3 Spin, Statistics and Correlations

We shall discuss in this chapter the basic idea of a relativistic particle with spin, based on Wigner's seminal work (Wigner 1939). The theory is adapted here to be applicable to relativistic quantum theory; in this form, Wigner's theory, together with the requirements imposed by the observed correlation between spin and statistics in nature for identical particle systems, makes it possible to define the total spin of a state of a relativistic many body system.

We shall show, furthermore, that a generalization of the construction of Wigner yields, in the framework we shall present here, a representation for tensor operators corresponding to an *invariant* decomposition in terms of irreducible representations of $SU(2)$; this procedure may be applied as well to spinorial valued operators, such as Rarita-Schwinger fields (Rarita 1941).

3.1 Relativistic Spin and the Dirac Representation

The spin of a particle in a nonrelativistic framework corresponds to the lowest dimensional nontrivial representation of the rotation group; the generators are the Pauli matrices σ_i divided by two, the generators of the fundamental representation of the double covering of $SO(3)$. The self-adjoint operators that are the generators of this group measure angular momentum and are associated with magnetic moments. Such a description is not relativistically covariant, but Wigner (1939) has shown how to describe this dynamical property of a particle in a covariant way. The method developed by Wigner provided the foundation for what is now known as the theory of induced representations (Mackey 1968), with very wide applications, including a very powerful approach to finding the representations of noncompact groups.

We shall show here how Wigner's approach can be used to describe the spin of a particle in the framework of the manifestly covariant theory of Stueckelberg, Horwitz and Piron (SHP; Stueckelberg 1941; Horwitz 1973), and how this method can be extended to describe the combined spin states of a many body system.

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In the nonrelativistic quantum theory, the spin states of a two or more particle system are defined by combining the spins of these particles at equal time using appropriate Clebsch-Gordan coefficients (Clebsch 1872) at each value of the time. The restriction to equal time follows from the tensor product form of the representation of the quantum states for a many body problem (Baym 1969; Fetter1971). For two spin $1/2$ (Fermi-Dirac) particles, an antisymmetric space distribution would correspond to a symmetric combination of the spin factors, i.e. a spin one state, and a symmetric space distribution would correspond to an antisymmetric spin combination, a spin zero state.[1] This correlation is the source of the famous Einstein-Podolsky-Rosen discussion (Einstein 1935) and provides an important model for quantum information transfer. The experiment proposed by Palacios et al. (2009) suggests that spin entanglement can occur for two particles at non-equal times; the spin carried by wave fnctions of SHP type would naturally carry such correlations over the width in t of the wave packets, and therefore the formulation we shall present here would be appropriate for application to relativistic quantum information transfer (e.g., Aharonov 1982; Hu 2012; Lin 2009; Lizier 2013).

Wigner (1939) worked out a method for defining spin for relativistic particles. This formulation is not appropriate for application to quantum theory, since it does not preserve, as we shall explain below, the covariance of the expectation value of coordinate operators. Before constructing a generalization of Wigner's method which is useful in relativistic quantum theory we first review Wigner's method in its original form, and show how the difficulties arise.

To establish some notation and the basic method, we start with the basic principle of relativistic covariance for a scalar quantum wave function $\psi(p)$. In a new Lorentz frame described by the parameters Λ of the Lorentz group, for which $p'^\mu = \Lambda^\mu_\nu p^\nu$ (we work in momentum space here for convenience), the same physical point in momentum space described in different coordinates, by arguing that the probability density must be the same, is associated with the wave function

$$\psi'(p') = \psi(p) \tag{3.1}$$

up to a phase, which we take to be unity. It then follows that as a function of p,

$$\psi'(p) = \psi(\Lambda^{-1}p). \tag{3.2}$$

Since, in Dirac's notation,

$$\psi'(p) \equiv < p|\psi' >, \tag{3.3}$$

Equation (2.67) follows equivalently by writing

$$|\psi' >= U(\Lambda)|\psi > \tag{3.4}$$

[1] See also the very informative study of Jabs 2010, and the discussion of Bennett (2015).

so that

$$\begin{aligned} < p|\psi' > &=< p|U(\Lambda)|\psi > \\ &=< \Lambda^{-1}p|\psi > \\ &= \psi(\Lambda^{-1}p), \end{aligned} \tag{3.5}$$

where we have used

$$U(\Lambda)^\dagger|p >= U(\Lambda^{-1})|p >= |\Lambda^{-1}p > .$$

To discuss the transformation properties of the representation of a relativistic particle with spin, Wigner proposed that we consider a special frame in which $p_0^\mu = (m, 0, 0, 0)$; the subgroup of the Lorentz group that leaves this vector invariant is clearly $O(3)$, the rotations in the three space in which $\mathbf{p} = 0$, or its covering $SU(2)$. Under a Lorentz boost, transforming the system to its representation in a moving inertial frame, the rest momentum appears as $p_0^\mu \to p^\mu$, but under this unitary transformation, the subgroup that leaves p_0^μ invariant is carried to a form which leaves p^μ invariant, and the group remains $SU(2)$. The 2×2 matrices representing this group are altered by the Lorentz transformation, and are functions of the momentum p^μ. The resulting state then transforms by a further change in p^μ and an $SU(2)$ transformation compensating for this change. This additional transformation is called the "little group" of Wigner. The family of values of p^μ generated by Lorentz transformations on p_0^μ is called the "orbit" of the induced representation. This $SU(2)$, in its lowest dimensional representation, parametrized by p^μ and the additonal Lorentz transformation Λ, corresponds to Wigner's covariant relativistic definition of the spin of a relativistic particle (Wigner 1937).

