|s'_1\rangle = 0$, and so $\Gamma^{12}_{s,s'} = \langle s|s'\rangle_{12} = \langle s_1|s'_1\rangle_1 \langle s_2|s'_2\rangle_2 = 0$.

Therefore, $P_1 \otimes P_2$ is macroscopically local. □

We have just proven that ML cannot be activated by local wirings without communication. The next section shows, however, that one can distill macroscopic non-locality from ML boxes via a prior non-local engineering and/or local postselections.

2.4.2 Activation of ML

Consider a tripartite scenario where Alice, Bob and Charlie have each a pair of measurement settings with two possible outcomes. For clarity, let us return momentarily to the standard notation in non-locality, where probabilities are denoted as $P(a, b, c|X, Y, Z)$, and a, b, c, X, Y, Z take values in $\{0, 1\}$. Then one can check that the tripartite set of correlations

$$P \equiv P(a, b, c|X, Y, Z) = \frac{1}{4}\delta_{a\oplus b\oplus c=XYZ} \tag{17}$$

generates local intensities. Moreover, if several copies of P were distributed to Alice, Bob and Charlie, together with any amount of shared randomness, and the parties were not allowed to communicate, any wiring they performed on their subsystems would not allow them to distill macroscopic non-locality. The reason is that the (gaussian) marginal distributions of the macroscopic intensity fluctuations observed by the three parties are completely determined by the bipartite correlations between different intensities. These, in turn, just depend on the bipartite probability distributions $P(a, b|X, Y)$, $P(a, c|X, Z)$, $P(b, c|Y, Z)$. Since such bipartite distributions also arise from the local tripartite distribution $L(a, b, c|X, Y, Z) = 1/8, \forall a, b, c$, any wiring $\mathcal{W}$ of m copies of P will be macroscopically indistinguishable from $\mathcal{W}(L^{\otimes m})$, and thus macroscopically local.

However, if Charlie measures $Z = 1$ and announces his outcome, then Alice and Bob would be sharing a perfect PR box [8], which, as we saw in Sect. 2, is macroscopically non-local [12].

Also, suppose that Alice's and Charlie's *separate degrees of freedom* are integrated into just one particle, call it AC, and imagine a macroscopic experiment

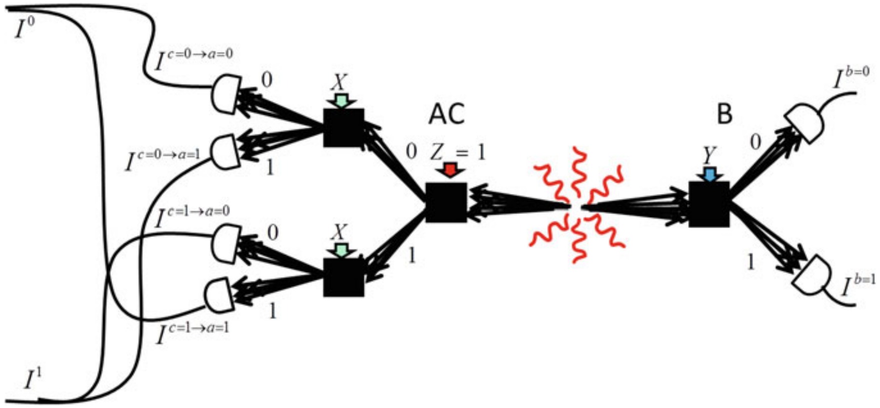


Fig. 5 Activation of macroscopic non-locality. By summing pairs of intensities, Alice and Bob can reproduce the macroscopic correlations generated by perfect PR boxes

where several independent pairs of AC/B particles are generated and sent to Alice and Bob. Then Alice could apply two consecutive interactions over her particle beam, as shown in Fig. 5. The first of such interactions, $Z = 1$, would address Charlie's degree of freedom, and split the particle beam into two different sub-beams. The subsequent application of an arbitrary interaction X to Alice's degree of freedom in AC, would subsequently split each sub-beam, thus ending up with four intensities on Alice's lab, each with mean value proportional to $NP(a, c|X, Z)$. Defining $\vec{I}_A \equiv (I^{c=0\to a=0}, I^{c=0\to a=1}, I^{c=1\to a=0}, I^{c=1\to a=1})$ and $\vec{I}_B \equiv (I^{b=0}, I^{b=1})$, it can be verified that the observed macroscopic distributions $P(\vec{I}_A, \vec{I}_B|X, Y)$ do not admit a LVHM, see Fig. 5.

P thus contains some *hidden macroscopic non-locality*, that can be activated either with one bit of communication or by joining two separate degrees of freedom into one.

