9 Hamiltonian Map to Conformal Modification of Spacetime Metric: Kaluza-Klein and TeVeS

In this chapter, we discuss the cosmological problem of accounting for the radiation curves of galaxies. It has commonly been assumed that the disagreement of simulations using the Newtonian form for gravitational attraction (with forces proportional to $1/r^2$ between stellar bodies) with the Tulley-Fisher radiation curves (Tulley 1977) is due to a matter distribution that is not visible through emitted light (so-called "dark matter"), but it has been difficult to find a viable candidate for what that matter should be. Milgrom (1983) has proposed (MOND) that the Newton law be modified by a law which coincides with Newton's for large accelerations, but differs from it when the accelerations are small. This suggestion has resulted in models which have been very successful in describing the galaxy radiation curves (e.g. Famaey 2012). However, as emphasized by Bekenstein (2004), it is difficult to change the basic Newton law without changing Einstein's formulation of gravity in the framework of general relativity (e.g. Weinberg 1972). He proposed that the Einstein metric $g_{\mu\nu}$ be replaced by a conformal modification $e^{-2\phi}g_{\mu\nu}$, where ϕ is a scalar field; in this way the modification proposed by Milgrom can be achieved in the post-Newtonian limit.

Although a suitable choice of ϕ has been shown to account well for the radiation curves, the gravitational distortion of light rays from other stars passing the galaxy is not described properly in this model; it would appear that the unaccounted for "matter" in the galaxy could be responsible. Bekenstein and Sanders (1994, 2004), however, have shown that the introduction of a field which, we shall call $\mathcal{U}_\mu(x)$, satisfying the normalization requirement

$$\mathcal{U}_\mu \mathcal{U}^\mu = -1 \tag{9.1}$$

permits the construction of a metric of the form

$$e^{-2\phi}(g_{\mu\nu} + \mathcal{U}_\mu \mathcal{U}_\nu) - e^{-2\phi}\mathcal{U}_\mu \mathcal{U}_\nu \tag{9.2}$$

which does make it possible to describe the bending of light passing by the galaxy as well as the galactic rotation curves without the addition of very much "dark matter" (Bekenstein 2004). It was pointed out by Contaldis et al. (2008) that if the fields $\mathcal{U}_\mu$ were taken to be gauge fields, they would suffer caustic singularities near large bodies. We show here that in the framework of the SHP theory these fields can be

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taken to be gauge fields which are nonabelian, and in the Abelian limit there are residual terms which may cancel the caustic singularities.

We start by discussing the application, originally developed to study the stability of nonrelativistic Hamiltonian dynamical systems (Horwitz 2007), by means of the introduction of a conformal metric, to the relativistic case. Introduction of the conformal modification of the metric in the relativistic framework provides a basis for Bekenstein's model. We remark that this can provide a relationship between the "dark matter" and "dark energy" (presumed responsible for the anomalous expansion of the universe) (Overduin 2008) problems. We discuss, furthermore, how the introduction of gauge fields can be taken into account in this framework and how, in the conformally modified structure, they emerge as (nonabelian in this context) Kaluza-Klein fields (Kaluza 1921). The Lorentz force due to such non-Abelian fields is computed by Hamiltonian methods (see also Land 1995), and it is suggested that small deviations of the orbits of satellites from the Newtonian orbits, such as the Pioneer (Turyshev 2006) (although some thermal effects have been implicated Turyshev 2012) may be accounted for by such nonabelian gauge fields.

9.1 Dynamics of a Relativistic Geometric Hamiltonian System

The Hamiltonian (Misner 1970)

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu, \tag{9.3}$$

with Hamilton equations (written in terms of derivatives with respect to the invariant world time τ)

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = \frac{1}{m} g^{\mu\nu} p_\nu \tag{9.4}$$

and

$$\dot{p}_\mu = -\frac{\partial K}{\partial x^\mu} = -\frac{1}{2m} \frac{\partial g^{\lambda\gamma}}{\partial x^\mu} p_\lambda p_\gamma \tag{9.5}$$

lead to the geodesic equation

$$\ddot{x}^\rho = -\Gamma^\rho{}_{\mu\nu} \dot{x}^\nu \dot{x}^\mu, \tag{9.6}$$

where what has appeared as a compatible connection form $\Gamma_\rho{}^{\mu\nu}$ is given by

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \Big(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \Big). \tag{9.7}$$

These results can be taken to be tensor relations with respect to diffeomorphisms admitted by the manifold $\{x^\mu\}$; writing the Hamiltonian in terms of (9.3), we see that the square of the invariant interval on an orbit is proportional, through the constant Hamiltonian, to the square of the corresponding interval world time, i.e.,

$$ds^2 = \frac{2}{m} K d\tau^2. \tag{9.8}$$

We shall study, in the following, a generalization of (9.3) consisting of the addition of a scalar field $\Phi(x)$. The presence of such a scalar field may be considered as a gauge compensation field for the τ derivative in the evolution term of the covariant generalization of (9.3) in the Stückelberg-Schrödinger equation (Saad 1989), an energy distribution not directly associated with electromagnetic radiation in the usual sense. We then follow the method of Horwitz (2007) to show that there is a corresponding Hamiltonian $\hat{K}$ with a conformally modified metric, and no explicit additive scalar field, which has the form of the construction of Bekenstein and Milgrom (1983), Bekenstein (2004) for the realization of Milgrom's MOND program (modified Newtonian dynamics) (Milgrom 1983) for achieving the observed galactic rotation curves. This simple form of Bekenstein's theory (called RAQUAL), which we discuss in some detail below for the sake of simplicity and clarity in the development of the mathematical method, does not properly account for causality and gravitational lensing; the theory has been further developed to include vector fields (which we shall call Bekenstein-Sanders fields) as well (*TeVeS*) (Bekenstein 2004), which has been relatively successful in accounting for these problems. It has been shown Gershon (2009), moreover, that a gauge type Hamiltonian, with Minkowski metric and both vector and scalar fields results, under a conformal map, in an effective Kaluza-Klein theory. We shall indicate here (using a general Einstein metric) how the *TeVeS* structure can emerge in terms of a Kaluza-Klein theory in this way, for which the Bekenstein-Sanders fields are considered as gauge fields. As a realization of this possibility, we exhibit a gauge transformation on the underlying quantum theory for which the vector fields, (Bekenstein 1994) which we shall call $\mathcal{U}^\mu(x)$, emerge as gauge compensation fields, such that, as required by the *TeVeS* theory, the property $\mathcal{U}^\mu \mathcal{U}_\mu = -1$ is preserved under such gauge transformations. The corresponding quantum theory then has the form of a Hilbert bundle and, in this framework, the gauge fields are of (generalized) Yang-Mills type (Yang 1954). Working in the infinitesimal neighborhood of a gauge in which the fields are Abelian, we show that in the limit the contributions from the nonabelian sector provide nonlinear terms in the field equations which may avoid the caustic singularity found by Contaldi et al. (Contaldi 2008).

