

Fundamental Theories of Physics 180

Lawrence P. Horwitz

# Relativistic Quantum Mechanics

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Volume 180

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# Relativistic Quantum Mechanics

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# Introduction and Some Problems Encountered in the Construction of a Relativistic Quantum Theory

# 1

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## 1.1 States in Relativistic Quantum and Classical Mechanics

One of the deepest and most difficult problems of theoretical physics in the past century has been the construction of a simple, well-defined one-particle theory which unites the ideas of quantum mechanics and relativity. Early attempts, such as the construction of the Klein-Gordon equation and the Dirac equation were inadequate to provide such a theory since, as shown by Newton and Wigner (1949), they are intrinsically non-local, in the sense that the solutions of these equations cannot provide a well-defined local probability distribution. This result will be discussed in detail below. Relativistic quantum field theories, such as quantum electrodynamics, provide a manifestly covariant framework for important questions such as the Lamb shift and other level shifts, the anomalous moment of the electron and scattering theory, but the discussion of quantum mechanical interference phenomena and associated local manifestations of the quantum theory are not within their scope; the one particle sector of such theories display the same problem pointed out by Newton and Wigner since they satisfy the same one-particle field equations.

On the other hand, the nonrelativistic quantum theory carries a completely local interpretation of probability density; it can be used as a rigorous basis for the development of nonrelativistic quantum field theory, starting with the construction of tensor product spaces to build the Fock space, and on that space to define annihilation and creation operators (e.g., Baym 1969). The development of a manifestly covariant single particle quantum theory, with local probability interpretation, could be used in the same way to develop a rigorous basis for a relativistic quantum field theory which carries such a local interpretation. A central problem in formulating such a theory is posed by the requirement of constructing a description of the quantum state of an elementary system (e.g., a “particle”) as a manifestly covariant function on a manifold of observable coordinates which belongs to a Hilbert space. The essential properties of the quantum theory, such as the notions of probability, transition amplitudes, linear superposition, observables and their expectation values, are realized in terms of the structure of a Hilbert space.

Nonrelativistic quantum mechanics, making explicit use of the Newtonian notion of a universal, absolute time, provides such a description in terms of a square integrable function over spatial variables at a given moment of this Newtonian time. This function is supposed to develop dynamically, from one moment of time to another, according to Schrödinger's equation, with some model Hamiltonian operator for the system. The theory furthermore satisfies the property of manifest covariance under the Galilean group.

This non-relativistic description of a state is, however, inconsistent with special relativity from both mathematical and physical points of view. The wave function  $\psi_t(\mathbf{x})$ , as a function of spatial variables, and parametrized by the Newtonian time, described in a frame in inertial motion with respect to another and related to it by a Lorentz transformation, undergoes a transformation which makes its interpretation in the new frame very difficult. In particular, if an event is predicted by this function with a certain probability to take place at the point  $\mathbf{x}$  at the time  $t$  in the original frame, that event should occur with the same probability, as seen in the new frame, at the point  $\mathbf{x}'$  at the time  $t'$ . According to the structure of the Lorentz transformation, the time  $t'$  depends on the location of the point  $\mathbf{x}$  in the original frame as well as  $t$ , so that it is inconsistent to label the wave functions in the new frame according to  $t'$ , now no longer a parameter, but partly dependent on the variable  $\mathbf{x}$ , with a value associated with the probability distribution defined by the original wave function. Since the Hilbert spaces associated with different times are distinct, the transformed function therefore loses its interpretation as the description of a state.

The situation for classical nonrelativistic mechanics is quite analogous; the state of a system is described by a set of canonical coordinates and momenta (the variables of the phase space) at a given time. These canonical variables develop in time according to the first order Hamilton equations of motion. The variables of the phase space, under the transformations of special relativity, are mapped into a new set in which the time parameter for each of them depends on the spatial location of the points; in addition, there is a structural lack of covariance of the phase space variables themselves (as for the quantum wave function, they become mixed with the time parameter).

On the other hand, observed interference phenomena, such as the Davisson-Germer experiment (Davisson 1927), showing the interference pattern due to the coherence of the wave function over the spatial variables at a given time, clearly should remain when observed from a moving frame. In this case, the parts of the wave function that interfere appear to pass the scattering centers (or slits, in a double slit experiment) at different times, and would not be coherent in the framework of the nonrelativistic theory. Hence one would expect that there is a more general, covariant, description of the state of a system, with Hilbert space based on a scalar product of the form, for example, for scalar wave functions,  $\int d^4x \psi_\tau^*(\mathbf{x}, t) \psi_\tau(\mathbf{x}, t)$  (for which the time  $t$  is considered as an observable), instead of the nonrelativistic form  $\int d^3\mathbf{x} \psi_\tau^*(\mathbf{x}) \psi_\tau(\mathbf{x})$ , where  $\tau$  is a parameter that we shall discuss below, which would predict such an interference pattern, modified only by the laws of special relativity when observed from a moving frame. In the succeeding chapters, I shall

discuss such a theory based on the original work of Stueckelberg (1941) and Horwitz and Piron (1973), and describe some important results that have been achieved in this framework.

## 1.2 The Problem of Localization for the Solutions of Relativistic Wave Equations

Attempts to take into account the required relativistic covariance of the quantum theory by means of relativistic wave equations such as the Klein-Gordon equation (Schrödinger 1926) for spin zero particles, and the Dirac equation (Dirac 1930) for spin 1/2 particles, have not succeeded in resolving the difficulties associated with the definition and evolution of quantum states. These equations are of manifestly covariant form, with the potential interpretation of providing a description of a quantum state, with spatial properties, in each frame, evolving according to the time parameter associated with that frame. The well-known problem posed by the lack of a positive definite probability density for the Klein-Gordon equation (Schweber 1964) is formally managed by passing to the second quantized formalism (Pauli 1934); the Dirac equation admits a positive definite density, but the problem of localization remains. In both cases, in the second quantized formalism, the vacuum to one particle matrix element of the field operator, which should have a quantum mechanical interpretation (the one-particle sector), poses the same problem of localization. Predictions of particle detection which follow from the formation of interference patterns remain ambiguous in this framework.

Foldy-Wouthuysen type (Foldy 1950) transformations (for both spin zero and spin 1/2 cases) restore the local property of the wave functions, but in this representation, manifest Lorentz covariance is lost. It is clear that the problem of localization is a fundamental difficulty in realizing a covariant quantum theory by means of the usual wave equations confining the energy momentum to a definite value of mass  $m$ . I describe the problem of localization in the following.

Newton and Wigner (1949), showed that the solution  $\phi(x)$ , for example, of the Klein-Gordon equation, cannot have the interpretation of an amplitude for a *local* probability density. The function  $\phi_0(x)$ , corresponding to a particle localized at  $\mathbf{x} = 0$ , at  $t = 0$ , has support in a range of  $\mathbf{x}$  of order  $1/m$ , where  $m$  is the mass of the particle. The argument of Newton and Wigner is as follows. The Klein-Gordon equation (we use indices  $\mu = 0, 1, 2, 3$  for time and space, with Minkowski metric  $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ ;  $x \equiv t, \mathbf{x}$ ), and  $\partial_\mu = \partial/\partial x^\mu$ , with  $\hbar = c = 1$ )

$$(-\partial_\mu \partial^\mu + m^2)\phi(x) = (\partial_t^2 - \nabla^2 + m^2)\phi(x) = 0 \quad (1.1)$$

has the conserved current

$$J_\mu(x) = \frac{i}{2}(\phi^*(x)\partial_\mu\phi(x) - (\partial_\mu\phi^*(x))\phi(x)). \quad (1.2)$$

The scalar function  $\phi(x)$  has the Fourier representation

$$\begin{aligned}\phi(x) &= \int d^4 p \delta(-p_\mu p^\mu - m^2) e^{i p^\mu x_\mu} \phi(p) \\ &= \int d^4 p \delta\left(\left(E - \sqrt{\mathbf{p}^2 + m^2}\right)\left(E + \sqrt{\mathbf{p}^2 + m^2}\right)\right) e^{i \mathbf{p} \cdot \mathbf{x} - i E t} \phi(\mathbf{p}, E) \quad (1.3) \\ &= \int \frac{d^3 \mathbf{p}}{2E} \{e^{i \mathbf{p} \cdot \mathbf{x} - i E t} \phi(\mathbf{p}, E) + e^{i \mathbf{p} \cdot \mathbf{x} + i E t} \phi(\mathbf{p}, -E)\},\end{aligned}$$

where in the last equality,  $E \equiv +\sqrt{\mathbf{p}^2 + m^2}$ , and, with the  $\delta$ -function in the first term, we have confined the integration to the “mass shell”  $m$ . The two terms in the first equality correspond to the contributions from the positive and negative values of energy in the integration. Assuming that the wave function has contributions only from positive energy, the scalar product may be derived from the positive definite norm obtained by integrating the fourth component of the current (1.2) over all space (a Lorentz invariant construction), i.e., using just the first term of (1.3) (containing the positive energy part)

$$\int d^3 \mathbf{x} J_0(x) = \int \frac{d^3 \mathbf{p}}{2E} |\phi(\mathbf{p}, E)|^2, \quad (1.4)$$

This norm is associated with a scalar product

$$(\phi_1, \phi_2) = \int \frac{d^3 \mathbf{p}}{2E} \phi_1^*(\mathbf{p}) \phi_2(\mathbf{p}). \quad (1.5)$$

Newton and Wigner then assume that  $\phi_1 \equiv \phi_0$  corresponds to the wave function in momentum space describing a particle known to be with certainty at the point  $\mathbf{x} = 0$ , and  $\phi_2 = e^{i \mathbf{p} \cdot \mathbf{a}} \phi_0$ , i.e., translated by  $\mathbf{a}$ . Since the two points are separated for  $\mathbf{a} \neq 0$ , the scalar product must be zero. It is a basic theorem in quantum mechanics that two macroscopically separated systems are in orthogonal quantum states. It then follows from (1.5) that

$$\int \frac{d^3 \mathbf{p}}{2E} |\phi_0(\mathbf{p})|^2 e^{i \mathbf{p} \cdot \mathbf{a}} = 0. \quad (1.6)$$

This result has the form of a Fourier transform of the function  $|\phi(\mathbf{p})|^2/2E$  which must vanish for all  $\mathbf{a} \neq 0$ , and therefore it must be a constant. Newton and Wigner argue that since it must be a representation of the Poincaré group, up to an overall constant phase,

$$\phi_0(\mathbf{p}) = C \sqrt{2E},$$

where  $C$  is some constant, and therefore the state of a particle known to be precisely at the point  $\hat{\mathbf{x}}$  is

$$\phi_{\hat{\mathbf{x}}}(\mathbf{p}) = C \sqrt{2E} e^{-i \mathbf{p} \cdot \hat{\mathbf{x}}}. \quad (1.7)$$

This implies that the wave function in space, the inverse Fourier transform (with weight factor  $1/2E$  is given by

$$\phi_{\mathbf{x}}(\mathbf{x}) = C \int \frac{d^3 \mathbf{p}}{\sqrt{2E}} e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (1.8)$$

This function, due to the momentum dependent denominator, is not localized, but rather spread out in a somewhat oscillatory way (a form of Bessel function), with a width for the central peak of the order of  $1/m$ . One learns from this that a particle with very small mass is very nonlocalized (this result gives rise to the common statement that the photon is not a localized particle). The operator for which the wave function (1.7) has eigenvalue  $\mathbf{x}$  is

$$\mathbf{x}_{NW} = i \left( \frac{\partial}{\partial \mathbf{p}} - \frac{\mathbf{p}}{2E^2} \right), \quad (1.9)$$

known as the Newton-Wigner operator. It is a Hermitian operator in the scalar product (1.5), the second term compensating for the derivative of the weight factor in the process of integrating by parts. We remark that for the scalar function discussed above, the Foldy Wouthuysen transformation corresponds to a map on the vector space by the factor  $1/\sqrt{2E}$ , which returns the scalar product to the usual form, and the representation of  $\mathbf{x}$  as  $i\partial/\partial\mathbf{p}$  as well as the locality of the theory, but, as in the case of the Dirac spinor theory, destroys its covariance. We remark that in the limit  $c \rightarrow \infty$ , i.e., the nonrelativistic limit, the momentum dependence in the denominator of (1.8) becomes negligible, and the wave function goes over to the local Schrödinger form.

One concludes from this discussion that the Klein Gordon wave function cannot represent a proper quantum theory, since the square of the wave function, which should correspond to a probability distribution, does not vanish in regions where the particle is known with certainty not to be present.

A similar conclusion was found for the solutions of the spin 1/2 Dirac equation (Newton 1949).

In this chapter we have discussed some of the fundamental issues involved in developing a relativistic quantum theory which have been encountered historically. We shall see in the next chapter that these difficult conceptual problems have a simple and natural resolution in the framework of a consistent manifestly covariant quantum theory. We furthermore discuss a relation closely related to the Newton-Wigner problem derived by Landau and Peierls (1931) that further illustrates the utility and content of the relativistic theory.

In Chap. 3 we treat the induced representation for the spin of a relativistic particle in the framework of the relativistic quantum theory, and discuss the associated quantum field theory for identical particles. It is shown that there is necessarily a universality of the orbit parameter on the whole set of identical particles, and that the nonrelativistic Clebsch-Gordan coefficients may be applied to compute angular momentum states independently of the state of motion of the particles.

In Chap. 4, we discuss the 5D gauge fields associated with the Stueckelberg theory. Along with the current of charged events, the field equations of the 5D theory reduce

to the standard Maxwell form under integration over the invariant world time parameter. The Lorentz force, however, is not of linear form, and therefore integration over  $\tau$  introduces a convolution, indicating that the particle does not stay on “mass shell” during the interaction.

The generalization of the classical radiation reaction problem for the relativistic charged particle is then formulated, and it is shown that the resulting Abraham-Dirac-Lorentz (Abraham 1903) equation is highly nonlinear, and the solution has chaotic behavior. Although it is highly unstable, as is the solution of the standard Abraham-Dirac-Lorentz equation (which has the so-called runaway solutions), the attractor that exists in this very non-linear equation appears to stabilize the macroscopic behavior of the classical solutions, as well as to provide a mechanism, under certain conditions, for the stability of the observed mass of a charged particle.

We also show in Chap. 4 how a simple description of flavor oscillations for neutrinos can emerge from a simple extension of the basic Stueckelberg semi-classical picture. The Lorentz force for both abelian and non-abelian gauge fields are treated.<sup>1</sup>

In Chap. 5, we shall show that the two-body bound state in an invariant phenomenological action-at-a-distance potential has a solution with spectrum in agreement with the corresponding non-relativistic two body problems, up to relativistic corrections, showing that the theory is a proper generalization of the non-relativistic Schrödinger quantum theory. The two body scattering amplitude is discussed in Chap. 7, providing further insights into how the relativistic theory can provide results consistent with the usual nonrelativistic structure.

In Chap. 6, we describe the experiment of Lindner et al. (2005) which demonstrates the existence of coherence in time, a fundamental property of the covariant relativistic theory. Calculating the effect in the framework of the covariant quantum theory, using the conditions of the experiment, one finds very good agreement with experimental results. We discuss in some depth as well why this result is not consistent with the nonrelativistic quantum theory. The formulas were actually obtained many years earlier by Horwitz and Rabin (1976) in their early investigations of the consequences of the relativistic theory, but at that time the necessary experimental tools for confirming the predictions were not available. A similar, but somewhat more complex problem occurs in the proposed experiment of Palacios et al. (2009) where spin correlations presumed to be maintained between particles at different times. The application of Wigner’s induced representation theory (Wigner 1939), discussed on Chap. 3, to the spin of a many body system, accounting for correlations between spins of particle at different times, may be applied to discuss this experiment in much the same way as the description of the Lindner et al. experiment; the point is that the wave functions, carrying information on the particle spin, are extended in time as well as space, and therefore entanglement can occur between particles located at different times  $t$ .

---

<sup>1</sup>Shnerb and Horwitz (1993) have carried out the full canonical second quantization of the  $U(1)$  gauge theory following the methods discussed by Henneaux and Teitelboim (1992) and Haller (1972).

In Chap. 6, I also discuss the consequences of the construction of a *spacetime lattice*, which one might imagine as the picture of an electromagnetic standing wave in a cavity, periodic in both space and time; the corresponding Stueckelberg wave function, like the Bloch waves in a crystal, has forbidden bands which could, in principle, be seen experimentally (Engelberg 2009).

Chapter 7 discusses scattering theory. Since the structure of the Stueckelberg theory is based on the existence of a Hamiltonian, the scattering theory has a very strong parallel to the nonrelativistic scattering theory, and in the same way makes accessible the use of rigorous mathematical techniques. We show that the partial wave expansion for scattering theory for potential models can be achieved in a form close to that of the non-relativistic theory. The problem of describing resonances in scattering theory for which a semigroup decay law can be achieved is described in the framework of the relatively recently developed theory of Lax and Phillips (1967), Strauss (2000), is here extended to systems of relativistic particles. A relativistic Lee-Friedrichs model (Horwitz 1995) is worked out (Strauss 2000a) as an illustration of this very powerful technique.

Since the Stueckelberg quantum theory is covariant, there is an open and important question of how the theory can be applied to problems previously only accessible to quantum field theory.

In Chap. 8, we show that the anomalous moment of the electron can be computed in this framework without resort to quantum yield theory (Bennett 2012), and that it therefore carries some of the information usually attributed to the effects of vacuum polarization. Some further results of this type are also discussed. In this chapter, we discuss also the existence of Berry (1984), Bahar (2014) phases for the perturbed relativistic oscillator problem.

Chapter 9 discusses the existence of a conformal map in the framework of general relativity that results in a description of Milgrom's approach (Milgrom 1983) to the modification of Newton's law to account for the radiation curves of galaxies as an alternative to dark matter; the TeVeS theory of Bekenstein and Sanders (1994), Bekenstein (2004) emerges from a nonabelian gauge construction in the Stueckelberg theory (Horwitz 2010).

In Chap. 10, the statistical mechanics of the N-body problem is worked out, discussing both the Gibbs ensembles and the non-equilibrium generalization to the Boltzmann equation (Horwitz 1981). The general  $H$ -theorem that follows from this equation shows that there is an entropy increase monotonically in  $\tau$ ; an increase in entropy in the Einstein  $t$  variable follows, in general, only if there is no pair formation or annihilation. All of the standard thermodynamic relations are obtained in this framework, with some new features. In particular, there may be a high temperature Bose-Einstein condensation (Burakovsky 1996) to a state with a sharp (average) mass determined by a chemical potential.

In Chap. 11, there is a review of the main ideas underlying the theory and their phenomenological basis, and some discussion pointing to possible future developments.



To develop the foundations of a manifestly covariant mechanics, we must first examine the Einstein notion of time and its physical meaning. We will then be in a position to introduce the relativistic quantum theory developed by Stueckelberg (1941) and Horwitz and Piron (1973). We describe in this chapter a simple and conceptual understanding of the Newton-Wigner problem (Newton 1949) presented above, a rigorous basis for the energy time uncertainty relation, as well as a simple explanation of the Landau-Peierls (Landau 1931) uncertainty relation between momentum and time. These applications provide a good basis for understanding the basic ideas of the relativistic quantum theory. Schieve and Trump (1999) have discussed at some length the associated manifestly covariant classical theory, but some basic aspects will be discussed here as well.

---

## 2.1 The Einstein Notion of Time

In this section, we shall carefully study the Einstein notion of time, the variable  $t$  which occurs in the Minkowski space and the Lorentz transformation.

We begin our study by returning to the basic thought experiment of Einstein (1922) Born (1962). Imagine a frame  $F$  with a set of synchronized clocks embedded, and a second frame  $F'$  with clocks embedded in it. Let us suppose that signals are sent, according to the clocks in  $F$ , at times  $\tau_1$  and  $\tau_2$  from  $F$  to  $F'$ . These signals are received by detectors in  $F'$  at times  $\tau'_1$  and  $\tau'_2$  according to the clocks embedded in  $F'$ . Then, we know, according to the phenomenology of the Michelson-Morley experiment and the formulation of the Lorentz transformation by Einstein, that

$$\tau'_2 - \tau'_1 = \frac{\tau_2 - \tau_1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2.1)$$

The interval  $\tau_1 - \tau_2$  is called the proper time interval for the transmitter of the signals according to the clock interval in the frame  $F$ . The interval recorded in the relatively moving frame  $F'$  is the *Einstein time*  $\Delta t = \tau'_2 - \tau'_1$ , corresponding to the time interval *observed* in the frame  $F'$  for the two events in  $F$ ; the values assigned to the time of arrival of these events are read on clocks in the frame  $F'$ . It is therefore essential in this construction that the clocks embedded in  $F'$  be identical to the clocks in  $F$ , running at the same rate, or there would be no basis for comparison; the numbers  $\tau'_1$  and  $\tau'_2$ , read off the clocks embedded in  $F'$  could otherwise be arbitrary.

We remark that if the clocks in  $F$  and  $F'$  that we consider have a varying self-energy caused by springs under tension or batteries with stored chemical energy, the rate of recording time of these clocks may be affected by the corresponding local concentration of energy density (as one may see from (2.12)). The standard universal clocks that we visualize as imbedded in each inertial frame must therefore be *ideal* clocks, in the sense that they contain no self-energy induced frequency shifts.

It is instructive, in this respect, to consider the gravitational redshift observed on a clock located at some point in the neighborhood of a very heavy planet, such as Jupiter. An interval of the time read on the face of such a clock  $\Delta t_J$ , its proper time, is determined, in general relativity, by the Einstein metric relation (we shall use units for which  $\hbar = c = 1$  in the following)

$$\Delta s^2 = -g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

where, at rest, the spatial interval  $\Delta \mathbf{x}$  is understood to be zero, and  $\Delta s$  is the corresponding free fall proper time. Then,

$$\frac{\Delta t_J}{\Delta s} = \sqrt{-\frac{1}{g_{00}^J}};$$

the ratio of such a reading on Jupiter to that taken of a similar system (say, an ammonia molecule) on Earth is then, assuming that the corresponding interval  $\Delta s$  is the same at both locations (Weinberg 1972),

$$\frac{\Delta t_J}{\Delta t_E} = \sqrt{\frac{g_{00}^E}{g_{00}^J}},$$

in good agreement with experiment.

This calculation is remarkable in two respects; first, in that the interval of proper time between pulses of these clocks on Jupiter and the Earth must be the same for the cancellation of  $\Delta s$  when the two equations are divided one by the other, and second, in that somehow these clock mechanisms are responsive to a proper time that could be physically effective only if they were freely falling. Neither of the two systems are freely falling in this example, but are fixed in their respective gravitational fields.

The conceptual difficulties raised by this description of the phenomenon of the redshift may be resolved by considering the clocks in the two environments, on Jupiter and on the Earth, as machines evolving according to a universal time  $\tau$ . The different gravitational field in the two cases causes the clocks to emit signals at different frequencies, according to the Einstein metric, as a result of the effect of the gravitational force on the equations of motion. Freely falling clocks may also be considered to be machines running according to this universal time. The absence of any gravitational (or other) force admits solutions which are a direct reflection of the universal time; we may therefore identify  $\Delta s$ , in this case, in the metric relation, with  $\Delta\tau$ , the universal time interval referred to in the thought experiment discussed above

We learn two essential points from these simple experiments. The first is that the Einstein time is defined as the result of measurement, and the second is that there must be an underlying time which is common to both frames in the first example in order to assign numerical values to the observed times that can be compared to the times associated with the emitted signals, and in the second example, to govern the dynamics of the clocks.

There appear, therefore, to be two types of time, an absolute time of clocks embedded in any system, independent of the state of the motion, and the second, the time that is the *outcome of a measurement*, as recorded in the detector (i.e., by the “observer”) (Horwitz 1988). The notion of the Einstein time as an observable is completely analogous to the property  $\mathbf{x}$  of location, corresponding to the position of a particle. When the particle is detected, the value  $\mathbf{x}$  assigned to its position is given by the corresponding location on a standard ruler. For the Lorentz transformation relating intervals in space, the measure of length must be universally embedded in each frame, and the difference  $\Delta\mathbf{x}'$ , detected in a relatively moving frame, corresponding to an interval  $\Delta\mathbf{x}$  in the original frame, is the outcome of measurement, induced by the dynamics of the relative motion. The spacetime coordinates of general relativity correspond to quantities that are observed by detectors; the general tensor properties under local diffeomorphisms, reflecting the covariance assumptions underlying general relativity, correspond to different physical situations, as for example, the Schwarzschild and Friedman-Robertson-Walker solutions of the Einstein equations (Schwarzschild 1916; Friedman 1924), where the coordinates are considered to be actual outcomes of measurement.<sup>1</sup>

These are the essential ingredients from which a manifestly covariant classical and quantum mechanics can be constructed (we shall confine ourselves here, for the most part, to the covariance characteristic of special relativity, although in a later chapter our considerations will be extended to applications in general relativity).

In classical nonrelativistic mechanics, the fact that the value assigned to the position of a particle  $\mathbf{x}$  and the value of the momentum  $\mathbf{p}$  are the outcomes of measurement

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<sup>1</sup>Note that both time intervals, as well as space intervals, must be thought of as measured by geodesic projection (e.g. Weinberg 1972) since clocks and rulers brought to the location of the events would suffer distortion due to the gravitational field as well.

gives rise to the notion of a point in phase space describing the *state* of the particle. The state evolves, according to the theory of Hamilton and Lagrange by means of an evolution determined by the Hamilton equations, an elegant formulation of Newton's laws of motion (we denote the gradient formally by a partial derivative with respect to a vector),

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{p}} \\ \frac{d\mathbf{p}}{dt} &= -\frac{\partial H(\mathbf{x}, \mathbf{p})}{\partial \mathbf{x}},\end{aligned}\tag{2.2}$$

where  $H(\mathbf{x}, \mathbf{p})$  is the Hamiltonian of the system. Here the variables  $\mathbf{x}$  and  $\mathbf{p}$  are functions of the time  $t$ . These equations can be directly generalized to  $N$  particles, writing  $\mathbf{x}_i$  in place of  $\mathbf{x}$  and  $\mathbf{p}_i$  in place of  $\mathbf{p}$  for  $i = 1, 2, 3, \dots, N$ , and the Hamiltonian is generally a function of all  $6N$  variables. This structure, sometimes called *symplectic* because the formulas (2.2) have the symmetry of the symplectic group, is made possible due to the correlation between the variables established by the existence of the universal Newtonian time  $t$ .

Similarly, in the quantum theory, where a pure state of the system (in the simplest case) is determined by a wave function  $\psi_t(\mathbf{x})$  (or  $\psi_t(\mathbf{p})$ ), the Schrödinger equation governs the evolution of the system according to

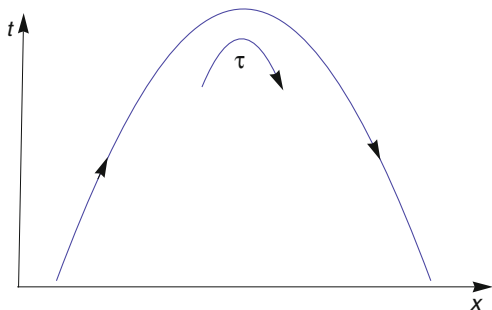
$$i \frac{\partial \psi_t}{\partial t} = H(\mathbf{x}, \mathbf{p})\psi_t,\tag{2.3}$$

where the Hamiltonian is a function of the observables, i.e., the Hermitian operators,  $\mathbf{x}$  and  $\mathbf{p}$ . This equation can be written in a representation (called the  $\mathbf{x}$ -representation) in which  $\mathbf{x}$  is diagonal, i.e. numerical valued, and  $\mathbf{p}$  is represented by  $-i$  times the partial derivative with respect to  $\mathbf{x}$ , or conversely, in a representation called the  $\mathbf{p}$ -representation) in which  $\mathbf{p}$  is diagonal, i.e. numerical valued, and  $\mathbf{x}$  is represented by  $i$  times the partial derivative with respect to  $\mathbf{p}$ . This structure may be generalized to an  $N$  body system in the same way, for which the wave function in the  $\mathbf{x}$  representation is a function of all the positions  $\mathbf{x}_i$ ,  $i = 1, 2, 3, \dots, N$  at a given value of the universal Newtonian time  $t$ .

This description of the dynamics of systems of particles rests on the identification of the observables. In nonrelativistic dynamics,  $t$  is a *parameter* providing a framework for the correlation of different parts of a system as well as for its dynamical development.

If, as we have argued above, the time  $t$  is understood as an observable in relativistic dynamics, the set of observables assigned to each particle (often called an *event*) is comprised of all four Minkowski coordinates  $x^\mu \equiv (t, x_1, x_2, x_3)$  as well as  $p^\mu \equiv (E, p_1, p_2, p_3)$ , along with others, such as the relativistic generalization of angular momentum (the Casimir operators of the Lorentz group, as we shall discuss further in Chap. 5). The construction of a dynamics to describe the motion of these fundamental objects, and some selected important applications of this dynamics, will be the subject of this book. I review in the following the arguments of Stueckelberg (1941) for the

**Fig. 2.1** Stueckelberg  
classical pair annihilation



construction of this theory and comment on an alternative, complementary, view (Horwitz 1973) leading to the same conclusions.

Stueckelberg (1941) first considered the classical spacetime diagram of the orbit, called a “worldline”, of a free particle, expected to be simply a straight line. He then supposed that there is some force acting on the particle that makes the worldline bend during the interaction. He further supposed that the interaction may be strong enough to make the world line turn back and run in a direction opposite to that of the  $t$  axis, as shown in Fig. 2.1.

It is clear that Stueckelberg was thinking of this process as reflecting the effect of some dynamical laws on the evolution of the sequence of events constituting the worldline rather than a global manifestation of the worldline (the later work of Currie et al. (1963) showed that such a global dynamics of worldlines would, with some assumptions, suffer from a no-go theorem). In contrast to the view of Weyl (1952), who suggested that the particles we see are the intersection of the observer’s plane of time with pre-existing world lines, comprising a static universe (see also discussion in Horwitz (1988)), with apparent motion generated by the effect of this plane cutting the worldlines at a succession of points in  $t$ , the worldline is envisaged here as generated by the motion of a single event moving according to dynamical laws, in a similar way to the formation of the orbit of a particle in nonrelativistic mechanics, generated as a function of the Newtonian time. Stueckelberg observed that in the extreme case of a reversal in the sense of time of this motion, the physical process of pair annihilation could be represented in the framework of *classical* mechanics if the path running backward in time were considered as an antiparticle. He, moreover, noted that the use of  $t$  as a parameter would be inadequate to describe this curve, but that an invariant parameter, which he called  $\tau$ , along the curve, had to be introduced to construct a consistent description. Feynman (1950) followed a structure of this type in the construction of his spacetime diagrammatic approach to perturbation expansions in quantum electrodynamics, elegantly explained in a paper by Nambu (1950).

Horwitz and Piron (1973) further assumed, in order to treat many body systems, that this parameter is universal, as for the Newtonian time; it, in fact, plays the role of the universal time postulated by Newton in his Principia (Newton 1687).

The concept of a world time controlling the dynamical evolution in contrast to evolution in  $t$  is illustrated in Fig. 2.1. Along the curve, the parameter  $\tau$  increases monotonically. The  $t$  axis of the diagram, however, consistently with the definition of  $t$  as the *measured* time in the laboratory, records the time on the clock in the laboratory at the moment when the signal is detected, which runs (in the absence of any other forces and on its mass shell) with  $\tau$ ; thus, the  $t$  measured in the laboratory records its evolution in  $\tau$ . The sequence of  $\tau$  values parametrizing the motion of the event along its world line in Fig. 2.1 is the same sequence along the  $t$  axis, reflected by values of  $t$  in the laboratory that coincide with  $\tau$ . Close to the initial condition, the corresponding points (i.e. equal  $\tau$  points) run along essentially the same  $t$  values, but as the system develops, the  $t$  values recorded in the laboratory as observed on the laboratory clock, and the  $t$  values detected as signals from the system under observation (with values read on the laboratory clock as well) diverge significantly. Thus, the character of the observable  $t$  becomes manifest as a consequence of the dynamics that affect its measured value.<sup>2</sup>

There is, however, another phenomenon illustrated in this diagram. During the period that the world line is deflected and curving, it passes through the light cone, becomes spacelike, and then becomes straight again in the final force-free region, but nevertheless, moving *backwards* in time. This inversion in the sequence in the final state cannot be attributed directly to forces acting on the system, but rather must be thought of as the positive monotonic evolution of the *antiparticle* in  $\tau$ , forward in  $t$ . The figure therefore illustrates a profound physical transition. In the asymptotic region after the interaction, it represents the motion of an antiparticle in the positive direction of time, as maintained by Stueckelberg (1941) in agreement with the view adopted by Feynman (1950) and associated with CPT conjugation. Since CPT conjugation, as we shall discuss later in more detail, reverses the sign of momentum and energy (as well as the charge), the positive monotonic evolution of the antiparticle in  $\tau$  is forward in  $t$  along the “outgoing” line. In this CPT conjugate picture the entire world line (taking into account the properties of CPT conjugation, such as a change in sign of the charge) is reversed in  $\tau$  ordering, and the previously “incoming” line now runs backward in  $t$ ; its CPT conjugate then runs forward in  $t$ , corresponding to the original incoming particle. The particle-antiparticle interpretation is not easily accessible in the interaction region, where the world line may be spacelike; the dynamics of the motion, however, is smoothly and unambiguously represented as a motion on spacetime according to  $\tau$  (such a process can occur repeatedly as in neutrino oscillations and the evolution of the  $K$  and  $B$  meson systems; we shall study these processes in Chap. 4).

To pose an apparent paradox, one may think of cutting the world line at some point on the incoming line, absorbing the particle entirely, as suggested by Havas (1956); he remarked that this would destroy its continuation. However, that continuation is in

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<sup>2</sup>This discussion is fundamental in understanding the essential distinction between the *measured* time of Einstein, which plays the role of a *coordinate* of a physical event, and the underlying absolute time  $\tau$  governing the dynamical processes of evolution.

the *past* of  $t$ , leading to an apparent contradiction. He resolved this paradox by noting that the instrument that absorbed the particle is located at this point in spacetime for all  $\tau$ , and therefore constitutes a change in initial conditions for the generation of this history. The antiparticle would therefore never have been produced. If the experiment records a particle and antiparticle, that antiparticle would have had to be generated elsewhere (e.g. at  $t \rightarrow +\infty$ ) and would not be associated with this annihilation diagram.

One can approach the theory as we have presented it above from a somewhat different point of view (Horwitz 1973). We observe that in nature the mass of a particle generally depends on its state. One understands the decay of a neutron into proton, electron and neutrino ( $\beta$  decay) as associated with the fact that the neutron is heavier than the proton in free space. In a nucleus, however, the neutron generally does not decay. Moreover, the proton in a nucleus may decay into a neutron, positron and neutrino (inverse  $\beta$  decay), indicating that the proton is more massive than the neutron in that environment. As another example, calculations in quantum electrodynamics show that the difference between the mass of an electron in free space and in a Coulomb potential is not zero, making a contribution to the Lamb shift (Lamb 1947) (see also the work of Davidson 2014, examining mass shifts in nuclei).

We therefore conclude that the observable mass of a particle, from the point of view of a *particle* theory (rather than investing the mass change in the surrounding fields) should be treated as a dynamical variable. Thus, in the momentum four vector,  $\mathbf{p}$  and  $E$  should be considered as independent dynamical variables. The Fourier complement of this picture corresponds to the time  $t$  and position  $\mathbf{x}$  necessarily being dynamical variables also (in accordance with our discussion of the physical meaning of these variables in special relativity above). Equations describing the distribution of these dynamical variables would then be static, with no parameter for the evolution of a state, and one must therefore introduce the notion of an invariant (universal in order to be able to treat the many body problem) variable  $\tau$  with which to generate dynamical change. The resulting theory is then identical to that of Stueckelberg, with the additional postulate that  $\tau$  is universal.

As a model for the structure of the dynamical laws that might be considered, Stueckelberg proposed a Lorentz invariant Hamiltonian for free motion of the form

$$K = \frac{p^\mu p_\mu}{2M}, \quad (2.4)$$

where  $M$  is considered a parameter, with dimension mass, associated with the particle being described, but is not necessarily its measured mass. In fact, the numerator (with metric  $- + + +$ ),

$$p^\mu p_\mu = -m^2, \quad (2.5)$$

corresponds to the actual observed mass (according to the Einstein relation  $E^2 = \mathbf{p}^2 + m^2$ ), where, in this context,  $m^2$  is a dynamical variable.

The Hamilton equations, generalized covariantly to four dimensions, are then

$$\begin{aligned}\dot{x}^\mu &\equiv \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \\ \dot{p}_\mu &\equiv \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu}.\end{aligned}\tag{2.6}$$

These equations are postulated to hold for any Hamiltonian model, including many types of interaction such as additive potentials or gauge fields (to be discussed in later chapters), and therefore a Poisson bracket may be defined in the same way as for the nonrelativistic theory. The construction is as follows. Consider the  $\tau$  derivative of a function  $F(x, p)$ , i.e.,

$$\begin{aligned}\frac{dF}{d\tau} &= \frac{\partial F}{\partial x^\mu} \frac{dx^\mu}{d\tau} + \frac{\partial F}{\partial p^\mu} \frac{dp^\mu}{d\tau} \\ &= \frac{\partial F}{\partial x^\mu} \frac{\partial K}{\partial p_\mu} - \frac{\partial F}{\partial p^\mu} \frac{\partial K}{\partial x_\mu} \\ &= \{F, K\},\end{aligned}\tag{2.7}$$

thus defining a Poisson bracket  $\{F, G\}$  quite generally. The arguments of the non-relativistic theory then apply, i.e., that functions which obey the Poisson algebra isomorphic to their group algebras will have vanishing Poisson bracket with the Hamiltonian which has the symmetry of that group, and are thus conserved quantities, and the Hamiltonian itself is then (identically) a conserved quantity.

It follows from the Hamilton equations that for the free particle case

$$\dot{x}^\mu = \frac{p^\mu}{M}\tag{2.8}$$

and therefore, dividing the space components by the time components, cancelling the  $d\tau$ 's ( $p^0 = E$  and  $x^0 = t$ ),

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{E},\tag{2.9}$$

the Einstein relation for the observed velocity. Furthermore, we see that

$$\dot{x}^\mu \dot{x}_\mu = \frac{p^\mu p_\mu}{M^2};\tag{2.10}$$

with the definition of the invariant

$$ds^2 = -dx^\mu dx_\mu,\tag{2.11}$$

corresponding to proper time squared (for a timelike interval), this becomes

$$\frac{ds^2}{d\tau^2} = \frac{m^2}{M^2}.\tag{2.12}$$



Therefore, the proper time interval  $\Delta s$  of a particle along a trajectory parametrized by  $\tau$  is equal to the corresponding interval  $\Delta\tau$  only if  $m^2 = M^2$ , a condition we shall call “on mass shell”.<sup>3</sup> The theory is, however, generally intrinsically “off-shell”. We shall see that this property is essential for the resolution of the Newton-Wigner problem, and therefore for the possibility that the wave function has a local probability interpretation, and can be a candidate for a consistent relativistic quantum theory. There is, however, no obvious constraint, even for simple interacting models, such as in the potential model

$$K = \frac{p^\mu p_\mu}{2M} + V(x), \quad (2.13)$$

that would insure that the particle maintains a physical mass in the small neighborhood of some given value.<sup>4</sup> One might suppose that an electron, after interaction that could perturb the value of  $m^2$ , would result in a particle with a different mass; it has therefore been an explicit assumption in many successful applications (for example, in the two body bound state that we shall treat in Chap. 5) that there is a mechanism for returning the particle to the neighborhood of some equilibrium value of mass, such as a relaxation of free energy of the system in interaction with other particles or fields (e.g., a suggestion of Jordan 1980). It was found by Burakovsky and Horwitz (1996) that there may be a high temperature Bose-Einstein condensation, in the framework of statistical mechanics (to be discussed in Chap. 8), that causes a particle to stabilize its mass at some value determined by a chemical potential. More recently, Aharonovich and Horwitz (2011) have found that the electromagnetic self interaction of a charged particle can dynamically drive the particle to its mass shell. We shall assume in the following that there exists such a mechanism for every object that is recognized as a “particle” (even for reasonably sharp resonances) which stabilizes its mass, and discuss this question in more detail in later chapters. However, for the theory to be effective, this mass shell property can only be approximate, i.e., an absolutely sharp mass value would not be compatible with the structure of the theory, as will become clear below.