2.4.3 The Set Q^{ml}

Theorem 5 shows that, once several macroscopically local systems have been distributed, the separated parties are not able to distill macroscopic non-locality. Now, a *general* wiring of a finite set of distributions $\{P_i\}_{i=1}^m$ will start (or not) with some post-selective measurements[7] (e.g.: in the previous section, in order to activate macroscopic non-locality, Charlie had to measure his subsystem and announce his measurement outcome). Afterwards, the remaining separate degrees of freedom will be distributed between the different parties, wirings will be made and a macroscopic

[7]Here, by a post-selective measurement, we understand the generation of a new state or set of correlations by preparing a (non-local) experimental setting and conditioning the final state on a specific arc.

experiment will take place. Invoking Theorem 5, it follows that, if such post-selected systems are already macroscopically local, then any wiring of them will be macroscopically local as well. On the other hand, any distribution P compatible with macroscopic locality has to remain macroscopically local under postselection. These considerations lead us to the following set:

Definition 6 A probability distribution P belongs to Q^{ml} iff, for any previous post-selective measurement and subsequent distribution in space of its separate degrees of freedom, the corresponding K-partite system is macroscopically local.

Following Remark 3, the necessary and sufficient conditions for a (conditional) set of correlations to be macroscopically local amount to the possibility to complete a sequence of growing matrices (whose determined entries are *linear*[8] in the microscopic probabilities and whose undetermined entries satisfy certain linear relations) in such a way that all of them are positive semidefinite. It follows that Q^{ml} is a convex set.

From the previous observations, it is clear that Q^{ml} is closed under wirings when the postselection part is deterministic (i.e., when the state distributed to the parties has the form $\otimes_i P_{i|a_i}$, where $P_{i|a_i}$ is the distribution P_i conditioned on the outcome(s) a). That Q^{ml} is closed under wirings in general is due to the fact that, after a generic postselection phase, the parties will be distributed convex combinations of boxes of the form $\otimes_{i,a} P_{i|a}$. Any experimental setting S_i (or wiring) on each side i will then only produce a convex combination of that same setting applied to the boxes $\otimes_{i,a} P_{i|a}$. Closure under wirings follows then from the convexity of the set Q^{ml}.

In sum, Q^{ml} is the maximal set of macroscopically local correlations that is closed under wirings.

Note that, in order to determine if a given set of probabilities $\{P(s)\}_s$ belongs to Q^{ml}, one would have to check for the existence of infinite dimensional covariance matrices for any possible postselection $\{P(s|s') : s\}$. That amounts to look for an infinite number of infinitely-sized matrices, not an easy task! In Sect. 4, however, we will see that in standard scenarios one just has to consider a finite set of finite-dimensional matrices.

3 Quantum Mechanics Satisfies Macroscopic Locality

In the last sections, we have defined ML and provided a semidefinite programming characterization of the set of microscopic correlations compatible with this principle. It is now time to prove that Quantum Mechanics satisfies ML.

---

[8] In principle, according to Remark 3, for a set of correlations conditioned on $\tilde{s}$ the determined entries of Γ should be of the form $P(s|\tilde{s}) = P(s, \tilde{s})/P(\tilde{s})$ and thus highly non-linear in P. Note, though, that we can always redefine such a matrix Γ as $\Gamma' = P(\tilde{s})\Gamma$. Obviously, as long as $P(\tilde{s}) \neq 0$, the positivity of Γ' is equivalent to that of Γ, but Γ' depends linearly on P.

Let $\mathcal{S}_{\text{acc}}$ be any set of experimentally accessible settings, and let $|\psi\rangle \in \mathcal{H}$ be the joint state of the corresponding quantum microscopic experiment (w.l.o.g., we can assume it to be pure). For any interaction X, call E_a the projector operator corresponding to outcome $a \in X$. Clearly, $E_a E_{a'} = \delta_{aa'} E_a$, for $a, a' \in X$ and $\sum_{a \in X} E_a = \mathbb{I}$. Also, if $O(a) \neq O(b)$, $[E_a, E_b] = 0$, i.e., observables corresponding to different parties commute. One can argue that during the course of the experiment the quantum system could experience some evolution U, perhaps depending on the sequence of past interactions effected on the particle. We will solve this issue by switching to the Heisenberg picture and redefining the measurement operators at each point of each arc via $E \to UEU^\dagger$. Since the experiment is assumed to be performed under space-like separation, operators belonging to different parties will still commute.