We now apply this method to review Wigner's construction based on a representation induced on the momentum p^μ. Let us define the momentum-spin ket

$$|p, \sigma >\equiv U(L(p))|p_0, \sigma >, \tag{3.6}$$

where $U(L(p))$ is the unitary operator inducing a Lorentz transformation of the timelike $p_0 = (m, 0, 0, 0)$ (rest frame momentum) to the general timelike vector p^μ. The effect of a further Lorentz transformation parameterized by Λ, induced by $U(\Lambda^{-1})$, can be written as

$$U(\Lambda^{-1})|p, \sigma >= U(L(\Lambda^{-1}p))U^{-1}(L(\Lambda^{-1}p))U(\Lambda^{-1})U(L(p))|p_0, \sigma > \tag{3.7}$$

The product of the last three unitary factors

$$U^{-1}(L(\Lambda^{-1}p))U(\Lambda^{-1})U(L(p)) \tag{3.8}$$

has the property that under this combined unitary transformation, the ket is transformed so that $p_0 \to p_0$, and thus corresponds to just a rotation (called the Wigner rotation), the stability subgroup of the vector p_0. This rotation can be represented by a 2×2 matrix acting on the index σ, i.e., so that

$$U(\Lambda^{-1})|p, \sigma >= U(L(\Lambda^{-1}p))|p_0, \sigma' > D_{\sigma,\sigma'}(\Lambda, p) = |\Lambda^{-1}p, \sigma' > D_{\sigma,\sigma'}(\Lambda, p). \tag{3.9}$$

where, as a representation of rotations, D is unitary. Therefore, taking the complex conjugate of

$$< \psi|U(\Lambda^{-1})|p, \sigma >=< \psi|\Lambda^{-1}p, \sigma' > D_{\sigma,\sigma'}(\Lambda, p),$$

one obtains

$$< p, \sigma|U(\Lambda)\psi >=< \Lambda^{-1}p, \sigma'|\psi > D_{\sigma',\sigma}(\Lambda p), \tag{3.10}$$

where, in this construction, we have

$$D_{\sigma',\sigma}(\Lambda, p) = \big((L(p)^{-1}\Lambda L(\Lambda^{-1}p))\big)_{\sigma',\sigma}, \tag{3.11}$$

expressed in terms of the $SL(2, C)$ matrices corresponding to the unitary transformation (3.8). This representation of the unitary transformation is a homomorphism due to the fact that this subgroup is compact, and has finite dimensional unitary representations, in particular, the one we use here (we could have chosen other representations corresponding to particles carrying intrinsic angular momentum not equal to $1/2$). The result (3.10) can be written as

$$\psi'(p, \sigma) = \psi(\Lambda^{-1}p, \sigma')D_{\sigma',\sigma}(\Lambda, p), \tag{3.12}$$

in accordance with (3.2), generalized to take into account the spin degrees of freedom of the wavefunction. The algebra of the 2×2 matrices of the fundamental representation of the group $SL(2, C)$ are isomorphic to that of the Lorentz group, and the product of the corresponding matrices provide the 2×2 matrix representation of $D_{\sigma',\sigma}(\Lambda, p)$; we may therefore write (2.77) as

$$D_{\sigma',\sigma}(\Lambda, p) = \big(L^{-1}(p)\Lambda L(\Lambda^{-1}p)\big)_{\sigma',\sigma}, \tag{3.13}$$

where L and Λ are the 2×2 matrices of $SL(2, C)$. We discuss these matrices (2×2 matrices of complex numbers with determinant unity) and the representation they provide for the Lorentz group in Appendix B.

As we have mentioned above, the presence of the p-dependent matrices representating the spin of a relativistic particle in the transformation law of the wave function destroys the covariance, in a relativistic quantum theory, of the expectation value of the coordinate operators. To see this, consider the expectation value of the dynamical variable x^μ, i.e.

$$< x^\mu >= \Sigma_\sigma \int d^4p\psi(p, \sigma)^\dagger i\frac{\partial}{\partial p_\mu}\psi(p, \sigma).$$

A Lorentz transformation would introduce the p-dependent 2×2 unitary transformation on the function $\psi(p)$, and the derivative with respect to momentum would destroy the covariance property that we would wish to see of the expectation value $< x^\mu >$.

It is also not possible, in this framework, to form wave packets of definite spin by integrating over the momentum variable, since this would add functions over different parts of the orbit, with a different $SU(2)$ at each point.

As will be described in the following, these problems were solved by inducing a representation of the spin on a timelike unit vector n^μ in place of the four-momentum, using a representation induced on a timelike vector, say, n^μ, which is independent of x^μ or p^μ (Horwitz 1975; Arshansky 1982). This solution also permits the linear superposition of momentum states to form wave packets of definite spin, and admits the construction of definite spin states for many body relativistic systems and its consequences for entanglement. In the following, we show how such a representation can be constructed, and discuss some of its dynamical implications.

To carry out this program, let us define, as in (3.6),

$$|n, \sigma >= U((L(n))|n_0, \sigma > \tag{3.14}$$

The generators of the transformations $U(\Lambda)$ act on the full vector space of both the n^μ and the x^μ (as well as p^μ). In terms of the canonical variables,

$$M^{\mu\nu} = M_n^{\mu\nu} + (x^\mu p^\nu - x^\nu p^\mu). \tag{3.15}$$

where

$$M_n^{\mu\nu} = -i\left(n^\mu \frac{\partial}{\partial n_\nu} - n^\nu \frac{\partial}{\partial n_\mu}\right) \tag{3.16}$$

The two terms of the full generator commute. Following the method outlined above, we now investigate the properties of a total Lorentz transformation, i.e.

$$U(\Lambda^{-1})|n, \sigma >= U(L(\Lambda^{-1}n))(U^{-1}(L(\Lambda^{-1}n))U(\Lambda^{-1})U(L(n)))|n_0, \sigma >. \tag{3.17}$$

Now, consider the conjugate of (3.17),

$$< n, \sigma|U(\Lambda) =< n_0, \sigma|(U(L^{-1}(n))U(\Lambda)U(L(\Lambda^{-1}n)))U^{-1}(L(\Lambda^{-1}n)). \tag{3.18}$$

The operator in the first factor (in parentheses) preserves n_0, and therefore, as before, contains an element of the little group associated with n^μ which may be represented by the matrices of $SL(2, C)$. We now define a state vector in terms of a vector-valued function $\Psi(x) \in L^2(R^4)$ for which $< n, \sigma|\Psi(x) >= \psi_{n\sigma}(x)$, so that

$$< n^0\sigma|U^{-1}(L(\Lambda^{-1}n))\Psi(x) >= \psi_{\Lambda^{-1}n\sigma}(x). \tag{3.19}$$

For $\Psi'(x) \equiv U(\Lambda)\Psi(x)$, contracting both sides of (3.18) with $\Psi(x)$, we obtain

$$\psi'_{n,\sigma}(x) = \psi_{\Lambda^{-1}n,\sigma'}(\Lambda^{-1}x)D_{\sigma',\sigma}(\Lambda, n). \tag{3.20}$$

where

$$D(\Lambda, n) = L^{-1}(n)\Lambda L(\Lambda^{-1}n), \tag{3.21}$$

with Λ and $L(n)$ the corresponding 2×2 matrices of $SL(2, C)$. Λ and $L(n)$ to be the corresponding 2×2 matrices of $SL(2, C)$.