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<sup>3</sup>In Galilean mechanics, due to the existence of a cohomology in the Lie algebra of the Galilean group, a definite value must be assigned to the value of the mass to achieve an irreducible representation (Sudarshan 1974). The Poincaré group does not have such a cohomology, and thus admits the full generality of the Stueckelberg theory. We discuss the Galilean limit in more detail in Chap. 10.

<sup>4</sup>An alternative covariant structure for a relativistic quantum theory, the so-called *constraint mechanics*, discussed in Appendix A of this chapter, based on the constraint theory developed by Dirac (1966) to deal with the quantization of gravity and gauge fields, extensively studied by Sudarshan et al. (1981a), Rohrlich (1981) and others (Llosa 1982), does have a mechanism for enforcing the asymptotic return of a particle to a given mass shell. This theory, however, necessarily makes use of a system of constraints of the *first class* (Itzykson 1980), a condition that makes the construction of a useful quantum theory very difficult (Horwitz 1982).

## 2.2 Classical Mechanics

To illustrate some of the properties of the covariant classical mechanics, consider the two body problem with invariant relative potential  $V(x_1 - x_2)$ , a Poincaré invariant potential (invariant under both the Lorentz group and translations); such a potential must be a function of  $x^\mu x_\mu = \mathbf{x}^2 - t^2$ , where we have called

$$x^\mu = x_1^\mu - x_2^\mu, \quad (2.14)$$

the relative spacetime coordinate, which we shall call  $x$ . The Stueckelberg Hamiltonian corresponding to this problem is (Horwitz 1973) (the assumption of the universality of  $\tau$  made in this work, not explicitly made by Stueckelberg, is essential to the formulation of this problem)

$$K = \frac{p_1^\mu p_{1\mu}}{2M_1} + \frac{p_2^\mu p_{2\mu}}{2M_2} + V(x). \quad (2.15)$$

Since  $K$  does not depend on the total (spacetime) “center of mass”

$$X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2}, \quad (2.16)$$

the two body Hamiltonian can be separated into the sum of two Hamiltonians, one for the “center of mass” motion and the second for the relative motion, by defining the total momentum, which is absolutely conserved,

$$P^\mu = p_1^\mu + p_2^\mu \quad (2.17)$$

and the relative motion momentum

$$p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2} \quad (2.18)$$

Then, it is an identity that (as in the nonrelativistic two body problem)

$$\begin{aligned} K &= \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2M} + V(x), \\ &\equiv K_{CM} + K_{rel}, \end{aligned} \quad (2.19)$$

where  $M = M_1 + M_2$  and  $x = x_1 - x_2$ ; both  $K_{CM}$  and  $K_{rel}$  are constants of the motion.

We see in this construction the significance of defining  $\tau$  as a universal parameter (Horwitz 1973). The potential function  $V(x_1 - x_2)$  implicitly carries in it the information that the points  $x_1$  and  $x_2$  are at equal  $\tau$ ; this correlation makes it possible to consider pairs of points along the two world lines of the two particles as having well-defined interaction. A similar assumption is made in ordinary nonrelativistic dynamics; the implicit assumption in writing a potential function as  $V(\mathbf{x}_1 - \mathbf{x}_2)$  is that the points  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are taken at equal  $t$  along the orbits. This assumption is usually not made explicit, since the nonrelativistic Galilean world is always assumed to be at equal universal Newtonian time. Thus we see that the parameter  $\tau$ , with the assumption of universality (Horwitz 1973), corresponds to the Newtonian time.

Although the  $t$  and  $x$  of Einstein undergo Lorentz transformations as they are perceived and measured in relatively moving inertial frames, the dynamical correlation provided by the invariant universal parameter  $\tau$  is maintained independently of the state of motion.

Models parallel to those of the nonrelativistic theory can be constructed, for example, by replacing the  $V(r)$  of nonrelativistic spherically symmetric models by  $V(\rho)$ , where  $\rho = \sqrt{x^\mu x_\mu}$  for the relative coordinate  $x^\mu$  spacelike, in accordance with our experience of the nonrelativistic two body problem. Moreover, for two timelike momenta, corresponding to particles with positive  $m^2$ , the relative momentum defined in (2.18) is generally spacelike since for not too large space components of the momenta, and for particles not too far from mass shell, the fourth components are then large and approximately equal to  $M_1$  and  $M_2$  respectively; the fourth component of the relative momentum carries a near cancellation, and the resulting vector is generally spacelike. The relativistic two-body problem therefore differs fundamentally from the nonrelativistic two body problem; in the latter case, separation of variables results in a center of mass motion accompanying what appears to be one particle in an external potential. In the relativistic case, the relative motion system is essentially *tachyonic*, i.e., it appears to describe a “particle” with spacelike momentum (for which  $\mathbf{p}/E > 1$ , and thus light speed would be exceeded). The situation is not unphysical; we must realize that this is a relative motion of a two body system, and that the two particles being described can be properly timelike. If the theory were designed to rule out such tachyonic systems, we would not be able to study the two body case in the way we have described above.

For such a class of models, one may choose, for example,

$$V(\rho) = \frac{k}{\rho}, \quad (2.20)$$

corresponding to a Coulomb potential for  $k = \pm e^2$ , or a gravitational Kepler problem for  $k = -GM_1M_2$ . Since, according to (2.6), the Hamilton equations (written for each particle),

$$\frac{dt_i}{d\tau} = \frac{E_i}{M_i}, \quad (2.21)$$

if the  $\{M_i\}$  are identified as the Galilean target masses of the particles (the Galilean group, as will be discussed further in Chap. 5, admits only a sharp mass, whereas the Poincaré group admits a continuum of possibilities (Sudarshan 1974), as occurs in the Stueckelberg theory), then the  $t$  values of all the particles may become identical in this limit, and the relative coordinate  $\rho$  goes over into the coordinate  $r = |\mathbf{x}|$ . Thus the Coulomb and Kepler models go over, as  $c \rightarrow \infty$ , precisely to the corresponding problems in the nonrelativistic theory.

We shall show in Chap. 5 that the corresponding relativistic quantum two body Coulomb problem can be solved exactly, and yields the nonrelativistic Schrödinger spectrum up to relativistic corrections ( $O(1/c^2)$ ).

For the classical case, a Lorentz invariant potential implies that the function

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad (2.22)$$

for which the Poisson bracket algebra is that of the Lorentz group, is conserved (its Poisson bracket with  $K$  vanishes). Therefore, the four linear cyclic combinations of  $\{x_\lambda M_{\mu\nu}\}$  which vanish identically provide constraints on the orbits. Two of these relations are degenerate, and the remaining two restrict the Kepler motion to a plane. One finds, in contrast to Sommerfeld's (1921) conclusion, the resulting ellipse does not precess. The precession which Sommerfeld found in his search for the origin of the precession of the orbit of Mercury was due to his use of the noncovariant form  $1/r$  for the potential. This problem is discussed in detail in Horwitz (1973) and in Trump (1999).

Another model of interest is that of the covariant harmonic oscillator, for which (for  $k$  some positive constant) (Feynman 1971; Kim 1977; Leutwyler 1977)

$$V = k\rho^2 = kx^\mu x_\mu. \quad (2.23)$$

The equations of motion separate into four independent second order equations, each of which correspond to a one dimensional oscillator, each following some elliptical path on spacetime, constituting an orbit which is bounded in the  $t$  direction; one may think of this as a continuing sequence of pair annihilation and creation processes (in relative motion) from the point of view of Stueckelberg's classical pair annihilation picture. In the corresponding quantum theory, this separation of variables leads to "ghost" states which must be suppressed by constraints. We shall see in Chap. 5 that this problem can be solved with no "ghost" states, obtaining the nonrelativistic oscillator spectrum (up to relativistic corrections).

From the point of view developed here, one sees the classical wave equations, such as the Klein-Gordon equation and the Dirac equation, as well as Maxwell's equations, as being essentially geometrical constraints rather than dynamical in this context. We shall be concerned here with developing the dynamics of systems evolving in a covariant way in spacetime.

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## 2.3 The Quantum Theory

In this section we shall study the form of the quantum theory associated with Stueckelberg's dynamics in spacetime (Stueckelberg 1941; Horwitz 1973, to be called SHP in the following). We have argued that in a relativistically covariant theory, the space and time variables are observable, and therefore correspond to Hermitian operators in the quantum theory. The operator commutation relations are taken to be

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad (2.24)$$

consistently with the Poisson bracket for the classical case, and the Lorentz covariant generalization of the nonrelativistic commutation relations  $[x_i, p_j] = i\delta_{ij}$ . With these commutation relations, the operator form of the definition (2.22) satisfies the commutation relations of the Lorentz group, just as the Poisson bracket relation for the classical case. To achieve this simple form for the generators of the Lorentz group in the quantum case, it is necessary that the Hilbert space be defined as  $L^2(R^4, d^4x)$ ,

as we define formally below; only in this way can the operator

$$E \rightarrow i \frac{\partial}{\partial t}$$

be considered as essentially self-adjoint. We shall discuss this operator form of the Lorentz group further in detail in Chap. 5.

The spectral decompositions of the self-adjoint operators  $x^\mu$  or  $p^\mu$  then provide representations of the quantum state, as explained, for example, in Dirac's book (Dirac 1930). The wave function is then a square integrable function on spacetime  $x$  (or  $p$ ); its square modulus corresponds to the probability of finding an event per unit spacetime volume  $d^4x$  (or energy momentum space  $d^4p$ ) at the point  $x$  (or  $p$ ). In the  $x$  representation, for which  $x_\mu$  is numerical valued,  $p^\mu$  is represented by  $-i\partial/\partial x_\mu$ , and in the  $p$  representation, for which  $p^\mu$  is numerical valued,  $x_\mu$  is represented by  $i\partial/\partial p^\mu$ .

Stueckelberg assumed that the dynamical development of the wave function is governed by a Schrödinger-like equation, which we shall call the Stueckelberg-Schrödinger equation

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x) \quad (2.25)$$

where in the notation of Dirac (1930),

$$\psi_\tau(x) = \langle x | \psi_\tau \rangle \quad (2.26)$$

and  $K$  is an operator function of  $x$ ,  $p$ , which may correspond to the classical models discussed above. The wave function is assumed to be scalar; the representation of a particle with spin will be discussed in Chap. 3. Gauge field interactions, such as electromagnetism, can be accounted for by imposing gauge invariance, as we shall discuss in later chapters.

The Eq. (2.25) corresponds to unitary evolution, as for the nonrelativistic Schrödinger equation, where the evolution is generated by the operator (for  $K$  not explicitly dependent on  $\tau$ )

$$U(\tau) = e^{-iK\tau}, \quad (2.27)$$

for which  $\psi_\tau(x) = U(\tau)\psi(x)$ .

The derivative of an expectation value of the observable  $F$  is then, as in the nonrelativistic quantum theory, consistent with the Poisson bracket formulation, i.e.

$$\frac{d}{d\tau} \langle \psi_\tau, F \psi_\tau \rangle = -i \langle \psi_\tau, [F, K] \psi_\tau \rangle, \quad (2.28)$$

where  $[F, K]$  is the commutator, with the correspondence defined by Dirac (1930)<sup>5</sup>

$$\{F, K\}_{PB} \rightarrow -i[F, K]. \quad (2.29)$$

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<sup>5</sup>As Van Hove (1951) has pointed out, this correspondence is not applicable for higher order polynomials; both the Poisson bracket and the commutators are distributive in the Leibniz sense, but in the quantum case the algebra is not commutative, and it is not always possible to regroup factors as in the classical, commutative, case. The problem of consistent quantization has been studied under the name "geometric quantization" (Kostant 1970).

Since the “standard” bras and kets correspond to representations of the self-adjoint operators  $\mathbf{x}$  and  $t$ , they are complete, and the scalar product (as for the expectation value in (2.28)) is given by

$$\begin{aligned} \langle \chi | \psi \rangle &= \int d^4x \langle \chi | x \rangle \langle x | \psi \rangle \\ &= \int d^4x \chi(x)^* \psi(x). \end{aligned} \quad (2.30)$$

This is clearly a positive scalar product, defining the norm

$$\|\psi\|^2 = \int d^4x \psi(x)^* \psi(x) = \int d^4x |\psi(x)|^2, \quad (2.31)$$

as previously discussed in Chap. 1. This property, together with linear superposition over the complex numbers (which follows from the linearity of the scalar product) and boundedness of the norm, consistent with the Born probability interpretation, results in the proper structure of a Hilbert space and a consistent quantum theory.

The momentum representation, as in the nonrelativistic theory, is constructed from the Fourier transform

$$\psi(p) = \frac{1}{(2\pi)^2} \int d^4x e^{ip^\mu x_\mu} \psi(x), \quad (2.32)$$

with inverse

$$\psi(x) = \frac{1}{(2\pi)^2} \int d^4p e^{-ip^\mu x_\mu} \psi(p). \quad (2.33)$$

As we have noted in Sect. 1.2, the interpretation of the solutions of the Klein-Gordon equation as wave functions in a quantum theory encounters serious problems with localizability. In the theory of Stueckelberg, we have interpreted the wave function as the amplitude for the local probability density. It is therefore important to discuss the Newton-Wigner problem in the context of the Stueckelberg theory, and we turn to this question in the next section.

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## 2.4 The Newton-Wigner Problem

Having defined the manifestly covariant quantum theory, we are now in a position to re-examine the Newton-Wigner problem (Newton 1949). From the viewpoint of this theory, we shall be able to understand the way the problem arises in the framework of theories which use equations of the type of those of Klein-Gordon and Dirac that impose a strict mass shell requirement.

We will show that the  $\mathbf{x}$  operator in the Stueckelberg theory, corrected to extrapolate the occurrence of an event at some point in spacetime back to  $t = 0$ , as sought by Wigner and Newton, is exactly the Newton-Wigner position operator on each mass value (in the sense of a direct sum) under the integral defining the expectation value (Horwitz 1973).

Consider the expectation value of  $\mathbf{x}$ :

$$\langle x \rangle = \int d^4 p \psi^*(\mathbf{p}, E) i \frac{\partial}{\partial \mathbf{p}} \psi(\mathbf{p}, E) \quad (2.34)$$

We now change variables, considering only  $E \geq 0$ , using the relation

$$E = \sqrt{\mathbf{p}^2 + m^2} \quad (2.35)$$

for  $m$  a new variable. Then,

$$dE = \frac{dm^2}{2E}, \quad (2.36)$$

where now  $E$  stands for the relation (2.35). Furthermore, if we want to think of the derivative in (2.34) as a straightforward derivative (it only acted on the first three arguments in  $\psi$  before the change of variables), we have to correct for its action on the fourth argument  $E$ , i.e., we must now write

$$\begin{aligned} i \frac{\partial}{\partial \mathbf{p}} &\rightarrow i \frac{\partial}{\partial \mathbf{p}} - i \frac{\partial E}{\partial \mathbf{p}} \frac{\partial}{\partial E} \\ &= i \frac{\partial}{\partial \mathbf{p}} - i \frac{\mathbf{p}}{E} \frac{\partial}{\partial E} \end{aligned} \quad (2.37)$$

when acting on  $\psi(\mathbf{p}, E = \sqrt{p^2 + m^2})$ .

We recognize that this extra term looks like velocity times time, the operator  $i\partial/\partial E$ . This corresponds to the displacement to get back to where a (virtual) world line would be at  $t = 0$ , if one imagines the semiclassical picture of a world line running through the point  $(\mathbf{x}, t)$ . This semiclassical interpretation of these operators, where the real information is encoded in the wave function, appears to be consistent. This extra term, however, in the quantum theory, should be symmetrized, so let us define the relativistic operator form of the Newton-Wigner operator in the context of the Stueckelberg theory as

$$x_{NW} = i \frac{\partial}{\partial \mathbf{p}} - \frac{1}{2} \{\mathbf{v}, t\}, \quad (2.38)$$

where  $\mathbf{v} = \mathbf{p}/E$  and  $t = i\partial/\partial E$ . One must use the fact that when  $\partial/\partial E$  acts on  $\mathbf{p}/E$ , it differentiates both this factor and the wave function that implicitly follows it. The last term in (2.38) is then

$$\frac{1}{2} \{\mathbf{v}, t\} = i \frac{\mathbf{p}}{E} \frac{\partial}{\partial E} - i \frac{\mathbf{p}}{2E^2}.$$

This is just the extra piece that came from the change of variables, plus a new term, which we saw is part of the Newton-Wigner operator displayed in Eq. (1.9). Thus, our operator (2.38), put into expectation value, can be seen as the expectation value of

$$\mathbf{x} \rightarrow \mathbf{x} - i \frac{\mathbf{p}}{2E^2},$$

as required by Newton and Wigner, but under the integral over all mass shells.

Therefore, the operator (2.38) may be represented as the Newton-Wigner operator under the integration over masses of an expectation value at each value of  $m$ .

The semiclassically expected value of the position of a particle as it passes  $t = 0$  corresponds in this way to the Newton-Wigner operator.

We can understand from the point of view of the relativistic theory that position and mass, as the operator  $\mathbf{x}$  and  $m = \sqrt{E^2 - \mathbf{p}^2}$ , are not compatible. The Klein Gordon theory does not consider the mass to be an operator; it is just a given number, corresponding to a point on the continuous spectrum of  $m$ . The Stueckelberg theory is completely local, consistent with our construction (2.38), and the interference phenomena we describe with the associated wave functions should predict the actual outcome of experiments. Such interference effects, predicted by Horwitz and Rabin (1976), have indeed been observed, as we shall discuss in Chap. 3 (the experiment of Lindner et al. 2005).

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## 2.5 The Landau-Peierls Problem

In 1931, Landau and Peierls (1931) deduced a relation between dispersion in momentum and time of the form (we restore  $\hbar$  and  $c$  in several formulas of this section to make the units clear)

$$\Delta p \Delta t \geq \hbar/c \quad (2.39)$$

concerning the time interval  $\Delta t$  during which the momentum of a particle is measured and the momentum dispersion of the state. According to Landau and Peierls, for any given dispersion of momentum in the state, there is a minimum interval of time necessary for measuring the outcomes predicted by knowledge of the state consistent with the relativistic bound on the velocities.

Landau and Peierls begin with the estimates of first order perturbation theory for the “almost conservation of energy”, i.e.

$$|E - E'| \sim \hbar/\Delta t; \quad (2.40)$$

where, in perturbation theory, one argues that in sufficient time  $\Delta t$ , the initial energy  $E$  and the final energy  $E'$  after the transition are close. This relation corresponds to the well known estimate for the nonrelativistic energy time uncertainty relation.

Landau and Peierls, however, use this result, not a rigorous property of the wave functions of a particular state, to argue that if there is a dispersion in energy in the incoming state, and a dispersion in the outgoing state, the two sets of values must be restricted by this relation, for which the central values essentially cancel. Thus, one obtains

$$|\Delta E - \Delta E'| \sim \hbar/\Delta t. \quad (2.41)$$

They then use the relation (valid for both nonrelativistic and relativistic kinematics)

$$\Delta E = \frac{dE}{dP} \Delta P = v \Delta P; \quad (2.42)$$

using absolute conservation of momentum to assert that

$$\Delta P = \Delta P',$$



they then obtain

$$|(v - v')|\Delta P \sim \hbar/\Delta t. \quad (2.43)$$

This result implies a change in velocity from incoming to outgoing states. For a given  $\Delta P$ , the smaller the time interval of measurement, the larger this velocity change must be. It is however, bounded by the velocity of light  $c$ , and one therefore obtains the relation (2.39).

Aharonov and Albert (1981) have understood this result in terms of causality. They argue that if a measurement is made in a short time  $\Delta t$  which restricts the particle to a range of momenta  $\Delta P$ , the wave function must extend to  $\Delta x \sim (\hbar/2\Delta P)$ . The Landau-Peierls result then assures that  $\Delta x \leq (c/2)\Delta t$ . From the point of view of Aharonov and Albert, involving causality, as well as the use of a relativistic bound by Landau and Peierls, it is clear that the relation (2.39) should be associated with relativity.

Following the method used by Landau and Peierls for the relativistic Stueckelberg-Schrödinger equation (2.25), it would follow in the same way from first order perturbation theory that

$$|K - K'| \sim \hbar/\Delta\tau \quad (2.44)$$

Since  $p^\mu p_\mu = -(E/c)^2 - \mathbf{p}^2 = m^2 c^2$ , where  $m$  is the mass of the particle measured in the laboratory. The initial and final free Hamiltonians have the form

$$K = \frac{p^\mu p_\mu}{2Mc^2} = -\frac{m^2 c^2}{2Mc^2} = -\frac{m^2}{2M}$$

and therefore the relation (2.44) becomes, for small  $\Delta m$ ,

$$\begin{aligned} \left| \frac{m^2 - m'^2}{2M} \right| &\sim \frac{\hbar}{\Delta\tau} \\ &= \frac{|(m - m')(m + m')|}{2M} \cong |\Delta m|, \end{aligned}$$

for  $m$  close to its “mass shell” value  $M$ . We therefore find the relation (Burakovsky 1996)

$$\Delta m \Delta\tau \cong \hbar, \quad (2.45)$$

a mass- $\tau$  uncertainty relation. This result provides a justification for the generally assumed relation that the width of the mass dispersions of elementary particles as seen in decay modes is associated with the lifetime of the particle in its proper frame. If the particle is off shell due to additional interactions during the decay process, there would clearly be corrections.

As we have noted, such estimates are not rigorous, but carry the same semi-quantitative arguments used by Landau and Peierls, based on first order perturbation theory.

The  $\Delta E \Delta t$  uncertainty relation in the SHP relativistic theory, on the other hand, follows rigorously from the commutation relation

$$[E, t] = i\hbar. \quad (2.46)$$

It is a general theorem in quantum mechanics that the dispersions of two self adjoint operators  $A$  and  $B$  in a given quantum state, defined by

$$\Delta A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$$

and

$$\Delta B = \sqrt{\langle (B - \langle B \rangle)^2 \rangle}$$

are related by

$$\Delta A \Delta B \geq \frac{\hbar}{2} | \langle [A, B] \rangle |.$$

It therefore follows from (2.46) that, as a rigorous property of the wave function representing the state of the system,

$$\Delta E \Delta t \geq \hbar/2. \quad (2.47)$$

In a similar way, it is possible to show that there is a simple and rigorous derivation of (2.39) in the framework of the manifestly covariant quantum theory we are working with here.

We have seen that the results of Newton and Wigner can be obtained in a straightforward way by defining an effective Newton-Wigner operator as in (2.38), with the semiclassical meaning of an extrapolation of the event position back to the value it would have at  $t = 0$ , interpreting the virtual velocity field contained in the wave function as associated (in expectation value) with an actual distribution that could be thought of as a collection of possible world lines. In the same way, we can construct an *effective time operator* by extrapolating the time of observation of an event back to the  $x = 0$  axis, which one might think of as the location of a Geiger counter triggered by the passage of a world line through its position at  $x = 0$ . We therefore define a Landau-Peierls time operator as (Arshansky 1985)

$$t_{LP} = t - \frac{1}{2} \{ \mathbf{x}; \frac{\mathbf{p}E}{p^2} \} \quad (2.48)$$

where  $\frac{\mathbf{p}E}{p^2}$  is an *inverse velocity operator*, providing a shift in time for a virtual worldline (the semicolon implies both dot product as well as anticommutator). It then follows that

$$[t_{LP}, p] = -[\mathbf{x}, p] \cdot \frac{\mathbf{p}E}{p^2}.$$

But ( $p \equiv \sqrt{\mathbf{p}^2}$ )

$$[x_i, p] = i\hbar \frac{p_i}{p},$$

so that

$$[t_{LP}, p] = -i\hbar \frac{E}{p}. \quad (2.49)$$

It therefore follows from (2.49) that

$$\Delta t_{LP} \Delta p \geq \frac{1}{2} \hbar \langle E/p \rangle. \quad (2.50)$$

The quantity  $E/p$  is the magnitude of the inverse velocity operator; if the virtual velocity  $p/E$  is bounded within the wave packet by the velocity of light  $c$ , we obtain the Landau-Peierls bound (2.39) as a rigorous property of the wave function describing the state of the system. There is, in principle, however, no bound on the occurrence of components of the wave function with values of  $p/E$  greater than one. On the other hand, application of the Ehrenfest theorem (Ehrenfest 1927), when it is valid, would rule out this possibility for the same causal reasons given by Landau and Peierls. The Ehrenfest theorem for the relativistic theory has the same structure as in the nonrelativistic theory, resulting in the classical Hamilton equations for the motion of the peak of the wave packet in spacetime. We review the argument in the following.

Consider a wave packet of the form (for free evolution)

$$\psi_\tau(x) = \frac{1}{(2\pi)^2} \int e^{ip^\mu x_\mu - i \frac{p^\mu p_\mu}{2M} \tau} \chi(p), \quad (2.51)$$

where  $\chi(p)$ , the momentum representation of the state, is a fairly sharp distribution in  $p^\mu$ . The function  $\chi(p)$  is modulus square normalized to one over integration on all four momenta if  $\psi(x)$  is modulus square normalized to one over spacetime. For large  $\tau$ , if one may assume that the values of  $x^\mu$  also become large, the stationary phase values

$$x^\mu \sim \frac{p^\mu}{M} \tau \quad (2.52)$$

make the primary contribution, as in the nonrelativistic argument. The value of  $p^\mu$  under the integral that contributes corresponds to the sharp peak value of the momentum space wave function, and the corresponding peak in the  $x^\mu$  wave function describes the motion of a classical event, as described above in Eq. (2.8). In this case, a strong presence of spacelike momenta in the wavepacket could result in the evolution of the worldline in a spacelike direction, i.e., with  $\frac{p}{E}$  exceeding light velocity. We could therefore, on the same causal grounds as Landau and Peierls, arguing that  $\langle E/p \rangle$  must be greater than  $1/c$ , rule out such a configuration, and arrive at the Landau-Peierls relation from (2.50).

However, as Zaslavsky (1985) has pointed out in the context of the nonrelativistic theory, the conditions for the validity of the Ehrenfest theorem degrade (in this case as a function of  $\tau$ ) due to the spreading of the wave packet as well as the effect of interactions on the structure of  $\chi(p)$ . Zaslavsky (1985) called the time for validity of the Ehrenfest theorem the ‘‘Ehrenfest time’’, and argued that for quantum systems for which the classical Hamiltonian induces chaotic behavior the Ehrenfest time is less. Therefore, dynamical effects may occur in the relativistic theory which could result in deviations from the Landau-Peierls bound. We shall discuss this subject further in Chap. 4.

In the classical construction of Stueckelberg (1941) in Fig. 2.1, the worldline of the particle passes through a region which is spacelike. In this region, the corresponding Landau-Peierls bound would be violated, with the contrary inequality

$$\Delta p \Delta t < \hbar/c, \quad (2.53)$$

implying that the wave function could be arbitrarily narrow in the  $t$ -direction for a given  $p$  distribution. Thus, this diagram could be described by a quantum wave packet which has normal Ehrenfest form for the incoming and outgoing lines, but may have a vertex which is very sharp in  $t$  over a small but finite distance. The spacetime diagrams discussed by Feynman (1949) may be thought of as an idealization of this limit. The example of neutrino oscillations and similar phenomena in the  $K$  and  $B$  meson systems, also providing an illustration of this effect, are discussed in Chap. 4.

The relation (2.48) was constructed from a semiclassical interpretation of the quantum observables, a procedure that was justified in our study of the Newton-Wigner problem. In that case, we began with the straightforward computation of the expectation value of the  $\mathbf{x}$  operator, which has the same representation as in the nonrelativistic quantum mechanics. However, there is no corresponding analog in nonrelativistic quantum mechanics for a *time operator*; in the nonrelativistic quantum theory,  $t$  is a parameter of evolution, and its expectation value is a trivial identity (Ludwig 1982; Dirac 1930). We can, however, construct an argument analogous to that used for the Newton-Wigner problem within the framework of the relativistic theory, and show in the same way that the Landau-Peierls time operator (2.48) emerges from the mass-shell restriction of the expectation value of the relativistic time operator. To see this, consider the expectation value

$$\langle t \rangle = \int d^4 p \psi^*(\mathbf{p}, E) \left(-i \frac{\partial}{\partial E}\right) \psi(\mathbf{p}, E), \quad (2.54)$$

where we shall consider, for each value of  $m$  the magnitude of the momentum to be a function of  $E$ . Let us change the variables  $p^\mu$  to the form  $(\Omega, p, E)$ , where  $\Omega$  corresponds to the angular coordinate variables of  $\mathbf{p}$ , and define

$$p = \sqrt{E^2 - m^2} \quad (2.55)$$

Then,

$$d^4 p = p^2 d\Omega dp dE = -\frac{1}{2} p d\Omega dE dm^2. \quad (2.56)$$

We may then write

$$\langle t \rangle = -\frac{1}{2} \int p d\Omega dE dm^2 \psi^*(\sqrt{E^2 - m^2}, \Omega, E) \left[ -i \frac{\partial}{\partial E} \psi(\sqrt{E^2 - m^2}, \Omega, E) + i \frac{E}{p} \frac{\partial}{\partial p} \psi(p, \Omega, E) \Big|_{p=\sqrt{E^2 - m^2}} \right], \quad (2.57)$$

where the last term (containing the factor  $(\partial p / \partial E = E/p)$ ) compensates for the fact that after the change of variables,  $i \partial / \partial E$  acts on  $p$  as well as the last argument.

We now note that the Landau-Peierls operator (2.48) can be written as

$$t_{LP} = t - \frac{1}{2} \left[ i \frac{\partial}{\partial \mathbf{p}} \cdot \frac{\mathbf{p}E}{p^2} + i \frac{\mathbf{p}E}{p^2} \frac{\partial}{\partial \mathbf{p}} \right] \quad (2.58)$$

$$= -i \frac{\partial}{\partial E} - \frac{i}{2} \frac{E}{p^2} - i \frac{\mathbf{p}E}{p^2} \cdot \frac{\partial}{\partial \mathbf{p}},$$

where we have used the fact that (most simply, carrying this out component by component)

$$\frac{\partial}{\partial \mathbf{p}} \cdot \frac{\mathbf{p}E}{p^2} = \frac{E}{p^2}.$$

If we take the expectation value of  $t_{LP}$  in place of  $t$  as in (2.53), one sees that the last term in (2.56) cancels with the last term in (2.58), resulting in

$$\langle t_{LP} \rangle = -\frac{1}{2} \int p d\Omega dE dm^2 \psi^* (\sqrt{E^2 - m^2}, \Omega, E) \left[ -i \frac{\partial}{\partial E} - \frac{i}{2} \frac{E}{p^2} \right] \psi (\sqrt{E^2 - m^2}, \Omega, E) \quad (2.59)$$

We now follow an argument similar to that used above for the Newton-Wigner problem to find the wave function of an event which occurs at a definite sharp *time*.

If  $\psi_{t=0}(p)$  corresponds to a state for which an event is strictly localized to a point in time  $t = 0$ , the wave function  $\psi_{t=t_0}$  must be orthogonal to it for  $t_0 \neq 0$ . Therefore,

$$\int d^4 p \psi_{t=t_0}^*(p) \psi_{t=0}(p) = 0 \quad (2.60)$$

for  $t_0 \neq 0$ . However, using the Poincaré group property  $\psi_{t=t_0}(p) = e^{iEt_0} \psi_{t=0}$ , we have

$$\int d^4 p e^{-iEt_0} |\psi_{t=0}(p)|^2 = 0, \quad (2.61)$$

implying that

$$\int d^3 p |\psi_{t=0}(p)|^2 = \text{const} \times (E),$$

or,

$$\int d\Omega p^2 dp |\psi_{t=0}(p)|^2 = \text{const} \times E \quad (2.62)$$

But, as pointed out above,  $p^2 dp = -(1/2) p dm^2$ , so that (2.62) becomes

$$-\frac{1}{2} \int p dm^2 d\Omega |\psi_{t=0}(p)|^2 = \text{const} \times E. \quad (2.63)$$

If the mass of the particle is concentrated at some value of  $m$  we conclude that

$$\int d\Omega |\psi_{t=0}(p)|^2 = \frac{1}{p} \times \text{const}, \quad (2.64)$$

or, for a spherically symmetric wave function,

$$\psi_{t=0}(p) \propto \frac{1}{\sqrt{p}}.$$

Shifting by translation in  $t$ , we see that

$$\psi_t(p) \propto (E^2 - m^2)^{-\frac{1}{4}} e^{iEt}. \quad (2.65)$$

This result corresponds to the necessary form of a wave function at some given value of  $m$  and concentrated at some value of  $t$ , the analog of the Newton-Wigner wave function for a particle concentrated at a given point  $\mathbf{x}$ . A simple computation shows that

$$-i \left( \frac{\partial}{\partial E} - \frac{iE}{2p^2} \right) \psi_t(p) = t \psi_t(p). \quad (2.66)$$

Thus, the operator that appears in the expectation value in (2.59) at each value of  $m$  in the foliation induced by the change of variables (2.56) corresponds to the analog of the Newton-Wigner position operator (1.9) for time, restricted to a given mass value.

Clearly, the Fourier transform of the function  $\psi_{t_0}(p)$  of (2.65) (picking the localization point to be  $t = t_0$ ) into the time domain by the kernel  $\exp -iEt$  would not be localized in  $t$ , as for the Newton-Wigner problem in  $\mathbf{x}$ , and would therefore not form a viable quantum theory if, as we have assumed, the mass is concentrated at a fixed point. One could not use such wave functions to compute interference phenomena in time, as we shall discuss in Chap. 6.

We remark that, as for  $\mathbf{x}_{NW}$ , the Landau-Peierls operator  $t_{LP}$  is a constant of the free motion (as can be easily verified by computing their commutator with the free Hamiltonian). The (mean) intercepts of the virtual motions contained in the wave function, respectively to  $t = 0$  and to  $x = 0$  do not change under the free motion.

In the next chapter, we describe the basis for the construction of quantum states of particles with spin.

## Appendix A

We describe here the basic ideas of the so-called constraint theory formulation of a many particle (many *event*) relativistic mechanics. In this theory, describing the positions  $\{x_i^\mu\}$ , and momenta  $\{p_i^\mu\}$  for  $i = 1, 2, 3, \dots, N$  of the particles, a constraint is defined for each of the particles of the form (we use the metric  $(-, +, +, +)$ )

$$K_i = p_i^\mu p_{\mu i} + m_i^2 + \phi_i(x, p), \quad (2.67)$$

where, on the constraint hypersurface  $K_i \approx 0$ , the  $\phi_i(x, p)$  are functions of all the  $x$ 's and  $p$ 's, and the  $\{m_i\}$  are the given masses of the particles. This set of  $N$  constraints restricts the motion to an  $N$  dimensional hypersurface in the  $8N$  dimensional phase space.

The "first class" constraints  $K_i$  may act as generators of motion under Poisson bracket action (e.g. Itzyson 1980), thus defining the infinitesimal variations with

respect to the corresponding parameters  $\tau_i$  of the infinitesimal transformations of the coordinates and momenta by

$$\begin{aligned}\frac{dx_i}{d\tau_i} &= i\{K_i, x_i\}_{PB} \\ \frac{dp_i}{d\tau_i} &= i\{K_i, p_i\}_{PB},\end{aligned}\tag{2.68}$$

providing a set of first order equations describing the motion on this hypersurface. This manifestly covariant formalism has the advantage that one may assume the interaction terms  $\phi_i$  vanish asymptotically when the particles are far apart; the constraint conditions then enforce the particles to lie on mass shell ( $p^\mu_i p_{\mu_i} + m_i^2 = 0$ ).

In order to construct a world line for the system on the range of these motions, one generally introduces another set of  $N - 1$  constraints, called *second class constraints*, forming surfaces with intersection along a line on the  $N$  dimensional hypersurface, and an  $N$ th constraint which cuts this line and is a function of a single parameter  $\tau$ , thus describing motion along this world line (Sudarshan 1981a). It is possible, however, to define these constraints in another way, by constructing a Hamiltonian of the form (Rohrlich 1981)

$$K = \Sigma_i \omega_i(x, p) K_i.\tag{2.69}$$

The Poisson bracket of this Hamiltonian with any observable  $\mathcal{O}(x, p)$  then forms a linear combination

$$\frac{d\mathcal{O}}{d\tau} = \Sigma_i \omega_i \frac{d\mathcal{O}}{d\tau_i},\tag{2.70}$$

where we have taken into account that the  $K_i$  vanish on the constraint hypersurface; the  $\omega_i$  are then identified with  $d\tau_i/d\tau$ , with the  $\tau_i$  considered as functions of the overall evolution parameter  $\tau$ .

Although this approach is very elegant on a classical level, there are some difficulties in passing to the quantum theory. The condition  $K_i = 0$  poses a difficult problem since, in general, the  $K_i$  have continuous spectrum, and the eigenstates would lie outside the Hilbert space. This problem can be treated by defining  $N$  Schrödinger type equations of the form (as for the treatment of cases with states in the continuous spectrum in the nonrelativistic theory)

$$i \frac{\partial \psi_{\tau_1, \tau_2, \dots}}{\partial \tau_i} = K_i \psi_{\tau_1, \tau_2, \dots}\tag{2.71}$$

but the combination  $\Sigma_i \omega_i(x, p) K_i$  would, in general, not be Hermitian. The symmetric product with the  $\omega_i$ 's would not be useful, since the functions  $\omega_i$  have no well-defined action on  $\psi_{\tau_1, \tau_2, \dots}$ . Nevertheless, Rohrlich and the author succeeded in formulating a viable scattering theory in this framework (see references under Llosa 1982).

We shall discuss in this chapter the basic idea of a relativistic particle with spin, based on Wigner's seminal work (Wigner 1939). The theory is adapted here to be applicable to relativistic quantum theory; in this form, Wigner's theory, together with the requirements imposed by the observed correlation between spin and statistics in nature for identical particle systems, makes it possible to define the total spin of a state of a relativistic many body system.

We shall show, furthermore, that a generalization of the construction of Wigner yields, in the framework we shall present here, a representation for tensor operators corresponding to an *invariant* decomposition in terms of irreducible representations of  $SU(2)$ ; this procedure may be applied as well to spinorial valued operators, such as Rarita-Schwinger fields (Rarita 1941).

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## 3.1 Relativistic Spin and the Dirac Representation

The spin of a particle in a nonrelativistic framework corresponds to the lowest dimensional nontrivial representation of the rotation group; the generators are the Pauli matrices  $\sigma_i$  divided by two, the generators of the fundamental representation of the double covering of  $SO(3)$ . The self-adjoint operators that are the generators of this group measure angular momentum and are associated with magnetic moments. Such a description is not relativistically covariant, but Wigner (1939) has shown how to describe this dynamical property of a particle in a covariant way. The method developed by Wigner provided the foundation for what is now known as the theory of induced representations (Mackey 1968), with very wide applications, including a very powerful approach to finding the representations of noncompact groups.

We shall show here how Wigner's approach can be used to describe the spin of a particle in the framework of the manifestly covariant theory of Stueckelberg, Horwitz and Piron (SHP; Stueckelberg 1941; Horwitz 1973), and how this method can be extended to describe the combined spin states of a many body system.