For both the RAQUAL and the *TeVeS* theories, the correspondence between K and $\hat{K}$ implies a relation between the conformal factor in $\hat{K}$ and the world scalar field Φ, and thus a possible connection between the so-called dark matter problem and a dark energy distribution represented by Φ (which could be put into correspondence with the fifth gauge field (see Chap. 4) of the general Stueckelberg theory).

9.2 Addition of a Scalar Potential and Conformal Equivalence

The addition of a scalar potential to the Hamiltonian (9.3), in the form

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu + \Phi(x), \tag{9.9}$$

leads, according to the Hamilton equations, to the geodesic equation[1]

$$\ddot{x}^\rho = -\Gamma^\rho{}_{\mu\nu}\dot{x}^\nu\dot{x}^\mu - \frac{1}{m}g_{\rho\nu}\frac{\partial\Phi}{\partial x^\nu}. \tag{9.10}$$

Now, consider the Hamiltonian (we carry out the calculations explicitly here since we shall have need of some of the intermediate results)

$$\hat{K} = \frac{1}{2m}\hat{g}^{\mu\nu}(y)p_\mu p_\nu. \tag{9.11}$$

It follows from the Hamilton equations that

$$\dot{y}^\mu = \frac{\partial\hat{K}}{\partial p_\mu} = \frac{1}{m}\hat{g}^{\mu\nu}p_\nu,$$

so that

$$p_\nu = m\hat{g}_{\mu\nu}\dot{y}^\mu \tag{9.12}$$

and

$$\dot{p}_\mu = -\frac{\partial\hat{K}}{\partial y^\mu} = -\frac{1}{2m}\frac{\partial\hat{g}^{\lambda\gamma}}{\partial y^\mu}p_\lambda p_\gamma.$$

As in (9.6), it then follows that

$$\ddot{y}^\mu = -\hat{\Gamma}^\mu_{\lambda\sigma}\dot{y}^\lambda\dot{y}^\sigma, \tag{9.13}$$

where, as for (9.6),

$$\hat{\Gamma}^\mu_{\lambda\sigma} = \frac{1}{2}\hat{g}^{\mu\nu}\{\frac{\partial\hat{g}_{\nu\sigma}}{\partial y^\lambda} + \frac{\partial\hat{g}_{\nu\lambda}}{\partial y^\sigma} - \frac{\partial\hat{g}_{\lambda\sigma}}{\partial y^\nu}\}. \tag{9.14}$$

We now establish an equivalence between the Hamiltonians (9.9) and (9.11) by assuming the momenta p_μ equal at every moment τ in the two descriptions. With the constraint

$$\hat{K} = K = k, \tag{9.15}$$

if we assume the conformal form

$$\hat{g}^{\nu\sigma}(y) = \phi(y)g^{\nu\sigma}(x), \tag{9.16}$$

it follows that

$$\phi(y)(k - \Phi(x)) = k. \tag{9.17}$$

The relation (9.17) is not sufficient to construct y as a function of x, but if we impose the relation (this relation follows from requiring the momenta in each picture to be equal for all τ (Horwitz 2015a))

$$\delta x^\mu = \phi^{-1}(y)\delta y^\mu \tag{9.18}$$

[1]Note that (9.10) does not admit an equivalence principle, but (9.14), arising from (9.11) does.

between variations generated on position in the two coordinate systems, it is sufficient to evaluate derivatives of $\phi(y)$ in terms of derivatives with respect to x of the scalar field $\Phi(x)$ (Horwitz 2015a; see also Calderon 2013). We review this construction briefly below.

We remark that the construction based on Eqs. (9.11) and (9.16) admits the same family of diffeomorphisms as that of (9.9), since ϕ is scalar. Under these diffeomorphisms, both $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ are second rank tensors, and by construction of the connection forms, (9.64) and (9.13) are covariant relations.

To see how these derivatives are constructed on the constraint hypersurface determined by (9.17), let us, for brevity, define

$$F(x) \equiv \frac{k}{k - \Phi(x)}, \tag{9.19}$$

so that the constraint relation (9.15) reads

$$\phi(y) = F(x). \tag{9.20}$$

Then, since variations in x and y are related by (9.18),

$$\phi(y + \delta y) = F(x + \delta x) \cong F(x) + \delta x^{\mu} \frac{\partial F(x)}{\partial x^{\mu}}. \tag{9.21}$$

To first order in Taylor's series on the left, we obtain the relation

$$\frac{\partial \phi(y)}{\partial y^{\mu}} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^{\mu}}, \tag{9.22}$$

In agreement with the requirement that the momenta are equal for all τ (Horwitz 2015a). We may therefore define a derivative, restricted to the constraint hypersurface

$$\frac{\tilde{\partial} F(x)}{\tilde{\partial} y^{\mu}} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^{\mu}} \tag{9.23}$$

The Leibniz relation follows easily for the product of functions, it e.g., for $\phi(y)$ $g^{\mu\nu}(x)$.