Associate to each measurement X an auxiliary Hilbert space $\mathbb{C}^{|X|}$. We will call such systems *registers*, and use $\mathcal{H}_r = \bigotimes_X \mathbb{C}^{|X|}$ to denote the space of all of them. Intuitively, the registers are going to hold a record of the outcomes we observe when we interact with particle p_k sequentially. Let $\{|j\rangle\}_{j=0}^{|X|-1}$ be an orthonormal basis of $\mathbb{C}^{|X|}$; and ϕ, a function which maps any outcome $a \in X$ to a natural number between 0 and $|X|-1$ in such a way that $\phi(a) \neq \phi(a')$, for $a \neq a'$, $a, a' \in X$. Now, consider the unitary $U_X \in B(\mathcal{H}_r \otimes \mathcal{H})$ given by

$$U_X = \sum_{a \in X} V_X^{\phi(a)} \otimes E_a, \tag{18}$$

where V_X is a unitary which acts non-trivially only over the register X as $V|j\rangle = |j+1 \pmod{|X|}\rangle$. For any measurement outcome $a \in X$, call $\bar{\Pi}_a \in B(\mathcal{H}_r \otimes \mathcal{H})$ the projector that acts non-trivially over register X as $|\phi(a)\rangle\langle\phi(a)|$. For any *fragment* of an arc $s = a_1 \to a_2 \to \ldots \to a_m$, $\bar{\Pi}_s$ will denote the projector

$$\bar{\Pi}_s = \bar{\Pi}_{a_m} \bar{\Pi}_{a_{m-1}} \ldots \bar{\Pi}_{a_1}. \tag{19}$$

Analogously, U_s and E_s will represent the unitary operator $U_{X(a_m)} U_{X(a_{m-1})} \ldots U_{X(a_1)}$ and the non-hermitian operator $E_{a_m} \ldots E_{a_1}$, respectively. Now, define the projector

$$\Pi_s = U_s^\dagger \bar{\Pi}_s U_s, \tag{20}$$

for $s \neq \mathbb{I}$ and $\mathbb{I}_{\mathcal{H}_r} \otimes \mathbb{I}_{\mathcal{H}}$ otherwise, and denote the state $\bigotimes_X |0\rangle_X \in \mathcal{H}_r$ by $|\vec{0}\rangle$. We claim that the positive semidefinite matrix

$$\Gamma_{ss'} = \langle\vec{0}|\langle\psi|\Pi_s \Pi_{s'}|\vec{0}\rangle|\psi\rangle. \tag{21}$$

satisfies the conditions (3).

Indeed:

1. $\Gamma_{\mathbb{I}\mathbb{I}} = \langle\vec{0}|\vec{0}\rangle\langle\psi|\psi\rangle = 1$
2. $\Gamma_{\mathbb{I}s} = \Gamma_{\mathbb{I}s} = \Gamma_{ss} = \langle\vec{0}|\langle\psi|\Pi_s|\vec{0}\rangle|\psi\rangle = \langle\psi|E_s^\dagger E_s|\psi\rangle = P(s)$.

3. Let s, s' be space-like separated. Then, the operators $\bar{\Pi}_s, U_s$ commute with $\bar{\Pi}_{s'}, U_{s'}$. It follows that

$$\Gamma_{ss'} = \langle\vec{0}|\langle\psi|\Pi_s\Pi_{s'}|\vec{0}\rangle|\psi\rangle = \langle\vec{0}|\langle\psi|U_s^\dagger U_{s'}^\dagger\bar{\Pi}_s\bar{\Pi}_{s'}U_sU_{s'}|\vec{0}\rangle|\psi\rangle = \\ = \langle\psi|E_{s'}^\dagger E_s^\dagger E_s E_{s'}|\psi\rangle = P(s,s'). \tag{22}$$

4. Let s, s' be locally orthogonal. Then, $s = s_1 \to a \to s_2$, $s' = s_1 \to a' \to s_2'$, with $a, a' \in X(a)$, $a \neq a'$, and so,

$$\Pi_s = U_{s_1\to a}^\dagger U_{s_2}^\dagger \bar{\Pi}_{s_1\to a}\bar{\Pi}_{s_2}U_{s_2}U_{s_1\to a}, \\ \Pi_{s'} = U_{s_1\to a'}^\dagger U_{s_2'}^\dagger \bar{\Pi}_{s_1\to a'}\bar{\Pi}_{s_2'}U_{s_2'}U_{s_1\to a'}. \tag{23}$$

Note that $U_{s_1\to a} = U_{s_1\to a'}$. Also, $\bar{\Pi}_{s_1\to a}, \bar{\Pi}_{s_1\to a'}$ commute with $U_{s_2}, U_{s_2'}$ (because they act over different subsystems). This, together with the relation $\bar{\Pi}_{s_1\to a}\bar{\Pi}_{s_1\to a'} = 0$, implies that $\Pi_s\Pi_{s'} = 0$, and, consequently, $\Gamma_{ss'} = \langle\vec{0}|\langle\psi|\Pi_s\Pi_{s'}|\vec{0}\rangle|\psi\rangle = 0$.