In the nonrelativistic quantum theory, the spin states of a two or more particle system are defined by combining the spins of these particles at equal time using appropriate Clebsch-Gordan coefficients (Clebsch 1872) at each value of the time. The restriction to equal time follows from the tensor product form of the representation of the quantum states for a many body problem (Baym 1969; Fetter 1971). For two spin 1/2 (Fermi-Dirac) particles, an antisymmetric space distribution would correspond to a symmetric combination of the spin factors, i.e. a spin one state, and a symmetric space distribution would correspond to an antisymmetric spin combination, a spin zero state.<sup>1</sup> This correlation is the source of the famous Einstein-Podolsky-Rosen discussion (Einstein 1935) and provides an important model for quantum information transfer. The experiment proposed by Palacios et al. (2009) suggests that spin entanglement can occur for two particles at non-equal times; the spin carried by wave functions of SHP type would naturally carry such correlations over the width in  $t$  of the wave packets, and therefore the formulation we shall present here would be appropriate for application to relativistic quantum information transfer (e.g., Aharonov 1982; Hu 2012; Lin 2009; Lizier 2013).

Wigner (1939) worked out a method for defining spin for relativistic particles. This formulation is not appropriate for application to quantum theory, since it does not preserve, as we shall explain below, the covariance of the expectation value of coordinate operators. Before constructing a generalization of Wigner's method which is useful in relativistic quantum theory we first review Wigner's method in its original form, and show how the difficulties arise.

To establish some notation and the basic method, we start with the basic principle of relativistic covariance for a scalar quantum wave function  $\psi(p)$ . In a new Lorentz frame described by the parameters  $\Lambda$  of the Lorentz group, for which  $p'^{\mu} = \Lambda^{\mu}_{\nu} p^{\nu}$  (we work in momentum space here for convenience), the same physical point in momentum space described in different coordinates, by arguing that the probability density must be the same, is associated with the wave function

$$\psi'(p') = \psi(p) \tag{3.1}$$

up to a phase, which we take to be unity. It then follows that as a function of  $p$ ,

$$\psi'(p) = \psi(\Lambda^{-1}p). \tag{3.2}$$

Since, in Dirac's notation,

$$\psi'(p) \equiv \langle p | \psi' \rangle, \tag{3.3}$$

Equation (2.67) follows equivalently by writing

$$|\psi' \rangle = U(\Lambda) |\psi \rangle \tag{3.4}$$

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<sup>1</sup>See also the very informative study of Jabs 2010, and the discussion of Bennett (2015).

so that

$$\begin{aligned}
 \langle p|\psi' \rangle &= \langle p|U(\Lambda)|\psi \rangle \\
 &= \langle \Lambda^{-1}p|\psi \rangle \\
 &= \psi(\Lambda^{-1}p),
 \end{aligned} \tag{3.5}$$

where we have used

$$U(\Lambda)^\dagger|p \rangle = U(\Lambda^{-1})|p \rangle = |\Lambda^{-1}p \rangle .$$

To discuss the transformation properties of the representation of a relativistic particle with spin, Wigner proposed that we consider a special frame in which  $p_0^\mu = (m, 0, 0, 0)$ ; the subgroup of the Lorentz group that leaves this vector invariant is clearly  $O(3)$ , the rotations in the three space in which  $\mathbf{p} = 0$ , or its covering  $SU(2)$ . Under a Lorentz boost, transforming the system to its representation in a moving inertial frame, the rest momentum appears as  $p_0^\mu \rightarrow p^\mu$ , but under this unitary transformation, the subgroup that leaves  $p_0^\mu$  invariant is carried to a form which leaves  $p^\mu$  invariant, and the group remains  $SU(2)$ . The  $2 \times 2$  matrices representing this group are altered by the Lorentz transformation, and are functions of the momentum  $p^\mu$ . The resulting state then transforms by a further change in  $p^\mu$  and an  $SU(2)$  transformation compensating for this change. This additional transformation is called the “little group” of Wigner. The family of values of  $p^\mu$  generated by Lorentz transformations on  $p_0^\mu$  is called the “orbit” of the induced representation. This  $SU(2)$ , in its lowest dimensional representation, parametrized by  $p^\mu$  and the additional Lorentz transformation  $\Lambda$ , corresponds to Wigner’s covariant relativistic definition of the spin of a relativistic particle (Wigner 1937).

We now apply this method to review Wigner’s construction based on a representation induced on the momentum  $p^\mu$ . Let us define the momentum-spin ket

$$|p, \sigma \rangle \equiv U(L(p))|p_0, \sigma \rangle, \tag{3.6}$$

where  $U(L(p))$  is the unitary operator inducing a Lorentz transformation of the timelike  $p_0 = (m, 0, 0, 0)$  (rest frame momentum) to the general timelike vector  $p^\mu$ . The effect of a further Lorentz transformation parameterized by  $\Lambda$ , induced by  $U(\Lambda^{-1})$ , can be written as

$$U(\Lambda^{-1})|p, \sigma \rangle = U(L(\Lambda^{-1}p))U^{-1}(L(\Lambda^{-1}p))U(\Lambda^{-1})U(L(p))|p_0, \sigma \rangle \tag{3.7}$$

The product of the last three unitary factors

$$U^{-1}(L(\Lambda^{-1}p))U(\Lambda^{-1})U(L(p)) \tag{3.8}$$

has the property that under this combined unitary transformation, the ket is transformed so that  $p_0 \rightarrow p_0$ , and thus corresponds to just a rotation (called the Wigner rotation), the stability subgroup of the vector  $p_0$ . This rotation can be represented by a  $2 \times 2$  matrix acting on the index  $\sigma$ , i.e., so that

$$U(\Lambda^{-1})|p, \sigma \rangle = U(L(\Lambda^{-1}p))|p_0, \sigma' \rangle D_{\sigma, \sigma'}(\Lambda, p) = |\Lambda^{-1}p, \sigma' \rangle D_{\sigma, \sigma'}(\Lambda, p). \tag{3.9}$$

where, as a representation of rotations,  $D$  is unitary. Therefore, taking the complex conjugate of

$$\langle \psi|U(\Lambda^{-1})|p, \sigma \rangle = \langle \psi|\Lambda^{-1}p, \sigma' \rangle D_{\sigma, \sigma'}(\Lambda, p),$$

one obtains

$$\langle p, \sigma | U(\Lambda) \psi \rangle = \langle \Lambda^{-1} p, \sigma' | \psi \rangle D_{\sigma', \sigma}(\Lambda p), \quad (3.10)$$

where, in this construction, we have

$$D_{\sigma', \sigma}(\Lambda, p) = \left( (L(p)^{-1} \Lambda L(\Lambda^{-1} p)) \right)_{\sigma', \sigma}, \quad (3.11)$$

expressed in terms of the  $SL(2, C)$  matrices corresponding to the unitary transformation (3.8). This representation of the unitary transformation is a homomorphism due to the fact that this subgroup is compact, and has finite dimensional unitary representations, in particular, the one we use here (we could have chosen other representations corresponding to particles carrying intrinsic angular momentum not equal to  $1/2$ ). The result (3.10) can be written as

$$\psi'(p, \sigma) = \psi(\Lambda^{-1} p, \sigma') D_{\sigma', \sigma}(\Lambda, p), \quad (3.12)$$

in accordance with (3.2), generalized to take into account the spin degrees of freedom of the wavefunction. The algebra of the  $2 \times 2$  matrices of the fundamental representation of the group  $SL(2, C)$  are isomorphic to that of the Lorentz group, and the product of the corresponding matrices provide the  $2 \times 2$  matrix representation of  $D_{\sigma', \sigma}(\Lambda, p)$ ; we may therefore write (2.77) as

$$D_{\sigma', \sigma}(\Lambda, p) = \left( L^{-1}(p) \Lambda L(\Lambda^{-1} p) \right)_{\sigma', \sigma}, \quad (3.13)$$

where  $L$  and  $\Lambda$  are the  $2 \times 2$  matrices of  $SL(2, C)$ . We discuss these matrices ( $2 \times 2$  matrices of complex numbers with determinant unity) and the representation they provide for the Lorentz group in Appendix B.

As we have mentioned above, the presence of the  $p$ -dependent matrices representing the spin of a relativistic particle in the transformation law of the wave function destroys the covariance, in a relativistic quantum theory, of the expectation value of the coordinate operators. To see this, consider the expectation value of the dynamical variable  $x^\mu$ , i.e.

$$\langle x^\mu \rangle = \Sigma_\sigma \int d^4 p \psi(p, \sigma)^\dagger i \frac{\partial}{\partial p_\mu} \psi(p, \sigma).$$

A Lorentz transformation would introduce the  $p$ -dependent  $2 \times 2$  unitary transformation on the function  $\psi(p)$ , and the derivative with respect to momentum would destroy the covariance property that we would wish to see of the expectation value  $\langle x^\mu \rangle$ .

It is also not possible, in this framework, to form wave packets of definite spin by integrating over the momentum variable, since this would add functions over different parts of the orbit, with a different  $SU(2)$  at each point.

As will be described in the following, these problems were solved by inducing a representation of the spin on a timelike unit vector  $n^\mu$  in place of the four-momentum, using a representation induced on a timelike vector, say,  $n^\mu$ , which is independent of  $x^\mu$  or  $p^\mu$  (Horwitz 1975; Arshansky 1982). This solution also permits the linear superposition of momentum states to form wave packets of definite spin, and admits the construction of definite spin states for many body relativistic systems and its consequences for entanglement. In the following, we show how such a representation can be constructed, and discuss some of its dynamical implications.

To carry out this program, let us define, as in (3.6),

$$|n, \sigma \rangle = U(L(n))|n_0, \sigma \rangle \quad (3.14)$$

The generators of the transformations  $U(\Lambda)$  act on the full vector space of both the  $n^\mu$  and the  $x^\mu$  (as well as  $p^\mu$ ). In terms of the canonical variables,

$$M^{\mu\nu} = M_n^{\mu\nu} + (x^\mu p^\nu - x^\nu p^\mu). \quad (3.15)$$

where

$$M_n^{\mu\nu} = -i \left( n^\mu \frac{\partial}{\partial n_\nu} - n^\nu \frac{\partial}{\partial n_\mu} \right) \quad (3.16)$$

The two terms of the full generator commute. Following the method outlined above, we now investigate the properties of a total Lorentz transformation, i.e.

$$U(\Lambda^{-1})|n, \sigma \rangle = U(L(\Lambda^{-1}n))(U^{-1}(L(\Lambda^{-1}n))U(\Lambda^{-1})U(L(n)))|n_0, \sigma \rangle. \quad (3.17)$$

Now, consider the conjugate of (3.17),

$$\langle n, \sigma | U(\Lambda) = \langle n_0, \sigma | (U(L^{-1}(n))U(\Lambda)U(L(\Lambda^{-1}n)))U^{-1}(L(\Lambda^{-1}n)). \quad (3.18)$$

The operator in the first factor (in parentheses) preserves  $n_0$ , and therefore, as before, contains an element of the little group associated with  $n^\mu$  which may be represented by the matrices of  $SL(2, C)$ . We now define a state vector in terms of a vector-valued function  $\Psi(x) \in L^2(R^4)$  for which  $\langle n, \sigma | \Psi(x) \rangle = \psi_{n\sigma}(x)$ , so that

$$\langle n^0 \sigma | U^{-1}(L(\Lambda^{-1}n))\Psi(x) \rangle = \psi_{\Lambda^{-1}n\sigma}(x). \quad (3.19)$$

For  $\Psi'(x) \equiv U(\Lambda)\Psi(x)$ , contracting both sides of (3.18) with  $\Psi(x)$ , we obtain

$$\psi'_{n,\sigma}(x) = \psi_{\Lambda^{-1}n,\sigma'}(\Lambda^{-1}x)D_{\sigma',\sigma}(\Lambda, n). \quad (3.20)$$

where

$$D(\Lambda, n) = L^{-1}(n)\Lambda L(\Lambda^{-1}n), \quad (3.21)$$

with  $\Lambda$  and  $L(n)$  the corresponding  $2 \times 2$  matrices of  $SL(2, C)$ .  $\Lambda$  and  $L(n)$  to be the corresponding  $2 \times 2$  matrices of  $SL(2, C)$ .

It is clear that, with this transformation law, one may take the Fourier transform to obtain the wave function in momentum space, and conversely. The matrix  $D$  is an element of  $SU(2)$ , and therefore linear superpositions over momenta or coordinates maintain the definition of the particle spin, and interference phenomena for relativistic particles with spin may be studied consistently. Furthermore, if two or more particles with spin are represented in representations induced on  $n^\mu$ , at a given value of  $n^\mu$  on their respective orbits, their spins can be added by the standard methods with the use of Clebsch-Gordan coefficients (Clebsch 1872). This method therefore admits the treatment of a many body relativistic system with spin.

Our assertion of the unitarity of the  $n$ -dependent part of the transformation has assumed that the integral measure on the Hilbert space, to admit integration by parts,

is of the form  $d^4 n d^4 x \delta(n^\mu n_\mu + 1)$ , i.e., although the timelike vector  $n^\mu$ , in many applications, is degenerate, it carries a probability interpretation under the norm, and may play a dynamical role.

There are two fundamental representations of  $SL(2, C)$  which are inequivalent (Boerner 1963). Multiplication by the operator  $\sigma \cdot p$  of a two dimensional spinor representing one of these results in an object transforming like the second representation. Such an operator could be expected to occur in a dynamical theory, and therefore the state of lowest dimension in spinor indices of a physical system should contain both representations. As we shall emphasize, however, in our treatment of the more than one particle system, for the rotation subgroup, both of the fundamental representations yield the same  $SU(2)$  matrices up to a unitary transformation, and therefore the Clebsch-Gordan decomposition of the product state into irreducible representations may be carried out independently of which fundamental  $SL(2, C)$  representation is associated with each of the particles.

We now discuss the construction of Dirac spinors. An approximate treatment of the Dirac equation in interaction with electromagnetism yields a connection with spin, identified through its interaction with the magnetic field (Bjorken 1964). As we shall see, however, the particle spin is already contained in the construction of the Dirac function through the fundamental construction of Wigner, combining the two fundamental representations of  $SL(2, C)$  (Arshansky 1982; Weinberg 1995).

We first remark that the defining relation for the fundamental  $SL(2, C)$  matrices is

$$\Lambda^\dagger \sigma^\mu n_\mu \Lambda = \sigma^\mu (\Lambda^{-1} n)_\mu, \quad (3.22)$$

where  $\sigma^\mu = (\sigma^0, \sigma)$ ;  $\sigma^0$  is the unit  $2 \times 2$  matrix, and  $\sigma$  are the Pauli matrices. Since the determinant of  $\sigma^\mu n_\mu$  is the Lorentz invariant  $n^{02} - \mathbf{n}^2$ , and the determinant of  $\Lambda$  is taken to be unity in  $SL(2, C)$ , the transformation represented on the left hand side of (3.22) must induce a Lorentz transformation on  $n^\mu$ . The inequivalent second fundamental representation may be constructed by using this defining relation with  $\sigma^\mu$  replaced by  $\underline{\sigma}^\mu \equiv (\sigma^0, -\sigma)$ . For every Lorentz transformation  $\Lambda$  acting on  $n^\mu$ , this defines an  $SL(2, C)$  matrix  $\underline{\Lambda}$  (we use the same symbol for the Lorentz transformation on a four-vector as for the corresponding  $SL(2, C)$  matrix acting on the 2-spinors).

Since both fundamental representations of  $SL(2, C)$  should occur in the general quantum wave function representing the state of the system, the norm in each  $n$ -sector of the Hilbert space must be defined as

$$N = \int d^4 x (|\hat{\psi}_n(x)|^2 + |\hat{\phi}_n(x)|^2), \quad (3.23)$$

where  $\hat{\psi}_n$  transforms with the first  $SL(2, C)$  and  $\hat{\phi}_n$  with the second. From the construction of the little group (3.21), it follows that  $L(n)\psi_n$  transforms with  $\Lambda$ , and  $\underline{L}(n)\phi_n$  transforms with  $\underline{\Lambda}$ ; making this replacement in (3.23), and using the fact, obtained from the defining relation (3.22), that  $L(n)^{\dagger-1}L(n)^{-1} = \mp \sigma^\mu n_\mu$  and  $\underline{L}(n)^{\dagger-1}\underline{L}(n)^{-1} = \mp \underline{\sigma}^\mu n_\mu$ , one finds that

$$N = \mp \int d^4 x \bar{\psi}_n(x) \gamma \cdot n \psi_n(x), \quad (3.24)$$

where  $\gamma \cdot n \equiv \gamma^\mu n_\mu$  (for which  $(\gamma \cdot n)^2 = -1$ ), and the matrices  $\gamma^\mu$  are the Dirac matrices as defined in the books of Bjorken and Drell (1964). Here, the four-spinor  $\psi_n(x)$  is defined by

$$\psi_n(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} L(n)\hat{\psi}_n(x) \\ L(n)\hat{\phi}_n(x) \end{pmatrix}, \quad (3.25)$$

and the sign  $\mp$  corresponds to  $n^\mu$  in the positive or negative light cone. The wave function defined by (3.25) transforms as

$$\psi'_n(x) = S(\Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \quad (3.26)$$

and  $S(\Lambda)$  is a (nonunitary) transformation generated infinitesimally, as in the standard Dirac theory (see, for example, Bjorken 1964; Weinberg 1995), by  $\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ .

The Dirac operator  $\gamma \cdot p$  is not Hermitian in the (invariant) scalar product associated with the norm (3.24). It is of interest to consider the Hermitian and anti-Hermitian parts

$$\begin{aligned} K_L &= \frac{1}{2}(\gamma \cdot p + \gamma \cdot n\gamma \cdot p\gamma \cdot n) = -(p \cdot n)(\gamma \cdot n) \\ K_T &= \frac{1}{2}\gamma^5(\gamma \cdot p - \gamma \cdot n\gamma \cdot p\gamma \cdot n) = -2i\gamma^5(p \cdot K)(\gamma \cdot n), \end{aligned} \quad (3.27)$$

where  $K^\mu = \Sigma^{\mu\nu}n_\nu$ , and we have introduced the factor  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ , which anticommutes with each  $\gamma^\mu$  and has square  $-1$  so that  $K_T$  is Hermitian and commutes with the Hermitian  $K_L$ . Since

$$K_L^2 = (p \cdot n)^2 \quad (3.28)$$

and

$$K_T^2 = p^2 + (p \cdot n)^2, \quad (3.29)$$

we may consider

$$K_T^2 - K_L^2 = p^2 \quad (3.30)$$

to pose an eigenvalue problem analogous to the second order mass eigenvalue condition for the free Dirac equation (the Klein Gordon condition). For the Stueckelberg equation of evolution corresponding to the free particle, we may therefore take

$$K_0 = \frac{1}{2M}(K_T^2 - K_L^2) = \frac{1}{2M}p^2. \quad (3.31)$$

In the presence of electromagnetic interaction, gauge invariance under a spacetime dependent gauge transformation (we discuss the more general case of a gauge transformation depending on  $\tau$  as well in the next chapter), the expressions for  $K_T$  and  $K_L$  given in (3.27), in gauge covariant form, then imply, in place of (3.31),

$$K = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}\Sigma_n^{\mu\nu}F_{\mu\nu}(x), \quad (3.32)$$

where

$$\Sigma_n^{\mu\nu} = \Sigma^{\mu\nu} + K^\mu n^\nu - K^\nu n^\mu \equiv \frac{i}{4}[\gamma_n^\mu, \gamma_n^\nu], \quad (3.33)$$

where the  $\gamma_n^\mu$  are defined in (3.37). The expression (3.32) is quite similar to that of the second order Dirac operator; it is, however, Hermitian and has no direct electric coupling to the electromagnetic field in the special frame for which  $n^\mu = (1, 0, 0, 0)$  in the minimal coupling model we have given here (note that in his calculation of the anomalous magnetic moment (Schwinger 1951), Schwinger puts the electric field to zero; a non-zero electric field would lead to a non-Hermitian term in the standard Dirac propagator, the inverse of the Klein-Gordon square of the interacting Dirac equation). The matrices  $\Sigma_n^{\mu\nu}$  are, in fact, a relativistically covariant form of the Pauli matrices.

To see this, we note that the quantities  $K^\mu$  and  $\Sigma_n^{\mu\nu}$  satisfy the commutation relations

$$\begin{aligned} [K^\mu, K^\nu] &= -i\Sigma_n^{\mu\nu} \\ [\Sigma_n^{\mu\nu}, K^\lambda] &= -i[(g^{\mu\lambda} + n^\nu n^\lambda)K^\mu - (g^{\nu\lambda} + n^\mu n^\lambda)K^\nu], \\ [\Sigma_n^{\mu\nu}, \Sigma_n^{\lambda\sigma}] &= -i[(g^{\nu\lambda} + n^\nu n^\lambda)\Sigma_n^{\mu\sigma} + (g^{\sigma\mu} + n^\sigma n^\mu)\Sigma_n^{\lambda\nu} \\ &\quad - (g^{\mu\lambda} + n^\mu n^\lambda)\Sigma_n^{\nu\sigma} + (g^{\sigma\nu} + n^\sigma n^\nu)\Sigma_n^{\lambda\mu}]. \end{aligned} \quad (3.34)$$

Since  $K^\mu n_\mu = n_\mu \Sigma_n^{\mu\nu} = 0$ , there are only three independent  $K^\mu$  and three  $\Sigma_n^{\mu\nu}$ . The matrices  $\Sigma_n^{\mu\nu}$  are a covariant form of the Pauli matrices, and the last of (3.34) is the Lie algebra of  $SU(2)$  in the spacelike surface orthogonal to  $n^\mu$ . The three independent  $K^\mu$  correspond to the non-compact part of the algebra which, along with the  $\Sigma_n^{\mu\nu}$  provide a representation of the Lie algebra of the full Lorentz group. The covariance of this representation follows from

$$S^{-1}(\Lambda)\Sigma_{\Lambda n}^{\mu\nu}S(\Lambda)\Lambda_\mu^\lambda\Lambda_\nu^\sigma = \Sigma_n^{\lambda\sigma}. \quad (3.35)$$

In the special frame for which  $n^\mu = (1, 0, 0, 0)$ ,  $\Sigma_n^{i,j}$  become the Pauli matrices  $\frac{1}{2}\sigma^k$  with  $(i, j, k)$  cyclic, and  $\Sigma_n^{0j} = 0$ . In this frame there is no direct electric interaction with the spin in the minimal coupling model (3.33). We remark that there is, however, a natural spin coupling which becomes pure electric in the special frame, given by

$$i[K_T, K_L] = -ie\gamma^5(K^\mu n^\nu - K^\nu n^\mu)F_{\mu\nu}. \quad (3.36)$$

It is a simple exercise to show that the value of this commutator reduces to  $\mp e\gamma^5\sigma \cdot \mathbf{E}$  in the special frame for which  $n^0 = -1$ ; this operator is Hermitian and would correspond to an electric dipole interaction with the spin.

Note that the matrices

$$\gamma_n^\mu = \gamma_\lambda \pi^{\lambda\mu}, \quad (3.37)$$

where the projection

$$\pi^{\lambda\mu} = g^{\lambda\mu} + n^\lambda n^\mu, \quad (3.38)$$

appearing in (3.34), play an important role in the description of the dynamics in the induced representation. In (3.32), the existence of projections on each index in the

spin coupling term implies that  $F^{\mu\nu}$  can be replaced by  $F_n^{\mu\nu}$  in this term, a tensor projected into the foliation subspace.

We further remark that in relativistic scattering theory, the  $S$ -matrix is Lorentz invariant (Bjorken 1964). The asymptotic states can be decomposed according to the conserved projection operators

$$\begin{aligned} P_{\pm} &= \frac{1}{2}(1 \mp \gamma \cdot n) \\ P_{E\pm} &= \frac{1}{2}\left(1 \mp \frac{p \cdot n}{|p \cdot n|}\right) \end{aligned} \quad (3.39)$$

and

$$P_{n\pm} = \frac{1}{2}\left(1 \pm \frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}}\right).$$

The operator

$$\frac{2i\gamma^5 K \cdot p}{[p^2 + (p \cdot n)^2]^{1/2}} \rightarrow \gamma^5 \sigma \cdot \mathbf{p}/|\mathbf{p}| \quad (3.40)$$

when  $n^\mu \rightarrow (1, 0, 0, 0)$ . i.e.,  $P_{n\pm}$  corresponds to a helicity projection. Therefore the matrix elements of the  $S$ -matrix at any point on the orbit of the induced representation is equivalent (by replacing  $S$  by  $U(L(n))SU^{-1}(L(n))$ ) to the corresponding helicity representation associated with the frame in which  $n$  is  $n_0$ .<sup>2</sup>

We shall show in a later chapter how the Lorentz force can be computed. We shall, furthermore, see that the anomalous magnetic moment of the electron can be computed in this framework (Bennett 2012) without appealing to the full quantum field theory of electrodynamics.

Note that the discrete symmetries act on the wavefunctions as

$$\begin{aligned} \psi_{\tau n}^C &= C\gamma^0\psi_{-\tau n}^*(x) \\ \psi_{\tau n}^P(x) &= \gamma^0\psi_{\tau, -\mathbf{n}, n_0}(-\mathbf{x}, t), \\ \psi_{\tau n}^T &= i\gamma^1\gamma^3\psi_{-\tau, \mathbf{n}, -n_0}^*(\mathbf{x}, -t), \\ \psi_{\tau n}^{CPT}(x) &= i\gamma^5\psi_{\tau, -\mathbf{n}}(-\mathbf{x}, -t), \end{aligned} \quad (3.41)$$

where  $C = i\gamma^2\gamma^0$ . The  $CPT$  conjugate wavefunction, according to its evolution in  $\tau$ , moves backwards in spacetime relative to the motion of  $\psi_{\tau n}$ . For a wave packet with  $E < 0$  components, which moves backwards in  $t$  as  $\tau$  goes forward, it is the  $CPT$  conjugate wavefunction which moves forward with charge  $-e$ , i.e., the observed antiparticle. No Dirac sea (Dirac 1932) is required for the consistency of the theory, since unbounded transitions to  $E < 0$  are prevented by conservation of  $K$ .

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<sup>2</sup>This result is consistent with the suggestion of Aharonov (1983) that  $n_0$  may be interpreted as corresponding to the frame of the Stern-Gerlach apparatus in which the spin state is prepared.



### 3.2 The Many Body Problem with Spin, and Spin-Statistics

As in the nonrelativistic quantum theory, one represents the state of an  $N$ -body system in terms of a basis given by the tensor product of  $N$  one-particle states, each an element of a one-particle Hilbert space. The general state of such an  $N$ -body system is given by a linear superposition over this basis (Fetter and Walecka 1971). Second quantization then corresponds to the construction of a Fock space, for which the set of all  $N$  body states, for all  $N$  are imbedded in a large Hilbert space, for which operators that change the number  $N$  are defined (Baym 1969). We shall discuss this structure in this section, and show, with our discussion of the relativistic spin given in the previous section, that the spin of a relativistic many-body system can be well-defined (see also, Bennett 2015).<sup>3</sup> In order to construct the tensor product space corresponding to the many-body system, we consider, as for the nonrelativistic theory, the product of wave functions which are elements of the same Hilbert space. In the nonrelativistic theory, this corresponds to functions at equal time; in the relativistic theory, the functions are taken to be at equal  $\tau$ . Thus, in the relativistic theory, there are correlations at unequal  $t$ , within the support of the Stueckelberg wave functions. Moreover, for particles with spin we argue that in the induced representation, these function must be taken at *identical values of  $n^\mu$* , i.e., taken at the same point on the orbits of the induced representation of each particle (Horwitz 2013):

*Identical particles must be represented in tensor product states by wave functions at equal  $\tau$  and equal  $n^\mu$ .*

The proof of this statement lies in the observation that the spin-statistics relation appears to be a universal fact of nature. The elementary proof of this statement, for example, for a system of two spin  $1/2$  particles, is that a  $\pi$  rotation of the system introduces a phase factor of  $e^{i\frac{\pi}{2}}$  for each particle, thus introducing a minus sign for the two body state. However, the  $\pi$  rotation is equivalent to an interchange of the two identical particles. This argument rests on the fact that each particle is in the same representation of  $SU(2)$ , which can only be achieved in the induced representation with the particles at the same point on their respective orbits. The same argument applies for bosons, which must be symmetric under interchange (in this case the phase of each factor in a pair is  $e^{i\pi}$ ). We therefore see that identical particles must carry the same value of  $n^\mu$ , and the construction of the  $N$ -body system must follow this rule. It therefore follows that the two body relativistic system can carry a spin computed by use of the usual Clebsch-Gordan coefficients, and entanglement would follow even at unequal time (within the support of the equal  $\tau$  wave functions), as

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<sup>3</sup>Jabs (2010) has noted that, with Jacob and Wick (1959) one can rotate the eigenfunctions of momentum separately so that the momenta are collinear and thus identify the Wigner little groups; this operation leaves the helicities invariant. The spin wave function would, however, develop phases that are not controlled by the helicities alone, so this procedure is not sufficient to provide a common  $SU(2)$ , as we shall see below.

in the proposed experiment of Palacios et al. (2009). This argument can be followed for arbitrary  $N$ , and therefore the Fock space of quantum field theory, as we show below, carries the properties usually associated with fermion (or boson) fields, with the entire Fock space foliated over the orbit of the inducing vector  $n^\mu$ .

We remark that since the relativistic  $S$ -matrix is Lorentz invariant, the matrix elements of the  $S$ -matrix in states labelled by the asymptotic projections  $P_{n^\pm}$  (defined in (3.39) can be replaced (by the substitution  $U(L(n)SU^{-1}(L(n)$  for  $S$ ) by helicities in the common frame in which  $n^\mu \rightarrow (1, 0, 0, 0)$ . The Lorentz transformation that achieves this acts in the same way on all of the momenta of the asymptotic states and the resulting measured cross sections for this helicity representation then correspond to a choice of frame in which the common orbit is specified to be at the point  $n^\mu = (1, 0, 0, 0)$ .<sup>4</sup>

Although, due to the Newton-Wigner problem discussed above, the solutions of the Dirac equation are not suitable for the covariant local description of a quantum theory, the functions constructed in (3.25), under the norm (3.24), can form the basis of a consistent covariant quantum theory; they describe the (off-shell) states of a *local* quantum theory.

We then start by constructing a two body Hilbert space in the framework of the relativistic quantum theory. The states of this two body space are given by linear combinations over the product wave functions, where the wave functions (for the spin (1/2) case) are given by the Dirac function of the type described in (3.25) (or, for integer spin functions), i.e.,

$$\psi_{ij}(x_1, x_2) = \psi_i(x_1) \times \psi_j(x_2), \quad (3.42)$$

where  $\psi_i(x_1)$  and  $\psi_j(x_2)$  are elements of the one-particle Hilbert space  $\mathcal{H}$ . Let us introduce the notation, often used in differential geometry, that

$$\psi_{ij}(x_1, x_2) = \psi_i \otimes \psi_j(x_1, x_2), \quad (3.43)$$

identifying the arguments according to a standard ordering. Then, without specifying the spacetime coordinates, we can write

$$\psi_{ij} = \psi_i \otimes \psi_j, \quad (3.44)$$

formally, an element of the tensor product space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . The scalar product is carried out by pairing the elements in the two factors according to their order, since it corresponds to integrals over  $x_1, x_2$ , i.e.,

$$(\psi_{ij}, \psi_{k\ell}) = (\psi_i, \psi_k)(\psi_j, \psi_\ell). \quad (3.45)$$

For two identical particle states satisfying Bose-Einstein or Fermi-Dirac statistics, we must write, according to our argument given above,

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{in} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{in}], \quad (3.46)$$

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<sup>4</sup>This result, as mentioned above, is in accordance with Aharonov's suggestion (Aharonov 1983) that the Stern-Gerlach apparatus for preparation of the spin state is labelled by this ("rest") value  $n_0$  of  $n$ .

where  $n \equiv n^\mu$  is the timelike four vector labelling the orbit of the induced representation. This expression has the required symmetry or antisymmetry only if both functions are on the same points of their respective orbits in the induced representation. Furthermore, they transform under the *same*  $SU(2)$  representation of the rotation subgroup of the Lorentz group, and thus for spin  $1/2$  particles, under a  $\pi$  spatial rotation (defined by the space orthogonal to the timelike vector  $n^\mu$ ) they both develop a phase factor  $e^{i\frac{\pi}{2}}$ . The product results in an over all negative sign. As in the usual quantum theory, this rotation corresponds to an interchange of the two particles, but here with respect to a “spatial” rotation around the vector  $n^\mu$ . The spacetime coordinates in the functions are rotated in this (foliated) subspace of spacetime, and correspond to an actual exchange of the positions of the particles on a spacelike hyperplane, as in the formulation of the standard spin-statistics theorem. It therefore follows that the interchange of the particles occurs in the foliated space defined by  $n^\mu$ , and, furthermore:

*The antisymmetry of identical spin  $1/2$  (fermionic) particles remains at unequal times (within the support of the wave functions). This is true for the symmetry of identical spin zero (bosonic) particles as well.*

The construction we have given enables us to define the spin of a many body system, even if the particles are relativistic and moving arbitrarily with respect to each other.

*The spin of an  $N$ -body system is well-defined, independent of the state of motion of the particles of the system, by the usual laws of combining representations of  $SU(2)$ , i.e., with the usual Clebsch-Gordan coefficients, if the states of all the particles in the system are in induced representations at the same point of the orbit  $n^\mu$ .*

Thus, in the quark model for hadrons (Gell-Mann 1962; Ne’eman 1961), the total spin of the hadron can be computed from the spins (and orbital angular momenta projected into the foliated space) of the individual quarks using the usual Clebsch-Gordan coefficients even if they are in significant relative motion, as part of the same  $SU(2)$ .

This result has important implications for the construction of the exchange interaction in many-body systems. Since there is no extra phase (corresponding to integer representations of the  $SU(2)$  for the Bose-Einstein case, the boson symmetry can then be extended to a covariant symmetry with important implications for Bose-Einstein condensation.

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### 3.3 Construction of the Fock Space and Quantum Field Theory

In the course of our construction, we have seen in detail that the foliation of the spacetime follows from the arguments based in the representations of a relativistic

particle with half-integer spin. However, our considerations of the nature of identical particles, and their association with the spin statistics properties observed in nature, require that the foliation persists in the bosonic sector as well, where a definite phase ( $\mp$ ) under  $\pi$  rotations, exchanging two particles, must be in a definite representation of the rotation group specified by the foliation vector  $n^\mu$ . We remark in this connection that the Cooper pairing (Cooper 1956) of superconductivity must be between electrons on the same point of their induced representation orbits, so that the superconducting state is defined on the corresponding foliation of spacetime as well. The resulting (quasi-) bosons have the identical particle properties inferred from our discussion of the boson sector.

The  $N$  body state of Fermi-Dirac particles can then be written as (the  $N$  body boson system should be treated separately since the normalization conditions are different, but we give the general result below)

$$\Psi_{nN} = \frac{1}{N!} \Sigma(-)^P P \psi_{nN} \otimes \psi_{nN-1} \otimes \cdots \psi_{n1}, \quad (3.47)$$

where the permutations  $P$  are taken over all possibilities, and no two functions are equal. By the arguments given above, any pair of particle states in this set of particles have the Fermi-Dirac properties. We may now think of such a function as an element of a larger Hilbert space, called the *Fock space* which contains all values of the number  $N$ . On this space, one can define an operator that adds another particle (by multiplication), performs the necessary antisymmetrization, and changes the normalization appropriately. This operator is called a *creation operator*, which we shall denote by  $a^\dagger(\psi_{nN+1})$  and has the property that

$$a^\dagger(\psi_{nN+1})\Psi_{nN} = \Psi_{nN+1}, \quad (3.48)$$

now to be evaluated on the manifold  $(x_{N+1}, x_N, x_{N-1} \dots x_1)$ . Taking the scalar product with some  $N + 1$  particle state  $\Phi_{nN+1}$  in the Fock space, we see that

$$(\Phi_{nN+1}, a^\dagger(\psi_{nN+1})\Psi_{nN}) \equiv (a(\psi_{nN+1})\Phi_{nN+1}, \Psi_{nN}), \quad (3.49)$$

thus defining the *annihilation operator*  $a^\dagger(\psi_{nN+1})$ .

The existence of such an annihilation operator, as in the usual construction of the Fock space, (e.g., Baym 1969) implies the existence of an additional element in the Fock space, the *vacuum*, or the state of no particles. The vacuum defined in this way lies in the foliation labelled by  $n^\mu$ . The covariance of the construction, however, implies that, since all sectors labelled by  $n^\mu$  are connected by the action of the Lorentz group, that this vacuum is an absolute vacuum for any  $n^\mu$ , i.e., the vacuum  $\{\Psi_{n0}\}$  over all  $n^\mu$  is Lorentz invariant.

The commutation relations of the annihilation- creation operators can be easily deduced from a low dimensional example, following the method used in the nonrelativistic quantum theory. Consider the two body state (3.44), and apply the creation operator  $a^\dagger(\psi_{n3})$  to create the three body state

$$\begin{aligned} \Psi(\psi_{n3}, \psi_{n2}, \psi_{n1}) &= \frac{1}{\sqrt{3!}} \{ \psi_{n3} \otimes \psi_{n2} \otimes \psi_{n1} + \psi_{n1} \otimes \psi_{n3} \otimes \psi_{n2} \\ &\quad + \psi_{n2} \otimes \psi_{n1} \otimes \psi_{n3} - \psi_{n2} \otimes \psi_{n3} \otimes \psi_{n1} \\ &\quad - \psi_{n1} \otimes \psi_{n2} \otimes \psi_{n3} - \psi_{n3} \otimes \psi_{n1} \otimes \psi_{n2} \} \end{aligned} \quad (3.50)$$

One then takes the scalar product with the three body state

$$\begin{aligned}\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) &= \frac{1}{\sqrt{3!}}\{\phi_{n3} \otimes \phi_{n2} \otimes \phi_{n1} + \phi_{n1} \otimes \phi_{n3} \otimes \phi_{n2} \\ &+ \phi_{n2} \otimes \phi_{n1} \otimes \phi_{n3} - \phi_{n2} \otimes \phi_{n3} \otimes \phi_{n1} \\ &- \phi_{n1} \otimes \phi_{n2} \otimes \phi_{n3} - \phi_{n3} \otimes \phi_{n1} \otimes \phi_{n2}\}\end{aligned}\quad (3.51)$$

Carrying out the scalar product term by term, and picking out the terms corresponding to scalar products of some functions with the two body state

$$\frac{1}{\sqrt{2}}\{\psi_{n2} \otimes \psi_{n1} - \psi_{n1} \otimes \psi_{n2}\}\quad (3.52)$$

one finds that the action of the adjoint operator  $a(\psi_{n3})$  on the state  $\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1})$  is given by

$$\begin{aligned}a(\psi_{n3})\Phi(\phi_{n3}, \phi_{n2}, \phi_{n1}) &= (\psi_{n3}, \phi_{n3})\phi_{n2} \otimes \phi_{n1} \\ &- (\psi_{n3}, \phi_{n2})\phi_{n3} \otimes \phi_{n1} + (\psi_{n3}, \phi_{n1})\phi_{n3} \otimes \phi_{n2},\end{aligned}\quad (3.53)$$

i.e., the annihilation operator acts like a derivation with alternating signs due to its fermionic nature; the relation of the two and three body states we have analyzed has a direct extension to the  $N$ -body case. The action of boson annihilation-creation operators can be derived in the same way.