In a similar way, one finds

$$\frac{\tilde{\partial}^2 F(x)}{\tilde{\partial} y^{\mu} \tilde{\partial} y^{\nu}} = \frac{\tilde{\partial}^2 F(x)}{\tilde{\partial} y^{\nu} \tilde{\partial} y^{\mu}} \tag{9.24}$$

This implies that the restricted derivative defined by (9.23) behaves as a *bona fide* derivative on the constraint hypersurface, admitting the consistent coexistence of the coordinates x and y related by (9.17). It has been shown (Horwitz 2015a) that all derivatives of $F(y)$ can be expressed in terms of $\phi(x)$ and its derivatives, and conversely, all derivatives of $\phi(x)$ can be expressed in terms of $F(y)$ and its derivatives.

In the following, we complete our argument of equivalence by reconstructing the equations of motion following from the Hamilton equations applied to (9.9), i.e., Eq. (9.10).

We begin our construction, in analogy with the procedure used in the nonrelativistic problem (Gershon 2009), by defining the new variable z_{μ} such that

$$\dot{z}_{\mu} = \hat{g}_{\mu\nu}(y) \dot{y}^{\nu}. \tag{9.25}$$

Substituting $\dot{y}^\nu = \hat{g}^{\mu\nu}(y)\dot{z}_\mu$ into (9.11), the τ derivatives of $\hat{g}^{\mu\nu}(y)$ generate terms that cancel two of the terms in $\hat{\Gamma}^\mu_{\lambda\sigma}$, leaving

$$\ddot{z}_\nu = \frac{1}{2}\frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu}\dot{y}^\lambda \dot{y}^\sigma. \tag{9.26}$$

Now, substituting for $\dot{y}^\lambda$ from (9.25), and using the identity

$$\hat{g}^{\gamma\lambda}\frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu}\hat{g}^{\sigma\rho} = -\frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}, \tag{9.27}$$

we find

$$\ddot{z}_\nu = -\frac{1}{2}\frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}\dot{z}_\gamma \dot{z}_\rho. \tag{9.28}$$

Finally, from the variational type argument we used above,

$$\begin{aligned}\hat{g}^{\rho\gamma}(y+\delta y) - \hat{g}^{\rho\gamma}(y) &= \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}\delta y^\nu \\ &= \frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu}\hat{g}^{\nu\lambda}\delta z_\lambda,\end{aligned} \tag{9.29}$$

so that

$$\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu}\hat{g}^{\nu\lambda} = \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda}$$

or

$$\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} = \hat{g}_{\nu\lambda}\frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \tag{9.30}$$

We therefore have the alternative form

$$\ddot{z}_\nu = -\frac{1}{2}\hat{g}_{\nu\lambda}\frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda}\dot{z}_\rho \dot{z}_\gamma. \tag{9.31}$$

This result constitutes a "geometric" embedding of the Hamiltonian motion induced by (9.9) in the same way as for the nonrelativistic case. Substituting the explicit form of $\hat{g}^{\rho\gamma}$ in terms of the original Einstein metric from (9.16), one obtains

$$\ddot{z}_\nu = -\frac{1}{2}g_{\nu\lambda}\frac{\partial g^{\rho\gamma}}{\partial z_\lambda}\dot{z}_\rho \dot{z}_\gamma - \frac{1}{2}\phi^{-1}g_{\nu\lambda}\frac{\partial \phi}{\partial z_\lambda}g^{\rho\gamma}\dot{z}_\gamma \dot{z}_\rho \tag{9.32}$$

The second term contains the potential field, as in the Hamilton equations, but the first term does not contain the full connection form. We may finally, however, define a "decontraction" of the connection in (9.30) using the Einstein metric. In fact, since according to (9.16), $\dot{y}^\nu = \phi \dot{x}^\nu$, and by (9.23),

$$\dot{z}_\mu = \hat{g}_{\mu\nu}\dot{y}^\nu = \phi^{-1}g_{\mu\nu}\dot{y}^\nu, \tag{9.33}$$

it follows that

$$\dot{z}_\mu = g_{\mu\nu}\dot{x}^\nu. \tag{9.34}$$

Making this substitution in (9.32) leads explicitly, taking into account the k shell constraint (9.15) and the form of (9.9), to the Eq. (9.10). We have thus completed our demonstration of the equivalence between the purely metric form of the Hamiltonian

(9.11) and the Hamilton (9.9), for which the relation (9.31) corresponds to a dynamics generated by a compatible connection form, and constitute a "geometric" embedding of the original Hamiltonian motion.

Our interest in this section has been in relating the Hamiltonian (9.9) to the simplest Bekenstein-Milgrom form of MOND, without concern in the development of this simplified case for lensing or causal effects, for which a *TeVeS* type theory would be required. In the next Section, we indicate how a *TeVeS* theory can be generated in this framework, i.e., as a result of a conformal map.

9.3 TeVeS and Kaluza-Klein Theory

In this section, we show that the *TeVeS* theory can be cast into the form of a Kaluza-Klein construction. There has recently been a discussion (Gershon 2009), from the point of view of conformal correspondence, of the equivalence of a relativistic Hamiltonian with an electromagnetic type gauge invariant form (Saad 1989; Oron 2001 and Chap. 4) (here $\eta^{\mu\nu}$ is the Minkowski metric $(-1,+1,+1,+1)$)

$$K = \frac{1}{2m}\eta^{\mu\nu}(p_\mu - ea_\mu)(p_\nu - ea_\nu) - ea_5, \tag{9.35}$$

where the $\{a_\mu\}$, as fields, may depend on the affine parameter τ as well as x^μ, and the a^5 field is necessary for the gauge invariance of the τ derivative in the quantum mechanical Stueckelberg-Schrödinger equation, with a Kaluza-Klein theory. As remarked in this work, Wesson (Overduin 2008; Liko 2005), as well as previous work on this structure (Oron 2001), have associated the source of the a_5 field with mass density. A Hamiltonian of the form

$$\hat{K} = \frac{1}{2m}\hat{g}^{\mu\nu}(p_\mu - ea_\mu)(p_\nu - ea_\nu) \tag{9.36}$$

can be put into correspondence, as in Sect. 9.2, with K by taking $\hat{g}^{\mu\nu}$ to be

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu}\frac{k}{k + ea_5}, \tag{9.37}$$

where k is the common (constant) value of K and $\hat{K}$. In this correspondence, the equations of notion generated by $\hat{K}$ through the Hamilton equations, have extra terms, beyond those provided by the connection form associated with $\hat{g}^{\mu\nu}$, due to the presence of the gauge fields. These additional terms can be identified as belonging to a connection form associated with a five dimensional metric, that of a Kaluza-Klein theory.