4 Predictions of ML

Note that up to now we have not discussed the dynamics of theories respecting ML. This is so because the formalism of black boxes only allows to speak about correlations between distant parties, independently of how those correlations originated. Consequently, the only predictions we can expect from the ML axiom are limits to the non-locality exhibited by the physical theories subject to them. We already saw, in Sect. 2, that supra-quantum isotropic PR boxes are not compatible with ML. In this section, we will explore further how ML constrains bipartite and tripartite correlations. We will see how these results compare to the no-signaling, quantum and classical cases. But first we have to point out a practical observation.

Currently, the state of the art in non-locality research is to consider scenarios where each party interacts with its subsystem only once, i.e., the length of all accessible arcs is 1. We will call this kind of scenario the *standard picture*. In the standard picture only probabilities of the type $P(a_1, \ldots, a_K)$ are considered. To determine if such distributions are ML, we just have to check the existence of LHVMs for a finite set of intensities $\{I^a\}$ in a finite set of experiments, i.e., ML can be certified in a finite number of steps. Along this Section we will always consider standard picture scenarios.

4.1 The Bipartite Case

In order to find out if a set of probabilities $P(a, b)$ is compatible with ML, it is enough to check for the existence of a LHVM for the intensity fluctuations $\{\bar{I}^a, \bar{I}^b : a \in X, b \in Y, \forall X, Y\}$. By Remark 3, we can therefore identify Q^{ml} as Q^1 [12], a set of correlations proposed in [17] as a first approximation to the set Q of quantum correlations. Q^1 is defined as the set of bipartite distributions $P(a, b)$ such that there exists a *positive semidefinite* matrix Γ, whose columns and rows are numbered by the symbol $\mathbb{I}$, Alice's outcomes a and Bob's outcomes b with the structure

$$\Gamma = \begin{pmatrix} 1 & \vec{P}(a)^T & \vec{P}(b)^T \\ \vec{P}(a) & A & P(a,b) \\ \vec{P}(a) & P(b,a) & B \end{pmatrix} \tag{24}$$

with $A_{a,a} = P(a)$, $B_{b,b} = P(b)$.
From Sect. 3, we know that $Q \subset Q^1$. Moreover, this inclusion is strict [12, 17]. However, in a sense, the two sets are quite close.

Consider, for instance, a scenario where both Alice and Bob perform s dichotomic measurements, and, for any pair of measurement settings X_i, Y_j, define the two-point correlators

$$E_{ij} \equiv \sum_{a\in X_i, b\in Y_j} P(a,b)(-1)^{\phi(a)\oplus\phi(b)}, \tag{25}$$

where ϕ is a function that assigns the values 0 and 1 to the two outcomes associated with each measurement. In [12] it was shown that the maximum of any Bell inequality of the form

$$\sum_{i,j} c_{i,j} E_{ij} \tag{26}$$

among all possible sets of correlations $P(a, b)$ compatible with ML is the same as the quantum optimum. This implies, as shown in Sect. 2.1, that the maximum violation of the Clauser-Horn-Shimony-Holt (CHSH) inequality [19]

$$CHSH \equiv E_{00} + E_{10} + E_{01} - E_{11} \leq 2. \tag{27}$$

allowed in ML theories is the Tsirelson bound, $2\sqrt{2}$ [20].

In this respect, a much more powerful and general result is derived in [21]: consider a bipartite non-locality scenario involving dichotomic observables, and let

$$f(E_i^A, E_j^B, E_{ij}) \leq R, \tag{28}$$

with $E_i^A = \sum_{a\in X_i}(-1)^{\phi(a)}P(a)$, $E_j^B = \sum_{b\in Y_j}(-1)^{\phi(b)}P(b)$, be a necessary condition for a microscopic distribution to be classical, i.e., let the former expression be a Bell inequality. Then, the relation

$$f\left(0, 0, \frac{2}{\pi}\arcsin\left[\frac{E_{ij} - E_i^A E_j^B}{\sqrt{1-(E_i^A)^2}\sqrt{1-(E_j^B)^2}}\right]\right) \leq R \tag{29}$$

holds for all distributions compatible with ML.

Given expression (28), inequality (29) is proven by considering the macroscopic intensity fluctuations $\bar{I}_A^i = \sum_{a\in X_i}(-1)^{\phi(a)}\bar{I}^a$, $\bar{I}_B^j = \sum_{b\in Y_j}(-1)^{\phi(b)}\bar{I}^b$ generated by many microscopic systems following a ML distribution $P(a, b)$ with two-point correlators $\{E_{ij}\}$ and mean values $\{E_i^A, E_j^B\}$. By hypothesis, $P(\bar{I}_A^i, \bar{I}_B^j)$ is a gaussian local distribution. It follows that the dichotomic distribution $P\left(\text{sgn}(\bar{I}_A^i), \text{sgn}(\bar{I}_B^j)\right)$, with two-point correlators

$$\tilde{E}_{ij} = \langle \text{sgn}(\bar{I}_A^i)\cdot \text{sgn}(\bar{I}_B^j)\rangle = \frac{2}{\pi}\arcsin\left(\frac{E_{ij} - E_i^A E_j^B}{\sqrt{1-(E_i^A)^2}\sqrt{1-(E_j^B)^2}}\right), \tag{30}$$

and average values $\langle \text{sgn}(\bar{I}_A^i)\rangle = \langle \text{sgn}(\bar{I}_B^j)\rangle = 0$, is also local and thus subject to (28).