Applying these operators to  $N$  and  $N + 1$  particle states, one finds directly their commutation and anticommutation relations

$$[a(\psi_n), a^\dagger(\phi_n)]_{\mp} = (\psi_n, \phi_n),\quad (3.54)$$

where the  $\mp$  sign, corresponds to commutator or anticommutator for the boson or fermion operators. If the functions  $\psi_n, \phi_n$  belong to a normalized orthogonal set  $\{\phi_{nj}\}$ , then

$$[a(\phi_{ni}), a^\dagger(\phi_{nj})]_{\mp} = \delta_{ij},\quad (3.55)$$

Let us now suppose that the functions  $\phi_{nj}$  are plane waves in spacetime, i.e., in terms of functions

$$\phi_{np}(x) = \frac{1}{(2\pi)^2} e^{-ip^\mu x_\mu}.\quad (3.56)$$

Then

$$(\phi_{np}, \phi_{np'}) = \delta^4(p - p').\quad (3.57)$$

The quantum fields are then constructed as follows. Define

$$\phi_n(x) \equiv \int d^4p a(\phi_{np}) e^{ip^\mu x_\mu}.\quad (3.58)$$

It then follows that, by the commutation (anticommutation) relations (3.52), these operators obey the relations

$$[\phi_n(x), \phi_n(x')]_{\mp} = \delta^4(x - x'),\quad (3.59)$$

corresponding to the usual commutation relations of bose and fermion *fields*. Under Fourier transform, one finds the commutation relations in momentum space

$$[\phi_n(\mathbf{p}), \phi_n(\mathbf{p}')]_{\mp} = \delta^4(\mathbf{p} - \mathbf{p}') \quad (3.60)$$

The relation of these quantized fields with those of the usual on-shell quantum field theories can be understood as follows. Let us suppose that the fourth component of the energy-momentum is  $E = \sqrt{\mathbf{p}^2 + m^2}$ , where  $m^2$  is close to a given number, the on-shell mass of a particle. Then, noting that  $dE = \frac{dm^2}{2E}$ , if we multiply both sides of (3.58) by  $dE$  and integrate over the small neighborhood of  $m^2$  occurring in both  $E$  and  $E'$ , the delta function  $\delta(E - E')$  integrates to unity. On the right hand side, there is a factor of  $1/2E$ , and we may absorb  $\sqrt{dm^2}$  in each of the field variables, obtaining

$$[\phi_n(\mathbf{p}), \phi_n(\mathbf{p}')]_{\mp} = 2E\delta(\mathbf{p} - \mathbf{p}'), \quad (3.61)$$

the usual formula for on-shell quantum fields. These algebraic results have been constructed in the foliation involved in the formulation of a consistent theory of relativistic spin, therefore admitting the action of the  $SU(2)$  group for a many body system, applicable for unequal times.

It is clear from the construction of the Fock space that fields associated with different values of  $n^\mu$  commute. The basis for the commutation relations is the creation and annihilation of (wave function) factors in the tensor product space; distinct values of  $n^\mu$  therefore correspond to different species.

In the scalar product between states in the Fock space, one must complete the scalar products between functions by integrating over  $\frac{d^3n}{n^0}$ . A single value of  $n^\mu$  in the product would have zero measure, so to compute probability amplitudes, one must construct wave packets over  $n^\mu$ ; these carry suitable weights for normalization. If the set  $\{n\}$  is not a superselection rule, there would be transition matrix elements of observable connecting different values, and the form of the wave packets could play a physical role.

### 3.4 Induced Representation for Tensor Operators

In the previous sections, we have discussed the induced representation for wave functions of a particle with spin, and for the associated quantum fields. The five dimensional electromagnetic field potentials, obtained as gauge compensation fields, contain a Lorentz scalar field and a Lorentz four vector. In our discussion of statistical mechanics in Chap. 10, we are obliged to consider the problem of black body radiation. As we shall see, the relativistic Bose-Einstein distribution has a very similar form to the distribution function obtained from nonrelativistic methods, and therefore the specific heat calculations are very similar. However, the usual argument for the number of polarizations of the field, based on dimensionality minus two, corresponding to the constraint of the Gauss law and a gauge condition, resulting in two

polarizations for the usual Maxwell field, but suggest three polarization states for the 5D fields. Indeed, in a discussion of the canonical second quantization of the 5D electromagnetic fields, it was found (Shnerb 1993) that there are three polarizations with either  $O(3)$  or  $O(2, 1)$  symmetry. We discuss in Chap. 10 a second asymptotic gauge condition for the induced representation for Lorentz tensor fields (leaving aside for the moment the Lorentz scalar component), which exhibits explicitly the  $SO(3)$  representations of the tensor operators in an invariant way, thus making the polarization states accessible for classification. In this way we shall be able to describe the black body radiation in a way consistent with experiment (with the two degrees of freedom corresponding to the physical intrinsic angular of the photons), as well as to be able to explicitly characterize higher rank tensors, and their associated second quantized forms, according to their angular momentum content (Horwitz 2015).

We concentrate in the following on the vector fields; higher rank tensors transform under the direct product of the representations contained in each of the indices.

The transformation law for a vector field is constructed by the Wigner type procedure for a general tensor operator  $A(x, n, \sigma)$  through the definition (we leave out the  $x$  dependence since it undergoes several transformations which must be followed eventually)

$$A(n, \sigma) \equiv U(L(n))A(n_0, \sigma)U^{-1}(L(n)), \quad (3.62)$$

where  $U(L(n))$ , as above, is the unitary representation of the Lorentz transformation  $L(n)$  taking  $n_0 = (1, 0, 0, 0)$  into the timelike vector  $n$ .

Then, as for the wave functions,

$$\begin{aligned} U(\Lambda)A(x, n, \sigma)U^{-1}(\Lambda) \\ = U(\Lambda)U(L(n))U^{-1}(L(\Lambda n))U(L(\Lambda n))A(n_0, \sigma) \\ U(L(\Lambda n))^{-1}U(L(\Lambda n))U^{-1}(L(n))U^{-1}(\Lambda) \end{aligned} \quad (3.63)$$

The first three unitary factors induce a rotation in  $SU(2)$  (we must remember that they act on the  $x$  variable as well; this can be taken into account separately). The  $\sigma$  index is transformed by the compact Wigner rotation (as an  $SL(2, C)$  matrix)

$$\mathcal{D}(\Lambda n) = L(\Lambda n)L(n)^{-1}\Lambda \quad (3.64)$$

Writing the  $\sigma$  index as the pair  $m', m''$ , if we use direct product of two  $SL(2, C)$ 's to represent this sequence, we may supply the appropriate Clebsch-Gordan coefficients (Edmonds, see Mackey (1968))  $C(1, m|\frac{1}{2}m', \frac{1}{2}m'')$  to form the angular momentum  $L = 1$  representation, and  $C(0, m|\frac{1}{2}m', \frac{1}{2}m'')$  to form the  $L = 0$  representation. These just correspond to predetermined linear combinations over the indices. In this way, we have constructed transformations of the tensor operator in terms of irreducible representations  $L = 1$  and  $L = 0$  of the rotation group in an invariant decomposition.

We may reconstitute the four vector by returning to the  $SL(2, C)$  representations through application of the inverse of the Clebsch Gordan coefficients, taking explicitly into account the fact that the  $\sigma$  index is really a pair of indices for the  $SL(2, C)$  representation of the tensor operator  $A_n$  on the orbit of the induced representation:

$$A(n) = \begin{pmatrix} A_0(n) + A_3(n) & A_1(n) - iA_2(n) \\ A_1(n) + iA_2(n) & A_0(n) - A_3(n) \end{pmatrix} \quad (3.65)$$

The determinant corresponds to the invariant  $A_{\mu}A^{\mu}$ . Left and right multiplying by the two by two nonunitary matrices of determinant unity,  $S^{\dagger}(\Lambda)$  and  $S(\Lambda)$  which are representations in  $SL(2, C)$  of the unitary Lorentz transformations, and include as well the generators of the transformation of  $n^{\mu}$  along the orbit one may reconstruct the representation of  $U^{\dagger}(\Lambda) = U^{-1}(\Lambda)$  and  $U(\Lambda)$  on the Hilbert space. Let us now define (here we write  $SL(2, C)$  symbols to stand for the full unitary action for brevity. i.e., including the transformation on  $n$ )

$$\hat{A}_n = L(n)A_nL^{-1}(n), \quad (3.66)$$

so that under a Lorentz transformation

$$\begin{aligned} \hat{A}_n &= L(n)A_nL^{-1}(n) \rightarrow \mathcal{D}^{-1}(\Lambda n)L(\Lambda n)A_{\Lambda n}L^{-1}(\Lambda n)\mathcal{D}^{-1}(\Lambda n) \\ &= \Lambda^{-1}L(n)L(\lambda n)^{-1}(L(\Lambda n)A_{\Lambda n}L^{-1}(\Lambda n))L(\Lambda n)L(n)^{-1}\Lambda \\ &= \Lambda^{-1}(L(n)A_nL^{-1}(n))\Lambda \\ &= \Lambda^{-1}\hat{A}_{\Lambda n}\Lambda, \end{aligned} \quad (3.67)$$

transforming under the  $SL(2, C)$  matrix  $\Lambda$  along the orbit. The matrix

$$\hat{A}(n) = \begin{pmatrix} \hat{A}_0(n) + \hat{A}_3(n) & \hat{A}_1(n) - i\hat{A}_2(n) \\ \hat{A}_1(n) + i\hat{A}_2(n) & \hat{A}_0(n) - \hat{A}_3(n) \end{pmatrix} \quad (3.68)$$

then corresponds to the four-vector  $\hat{A}(n)_{\mu}$ .

This construction may be directly applied to tensor operators of any rank (with mixed tensor-spinor indices as well), explicitly displaying the angular momentum content of such operators through the direct product of the invariant decomposition of each index into angular momentum one and zero (or half integer) components. The theory of recoupling of angular momentum states (Biedenharn (1981); Racah (1942)) applies to this construction as well.

## Appendix B

We describe here some of the essential properties of the  $2 \times 2$  matrices that constitute the fundamental representation of the group  $SL(2, C)$ . Consider the Hermitian matrix

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \quad (3.69)$$

The determinant of this matrix is

$$\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2, \quad (3.70)$$

This determinant is the invariant quadratic form of special relativity. The matrix  $X$  may be written as

$$X = x^{\mu}\sigma_{\mu}, \quad (3.71)$$

where (1 is the unit matrix)

$$\sigma_{\mu} = (1, \sigma), \quad (3.72)$$



where  $\sigma$  corresponds to the vector constructed of the three Pauli matrices. The matrix

$$\tilde{X} = x^\mu \tilde{\sigma}_\mu, \quad (3.73)$$

where

$$\tilde{\sigma}_\mu = (1, -\sigma) \quad (3.74)$$

clearly has the same determinant as  $X$ . However, there is no unitary transformation that can map  $\sigma_\mu$  to  $\tilde{\sigma}_\mu$ . Unitary  $2 \times 2$  transformations leave the unit matrix invariant, and  $\sigma$  can be rotated, but not reflected in the sign of all three components (this discrete operation is a parity reflection). The two fundamental representations that we can construct in this way for the Lorentz group are therefore inequivalent. If we multiply  $X$  by some matrix in a congruency

$$X' = SXS^\dagger \quad (3.75)$$

we obtain a matrix of the same form as in Eq. (3.69), but with  $x^\mu$  replaced by  $x^{\mu'}$ . This follows from the fact that an arbitrary  $2 \times 2$  Hermitian matrix, say, where

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}, \quad (3.76)$$

where  $a$  and  $c$  are real, can always be expressed in the form of (3.69), where

$$\begin{aligned} x^0 &= \frac{1}{2}(a + c) \\ x^1 &= \frac{1}{2}(b^* + b) \\ x^2 &= \frac{1}{2i}(b^* - b). \\ x^3 &= \frac{1}{2}(a - c) \end{aligned} \quad (3.77)$$

For the second representation, defined by (3.73), (3.74), we have

$$\begin{aligned} x^0 &= \frac{1}{2}(a + c) \\ x^1 &= -\frac{1}{2}(b^* + b) \\ x^2 &= \frac{1}{2i}(b - b^*). \\ x^3 &= \frac{1}{2}(c - a) \end{aligned} \quad (3.78)$$

The conjugacy (3.75) can therefore only change  $x^\mu$  to  $x^{\mu'}$ .

These matrices therefore form a representation of  $SL(2, C)$  if they have determinant unity, since this implies that

$$\det X = \det X', \quad (3.79)$$

i.e., the two quadratic forms satisfy (equally valid for both representations)

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0'})^2 - (x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2, \quad (3.80)$$

corresponding to the defining invariance of the Lorentz group.

These two inequivalent representations, as explained in the chapter, enter into the construction of the four dimensional spinor representation of Dirac.

In this chapter we discuss the general formulation of gauge fields in the quantum theory, both abelian and nonabelian. A generalization of the elementary Stueckelberg diagram (Fig. 2.1), demonstrating a “classical” picture of pair annihilation and creation, provides a similar picture of a process involving two or more vertices (diagrams of this type appear in Feynman’s paper in 1949 (Feynman 1949) with sharp instantaneous vertices). A single vertex, as in Stueckelberg’s original diagram, in the presence of a nonabelian gauge field, can induce a flavor change on the particle line, resulting in a transition to an antiparticle with different identity. An even number of such transitions can result in flavor oscillations, such as in the simple case of neutrino oscillations. On the quark constituent level, such transitions can be associated with  $K$ ,  $B$  or  $D$  meson oscillations as well. The construction of the Lorentz force acting on particles with abelian or nonabelian gauge will also be discussed, with results consistent with the assumptions for the semiclassical model. In view of our discussion of the previous chapter, it will also be shown that this picture could provide a fundamental mechanism for  $CP$  violation.

## 4.1 Abelian Gauge Fields

In his original paper Stueckelberg (1941) introduced the electromagnetic vector gauge fields, as we shall explain below, as compensation fields for the derivatives on the wave functions representing the four-momenta. For a Hamiltonian of the form (2.4), i.e.,

$$K = \frac{p^\mu p_\mu}{2M}, \quad (4.1)$$

for which the Stueckelberg-Schrödinger equation is

$$i \frac{\partial}{\partial \tau} \psi_\tau(x) = K \psi_\tau(x), \quad (4.2)$$

one must introduce so-called compensation fields to retain the form of the equation when the wave function is modified by a (differentiable) phase function at every point. Thus, for

$$\psi(x)' = e^{ie\Lambda(x)}\psi(x), \quad (4.3)$$

the relation

$$(p^\mu - eA^\mu(x)')\psi(x)' = e^{ie\Lambda}(p^\mu - eA^\mu(x))\psi(x), \quad (4.4)$$

is satisfied if

$$A^\mu(x)' = A^\mu(x) + \partial^\mu \Lambda. \quad (4.5)$$

One sees that the gauge transformation induced on the compensation field is of the same form as the gauge transformations of the Maxwell potentials, and therefore this procedure may be thought of as an underlying theory for electromagnetism (Wu 1975). Stueckelberg (1941) noted that he was unable to explain the diagram of Fig. 2.1 with this form of the electromagnetic interaction. The reason is that the canonical velocity is

$$\dot{x}^\mu = \frac{p^\mu - eA^\mu}{M}, \quad (4.6)$$

so that

$$\dot{x}^\mu \dot{x}_\mu = -\left(\frac{ds}{d\tau}\right)^2 = \frac{(p^\mu - eA^\mu)(p_\mu - eA_\mu)}{M^2}. \quad (4.7)$$

This expression is proportional to the conserved Hamiltonian (for a closed system), so that the proper time cannot go through zero. To avoid this difficulty, he added an extra force term in the equations of motion. However, this construction did not take into account the compensation field required for the  $\tau$  derivative in the Stueckelberg-Schrödinger equation.

Applying the same procedure to the nonrelativistic Schrödinger equation, the  $t$  derivative in the equation requires a compensation field  $A^0$  (in addition to the  $\mathbf{A}$  fields compensating for the action of the derivative  $-i\frac{\partial}{\partial\tau}$ ), thus providing the full set of Maxwell fields. Taking this requirement into account in the Stueckelberg-Schrödinger equation, we arrive at a *five dimensional generalization* of the Maxwell theory (Saad 1989; see also Wesson 2006). We furthermore recognize that since the gauge phase depends, in general, on  $\tau$ , the compensation fields, which we shall denote by  $a_\mu, a_5$ , must also depend on  $\tau$ . We shall see that under integration over  $\tau$ , i.e., the zero mode, the fields  $a_\mu$  reduce to the usual Maxwell fields satisfying the usual Maxwell equations, and the  $a_5$  field decouples. The more general theory therefore properly contains the Maxwell theory.

We first remark that  $a_5$  and  $a_\mu$  must transform under a gauge change according to

$$\begin{aligned} a_5(x, \tau)' &= a_5(x, \tau) + \frac{\partial\Lambda}{\partial\tau} \\ a_\mu(x, \tau)' &= a_\mu(x, \tau) + \frac{\partial\Lambda}{\partial x^\mu}, \end{aligned} \quad (4.8)$$

or, with  $\alpha = (0, 1, 2, 3, 5)$ , and  $x^5 \equiv \tau$ ,

$$a_\alpha(x, \tau)' = a_\alpha(x, \tau) + \frac{\partial \Lambda}{\partial x^\alpha}. \quad (4.9)$$

The Stueckelberg-Schrödinger evolution operator in the presence of this 5D gauge field must therefore have, minimally, the form

$$i \frac{\partial \psi_\tau(x)}{\partial \tau} = \left\{ \frac{(p^\mu - e' a^\mu)(p_\mu - e' a_\mu)}{2M} - e' a^5(x) \right\} \psi_\tau(x), \quad (4.10)$$

where  $e'$  is related to the Maxwell elementary charge  $e$ , as we shall see, by a dimensional scale factor.

One may extract from (4.10) the form for the corresponding classical Hamiltonian,

$$K = \frac{(p^\mu - e' a^\mu)(p_\mu - e' a_\mu)}{2M} - e' a^5(x). \quad (4.11)$$

In this form, the Stueckelberg line drawn in Fig. 2.1 is, in principle, realizable. If  $-e' a^5$  reaches a value equal to  $K$ , these terms can cancel; at these points the proper time interval can pass through zero, and the semiclassical picture of pair annihilation becomes consistent in a simple way. We shall discuss an example of this mechanism in a semiclassical mechanism for neutrino oscillations to be given below.

It follows from the transformation laws (4.9) that the quantities (we use  $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$ )

$$f_{\alpha\beta}(x, \tau) = \partial_\alpha a_\beta - \partial_\beta a_\alpha \quad (4.12)$$

are gauge invariant, and may be considered, in analogy to the Maxwell case, as field strengths. To consider these quantities as *tensors* requires an additional, very strong assumption, i.e., that the five variables  $\{x^\mu, x^5\} \equiv \{x^\alpha\}$ , where  $x^5 \equiv \tau$  transform together under some group such as  $O(3, 2)$  or  $O(4, 1)$ . An examination of the field equations suggest that there may be such a symmetry, as one sees in the parallel derivation of the Maxwell equations from the gauge invariant nonrelativistic Schrödinger equation. For the latter, the explicit invariance which is evident in the homogeneous equations, that of the Lorentz group, had significant experimental evidence to justify such an assumption; at the present time there is some evidence for such a larger symmetry as  $O(3, 2)$  or  $O(4, 1)$ , as we shall see in the discussion of the applications of the five dimensional generalization of Maxwell's theory below, but it is not yet definitive. We therefore do not assume, *a priori*, the full symmetry under  $O(3, 2)$  or  $O(4, 1)$ . It is sufficient for our purposes to achieve manifest Lorentz covariance (and Poincaré symmetry for the equations of motion). We first demonstrate this argument with an analysis of the gauge theory for the nonrelativistic Schrödinger equation. Nevertheless, we shall refer to quantities such as  $f_{\alpha\beta}(x, \tau)$  as *tensors* as a matter of notation.

The nonrelativistic fully gauge invariant Schrödinger equation is

$$i \frac{\partial}{\partial t} \psi_t(\mathbf{x}) = \frac{(\mathbf{p} - e\mathbf{A}(\mathbf{x}, t))^2}{2M} \psi_t(\mathbf{x}) - eA_0; \quad (4.13)$$

the current  $\mathbf{J}$  and the charge density  $J^0 \equiv \rho$  satisfy the conservation law

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \quad (4.14)$$

where

$$\mathbf{J} = \frac{ie}{2M} [\psi^*(\nabla - ie\mathbf{A})\psi - \psi(\nabla + ie\mathbf{A})\psi^*], \quad (4.15)$$

and

$$J^0 = \rho = e\psi^*\psi. \quad (4.16)$$

The inhomogeneous Maxwell field equations, written formally in terms of the four-vector indices, are (e.g. Jackson 1974; Landau 1951)

$$\partial_\nu F^{\mu\nu} = eJ^\mu; \quad (4.17)$$

they may be obtained from a Lagrangian providing the Schrödinger equation as a field equation, with a gauge invariant term proportional to  $F_{\mu\nu}F^{\mu\nu}$ , as we shall describe below in our discussion of the  $5D$  fields.

There is clearly no linear coordinate transformation that can generate a linear combination of  $\mathbf{J}$  and  $J^0$ , and therefore the relation (4.17) is not covariant. The relativistic covariance of the Maxwell equations, as discussed by Einstein (1905) is based on the assumption that the current is a covariant four vector. As we have seen, this does not hold for the gauge field construction based on the nonrelativistic Schrödinger equation.

For the relativistic case, Jackson (1974) has shown how one can construct a covariant four vector current from a sequence of elementary charged *events* in spacetime, which we shall refer to again below. It is, however, important to note that the *homogeneous* equations corresponding to (4.17), i.e., for  $J^\mu = 0$ , do reflect the Lorentz symmetry, suggesting that such a symmetry may indeed be a symmetry of the world. To realize this symmetry consistently, one must use a form of the quantum theory that gives rise to a covariant four current based, as we see above, on (4.10).

A simple set of field equations, providing second order derivatives of the potentials, is obtained by considering the Lagrangian density due to the field variables to be of the form  $f_{\alpha\beta}f^{\alpha\beta}$ , where we leave open for now the question of choosing a signature for raising and lowering the index of the fifth component. Writing a Lagrangian density for which setting the coefficient of the variation of  $\psi^*$  equal to zero gives the Stueckelberg-Schrödinger equation,<sup>1</sup> with this additional term for the gauge fields of the form

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<sup>1</sup>Gottfried (1966) has pointed out that this procedure is not completely consistent since the Schrödinger wave  $\psi$  is not a mechanical quantity; it is, however, consistent for quantum field theory, and provides a convenient procedure to generate field equations for the first quantized theory under discussion here. The method is widely used as a heuristic tool (for example, Bjorken 1964).

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2} \left( i \frac{\partial \psi}{\partial \tau} \psi^* - i \psi \frac{\partial \psi^*}{\partial \tau} \right) \\
& - \frac{1}{2M} \left[ (p^\mu - e' a^\mu) \psi ((p_\mu - e' a_\mu) \psi)^* \right] \\
& + e' (a^5 \psi \psi^*) - \frac{\lambda}{4} f^{\alpha\beta} f_{\alpha\beta},
\end{aligned} \tag{4.18}$$

where  $\lambda$  is, as we discuss below, an arbitrary real dimensional scale factor.

As for the 4D Maxwell fields, for which the Lagrangian does not contain  $\partial A_0/\partial t$ , the Lagrangian does not contain  $\partial a_5/\partial \tau$ , and therefore a full canonical quantization (as contrasted with path integral approaches such as Fadeev-Popov (Fadeev 1967)), which requires identification of a canonical momentum for the fields as the derivative of the Lagrangian density with respect to the time derivative of the field, is not easily accessible. Henneaux and Teitelboim (1992) and Haller (1972) have discussed methods for dealing with this problem; these methods have been applied by Horwitz and Shnerb (1993) to carry out the canonical quantization of the 5D fields (see Sect. 10.7). We just remark here that the three photon polarization states (in dimensionality the number of field components minus the two constraints due to Gauss's law and a gauge condition) may fall under the  $O(2, 1)$  or  $O(3)$  symmetry groups; as we discuss in Chap. 10, the two degrees of freedom of black body radiation is the result of the application of a second gauge condition on the asymptotic fields.

The variation of the potentials  $a^\alpha$  in (4.18) then provides the field equations

$$\lambda \partial^\alpha f_{\beta\alpha} = j_\beta \tag{4.19}$$

where

$$j_\mu = \frac{i e'}{2M} \{ (\partial_\mu - i e' a_\mu) \psi \psi^* - \psi ((\partial_\mu - i e' a_\mu) \psi)^* \}, \tag{4.20}$$

and

$$j_5 = e' \psi \psi^* \equiv \rho_5. \tag{4.21}$$

As for the nonrelativistic gauge theory based on the Schrödinger equation, there is no coordinate transformation which can induce a linear combination of  $j_\mu$  and  $j_5$ , and therefore these equations cannot be covariant under  $O(4, 1)$  or  $O(3, 2)$ , although the homogeneous form of (4.19) for  $j^\alpha = 0$  does admit such a higher symmetry.<sup>2</sup>

<sup>2</sup>If such a higher symmetry, such as  $O(3, 2)$  or  $O(4, 1)$  were to be found as a general property of particle kinematics, such as Lorentz covariance, in the framework of our present experimental knowledge, then a generalization of the Stueckelberg theory could be written with five momenta transforming under this group. The corresponding gauge fields would then be one dimension higher, to take into account the evolution of the system, and the resulting homogeneous field equations would appear to be  $O(4, 2)$ ,  $O(3, 3)$  or  $O(5, 1)$  invariant. The corresponding theory of spin, as worked out in the previous chapter, would then rest on the method of Wigner applied to the stability group of a five-vector. In this book, we shall restrict our analysis to systems which are manifestly covariant on the level of the Lorentz group.

Furthermore, the current  $j^\alpha$  satisfied, as follows from the Stueckelberg-Schrödinger equation, the conservation law

$$\partial_\alpha j^\alpha = 0 \quad (4.22)$$

In general, then, the current  $j^\mu$ , cannot be the conserved Maxwell current (Saad 1989; see also Stueckelberg 1941). Writing Eq. (4.22) in the form

$$\partial_\mu j^\mu + \frac{\partial \rho}{\partial \tau} = 0 \quad (4.23)$$

suggests taking the integral over all  $\tau$  (Stueckelberg 1941). If  $\rho_\tau(x) \rightarrow 0$  for  $\tau \rightarrow \pm\infty$ , that is, that the expectation of the occurrence of events in a finite region of  $x^\mu$  vanishes for large values of the evolution parameter (the physical system evolves out of the laboratory), then the second term vanishes under this integration, and one finds that

$$\partial_\mu J^\mu = 0, \quad (4.24)$$

where

$$J^\mu(x) = \int_{-\infty}^{+\infty} d\tau j_\tau^\mu(x) \quad (4.25)$$

can be identified as the Maxwell current (this procedure has been called “concatenation” (Horwitz 1982).

In his book on electrodynamics, Jackson (1974) provides a construction of a covariant current by starting with an elementary current element  $e\dot{x}^\mu\delta^4(x - x(s))$ , where  $s$  is considered to be some parameter along the worldline  $x^\mu(s)$  of the moving charged event, say, the proper time. He then asserts that

$$J^\mu(x) = e \int ds \dot{x}^\mu \delta^4(x - x(s)) \quad (4.26)$$

is conserved by noting that

$$\begin{aligned} \partial_\mu J^\mu(x) &= e \int ds \dot{x}^\mu \partial_\mu \delta^4(x - x(s)) \\ &= -e \int \frac{d}{ds} \delta^4(x - x(s)), \end{aligned} \quad (4.27)$$

which vanishes if the worldline moves out of the range of the laboratory as  $s \rightarrow \pm\infty$ . The transition from (4.25) to (4.26) is achieved by noting the identity

$$-\frac{d}{ds} \delta^4(x - x(s)) = \dot{x}^\mu \partial_\mu \delta^4(x - x(s)); \quad (4.28)$$

this is, however, precisely the conservation law (2.21) for the case  $\rho(x) = \delta^4(x - x')$  for a charged event at the point  $x'$ . It follows from Jackson’s construction, as well as the argument leading to (4.25), that what is considered a “particle”, in electromagnetism, but also in the probability theory associated with quantum mechanics, i.e. an object which satisfies a law of conserved current and charge (or probability density), corresponds to at least a large segment of a worldline (Land 1998), an essentially

nonlocal object in the Minkowski space. The nonrelativistic Schrödinger equation has a locally defined conserved current; the bilinear density  $\psi_{NR}(\mathbf{x})^* \psi_{NR}(\mathbf{x})$  contains the product of wave functions of precisely equal mass. As we have seen in Chap. 2, e.g. (2.27), the Stueckelberg wave function for a free particle evolves according to

$$\psi_\tau(x) = U(\tau)\psi(x) = \frac{1}{(2\pi)^2} \int d^4 p e^{-i \frac{p^\mu p_\mu}{2M} \tau} e^{-i p^\mu x_\mu} \psi(p); \quad (4.29)$$

since  $p^\mu p_\mu = -m^2$ , the variable corresponding to the measured mass, the  $\tau$  integration of the bilinear has the effect of reducing this form to an integral over a bilinear diagonal in the mass. Thus, the  $\tau$  integration is associated with the retrieval of “particle” properties, as in our discussion of the Newton-Wigner problem in Chap. 2, and Nambu’s (1950) reduction, by integrating the wave function over  $\tau$  with a factor  $e^{-i \frac{M\tau}{2}}$  with predetermined  $M$  of Feynman’s formulation (Feynman 1950) of perturbation theory to the particle mass shell.

Turning to the field equations (4.19), we see that an integral over  $\tau$ , assuming the asymptotic vanishing of the  $f_{\mu 5}$  field in  $\tau$ , results (for the  $\mu$  component) in

$$\partial^\nu \int d\tau f_{\mu\nu}(x, \tau) = \int d\tau j_\mu(x, \tau); \quad (4.30)$$

the right hand side corresponds, as we have argued, to the conserved current of Maxwell, so that we may identify, from (4.12),

$$\int d\tau a_\mu(x, \tau) = A_\mu(x), \quad (4.31)$$

i.e., the Maxwell  $\tau$ -independent field. Thus the Maxwell field emerges as the zero mode of the fields  $a_\mu(x, \tau)$ , which we have called the “pre-Maxwell” fields (Saad 1989). Due to the linearity of the field equations, the integral over the field equations (4.19) reduce precisely, as we have seen, to the standard Maxwell form (this remark does not hold, as we shall see, to the nonlinear equations of the nonabelian Yang-Mills fields).

The physical situation that we have described here corresponds to the emergence of the Maxwell fields from detection apparatus that intrinsically integrates over  $\tau$ . It would appear that there is, according to this theory, a high frequency modulation of the Maxwell field that is not easily observable in apparatus available in laboratories at the present time. There has been some indirect evidence, in connection with the self-interaction problem, for the existence of the classical  $5D$  fields in connection with an extensive investigation of the self-interaction problem (Aharonovich 2011). Furthermore, the fifth field, as we have pointed out above, can be responsible for the transition represented in Stueckelberg’s diagram Fig. 2.1; it also plays an essential role in the neutrino oscillation model that we shall describe below.

Equation (4.31) implies that the dimensionality of the pre-Maxwell fields must, since the Maxwell fields  $A$  have dimensionality  $L^{-1}$ , be  $L^{-2}$ . Thus the charge that we have called  $e'$  must have dimensionality  $L$  ( $p^\mu$  has dimensionality  $L^{-1}$ ). The gauge invariant field strengths then have dimensionality  $L^{-3}$ . The quadratic contribution of the field strengths to the Lagrangian,  $f^{\alpha\beta} f_{\alpha\beta}$  then has dimensionality  $L^{-6}$ . Since the action



is an integral of the Lagrangian density over  $d\tau d^4x$ , of dimension  $L^5$ , the quadratic field strength terms must have a dimensional factor  $\lambda$ . The current in the resulting field equations contains the factor  $e'$ , and the derivatives of the field strength on the right emerge with a factor  $\lambda$ ; thus we can identify

$$e = e'/\lambda \quad (4.32)$$

with the dimensionless Maxwell charge.

Assuming the analog of the Lorentz gauge for the five dimensional fields (4.12),

$$\partial^\alpha a_\alpha = 0, \quad (4.33)$$

the field equations (4.19) become

$$(-\partial_\tau^2 + \partial_t^2 - \nabla^2)a_\beta = j_\beta/\lambda, \quad (4.34)$$

where we have taken the  $O(4, 1)$  signature for the fifth variable  $\tau$ . Representing  $a_\beta(x, \tau)$  in terms of its Fourier transform  $a_\beta(x, s)$ , with

$$a_\beta(x, \tau) = \int ds e^{-is\tau} a_\beta(x, s), \quad (4.35)$$

one obtains

$$(s^2 + \partial_t^2 - \nabla^2)a_\beta(x, s) = j_\beta(x, s)/\lambda, \quad (4.36)$$

providing a relation between the off-shell mass spectrum of the  $a_\beta$  field and the quantum mechanical current source. As we have pointed out earlier, the solutions of wave equations with a definite mass  $m$  have, according to Newton and Wigner, contain nonlocality of the order of  $1/m$ ; thus (for application of their arguments, thinking of the field as the wave function of a quantum of the field) the massless particle would have a very large support. There is some difficulty in imagining the emission of a photon from an atom of the size  $10^{-8}$  cm which instantaneously has infinite support. However, if the photon being emitted is far off shell, and has an effective mass  $s$ , as in the equations above, which is fairly large, the particle being emitted can have very small spatial support, undergoing a relaxation process asymptotically to a particle with very small, essentially zero, mass.

A similar argument can be applied to the photoelectric effect; the energy  $\hbar\omega$  associated with a photon of frequency  $\omega$  is absorbed by a metal plate, and an electron emitted with exactly this energy (minus the work function to free the electron). The contraction of the energy of a highly nonlocalized radiation field into the very small region occupied by the electron is often attributed to ‘‘collapse of the wave function’’, but this statement does not account for the physical mechanism (even ‘‘collapse’’ mechanisms require the construction of a model (Hughston 1996; see also Silman 2008)). In this process, again one may think of the photon going far off mass shell to be able to be absorbed locally.

It has often been argued, moreover, that an experimental bound on the photon mass is provided by gauge invariance. This argument would, of course, provide a bound if the mass term in the field equations had some given constant value; then the

shift of the vector potential by a gradient term, even if the gauge function satisfied a homogeneous d'Alembert type equation, would leave an extra term in the equation that would not vanish. However, as we have seen, the field equation contains a second derivative with respect to  $\tau$ , and if the gauge function has a vanishing  $5D$  d'Alembertian of  $O(4, 1)$  or  $O(3, 2)$  type, gauge invariance would be maintained.

Finally, we remark that the foliation due to spin-statistics in the framework of Wigner's theory of induced representations, although worked out for the four-vector gauge fields in the previous chapter, remains valid for the  $O(3, 1)$  part of the  $5D$  gauge fields; only a fifth scalar field must be added to the Hamilton constructed, as we shall see in the next section.

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## 4.2 Nonabelian Gauge Fields and Neutrino Oscillations

In this section we extend the picture of Stueckelberg for pair annihilation in classical dynamics to a diagram with two (or more) maxima, such as shown in Fig. 4.1, in which the incoming line eventually continues to move in the positive direction of  $t$ . Recall that the diagram of Fig. 2.1 constitutes, in the simplest interpretation, in its application to electromagnetism, to particle-antiparticle annihilation. In the case of a system representing a higher symmetry group than the  $U(1)$  of electromagnetism, the two branches of the curve can correspond to two different "flavors", i.e. two different types of particles, each corresponding to a component of a vector-valued wave function, such as the nucleon, containing both the neutron and the proton. Yang and Mills (1954) thought of the nucleon as represented by such a wave function with two components corresponding to the neutron and the proton, which have different charge but almost the same mass, as a doublet state. Such a wave function would support the action of a higher symmetry group, in this case  $SU(2)$ . In our discussion of the gauge transformation  $\psi \rightarrow e^{i\Lambda}\psi$ , one may use a two by two matrix for the exponent  $\Lambda$ ; the resulting gauge compensation fields, which one might call  $b^\alpha$ , again for  $\alpha = 0, 1, 2, 3, 5$  would then be two by two matrices as well and noncommuting. The corresponding  $SU(2)$  group is called "isotopic spin," or "isospin," since the transition between neutrons and protons is involved in the generation of isotopes in nuclear physics.

Such fields are called nonabelian gauge fields, and play an important role in modern gauge theories. The importance of such theories lies largely in the fact that fields corresponding to gauge groups obey Ward identities (Kaku 1993; Peskin 1995) that control the singularities generated by the quantized fields and admit the application of the renormalization program (Bogliubov 1959; 't Hooft 1971). In such a construction, the vertex of the Stueckelberg diagram can contain not just a transition to antiparticle, but to an antiparticle with a different identity; the transition is induced through an interaction with a field that can connect different components of the incoming and outgoing (in  $\tau$ ) wave function. The diagram of Fig. 4.1 can then

corresponds to two such transitions, one at each vertex (such a diagram appears in Feynman's paper in 1949 (Feynman 1949) with the sharp vertices characteristic of a perturbation expansion), resulting in a change of components of the particle as it evolves in spacetime.

As a simple example of such a phenomenon, we discuss here the so-called neutrino oscillation. In the theory of weak and electromagnetic interactions of Glashow et al. (1967), the electron neutrino, observed, for example in neutron decay to proton, electron and (anti-)neutrino, and the  $\mu$ -neutrino, observed in muon decay to electron neutrino and antineutrino, form a doublet under a group called "weak isospin". There is an additional type of neutrino, the  $\tau$  neutrino, which occurs in the decay of the  $\tau$  meson, produced, for example, in high energy  $e^+e^-$  collisions (Henley 2007). Even though the masses of the three types of neutrinos are quite different they may be thought of as a triplet, with a gauge group  $SU(3)$  with good analogy to the  $SU(3)$  of quantum chromodynamics. Since the  $\tau$  neutrino appears to be much heavier, it is less likely to be involved in the neutrino oscillations, but certainly not ruled out. However, for simplicity, we shall restrict our attention here to the  $\nu_e$  and  $\nu_\mu$  oscillations, although the same qualitative picture would be applicable if all three neutrinos were taken into account. The corresponding gauge fields are called  $W$  and  $Z$ , after the particle resonances thought to play an important role in the mediation of the weak and electromagnetic (along with the electromagnetic potential field  $A$ ) interactions.

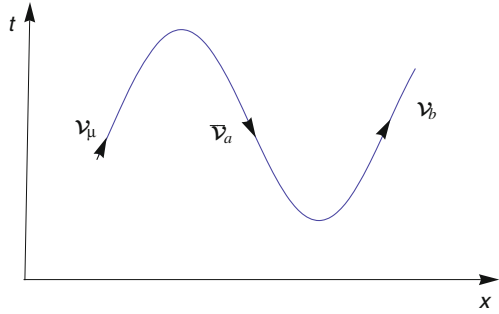
In the flavor oscillations of the neutrino system, interactions with the vector bosons of the Glashow-Salam-Weinberg (GSW) theory (Weinberg 1967) which induce the transition can produce, as we have pointed out, pair annihilation-creation events. In the framework of Stueckelberg theory, pair annihilation and creation events can be correlated, as shown in Fig. 4.1, by following the world line.<sup>3</sup> The methods of Feynman's original paper, based on a spacetime picture (Feynman 1949), closely related to Stueckelberg's earlier formulation, would admit such a construction as well. An "on-shell" version of our Fig. 4.1 appears, with sharp vertices, in Feynman (1949).

It may be noted from this figure that there is a net decrease in the time interval, possibly very small, observed for the particle to travel a certain distance. One might expect that over a long distance of transmission (long baseline experiments), neutrinos, due to this oscillation phenomenon, might arrive earlier at their destination than predicted by light speed estimates. The most recent experiments have shown that the arrival times are consistent with light speed; in the most recent OPERA experiment (Acquafredda(2009), Adam (2013)) over the 732 km distance from CERN to Gran Sasso, an arrival time of  $6.5 \pm 7.4 \left\{ \begin{smallmatrix} +8.3 \\ -8.0 \end{smallmatrix} \right\}$  ns less than light speed arrival is reported, certainly *consistent* with light speed. There is, however, some room in the distribution found for early arrival; it would require higher precision to rule out early arrival.

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<sup>3</sup>The curve shown in Fig. 4.1 should be thought of as corresponding to the expectation values computed with the local density matrix associated with the gauge structured wave function of the neutrino beam.