We may apply the same procedure to the Hamiltonian

$$K = \frac{1}{2m}g^{\mu\nu}(p_\mu - \epsilon\mathcal{U}_\mu)(p_\nu - \epsilon\mathcal{U}_\nu) + \Phi, \tag{9.38}$$

where $g^{\mu\nu}$ is an Einstein metric, Φ is a world scalar field, and $\mathcal{U}_\mu$ are to be identified with the Bekenstein-Sanders fields for which (Bekenstein 1994) $\mathcal{U}_\nu\mathcal{U}^\nu = -1$, with $\mathcal{U}^\mu = g^{\mu\nu}\mathcal{U}_\nu$.

We shall discuss in Sect. 9.4 a class of gauge transformations on the wave functions of the underlying quantum theory for which the $\mathcal{U}_\mu$ arise as gauge compensation fields.

Let us define, as in Eq. (9.37), the conformally modified metric

$$\begin{aligned}\hat{g}^{\mu\nu} &= g^{\mu\nu}\frac{k}{k-\Phi}\\ &\equiv e^{-2\phi}g^{\mu\nu}.\end{aligned} \tag{9.39}$$

The "equivalent" Hamiltonian

$$\hat{K} = \frac{1}{2m}\hat{g}^{\mu\nu}(p_\mu - \epsilon\mathcal{U}_\mu)(p_\nu - \epsilon\mathcal{U}_\nu) \tag{9.40}$$

then generates, through the Hamilton equations, an equation of motion which corresponds to the geodesic equation for an effective Kaluza-Klein metric, as in Gershon (2009).

Now, consider the Hamiltonian

$$K_K = \frac{1}{2m}\tilde{g}^{\mu\nu}p_\mu p_\nu, \tag{9.41}$$

with the Bekenstein-Sanders metric (Bekenstein 1994)

$$\tilde{g}^{\mu\nu} = e^{-2\phi}(g^{\mu\nu} + \mathcal{U}^\mu\mathcal{U}^\nu) - e^{2\phi}\mathcal{U}^\mu\mathcal{U}^\nu \tag{9.42}$$

The Hamiltonian K_K then has the form

$$K_K = e^{-2\phi}g^{\mu\nu}p_\mu p_\nu - 2\sinh 2\phi(\mathcal{U}^\mu p_\mu)^2, \tag{9.43}$$

Let us now define a Kaluza-Klein type metric of the form obtained in Gershon (2009), arising from the equations of motion generated by (9.40),

$$g^{AB} = \begin{pmatrix}\hat{g}^{\mu\nu} & \mathcal{U}^\nu\\ \mathcal{U}^\mu & g^{55}\end{pmatrix}. \tag{9.44}$$

Contraction to a bilinear form with the (5D) vectors $p_A = \{p_\lambda, p_5\}$, with indices $\lambda = \nu$ on the right and $\lambda = \mu$ on the left, one finds

$$g^{AB}p_A p_B = \hat{g}^{\mu\nu}p_\mu p_\nu + 2p_5(p_\mu\mathcal{U}^\mu) + (p_5)^2 g^{55}. \tag{9.45}$$

If we take

$$p_5 = -\frac{(p_\mu\mathcal{U}^\mu)}{g^{55}}\left(1 \pm \sqrt{1 - 2g^{55}\sinh 2\phi}\right), \tag{9.46}$$

then the Kaluza-Klein theory coincides with (9.41), i.e.,

$$K_K = \frac{1}{2m}g^{AB}p_A p_B. \tag{9.47}$$

As remarked by Wesson (Overduin 2008; Kaluza 1921), one can choose $g_{55} =$ const. for consistency with electromagnetism, while Wesson makes the more general choice of a world scalar field. Moreover, the value $g^{55} = 0$ is well defined (as in Gershon 2009).

Since the fields $\mathcal{U}^\mu$ are timelike unit vectors (Bekenstein 1994), $(p^\mu\mathcal{U}_\mu)$ corresponds, in an appropriate local frame, to the energy of the particle, close to its mass shell in the case of a nonrelativistic particle, or to the frequency in the case of on-shell photons. It clearly remains to understand more deeply the apparently *ad hoc* choice of p^5 in (9.46) in terms of a $5D$ canonical dynamics, along with the structure of the $5D$ Einstein equations for g_{AB} that follow from the geometry associated with (9.47).

9.4 The Bekenstein-Sanders Vector Field as a Gauge Field

Essential features of the Bekenstein-Sanders field (Bekenstein 1994) of the *TeVeS* theory are that it be a local field, i.e., $\mathcal{U}_\mu(x)$, and there is a normalization constraint

$$\mathcal{U}^\mu \mathcal{U}_\mu = -1, \tag{9.48}$$

so that the vector is timelike. To preserve the normalization condition (9.48) under gauge transformation, we shall study the construction of a class of gauge transformations which essentially moves the $\mathcal{U}(x)$ field on a hyperbolic surface with a Lorentz type transformation (at the point x).

If we think of our underlying quantum structure, which generates the gauge field, as a fiber bundle with base x^μ, then we must think of the transformation acting in such a way that the absolute square (norm) of the wave function attached to the base point x^μ preserves its value (Yang 1954).

An analogy can be drawn to the usual Yang-Mills gauge (Yang 1954) on $SU(2)$, where there is a two-valued index for the wave function $\psi_\alpha(x)$. The gauge transformation in this case is a two by two matrix function of x, and acts only on the indices α. The condition of invariant absolute square (probability) is

$$\sum_\alpha |\sum_\beta U_{\alpha\beta}\psi_\beta|^2 = \sum |\psi_\alpha|^2 \tag{9.49}$$

Generalizing this structure, one can take the indices α to be continuous, so that (9.49) becomes

$$\int (d\mathcal{U})| \int (d\mathcal{U}')U(\mathcal{U},\mathcal{U}')\psi(\mathcal{U}',x)|^2 = \int (d\mathcal{U})|\psi(\mathcal{U},x)|^2, \tag{9.50}$$

implying that $U(\mathcal{U},\mathcal{U}')$ is a unitary operator on a Hilbert space $L^2(d\mathcal{U})$. Since we are assuming that $\mathcal{U}_\mu$ lies on an orbit determined by (9.50), the measure is

$$(d\mathcal{U}) = \frac{d^3\mathcal{U}}{\mathcal{U}^0}, \tag{9.51}$$

i.e., a three dimensional Lorentz invariant integration measure (since $\mathcal{U}^\mu\mathcal{U}_\mu = -1$).