Applying the former result to the CHSH inequality (27), for example, we deduce that any microscopic distribution compatible with ML must satisfy

$$\frac{2}{\pi}\arcsin\left(\frac{E_{11} - E_1^A E_1^B}{\sqrt{1-(E_1^A)^2}\sqrt{1-(E_1^B)^2}}\right) + \frac{2}{\pi}\arcsin\left(\frac{E_{12} - E_1^A E_2^B}{\sqrt{1-(E_1^A)^2}\sqrt{1-(E_2^B)^2}}\right) - \frac{2}{\pi}\arcsin\left(\frac{E_{21} - E_2^A E_1^B}{\sqrt{1-(E_2^A)^2}\sqrt{1-(E_1^B)^2}}\right) - \frac{2}{\pi}\arcsin\left(\frac{E_{22} - E_2^A E_2^B}{\sqrt{1-(E_2^A)^2}\sqrt{1-(E_2^B)^2}}\right) \leq 2. \tag{31}$$

This is a strengthening of the non-linear condition discovered by Landau [30], which can be derived from Eq. (31) by taking $E_i^A = E_j^B = 0$ for all i, j. As shown in [17, 21], for this particular scenario of two settings and two outputs, this condition (and the ones derived by symmetry considerations) is also sufficient to single out all no-signaling correlations compatible with ML.

It turns out that there exist microscopic distributions with biased outcomes (i.e., with some $E_i \neq 0$) attaining the Tsirelson bound which are also compatible with this condition. On the other hand, any set of quantum correlations maximizing the CHSH violation can be shown to have unbiased outcomes [22]. We thus conclude that ML alone is not sufficient to characterize the bipartite quantum set of correlations.

How does ML compare with other physical axioms at the correlation level? Many physical principles have been proposed to constrain the set of all bipartite distributions beyond the non-signalling set, like non-trivial communication complexity [9], Non-local Computation [10] and Information Causality [11]. However, so far only those correlations compatible with Information Causality (IC) have been thoroughly studied [23, 24]. Although a concrete characterization of IC correlations is still missing, current literature suggests that IC imposes weaker constraints than ML when applied to scenarios with a small number of measurement outcomes, like the CHSH scenario. Note, indeed, that, when applied to single out the set of physical two-point correlators, IC does not seem to recover the quantum set [23]. It has been shown, nevertheless, that in setups with a large number of measurement outcomes, there exist distributions compatible with ML which would allow two parties to violate IC [24]. Both principles hence seem to be independent of one another.

4.2 The Tripartite Case

Here we will briefly analyze the scenario with three separate degrees of freedom (i.e., three particles), two settings and two outcomes. The correlations will thus have the form $P(a, b, c)$.

The tripartite case is the simplest scenario where one can study the phenomenon of monogamy of correlations [25]. Consider, for instance, how much Alice and Bob can violate the bipartite CHSH Bell inequality [19], see Eq. (27), for a fixed value $CHSH_{AC}$ of the CHSH parameter with respect to Alice and Charlie. From arguments of extensibility, we know that $CHSH_{AB}, CHSH_{AC}$ cannot be both non-local (i.e., greater than 2) at the same time [25]. Moreover, as shown in [26], the no-signaling condition alone implies that

$$|CHSH_{AB}| + |CHSH_{AC}| \leq 4. \tag{32}$$

Toner and Verstraete [27] found that, in quantum theories, this inequality can be replaced by a stronger one, namely,

$$|CHSH_{AB}|^2 + |CHSH_{AC}|^2 \leq 8, \tag{33}$$

that, in particular, allows recovering the original Tsirelson bound [20]. Both inequalities are tight in the no-signaling polytope and the set of quantum correlations, respectively.

It is thus intriguing how these inequalities evolve when we move from one theory to another following the inclusion chain Classical Physics $\subset$ Quantum Physics $\subset$ ML $\subset$ NS.