**Fig. 4.1** Semiclassical neutrino oscillation

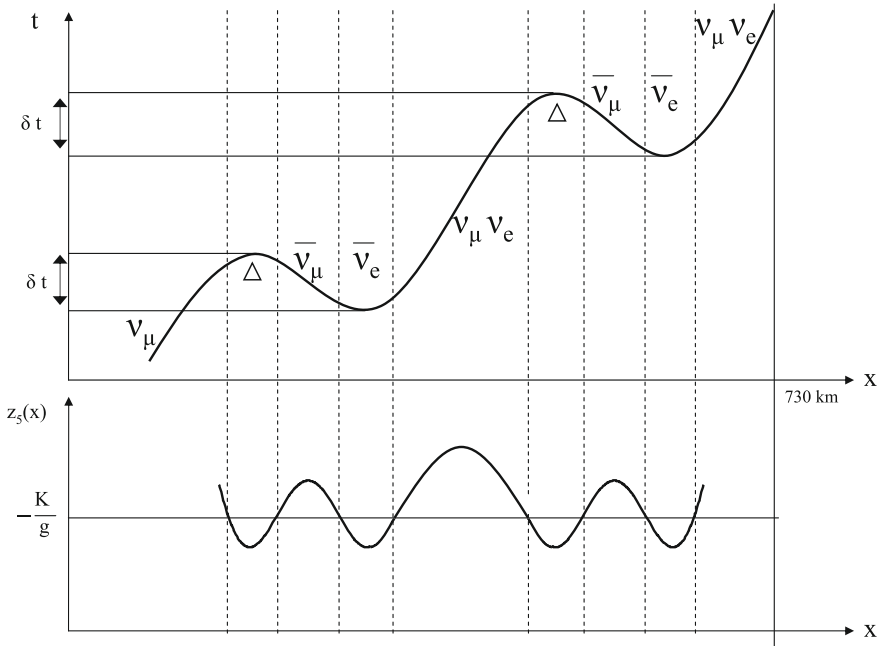


We remark that it has, however, been observed in the Supernova 1987a that the neutrinos arrive about 3 h before the light signal (Bahcall 1989). To show that a small advance in neutrino arrival times (“pull back” in time) could be consistent with this data as well, we make the following estimate.

An advanced arrival of the order of 6.5 ns in each 730 km (consistent with this data) would result in approximately  $3 \times 10^3$  h early arrival. However, as we shall see below, the mechanism for the oscillations associated with such a “pull-back” involves the participation of the fifth field in an essential way, expected to fall off far from sources. One may estimate on the basis of a 3 h early arrival the range of effectiveness of the fifth field, assuming an advance of 6.5 ns in each 730 km where effective. A simple estimate yields about 30 parsec (pc), as an effective size of the supernova. The Sun is about  $10^4$  pc from the center of the galaxy, so an effective range of about 30 pc is not unreasonable. This argument is certainly not a proof of a “pull back”; it is meant to show that a small effect of this type could be consistent with the supernova 1987a data (see, moreover, further discussion in Bahcall (1989)).

Suppose, for example, that such oscillations can occur twice during the transit (Kayser 2004) from CERN to the Gran Sasso detectors, as in Fig. 4.2. The particles (and antiparticles) have almost everywhere propagation speed less than light velocity (except for the vertices, which we estimate, based on the  $Z$ ,  $W$  lifetimes, to occur in about  $10^{-22}$  s); it is clear from Fig. 4.2 that an early arrival would *not* imply, in this model, that the neutrinos travel faster than light speed. The effect noted by Glashow and Cohen (2011), indicating that Čerenkov radiation would be seen from faster than light neutrinos, would likely not be observed from the very short lived vertices, involving interaction with the  $W$  and  $Z$  fields, without sensitive detectors placed appropriately on the track. The neutrino arrivals detected at Gran Sasso appear to be almost certainly normal particles. The ICARUS detector (Acquafredda et al. 2009) records no  $\gamma$ 's or  $e^+e^-$  pairs which would be expected from Čerenkov radiation from faster than light speed neutrinos (Cohen 2011).

A quantum mechanical counterpart of this model, in terms of Ehrenfest wave packets, is consistent with this conclusion. The derivation of the Landau-Peierls relation



**Fig. 4.2** Gauge condition for oscillations

$\Delta p \Delta t \geq \frac{\hbar}{2c}$  in the framework of the Stueckelberg theory discussed in Chap. 2, involves the assumption that the energy-momentum content of the propagating wave function contains predominantly components for which  $\frac{p}{E} < 1$ . Interactions, e.g., at the vertices of the curve in Fig. 4.2, can affect this distribution in such a way that, for some (small) interval of evolution, the wave packet can contain significant contributions to the expectation value of  $p/E$  much larger than unity, and thus the dispersion  $\Delta t$  in the Landau-Peierls relation can become very small without violating the uncertainty bound established by  $\langle E/p \rangle$ . The interaction vertex may then be very sharp in  $t$ , admitting a precise manifestation of the deficit time intervals (as in the corresponding Feynman diagram (Feynman (1948))).

The upper part of Fig. 4.2 shows schematically the orbit of a neutrino in spacetime during its transit, according to this theory, in which the first (annihilation) event results in the transition from a  $\nu_\mu$  to either a  $\nu_\mu$  or  $\nu_e$  through interaction with a GSW boson (for this simple illustration we consider only the  $\mu$  and  $e$  neutrinos, although there is no reason to exclude the  $\tau$  neutrino) and the second (creation) event involves a transition from either of these states back to a  $\nu_\mu, \nu_e$  state.

We now proceed to formulate the nonabelian gauge model; here, we call generically, the nonabelian gauge field  $z^\alpha$ .

The gauge covariant form of the Stueckelberg Hamiltonian, valid for the non-Abelian case as well as for the Abelian, with coupling  $g$  to the 5D fields, is

$$K = \frac{(p^\mu - gz^\mu)(p_\mu - gz_\mu)}{2M} - gz^5(x), \quad (4.37)$$

where the  $z^\mu$  fields are non-Abelian in the  $SU(2)$  sector of the electroweak theory. Since, as we shall below,  $\dot{x}^\mu$  is proportional to  $p^\mu - gz^\mu$ , the local expectation of the square of the “proper time” is proportional to that of the first term in the Hamiltonian. Therefore, see we see that the local expectation of  $z^5$  must pass through that of the conserved value of  $-K/g$  to admit passage of the orbit through the light cone. In the lower part of Fig. 4.2, we have sketched a form for a smooth  $z^5$  wave (in expectation value) that would satisfy this condition. Such a wave can be easily constructed as the superposition of a few harmonic waves with different wavelengths (originating in the spectral density of the neutrino wave functions [see Eq. (4.49) below]).

The occurrence of such a superposition can be understood from the point of view of the structure of the 5D GSW fields. Working in the context of the first quantized theory, where the functions  $\psi$  belong to a Hilbert space  $L^2(x, d^4x) \otimes d$ , with  $d$  the dimensionality of the gauge fields ( $d = 2$  corresponds to the Yang-Mills case (Yang 1954) and the  $SU(2)$  sector of the electroweak theory which we shall deal with here; our procedure for extracting the field equations and Lorentz force applies for any  $d$ ), the field equations can be derived from the Lagrangian density (we consider the case of particles with spin in the next section)

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \text{Tr} \left( i \frac{\partial \psi}{\partial \tau} \psi^\dagger - i \psi \frac{\partial \psi^\dagger}{\partial \tau} \right) \\ &\quad - \frac{1}{2M} \text{Tr} \left[ (p^\mu - gz^\mu) \psi ((p_\mu - gz_\mu) \psi)^\dagger \right] \\ &\quad + g \text{Tr} (z^5 \psi \psi^\dagger) - \frac{\lambda}{4} \text{Tr} f^{\alpha\beta} f_{\alpha\beta}, \end{aligned} \quad (4.38)$$

where  $\psi$  is a vector valued function representing the algebraic action of the gauge field, and  $\psi^\dagger$  is a 2-component (row) conjugate vector valued function;  $\mathcal{L}$  is a local scalar function. The operation  $\text{Tr}$  corresponds to a trace over the algebraic indices of the fields; the dimensional parameter  $\lambda$  arises from the relation of these fields to the zero mode fields of the usual 4D theory (Yang 1954), as for the electromagnetic fields discussed above. For the variation of the field strengths we take  $\delta z^\alpha$  to be general infinitesimal Hermitian algebra-valued functions. Extracting the coefficients of these variations, with the definition of the non-Abelian gauge invariant field strength tensor (Yang 1954)

$$f^{\alpha\beta} = \partial^\alpha z^\beta - \partial^\beta z^\alpha - ig[z^\alpha, z^\beta], \quad (4.39)$$

one obtains the field equations

$$\lambda [\partial^\alpha f_{\beta\alpha} - ig[z^\alpha, f_{\beta\alpha}]] = j_\beta \quad (4.40)$$

where

$$j_\mu = \frac{ig}{2M} \{ (\partial_\mu - igz_\mu) \psi \psi^\dagger - \psi ((\partial_\mu - igz_\mu) \psi)^\dagger \}, \quad (4.41)$$

and

$$j_5 = g\psi\psi^\dagger \equiv \rho_5. \quad (4.42)$$

Let us now impose, as done by Yang and Mills (1954), the subsidiary condition

$$\partial^\alpha z_\alpha = 0. \quad (4.43)$$

We then obtain from (4.40)

$$(-\partial_\tau^2 + \partial_t^2 - \nabla^2)z_\beta = j_\beta/\lambda + ig[z^\alpha, f_{\beta\alpha}], \quad (4.44)$$

where we have taken the  $O(4, 1)$  signature for the fifth variable  $\tau$ . Representing  $z_\beta(x, \tau)$  in terms of its Fourier transform  $z_\beta(x, s)$ , with

$$z_\beta(x, \tau) = \int ds e^{-is\tau} z_\beta(x, s), \quad (4.45)$$

one obtains

$$(s^2 + \partial_t^2 - \nabla^2)z_\beta(x, s) = j_\beta(x, s)/\lambda + ig \int d\tau e^{is\tau} [z^\alpha(x, \tau), f_{\beta\alpha}(x, \tau)], \quad (4.46)$$

providing a relation between the off-shell mass spectrum of the  $z_\beta$  field and the sources including the quantum mechanical current as well as the non-linear self-coupling of the fields.

Since the behavior of the  $z_5$  field plays an essential role in the immediately applicable predictions of our model, consider the Eq. (4.46) for  $\beta = 5$ ,

$$(s^2 + \partial_t^2 - \nabla^2)z_5(x, s) = j_5(x, s)/\lambda + ig \int d\tau e^{is\tau} [z^\nu(x, \tau), f_{5\nu}(x, \tau)]. \quad (4.47)$$

In a zeroth approximation, neglecting the nonlinear coupling term, we can study the equation

$$(s^2 + \partial_t^2 - \nabla^2)z_5(x, s) \cong j_5(x, s)/\lambda. \quad (4.48)$$

The source term is a convolution of the lepton wave functions in the Fourier space, so that

$$(s^2 + \partial_t^2 - \nabla^2)z_5(x, s) \cong \frac{g}{2\pi\lambda} \int ds' \psi(x, s')\psi^\dagger(x, s' - s). \quad (4.49)$$

The Fourier representation over  $s$  of the wave function corresponds to the set of probability amplitudes for finding the particle in the corresponding mass states; we expect these functions to peak in absolute value, in free motion, at the measured neutrino masses. There is therefore the possibility of several mass values contributing to the frequency of the spectrum of the  $z_5$  field (the diagonal contributions contribute only to its zero mode, a massless radiative field of essentially zero measure). In order for the forces to give rise to a form for the  $z_5$  field of the type illustrated in Fig. 4.2, there must be at least three peaks in the mass distribution of the wave functions, corresponding to three families of neutrinos. This condition has been noted in a somewhat different context (Nunokawa 2006) and in other studies (for example Refs. Fogli 1995; Bandyopadhyay 2002) discussing the three family structure).

We now turn to study the trajectories of the particles with non-Abelian gauge interactions to further check the consistency of our model. The Heisenberg equations

of motion are associated with expectation values for which the classical motion is a good approximation if the wave packets are fairly well localized.

From the Hamiltonian (4.37) one obtains

$$\begin{aligned}\dot{x}^\lambda &= i[K, x^\lambda] \\ &= \frac{1}{M}(p^\lambda - gz^\lambda),\end{aligned}\tag{4.50}$$

of the same form as the classical result.

The second derivative is defined by

$$\ddot{x}^\lambda = i[K, \dot{x}^\lambda] + \frac{\partial \dot{x}^\lambda}{\partial \tau},\tag{4.51}$$

where the last term is necessary because  $\dot{x}^\lambda$  contains, according to (4.50), an explicit  $\tau$  dependence which occurs in the fields  $z^\lambda$ . One then obtains (the Lorentz force for the non-Abelian case was also obtained, using an algebraic approach, in Land 1995)

$$\ddot{x}^\lambda = -\frac{g}{2M}\{\dot{x}^\mu, f^\lambda{}_\mu\} - \frac{g}{M}f^{5\lambda}.\tag{4.52}$$

Let us make here the crude approximation that was used in obtaining (4.48), i.e., neglecting the nonlinear coupling to the spacetime components of the field. Then, (4.52) becomes, for the time component,

$$\ddot{i} \cong -\frac{g}{M} \frac{\partial z^5}{\partial t}.\tag{4.53}$$

The rising  $z^5$  field (Fig. 4.2), before the first passage through the light cone, would imply a negative curvature, as required. This consistency persists through the whole process.

We further note that

$$-\frac{ds^2}{d\tau^2} = \frac{2}{M}(K + gz^5),\tag{4.54}$$

so that

$$\frac{d}{d\tau} \frac{ds^2}{d\tau^2} = -\frac{2g}{M} \frac{dz^5}{d\tau},\tag{4.55}$$

consistent as well with the form of Fig. 4.2.

### 4.3 The Hamiltonian for the Spin $\frac{1}{2}$ Neutrinos

The Lorentz force for the Abelian case with spin can be computed in exactly the same way from (3.32) with the additional term  $-e'a_5$  as in (4.11). Note that the Lorentz force is not linear, so it cannot be mapped back to the Maxwell Lorentz force directly by concatenation.



Following the method of Chap. 3 for the non-Abelian case, we find a Hamiltonian of the form

$$K = \frac{1}{2M}(p - gz)_\mu(p - gz)^\mu - \frac{g}{2M} f_{\mu\nu} \Sigma_n^{\mu\nu} - gz^5, \quad (4.56)$$

where  $\Sigma_n^{\mu\nu}$  is defined by (3.33).

Since  $\gamma^5$  commutes with this Hamiltonian, there is a chiral decomposition (true for (3.32) as well), *independently of the mass of the neutrinos*, which admits the usual construction of the  $SU(2) \times U(1)$  electroweak gauge theory. The  $SU(2)$  sector that we discuss below would then apply to the left handed leptons. The asymptotic (free) solutions also admit a (foliated) helicity decomposition (Arshansky 1982).

We shall discuss the possibilities of  $CP$  violation provided by this structure below.

To compute the Lorentz force, as in (4.50), one obtains the particle velocity

$$\dot{x}^\lambda = i[K, x^\lambda] = \frac{1}{M}(p^\lambda - gz^\lambda). \quad (4.57)$$

For the second derivative, from (4.51) and (3.34), we obtain

$$\begin{aligned} \ddot{x}^\lambda = & -\frac{g}{2M}\{f^{\lambda\mu}, \dot{x}_\mu\} - \frac{g}{M}f^{5\lambda} \\ & + \frac{g}{2M^2}\partial^\lambda f^n_{\mu\nu}\Sigma_n^{\mu\nu} + \frac{ig^2}{2M^2}[f^n_{\mu\nu}, z^\lambda]\Sigma_n^{\mu\nu}. \end{aligned} \quad (4.58)$$

The third term of (4.58) corresponds to a Stern-Gerlach type force. Note that we have included the subscript or superscript  $n$  to the quantities that are transverse in the foliation.

Under the assumption that the fields are not too rapidly varying, and again neglecting coupling to the spacetime components of the field  $z^\alpha$ , we see that the acceleration of the time variable along the orbit may again be approximated by (4.53).

We are now in a position to write the Lagrangian for the full theory with spin. We take for the Lagrangian the form (4.38) with an additional term for the spin interaction and factors of  $\gamma^0(\gamma \cdot n)$  to assure covariance, yielding under variation of  $\psi^\dagger$  the Stueckelberg equation for  $\psi$  with Hamiltonian (4.56):

$$\begin{aligned} \mathcal{L}_n = & \frac{1}{2}\text{Tr}\left(i\frac{\partial\psi}{\partial\tau}\bar{\psi} - i\psi\frac{\partial\bar{\psi}}{\partial\tau}\right)(\gamma \cdot n) \\ & - \frac{1}{2M}\text{Tr}\left[(p^\mu - gz^\mu)\psi\overline{(p_\mu - gz_\mu)\psi}\right](\gamma \cdot n) \\ & + g\text{Tr}(z^5\psi\bar{\psi})(\gamma \cdot n) - \frac{\lambda}{4}\text{Tr}f^{\alpha\beta}f_{\alpha\beta} \\ & + \frac{g}{2M}\text{Tr}(f_{\mu\nu}\Sigma_n^{\mu\nu}\psi\bar{\psi})(\gamma \cdot n). \end{aligned} \quad (4.59)$$

Defining  $j_\alpha$  as in (4.41), (4.42), but with the factor  $\gamma^0\gamma \cdot n$ , required for covariance, i.e.,

$$j_{n\mu} = \frac{ig}{2M}\{(\partial_\mu - igz_\mu)\psi\bar{\psi} - \psi\overline{(\partial_\mu - igz_\mu)\psi}\}(\gamma \cdot n), \quad (4.60)$$

and

$$j_{n5} = g\psi\bar{\psi}(\boldsymbol{\gamma} \cdot \mathbf{n}) \equiv \rho_n, \quad (4.61)$$

the variation of the Lagrangian with respect to the  $z$ -fields, where we have used the cyclic properties of matrices under the trace, yields, setting the coefficients of  $\delta z^\nu$ ,  $\delta z^5$  equal to zero, the field equations

$$\lambda(\partial_\beta f^{5\beta} - ig[z_\beta, f^{5\beta}]) = \rho_n \quad (4.62)$$

and

$$\begin{aligned} \lambda(\partial_\beta f^{\nu\beta} - ig[z_\beta, f^{\nu\beta}]) \\ = j_n^\nu + \frac{g}{M} \Sigma_n^{\mu\nu} \{ \partial_\mu \rho_n - ig[z_\mu, \rho_n] \}. \end{aligned} \quad (4.63)$$

Equation (4.63) corresponds to a Gordon type decomposition of the current, here projected into the foliation space (spacelike) orthogonal to  $n^\mu$ . Note that the covariant derivative of  $\rho_n$  in the last term is also projected into the foliation space.

With the subsidiary condition  $\partial^\beta z_\beta = 0$ , as before, we may write the field equations as

$$\lambda(-\partial^\beta \partial_\beta z^5 - ig[z_\mu, f^{5\mu}]) = \rho_n \quad (4.64)$$

and

$$\lambda(-\partial^\beta \partial_\beta z^\nu - ig[z_\beta, f^{\nu\beta}]) = j_n^\nu + \frac{g}{M} \Sigma_n^{\mu\nu} \{ \partial_\mu \rho_n - ig[z_\mu, \rho_n] \}. \quad (4.65)$$

Note that the spin coupling is not explicit in (4.65). Neglecting, as before, coupling to the spacetime components, one reaches the same conclusions for the approximate behavior of the  $z^5$  field, i.e., as determined by Eq. (4.49) with  $\psi^\dagger$  replaced by  $\bar{\psi}\boldsymbol{\gamma} \cdot \mathbf{n}$ . The latter reduces to the same expression for  $n^\mu \rightarrow (-1, 0, 0, 0)$ .

## 4.4 CP and T Conjugation

The association of this timelike vector with the spacelike surfaces used by Schwinger and Tomonaga (1948) for the quantization of field theories has been recently discussed (Horwitz 2013). These spacelike surfaces form the support of a complete set of commuting local observables on which the Hilbert space of states is constructed. It follows from the properties of the wave functions for a particle with spin, discussed in Chap. 3, that the *CPT* conjugate theory would be associated with the same spacelike surface, corresponding to  $\pm n^\mu$ . However, the *CP* conjugate, taking  $\mathbf{n} \rightarrow -\mathbf{n}$  and  $n^0 \rightarrow n^0$  refers to an entirely different spacelike surface (the time reversed states, for which  $\mathbf{n} \rightarrow \mathbf{n}$  and  $n^0 \rightarrow -n^0$  are associated with this spacelike surface as well, with reflected unit timelike vector). The equivalence of the physical processes described in these two frameworks would depend on the existence of an isometry (including both unitary and antiunitary transformations) changing the basis of the space from

the set of local observables on the first spacelike surface to those defined on the conjugated surface as well as the equivalence of the physics evolving from it after the  $CP$  (or  $T$ ) conjugation.

The spin coupling term in (4.56) contains the possibility of  $CP$  violation in generating a physics that is inequivalent on the new spacelike surface. The nonrelativistic quantum theory with Zeeman type  $\boldsymbol{\sigma} \cdot \mathbf{H}$  coupling is, of course, not invariant under  $T$  conjugation. Precisely the same situation is true in the corresponding relativistic equation (4.56); as we have pointed out, in the special frame in which  $n^\mu = (-1, 0, 0, 0)$ , the matrices  $\Sigma_n^{\mu\nu}$  reduce to Pauli matrices. Under Lorentz transformation they still generate the algebra of  $SU(2)$  in a fundamental representation, and therefore still contain the imaginary unit. Therefore, the physical evolution on the  $CP$  conjugate spacelike surface is not, in general, equivalent to the original evolution. For this phenomenon to occur, it is necessary that there be present an  $f_{\mu\nu}$  field. In addition to self-interaction effects, for which the intrinsic  $CP$  violation can be expected to cancel, the Stueckelberg oscillation diagram of Fig. 4.1 suggests the existence of fields present in the equations of motion of the second branch due to the proximity of the accelerated motion in the first branch, thus providing a fundamental mechanism for  $CP$  violation. A consequence of this structure is that the physics in the corresponding  $CP$  conjugated system of the quantum fields, evolving from the  $CP$  conjugate spacelike surface, could be inequivalent.

In this chapter, we have argued that, according to the derivation of the Landau-Peierls relation given in Arshansky (1985), the vertices of the neutrino-antineutrino transitions may be very sharp, and provide for a rather precise “pull back” of the time interval. Significantly higher precision than available in the present experiments would be necessary to see such an effect.

We have worked out the equations describing the Lorentz forces and the field equations of the corresponding (5D) non-Abelian gauge theory, with the help of Stueckelberg type Hamiltonians both for the spinless case and for the case of relativistic particles with spin in interaction with such a nonabelian gauge field, and have shown that the conclusions reached are, in lowest approximation, consistent with our simple model. We emphasize that, in the framework of the Stueckelberg model, the dynamics of the fifth gauge field, modulated by the particle mass spectrum contained in the wave function (as in Eq. (4.49)), plays an essential role for the oscillation process.

The presence of spin, described in the relativistic framework of Wigner (1939), as in Arshansky (1982), Horwitz (2013), introduces a foliation in the Hilbert space and in the structure of the fields, both classical and quantum. Since, in Tomonaga-Schwinger quantization (Tomonaga 1948) of the fields, the spacelike surface constructed to define a complete set of local observables is characterized by being orthogonal to a timelike vector  $n$  of the foliation (Horwitz 2013), the actions of the discrete  $CP$  or  $T$  transformations change the basis for the construction of the Hilbert space to essentially different spacelike surfaces. Along with the form of the spin coupling

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term in (4.56), this suggests a model for  $CP$  or  $T$  violation on the first quantized level.

We furthermore remark that our model would be applicable to the  $K$ ,  $B$  and  $D$  systems (Kayser 2004) as well, manifested by the quark gluon interactions in their substructure.

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## The Relativistic Action at a Distance Two Body Problem

# 5

Models with action at a distance potentials, such as the Coulomb potential, have been very useful in nonrelativistic mechanics. They provide a simpler framework than the perhaps more fundamental field mediated models for interaction, and are also straightforwardly amenable to rigorous mathematical analysis. In this Newtonian-Galilean view, all events directly interacting dynamically occur simultaneously; the dynamical phase space of  $N$  particles contains the points  $\mathbf{x}_n(t)$  and  $\mathbf{p}_n(t)$ , for  $n = 1, 2, 3, \dots, N$ ; these points move through the phase space as a function of the parameter  $t$ , following some prescribed equations of motion. Two particles may be thought of as interacting through a potential function  $V(\mathbf{x}_1(t), \mathbf{x}_2(t))$ ; for Galilean invariance,  $V$  may be a scalar function of the difference, i.e.,  $V(\mathbf{x}_1(t) - \mathbf{x}_2(t))$ . It is usually understood that  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are taken to be at equal time, corresponding to a correlation between the two particles consistent with the Newtonian-Galilean picture. With the advent of special relativity, it became a challenge to formulate dynamical problems on the same level as that of the nonrelativistic theory.

For the relativistic theory, one might think of two world lines with action at a distance interaction, but the correlation that could be used between the two points  $x_1^\mu$  and  $x_2^\mu$  cannot be maintained by the variable  $t$  in every frame. Dirac (1932) introduced a “many time” theory to describe the dynamics of an  $N$  body system, maintaining the notion of the  $t$  component of the four vector position as associated with evolution; later, his lectures at the Belfer School (Dirac 1966) laid the foundations for the constraint dynamics mentioned in Chap. 2, with some details given in the Appendix of that chapter, for which each particle has its own effective invariant evolution parameter based on the canonical transformation properties induced by the constraints of an  $8N$  dimensional phase space; attempts to imbed this classical approach into a quantum theory met some difficulties, but some progress was made in developing a useful scattering theory (Horwitz 1982). As we have pointed out the Stueckelberg [SHP] theory provides an effective and systematic way of dealing with the  $N$  body problem, and has been applied in describing relativistic fluid mechanics (Sklarz 2001), the Gibbs ensembles in statistical mechanics and the Boltzmann equation (Horwitz 1981), systems of many identical particles, as described in Chap. 3, and other applications. The essential ingredient in developing these applications is the

use of a single invariant parameter (Horwitz 1973 and remark of Sudarshan 1981), to define the correlated interactions of a many body system.

We study in this chapter the relativistic two body problem with invariant action at a distance potentials, for bound states. The two body quantum relativistic scattering problem provides an important and informative application for the methods developed in this chapter; we shall treat it in detail in a later chapter devoted to general scattering theory.

The harmonic string, involving only nearest neighbor interactions, can be decomposed into a set of two body problems, and we therefore study this structure as an example, with interesting connections to the starting points of the development of modern string theory (Suleymanov 2015).

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## 5.1 The Two Body Bound State for Scalar Particles

As a candidate for an invariant action at a distance potential for the two body relativistic bound state (we shall also discuss the scattering states) the function  $V(\rho)$ , for

$$\rho^2 = (\mathbf{x}_1 - \mathbf{x}_2)^2 - (t_1 - t_2)^2 \equiv \mathbf{x}^2 - t^2, \quad (5.1)$$

where  $x_1^\mu$  and  $x_2^\mu$  are taken at equal  $\tau$ , acting as a correlation parameter as well as the global generating parameter of evolution. This “relative coordinate” (squared) reduces to  $(\mathbf{x}_1 - \mathbf{x}_2)^2 \equiv \mathbf{x}^2$  at equal time for the two particles, in the nonrelativistic limit, so that  $1/\rho$  becomes the Coulomb radial dependence  $1/r$  in this limit. Clearly, the solutions of a problem with this potential must then reduce to the solutions of the corresponding nonrelativistic problem in that limit, and therefore this problem poses an important challenge to the theory.

As pointed out in Chap. 2, the two body Stueckelberg Hamiltonian, is

$$K = \frac{p_1^\mu p_{1\mu}}{2M_1} + \frac{p_2^\mu p_{2\mu}}{2M_2} + V(x). \quad (5.2)$$

Since  $K$  does not depend on the total (spacetime) “center of mass”

$$X^\mu = \frac{M_1 x_1^\mu + M_2 x_2^\mu}{M_1 + M_2}, \quad (5.3)$$

the two body Hamiltonian can be separated into the sum of two Hamiltonians, one for the “center of mass” motion and the second for the relative motion, by defining the total momentum, which is absolutely conserved,

$$P^\mu = p_1^\mu + p_2^\mu \quad (5.4)$$

and the relative motion momentum

$$p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M_1 + M_2}. \quad (5.5)$$

This separation is canonical, i.e., the pairs  $P^\mu, X^\mu$  and  $p^\mu, x^\mu$  satisfy separately the canonical Poisson bracket (classically) and commutation relations (quantum

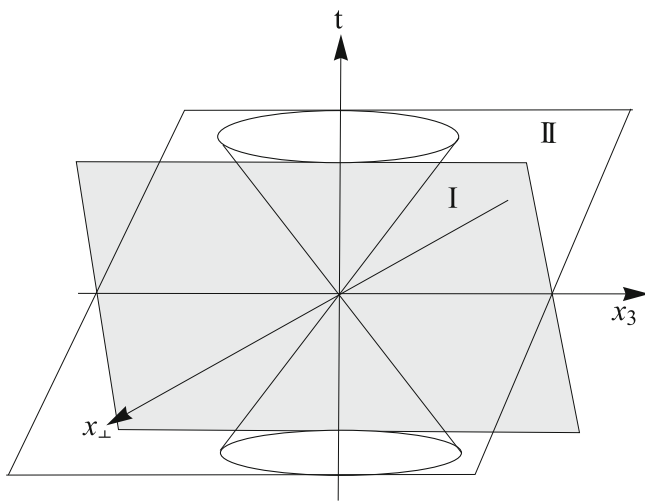
mechanically), and commute with each other. Then, it is an identity that (as in the nonrelativistic two body problem)

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(x), \quad (5.6)$$

$$\equiv K_{CM} + K_{rel},$$

where  $M = M_1 + M_2$ ,  $m = M_1 M_2 / (M_1 + M_2)$ , and  $x = x_1 - x_2$ . Both  $K_{CM}$  and  $K_{rel}$  are constants of the motion; the total and relative momenta for the quantum case may be represented by partial derivatives with respect to the corresponding coordinates. This problem was solved explicitly for the classical case by Horwitz and Piron (1973), where it was shown that there is no precession of the type predicted by Sommerfeld (1939), who used the nonrelativistic form  $1/r$  for the potential (and obtained a period for the precession of Mercury that does not fit the data).

The corresponding quantum problem was solved by Cook (1972), with support for the wave functions in the full spacelike region; however, he obtained a spectrum that did not agree with the Balmer spectrum for hydrogen, i.e. a spectrum of the form  $1/(n + \frac{1}{2})^2$ , with  $n$  an integer. Zmuidzinis (1966), brought to our attention by Winternitz (1985), however, proved that there is *no complete set of functions* in the full spacelike region, and separated the spacelike region into two submanifolds, in each of which there could be complete orthogonal sets. This construction is shown in Fig. 5.1. The region for which  $x^2 > t^2$ , in particular, permits the solution of the differential equations corresponding to the problem posed by (5.2) by separation of variables and provides spectra that coincide exactly with the corresponding nonrelativistic problems with potentials depending on  $r$  alone. We shall call this sector the RMS (reduced Minkowski space) (Arshansky 1989).



**Fig. 5.1** The Reduced Minkowski Space (RMS) support

We may see, moreover, that the RMS carries an important physical interpretation for the nature of the solutions of the differential equations by examining the appropriate variables describing the full spacelike and RMS regions. The full spacelike region is spanned by

$$\begin{aligned} x^0 &= \rho \sinh \beta, & x^1 &= \rho \cosh \beta \cos \phi \sin \theta \\ x^2 &= \rho \cosh \beta \sin \phi \sin \theta, & x^3 &= \rho \cosh \beta \cos \theta \end{aligned} \quad (5.7)$$

over all  $\rho$  from 0 to  $\infty$ ,  $\beta$  in  $(-\infty, \infty)$ ,  $\phi$  in  $(0, 2\pi)$  and  $\theta$  in  $(0, \pi)$ . In general, separation of variables in a second order equation of d'Alembert type follows the method of separating out the least frequent variable on the right hand side of (5.7) first; the separation constant then enters into the second step, involving the separation of the next most frequent variable; this process is continued until all degrees of freedom have been accounted for. In the definitions given in Eq. (5.7), we see that the first separation constant, obtained from the equation for  $\phi$ , which we could call  $m$ , enters into the equation for  $\theta$ . The separation constant for the  $\theta$  equation can be related to  $\ell$ , as it occurs in the usual approach to the solution of the Laplace equation for the nonrelativistic case, and this constant then enters into the equation for  $\beta$ , a physical quantity not recognizable from the nonrelativistic theory (it is associated with hyperbolic functions involved in the Lorentz boost); the resulting quantum number, which we shall call  $n$ , becomes involved in the equation involving the "radial" coordinate. The solution of the equation for the radial function provides the spectrum, in this case, with dependence on a variable not easily associated with the nonrelativistic (and experimentally satisfactory) results. On the other hand, the set of variables describing the RMS, running over the same range of parameters,

$$\begin{aligned} x^0 &= \rho \sin \theta \sinh \beta, & x^1 &= \rho \sin \theta \cosh \beta \cos \phi \\ x^2 &= \rho \sin \theta \cosh \beta \sin \phi, & x^3 &= \rho \cos \theta, \end{aligned} \quad (5.8)$$

cover the entire space external to the planes shown in Fig. 5.1 (for  $x_1^2 + x_2^2 > t^2$ ).

As for (5.7), for  $\beta \rightarrow 0$ , these coordinates become the standard spherical representation of the three dimensional space (at the "simultaneity" point  $t = 0$ , where  $\rho$  becomes  $r$ ). However, the sequence of separation of variables corresponds to the assignment of the constant  $m$ , then  $n$ , and finally  $\ell$ , which is, in this case, the only separation constant entering the spectrum determining equation for the  $\rho$  dependence in the solution. This constant, associated with the Legendre functions, then carries the physical meaning of angular momentum, entering in the same way in the spectrum, as we shall see. Independently of the form of the potential  $V(\rho)$ , one obtains the same radial equation (in  $\rho$ ) as for the nonrelativistic Schrödinger equation (in  $r$ ), and therefore the same spectra for the reduced Hamiltonian. We shall discuss the relation of these results to the energy spectrum after writing the solutions. We summarize in the following the basic mathematical steps.

Assuming the total wavefunction (for  $P \rightarrow P'$ , a point on the continuum of the spectrum of the conserved operator  $P$ )

$$\Psi_{P'\tau}(X, x) = e^{iP'\mu X_\mu} \psi_{P'\tau}(x), \quad (5.9)$$



the evolution equation for each value of the total energy momentum of the system then becomes

$$i \frac{\partial}{\partial \tau} \Psi_{P'\tau}(X, x) = (K_{CM} + K_{rel}) \Psi_{P'\tau}(X, x) = \left[ \frac{P'^2}{2M} + K_{rel} \right] \Psi_{P'\tau}(X, x). \quad (5.10)$$

For the case of discrete eigenvalues  $K_a$  of  $K_{rel}$  (we suppress reference to the value of  $P'$  in the following; due to the complete separation of variables into total and relative motion, the discrete eigenfunctions do not depend on  $P'$ , although the Hilbert space of the reduced motion is attached to the point  $P'$  on the spectrum of  $K_{CM}$ ), we then have the eigenvalue equation (cancelling the center of mass wave function factor and  $K_{CM}$  on both sides)

$$\begin{aligned} K_{rel} \psi^{(a)}(x) &= K_a \psi^{(a)}(x) \\ &= -(1/2m) \partial_\mu \partial^\mu + V(\rho) \psi^{(a)}(x), \end{aligned} \quad (5.11)$$

resembling a Klein Gordon type equation with a spacetime dependent “mass” term. Using the  $O(3, 1)$  Casimir operator, in a way quite analogous to the use of the square of the total angular momentum operator, the Casimir operator of the rotation group  $O(3)$  in the nonrelativistic case, we may separate the angular and hyperbolic angular degrees of freedom from the  $\rho$  dependence. There are two Casimir operators defining the representations of  $O(3, 1)$  (Naimark 1964). The first Casimir operator is

$$\Lambda = \frac{1}{2} M_{\mu\nu} M^{\mu\nu}; \quad (5.12)$$

the second Casimir operator  $\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} M_{\mu\nu} M_{\lambda\sigma}$  is identically zero for two particles without spin. Recalling that our separation into center of mass and relative motion is canonical, and that

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad (5.13)$$

it is straightforward to show, using the canonical commutation relations, that

$$\Lambda = x^2 p^2 + 2ix \cdot p - (x \cdot p)^2. \quad (5.14)$$

Since

$$x \cdot p \equiv x^\mu p_\mu = -i\rho \frac{\partial}{\partial \rho}, \quad (5.15)$$

we see that

$$\Lambda = -\rho^2 \partial^\mu \partial_\mu + 3\rho \frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \rho^2},$$

or

$$-\partial_\mu \partial^\mu = -\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda}{\rho^2}. \quad (5.16)$$

It now follows that (5.11) can be written as

$$K_a \psi^{(a)}(x) = \left\{ \frac{1}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda}{\rho^2} \right] + V(\rho) \right\} \psi^{(a)}(x). \quad (5.17)$$

Further separation of variables depends strongly on the sector of the Minkowski space that we are using. Choosing the RMS variables as we have defined them in (5.8), and with the help of the decomposition (to simplify the calculations a little)

$$L_i = \frac{1}{2} \epsilon_{ijk} (x^j p^k - x^k p^j), \quad (5.18)$$

corresponding to the definition of the nonrelativistic angular momentum  $\mathbf{L}$ , and

$$A^i = x^0 p^i - x^i p^0, \quad (5.19)$$

corresponding to the boost generator  $\mathbf{A}$ , so that

$$\Lambda = \mathbf{L}^2 - \mathbf{A}^2, \quad (5.20)$$

one finds (after writing the time and spatial derivatives in terms of definitions that we have given in (5.8), and some careful computation),

$$\Lambda = -\frac{\partial^2}{\partial \theta^2} - 2 \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} N^2, \quad (5.21)$$

where

$$N^2 = L_3^2 - A_1^2 - A_2^2 \quad (5.22)$$

is the Casimir operator of the  $O(2, 1)$  subgroup of  $O(3, 1)$  leaving the  $z$  axis (and the RMS submanifold) invariant (Bargmann 1947). In terms of the RMS variables that we have defined above,

$$N^2 = \frac{\partial^2}{\partial \beta^2} + 2 \tanh \beta \frac{\partial}{\partial \beta} - \frac{1}{\cosh^2 \beta} \frac{\partial^2}{\partial \phi^2}. \quad (5.23)$$

We emphasize that these operators act freely on their complete natural domain on the whole range of the coordinate parametrizations. Except for the derivatives on  $\beta$ , which runs on the whole real line, derivatives with respect to  $\rho$ ,  $\theta$  and  $\phi$  have the problems of Hermiticity as for the nonrelativistic spherical coordinates (these variables are bounded or semibounded, and the derivatives are not *a priori* defined at the end points); as in the nonrelativistic case, the second order operators we shall use to characterize the quantum states are essentially self-adjoint, and one obtains real spectra.

We now proceed to separate variables and find the eigenfunctions. The solution of the general eigenvalue problem (5.17) can be written

$$\psi(x) = R(\rho) \Theta(\theta) B(\beta) \Phi(\phi), \quad (5.24)$$

with invariant measure in the  $L^2(R^4)$  of the RMS

$$d\mu = \rho^3 \sin^2 \theta \cosh \beta d\rho d\phi d\beta d\theta. \quad (5.25)$$

The functions which are factors of the separated solution (5.24) must each be normalized on its range. To satisfy the  $\phi$  derivatives in (5.23), it is necessary to take

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{i[m+\frac{1}{2}]\phi}, \quad 0 \leq \phi < 2\pi, \quad (5.26)$$

where we have indexed the solutions by the separation constant  $m$ . For the case  $m$  an integer, this is a double valued function. To be compatible with the conditions on the other functions, this is the necessary choice; one must use, in fact,  $\Phi_m(\phi)$  for  $m \geq 0$  and  $\Phi_m^*(\phi)$  for  $m < 0$ .