Moreover, the Lorentz transformation on $\mathcal{U}_\mu$ is generated by a non-commutative operator, and therefore the gauge transformation is non-Abelian. We demonstrate the resulting noncommutativity of the operator valued fields, $\mathcal{U}'$, after an infinitesimal gauge transformation of this type, explicitly below.

This construction is somewhat similar to the treatment of the electromagnetic potential vector and its time derivative as oscillator variables in the process of second quantization of the radiation field (the energy density of the field is given by these variables in the form of an oscillator). One can think of such a structure as a *Hilbert bundle* (Dixmeier 1959).

We now examine the gauge condition:

$$(p_\mu - \epsilon\mathcal{U}'_\mu)U\psi = U(p_\mu - \epsilon\mathcal{U}_\mu)\psi \tag{9.52}$$

Identifying p_μ with $-i\partial/\partial x^\mu$, and cancelling the terms $Up_\mu\psi$ on both sides, we obtain

$$\mathcal{U}'_\mu = U\mathcal{U}_\mu U^{-1} - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}U^{-1}, \tag{9.53}$$

in the same form as the Yang-Mills theory (Yang 1954). It is evident in the Yang-Mills theory, that due to the matrix nature of the second term, the field will be algebra-valued, resulting in the usual structure of the Yang-Mills nonabelian gauge theory. Here, if the transformation U is a Lorentz transformation, the numerical valued field $\mathcal{U}_\mu$ would be carried, in the first term, to a new value on a hyperbolic surface. However, the second term may well be operator valued on $L^2(d\mathcal{U})$, and thus, as in the Yang-Mills theory, $\mathcal{U}'^\mu$ would become nonabelian, implying, in general, that $\mathcal{U}$ is a nonabelian field.

It follows from (9.51) that the field strengths

$$f_{\mu\nu} = \frac{\partial\mathcal{U}_\mu}{\partial x^\nu} - \frac{\partial\mathcal{U}_\nu}{\partial x^\mu} + i\epsilon[\mathcal{U}_\mu, \mathcal{U}_\nu] \tag{9.54}$$

are related to the the field strengths in the transformed form

$$f'_{\mu\nu} = \frac{\partial\mathcal{U}'_\mu}{\partial x^\nu} - \frac{\partial\mathcal{U}'_\nu}{\partial x^\mu} + i\epsilon[\mathcal{U}'_\mu, \mathcal{U}'_\nu] \tag{9.55}$$

according to

$$f'_{\mu\nu}(x) = Uf_{\mu\nu}(x)U^{-1}, \tag{9.56}$$

just as in the finite dimensional Yang-Mills theories.

This result follows from writing out, from (9.51),

$$\begin{aligned}\frac{\partial\mathcal{U}'_\mu}{\partial x^\nu} &= \frac{\partial U}{\partial x^\nu}\mathcal{U}_\mu U^{-1} + U\frac{\partial\mathcal{U}_\mu}{\partial x^\nu}U^{-1} + U\mathcal{U}_\mu\frac{\partial U^{-1}}{\partial x^\nu} \\ &\quad -\frac{i}{\epsilon}\frac{\partial^2 U}{\partial x^\mu\partial x^\nu}U^{-1} - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}\frac{\partial U^{-1}}{\partial x^\nu},\end{aligned} \tag{9.57}$$

and subtracting the same expression with μ, ν reversed. Then add the result to

$$\begin{aligned}i\epsilon[\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= i\epsilon U[\mathcal{U}_\mu, \mathcal{U}_\nu]U^{-1} + [U\mathcal{U}_\mu U^{-1}, \frac{\partial U}{\partial x^\nu}U^{-1}] \\ &\quad +[\frac{\partial U}{\partial x^\mu}U^{-1}, U\mathcal{U}_\nu U^{-1}] - \frac{i}{\epsilon}[\frac{\partial U}{\partial x^\mu}U^{-1}, \frac{\partial U}{\partial x^\nu}U^{-1}]\end{aligned} \tag{9.58}$$

Whenever the combination

$$U^{-1}\frac{\partial U}{\partial x^\mu}U^{-1}$$

appears, it should be replaced by

$$-\frac{\partial U^{-1}}{\partial x^\mu}.$$

The result (9.56) then follows after a little manipulation.

Now, consider the possibility that this finite gauge transformation leaves $\mathcal{U}_\mu\mathcal{U}^\mu = -1$.

We write out

$$\begin{aligned}(U\mathcal{U}_\mu U^{-1} - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}U^{-1})(U\mathcal{U}^\mu U^{-1} - \frac{i}{\epsilon}\frac{\partial U}{\partial x_\mu}U^{-1}) &= -1 - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}\mathcal{U}^\mu U^{-1} \\ &\quad - \frac{i}{\epsilon}U\mathcal{U}_\mu U^{-1}\frac{\partial U}{\partial x_\mu}U^{-1} \\ &\quad - \frac{1}{\epsilon^2}\frac{\partial U}{\partial x^\mu}U^{-1}\frac{\partial U}{\partial x_\mu}U^{-1} \\ &= -1 - \frac{i}{\epsilon}\frac{\partial U}{\partial x^\mu}\mathcal{U}^\mu U^{-1} \\ &\quad + \frac{i}{\epsilon}U\mathcal{U}_\mu\frac{\partial U^{-1}}{\partial x_\mu} \\ &\quad + \frac{1}{\epsilon^2}\frac{\partial U}{\partial x^\mu}\frac{\partial U^{-1}}{\partial x_\mu}. \end{aligned} \tag{9.59}$$