To find the solution, we had to perform linear optimizations over the set of all tripartite distributions compatible with ML. In order to prove that $P(a, b, c)$ is compatible with ML, it is enough to check that the intensities generated in the three scenarios depicted in Fig. 6 (tripartite case, bipartite with recombination of separate degrees of freedom and bipartite with post-selection) admit a LHVM. This implies checking the positivity of 10 covariance matrices. We performed the corresponding SDP calculations with the MATLAB package *YALMIP* [28] in combination with *SeDuMi* [29].

The results can be seen in Fig. 7, that shows the trade-off between $CHSH_{AB}$ and $CHSH_{AC}$ for different classes of theories. The predictions of ML are disappointing for their simplicity: the no-signaling bound is just complemented with the requirement that ML only allows violations of the CHSH inequality up to $2\sqrt{2}$.

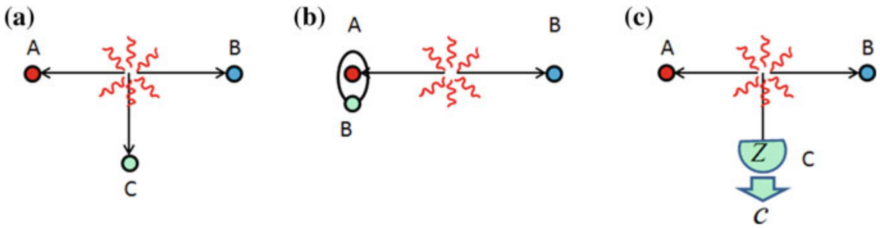
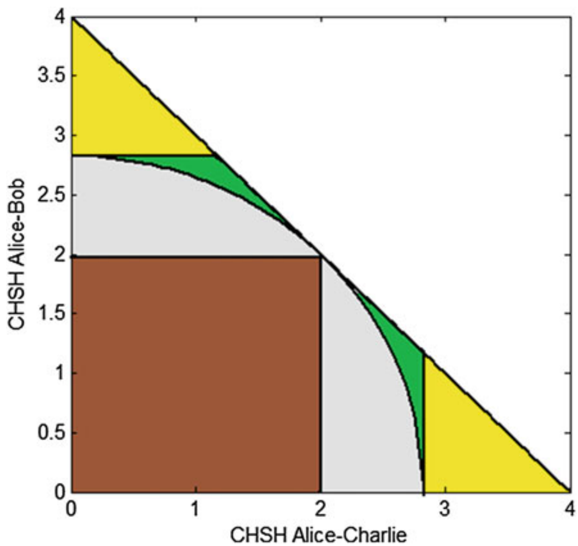


Fig. 6 The GHZ scenario. These are the only three non-trivial ways (modulo permutations of the parties) in which three separate degrees of freedom can be distributed

Fig. 7 Monogamy of bipartite correlations. The plot shows the trade-off between Alice and Bob's and Alice and Charlie's CHSH parameter in different theories. The *yellow regions* corresponds to the accessible points exclusive to the no-signaling polytope (bound (32)). The *green zone* shows the limits compatible with ML. The predictions of QM (Eq. (33)) and classical physics are denoted in *grey* and *brown*, respectively



5 Conclusion

In this chapter we have introduced the axiom of Macroscopic Locality as a fundamental principle to be satisfied by future physical theories that aim at describing our Universe. We derived a consistent set Q^{ml} of ML multipartite correlations, which we showed to contain strictly the set Q of quantum correlations. In the process, we noted the phenomenon of macroscopic non-locality activation, whereby K parties sharing a macroscopically local multipartite distribution can generate macroscopic non-locality via clustering or classical communication. We showed how to compute the boundaries of ML in standard nonlocality scenarios and connected our results with previous works on quantum correlations. Our analysis revealed that, in spite of the similarities between Q^{ml} and Q, there exist bipartite correlations compatible with ML which are impossible to approximate by means of quantum systems. This offers some hope to the possibility that quantum mechanics is experimentally falsified in the future via bipartite Bell-type experiments.

Appendix 1: Local Gaussian Distributions

In this appendix, we will show a simple criterion to decide when a set of gaussian marginal distributions admits a local hidden variable model. Let, then, Ξ be a set of variables $\Xi = \{\xi_1, \xi_2, \ldots, \xi_M\}$, and let $\{\Xi_i\}_i$ be a collection of subsets of Ξ such that, for any i, there exists a gaussian probability distribution $P_i(\Xi_i)$ with zero mean and covariance matrix γ^i for all $\xi \in \Xi_i$. We remind the reader that the covariance matrix of a set of variables $(x_1, \ldots, x_n)$ is a matrix whose entries are labeled by the variable indices and given by the expression $\gamma_{ij} \equiv \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$. The following theorem provides a characterization of all marginal probability distributions $P_i(\Xi_i)$ that arise from a global probability distribution $P(\Xi)$.