It has been suggested by Bacry (1990) that the occurrence of the half-integer in the phase is associated with the fact that the RMS is a connected, but not simply connected manifold. One can see this by considering the projective form of the restrictions

$$x^2 + y^2 + z^2 - t^2 > 0 \quad (5.27)$$

assuring that the events are relatively spacelike, and

$$x^2 + y^2 - t^2 > 0, \quad (5.28)$$

assuring, in addition, that the relative coordinates lie in the RMS. Dividing (5.27) and (5.28) by  $t^2$ , and calling the corresponding projective variables  $X, Y, Z$ , we have from (5.27)

$$X^2 + Y^2 + Z^2 > 1, \quad (5.29)$$

the exterior of the unit sphere in the projective space, and from (5.28),

$$X^2 + Y^2 > 1, \quad (5.30)$$

the exterior of the unit cylinder along the  $z$ -axis. Since the space is projective, we can identify the points at infinity of the cylinder, and see that this corresponds to a torus with the unit sphere imbedded in the torus at the origin. Thus, a closed curve around the torus, passing through the central region, cannot be homotopically contracted to a point, indicating the region is not simply connected. Characteristically, such a topological structure is associated with half integer phase (e.g. Shapere 1989). This picture also provides a simple interpretation of what would happen to a quantum state with wave packet inside the torus (in the region  $X^2 + Y^2 < 1$ ); it could tunnel through the imbedded sphere, continuously connected to this interior solution, which would then be a scattering state, not a bound state. Furthermore, Naimark (1964) discusses the mapping of such a wave function into a spinorial representation, suggesting an implicit spin structure in the properties of the bound states of the type we shall discuss. A complete mathematical formulation of this structure has not yet been worked out (I thank Michel 1991 for discussions of this problem), but investigations are continuing.

We now continue with our discussion of the structure of the solutions.

The operator  $\Lambda$  contains the  $O(2, 1)$  Casimir  $N^2$ ; with our solution (5.23), we then have

$$\begin{aligned} N^2 B_{mn}(\beta) &= \left[ \frac{\partial^2}{\partial \beta^2} + 2 \tanh \beta \frac{\partial}{\partial \beta} + \frac{(m + \frac{1}{2})^2}{\cosh^2 \beta} \right] B_{mn}(\beta) \\ &\equiv \left( n^2 - \frac{1}{4} \right) B_{mn}(\beta), \end{aligned} \quad (5.31)$$

where  $n^2$  is the separation constant for the variable  $\beta$ . The term  $(m + \frac{1}{2})^2$  must be replaced by  $(m - \frac{1}{2})^2 = (|m| + (\frac{1}{2}))^2$  for  $m < 0$ . We study only the case  $m \geq 0$  in what follows. The remaining equation for  $\Lambda$  is then

$$\Lambda \Theta(\theta) = \left[ -\frac{\partial^2}{\partial \theta^2} - 2 \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( n^2 - \frac{1}{4} \right) \right] \Theta(\theta). \quad (5.32)$$

For the treatment of Eq.(5.31), it is convenient to make the substitution

$$\zeta = \tanh \beta, \quad (5.33)$$

so that  $-1 \leq \zeta \leq 1$ . One then finds that for

$$B_{mn}(\beta) = (1 - \zeta^2)^{1/4} \hat{B}_{mn}(\zeta), \quad (5.34)$$

(5.31) becomes the well-known equation

$$(1 - \zeta^2) \frac{\partial^2 \hat{B}_{mn}(\zeta)}{\partial \zeta^2} - 2\zeta \frac{\partial \hat{B}_{mn}(\zeta)}{\partial \zeta} + [m(m+1) - \frac{n^2}{1-\zeta^2}] \hat{B}_{mn}(\zeta) = 0. \quad (5.35)$$

The solutions are the associated Legendre functions of the first and second kind (Gel'fand 1963; see also Merzbacher 1970),  $P_m^n(\zeta)$  and  $Q_m^n(\zeta)$ . The normalization condition on these solutions, with the measure (5.25) is

$$\int \cosh \beta |B(\beta)|^2 < \infty,$$

or, in terms of the variable  $\zeta$

$$\int_{-1}^1 (1 - \zeta^2)^{-1} |\hat{B}(\zeta)|^2 d\zeta < \infty. \quad (5.36)$$

The second kind Legendre functions do not satisfy this condition. For the condition on the  $P_m^n(\zeta)$ , it is simplest to write the known result (Gradshteyn (2003))

$$\int_{-1}^1 (1 - \zeta^2)^{-1} |P_{\mu+\nu}^{-\nu}(\zeta)|^2 d\zeta = \frac{1}{\nu} \frac{\Gamma(1 + \mu)}{\Gamma(1 + \mu + 2\nu)} \quad (5.37)$$

The normalized solutions (it is sufficient to consider  $n \geq 0$ ) may be written as

$$\hat{B}_{mn}(\zeta) = \sqrt{n} \sqrt{[\Gamma(1 + m + n) / \Gamma(1 + m - n)]} \times P_m^{-n}(\zeta), \quad (5.38)$$

where  $m \geq n$ .

It is very significant for the structure of the theory that the case  $n = 0$  must be treated with special care; it requires a regularization. For  $n = 0$ , the associated Legendre functions become the Legendre polynomials  $P_m(\zeta)$ . In terms of the integration on  $\beta$ , the factor  $\cosh \beta = (1 - \zeta^2)^{-1/2}$  in the measure is cancelled by the square of the factor  $(1 - \zeta^2)^{1/4}$  in the norm, so that the integration appears as

$$\int_{-\infty}^{\infty} |\hat{B}_m(\zeta)|^2 d\beta.$$

The Legendre polynomials do not vanish at  $\zeta = \pm 1$ , so if  $\hat{B}_m$  and  $P_m$  are related by a finite coefficient, the integral would diverge. When  $n$  goes to zero, as we shall see, associated with the ground state, the wave function spreads along the hyperbola labelled by  $\rho$ , going asymptotically to the light plane; the probability density with respect to intervals of  $\beta$  becomes constant for large  $\beta$ . Events associated with the two particles may be found (for sufficiently large space separation) with  $2 + 1$  lightlike separation out to remote regions of the tangent planes. There is no current associated with such a bound state, and therefore one would not see actual particles. The configuration provides a spacetime structure to the ground state. The (regularized) expectation values reproduce the distribution of the lowest Schrödinger bound state, although the spacetime wave function approaches that of a generalized eigenfunction.

To carry out the regularization, we take the limit as  $n$  goes continuously to zero after computation of scalar products. Thus, we assume the form

$$\hat{B}_m(\zeta) = \sqrt{\epsilon}(1 - \zeta^2)^{\epsilon/2} P_m(\zeta), \quad (5.39)$$

with  $\epsilon \rightarrow 0$  after computation of scalar products. This formula is essentially a residue of the Rodrigues formula

$$P_m^{-n}(\zeta) = (-1)^n (1 - \zeta^2)^{n/2} \frac{d^n}{d\zeta^n} P_m(\zeta) \quad (5.40)$$

for  $n \rightarrow 0$ .

These remarks are, as indicated above, important for the structure of the theory. The operator for the differential equation (5.17) for the eigenvalue of the reduced motion is invariant under the action of the Lorentz group. It follows from acting on the equation with the unitary representation of the Lorentz group that the eigenfunctions must be representations of that group (Wigner 1931) for each value of the eigenvalue. However, as one can easily see, the solutions that we found are, in fact, irreducible representations of  $O(2, 1)$ , not, *a priori*, representations of the Lorentz group  $O(3, 1)$ .

We have required that the wave functions be eigenfunctions of the Casimir operator (5.22) of the  $O(2, 1)$  subgroup. For the generators of  $O(2, 1)$ , we note that

$$\begin{aligned} H_{\pm} &\equiv A_1 \pm iA_2 = e^{\pm i\phi} \left( -i \frac{\partial}{\partial \beta} \pm \tanh \beta \frac{\partial}{\partial \phi} \right), \\ L_3 &= -i \frac{\partial}{\partial \phi}, \\ A_3 &= -i \left( \cot \theta \cosh \beta \frac{\partial}{\partial \beta} - \sinh \beta \frac{\partial}{\partial \theta} \right) \\ L_{\pm} &= L_1 \pm iL_2 \\ &= e^{\pm i\phi} \left( \pm \cosh \beta \frac{\partial}{\partial \theta} - \sinh \beta \cot \theta \frac{\partial}{\partial \beta} \right. \\ &\quad \left. + i \frac{\cot \theta}{\cosh \beta} \frac{\partial}{\partial \phi} \right). \end{aligned} \quad (5.41)$$

It then follows that  $H_{\pm}$  are raising and lowering operators for  $m$  on the functions

$$\begin{aligned}\xi_{n+k}^{-n}(\zeta, \phi) &\equiv B_{n+k,n}(\beta)\Phi_{n+k}(\phi) \\ &= (1 - \zeta^2)^{1/4} \hat{B}_{n+k,n}(\zeta)\Phi_{n+k}(\phi),\end{aligned}\tag{5.42}$$

where it is convenient to replace  $m$  by  $n + k$ . With the relation

$$[L_3, H_{\pm}] = \pm H_{\pm}\tag{5.43}$$

one can show (Arshansky 1989) that

$$H_+\chi_{n+k}^{-n}(\zeta, \phi) = i\sqrt{(k+1)(2n+k+1)}\chi_{n+k+1}^{-n}(\zeta, \phi)\tag{5.44}$$

and that

$$H_-\chi_{n+k+1}^{-n}(\zeta, \phi) = -i\sqrt{(k+1)(2n+k+1)}\chi_{n+k}^{-n}(\zeta, \phi).\tag{5.45}$$

The complex conjugate of  $\chi_{n+k}^{-n}$  transforms in a similar way, resulting in a second (inequivalent) representation of  $O(2, 1)$  with the same value of the  $O(2, 1)$  Casimir operator (these states correspond to replacement of  $m + \frac{1}{2}$  by  $m - \frac{1}{2}$  for  $m < 0$ , and are the result of charge conjugation. Since the operators  $A_1, A_2$  and  $L_3$  are Hermitian, complex conjugation is equivalent to the transpose. Replacing these operators by their negative transpose (to be defined by  $C$ ), leaves the commutation relations invariant. Thus the action on the complex conjugate states involves

$$\begin{aligned}H_-^C &= -H_+^* = H_-, & H_+^C &= -H_-^* = H_+, \\ L_3^C &= -L_3^* = L_3;\end{aligned}\tag{5.46}$$

These are precisely the operators under which the complex conjugate states transform, and thus corresponds to charge conjugation.

We have therefore determined that the wave functions we have obtained are irreducible representations of  $O(2, 1)$ . To construct representations of  $O(3, 1)$ , let us consider the well established method which is effective in constructing representations of  $O(3, 1)$  from representations of  $O(3)$ , a group that we would have found if we were working with solutions in the timelike region (Naimark 1964; Gel'fand 1963), called the *ladder representation*. It follows from the Lie algebra of  $O(3, 1)$  that the  $O(3)$  subgroup Casimir operators  $\ell(\ell + 1)$  are stepped by  $\ell \rightarrow \ell \pm 1$  under the action of the boost from  $O(3, 1)$ . Thus the whole set of representations of  $O(3)$ , from  $\ell = 0$  to  $\infty$  form a representation of  $O(3, 1)$ . Each of the representations of  $O(3)$  entering this tower are trivially normalizable, since they are of dimension  $(2\ell + 1)$ . However, attempting to apply this method to the representations of  $O(2, 1)$  fails, since the normalization of these representations is far from trivial; they are represented in infinite dimensional Hilbert spaces, since there are no unitary finite dimensional representations of a noncompact group such as  $O(2, 1)$ . The application of the Lie algebra to this set connects the lowest state of the tower with the ground state which, as we have shown, requires regularization. The action of the algebra does not provide such a regularization, and therefore the method is inapplicable. We have discussed in Chap. 3 the idea of the *induced representation*, there, applied to representations of spin based on a timelike vector. We may apply this method to

constructing the representations of  $O(3, 1)$  based on an induced representation with the  $O(2, 1)$  “little group”, based on a spacelike vector corresponding to the choice of the  $z$  axis. We shall follow this method in the next section.

We now return to record the solutions of (5.32).

Defining

$$\xi = \cos \theta \quad (5.47)$$

and the functions

$$\hat{\Theta}(\theta) = (1 - \xi^2)^{1/4} \Theta(\theta), \quad (5.48)$$

Equation (5.32) becomes

$$\frac{d}{d\xi} \left( (1 - \xi^2) \frac{d}{d\xi} \hat{\Theta}(\theta) \right) + \left( \ell(\ell + 1) - \frac{n^2}{1 - \xi^2} \right) \hat{\Theta}(\theta), \quad (5.49)$$

where we have defined

$$\Lambda = \ell(\ell + 1) - \frac{1}{4}. \quad (5.50)$$

The solutions of (5.49) are proportional to the associated Legendre functions of the first or second kind,  $P_\ell^n(\xi)$  or  $Q_\ell^n(\xi)$ . For  $n \neq 0$ , the second kind functions are not normalizable (the measure we have specified in (5.25) is the usual one for the Legendre functions), and we therefore reject these.

The unitary irreducible representations of  $O(2, 1)$  are single or double valued, and hence  $m$  must be integer or half integer. As we have seen,  $k$  is integer valued, and therefore  $n$  must be integer or half integer also. Normalizability conditions on the associated Legendre functions then require that  $\ell$  be respectively, positive half-integer or integer. The lowest mass state, as we shall see from the spectral results, corresponds to  $\ell = 0$ , and hence we shall consider only integer values of  $\ell$ . Note that in the nonrelativistic quantum theory the angular momentum quantum number  $\ell$  is chosen to be integer to provide correct representations for the rotation group (Gottfried 1962). Therefore,  $n$  and  $m$  must be integer.

If we take  $\beta$  to zero in the RMS variables of (5.8), this set of variables, as we pointed out, reduces to the standard spherical coordinates on 3D. The factor

$$Y_\ell^n(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{i\phi n} \hat{\Theta}_\ell^n(\theta) \quad (5.51)$$

in the separated solution, where

$$\hat{\Theta}_\ell^n(\theta) = \left( \frac{2\ell + 1}{2} \frac{(\ell - n)!}{(\ell + n)!} \right)^{1/2} P_\ell^n(\cos \theta) \quad (5.52)$$

transforms as

$$Y_\ell^n(\theta, \phi) = \sum^{n'} D_{mn'}^\ell(\eta_1, \eta_2, \eta_3) Y_\ell^{n'}(\theta', \phi'), \quad (5.53)$$

where the  $D_{mn'}^\ell$  are the Wigner rotation functions (Wigner 1931) of the Euler angles  $\eta_1, \eta_2, \eta_3$ .

We now turn to the solution of the radial equations, containing the spectral content of the theory. With the evaluation of  $\Lambda$  in (5.50), we may write the radial equation as

$$\left[ \frac{1}{2m} \left( -\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\ell(\ell+1) - \frac{3}{4}}{\rho^2} \right) + V(\rho) \right] R^{(a)}(\rho) = K_a R^{(a)}(\rho). \quad (5.54)$$

If we put

$$R^{(a)}(\rho) = \frac{1}{\sqrt{\rho}} \hat{R}^{(a)}(\rho), \quad (5.55)$$

Equation (5.54) becomes precisely the nonrelativistic Schrödinger equation for  $\hat{R}^{(a)}$  in the variable  $\rho$ , with potential  $V(\rho)$  (the measure for these functions is, from (5.25), just  $\rho^2 d\rho$ , as for the nonrelativistic theory)

$$\begin{aligned} \frac{d^2 \hat{R}^{(a)}(\rho)}{d\rho^2} + \frac{2}{\rho} \frac{d\hat{R}^{(a)}(\rho)}{d\rho} - \frac{\ell(\ell+1)}{\rho^2} \hat{R}^{(a)}(\rho) + 2m(K_a - V(\rho)) \hat{R}^{(a)}(\rho) &= 0. \end{aligned} \quad (5.56)$$

The lowest eigenvalue  $K_a$ , as for the energy in the nonrelativistic Schrödinger equation, corresponds to the  $\ell = 0$  state of the sequence  $\ell = 0, 1, 2, 3, \dots$ , and therefore the quantum number  $\ell$  plays a role analogous to the orbital angular momentum. This energy is of a lower value than achievable with wave functions with support in the full spacelike region (Cook 1972) and the relaxation of the system to wave functions with support in the RMS may be thought of, in this sense, as a spontaneous symmetry breaking (I thank A. Ashtekar for his remark on this point (Ashtekar 1982)).

We emphasize that this type of solution is available for *every nonrelativistic problem with spherically symmetry potential  $V(r)$* ; all of the details of our derivation depended only on the angular and hyperangular properties of the Stueckelberg-Schrödinger operator.

The value of the full generator  $K$  is then determined by these eigenvalues and the value of the center of mass total mass squared operator, i.e.,

$$K = \frac{P^\mu P_\mu}{2M} + K_a. \quad (5.57)$$

The first term corresponds to the total effective rest mass of the system, and contains the observable energy spectrum through the mass energy relation of Einstein. In particular, the invariant mass squared of the system is given by (sometimes called the Mandelstam variable  $s$  (Eden (1967), Chew (1966))

$$s_a \equiv -P_a^2 = 2M(K_a - K). \quad (5.58)$$

This total center of mass momentum is observed in the laboratory in scattering and decay processes, where it is defined as the sum of the outgoing momenta squared. In the case of two particles, it would be given by  $-(p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu})$ , as



we have defined it in (5.58). This quantity is given in terms of total energy and momentum by

$$s_a = E_T^2 - \mathbf{P}_T^2, \quad (5.59)$$

and in the center of momentum frame, for  $\mathbf{P}_T = 0$ , is just  $E_T^2$ .

In order to extract information about the *energy* spectrum, we must therefore make some assumption on the value of the conserved quantity  $K$ . In the case of a potential that vanishes for large  $\rho$ , we may consider the two particles to be asymptotically free, so the effective Hamiltoniana in this asymptotic region

$$K \cong \frac{p_1^\mu p_{1\mu}}{2M_1} + \frac{p_2^\mu p_{2\mu}}{2M_2}. \quad (5.60)$$

Further, assuming that the two particles at very large distances, in accordance with our experience, undergo a relaxation to their mass shells, so that  $p_i^2 \cong -M_i^2$  (although the mechanism for this effect is not definitively known, the radiation reaction problem and some results in statistical mechanics, to be discussed in Chap. 10, provide examples of how such a mechanism could work). In this case,  $K$  would be assigned the value

$$K \cong -\frac{M_1}{2} - \frac{M_2}{2} = -\frac{M}{2}. \quad (5.61)$$

The two particles in this asymptotic state would, for the bound state problem, be at the ionization point. The process of bringing the two particles together by some interaction localized remotely far in a timelike direction, say, for large negative values of  $\tau$ , which would not influence the solutions of the bound state problem appreciably, we could then, adiabatically bring the particles into their bound states without affecting the conserved total  $K$ . If these assumptions are approximately valid, we find for the total energy, which we now label  $E_a$ ,

$$E_a/c \cong \sqrt{M^2 c^2 + 2MK_a}, \quad (5.62)$$

where we have restored the factors  $c$ .

In the case of excitations small compared to the total mass of the system, we may factor out  $Mc$  and represent the result in a power series expansion

$$E_a \cong Mc^2 + K_a - \frac{1}{2} \frac{K_a^2}{Mc^2} + \dots, \quad (5.63)$$

so that the energy spectrum is just the set  $\{K_a\}$  up to relativistic corrections. Thus, the spectrum for the  $1/\rho$  potential is just that of the nonrelativistic hydrogen problem up to relativistic corrections, of order  $1/c^2$ .

If the spectral set  $\{K_a\}$  includes large negative values, the result (5.62) could become imaginary, indicating the possible onset of instability. However, the asymptotic condition imposed on the evaluation of  $K$  must be re-examined in this case. If the potential grows very rapidly as  $\rho \rightarrow 0$ , then at large spacelike distances, where the hyperbolic surfaces  $\rho = \text{const}$  approach the lightcone, the Euclidean measure  $d^4x$  (thought of, in this context, as small but finite) on the  $R^4$  of spacetime starts to cover very singular values and the expectation values of the Hamiltonian at large spacelike distances may not permit the contribution of the potential to become

negligible; it may have an effectively very long range. This effect can occur in the transverse direction to the  $z$  axis along the tangent to the light cone; the hyperbolas cannot reach the light cone in the  $z$  direction, which, as we shall see, corresponds to the direction of a scattering beam in the standard phase shift analysis. It may play an important role in modelling the behavior of the transverse scattering amplitudes in high energy scattering studied, for example, by Hagedorn (1965).

The resulting value of  $K$ , perhaps at this level necessarily chosen phenomenologically to fit the data, may therefore nevertheless maintain real values for  $E_a$ . This question constitutes an interesting field for rigorous analysis (I thank S. Nussinov for a discussion of this point), somewhat related to the existence of the wave operator in scattering theory, which we will discuss in a later chapter.

## 5.2 Some Examples

In this section we give some examples. We will treat the Coulomb potential, the oscillator and the analog of the three dimensional square well.

For the analog of the Coulomb potential, we take

$$V(\rho) = -\frac{Ze^2}{\rho}. \quad (5.64)$$

In this case the spectrum, according to the solutions above, is given by

$$K_a = -\frac{Z^2 m e^4}{2\hbar^2(\ell + 1 + n_a)^2}, \quad (5.65)$$

where  $n_a = 0, 1, 2, 3, \dots$ . The wave functions  $\hat{R}(\rho)^a$  are the usual hydrogen functions (e.g. Landau 1965)

$$\hat{R}_{n_a \ell}(\rho) = \sqrt{\frac{Z n_a!}{(n_a + \ell + 1)^2 (n_a + 2\ell + 1)}} e^{-x/2} x^{\ell+1} L_{n_a}^{2\ell+1}(x), \quad (5.66)$$

where  $L_{n_a}^{2\ell+1}$  are the Laguerre polynomials, and the variable  $x$  is defined by

$$x = \frac{(2Z\rho/a_0)}{(n_a + \ell + 1)}, \quad (5.67)$$

and  $a_0 = \hbar^2/m e^2$ . The *size* of the bound state, which is related to the atomic form factor, is measured according to the variable  $\rho$  (Hofstadter 1958). For the lowest level (using the regularized functions)  $n_a = \ell = 0$ ,

$$\langle \rho \rangle_{n_a=\ell=0} = \frac{3}{2} a_0. \quad (5.68)$$

The total mass spectrum, given by (5.58), is then

$$s_{n_a, \ell} \cong M^2 c^2 - \frac{m M Z^2 e^4}{\hbar^2 (n_a + \ell + 1)^2}. \quad (5.69)$$

For the case that the nonrelativistic spectrum has value small compared to the sum of the particle rest masses, we may use the approximate relation (5.63) to obtain

$$E_{a,\ell} \cong Mc^2 - \frac{Z^2 me^4}{2\hbar^2(n_a + \ell + 1)^2} - \frac{1}{8} \frac{Z^4 m^2 e^8}{Mc^2 \hbar^4 (n_a + \ell + 1)^4} + \dots \quad (5.70)$$

The lowest order relativistic correction to the rest energy of the two body system with Coulomb like potential is then

$$\frac{\Delta(E_{a,\ell} - Mc^2)}{E_{a,\ell} - Mc^2} = \frac{Z\alpha^2}{4} \left(\frac{m}{M}\right) \frac{1}{(n_a + \ell + 1)^2}. \quad (5.71)$$

For spinless atomic hydrogen ( $Z = 1$ ),  $\Delta(E - Mc^2) \cong 9.7 \times 10^{-8}$  eV and  $E - Mc^2 \cong 13.6$  eV for the ground state. The relativistic correction is therefore of the order of one part in  $10^8$ , about 10% of the hyperfine splitting. For positronium,  $\Delta(E - Mc^2) \sim 2 \times 10^{-5}$  eV it is about one part in  $10^5$ , about 2% of the positronium hyperfine splitting of  $8.4 \times 10^{-4}$  eV (Itzykson 1980). We see quantitatively that the relativistic theory gives results that are consistent with the known data on these experimentally well studied bound state systems.

For the four dimensional oscillator, with  $V(\rho) = \frac{1}{2}m\omega^2\rho^2$ , Eq. (5.56) takes the form

$$\begin{aligned} \frac{d^2 \hat{R}^{(a)}(\rho)}{d\rho^2} + \frac{2}{\rho} \frac{d\hat{R}^{(a)}(\rho)}{d\rho} - \frac{\ell(\ell+1)}{\rho^2} \hat{R}^{(a)}(\rho) \\ + 2m \left( K_a - \frac{m^2 \omega^2}{\hbar^2} \rho^2 - \frac{\ell(\ell+1)}{\rho^2} \right) \hat{R}^{(a)}(\rho) = 0. \end{aligned} \quad (5.72)$$

With the transformation

$$\hat{R}^{(a)}(\rho) = x^{\ell/2} e^{-x/2} w^{(a)}(x), \quad (5.73)$$

for

$$x = \frac{m\omega}{\hbar} \rho^2, \quad (5.74)$$

we obtain the equation

$$\begin{aligned} x \frac{d^2 w^{(a)}}{dx^2} + \left( \ell + \frac{3}{2} - x \right) \frac{dw^{(a)}}{dx} \\ + \frac{1}{2} \left( \ell + \frac{3}{2} - \frac{K_a}{\hbar\omega} \right) w^{(a)} = 0 \end{aligned} \quad (5.75)$$

Normalizable solutions, the Laguerre polynomials  $L_{n_a}^{\ell+1/2}(x)$ , exist (Landau 1965) when the coefficient of  $w^{(a)}(x)$  is a negative integer, so that the eigenvalues are

$$K_a = \hbar\omega \left( \ell + 2n_a + \frac{3}{2} \right), \quad (5.76)$$

where  $n_a = 0, 1, 2, 3, \dots$ . The total mass spectrum is given by (5.58) as

$$s_{n_a, \ell} = -2MK + 2M\hbar\omega\left(\ell + 2n_a + \frac{3}{2}\right), \quad (5.77)$$

Note that the “zero point” term is  $\frac{3}{2}$ , indicating that in the RMS, in the covariant equations there are effectively three intrinsic degrees of freedom, as for the nonrelativistic oscillator.

The choice of  $K$  is arbitrary here, since there is no ionization point for the oscillator, and no *a priori* way of assigning it a value; setting  $K = -\frac{Mc^2}{2}$  as for the Coulomb problem (a choice that may be justified by setting the spring constant equal to zero and adiabatically increasing it to its final value), one obtains, for small excitations relative to the particle masses,

$$E_a \cong Mc^2 + \hbar\omega\left(\ell + 2n_a + \frac{3}{2}\right) - \frac{1}{2} \frac{\hbar^2\omega^2\left(\ell + 2n_a + \frac{3}{2}\right)^2}{Mc^2} + \dots \quad (5.78)$$

Feynman et al. (1971), Kim and Noz (1977) and Leutwyler and Stern (1977) have studied the relativistic oscillator and obtained a positive spectrum (as in (5.76)) by imposing a subsidiary condition suppressing timelike excitations, which lead, in the formalism of annihilation-creation operators to generate the spectrum, to negative norm states (“ghosts”). There are no ghost states in the covariant treatment, and no extra constraints invoked in finding the spectrum. The solutions are given in terms of Laguerre polynomials, but unlike the case of the standard treatment of the 4D oscillator, in which  $x^\mu \pm ip^\mu$  are considered annihilation-creation operators, the spectrum generating algebra (for example, Dothan 1965) for the covariant SHP oscillator has been elusive (Land 2011).

We now turn to the  $O(3, 1)$  invariant square well.<sup>1</sup> In this case the radial Eq. (5.56) must be solved with the potential given by

$$V(\rho) = -U \quad \rho \leq a \quad (5.79)$$

and

$$V(\rho) = 0 \quad \rho > a \quad (5.80)$$

The solutions have the form (for  $-U \leq K_a \leq 0$ ) (Merzbacher 1970)

$$\hat{R}^{(a)}(\rho) = A j_\ell(\kappa_1 \rho), \quad \rho \leq a, \quad (5.81)$$

and

$$\hat{R}^{(a)}(\rho) = B h_\ell^{(1)}(i\kappa_2 \rho), \quad \rho > a \quad (5.82)$$

<sup>1</sup>The  $O(1, 1)$  invariant square well has a completely different character. The boundaries are necessarily hyperbolic, and tunneling through the lightlike regions between the spacelike regions of relative motion, as in the interior region of the RMS (for  $\mathbf{x}^2 - t^2 < 0$ ) prevents the formation of bound states. The scattering theory was worked out by Arshansky and Horwitz (1984).

where  $j_\ell$  are spherical Bessel functions and  $h_\ell^{(1)}$  are spherical Hankel functions of the first kind; here

$$\kappa_1 = \sqrt{2m(K_a + U)/\hbar^2}, \quad \kappa_2 = \sqrt{(-2mK_a)/\hbar^2} \quad (5.83)$$

The radial measure for  $\hat{R}^{(a)}(\rho)$  is the same as for the nonrelativistic case (in  $r$ ), i.e.,  $\rho^2 d\rho$ . Continuity of the wave function at the boundary  $\rho = a$  provides the condition for the values of  $K_a$ . Note that the boundary at  $\rho = a$  for the square well is the apex of the hyperbola; at  $t = 0$ , it lies at  $r = a$ . For the arguments of the Bessel functions large enough (high excitations,  $a$  sufficiently large) the asymptotic form of the Bessel functions may be used to investigate some analytic properties of the wave functions. Since  $\kappa_1^2 + \kappa_2^2 = 2mU/\hbar^2$ , the large argument approximation is satisfied for

$$\xi^2 \equiv \frac{2mU}{\hbar^2} a^2 \gg 1. \quad (5.84)$$

Using the asymptotic forms

$$\begin{aligned} j_\ell(z) &\sim \frac{1}{z} \cos(z - \pi\ell/2 - \pi/2) \\ h_\ell^{(1)} &\sim \frac{1}{z} e^{i(z - \pi\ell/2 - \pi/2)}, \end{aligned} \quad (5.85)$$

we obtain the eigenvalue relations

$$\begin{aligned} -\cot \kappa_1 a &\cong \frac{\kappa_0}{\kappa_1} \quad (\text{l even}) \\ \tan \kappa_1 a &\cong \frac{\kappa_0}{\kappa_1} \quad (\text{l odd}) \end{aligned} \quad (5.86)$$

In the approximation (5.84), for  $n\pi/\xi \cong 1/\sqrt{2}$ , i.e.  $n$  large, the spectrum is approximately given by

$$K_a \cong -\left\{U - \frac{n_a^2 \pi^2 \hbar^2}{2ma^2}\right\} \quad (5.87)$$

so that

$$\begin{aligned} E_a &\cong Mc^2 - \left(U - \frac{n_a^2 \pi^2 \hbar^2}{2ma^2}\right) \\ &\quad - \frac{1}{2Mc^2} \left(U - \frac{n_a^2 \pi^2 \hbar^2}{2ma^2}\right)^2 + \dots \end{aligned} \quad (5.88)$$

The second order relativistic correction to the relativistic spectrum is therefore

$$\frac{\Delta(E_a - Mc^2)}{(E_a - Mc^2)} \cong \frac{1}{2Mc^2} \left(U - \frac{n_a^2 \pi^2 \hbar^2}{2ma^2}\right)^2. \quad (5.89)$$

In (5.87) we see a simple example of the phenomenon described at the end of the previous section, i.e., for sufficiently large well-depth  $U$ , the eigenvalue  $K_a$  can become very large and negative; an argument leading to  $K \cong Mc^2/2$ , as in the Coulomb case (one could conjecture that there should be asymptotic states for the

barrier problem), a negative value can be reached for  $U \geq Mc^2$ . Thus one cannot argue that the value of  $K$  is determined by the asymptotic states; for sufficiently large  $t$  there would always be some support for the wavefunction close (Euclidean, i.e. with measure  $d^4x$ ) enough to the light cone to be sensitive to the very large potential, so the idea of a bound state as a composite system of two particles defined asymptotically as free would become untenable. For  $c \rightarrow \infty$  there is no  $U$  large enough for this effect to occur, so that the phenomenon is intrinsically relativistic.

For the Coulomb case, the assignment of  $K \cong -Mc^2/2$  becomes untenable at

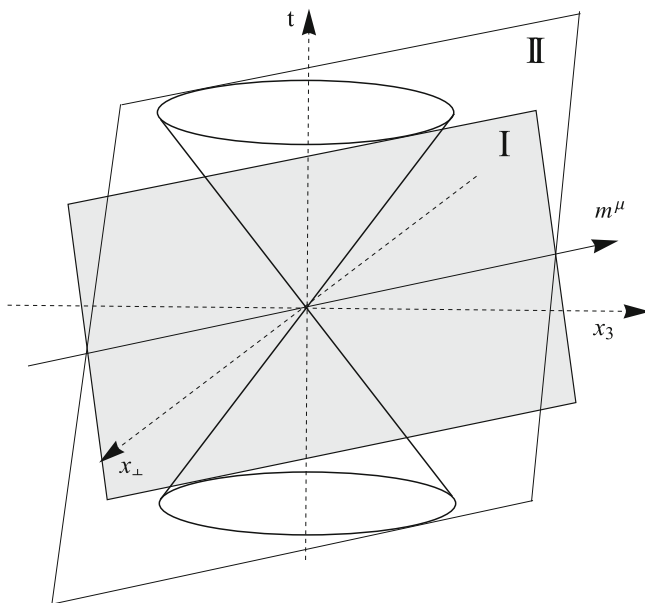
$$Z \geq \frac{M}{\sqrt{M_1 M_2}} (1/\alpha) \quad (5.90)$$

For  $M_1 \ll M_2$ , for example for one electron in the field of a nucleus, e.g.  $M_2 = 2ZM_p$ , the bound on  $Z$  is very high, about  $5 \times 10^5$ . For equal mass, for example two ions, the bound is at  $Z \geq 2/\alpha$ , which is the order of magnitude of the the value for which the Dirac equation becomes unstable. For a Coulomb type strong interaction, where  $\alpha \sim 1$ , the simple picture of compositeness becomes questionable for any  $Z \geq 1$ .

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### 5.3 The Induced Representation

We have remarked that the solutions of the invariant two body problem results in solutions that are irreducible representations of  $O(2, 1)$ , in fact, the complex representations of its covering group  $SU(1, 1)$ , and pointed out that the ladder representations generated by the action of the Lorentz group on these states cannot be used to obtain representations of the full Lorentz group  $O(3, 1)$  or its covering  $SL(2, C)$ . Since the differential equations defining the physical states are covariant under the action of  $O(3, 1)$ , the solutions must be representations of  $O(3, 1)$ . To solve this problem, one observes (Arshansky 1989) that the  $O(2, 1)$  solutions are constructed in the RMS which is referred to the spacelike  $z$  axis. Under a Lorentz boost, the entire RMS turns, leaving the light cone invariant, as shown in Fig. 5.2. After this transformation the new RMS is constructed on the basis of a new spacelike direction which we call here  $m^\mu$  (to distinguish from the timelike  $n^\mu$  of the induced representation for spin treated in Chap. 3). However, the differential equations remain identically the same in the covariant notation, since the operator form of these equations is invariant. The change of coordinates to RMS variables, although now with new geometrical meaning, has the same form as well, and therefore the set of solutions of these equations have the same structure. These functions are now related to the new “ $z$ ” axis, and their transformation properties follow the same general rules that we have explained for the induced representations of spin in this Chap. 5. Under the action of the full Lorentz group the wave functions then undergo a transformation involving a linear combination of the set of eigenfunctions found in the previous section; this action does not change the value of the  $SU(1, 1)$  (or  $O(2, 1)$ ) Casimir operator; together with the change in direction of the vector  $m^\mu$ , they provide an induced representation of  $SL(2, C)$  (or  $O(3, 1)$ ) with little group  $SU(1, 1)$  in the same way that our previous



**Fig. 5.2** The Reduced Minkowski Space (RMS) support

study of the representations of relativistic spin provided a representation of  $SL(2, C)$  with  $SU(2)$  little group. The coefficients in this superposition then play the role of the Wigner  $D$  functions in the induced representation of relativistic particles with spin.

Let us define the coordinates  $\{y_{\mu}\}$ , isomorphic to the set  $\{x_{\mu}\}$ , in an accompanying frame for the  $RMS(m_{\mu})$ , with  $y_3$  along the axis  $m_{\mu}$ . Along with infinitesimal operators of the  $O(2, 1)$  generating changes within the  $RMS(m_{\mu})$ , there are generators on  $O(3, 1)$  which change the direction of  $m_{\mu}$ ; as for the induced representations for systems with spin, the Lorentz group contains these two actions, and therefore both Casimir operators are essential to defining the representations, i.e., both

$$c_1 \equiv \mathbf{L}(m)^2 - \mathbf{A}(m)^2 \quad (5.91)$$

and

$$c_2 \equiv \mathbf{L}(m) \cdot \mathbf{A}(m), \quad (5.92)$$

which is not identically zero, and commutes with  $c_1$ .

In the following, we construct functions on the orbit of the  $SU(1, 1)$  little group representing the full Lorentz group; along with the designation of the point on the orbit, labelled by  $m_{\mu}$ , these functions constitute a description of the physical state of the system.

It is a quite general result that the induced representation of a noncompact group contains all of the irreducible representations. We decompose the functions along the orbit into basis sets corresponding to eigenfunctions for the  $O(3)$  subgroup Casimir

operator  $\mathbf{L}(m)^2 \rightarrow L(L+1)$  and  $L_1 \rightarrow q$  that take on values that persist along the orbit; these solutions correspond to the *principal series* of Gel'fand (1963). These quantum numbers for the induced representation do not correspond directly to the observed angular momenta of the system. The values that correspond to spectra and wavefunctions with nonrelativistic limit coinciding with those of the nonrelativistic problem, are those with  $L$  half-integer for the lowest Gel'fand  $L$  level. The partial wave expansions in scattering theory, which we discuss in a later chapter (for the continuous spectrum of  $K_{rel}$ ), depend on the quantum number  $\ell$  of the  $O(3, 1)$  defined on the whole space by the quantum form of (5.13), and a magnetic quantum number, which we shall call  $n$ , associated with the Casimir of the  $SU(1, 1)$  discussed above, then playing the role of the magnetic quantum number, as discussed in the previous section for the bound state problem. In fact, in the Gel'fand classification, the two Casimir operators take on the values  $c_1 = L_0^2 + L_1^2 - 1$ ,  $c_2 = -iL_0L_1$ , where  $L_1$  is pure imaginary and, in general,  $L_0$  is integer or half-integer. In the nonrelativistic limit, the action of the group on the relative coordinates becomes deformed in such a way that the  $O(3, 1)$  goes into the nonrelativistic  $O(3)$ , and the  $O(2, 1)$  into the  $O(2)$  subgroup in the initial configuration of the RMS based on the  $z$  axis.

The representations that we shall obtain, in the principal series of Gel'fand (1963), are unitary in a Hilbert space with scalar product product that is defined by an integration invariant under the full  $SL(2, C)$ , including an integration over the the measure space of  $SU(1, 1)$ , carried out in the scalar product in  $L^2(R^4 \subseteq \text{RMS}(m_\mu))$ , for each  $m_\mu$  (corresponding to the orientation of the new  $z$  axis, and an integration over the measure of the coset space  $SL(2, C)/SU(1, 1)$ ; the complete measure is  $d^4y d^4m \delta(m^2 - 1)$ , i.e., a probability measure on  $R^7$ , where  $y_\mu \in \text{RMS}(m_\mu)$ . The coordinate description of the quantum state therefore corresponds to an ensemble of (relatively defined) events lying in a set of  $\text{RMS}(m_\mu)$ 's over all possible spacelike  $\{m_\mu\}$ .