It may be possible that U can be chosen to make all but the first term in (9.59) vanish, but in the case of finite gauge transformations, it is not so easy to see how to construct examples. For the infinitesimal case, it is, however, straightforward to construct a gauge function with the required properties. For

$$U \cong 1 + iG, \tag{9.60}$$

where G is infinitesimal, (9.53) becomes

$$\mathcal{U}'_\mu = \mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x^\mu} + O(G^2). \tag{9.61}$$

Then,

$$\begin{aligned}\mathcal{U}'_\mu\mathcal{U}'^\mu &\cong \mathcal{U}_\mu n_\mu + i(\mathcal{U}_\mu[G, \mathcal{U}^\mu] + [G, \mathcal{U}_\mu]\mathcal{U}^\mu) \\ &\quad + \frac{1}{\epsilon}(\frac{\partial G}{\partial x^\mu}\mathcal{U}^\mu + \mathcal{U}_\mu\frac{\partial G}{\partial x_\mu}).\end{aligned} \tag{9.62}$$

Let us take

$$\begin{aligned}G &= -\frac{i\epsilon}{2}\sum\{\omega_{\lambda\gamma}(\mathcal{U}, x), (\mathcal{U}^\lambda\frac{\partial}{\partial\mathcal{U}_\gamma} - \mathcal{U}^\gamma\frac{\partial}{\partial\mathcal{U}_\lambda})\} \\ &\equiv \frac{\epsilon}{2}\sum\{\omega_{\lambda\gamma}(\mathcal{U}, x), N^{\lambda\gamma}\}\end{aligned} \tag{9.63}$$

where symmetrization is required since $\omega_{\lambda\gamma}$ is a function of $\mathcal{U}$ as well as x, and

$$N^{\lambda\gamma} = -i(\mathcal{U}^\lambda\frac{\partial}{\partial\mathcal{U}_\gamma} - \mathcal{U}^\gamma\frac{\partial}{\partial\mathcal{U}^\lambda}). \tag{9.64}$$

This construction is valid in the initially special gauge, which we shall call the "special abelian gauge", in which the components of $\mathcal{U}^\mu$ commute. The appearance of $\mathcal{U}^\mu$ in the gauge functions is then admissible since this quantity acts on the wave functions $<\mathcal{U}, x|\psi) = \psi(\mathcal{U}, x)$ at the point x, in the representation in which the operator $\mathcal{U}^\mu$ on $L^2(d\mathcal{U})$ is diagonal.

Our investigation in the following will be concerned with a study of the infinitesimal gauge neighborhood of this limit, where the components of $\mathcal{U}^\mu$ do not commute, and therefore constitute a Yang Mills type field. We shall show in the limit that the corresponding field equations acquire nonlinear terms, and may therefore suppress the caustic singularities found by Contaldi et al. (Contaldi 2008). They found that nonlinear terms associated with a non-Maxwellian type action, such as $(\partial_\mu \mathcal{U}^\mu)^2$, could avoid this caustic singularity, so that the nonlinear terms we find as a residue of the Yang-Mills structure induced by our gauge transformation might achieve this effect in a natural way.

The second term of (9.62), which is the commutator of G with $\mathcal{U}^\mu \mathcal{U}_\mu$ vanishes, since this product is Lorentz invariant (the symmetrization in G does not affect this result).

We now consider the third term in (9.62).

$$\begin{aligned}\frac{1}{\epsilon}\Big(\frac{\partial G}{\partial x^\mu}\mathcal{U}^\mu + \mathcal{U}_\mu \frac{\partial G}{\partial x_\mu}\Big) &= \frac{1}{2}\Big\{\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}, N^{\lambda\gamma}\Big\}\mathcal{U}^\mu + \mathcal{U}^m u\Big\{\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}, N^{\lambda\gamma}\Big\} \\ &= \frac{1}{2}\Big\{N^{\lambda\gamma}\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}\mathcal{U}^\mu + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}N^{\lambda\gamma}\mathcal{U}^\mu \\ &\quad + \mathcal{U}^\mu N^{\lambda\gamma}\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \mathcal{U}^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}N^{\lambda\gamma}\Big\}\end{aligned} \tag{9.65}$$

There are two terms proportional to

$$\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}\mathcal{U}^\mu.$$

If we take (locally)

$$\omega_{\lambda\gamma}(\mathcal{U}, x) = \omega_{\lambda\gamma}(k_\nu x^\nu), \tag{9.66}$$

where $k_\nu \mathcal{U}^\nu = 0$, then

$$\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}\mathcal{U}_\mu = k_\mu \mathcal{U}^\mu \omega'_{\lambda\gamma} = 0. \tag{9.67}$$

For the remaining two terms,

$$\begin{aligned}\mathcal{U}^\mu N^{\lambda\gamma}\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}N^{\lambda\gamma}\mathcal{U}^\mu \\ &= N^{\lambda\gamma}\mathcal{U}^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \\ &\quad + [\mathcal{U}^\mu, N^{\lambda\gamma}]\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}\mathcal{U}^\mu N^{\lambda\gamma} \\ &\quad + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}[N^{\lambda\gamma}, \mathcal{U}^\mu].\end{aligned} \tag{9.68}$$

Since the commutators contain only terms linear in $\mathcal{U}_\mu$ and they have opposite sign, and cancel. The remaining terms are zero by the argument (9.67). The condition $\mathcal{U}_\mu \mathcal{U}^\mu = -1$ is therefore invariant under this gauge transformation, involving the coefficient $\omega_{\lambda\gamma}$ which is a function of the projection of x^μ onto a hyperplane orthogonal to $\mathcal{U}_\mu$, i.e., a function of $k_\mu x^\mu$, where $k_\mu \mathcal{U}^\mu = 0$. The vector k_μ, of course, depends on $\mathcal{U}_\mu$ (for example, $k_\mu = \mathcal{U}_\mu(\mathcal{U} \cdot b) + b_\mu$, for some $b_\mu \neq 0$).