It is clear that, with this transformation law, one may take the Fourier transform to obtain the wave function in momentum space, and conversely. The matrix D is an element of $SU(2)$, and therefore linear superpositions over momenta or coordinates maintain the definition of the particle spin, and interference phenomena for relativistic particles with spin may be studied consistently. Furthermore, if two or more particles with spin are represented in representations induced on n^μ, at a given value of n^μ on their respective orbits, their spins can be added by the standard methods with the use of Clebsch-Gordan coefficients (Clebsch 1872). This method therefore admits the treatment of a many body relativistic system with spin.

Our assertion of the unitarity of the n-dependent part of the transformation has assumed that the integral measure on the Hilbert space, to admit integration by parts, is of the form $d^4nd^4x\delta(n^\mu n_\mu + 1)$, i.e., although the timelike vector n^μ, in many applications, is degenerate, it carries a probability interpretation under the norm, and may play a dynamical role.

There are two fundamental representations of $SL(2, C)$ which are inequivalent (Boerner 1963). Multiplication by the operator $\sigma \cdot p$ of a two dimensional spinor representating one of these results in an object transforming like the second representation. Such an operator could be expected to occur in a dynamical theory, and therefore the state of lowest dimension in spinor indices of a physical system should contain both representations. As we shall emphasize, however, in our treatment of the more than one particle system, for the rotation subgroup, both of the fundamental representations yield the same $SU(2)$ matrices up to a unitary transformation, and therefore the Clebsch-Gordan decomposition of the product state into irreducible representations may be carried out independently of which fundamental $SL(2, C)$ representation is associated with each of the particles.

We now discuss the construction of Dirac spinors. An approximate treatment of the Dirac equation in interaction with electromagnetism yields a connection with spin, identified through its interaction with the magnetic field (Bjorken 1964). As we shall see, however, the particle spin is already contained in the construction of the Dirac function through the fundamental construction of Wigner, combining the two fundamental representations of $SL(2, C)$ (Arshansky 1982; Weinberg 1995).

We first remark that the defining relation for the fundamental $SL(2, C)$ matrices is

$$\Lambda^\dagger \sigma^\mu n_\mu \Lambda = \sigma^\mu (\Lambda^{-1} n)_\mu, \tag{3.22}$$

where $\sigma^\mu = (\sigma^0, \sigma)$; σ^0 is the unit 2×2 matrix, and σ are the Pauli matrices. Since the determinant of $\sigma^\mu n_\mu$ is the Lorentz invariant $n^{0^2} - \mathbf{n}^2$, and the determinant of Λ is taken to be unity in $SL(2, C)$, the transformation represented on the left hand side of (3.22) must induce a Lorentz transformation on n^μ. The inequivalent second fundamental representation may be constructed by using this defining relation with σ^μ replaced by $\underline{\sigma}^\mu \equiv (\sigma^0, -\sigma)$. For every Lorentz transformation Λ acting on n^μ, this defines an $SL(2, C)$ matrix $\underline{\Lambda}$ (we use the same symbol for the Lorentz transformation on a four-vector as for the corresponding $SL(2, C)$ matrix acting on the 2-spinors).

Since both fundamental representations of $SL(2, C)$ should occur in the general quantum wave function representing the state of the system, the norm in each n-sector of the Hilbert space must be defined as

$$N = \int d^4x (|\hat{\psi}_n(x)|^2 + |\hat{\phi}_n(x)|^2), \tag{3.23}$$

where $\hat{\psi}_n$ transforms with the first $SL(2, C)$ and $\hat{\phi}_n$ with the second. From the construction of the little group (3.21), it follows that $L(n)\psi_n$ transforms with Λ, and $\underline{L}(n)\phi_n$ transforms with $\underline{\Lambda}$; making this replacement in (3.23), and using the fact, obtained from the defining relation (3.22), that $L(n)^{\dagger^{-1}} L(n)^{-1} = \mp\sigma^\mu n_\mu$ and $\underline{L}(n)^{\dagger^{-1}} \underline{L}(n)^{-1} = \mp\underline{\sigma}^\mu n_\mu$, one finds that

$$N = \mp \int d^4x \bar{\psi}_n(x) \gamma \cdot n \psi_n(x), \tag{3.24}$$

where $\gamma \cdot n \equiv \gamma^\mu n_\mu$ (for which $(\gamma \cdot n)^2 = -1$), and the matrices γ^μ are the Dirac matrices as defined in the books of Bjorken and Drell (1964). Here, the four-spinor $\psi_n(x)$ is defined by

$$\psi_n(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} L(n)\hat{\psi}_n(x) \\ \underline{L}(n)\hat{\phi}_n(x) \end{pmatrix}, \tag{3.25}$$

and the sign $\mp$ corresponds to n^μ in the positive or negative light cone. The wave function defined by (3.25) transforms as

$$\psi'_n(x) = S(\Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \tag{3.26}$$

and $S(\Lambda)$ is a (nonunitary) transformation generated infinitesimally, as in the standard Dirac theory (see, for example, Bjorken 1964; Weinberg 1995), by $\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

The Dirac operator $\gamma \cdot p$ is not Hermitian in the (invariant) scalar product associated with the norm (3.24). It is of interest to consider the Hermitian and anti-Hermitian parts

$$\begin{aligned} K_L &= \tfrac{1}{2}(\gamma \cdot p + \gamma \cdot n\gamma \cdot p\gamma \cdot n) = -(p \cdot n)(\gamma \cdot n) \\ K_T &= \tfrac{1}{2}\gamma^5(\gamma \cdot p - \gamma \cdot n\gamma \cdot p\gamma \cdot n) = -2i\gamma^5(p \cdot K)(\gamma \cdot n), \end{aligned} \tag{3.27}$$

where $K^\mu = \Sigma^{\mu\nu} n_\nu$, and we have introduced the factor $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which anticommutes with each γ^μ and has square -1 so that K_T is Hermitian and commutes with the Hermitian K_L. Since

$$K_L^2 = (p \cdot n)^2 \tag{3.28}$$

and

$$K_T^2 = p^2 + (p \cdot n)^2, \tag{3.29}$$

we may consider

$$K_T^2 - K_L^2 = p^2 \tag{3.30}$$

to pose an eigenvalue problem analogous to the second order mass eigenvalue condition for the free Dirac equation (the Klein Gordon condition). For the Stueckelberg equation of evolution corresponding to the free particle, we may therefore take