A coordinate system oriented with its  $z$  axis along the direction  $m_\mu$ , as referred to above, can be constructed by means of a coordinate transformation of Lorentz type (here  $m$  represents the spacelike orientation of the transformed RMS, not to be confused with a magnetic quantum number),

$$y_\mu = L(m)_\mu{}^\nu x_\nu. \quad (5.93)$$

For example, if we take a vector  $x_\mu$  parallel to  $m_\mu$ , with  $x_\mu = \lambda m_\mu$ , then the corresponding  $y_\mu$  is  $\lambda m_\mu^0$ , with  $m_\mu^0$  in the direction of the initial orientation of the orbit, say, the  $z$  axis. This definition may be replaced by another by right multiplication of an element of the stability group of  $m_\mu$  and left multiplication by an element of the stability group of  $m_\mu^0$ , constituting an isomorphism in the RMS.

The variables  $y_\mu$  may be parametrized by the same trigonometric and hyperbolic functions as in (5.8) since they span the RMS, and provide a complete characterization of the configuration space in the  $\text{RMS}(n_\mu)$  that is universal in the sense that it is the same in every Lorentz frame. It is convenient to define the functions

$$\psi_m(y) = \phi_m(L^T(m)y) = \phi_m(x) \quad (5.94)$$



In a similar way to our previous treatment of the induced representation for spin, we can then define the map of the Hilbert spaces associate with each  $m_\mu$  in the foliation  $\mathcal{H}_m \rightarrow \mathcal{H}_{\Lambda m}$  such that the state vectors are related by the norm preserving transformation

$$\Psi_{\Lambda m}^\Lambda = U(\Lambda)\Psi_m. \quad (5.95)$$

In the new Lorentz frame (with  $y = L(\Lambda m)x$ ),

$$\begin{aligned} \phi_{\Lambda m}^\Lambda(x) &= \Lambda m \langle x | \Psi_{\Lambda m}^\Lambda \rangle \\ &= \Lambda m \langle x | U(\Lambda)\Psi_m \rangle = \phi_{\Lambda m}^\Lambda(L^T(\Lambda m)y) \\ &= \psi_{\Lambda m}^\Lambda(y). \end{aligned} \quad (5.96)$$

If  $\phi_m(x)$  is scalar under Lorentz transformation, so that (we assume no additional phase)

$$\phi_{\Lambda m}^\Lambda(\Lambda x) = \phi_m(x), \quad (5.97)$$

it follows from (5.96) that

$$U(\Lambda)|x \rangle_m = |\Lambda x \rangle_{\Lambda m}. \quad (5.98)$$

The wave function  $\phi_{\Lambda m}^\Lambda(x)$  describes a system in a Lorentz frame in motion with respect to the frame in which the state is described by  $\phi_m(x)$ , and for which the support is in the RMS  $((\Lambda m)_\mu)$ . The value of this function at  $x$  in the new frame is determined by its value at  $\Lambda^{-1}x$  in the original frame; moreover, the subensemble associated with values of  $m_\mu$  over the orbit in the new frame is determined by the subensemble associated with the values of  $(\Lambda^{-1}m)_\mu$  in the old frame. We define the description of the state of the system in the new frame in terms of the set (over  $\{m_\mu\}$ ) of transformed wave functions

$$\begin{aligned} \psi_m^\Lambda(y) &\equiv \phi_{\Lambda^{-1}m}(\Lambda^{-1}x) \\ &= \psi_m^\Lambda(D^{-1}(\Lambda, m)y) \end{aligned} \quad (5.99)$$

where we have used (5.93) (the transformed function has support oriented with  $m_\mu$ ) and defined the (pseudo) orthogonal matrix (we define a ‘‘matrix’’  $A$  as  $\{A_\mu^\nu\}$ )

$$D(\Lambda, m) = L(m)\Lambda L^T(\Lambda^{-1}m). \quad (5.100)$$

This matrix is analogous to the ‘‘little group’’ acting on the  $SU(2)$  of the rotation subgroup discussed in Chap. 3. The transformation  $D^{-1}(\Lambda, m)$  stabilizes  $m_\mu^0$ , and is therefore in the  $O(2, 1)$  subgroup that leaves the RMS of the original system invariant. Equation (5.99) defines an induced representation of  $SL(2, C)$ , the double covering of  $O(3, 1)$ .

Classification of the orbits of the induced representation are determined by the Casimir operators of  $SL(2, C)$ , defined as differential operators on the functions  $\psi_m(y)$  of (5.94), i.e., the operators defined in (5.91) and (5.92). To define these variables as differential operators on the space  $\{y\}$ , we study the infinitesimal Lorentz transformations

$$\Lambda \cong 1 + \lambda, \quad (5.101)$$

for which

$$\psi^{1+\lambda}_m(y) = \psi_{m-\lambda m}(D^{-1}(1 + \lambda, n)y), \quad (5.102)$$

and  $\lambda$  is an infinitesimal Lorentz transformation (antisymmetric). To first order, the little group transformation is

$$D^{-1}(1 + \lambda, n) \cong 1 - (d_m(\lambda)L(m))L^T(m) - L(m)\lambda L^T(m), \quad (5.103)$$

where  $d_m$  is a derivative with respect to  $m_\mu$  holding  $y_\mu$  fixed,

$$d_m(\lambda) = \lambda_\mu{}^\nu m_\nu \frac{\partial}{\partial n_\mu}. \quad (5.104)$$

From the property  $L(m)L^T(m) = 1$ , it follows that

$$(d_m(\lambda)L(m))L^T(m) = -L(m)(d_m(\lambda)L^T(m)), \quad (5.105)$$

so that (5.103) can be written as

$$\begin{aligned} D^{-1}(1 + \lambda, n) &\cong 1 + L(m)(d_n(\lambda)L^T(m) - \lambda L^T(m)) \\ &\equiv 1 - G_m(\lambda). \end{aligned} \quad (5.106)$$

For the transformation of  $\psi_m$  we then obtain

$$\psi^{1+\lambda}_m(y) \cong \psi_m(y) - d_m(\lambda + g_m(\lambda))\psi_m(y), \quad (5.107)$$

where

$$g_m(\lambda) = G_m(\lambda)_\mu{}^\nu y_\nu \frac{\partial}{\partial y_\mu}. \quad (5.108)$$

Equation (5.107) displays explicitly the effect of the transformation along the orbit and the transformation within the little group.

The algebra of these generators of the Lorentz group are investigated in Arshansky (1989); the closure of this algebra follows from the remarkable property of compensation for the derivatives of the little group generators along the orbit (behaving as a covariant derivative in differential geometry). The general structure we have exhibited here is a type of fiber bundle, sometimes called a *Hilbert bundle*, consisting for a set of Hilbert spaces on the base space of the orbit; in this case, the fibers, corresponding to these Hilbert spaces, transform under the little group  $O(2, 1)$ .

There are functions on the orbit with definite values of the two Casimir operators, as well as  $\mathbf{L}(m)^2$  and  $L_1(m)$ ; one finds the Gel'fand Naimark canonical representation with decomposition over the  $SU(2)$  subgroup of  $SL(2, C)$ , enabling an identification of the angular momentum content of the representations (Arshansky 1989). With a consistency relation between the Casimir operators (for the solution of the finite set of equations involving functions on the hyperbolic parameters of the spacelike four vector  $m_\mu$ ), we find that we are dealing with the principal series of Gel'fand and Naimark.

## 5.4 The Stueckelberg String

An interesting example utilizing the structure discussed above is a many body problem with two body forces, for example, a harmonic spacetime string. The many body string can be resolved by separation of variables in the Fourier space into a direct sum of oscillators of the type discussed in Sect. 5.2 (Suleymanov 2015).

Consider the model

$$K = \sum_{i=1}^N \left\{ \frac{p_i^\mu p_{i\mu}}{2m} + \frac{\kappa}{2} (x_{i+1}^\mu - x_i^\mu)(x_{i+1\mu} - x_{i\mu}) \right\}, \quad (5.109)$$

where we have taken equal mass parameter (here called  $m$ ) for all the particles in the string. It follows from Hamilton's equations that

$$\dot{x}_i^\mu = \frac{p_i^\mu}{m} \quad (5.110)$$

and that

$$\dot{p}_i^\mu = \kappa(x_{i+1}^\mu + x_{i-1}^\mu - 2x_i^\mu), \quad (5.111)$$

since the index  $i$  occurs in neighboring terms. We then obtain

$$\ddot{x}_i^\mu = \frac{\kappa}{m} (x_{i+1}^\mu + x_{i-1}^\mu - 2x_i^\mu). \quad (5.112)$$

In the continuum limit, the right hand side goes to a second derivative in a continuous body coordinate which we shall call  $\sigma$ . The usual procedure (Mahan 1990; Huang 1967) is then to replace the string function  $x^\mu(\sigma, \tau)$  by a Fourier expansion in which the coefficients become the amplitudes for the modes. The resulting expression appears to be that of a family of oscillators, for which the quantum problem can be solved for a bound state spectrum in each mode provided that the amplitudes lie in an RMS, as pointed out in the previous section. In this procedure, one notices that the linear combinations of discrete string coordinates that go in the limit to a second derivative of the function  $x^\mu(\sigma, \tau)$  involve a sequence of second differences of neighboring coordinates along the string. Thus the assignment of the Fourier amplitudes of these functions to an RMS implies that these vector valued functions are oriented according to the same RMS along the string; both the amplitudes and functions  $x^\mu(\sigma, \tau)$  transform as tensors under the  $O(2, 1)$  of the associated RMS, corresponding to a given  $m_\mu$ . A change in orientation along the string, corresponding to differing physical meanings for the points  $\{x_i\}$  would not be consistent with the identification of the Fourier amplitudes with an RMS. In the induced representation for the bound state eigenfunctions, we therefore conclude that *all of the modes of the quantized string must be oriented according same RMS, i.e., on the same point of the orbit  $m_\mu$* . This result is analogous to the universality of the location on the orbit of the induced representation for all particles in a many body system with spin that was discussed in Chap. 3, and would be applicable for the perturbative treatment of general many body systems with symmetric two body forces for which one seeks effective quasiparticle modes.

To illustrate these remarks explicitly, we give here the general procedure (Suleymanov 2015) (in the framework of the SHP theory) for the quantized string; models

of this type have been used as a basis for the development of modern string theory (Nambu 1970; Polchinski 1998; Green 1986). Defining  $\rho d\sigma \cong m$  and  $\kappa d\sigma \cong \eta$  in taking the limit to the continuous problem, one finds, from (5.109), the Hamiltonian

$$K = \frac{1}{2} \int_0^L d\sigma \left\{ \rho \dot{x}^\mu \dot{x}_\mu + \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x_\mu}{\partial \sigma} \right\}, \quad (5.113)$$

where we have taken  $L$  to be the length (on the body coordinate  $\sigma$ ) of the string; this notation does not preclude a closed string, which would imply additional boundary conditions for continuity. The Eqs. (5.112) then become (they may be directly derived from (5.113))

$$\rho \frac{\partial^2 x^\mu}{\partial \tau^2} = \eta \frac{\partial^2 x^\mu}{\partial \sigma^2}. \quad (5.114)$$

Defining  $q_n = n(2\pi/L)$ , one may write the Fourier expansion

$$x^\mu(\sigma, \tau) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} (A_n^\mu(\tau) \cos(q_n \sigma) + B_n^\mu(\tau) \sin(q_n \sigma)). \quad (5.115)$$

Although we shall work here in the framework of the modes to describe the quantum system, it is clear that there is a wavefunction  $\psi_{\sigma, \tau}(x)$ , constructed from the quantum state vector using the spectral representation of  $x^\mu(\sigma, \tau)$  as a set of self-adjoint operators (Dirac 1930; von Neumann 1955) with the interpretation that for each  $\tau$  in the evolution of the string, and choice of body coordinate along the string, there is a likelihood of finding an event at the point  $x^\mu$ . In second quantization, this structure corresponds well to that of the structure of conformal field theory (e.g. Di Francesco 1997).

From Eq. (5.115) we have that

$$A_n^\mu = \sqrt{\frac{2}{L}} \int_0^L d\sigma \cos(q_n \sigma) x^\mu(\sigma, \tau) \quad B_n^\mu = \sqrt{\frac{2}{L}} \int_0^L d\sigma \sin(q_n \sigma) x^\mu(\sigma, \tau). \quad (5.116)$$

The canonical momentum density is then given by

$$p^\mu = \rho \dot{x}^\mu(\sigma, \tau). \quad (5.117)$$

One then obtains

$$\int_0^L d\sigma \dot{x}^\mu \dot{x}_\mu = \sum_{n=0}^{\infty} (\dot{A}_n^\mu \dot{A}_{n\mu} + \dot{B}_n^\mu \dot{B}_{n\mu}) \quad (5.118)$$

and

$$\int_0^L d\sigma \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x_\mu}{\partial \sigma} = \sum_{n=0}^{\infty} q_n^2 (A_n^\mu A_{n\mu} + B_n^\mu B_{n\mu}) \quad (5.119)$$

With these results, we have from (5.113) (and the definitions of  $\rho$  and  $\eta$ ) that

$$K = \frac{1}{2} \rho \sum_{n=0}^{\infty} (\dot{A}_n^\mu \dot{A}_{n\mu} + \dot{B}_n^\mu \dot{B}_{n\mu} + \omega_n^2 (A_n^\mu A_{n\mu} + B_n^\mu B_{n\mu})), \quad (5.120)$$

so that  $K$  can be written as a sum over the Hamiltonians for each mode, i.e.,

$$K = \sum_{n=0}^{\infty} K_n \quad (5.121)$$

with

$$K_n = \frac{1}{2}\rho(\dot{A}_n^\mu \dot{A}_{n\mu} + \dot{B}_n^\mu \dot{B}_{n\mu} + \omega_n^2(A_n^\mu A_{n\mu} + B_n^\mu B_{n\mu})). \quad (5.122)$$

For the quantum treatment, for which we impose equal  $\tau$  commutation relations

$$[x^\mu(\sigma, \tau), p^\nu(\sigma', \tau)] = i\delta(\sigma - \sigma')g^{\mu\nu}, \quad (5.123)$$

or,

$$[x^\mu(\sigma, \tau), \dot{x}^\nu(\sigma', \tau)] = \frac{1}{\rho}i\delta(\sigma - \sigma')g^{\mu\nu}, \quad (5.124)$$

it follows that

$$[A_n^\mu(\tau), \dot{A}_n^\nu(\tau)] = \frac{2}{L} \int_0^L d\sigma d\sigma' \cos(q_n\sigma) \cos(q_n\sigma') [x^\mu(\sigma, \tau), \dot{x}^\nu(\sigma', \tau)] = \frac{1}{\rho}i\delta_{mn}g^{\mu\nu}. \quad (5.125)$$

We can therefore define a conjugate momentum for  $A_n^\mu$  as

$$\pi_{n,A}^\mu = \rho \dot{A}_n^\mu; \quad (5.126)$$

in the representation in which  $A_n^\mu$  is diagonal, it acts as  $-i\partial/\partial A_{n\mu}$ . With a similar definition for  $\pi_{n,B}^\mu$ , we have the pair of canonical sets (the  $A$  and  $B$  variables commute with each other)

$$[A_n^\mu(\tau), \pi_{nA}^\nu(\tau)] = i\delta_{nm}g^{\mu\nu} \quad [B_n^\mu(\tau), \pi_{nB}^\nu(\tau)] = i\delta_{nm}g^{\mu\nu} \quad (5.127)$$

With these definitions, (5.122) takes the form of a Hamiltonian for two decoupled oscillators

$$K_n = \frac{1}{2\rho}(\pi_{nA}^\mu \pi_{nA\mu} + \pi_{nB}^\mu \pi_{nB\mu}) + \frac{1}{2}\rho\omega_n^2(A_n^\mu A_{n\mu} + B_n^\mu B_{n\mu}). \quad (5.128)$$

On the manifolds  $\{A_{n\mu}\}$  and  $\{B_{n\mu}\}$ , each of the decoupled problems is precisely that which was solved for the 4D oscillator in the previous sections; the RMS is now defined in terms of the ‘‘coordinates’’  $\{A_{n\mu}\}$  and  $\{B_{n\mu}\}$ . Since, as we have argued above, both manifolds are derived through linear transforms of the functions  $x^\mu(\sigma, \tau)$ , the decoupled solutions must nevertheless lie in their associated RMS at the same point of the orbit labelled by  $m_\mu$ . The mode eigenfunctions satisfy

$$\left[ \frac{1}{2\rho}\pi_{nA}^\mu \pi_{nA\mu} + \frac{1}{2}\rho\omega_n^2(A_n^\mu A_{n\mu}) \right] \varphi_n^{\lambda_A(n)}(A_n) = k_n^{\lambda_A(n)} \varphi_n^{\lambda_A(n)}(A_n) \quad (5.129)$$

and

$$\left[ \frac{1}{2\rho}\pi_{nB}^\mu \pi_{nB\mu} + \frac{1}{2}\rho\omega_n^2(B_n^\mu B_{n\mu}) \right] \varphi_n^{\lambda_B(n)}(B_n) = k_n^{\lambda_B(n)} \varphi_n^{\lambda_B(n)}(B_n) \quad (5.130)$$

The general wave function includes all of the modes in the form

$$\varphi(A_1, A_2, \dots, B_1, B_2, \dots) = \prod_{n=1}^{\infty} \varphi_n^{\lambda_A(n)}(A_n) \varphi_n^{\lambda_B(n)}(B_n), \quad (5.131)$$

for

$$\{\lambda\} = \lambda_A(1), \lambda_B(1), \lambda_A(2), \lambda_B(2) \dots, \quad (5.132)$$

and the spectrum, given for each mode (for  $A$  or  $B$ ) by (5.76) as

$$k_n^\lambda \equiv k_n^{\ell r} = \hbar\omega_n(\ell_n + 2r_n + \frac{3}{2}), \quad (5.133)$$

The complete relative spectrum is the sum over the  $A$  and  $B$  modes.

The total energy of the system can be computed in a way analogous to that of the two body bound state. The potential is Poincaré invariant, and therefore the total momentum of the string is absolutely conserved. We may therefore write the total Hamiltonian of the system as

$$K_T = \frac{P^\mu P_\mu}{2M} + K_{Osc}, \quad (5.134)$$

where  $K_{Osc}$  is the string Hamiltonian (5.109). The total invariant energy of the system is then given by (5.58),

$$s_a \equiv -P^2 = 2M(K_{Osc} - K). \quad (5.135)$$

where  $K_{Osc}$  is the set of string eigenvalues given in terms of (5.133) (including both  $A$  and  $B$  contributions), which, for small excitations, provides essentially the nonrelativistic oscillator spectra. The total rotational degrees of freedom can also be added to the total Hamiltonian in a covariant way, and its quantization would lead to additional rotator levels.

The eigenfunctions, also given by our previous discussion of the oscillator, are normalizable and can be used for the computation of the expectation values of string observables in general string states. Relations of this construction to the usual approach to string theory, and some applications, can be found in Suleymanov (2015).

In this chapter we shall discuss the notion of coherence in time, and, in particular, describe in some detail the experiment of Lindner et al. (2005) in which it was demonstrated that an electron wave packet undergoing a sequential ionizing perturbation in time (from Argon gas) undergoes interference phenomena, and the careful analysis and design of an experiment by Palacios et al. (2009) to show that a two electron spin state could be formed coherently from two electrons at different times. We explain that neither of these experiments can be understood consistently in the framework of standard nonrelativistic quantum theory (or its extension to the nonrelativistic Floquet theory (Floquet 1883)), but that the Stueckelberg-Horwitz-Piron theory can describe these phenomena in a simple way when the time intervals involved lie within the spread in time of the wave packets.

In this chapter, we also discuss the phenomenon of the *spacetime lattice*. For example, a standing electromagnetic wave in a cavity is periodic in space, but at each position, there is a time oscillation as well. Plotting this picture on a diagram of amplitude in space versus time, we see a spacetime lattice. Since the Stueckelberg wave function is a function on spacetime, this lattice has its expression in terms of Bloch waves, just as for a space lattice in the three dimensional nonrelativistic theory. It has been shown (Engelberg 2009) that with suitable lattice spacing, there are bands and mass gaps just as in the nonrelativistic theory, and that the detection of these gaps might be accessible experimentally.

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## 6.1 General Problem of Coherence in Time

The standard nonrelativistic quantum theory does not have the property of coherence in time. We discuss in the following some theoretical structures which consider the time as an observable, and thus could, in principle, have coherence in time, and show that among these, only the Stueckelberg relativistically covariant quantum theory can provide results applicable to, and consistent with, these experiments.

In the book edited by Muga et al. (2009), much discussion is devoted to the difficulties of treating time as an operator in the standard framework, and some constructions are given for which certain operator valued functions of the phase space variables  $q, p$  can be interpreted as “time operators” (Galapon 2002). The spectra of these operators do not, however, enter into a propagator in a way that can naturally generate interference effects of the type observed in Lindner et al. (2005).

A time operator, corresponding to the variable  $t$  that we recognize as time in the laboratory, is constructed in the work of Hahne (2006), in which he permits a space coordinate, say,  $z$  to act as an evolution parameter, and  $x, y, t$  become operator valued. Although coherence in  $t$  can be achieved, the use of  $z$  as an evolution parameter for the flow of the wave packet makes it difficult to formulate a description of the Lindner et al., experiment or the proposed experiment of Palacios et al. (2009) in these terms.

Bauer (2013) among others, has discussed the existence of functions of the variables  $p, q$  that can be translated by the Hamiltonian and may be effective in defining a time of flight measure for wave packets (see also discussion of the construction of POV operators in Strauss et al. (2011) where a Lyapunov function is rigorously constructed recording the passage of time in the Schrödinger evolution, remarkable in its power of description of resonant states, but does not define a time operator).

On the the other hand, Floquet theory (Floquet 1883), a method for treating time as a dynamical variable in a nonrelativistic framework and the manifestly covariant quantum theory of Stueckelberg, as discussed in the previous chapters, are, in principle, directly applicable.

The Floquet theory, although originally developed in the quantum theory for a very different purpose, for the development of techniques of handling time dependent (in particular, with periodic time dependence) Hamiltonian systems, actually emerges in the nonrelativistic limit of the Stueckelberg theory. We shall show that it cannot account for the results of the Lindner experiment. However, the Stueckelberg theory, in its full relativistic form, does account for the results in a simple and consistent way, implying that the effect is essentially relativistic.

In the following, we describe the Lindner et al. experiment and the proposed experiment of Palacios et al.

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## 6.2 The Lindner Experiment

We begin with a discussion of the Lindner experiment and its implications for the nature of the observed time. In this experiment, laser light of about 850 nm wavelength is radiated onto a sample of Argon gas in a short pulse of one and a half wavelengths, constituting two peaks in the electric field in one direction, and one, in between, in the opposite direction. An electron may be emitted as a result of interaction with the first peak, or the third, separated by about one femtosecond in time. At the detector, one sees an interference pattern between the two possibilities corresponding to ejection at the first or third maximum in the wave, much like the double slit experiment in space (Merzbacher 1970). The second peak in the opposite



direction, which exhibits no perceptible interference effect, was used to confirm that just a single electron was involved in the process.

The interference observed in the spacial double slit experiment is accounted for by the coherence of the wave function in space, and was one of the earliest experimental confirmations (Davisson 1927) of the structure of the quantum theory as it emerged from its formulation in Hilbert space. In view of the development of recent technologies, it was natural for the group at the Max Planck Institute (Lindner et al.) to ask whether one could see interference in time. Their experiment was remarkably successful, but raised fundamental questions on the role of time in the quantum theory.

This experiment clearly shows the effect of quantum interference in time. The results are discussed in their paper in terms of a very precise solution of the time-dependent nonrelativistic Schrödinger equation. However, the nonrelativistic Schrödinger theory cannot, according to the basic principles of the quantum theory, be used to predict interference phenomena in time, and therefore the very striking results of this beautiful experiment have a fundamental importance which goes beyond the technical advances which they represent. These results imply, in fact, that the time variable  $t$  must be adjoined to the set of standard quantum variables so that the standard ket  $|x, t\rangle$  for the representation of the quantum state (in Dirac's terminology (Dirac 1930)) can be constructed (Horwitz 2006). It is this structure for the wave function  $\psi(x, t) \equiv \langle x, t | \psi \rangle$ , where  $x$  and  $t$  are the spectra of self-adjoint operators, that provides the possibility of coherence in  $t$ , and therefore, interference phenomena. If the quantum theory is to remain symplectic in form, moreover, the variable  $E$  (in addition to  $t$ ) must also be adjoined.

Let us examine the reasons why the standard nonrelativistic quantum theory cannot be used to predict interference in time. For example, Ludwig (1982) has pointed out that "time" cannot be a quantum observable, since there is no imprimitivity system involving this variable (i.e., no observable exists that does not commute with  $t$  in the nonrelativistic theory). Note that the Hamiltonian of the standard theory evolves quantum states in time, but does not act as a shift operator since it commutes with  $t$ . Dirac (1930) has argued that if  $t$  were an operator, then the resulting  $t, E$  commutation relation would imply that the energy of the system is unbounded below (with no gaps), from which he concluded that the time cannot be an observable in the nonrelativistic quantum theory. Moreover, in the general practice of the use of the nonrelativistic quantum theory (Merzbacher 1970), and as the axiomatic treatment of Piron (1976) (see also, Jauch 1968) shows, the Hilbert space of the quantum theory is constructed of a set of wave functions satisfying a normalization condition based on integration over all space, e.g., for a single particle,  $\int |\psi_t(x)|^2 d^3x \leq \infty$ , for each value of the parameter  $t$ . Since  $t$  is not integrated over,  $\psi_t$  does not carry a probability distribution for values of  $t$ . There is a distinct Hilbert space for each value of the parameter  $t$ .

Moreover, as pointed out by Wick et al. (1952), a Hilbert space decomposes into incoherent sectors if there is no observable that connects these sectors; hence, if there were a larger Hilbert space containing a representation for  $t$ , the absence of any observable that connects different values of  $t$  in the standard nonrelativistic physics would induce a decomposition of the Hilbert space into a (continuous) direct sum

of superselection sectors (Piron 1976). Therefore, no superposition of vectors for different values of  $t$  would be admissible. This argument would also would exclude the interpretation of the experiment given by Lindner et al. (2005) forming the basis of the analysis carried out by the authors involving the linear superposition of two parts of a particle wave function arriving at the detector simultaneously, but originating at two different times, in the framework of the standard nonrelativistic quantum theory.

The situation for particles, in this respect, is very different from that of electromagnetic waves, for which the second order equations imply coherence in time as well as space (the coherence time for light waves is a commonly measured characteristic of light sources). It is clear from the (spatial) double slit interference of light, which travels at a fixed velocity, that the sections of a wave front passing through the two slits must pass at different times if they are to arrive simultaneously at the detection plane off-center. The arrival of pieces of a *particle* wave packet which have passed through two spatially separated slits simultaneously on a screen off center is made possible by the dispersion of momenta in the wave packet, permitting a range of velocities. If the two contributions to the linear superposition on the screen were not taken to be simultaneous at the two slits, they would not interfere, since they would have originated on wave packets at different values of time. If, indeed, such interference could take place, we would have to add up the contributions passing each slit for all times, and this would destroy the interference pattern (one can see in the standard calculation in every textbook (e.g. Merzbacher 1970) that the two pieces of the wave packet that contribute to the interference are taken at equal time, i.e., from a single wave packet arriving at the slits).

A simple argument based on the propagator for the Schrödinger equation demonstrates that interference in time could not occur in the standard Schrödinger theory in a simple and general way, even if we *a priori* arbitrarily assume coherence, although for very short time intervals, as we show below, an approximate interference type pattern can be extracted.

The free propagator for wave functions in the standard Schrödinger treatment is given by

$$\begin{aligned} \langle x|U(t)|x' \rangle &= \left(\frac{m}{2\pi i t}\right)^{\frac{3}{2}} e^{i\frac{m}{2(t-t')}(x-x')^2} \\ &= G(x-x', t-t'), \end{aligned} \quad (6.1)$$

where  $x$  is here a three dimensional variable. The action of this propagator, a transcription of the Schrödinger evolution  $e^{-iH(t-t')}$  into coordinate representation, is

$$\psi_t(x) = \int dx' G(x-x', t-t')\psi_{t'}(x'). \quad (6.2)$$

The integration over  $x$  makes possible the description of the double slit experiment in space (by coherently adding up contributions from two or more locations in  $x$  at a given  $t$  to the wave arriving at a screen over  $\{x'\}$  at the time  $t'$ ); there is, however, no integration over  $t$ , and therefore no mechanism for constructing interference in time. This result, obvious from the form of (6.2), is a reflection of the arguments of Ludwig (1982) cited above, and is fundamental to the standard nonrelativistic quantum theory.

This structure constitutes a formal argument that no interference effect in time is predicted by the standard nonrelativistic Schrödinger theory. Introducing two packets into the beam of an experiment at two different times  $t_1$  and  $t_2$  would result in the *direct sum* of the two packets at some later time, say  $t_3$ , if one propagates the first from  $t_1$  to  $t_3$ , and the second from  $t_2$  to  $t_3$ . This would constitute, by construction, a mixed state, for which no interference would take place, just as the construction of a beam of  $n + m$  particles by adding a set of  $n$  particles with definite spin up to another set of  $m$  particles with spin down results in a mixed state described by a diagonal density matrix with *a priori* probability  $n/(n + m)$  for outcome spin up, and  $m/(n + m)$  for outcome spin down. There is no coherent superposition which would result in some spin with certainty in *any* direction (Jauch 1968).

To understand the apparent qualitative success of the calculations performed by the Lindner group (Lindner 2005), suppose we assume that we are dealing with ordinary functions (as distinguished from the rays in Hilbert space describing a quantum state (Wigner 1931; see also Weinberg 1995)) described by the Schrödinger evolution, and carry out an approximate calculation for short time intervals, ignoring the fact that these contributions cannot be coherent.

If we were to assume, arbitrarily, that the waves from sources at two different times could be coherently added, the propagator above could be expanded in a power series for small variations in the final time, and one would find some semblance of an interference pattern for a few maxima before distortion would set in. For  $\varepsilon$  the time between peaks of the laser beam,  $\mathcal{T}$  the time between peaks on the predicted “interference pattern” on a screen at a distance  $L$  from the emission source,  $m$  the mass of the electron, one finds a crude estimate (in agreement with the calculation of Lindner et al.)

$$\varepsilon\mathcal{T} \cong \pi\hbar\sqrt{\frac{m}{2}}L(E_{kin}^e)^{-\frac{3}{2}}, \quad (6.3)$$

where  $E_{kin}^e$  is the kinetic energy of the emitted electron, and  $L$  is the distance from emitter to the detector. It is instructive to understand how this formula is obtained from the nonrelativistic Schrödinger equation,

$$\frac{i}{\hbar} \frac{\partial\psi_t(\mathbf{x})}{\partial t} = H\psi_t(\mathbf{x}), \quad (6.4)$$

where we shall take  $H$  to be the free Hamiltonian  $\mathbf{p}^2/2m$ . The formal solution of this equation is

$$\psi_t(\mathbf{x}) = \left( e^{\frac{-iH(t-t')}{\hbar}} \psi_{t'} \right) (\mathbf{x}). \quad (6.5)$$

where  $\psi_{t'}$  represents the state at time  $t'$ . The matrix element of the unitary transformation appearing in Eq. (6.5),

$$\langle \mathbf{x} | e^{\frac{-iH(t-t')}{\hbar}} | \mathbf{x}' \rangle = \frac{1}{(2\pi\hbar)^3} \int d^3 p e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')/\hbar} e^{-i\frac{\mathbf{p}^2}{2m}(t-t')/\hbar} \quad (6.6)$$

where we have used the transformation function

$$\langle \mathbf{x} | \mathbf{p} \rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \quad (6.7)$$

to transform from the  $\mathbf{x}$  representation to the  $\mathbf{p}$  representation, for which the Hamiltonian is diagonal. The resulting integration can be easily carried out to yield (as in (6.1))

$$\langle \mathbf{x} | e^{\frac{-iH(t-t')}{\hbar}} | \mathbf{x}' \rangle = \left( \frac{m}{2\pi i \hbar (t-t')} \right)^{\frac{3}{2}} e^{i \frac{(\mathbf{x}-\mathbf{x}')^2}{2\hbar} \frac{m}{t-t'}}. \quad (6.8)$$

We first use this result to compute the standard interference pattern observed from two slits separated in space in order to be able to directly compare the computation of space interference and time interference (assuming coherence in time as well). We use this propagator, or Green's function, to propagate the waves from two slits set apart a short distance in space at the points  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . Assuming equal phase of the wave function at the two points, the resulting wave at the point  $\mathbf{x}$  in the screen will be proportional to

$$e^{i \frac{(\mathbf{x}-\mathbf{x}_1)^2}{2\hbar} \frac{m}{t-t'}} + e^{i \frac{(\mathbf{x}-\mathbf{x}_2)^2}{2\hbar} \frac{m}{t-t'}}. \quad (6.9)$$

Factoring out the overall phase given by the first term, one is left with a formula for the intensity on the screen proportional to

$$\left| 1 + e^{i \left( \frac{(\mathbf{x}-\mathbf{x}_2)^2}{2\hbar} - \frac{(\mathbf{x}-\mathbf{x}_1)^2}{2\hbar} \right) \frac{m}{t-t'}} \right|^2 \quad (6.10)$$

Writing the difference between two squares in the exponent as the sum and difference, we see that the  $\mathbf{x}$  dependence cancels from the difference, and in the sum factor, it is doubled, i.e., the phase is given by

$$e^{i \frac{(2\mathbf{x} - (\mathbf{x}_1 + \mathbf{x}_2))(\mathbf{x}_1 - \mathbf{x}_2)}{2\hbar} \frac{m}{t-t'}}. \quad (6.11)$$

The cross term in the absolute square in Eq.(6.10) is twice the cosine of the angle appearing in the exponent, and we see that as a function of  $\mathbf{x}$  on the image plane, there is an oscillation frequency of

$$\frac{2(\mathbf{x}_1 - \mathbf{x}_2)}{2\hbar} \frac{m}{t-t'}, \quad (6.12)$$

where the quantity  $(t-t')$  can be estimated by the distance to the screen divided by the average velocity of the wave packets.

In this calculation, it is clearly seen that the formation of the interference pattern is a simple consequence of the factorization of the phase factors in the numerator of the exponent.

The expected pattern to be obtained from Eq.(6.8) in the superposition of sources at two times, following the assumption of coherence, is not quite so straightforward to obtain. To do this, let us examine the corresponding linear superposition of waves, assuming coherence, from two gates in time at  $t_1$  and  $t_2$ . Neglecting the modification

of the coefficients of the wave functions that pass at these two times, the superposed wave is approximately proportional to

$$e^{-i \frac{(\mathbf{x}-\mathbf{x}')^2 m}{2\hbar(t_1-t)}} - e^{-i \frac{(\mathbf{x}-\mathbf{x}')^2 m}{2\hbar(t_2-t)}}$$

so that, factoring out the first term, up to a phase we obtain

$$1 + e^{i \frac{(\mathbf{x}-\mathbf{x}')^2 m}{2\hbar} \left( \frac{1}{(t_1-t)} - \frac{1}{(t_2-t)} \right)} \quad (6.13)$$

This function is clearly not simply related to what might be expected to be an interference phenomenon, i.e., as in (6.11). However, let us expand this result to first order in both of the small quantities  $\varepsilon = t_2 - t_1$  and in the deviation  $T - T_0$  from the classical flight pattern, in which both of the signals would arrive at the screen at time  $T_0$ . Let us first define

$$\begin{aligned} \Delta(t') &= \frac{1}{t_1 - t'} - \frac{1}{t_2 - t'} \\ &= \frac{1}{t_1 - t'} - \frac{1}{t_1 + \varepsilon - t'} \\ &\cong \frac{\varepsilon}{(t_1 - t')^2}. \end{aligned} \quad (6.14)$$

Now, consider the arrival times  $t' = T, T_0$ ; the difference in the resulting phases is given by

$$\Delta(T) - \Delta(T_0) \cong -\frac{2T}{(t_1 - T_0)^3}, \quad (6.15)$$

where  $\mathcal{T} = T - T_0$ . For a  $2\pi$  shift in overall phase, we must therefore have

$$(\mathbf{x} - \mathbf{x}')^2 \frac{m}{2\hbar} \left( \frac{2T\varepsilon}{(t_1 - T_0)^3} \right) = 2\pi,$$

or,

$$\varepsilon \mathcal{T} \cong \frac{2\pi\hbar}{m} \frac{(t_1 - T_0)^3}{(\mathbf{x} - \mathbf{x}')^2}. \quad (6.16)$$

We now estimate  $T_0 - t_1 \sim L/v = \frac{L}{\sqrt{\frac{2E_{kin}^e}{m}}}$ , for  $L = |\mathbf{x} - \mathbf{x}'|$ , obtaining the result Eq. (6.3).

Salières et al. (2001), remark that in the very nonlinear emission of an electron by a high energy laser beam, several hundred photons should be absorbed to be converted to kinetic energy. If we take the electron kinetic energy as equal to the energy  $\hbar\omega$  of the laser beam, for this experiment, approximately 1.46 eV, with a factor of 300, and  $L \sim 1$  cm, we obtain for the first several predicted peaks (before distortion due to nonconvergence of the power series expansion) a value of  $\varepsilon\mathcal{T} \sim 9 \times 10^{-28} \text{ s}^2$ , or, for  $\varepsilon = t_1 - t_2 \cong 4$  fs, a diffraction spacing of  $\mathcal{T} \sim 2 \times 10^{-16} = 0.2$  as. This calculation indicates how an exact solution of the time dependent Schrödinger equation, as carried out by the Lindner et al. group could exhibit an interference-like pattern for several peaks on the detector plane in approximate agreement with

the experimental results. These numerical computations do not, however, provide reliable evidence that the procedure is consistent with the theoretical basis of the standard nonrelativistic quantum theory.

The linear terms that generate interference like effects form a fairly good approximation for short time intervals, but the higher order terms in the expansion become important for longer time intervals, and the pattern would become distorted (Engelberg 2009).

The significance of the experiment of Lindner et al. is that it demonstrates at least one class of phenomena actually seen to occur in nature at low energies (but high frequency), for which the standard nonrelativistic quantum theory does not provide an adequate description, and therefore demands the development of some new theoretical tools which are a proper generalization of the standard theory.

Moshinsky (1952), in 1952, raised the question of interference in time. His calculation, however, was concerned with the evolution of a single wave packet, passed through a spatial slit opened at time  $t = 0$ . The transient form of the wave function was then calculated; it has the appearance of a Fresnel interference pattern. Using semiclassical time of flight arguments, it was deduced that this behavior could be thought of as an interference in time. The actual superposition of wave functions at two different times was not considered.

There appear to be several types of theories which accommodate time as an observable. These are, in a nonrelativistic framework, Floquet theory (Floquet 1883), for which the evolution operator (Howland 1974) has the form  $E + H$ , where  $H$  is a standard Hamiltonian model (which may be  $t$  dependent) and  $E$  corresponds to the operator  $i\partial/\partial t$ , a formulation of quantum theory in which one of the space variables becomes the evolution parameter and the time variable becomes an observable (Piron 1972), the quantum Lax-Phillips theory for irreversible processes (also applicable to relativistic quantum theory; Strauss 2000a, b), and in a relativistic framework, the so-called constraint theories (Llosa 1982) in addition to the theory developed by Stueckelberg (1941).

The quantum Lax-Phillips theory (Strauss 2000a) provides a systematic and rigorous description of irreversible processes and resonances. In this theory, a unitary evolution by a parameter, say  $s$ , is introduced on a Hilbert space  $\bar{H}$ , which is foliated along the spectrum  $t$  of a “time” variable which is a self-adjoint operator on  $\bar{H}$  into a set of Hilbert spaces  $H_t$ , which may be identified with the Hilbert spaces of the ordinary quantum theory, but maintains its coherence in  $t$ . The existence of invariant subspaces delimited by time intervals makes it possible to construct semigroups in these subspaces. A relativistic Lee model (Horwitz 1995), constructed in the framework of the Stueckelberg theory has been treated in Strauss (2000); we shall discuss this in a later chapter. The Stueckelberg theory, which goes over in its nonrelativistic limit to the Floquet theory, has a structure somewhat similar the Lax-Phillips theory (Flesia 1984). Furthermore, the constraints formalism (in quantized form (Horwitz 1982); see Appendix Chap. 2) for the one and two body case is equivalent to the Stueckelberg construction. although the theory takes a very different form for the many body case (Horwitz 1982).