We now demonstrate explicitly the nonabelian nature of the gauge fields after infinitesimal gauge transformation. With (9.61), the commutator term in (9.55) is

$$\begin{aligned}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= (\mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x^\mu})(\mathcal{U}_\nu + i[G, \mathcal{U}_\nu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x^\nu}) \\ &\quad - (\mathcal{U}_\nu + i[G, \mathcal{U}_\nu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x^\nu})(\mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon}\frac{\partial G}{\partial x^\mu}) \\ &= \frac{1}{\epsilon}\{[\mathcal{U}_\mu, \frac{\partial G}{\partial x^\nu}] - [\mathcal{U}_\nu, \frac{\partial G}{\partial x^\mu}]\} \\ &\quad + i[\mathcal{U}_\mu, [G, \mathcal{U}_\nu]] - i[\mathcal{U}_\nu, [G, \mathcal{U}_\mu]],\end{aligned} \tag{9.69}$$

where the remaining terms have identically cancelled. Note that this expression does not contain any noncommutative quantities. Now,

$$[G, \mathcal{U}_\nu] = 2i\epsilon\omega_\nu{}^\gamma \mathcal{U}_\gamma \tag{9.70}$$

and

$$[\mathcal{U}_\mu, \frac{\partial G}{\partial x^\nu}] = 2i\epsilon \mathcal{U}_\lambda \frac{\partial \omega^\lambda{}_\mu}{\partial x^\nu}. \tag{9.71}$$

The terms involving $[G, \mathcal{U}_\nu]$ and $[G, \mathcal{U}_\mu]$ therefore cancel, so that

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i\mathcal{U}_\lambda\left(\frac{\partial \omega^\lambda{}_\mu}{\partial x^\nu} - \frac{\partial \omega^\lambda{}_\nu}{\partial x^\mu}\right) \tag{9.72}$$

We have taken $\omega^\lambda{}_\mu = \omega^\lambda{}_\mu(k_\sigma x^\sigma)$, so that

$$\frac{\partial \omega^{\lambda\mu}}{\partial x^\nu} = k_\nu \omega'^\lambda{}_\mu, \tag{9.73}$$

and therefore

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i(k_\nu \omega'^\lambda{}_\mu - k_\mu \omega'^\lambda{}_\nu)\mathcal{U}_\lambda, \tag{9.74}$$

generally not zero. This demonstrates the nonabelian character of the fields. In the Abelian limit, we may take $\omega' \to 0$, but as we shall see, there is a residual nonlinearity, which depends on ω'' may remain in the field equations.

We now consider the derivation of field equations from a Lagrangian constructed with the ψ's and $f^{\mu\nu} f_{\mu\nu}$. We take the Lagrangian to be of the form (the indices are raised and lowered with $g_{\mu\nu}$)

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_m, \tag{9.75}$$

where

$$\mathcal{L}_f = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} \tag{9.76}$$

and

$$\mathcal{L}_m = \psi^*\big(i\frac{\partial}{\partial \tau} - \frac{1}{2M}(p_\mu - \epsilon\mathcal{U}_\mu)g^{\mu\nu}(p_\nu - \epsilon\mathcal{U}_\nu) - \Phi\big)\psi \quad + \quad \text{c.c.} \tag{9.77}$$

We shall be working in the infinitesimal neighborhood of the special gauge for Abelian $\mathcal{U}_\mu$, for which it has the form given in (9.59) for infinitesimal G. It is therefore not Abelian to first order, but we take its variation $\delta\mathcal{U}$ to be a c-number function, carrying the variation, to lowest order, by variation of the first term in (9.61), and not varying the part of $\mathcal{U}$ introduced by the infinitesimal gauge transformation (evaluated on the original value of $\mathcal{U}$).

In carrying out the variation of $\mathcal{L}_m$, the contributions of varying the ψ's with respect to $\mathcal{U}$ vanish due to the field equations (Stueckelberg-Schrödinger equation) obtained by varying ψ^* (or ψ), and therefore in the variation with respect to $\mathcal{U}$, only the explicit presence of $\mathcal{U}$ in (9.77) need be taken into account.

Note that for the general case of $\mathcal{U}$ operator valued, we can write

$$\psi^*(p_\mu - \epsilon\mathcal{U}_\mu)g^{\mu\nu}(p_\nu - \epsilon\mathcal{U}_\nu)\psi = g^{\mu\nu}\big((p^\mu - \epsilon\mathcal{U}^\mu)\psi\big)^*(p_\nu - \epsilon\mathcal{U}_\nu)\psi, \tag{9.78}$$

since the Lagrangian density (9.75) contains an integration over $(d\mathcal{U}')(d\mathcal{U}'')$ (considered in lowest order) as well as an integration over (dx) in the action and the operators $\mathcal{U}$ are Hermitian. In the limit in which $\mathcal{U}$ is evaluated in the special Abelian gauge (real valued), and noting that p_μ is represented by an imaginary differential operator, we can write this as

$$g^{\mu\nu}\psi^*(p_\mu - \epsilon\mathcal{U}_\mu)(p_\nu - \epsilon\mathcal{U}_\nu)\psi = -g^{\mu\nu}(p_\mu + \epsilon\mathcal{U}_\mu)\psi^*(p_\nu - \epsilon\mathcal{U}_\nu)\psi, \tag{9.79}$$

i.e., replacing explicitly p_μ by $-i(\partial/\partial x^\mu) \equiv -i\partial_\mu$, we have

$$\delta_\mathcal{U}\mathcal{L}_m = -i\frac{\epsilon}{2M}\big\{\psi^*(\partial_\mu - i\epsilon\mathcal{U}_\mu)\psi - ((\partial_\mu + i\epsilon\mathcal{U}_\mu)\psi^*)\psi\big\}\delta\mathcal{U}^\mu, \tag{9.80}$$

where we have called $g^{\mu\nu}\delta\mathcal{U}_\nu = \delta\mathcal{U}^\mu$, or

$$\delta_\mathcal{U}\mathcal{L}_m = j_\mu(\mathcal{U}, x)\delta\mathcal{U}^\mu, \tag{9.81}$$

where j_μ has the usual form of a gauge invariant current.

For the calculation of the variation of $\mathcal{L}_f$ we note that the commutator term in (9.54) is, in lowest order, a c-number function, as given in (9.74).