$$K_0 = \frac{1}{2M}(K_T^2 - K_L^2) = \frac{1}{2M}p^2. \tag{3.31}$$

In the presence of electromagnetic interaction, gauge invariance under a spacetime dependent gauge transformation (we discuss the more general case of a gauge transformation depending on τ as well in the next chapter), the expressions for K_T and K_L given in (3.27), in gauge covariant form, then imply, in place of (3.31),

$$K = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}\Sigma_n^{\mu\nu} F_{\mu\nu}(x), \tag{3.32}$$

where

$$\Sigma_n^{\mu\nu} = \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu \equiv \frac{i}{4}[\gamma_n^\mu, \gamma_n^\nu], \tag{3.33}$$

where the γ_n^μ are defined in (3.37). The expression (3.32) is quite similar to that of the second order Dirac operator; it is, however, Hermitian and has no direct electric coupling to the electromagnetic field in the special frame for which $n^\mu = (1, 0, 0, 0)$ in the minimal coupling model we have given here (note that in his calculation of the anomalous magnetic moment (Schwinger 1951), Schwinger puts the electric field to zero; a non-zero electric field would lead to a non-Hermitian term in the standard Dirac propagator, the inverse of the Klein-Gordon square of the interacting Dirac equation). The matrices $\Sigma_n^{\mu\nu}$ are, in fact, a relativistically covariant form of the Pauli matrices.

To see this, we note that the quantities K^μ and $\Sigma_n^{\mu\nu}$ satisfy the commutation relations

$$\begin{aligned}
[K^\mu, K^\nu] &= -i\Sigma_n^{\mu\nu} \\
[\Sigma_n^{\mu\nu}, K^\lambda] &= -i[(g^{\mu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\mu\lambda} + n^\mu n^\lambda)K^\nu, \\
[\Sigma_n^{\mu\nu}, \Sigma_n^{\lambda\sigma}] &= -i[(g^{\nu\lambda} + n^\nu n^\lambda)\Sigma_n^{\mu\sigma} + (g^{\sigma\mu} + n^\sigma n^\mu)\Sigma_n^{\lambda\nu} \\
&\quad - (g^{\mu\lambda} + n^\mu n^\lambda)\Sigma_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\Sigma_n^{\lambda\nu}].
\end{aligned} \tag{3.34}$$

Since $K^\mu n_\mu = n_\mu \Sigma_n^{\mu\nu} = 0$, there are only three independent K^μ and three $\Sigma_n^{\mu\nu}$. The matrices $\Sigma_n^{\mu\nu}$ are a covariant form of the Pauli matrices, and the last of (3.34) is the Lie algebra of $SU(2)$ in the spacelike surface orthogonal to n^μ. The three independent K^μ correspond to the non-compact part of the algebra which, along with the $\Sigma_n^{\mu\nu}$ provide a representation of the Lie algebra of the full Lorentz group. The covariance of this representation follows from

$$S^{-1}(\Lambda)\Sigma_{\Lambda n}^{\mu\nu} S(\Lambda)\Lambda_\mu^\lambda \Lambda_\nu^\sigma = \Sigma_n^{\lambda\sigma}. \tag{3.35}$$

In the special frame for which $n^\mu = (1, 0, 0, 0))$, $\Sigma_n^{i,j}$ become the Pauli matrices $\frac{1}{2}\sigma^k$ with (i, j, k) cyclic, and $\Sigma_n^{0j} = 0$. In this frame there is no direct electric interaction with the spin in the minimal coupling model (3.33). We remark that there is, however, a natural spin coupling which becomes pure electric in the special frame, given by

$$i[K_T, K_L] = -ie\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \tag{3.36}$$

It is a simple exercise to show that the value of this commutator reduces to $\mp e\gamma^5\sigma \cdot \mathbf{E}$ in the special frame for which $n^0 = -1$; this operator is Hermitian and would correspond to an electric dipole interaction with the spin.

Note that the matrices

$$\gamma_n^\mu = \gamma_\lambda \pi^{\lambda\mu}, \tag{3.37}$$

where the projection

$$\pi^{\lambda\mu} = g^{\lambda\mu} + n^\lambda n^\mu, \tag{3.38}$$

appearing in (3.34), play an important role in the description of the dynamics in the induced representation. In (3.32), the existence of projections on each index in the spin coupling term implies that $F^{\mu\nu}$ can be replaced by ${F_n}^{\mu\nu}$ in this term, a tensor projected into the foliation subspace.

We further remark that in relativistic scattering theory, the S-matrix is Lorentz invariant (Bjorken 1964). The asymptotic states can be decomposed according to the conserved projection operators

$$\begin{aligned} P_{\pm} &= \frac{1}{2}(1 \mp \gamma \cdot n) \\ P_{E\pm} &= \frac{1}{2}(1 \mp \frac{p \cdot n}{|p \cdot n|}) \\ &\text{and} \\ P_{n\pm} &= \frac{1}{2}(1 \pm \frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}}). \end{aligned} \tag{3.39}$$

The operator

$$\frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}} \rightarrow \gamma^5 \sigma \cdot \mathbf{p}/|\mathbf{p}| \tag{3.40}$$

when $n^\mu \rightarrow (1, 0, 0, 0)$. i.e., $P_{n\pm}$ corresponds to a helicity projection. Therefore the matrix elements of the S-matrix at any point on the orbit of the induced representation is equivalent (by replacing S by $U(L(n))SU^{-1}(L(n))$) to the corresponding helicity representation associated with the frame in which n is n_0.[2]

We shall show in a later chapter how the Lorentz force can be computed. We shall, furthermore, see that the anomalous magnetic moment of the electron can be computed in this framework (Bennett 2012) without appealing to the full quantum field theory of electrodynamics.

Note that the discrete symmetries act on the wavefunctions as

$$\begin{aligned} \psi^C_{\tau n} &= C\gamma^0 \psi^*_{-\tau n}(x) \\ \psi^P_{\tau n}(x) &= \gamma^0 \psi_{\tau, -\mathbf{n}, n^0}(-\mathbf{x}, t), \\ \psi^T_{\tau n} &= i\gamma^1\gamma^3 \psi^*_{-\tau, \mathbf{n}, -n^0}(\mathbf{x}, -t), \\ \psi^{CPT}_{\tau n}(x) &= i\gamma^5 \psi_{\tau, -n}(-\mathbf{x}, -t), \end{aligned} \tag{3.41}$$

where $C = i\gamma^2\gamma^0$. The CPT conjugate wavefunction, according to its evolution in τ, moves backwards in spacetime relative to the motion of $\psi_{\tau n}$. For a wave packet with $E < 0$ components, which moves backwards in t as τ goes forward, it is the CPT conjugate wavefunction which moves forward with charge $-e$, i.e., the observed antiparticle. No Dirac sea (Dirac 1932) is required for the consistency of the theory, since unbounded transitions to $E < 0$ are prevented by conservation of K.