Theorem 7 *Let Ξ_i be sets of continuous variables $\xi_1, \xi_2, \ldots$, as defined above, and let $\Xi = \bigcup_i \Xi_i$. Then, there exists a joint probability density $P(\Xi)$ such that $P(\Xi_i) d\Xi_i = d\Xi_i \int P(\Xi) \prod_{\xi \in \Xi \setminus \Xi_i} d\xi$ holds for all i iff the following conditions are satisfied.*

1. *For all $\xi, \xi' \in \Xi_i \cap \Xi_j$, $\gamma^i_{\xi\xi'} = \gamma^j_{\xi\xi'}$, that is, covariance matrix entries corresponding to the same two variables have the same value.*
2. *There exists a positive semidefinite matrix γ whose entries are labeled by the elements of Ξ and such that, for any i and $\xi, \xi' \in \Xi_i$,*

$$\gamma_{\xi\xi'} = \gamma^i_{\xi\xi'}. \tag{34}$$

Notice that, in case ξ, ξ' do not both belong to one of the sets Ξ_i, the coefficient $\gamma_{\xi\xi'}$ does not appear among the entries of $\{\gamma^j\}_j$.

Proof We will first prove that, if $P(\Xi)$ exists, then conditions 1 and 2 are satisfied. First of all, if there exists a joint probability distribution for the variables in Ξ, then, for any pair of variables $\xi, \xi' \in \Xi$ the mean value $\langle \xi\xi' \rangle$ is uniquely defined, and so, if $\xi, \xi' \in \Xi_i \cap \Xi_j$, then $\gamma^i_{\xi\xi'} = \langle \xi\xi' \rangle = \gamma^j_{\xi\xi'}$. Condition 1 is thus satisfied. To see that condition 2 is also respected define the symmetric real matrix

$$\gamma_{\xi\xi'} \equiv \int \xi\xi' P(\Xi) d\Xi. \tag{35}$$

From previous considerations, it is clear that Eq. (34) applied to γ holds. To see that γ is positive semidefinite, multiply γ on both sides by an arbitrary vector $\vec{v}$. We have that

$$\vec{v}^T \gamma \vec{v} = \sum_{\xi,\xi'} v_\xi v_{\xi'} \gamma_{\xi\xi'} = \int \left(\sum_\xi v_\xi \xi \right)^2 P(\Xi) d\Xi \geq 0. \tag{36}$$

Since $\vec{v}$ was an arbitrary vector, it follows that, indeed, $\gamma \geq 0$.

Now we will prove the opposite implication: suppose that there exists a positive semidefinite matrix γ fulfilling Eq. (34). One can then check that the gaussian distribution $P(\Xi) \propto e^{-\vec{\xi}^T \gamma^{-1} \vec{\xi}/2}$ admits $P_i(\Xi_i)$ as marginals, as long as γ is invertible.

In case γ is not invertible, let $r = \text{rank}(\gamma)$, let $\{\vec{u}^i\}_{i=1}^r$ be a basis for its range; and $\{\vec{v}^i\}_{i=r+1}^M$, a basis for its kernel. Now, perform a change of variables $\xi'_i = \sum_i u^i_j \xi_j$, for $i = 1, \ldots, r$ and $\xi'_i = \sum_i v^i_j \xi_j$, for $i = r+1, \ldots, M$. Since, for all $\vec{v}^i$, $(\vec{v}^i)^T \gamma \vec{v}^i = \langle (\xi'_i)^2 \rangle = 0$, it follows that $\xi'_i = 0$, for $i = r+1, \ldots, M$. The distribution of the remaining $\{\xi'_i\}_{i=1}^r$ is thus given by $P(\{\xi'_1, \ldots, \xi'_r\}) \propto e^{-\vec{\xi}'^T (\gamma')^{MP} \vec{\xi}'/2}$. Here the symbol MP denotes the Moore-Penrose inverse. □

Appendix 2: Macroscopic Locality

Here we will study the conditions under which the intensity fluctuations generated by independent sets of multipartite microscopic correlations admit a classical model.

As explained in the main text, a macroscopic experiment will involve a source of N identical and independent K-tuples of particles, all parties are allowed to perform identical microscopic interactions over the particle beams they receive, and their detectors have a resolution that only allows measuring intensity fluctuations of the order $O(\sqrt{N})$.