Calling

$$\omega'^\lambda{}_\mu\mathcal{U}_\lambda \equiv v_\mu, \tag{9.82}$$

we compute the variation of

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i(k_\nu v_\mu - k_\mu v_\nu) \tag{9.83}$$

Then, for

$$\delta_\mathcal{U}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = \delta_{\mathcal{U}_\gamma}\frac{\partial}{\partial\mathcal{U}_\gamma}[\mathcal{U}'_\mu, \mathcal{U}'_\nu], \tag{9.84}$$

we compute

$$\frac{\partial}{\partial\mathcal{U}_\gamma}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i(\frac{\partial k_\nu}{\partial\mathcal{U}_\gamma}v_\mu + k_\nu\frac{\partial v_\mu}{\partial\mathcal{U}_\gamma}) - (\mu \leftrightarrow \nu)). \tag{9.85}$$

With our choice of $k_\nu = \mathcal{U}_\nu(\mathcal{U}\cdot b) + b_\nu$,

$$\frac{\partial k_\nu}{\partial \mathcal{U}_\gamma} = \delta_\nu{}^\gamma(\mathcal{U}\cdot b) + \mathcal{U}_\nu b^\gamma, \tag{9.86}$$

so that

$$\begin{aligned}\frac{\partial}{\partial \mathcal{U}_\gamma}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= 2i(\delta_\nu{}^\gamma(\mathcal{U}\cdot b) + \mathcal{U}_\nu b_\gamma)v^\mu \\ &\quad + k_\nu \frac{\partial v_\mu}{\partial \mathcal{U}_\gamma} - (\mu \leftrightarrow \nu)) \\ &\equiv \mathcal{O}^\gamma{}_{\mu\nu},\end{aligned} \tag{9.87}$$

i.e.

$$\delta_{\mathcal{U}}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = \mathcal{O}^\gamma{}_{\mu\nu}\delta\mathcal{U}_\gamma \tag{9.88}$$

The quantity v_μ is proportional to the derivative of ω_μ^λ. In the limit that $\omega, \omega' \to 0$ (cf. (9.83)), the second derivative, ω'' which appears in $\mathcal{O}^\gamma{}_{\mu\nu}$ may not vanish (somewhat analogous to the case in gravitational theory when the connection form vanishes but the curvature does not), so that this term can contribute in limit to the special Abelian gauge.

Returning to the variation of $\mathcal{L}_f$ in (9.76), we see that

$$\delta\mathcal{L}_f = -\partial^\nu f_{\mu\nu}\delta\mathcal{U}^\mu + 2if_{\mu\nu}\delta[\mathcal{U}_\mu, \mathcal{U}_\nu], \tag{9.89}$$

where we have taken into account the fact that $[\mathcal{U}_\mu, \mathcal{U}_\nu]$ is a commuting function, and integrated by parts the derivatives of $\delta\mathcal{U}$. With (9.88) we obtain

$$\delta\mathcal{L}_f = -\partial^\nu f_{\mu\nu}\delta\mathcal{U}^\mu + 2i\epsilon f_{\lambda\sigma}\mathcal{O}^{\lambda\sigma}{}_\mu\delta\mathcal{U}^\mu \tag{9.90}$$

Since the coefficient of $\delta\mathcal{U}^\mu$ must vanish, we obtain, with (9.79), the Yang-Mills equations for the fields given the source currents

$$\partial^\nu f_{\mu\nu} = j_\mu - 2i\epsilon f_{\lambda\sigma}\mathcal{O}^{\lambda\sigma}{}_\mu, \tag{9.91}$$

which is nonlinear in the fields $\mathcal{U}_\mu$, as we have seen, even in the Abelian limit, where, from (9.80) and (9.81),

$$j_\mu = -i\frac{\epsilon}{2M}\{\psi^*(\partial_\mu - i\epsilon\mathcal{U}_\mu)\psi - ((\partial_\mu + i\epsilon\mathcal{U}_\mu)\psi^*)\psi\}. \tag{9.92}$$

We point out that this current corresponds to a flow of the matter field; the absolute square of the wave functions corresponds to an event density. The coupling ϵ is not necessarily the electron charge, and the fields $\mathcal{U}$ are not necessarily electromagnetic even in the Abelian limit. However, the Hamiltonian (9.38) leads directly to a Lorentz type force, similar in form to that generated by the Hilbert-Einstein action (see Chap. 4).

9.5 Summary

We have seen in this chapter that a map of the type discussed in Gershon (2009) of a Hamiltonian containing an Einstein metric, generating the connection form of general relativity, and a world scalar field, representing a distribution of energy on the spacetime manifold, into a corresponding Hamiltonian with a conformal metric (and compatible connection form), can account for the structure of the RAQUAL theory of Bekenstein and Milgrom (1983). Furthermore, applying this correspondence to a Hamiltonian with gauge-type structure, we have shown that one obtains a non-compact Kaluza-Klein effective metric which can account for the *TeVeS* structure of Bekenstein, Sanders and Milgrom (1989, 1994).

In order to maintain the constraint condition $\mathcal{U}_\mu \mathcal{U}^\mu = -1$ for the Bekenstein-Sanders fields, under local gauge transformations, we have introduced a class of gauge transformations on the underlying quantum theory which acts on the Hilbert bundle, quite analogous to that arising in the second quantization of the electromagnetic field (where the vector potentials and their time derivatives are considered as quantum oscillator variables) associated with the values of the gauge fields. The action of this class of gauges induces a nonabelian structure on the fields, which therefore satisfy Yang-Mills type field equations with source currents associated with matter flow. In the Abelian limit, these equations contain residual non-linear terms which may avoid the caustic singularities found by Contaldi et al. (2008) for an electromagnetic type gauge field.

The phenomenological constraints placed on the *TeVeS* variables in its astrophysical applications and on its MOND limit (Milgrom 1983) would, in principle, place constraints on the vector and scalar fields appearing in the corresponding Hamiltonian model, for which the additive world scalar field corresponds to an energy distribution, not associated with electromagnetic radiation, which could contribute to the anomalous expansion of the universe (Rañada (2003), (2004), Anderson (1998), Rosales (1999)).