[2]This result is consistent with the suggestion of Aharonov (1983) that n_0 may be interpreted as corresponding to the frame of the Stern-Gerlach apparatus in which the spin state is prepared.

3.2 The Many Body Problem with Spin, and Spin-Statistics

As in the nonrelativistic quantum theory, one represents the state of an N-body system in terms of a basis given by the tensor product of N one-particle states, each an element of a one-particle Hilbert space. The general state of such an N-body system is given by a linear superposition over this basis (Fetter and Walecka 1971). Second quantization then corresponds to the construction of a Fock space, for which the set of all N body states, for all N are imbedded in a large Hilbert space, for which operators that change the number N are defined (Baym 1969). We shall discuss this structure in this section, and show, with our discussion of the relativistic spin given in the previous section, that the spin of a relativistic many-body system can be well-defined (see also, Bennett 2015).[3] In order to construct the tensor product space corresponding to the many-body system, we consider, as for the nonrelativistic theory, the product of wave functions which are elements of the same Hilbert space. In the nonrelativistic theory, this corresponds to functions at equal time; in the relativistic theory, the functions are taken to be at equal τ. Thus, in the relativistic theory, there are correlations at unequal t, within the support of the Stueckelberg wave functions. Moreover, for particles with spin we argue that in the induced representation, these function must be taken at *identical values of* n^μ, i.e., taken at the same point on the orbits of the induced representation of each particle (Horwitz 2013):

> *Identical particles must be represented in tensor product states by wave functions at equal* τ *and equal* n^μ.

The proof of this statement lies in the observation that the spin-statistics relation appears to be a universal fact of nature. The elementary proof of this statement, for example, for a system of two spin 1/2 particles, is that a π rotation of the system introduces a phase factor of $e^{i\frac{\pi}{2}}$ for each particle, thus introducing a minus sign for the two body state. However, the π rotation is equivalent to an interchange of the two identical particles. This argument rests on the fact that each particle is in the same representation of $SU(2)$, which can only be achieved in the induced representation with the particles at the same point on their respective orbits. The same argument applies for bosons, which must be symmetric under interchange (in this case the phase of each factor in a pair is $e^{i\pi}$). We therefore see that identical particles must carry the same value of n^μ, and the construction of the N-body system must follow this rule. It therefore follows that the two body relativistic system can carry a spin computed by use of the usual Clebsch-Gordan coefficients, and entanglement would follow even at unequal time (within the support of the equal τ wave functions), as

[3] Jabs (2010) has noted that, with Jacob and Wick (1959) one can rotate the eigenfunctions of momentum separately so that the momenta are collinear and thus identify the Wigner little groups; this operation leaves the helicities invariant. The spin wave function would, however, develop phases that are not controlled by the helicities alone, so this procedure is not sufficient to provide a common $SU(2)$, as we shall see below.

in the proposed experiment of Palacios et al. (2009). This argument can be followed for arbitrary N, and therefore the Fock space of quantum field theory, as we show below, carries the properties usually associated with fermion (or boson) fields, with the entire Fock space foliated over the orbit of the inducing vector n^μ.

We remark that since the relativistic S-matrix is Lorentz invariant, the matrix elements of the S-matrix in states labelled by the asymptotic projections $P_{n\pm}$ (defined in (3.39) can be replaced (by the substitution $U(L(n)SU^{-1}(L(n)$ for S) by helicities in the common frame in which $n^\mu \to (1, 0, 0, 0)$. The Lorentz transformation that achieves this acts in the same way on all of the momenta of the asymptotic states and the resulting measured cross sections for this helicity representation then correspond to a choice of frame in which the common orbit is specified to be at the point $n^\mu = (1, 0, 0, 0)$.[4]

Although, due to the Newton-Wigner problem discussed above, the solutions of the Dirac equation are not suitable for the covariant local description of a quantum theory, the functions constructed in (3.25), under the norm (3.24), can form the basis of a consistent covariant quantum theory; they describe the (off-shell) states of a *local* quantum theory.

We then start by constructing a two body Hilbert space in the framework of the relativistic quantum theory. The states of this two body space are given by linear combinations over the product wave functions, where the wave functions (for the spin (1/2) case) are given by the Dirac function of the type described in (3.25) (or, for integer spin functions), i.e.,

$$\psi_{ij}(x_1, x_2) = \psi_i(x_1) \times \psi_j(x_2), \tag{3.42}$$

where $\psi_i(x_1)$ and $\psi_j(x_2)$ are elements of the one-particle Hilbert space $\mathcal{H}$. Let us introduce the notation, often used in differential geometry, that

$$\psi_{ij}(x_1, x_2) = \psi_i \otimes \psi_j(x_1, x_2), \tag{3.43}$$

identifying the arguments according to a standard ordering. Then, without specifying the spacetime coordinates, we can write

$$\psi_{ij} = \psi_i \otimes \psi_j, \tag{3.44}$$

formally, an element of the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The scalar product is carried out by pairing the elements in the two factors according to their order, since it corresponds to integrals over x_1, x_2, i.e.,

$$(\psi_{ij}, \psi_{k,\ell}) = (\psi_i, \psi_k)(\psi_j, \psi_\ell). \tag{3.45}$$

For two identical particle states satisfying Bose-Einstein of Fermi-Dirac statistics, we must write, according to our argument given above,

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{in} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{in}], \tag{3.46}$$

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[4]This result, as mentioned above, is in accordance with Aharonov's suggestion (Aharonov 1983) that the Stern-Gerlach apparatus for preparation of the spin state is labelled by this ("rest") value n_0 of n.

where $n \equiv n^u$ is the timelike four vector labelling the orbit of the induced representation. This expression has the required symmetry or antisymmetry only if both functions are on the same points of their respective orbits in the induced representation. Furthermore, they transform under the *same* $SU(2)$ representation of the rotation subgroup of the Lorentz group, and thus for spin 1/2 particles, under a π spatial rotation (defined by the space orthogonal to the timelike vector n^μ) they both develop a phase factor $e^{i\frac{\pi}{2}}$. The product results in an over all negative sign. As in the usual quantum theory, this rotation corresponds to an interchange of the two particles, but here with respect to a "spatial" rotation around the vector n^μ. The spacetime coordinates in the functions are rotated in this (foliated) subspace of spacetime, and correspond to an actual exchange of the positions of the particles on a spcelike hyperplane, as in the formulation of the standard spin-statistics theorem. It therefore follows that the interchange of the particles occurs in the foliated space defined by n^μ, and, furthermore:

The antisymmetry of identical spin 1/2 *(fermionic) particles remains at unequal times (within the support of the wave functions). This is true for the symmetry of identical spin zero (bosonic) particles as well.*

The construction we have given enables us to define the spin of a many body system, even if the particles are relativistic and moving arbitrarily with respect to each other.