Given a possible arc $s \in S \subset \vec{O}_i$, define the observable d^s_l as equal to 1 if party i's particle from the lth K-tuple impinges on detector $D(s)$ at the end of the arc s. If we label by $\bar{I}^s$ the intensity fluctuation measured by this party in detector $D(s)$, it is straightforward that

$$\bar{I}^s \propto \sum_{l=1}^{N} [d_l^s - P(s)]. \quad (37)$$

Since the precision of the party's detectors only allow it to detect fluctuations of the order $\sqrt{N}$, the ith experimentalist will be measuring a truncation (in principle, up to an arbitrary number of decimal places) of the variable

$$\bar{I}^s = \frac{\sum_{l=1}^{N} [d_l^s - P(s)]}{\sqrt{N}}. \quad (38)$$

Using the notation $P(s,s) = P(s)$, and $P(s,s') = 0$ if s, s' are locally orthogonal arcs, we have that, for any two space-like separated, locally orthogonal or identical arcs s, s',

$$\langle \bar{I}^s \bar{I}^{s'} \rangle = P(s,s') - P(s)P(s'). \quad (39)$$

By virtue of the Central Limit Theorem [31], in the limit $N \to \infty$, for any collection of local settings $\bar{S} = \{S_i\}_{i=1}^K$, the distribution of the variables $\{\bar{I}^s : s \in S_i,$ for some $i\}$ will converge to a multivariate gaussian distribution with zero mean and covariance matrix given by

$$\gamma_{ss'}^{\bar{S}} = P(s,s') - P(s)P(s'). \quad (40)$$

According to Appendix 1, for any finite number of local settings, the set of intensity fluctuations arising from a finite set of accessible local experimental settings $\mathcal{S}_{\text{acc}}$ will admit a LHVM iff there exists a positive semidefinite covariance matrix γ that has $\gamma^{\bar{S}}$ as a submatrix for all collections $\bar{S} \in \mathcal{S}_{\text{acc}}$.

References

1. J. Kofler, C. Brukner, Phys. Rev. Lett. **99**, 180403 (2007)
2. L. Hardy, arXiv:quant-ph/0101012
3. B. Dakic, C. Brukner, arXiv:0911.0695
4. Ll Masanes, M.P. Mueller, New J. Phys. **13**, 063001 (2011)
5. G. Chiribella, G.M. D'Ariano, P. Perinotti, Phys. Rev. A **84**, 012311 (2011)
6. L. Hardy, arXiv:1104.2066
7. G. de la Torre, Ll. Masanes, A.J. Short, M.P. Mueller, arXiv:1110.5482
8. D. Rohrlich, S. Popescu, Found. Phys. **24**(3), 279 (1995)
9. G. Brassard, H. Buhrman, N. Linden, A.A. Methot, A. Tapp, F. Unger, Limit on nonlocality in any world in which communication complexity is not trivial. Phys. Rev. Lett. **96**, 250401 (2006)
10. N. Linden, S. Popescu, A.J. Short, A. Winter, Phys. Rev. Lett. **99**, 180502 (2007)
11. M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, M. Zukowski, Nature **461**, 1101 (2009)
12. M. Navascués, H. Wunderlich, Proc. R. Soc. A **466**, 881–890 (2009)

13. N. Brunner, P. Skrzypczyk, Phys. Rev. Lett. **102**, 160403 (2009)
14. B.S. Tsirelson, J. Sov. Math. **36**, 557 (1987)
15. R.A. Horn, C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1999)
16. L. Vandenberghe, S. Boyd, SIAM Rev. **38**, 49 (1996)
17. M. Navascués, S. Pironio, A. Acín, Phys. Rev. Lett. **98**, 010401 (2007); M. Navascués, S. Pironio, A. Acín, New J. Phys. **10**(7), 073013 (2008)
18. J. Allcock, N. Brunner, N. Linden, S. Popescu, P. Skrzypczyk, T. Vertesi, Phys. Rev. A **80**, 062107 (2009)
19. J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett. **23**, 880 (1969)
20. B.S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980)
21. T.H. Yang, M. Navascués, L. Sheridan, V. Scarani, Phys. Rev. A **83**, 022105 (2011)
22. R. Werner, M. Wolf, QIC **1**(3), 1 (2001)
23. J. Allcock, N. Brunner, M. Pawlowski, V. Scarani, Phys. Rev. A **80**, 040103(R) (2009)
24. D. Cavalcanti, A. Salles, V. Scarani, Nat. Commun. **1**, 136 (2010)
25. Ll Masanes, A. Acín, N. Gisin, Phys. Rev. A. **73**, 012112 (2006)
26. B. Toner, Proc. R. Soc. A **465**, 59–69 (2009)
27. B. Toner, F. Verstraete, arXiv:quant-ph/0611001
28. J. Löfberg, YALMIP: A toolbox for modeling and optimization in MATLAB, http://control.ee.ethz.ch/joloef/yalmip.php
29. J.F. Sturm, SeDuMi, a MATLAB toolbox for optimization over symmetric cones, http://sedumi.mcmaster.ca
30. L. Landau, Found. Phys. **18**, 449–460 (1988)
31. H. Tijms, *Understanding Probability: Chance Rules in Everyday Life* (Cambridge University Press, Cambridge, 2004)