The spin of an N-body system is well-defined, independent of the state of motion of the particles of the system, by the usual laws of combining representations of $SU(2)$, i.e., *with the usual Clebsch-Gordan coefficients, if the states of all the particles in the system are in induced representations at the same point of the orbit* n^μ.

Thus, in the quark model for hadrons (Gell-Mann 1962; Ne'eman 1961), the total spin of the hadron can be computed from the spins (and orbital angular momenta projected into the foliated space) of the individual quarks using the usual Clebsch-Gordan coefficients even if they are in significant relative motion, as part of the same $SU(2)$.

This result has important implications for the construction of the exchange interaction in many-body systems. Since there is no extra phase (corresponding to integer representations of the $SU(2)$ for the Bose-Einstein case, the boson symmetry can then be extended to a covariant symmetry with important implications for Bose-Einstein condensation.

3.3 Construction of the Fock Space and Quantum Field Theory

In the course of our construction, we have seen in detail that the foliation of the spacetime follows from the arguments based in the representations of a relativistic particle with half-integer spin. However, our considerations of the nature of identical particles, and their association with the spin statistics properties observed in nature, require that the foliation persists in the bosonic sector as well, where a definite phase ($\mp$) under π rotations, exchanging two particles, must be in a definite representation of the rotation group specified by the foliation vector n^μ. We remark in this connection that the Cooper pairing (Cooper 1956) of superconductivity must be between electrons on the same point of their induced representation orbits, so that the superconducting state is defined on the corresponding foliation of spacetime as well. The resulting (quasi-) bosons have the identical particle properties inferred from our discussion of the boson sector.

The N body state of Fermi-Dirac particles can then be written as (the N body boson system should be treated separately since the normalization conditions are different, but we give the general result below)

$$\Psi_{nN} = \frac{1}{N!}\Sigma(-)^P P\psi_{nN} \otimes \psi_{nN-1} \otimes \cdots \psi_{n1}, \tag{3.47}$$

where the permutations P are taken over all possibilities, and no two functions are equal. By the arguments given above, any pair of particle states in this set of particles have the Fermi-Dirac properties. We may now think of such a function as an element of a larger Hilbert space, called the *Fock space* which contains all values of the number N. On this space, one can define an operator that adds another particle (by multiplication), performs the necessary antisymmetrization, and changes the normalization appropriately. This operator is called a *creation operator*, which we shall denote by $a^\dagger(\psi_{nN+1})$ and has the property that

$$a^\dagger(\psi_{nN+1})\Psi_{nN} = \Psi_{nN+1}, \tag{3.48}$$

now to be evaluated on the manifold $(x_{N+1}, x_N, x_{N-1} \dots x_1)$. Taking the scalar product with some $N+1$ particle state Φ_{nN+1} in the Fock space, we see that

$$(\Phi_{nN+1}, a^\dagger(\psi_{nN+1})\Psi_{nN}) \equiv (a(\psi_{nN+1})\Phi_{nN+1}, \Psi_{nN}), \tag{3.49}$$

thus defining the *annihilation* operator $a^\dagger(\psi_{nN+1})$.

The existence of such an annihilation operator, as in the usual construction of the Fock space, (e.g., Baym 1969) implies the existence of an additional element in the Fock space, the *vacuum*, or the state of no particles. The vacuum defined in this way lies in the foliation labelled by n^μ. The covariance of the construction, however, implies that, since all sectors labelled by n^μ are connected by the action of the Lorentz group, that this vacuum is an absolute vacuum for any n^μ, i.e., the vacuum $\{\Psi_{n0}\}$ over all n^μ is Lorentz invariant.

The commutation relations of the annihilation- creation operators can be easily deduced from a low dimensional example, following the method used in the nonrelativistic quantum theory. Consider the two body state (3.44), and apply the creation operator $a^\dagger(\psi_{n3})$ to create the three body state

$$\begin{aligned}\Psi(\psi_{n3}, \psi_{n2}, \psi_{n1}) = \frac{1}{\sqrt{3!}}\{&\psi_{n3} \otimes \psi_{n2} \otimes \psi_{n1} + \psi_{n1} \otimes \psi_{n3} \otimes \psi_{n2} \\ &+ \psi_{n2} \otimes \psi_{n1} \otimes \psi_{n3} - \psi_{n2} \otimes \psi_{n3} \otimes \psi_{n1} \\ &- \psi_{n1} \otimes \psi_{n2} \otimes \psi_{n3} - \psi_{n3} \otimes \psi_{n1} \otimes \psi_{n2}\}\end{aligned} \tag{3.50}$$

One then takes the scalar product with the three body state

$$\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = \frac{1}{\sqrt{3!}}\{\phi_{n3} \otimes \phi_{n2} \otimes \phi_{n1} + \phi_{n1} \otimes \phi_{n3} \otimes \phi_{n2} + \phi_{n2} \otimes \phi_{n1} \otimes \phi_{n3} - \phi_{n2} \otimes \phi_{n3} \otimes \phi_{n1} - \phi_{n1} \otimes \phi_{n2} \otimes \phi_{n3} - \phi_{n3} \otimes \phi_{n1} \otimes \phi_{n2}\} \tag{3.51}$$

Carrying out the scalar product term by term, and and picking out the terms corresponding to scalar products of some functions with the two body state

$$\frac{1}{\sqrt{2}}\{\psi_{n2} \otimes \psi_{n1} - \psi_{n1} \otimes \psi_{n2}\} \tag{3.52}$$

one finds that the action of the adjoint operator $a(\psi_{n3})$ on the state $\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1})$ is given by

$$a(\psi_{n3})\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) = (\psi_{n3}, \phi_{n3})\phi_{n2} \otimes \phi_{n1} - (\psi_{n3}, \phi_{n2})\phi_{n3} \otimes \phi_{n1} + (\psi_{n3}, \phi_{n1})\phi_{n3} \otimes \phi_{n2}, \tag{3.53}$$

i.e., the annihilation operator acts like a derivation with alternating signs due to its fermionic nature; the relation of the two and three body states we have analyzed has a direct extension to the N-body case. The action of boson annihilation-creation operators can be derived in the same way.

Applying these operators to N and $N + 1$ particle states, one finds directly their commutation and anticommutation relations

$$[a(\psi_n), a^\dagger(\phi_n)]_\mp = (\psi_n, \phi_n), \tag{3.54}$$

where the $\mp$ sign, corresponds to commutator or anticommutator for the boson or fermion operators. If the functions ψ_n, ϕ_n belong to a normalized orthogonal set $\{\phi_{nj}\}$, then

$$[a(\phi_{ni}, a^\dagger(\phi_{nj}]_\mp = \delta_{ij}, \tag{3.55}$$

Let us now suppose that the functions ϕ_{nj} are plane waves in spacetime, i.e., in terms of functions

$$\phi_{np}(x) = \frac{1}{(2\pi)^2} e^{-ip^\mu x_\mu}. \tag{3.56}$$

Then

$$(\phi_{np}, \phi_{np'}) = \delta^4(p - p'). \tag{3.57}$$

The quantum fields are then constructed as follows. Define

$$\phi_n(x) \equiv \int d^4 p a(\phi_{np}) e^{ip^\mu x_\mu}. \tag{3.58}$$

It then follows that, by the commutation (anticommutation) relations (3.52), these operators obey the relations

$$[\phi_n(x), \phi_n(x')]_\mp = \delta^4(x - x'), \tag{3.59}$$

corresponding to the usual commutation relations of bose and fermion *fields*. Under Fourier transform, one finds the commutation relations in momentum space

$$[\phi_n(p), \phi_n(p')]_{\mp} = \delta^4(p - p') \tag{3.60}$$

The relation of these quantized fields with those of the usual on-shell quantum field theories can be understood as follows. Let us suppose that the fourth component of the energy-momentum is $E = \sqrt{\mathbf{p}^2 + m^2}$, where m^2 is close to a given number, the on-shell mass of a particle. Then, noting that $dE = \frac{dm^2}{2E}$, if we multiply both sides of (3.58) by dE and integrate over the small neighborhood of m^2 occurring in both E and E', the delta function $\delta(E - E')$ integrates to unity. On the right hand side, there is a factor of $1/2E$, and we may absorb $\sqrt{dm^2}$ in each of the field variables, obtaining

$$[\phi_n(\mathbf{p}), \phi_n(\mathbf{p}')]_{\mp} = 2E\delta(\mathbf{p} - \mathbf{p}'), \tag{3.61}$$

the usual formula for on-shell quantum fields. These algebraic results have been constructed in the foliation involved in the formulation of a consistent theory of relativistic spin, therefore admitting the action of the $SU(2)$ group for a many body system, applicable for unequal times.

It is clear from the construction of the Fock space that fields associated with different values of n^μ commute. The basis for the commutation relations is the creation and annihilation of (wave function) factors in the tensor product space; distinct values of n^μ therefore correspond to different species.

In the scalar product between states in the Fock space, one must complete the scalar products between functions by integrating over $\frac{d^3n}{n^0}$. A single value of n^μ in the product would have zero measure, so to compute probability amplitudes, one must construct wave packets over n^μ; these carry suitable weights for normalization. If the set $\{n\}$ is not a superselection rule, there would be transition matrix elements of observable connecting different values, and the form of the wave packets could play a physical role.

3.4 Induced Representation for Tensor Operators

In the previous sections, we have discussed the induced representation for wave functions of a particle with spin, and for the associated quantum fields. The five dimensional electromagnetic field potentials, obtained as gauge compensation fields, contain a Lorentz scalar field and a Lorentz four vector. In our discussion of statistical mechanics in Chap. 10, we are obliged to consider the problem of black body radiation. As we shall see, the relativistic Bose-Einstein distribution has a very similar form to the distribution function obtained from nonrelativistic methods, and therefore the specific heat calculations are very similar. However, the usual argument for the number of polarizations of the field, based on dimensionality minus two, corresponding to the constraint of the Gauss law and a gauge condition, resulting in two polarizations for the usual Maxwell field, but suggest three polarization states for the 5D fields. Indeed, in a discussion of the canonical second quantization of the 5D electromagnetic fields, it was found (Shnerb 1993) that there are three polarizations with either $O(3)$ or $O(2, 1)$ symmetry. We discuss in Chap. 10 a second asymptotic gauge condition for the induced representation for Lorentz tensor fields (leaving aside for the moment the Lorentz scalar component), which exhibits explicitly the $SO(3)$ representations of the tensor operators in an invariant way, thus making the polarization states accessible for classification. In this way we shall be able to describe the black body radiation in a way consistent with experiment (with the two degrees of freedom corresponding to the physical intrinsic angular of the photons), as well as to be able to explicitly characterize higher rank tensors, and their associated second quantized forms, according to their angular momentum content (Horwitz 2015).

We concentrate in the following on the vector fields; higher rank tensors transform under the direct product of the representations contained in each of the indices.

The transformation law for a vector field is constructed by the Wigner type procedure for a general tensor operator $A(x, n, \sigma)$ through the definition (we leave out the x dependence since it undergoes several transformations which must be followed eventually)

$$A(n, \sigma) \equiv U(L(n))A(n_0, \sigma)U^{-1}(L(n)), \tag{3.62}$$

where $U(L(n))$, as above, is the unitary representation of the Lorentz transformation $L(n)$ taking $n_0 = (1, 0, 0, 0)$ into the timelike vector n.

Then, as for the wave functions,

$$\begin{aligned} &U(\Lambda)A(x, n, \sigma)U^{-1}(\Lambda) \\ &\quad = U(\Lambda)U(L(n))U^{-1}(L(\Lambda n))U(L(\Lambda n))A(n_0, \sigma) \\ &\quad U(L(\Lambda n))^{-1}U(L(\Lambda n))U^{-1}(L(n))U^{-1}(\Lambda) \end{aligned} \tag{3.63}$$

The first three unitary factors induce a rotation in $SU(2)$ (we must remember that they act on the x variable as well; this can be taken into account separately). The σ index is transformed by the compact Wigner rotation (as an $SL(2, C)$ matrix)

$$\mathcal{D}(\Lambda n) = L(\Lambda n)L(n)^{-1}\Lambda \tag{3.64}$$

Writing the σ index as the pair m', m'', if we use direct product of two $SL(2, C)$'s to represent this sequence, we may supply the appropriate Clebsch-Gordan coefficients (Edmonds, see Mackey (1968)) $C(1, m|\frac{1}{2}m', \frac{1}{2}m'')$ to form the angular momentum $L = 1$ representation, and $C(0, m|\frac{1}{2}m', \frac{1}{2}m'')$ to form the $L = 0$ representation. These just correspond to predetermined linear combinations over the indices. In this way, we have constructed transformations of the tensor operator in terms of irreducible representations $L = 1$ and $L = 0$ of the rotation group in an invariant decomposition.

We may reconstitute the four vector by returning to the $SL(2, C)$ representations through application of the inverse of the Clebsch Gordan coefficients,taking explicitly into account the fact that the σ index is really a pair of indices for the $SL(2, C)$ representation of the tensor operator A_n on the orbit of the induced representation:

$$A(n) = \begin{pmatrix} A_0(n) + A_3(n) & A_1(n) - iA_2(n) \\ A_1(n) + iA_2(n) & A_0(n) - A_3(n) \end{pmatrix} \tag{3.65}$$

The determinant corresponds to the invariant $A_\mu A^\mu$. Left and right multiplying by the two by two nonunitary matrices of determinant unity, $S^\dagger(\Lambda)$ and $S(\Lambda)$ which are representations in $SL(2, C)$ of the unitary Lorentz transformations, and include as well the generators of the transformation of n^μ along the orbit one may reconstruct the representation of $U^\dagger(\Lambda) = U^{-1}(\Lambda)$ and $U(\Lambda)$ on the Hilbert space.Let us now define (here we write $SL(2, C)$ symbols to stand for the full unitary action for brevity. i.e., including the transformation on n)

$$\hat{A}_n = L(n) A_n L^{-1}(n), \tag{3.66}$$

so that under a Lorentz transformation

$$\begin{aligned} \hat{A}_n &= L(n) A_n L^{-1}(n) \to \mathcal{D}^{-1}(\Lambda n) L(\Lambda n) A_\Lambda n L^{-1}(\Lambda n) \mathcal{D}^{-1}(\Lambda n) \\ &= \Lambda^{-1} L(n) L(\lambda n)^{-1} \big(L(\Lambda n) A_{\Lambda n} l(\Lambda n)^{-1} \big) L(\Lambda n) L(n)^{-1} \Lambda \\ &= \Lambda^{-1} \big(L(n) A_{\Lambda n} L^{-1}(n) \big) \Lambda \\ &= \Lambda^{-1} \hat{A}_{\Lambda n} \Lambda, \end{aligned} \tag{3.67}$$

transforming under the $SL(2, C)$ matrix Λ along the orbit. The matrix

$$\hat{A}(n) = \begin{pmatrix} \hat{A}_0(n) + \hat{A}_3(n) & \hat{A}_1(n) - i\hat{A}_2(n) \\ \hat{A}_1(n) + i\hat{A}_2(n) & \hat{A}_0(n) - \hat{A}_3(n) \end{pmatrix} \tag{3.68}$$

then corresponds to the four-vector $\hat{A}(n)_\mu$.

This construction may be directly applied to tensor operators of any rank (with mixed tensor-spinor indices as well), explicitly displaying the angular momentum content of such operators through the direct product of the invariant decomposition of each index into angular momentum one and zero (or half integer) components. The theory of recoupling of angular momentum states (Biedenharn (1981); Racah (1942)) applies to this construction as well.

Appendix B

We describe here some of the essential properties of the 2×2 matrices thaty constitute the fundamental representation of the group $SL(2, C)$. Consider the Hermitian matrix

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{3.69}$$

The determinant of this matrix is

$$\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2, \tag{3.70}$$

This determinant is the invariant quadratic form of special relativity. The matrix X may be written as

$$X = x^\mu \sigma_\mu, \tag{3.71}$$

where (1 is the unit matrix)

$$\sigma_\mu = (1, \sigma), \tag{3.72}$$

where σ corresponds to the vector constructed of the three Pauli matrices. The matrix

$$\tilde{X} = x^{\mu}\tilde{\sigma}_{\mu}, \tag{3.73}$$

where

$$\tilde{\sigma}_{\mu} = (1, -\sigma) \tag{3.74}$$

clearly has the same determinant as X. However, there is no unitary transformation that can map σ_μ to $\tilde{\sigma}_\mu$. Unitary 2×2 transformations leave the unit matrix invariant, and σ can be rotated, but not reflected in the sign of all three components (this discrete operation is a parity reflection). The two fundamental representations that we can construct in this way for the Lorentrz group are therefore inequivalent. If we multiply X by some matrix in a congruency

$$X' = SXS^{\dagger} \tag{3.75}$$

we obtain a matrix of the same form as in Eq. (3.69), but with x^μ replaced by $x^{\mu\prime}$. This follows from the fact that an arbitrary 2×2 Hermitian matrix, say, where

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \tag{3.76}$$

where a and c are real, can always be expressed in the form of (3.69), where

$$\begin{aligned} x^0 &= \frac{1}{2}(a+c) \\ x^1 &= \frac{1}{2}(b^* + b) \\ x^2 &= \frac{1}{2i}(b^* - b). \\ x^3 &= \frac{1}{2}(a-c) \end{aligned} \tag{3.77}$$

For the second representation, defined by (3.73), (3.74), we have

$$\begin{aligned} x^0 &= \frac{1}{2}(a+c) \\ x^1 &= -\frac{1}{2}(b^* + b) \\ x^2 &= \frac{1}{2i}(b - b^*). \\ x^3 &= \frac{1}{2}(c-a) \end{aligned} \tag{3.78}$$

The conjugacy (3.75) can therefore only change x^μ to $x^{\mu\prime}$.

These matrices therefore form a representation of $SL(2, C)$ if they have determinant unity, since this implies that

$$\det X = \det X', \tag{3.79}$$

i.e., the two quadratic forms satisfy (equally valid for both representations)

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2, \tag{3.80}$$

corresponding to the defining invariance of the Lorentz group.

These two inequivalent representations, as explained in the chapter, enter into the construction of the four dimensional spinor representation of Dirac.