In following, we discuss the Floquet theory and the Stueckelberg theory as somewhat categorical among the set of theories which may be used to describe the Lindner et al. experiment. We first discuss the application of the nonrelativistic Floquet theory, and show that it is not adequate.

Floquet theory (1883) was originally intended for the treatment of differential equations with periodic coefficients. It entered physics in an important way in solid state theory where the potential in a crystal is periodic in space. Utilizing the translation operator  $U(\mathbf{a}) = e^{i\mathbf{p}\cdot\mathbf{a}}$ , where  $\mathbf{a}$  is a crystal lattice vector, one can show that the solutions of the Schrödinger equation, as a representation of this translation group, take on the Bloch form.

The idea then arose that for a Hamiltonian periodic in time, the same method could be used. However, since the Hamiltonian commutes with  $t$ , to make the group action explicit, it was necessary to introduce a new variable  $E$  (the generator of translations in  $t$ ). The evolution operator was then defined as

$$K = E + H, \quad (6.17)$$

where  $E \equiv -i\hbar\partial_t$ . Then, clearly, for

$$U(s) = e^{-iKs} \quad (6.18)$$

the operator  $U(s)$  carries  $t \rightarrow t - s$ , translating functions of  $t$  to the right by  $s$ .

The introduction of this modification of the Hamiltonian was also suggested by Howland (1971) for both the classical and quantum theories for treating problems in which the Hamiltonian depends on time. For the classical theory, introducing a new parameter of evolution,  $s$ , the Hamilton equations would then include the relations

$$\frac{dt}{ds} = \frac{\partial K}{\partial E} \quad (6.19)$$

and

$$\frac{dE}{ds} = -\frac{\partial K}{\partial t} = -\frac{\partial H}{\partial t}, \quad (6.20)$$

thus providing some interpretation for  $E$ . Since then  $\frac{dt}{ds} = 1$ , by a change of variables, this formulation becomes completely equivalent to the standard form. However, in the quantum theory, the Hamilton equations, as operator equations, imply conditions on expectation values; the variables  $t$  and  $s$  are then no longer equivalent. In this case,  $s$  is the parameter of the motion, and  $t$  is a quantum operator, an observable. The wave functions are then coherent in  $t$ , making possible, in principle, interference phenomena in  $t$ .

The resulting theory is very different from the standard Schrödinger theory. To see this, let us write the corresponding evolution equation in what we shall nevertheless call "Floquet theory", since it has the same structure for the Hilbert space, but  $H$  does not necessarily depend on  $t$ . The mathematical framework is independent of this dependence (clearly the consequences of the theory, and the results one may obtain, can be very strong when  $H$  is periodic in  $t$ ).

The evolution equation has the form (Flesia 1984)

$$\begin{aligned} i \frac{\partial \psi_s}{\partial s} &= K \psi_s \\ &= (-i\hbar \partial_t + H) \psi_s \end{aligned} \quad (6.21)$$

where  $\psi_s$  is a function of  $x, t$ . The functions  $\psi_s$  have the property that

$$\|\psi_s\|^2 = \int d^3x dt |\psi_s(x, t)|^2 < \infty, \quad (6.22)$$

the condition that  $\psi_s$  belongs to a Hilbert space  $H_s$  (now labelled by  $s$ ). As pointed out by Kulander and Lewenstein (1996), if  $H$  (or  $K$ ) is periodic over some hundreds of cycles, it would be a good approximation to assume a “stationary state” in which the  $s$  derivative of  $\psi_s$  is replaced by an eigenvalue (their Eq. (72.31)). Such a state would be stationary in  $s$ , not  $t$ ; the idea is that the spacetime function  $\psi_s(x, t)$  reaches a steady form and no longer changes, on the spacetime manifold, as a function of  $s$  (up to a phase determined by the eigenvalue). In this case, the solution of Eq. (6.21) amounts formally to an integration of the time dependent Schrödinger equation over  $t$ , as carried out in the analysis of Lindner et al. (2005), with the Hamiltonian shifted by the Floquet eigenvalue (possibly zero). The theory would then predict coherence in  $t$  for such a solution. This is not, however, a valid procedure for the conditions of the experiment of Lindner et al. (Paulus 2005), since this experiment involves essentially just a couple of cycles.

In the following we calculate the propagator for the Floquet equation (6.21) for the case of a *free* particle. We will see that even though interference in  $t$  is, in principle, possible in the Floquet framework, two narrow segments, in time, of a particle wave function will not interfere unless (a) the segments initially overlap, or (b) there is a nontrivial  $t$ -dependence (but not necessarily periodic) in  $H$ , the latter certainly providing an interesting possibility for application to the experiment we are discussing.

To obtain the form of the propagator, let us consider the  $x, t$  matrix elements of the unitary evolution  $U(s)$  of  $\psi_s$ :

$$\langle x, t | U(s) | x', t' \rangle = \int dE' dE dp' dp'' \langle x, t | E' p' \rangle \langle E' p' | e^{-i(H-E)s} | E'' p'' \rangle \langle E'' p'' | x' t' \rangle \quad (6.23)$$

Here, the momenta and coordinates are three dimensional (the differentials are also  $dp \equiv d^3p$ ).

We now assume that  $H$  has the free particle form  $\mathbf{p}^2/2m$  and therefore commutes with  $E$ . Then, (6.23) becomes

$$\langle x, t | U(s) | x', t' \rangle = \delta(t' - t + s) \left( \frac{m}{2\pi i s} \right)^{\frac{3}{2}} e^{i \frac{m}{2s} (x-x')^2}. \quad (6.24)$$

Let us now call the coefficient of the  $\delta$  function  $G(x - x', s)$ . The propagation of  $\psi_{s'}(x', t')$  to  $\psi_s(x, t)$  is given by

$$\psi_s(x, t) = \int dx' dt' \delta(t' - t + (s - s')) G(x - x', s - s') \psi_{s'}(x', t') \quad (6.25)$$



clearly displaying the possibility of interference in  $t$ , i.e., there may be contributions at several different values of  $t'$  corresponding to the opening of time gates. However, there is no spreading, in this propagation, of the width of the time pulses, independently of the form of the wave packet  $\psi_{s'}(x', t')$ .

Consider the contribution of two gates at  $t_1$  and  $t'_1$ . In this case,

$$\psi_s(x, t) = \int dx' G(x - x', s - s') \{ \delta((s - s') - (t - t_1)) \psi_{s'}(x', t_1) \Delta t_1 + \delta((s - s') - (t - t'_1)) \psi_{s'}(x', t'_1) \Delta t'_1 \}, \quad (6.26)$$

where  $\Delta t_1$  and  $\Delta t'_1$  are the (narrow) widths of the gates.

One can evaluate  $s - s'$  approximately through the Hamilton equations. Since  $dx/ds = p/M$ , it follows that  $\Delta s \cong ML/p$ , where  $L$  is the distance from source to detector. Since  $d < t > /ds = 1$ , the expectation value of  $t(s)$  goes with  $s$ , so that  $s - s' \cong t - t'$ , the latter giving the time from the source gate to the time on the detector when the measurement is made. Due to the delta function constraint, we see that there can be no interference if the source pulses do not overlap (the effective source widths are of the order of 1 as), as is the case in the Lindner experiment.

Alternatively, the delta functions would not appear if the Hamiltonian had an explicit  $t$ -dependence, and the result would depend on the particular model. Interference in the framework of the Floquet structure, therefore, although in principle possible, would not occur for narrow source pulses in the absence of explicit (additional) time dependence in the Hamiltonian. We remark that the computations done by the experimental group associated with the Lindner experiment treated the gating process as an explicit time dependence itself, but there were insufficient cycles, according to the criterion cited above, to be able to treat the problem as an  $s$ -independent (stationary in this sense) Floquet process as we have assumed here. A more appropriate description would be to permit the process to be governed by the  $s$  dependence of the system, which we shall do in the relativistically covariant form of the Stueckelberg theory.

The Floquet theory is, in fact, a nonrelativistic limit of Stueckelberg's relativistic quantum theory (Stueckelberg 1941). To see this, let us define the nonrelativistic limit by noting first that in the Galilean-Newtonian theory, the particle must have a definite mass (Sudarshan 1974). We can then require that (Horwitz 1981)

$$E - Mc^2 < \infty \quad (6.27)$$

as  $c \rightarrow \infty$ . Defining

$$E \equiv c\sqrt{m^2c^2 + \mathbf{p}^2}, \quad (6.28)$$

where  $m$  is the dynamical off-shell mass, we may then extract the large factor  $c$  from the square root, and obtain the expansion

$$\varepsilon \equiv E - Mc^2 = (m - M)c^2 + \frac{\mathbf{p}^2}{2M} + O\left(\frac{1}{c^2}\right). \quad (6.29)$$

We therefore see that enforcing the limit (6.27), the first term of (6.29), which we shall call  $\eta$ , must be finite, and  $m \rightarrow M$  as  $c \rightarrow \infty$ .

We remark that we have treated the approximate expressions above as if they were simple numerical relations, a procedure entirely appropriate for classical mechanics. For the quantum theory, we must understand that the constraints implied by these relations correspond to the structure of the wave function of the quantum state. The wave function of the state corresponds to the *a priori* probability amplitude for finding quantities such as  $E$  and  $\mathbf{p}$  in the outcome of experiment; its structure therefore imposes bounds on the quantity  $m^2c^2 = (E/c)^2 - \mathbf{p}^2$ , going over in the Galilean limit to the Galilean target value  $m \rightarrow M$ .

The quantity  $\varepsilon$  clearly satisfies the commutation relation  $[\varepsilon, t] = i\hbar$ , and therefore  $\varepsilon$  retains its meaning as a dynamical variable conjugate to  $t$  (it also commutes with  $\mathbf{x}$ ). We can then write the free part of the Stueckelberg Hamiltonian as

$$\frac{\mathbf{p}^2 - \frac{E^2}{c^2}}{2M} = \frac{\mathbf{p}^2}{2M} - \frac{(\varepsilon + Mc^2)^2}{2Mc^2} \quad (6.30)$$

or, for  $c \rightarrow \infty$ , with the potential term  $V$ , the operator  $K$  becomes

$$K = \frac{\mathbf{p}^2}{2M} - \varepsilon + V - \frac{Mc^2}{2} \quad (6.31)$$

In the function  $V$ , generally depending on  $x, t$  in a covariant way, the variable  $t$ , as pointed out in the earlier chapters, goes over to  $\tau$ , since  $\frac{dt}{d\tau} = \frac{E}{M} \rightarrow 1$ . For the reduced two body problem,  $t$  corresponds to  $t_1 - t_2$  for the two particles, and this difference may go to zero. Thus  $V(x)$  goes over to a nonrelativistic potential, which may be time-dependent. Discarding the constant term  $\frac{Mc^2}{2}$ , we arrive at precisely the Floquet form

$$K = H - \varepsilon, \quad (6.32)$$

where  $\varepsilon$  is called just  $-E$  in the Floquet theory (corresponding to  $i\partial/\partial t$  in the quantum case).

Since the Stueckelberg theory reduces to the Floquet theory in the nonrelativistic limit, the effect observed in the Lindner experiment, as described in the Stueckelberg theory, is therefore necessarily relativistic. Although the Stueckelberg theory is essentially relativistic, and the energies of the macroscopic motions of the particles involved in this experiment are low, the very high frequencies used to establish excitations and pulse rates involve high energy components of the wave packets, and thus the use of a relativistically covariant theory is appropriate.

In 1976, Horwitz and Rabin (1976) pointed out that the relativistic quantum theory of Stueckelberg predicts interference in time. In this theory,  $t$  is treated as a quantum observable, since the Einstein variables  $x, t$  are considered, in relativity, as the nontrivial outcome of experiments measuring the place and time of occurrence of events. Their calculation will be briefly redone below for the parameters of the experiment of Lindner et al. In this theory, interference does not require initial overlap or an explicitly time dependent Hamiltonian. The estimate given below shows that the interference criteria are satisfied with numbers very close to the conditions and results of the experiment under discussion; the high frequencies required are due in this case to the large value of the velocity of light.

The Stueckelberg theory for the free particle introduces an equation quite similar to the Floquet equation (we use  $s$  as the parameter of evolution here for analogy to the Floquet formalism), but with an evolution operator that is Lorentz invariant:

$$i\hbar \frac{\partial \psi_s}{\partial s} = \frac{p^2 - (\frac{E}{c})^2}{2M} \psi_s, \quad (6.33)$$

where  $\psi_s(x, t)$  satisfies the same normalization condition as for the Floquet theory, on space and time, i.e.,  $\int |\psi_s(x, t)|^2 d^4x \leq \infty$ , and  $M$  is the Galilean target mass (the so-called mass shell value for  $m^2 c^2 = (E/c)^2 - \mathbf{p}^2$ ). The propagator has a similar form to that of the Floquet propagator, but is Gaussian in all four variables:

$$\langle x|U(s)|x' \rangle = \left(\frac{M}{2\pi i s \hbar}\right)^2 e^{i \frac{M}{2s\hbar} (x-x')^2}, \quad (6.34)$$

where now  $(x - x')^2$  is the invariant  $(\mathbf{x} - \mathbf{x}')^2 - c^2(t - t')^2$ ; we write  $x$  for  $(x, t)$ . It is the quadratic term in  $t - t'$  in the exponent which leads directly to interference in the same way as the double slit in space.

The diffraction formula, obtained from (6.15), using the Hamilton relations

$$\frac{dx}{ds} = \frac{p}{M}, \quad (6.35)$$

and (note that this relation allows for two pulses emitted at different times to arrive at a detector at the same time due to the spread in the spectrum of  $E$ )

$$\frac{dt}{ds} = \frac{E}{Mc^2}, \quad (6.36)$$

is (Horwitz 1976)

$$\varepsilon T \cong \frac{2\pi\hbar L}{< p > c^2}, \quad (6.37)$$

where  $\varepsilon$  is the gate spacing in time, and  $T$  is the time between diffraction peaks at a distance  $L$ .

For 850nm light, as utilized in the experiment under discussion, as remarked above,  $\hbar\omega$  is about 1.46 eV. Using the on-shell value for the electron mass, taking into account (as assumed above) that the electron may absorb about 300 photons during the emission,  $cp$  (for  $p$  in the beam direction) then has a value of  $1.21 \times 10^3$  eV. With these values, one finds that,

$$\varepsilon T \cong 6.9 \times 10^{-30} \text{ s}^2, \quad (6.38)$$

so that for  $\varepsilon \sim T$ ,  $T \sim 2.6 \times 10^{-15}$  s. This result, for the pulse rate and the observed diffraction pattern, is in good agreement with the results obtained in the experiment.

More precise estimates can be obtained by taking into account more details of the interaction, and the dependence on  $L$  can be used as a parameter to test the reliability of (6.17).

The relativistic model therefore seems to provide a simple and consistent description of the experimental results.

### 6.3 Experiment Proposed by Palacios et al.

A recently proposed experiment of Palacios et al. (2009) assumes that the spin of two particles at different times retain the entanglement characteristic of two particles at equal time. In their proposed experiment, they consider a situation in which two short ultraviolet pulses with different central frequencies doubly ionize a Helium atom to produce electrons with slightly different energies; the short subfemtosecond duration of each pulse gives it an appreciable energy bandwidth, so that the pulses overlap. Therefore, an electron with a given energy in the overlap region could have been emitted by either pulse. It is assumed that the electrons are indistinguishable after emission at two different times, and that the spin correlations between them remain as determined by their coupling in the initial state (singlet in their case). They argue that the resulting probabilities for ejecting two electrons restricted in their total energy by the total ionization energy of the He and the photon energies, should show interference oscillations that depend on the time delay between the pulses as well as their durations. Their study of this system is based on accurate solutions of the time dependent Schrödinger equation carried out, for this particular case, along much the same lines as the corresponding calculation of Lindner et al. However, as we have discussed above, the nonrelativistic quantum theory does not provide a framework for coherence of Schrödinger wave functions at different times, and, as discussed on Chap. 3, it does not provide a basis for spin correlations enabling the construction of total spin states for two subsystems defined at different times.

The detailed computations of Palacios et al. are based on the assumption of non-relativistic entanglement and the use of the nonrelativistic Schrödinger equation for the evolution of the state of the two body system, as done by the Lindner et al. group. The addition of angular momenta for a two or more body system, according to the use of the usual Clebsch-Gordan coefficients, however, is valid in the nonrelativistic theory only for systems at equal time. In the nonrelativistic theory, the correlation is destroyed by the successive emission of the electrons.

The tensor product space for the two body system is constructed for equal time, corresponding to the parameter labelling the Hilbert space under unitary evolution. This structure is not covariant.

The many body Hilbert space of the relativistic theory is constructed, as we have seen in Chap. 3, however, as a tensor product over one-body functions at equal  $\tau$  and  $n$ , the timelike inducing vector. In particular, as pointed out in Eq. (3.44), the covariant relativistic two body state

$$\psi_{ijn} = \frac{1}{\sqrt{2}}[\psi_{in} \otimes \psi_{jn} \pm \psi_{jn} \otimes \psi_{in}], \quad (6.39)$$

contains (with the negative sign for the two fermion system), precise antisymmetry and, since both functions are at the same point on their respective orbits of the induced representation, they transform under the full  $SU(2)$  acting on the two body state. Labelling each function with its (local on the orbit) spin indices  $a, b$  we see that

$$\psi_{ijnJc} = \frac{1}{\sqrt{2}}C(a, b|c; J, \frac{1}{2}, \frac{1}{2})[\psi_{ian} \otimes \psi_{jbn} - \psi_{jbn} \otimes \psi_{ian}], \quad (6.40)$$

where  $i, j$  are the two degenerate lowest He orbitals (according to our results in Chap. 5, these are equivalent to the nonrelativistic orbital functions in angular dependence), and  $C(a, b|c; J, \frac{1}{2}, \frac{1}{2})$  are the Clebsch-Gordan coefficients combining the spin states for  $a, b = \pm\frac{1}{2}$  to, in this case, the singlet  $J = 0, c = 0$  state; we have written this result in somewhat more general terms to illustrate the method.

The spin zero state is then seen to be maintained for particles that are entangled at different times; the support of the wave function lies in time differences of the order of  $2\Delta t$  where  $\Delta t$  is the time width of each of the factor functions. An estimate can be found for this correlation from the Lindner experiment where time interference was seen from a similar ejection of electrons (from Argon rather than Helium), for which  $\Delta t$  was of the order of 4 fs (or more), since the pulse spacings, which act as slits in time, were of that order, and interference requires that the slits lie in the coherence range of the wave function.

Equation (6.40) is of a general form of spin entanglement, enforcing correlations when one of the spins is measure (see Silman et al. 2008 on the collapse mechanism). In the nonrelativistic theory of quantum information transfer, models of this type play a fundamental role; it is clear that similar models may be constructed from spin entanglement for the relativistic case with the additional information that the orientation of the induced representation for the receiver must correspond to that of the transmitter, requiring an overlap in the density matrix over the induced representation orbit.

Relativistic scattering theory has been generally based on quantum field theory, providing methods of computing an  $S$  matrix (transition amplitude operator) by the semi-axiomatic approach of Lehmann et al. (1955) or through the use of interaction picture expansion of the perturbed field equations (Schweber 1964; Jauch and Rohrlich 1955; Schwinger-Tomonaga 1948). Feynman's approach to scattering in spacetime (Feynman 1949), using the method of propagators, is very close to the methods afforded by the covariant quantum theory that we shall discuss below, but the notion of invariant evolution is not used explicitly in those computations (Feynman 1950), however derived Stueckelberg's equation for free motion of a single particle, and Schwinger (1951) arrived at an evolution equation of Stueckelberg type in his treatment of the propagator in the derivation of the electron anomalous moment, to be discussed in the next chapter).

Nonrelativistic scattering theory is based on the unitary evolution generated by two Hamiltonians, one considered to be the full interacting Hamiltonian of the system, and the second, an unperturbed, or "free" Hamiltonian describing the asymptotic motion when the particles are separated sufficiently to be noninteracting. In this framework, rigorous conditions can be set which can assure the existence of wave operators relating free waves to the physical scattering states and prove asymptotic completeness (for which the range of the wave operators cover the whole space of scattering states for a given Hamiltonian); see, for example Reed (1979), Amrein (1977), Taylor (1972), Newton (1967), Sigal (1987) for the many body case, and Horwitz (1980) for the relativistic case.

A problem common to that of both the relativistic and nonrelativistic theories has been that of the description of *resonances*, states of matter that are not stable, and for which the apparently irreversible process of decay is of semigroup type (to be defined precisely below). Most of the known particles listed, for example, in the Particle Data Group publication (Olive 2014) are so short lived that their time evolution cannot be easily measured (their lifetimes are usually estimated by applying the energy time uncertainty relation to the measured widths of the mass distributions), and are considered to be resonances, a scattering in which there is a long time delay (Wigner 1955; Goldberger 1964); the time evolution of systems subject to weak decay, such as

the neutron, nuclear beta decay, the muon and  $K$  meson systems has been observed, and to very high precision appear to have exponential decay laws. Gamow (1928) proposed the use of a complex energy in the Schrödinger equation to account for the exponential decays observed in nuclear physics, and Wigner and Weisskopf (1930) provided a more complete quantum mechanical model which still forms the basis for computations; this theory results in exponential decay for large enough times for single channel decay, but has quadratic time dependence for very short times and does not give semigroup behavior even for intermediate times for decays with several channels (such as the  $K^0$  system (Lee 1956)).

Lax and Phillips in 1967 (Lax 1967) presented a description of resonance phenomena with exact semigroup behavior for classical wave equations (such as electromagnetic scattering) for which the resonant states are represented as elements of a Hilbert space. The description of Lax and Phillips has recently been formulated in the framework of quantum theory (Strauss 2000a) thus making the computation of expectation values of observables, as well as the many other properties of resonant quantum states, accessible. The method has been applied to the relativistic quantum theory, providing an effective description of  $K^0$  meson decay (Strauss 2002), and this work is summarized here as well.

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## 7.1 Foundations of Relativistic Scattering Theory

Since the covariant theory has the same Hamilton-Lagrange formulation as the non-relativistic theory, the relativistic quantum scattering theory has the same structure as the nonrelativistic theory. In the following, we formulate this structure in the relativistic framework discussed in the previous chapters.

Consider a system characterized by the Hamiltonians  $K$  and  $K_0$ , corresponding to the full and unperturbed Hamiltonians, e.g. for  $K = p^2/2M + V$  and  $K_0 = p^2/2M$ , where  $p \equiv p^\mu$  is the momentum four vector of a particle (or the reduced momentum of a two body system) and  $V$  is the potential. For the two body problem treated in Chap. 5,  $K$  and  $K_0$  may refer to the reduced motion (where  $M$  is replaced by the reduced “mass”  $m$  used in Eq. (5.6)). Then there is a state  $\psi$  which evolves according to the unitary operator  $U(\tau) = e^{-iK\tau}$  and an asymptotic state  $\phi$  that evolves according to the unitary operator  $U_0(\tau) = e^{-iK_0\tau}$ . The basic condition for scattering theory is that (Amrein 1977) for every  $\epsilon > 0$  there is a  $T$  and a  $\phi$  such that

$$\|U(\tau)\psi - U_0(\tau)\phi\| < \epsilon \quad (7.1)$$

for  $|\tau| > T$ . Note that this definition does not require that the two terms in (7.1) are identically equal, but only approach each other asymptotically. This formulation permits the establishment of many rigorous properties of scattering systems. Since the norm is invariant under multiplication by a unitary operator, let us multiply both terms by  $U^\dagger(\tau) \equiv U^{-1}(\tau)$  to obtain

$$\|\psi - U^\dagger(\tau)U_0(\tau)\phi\| < \epsilon. \quad (7.2)$$

The sequence (7.2) (in the index  $\tau$ ) has the form, for  $\{\varphi_n\}$  elements of a dense set,

$$\|\varphi - \varphi_n\| < \epsilon, \quad (7.3)$$

for which  $\varphi_n \rightarrow \varphi$ . so that, if the limit exists,

$$\lim_{\tau \rightarrow \pm\infty} U^\dagger(\tau)U_0(\tau)\phi = \psi \quad (7.4)$$

If the limit exists on a *dense set* $\{\phi\}$  (there are sufficient  $\phi$ 's to take limits of the type (7.3) to *any* vector on the Hilbert space), then the operators

$$\Omega_{\mp} = \lim_{\tau \rightarrow \pm\infty} U^\dagger(\tau)U_0(\tau) \quad (7.5)$$

are well-defined. We see this from the fact that the sequence  $\|\Omega\phi - \Omega\phi_n\| \leq \|\Omega\|\|\phi - \phi_n\|$ . Since  $\Omega$  is a bounded operator, and  $\phi_n$  converges to any vector (by the dense property)  $\phi$ , then  $\Omega$  is defined everywhere.

These operators, called the *wave operators* have the property that the full interacting wave function of the system can be expressed in terms of the non-interacting wave function by multiplication by a wave operator, as in (7.4).

Furthermore, by differentiating the operator appearing in (7.5) with respect to  $\tau$ , one finds, as in the nonrelativistic case, that

$$\lim_{\tau \rightarrow \pm\infty} (KU^\dagger(\tau)U_0(\tau) - U^\dagger(\tau)U_0(\tau)K_0) = 0$$

so that

$$K\Omega_{\mp} = \Omega_{\mp}K_0, \quad (7.6)$$

a remarkable property called *intertwining*.

A necessary condition for the existence of the wave operators is that

$$\frac{d}{d\tau}U^\dagger(\tau)U_0(\tau)\phi \rightarrow 0,$$

or,

$$e^{-iK\tau}(K - K_0)e^{-iK_0\tau}\phi \rightarrow 0.$$

Calling  $K - K_0 = V$ , we see that it is required that

$$\|Ve^{-iK_0\tau}\phi\| \rightarrow 0 \quad (7.7)$$

At this point, we notice an essential difference between the relativistic and nonrelativistic theories. If  $V$  is a local potential of the form  $V(\rho)$  as in the potential models considered in Chap. 5, then we see that the condition (7.7) can be satisfied if the free evolution carries wave packets  $\phi(x)$  deep into the spacelike region out of the range of the potential. In the nonrelativistic case, it is not so difficult for the free motion to bring the wave packet out of the range of a local potential  $V(\mathbf{x})$ , but large distances along the hyperbolas  $\rho = \text{const}$ , going asymptotically along the light cone, the size of the potential does not decrease, an effect we have commented on in our previous discussions of the energy spectrum. However, it has been shown that for spacelike momenta, careful estimates, essentially due to a diminishing Euclidean measure of the wave packet as it approaches the light cone where the potential remains large, does admit such a convergence (Horwitz 1980). This is an important property for the formulation of relativistic statistical mechanics, a point which we will return to in Chap. 10.



## 7.2 The $S$ Matrix

Since we have postulated that  $\tau$  is the ordering parameter for physical processes, the limit (7.2) for  $\tau \rightarrow -\infty$  corresponds to the transformation of the *in* state to the physical state  $\psi$ , and for  $\tau \rightarrow +\infty$ , the transformation from the asymptotic *out* state to  $\psi$ , so that we can write

$$\psi = \Omega_- \phi_{out} = \Omega_+ \phi_{in},$$

or

$$\phi_{out} = \Omega_-^{-1} \Omega_+ \phi_{in}, \quad (7.8)$$

defining the  $S$ -matrix (as it is commonly called)

$$S = \Omega_-^{-1} \Omega_+. \quad (7.9)$$

In conventional relativistic quantum field theory, the Lagrangian approach provides an interaction term which can be used in developing an interaction picture expansion corresponding to the sum over integrals expressed in terms of Feynman diagrams (Jauch 1955; Peskin 1995). A similar structure emerges from the relativistic quantum theory under discussion here. Starting with the Stueckelberg-Schrödinger equation, for  $K = K_0 + V$ , consider the evolution of the wave function

$$\psi_\tau = e^{-iK_0\tau} \chi_\tau. \quad (7.10)$$

Then,

$$\begin{aligned} i \frac{\partial}{\partial \tau} \psi_\tau &= K_0 \psi_\tau + e^{-iK_0\tau} i \frac{\partial}{\partial \tau} \chi_\tau \\ &= (K_0 + V) \psi_\tau. \end{aligned} \quad (7.11)$$

Cancelling  $K_0 \psi$  from both sides, we obtain, as in the nonrelativistic theory,

$$i \frac{\partial}{\partial \tau} \chi_\tau = V(\tau) \chi_\tau, \quad (7.12)$$

where  $V(\tau) \equiv e^{iK_0\tau} V e^{-iK_0\tau}$ . We can integrate (7.12) from zero to some  $\tau$

$$\chi_\tau = \chi_0 - i \int_0^\tau V(\tau') \chi_{\tau'} d\tau' \quad (7.13)$$

and, in case  $V$  is small (Kato 1980), iterate to get a convergent series expansion for an evolution  $U(0, \tau)$  for the interaction picture states  $\chi_\tau$ . In the first iteration, one replaces  $\chi_{\tau'}$  in the integrand by the form given by (7.13), with integration (on, say,  $\tau''$  running up to  $\tau'$ ). Thus successive iterations contain integrations up to the previous time  $\tau$ , and the result is a sum of  $\tau$ -ordered integrals. As in the nonrelativistic theory (Amrein 1977), the integrals can be formally completed to the endpoint  $\tau$ , after dividing by  $n!$  in each  $n$ th iterate, with the well-known (in nonrelativistic scattering theory) result

$$U(0, \tau) = \left( e^{-i \int_0^\tau V(\tau') d\tau'} \right)_+, \quad (7.14)$$

where the  $+$  subscript implies  $\tau$ -ordering in the series expansion. The starting point  $\tau = 0$  is arbitrary, and the definition can be extended to  $U(\tau_1, \tau_2)$ . These operators satisfy

$$U(\tau_1, \tau_2)U(\tau_2, \tau_3) = U(\tau_1, \tau_3). \quad (7.15)$$

Comparison of the definition of  $\chi_\tau$  with the definition of the wave operator (7.10)<sup>1</sup> shows that in fact

$$\lim_{t \rightarrow +\infty} U(0, \tau) = \Omega_-, \quad (7.16)$$

and the  $S$  matrix is then given by

$$S = \lim_{\tau_1 \rightarrow \infty, \tau_2 \rightarrow -\infty} U(\tau_1, \tau_2), \quad (7.17)$$

There is an alternative form for understanding the wave operators and their physical properties associated with the Green's function. Let us define for the relativistic theory the unperturbed and perturbed Green's operators (Taylor (1972), Amrein (1977), Newton (1967)) (often called "resolvents" in the mathematical literature)

$$\begin{aligned} G^0(z) &= (z - K_0)^{-1} \\ G(z) &= (z - K)^{-1}, \end{aligned} \quad (7.18)$$

where  $z$  may be real or complex (in the upper half plane, as we shall see below). In the nonrelativistic case, where the spectrum of  $H$  (and  $H_0$ ) is often bounded from below, the operators  $K$  and  $K_0$  are generally not (due to the hyperbolic differential operator for the free motion  $p^\mu p_\mu$ ). However, in the reduced two body problem with symmetric potential, as we have seen in Chap. 5, the reduced Hamiltonian  $K_{rel}$  is bounded from below, providing the Green's function for the relative motion  $G(z)$  (as well as the scattering operator  $T(z)$  to be defined in (7.19)) with simple properties for analytic continuation and causal structure relevant to the properties of resonances, to be discussed in a later section. Although we review the construction in the following, the development of the formal scattering theory is almost exactly the same as in the nonrelativistic theory (Amrein 1977; Taylor 1972; Newton 1967) primarily due to the fact that our formulation of relativistic quantum theory admits a Hamiltonian type structure.

For the scattering problem it is convenient to define another operator, called the " $T$ -matrix", by

$$T(z) = V + VG(z)V, \quad (7.19)$$

which has the same analytic properties as  $G(z)$ . Multiplying (7.19) by  $G_0$ , we obtain

$$G_0 T(z) = G_0 V + G_0 V G(z) V,$$

---

<sup>1</sup>Since

$$\chi_\tau = e^{iK_0\tau} e^{-iK\tau} \psi = U(\tau, 0)\psi,$$

as  $\tau \rightarrow \infty$ , this becomes  $\Omega_-^\dagger \psi$ , i.e.,  $U(\tau, 0) \rightarrow \Omega_-^\dagger$ .

but since, by (7.18), it is an identity (sometimes called the second resolvent equation) that

$$G = G_0 + G_0VG, \quad (7.20)$$

it follows that

$$G_0(z)T(z) = G(z)V. \quad (7.21)$$

Multiplying on the right by  $G_0$  one finds, similarly,

$$T(z)G_0(z) = VG(z) \quad (7.22)$$

A useful integral equation for the  $T$  operator can be obtained from (7.20) and (7.22); replacing  $VG$  in (7.20) by  $TG_0$  as in (7.22), we obtain

$$G(z) = G_0(z) + G_0(z)T(z)G_0(z). \quad (7.23)$$

Therefore, the information contained in  $T(z)$  (on the effect of interaction) is equivalent to that of  $G(z)$ .

Furthermore, if we replace  $GV$  in the definition of the  $T$  operator (7.19) by  $G_0T$  as in (7.21), we obtain

$$T(z) = V + VG_0(z)T(z). \quad (7.24)$$

This equation is known as the Lippmann-Schwinger equation for the computation of  $T(z)$ , and used in many applications (e.g., Adler (1965), Weisberger (1966)). If, for example,  $V$  is very small, one obtains the lowest Born approximation (we shall see below that  $T$  is directly related to the scattering transition amplitude in the form  $S = 1 - 2\pi iT$ ) for which  $T \sim V$ , and in general, by iteration, one obtains the full Born series

$$T(z) = V + VG_0V + VG_0VG_0V + \dots \quad (7.25)$$

It is interesting to note that, as is well known in the nonrelativistic case,  $G(z^*) = G(z)^\dagger$ , and therefore  $T(z^*) = T(z)^\dagger$ , i.e., the Hermitian conjugate, well defined in the Stueckelberg Hilbert space of states.

We now establish important connections between the Green's function, the  $T$ -matrix, and the  $S$  matrix. The limit (7.5) for the definition of the wave operator may be written in a different and useful way as (we write the result first for  $\Omega_-$ )

$$\Omega_- = \lim_{\tau \rightarrow \infty} U^\dagger(\tau)U_0(\tau) = \lim_{\epsilon \rightarrow 0} \int_0^\infty d\tau e^{-\epsilon\tau} U(\tau)^\dagger U_0(\tau), \quad (7.26)$$

with the limit in  $\epsilon$  taken from above. This result, a procedure introduced by Abel (Amrein 1977), is easily proved by assuming that there is a number  $T$  such that for  $\tau > T$  the product of unitaries under the limit has converged. The finite part of the integral vanishes when  $\epsilon \rightarrow 0$ . What remains is then an integral of the exponential from  $T$  to  $\infty$ ; this cancels the factor  $\epsilon$  and leaves unity in the limit  $\epsilon$  goes to zero. To work

with this formula, we note that the prefactor of  $\epsilon$  can be provided by differentiating the exponent with a minus sign, and then integrating by parts, one finds

$$\Omega_- = 1 + i \lim_{\epsilon \rightarrow 0} \int_0^\infty d\tau e^{-\epsilon\tau} U(\tau)^\dagger V U_0(\tau). \quad (7.27)$$

In a similar way, one obtains

$$\Omega_+ = 1 - i \lim_{\epsilon \rightarrow 0} \int_{-\infty}^0 d\tau e^{+\epsilon\tau} U(\tau)^\dagger V U_0(\tau). \quad (7.28)$$

Now, following the usual procedure, let  $\Omega_{\mp}$  act on  $\phi$  respectively, for  $\phi_{in}$  and  $\phi_{out}$ , represented as

$$|\phi\rangle = \int d^4 p |p\rangle \langle p|\phi\rangle \quad (7.29)$$

In its action on this state, in this representation, the operator  $K_0$ , which may depend on  $p^\mu$  alone, takes on the value  $K_p$ . There are cases for which  $K_0$  may be more involved; in such cases, one uses its spectral representation (usually absolutely continuous). Carrying out the integration over  $\tau$ , and using the definition (7.18) for the Green's function, we then have

$$\Omega_- |\phi\rangle = |\phi\rangle + \lim_{\epsilon \rightarrow 0} \int d^4 p G((K_p - i\epsilon)V) |p\rangle \langle p|\phi\rangle \quad (7.30)$$

and

$$\Omega_+ |\phi\rangle = |\phi\rangle + \lim_{\epsilon \rightarrow 0} \int d^4 p G((K_p + i\epsilon)V) |p\rangle \langle p|\phi\rangle. \quad (7.31)$$

Note that  $\Omega_{\pm}$  goes with the  $\pm i\epsilon$  in these expressions, sometimes stated as the reason for this notation. We also remark that the incoming or outgoing wave packets associated with these states may be moving forward or backward in  $t$  as a function of  $\tau$  (thus the interaction may be capable of inducing pair annihilation, as in Stueckelberg's original conception).

The important result of this computation is that the incoming and outgoing waves are now expressed in terms of the operators  $GV$ , enabling us to express, with (7.21), these waves in terms of  $G_0$  and the scattering operator  $T$ , i.e.,

$$\Omega_- |\phi\rangle = |\phi\rangle + \lim_{\epsilon \rightarrow 0} \int d^4 p G_0(K_p - i\epsilon) T(K_p - i\epsilon) |p\rangle \langle p|\phi\rangle \quad (7.32)$$

and

$$\Omega_+ |\phi\rangle = |\phi\rangle + \lim_{\epsilon \rightarrow 0} \int d^4 p G_0(K_p + i\epsilon) T(K_p + i\epsilon) |p\rangle \langle p|\phi\rangle \quad (7.33)$$

Using the same techniques, we now proceed to express the  $S$  matrix in terms of the operator  $T(z)$  as well. To do this, we write the matrix elements

$$\langle p' | S | p \rangle = \lim_{\tau' \rightarrow \infty, \tau \rightarrow -\infty} \langle p' | e^{iK_0\tau'} e^{-iK\tau'} e^{iK\tau} e^{-iK_0\tau} | p \rangle;$$

the two limits may be taken simultaneously since both factors are supposed convergent, so the calculation may be made for

$$\langle p' | S | p \rangle = \lim_{\tau \rightarrow \infty} \langle p' | e^{iK_0\tau} e^{-i2K\tau} e^{iK_0\tau} | p \rangle. \quad (7.34)$$

We now use the formula of Abel as before to write this as

$$\langle p' | S | p \rangle = \lim_{\epsilon \rightarrow 0} \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} \langle p' | e^{iK_0\tau} e^{-i2K\tau} e^{iK_0\tau} | p \rangle,$$

provide the factor  $\epsilon$  by differentiating the exponent, and integrate by parts (taking one factor of  $e^{-iK\tau}$  to the left and one to the right), to obtain

$$\begin{aligned} \langle p' | S | p \rangle &= \delta^4(p' - p) - i \lim_{\epsilon \rightarrow 0} \int_0^\infty d\tau \{ V e^{i(K_{p'} + K_p - 2K + i\epsilon)\tau} \\ &\quad + e^{i(K_{p'} + K_p - 2iK + i\epsilon)\tau} V \}. \end{aligned}$$

Carrying out the integral over  $\tau$ , one obtains

$$\begin{aligned} \langle p' | S | p \rangle &= \delta^4(p' - p) + \frac{1}{2} \lim_{\epsilon \rightarrow 0} \left\{ \langle p' | V G \left( \frac{K_{p'} + K_p}{2} + i\epsilon \right) \right. \\ &\quad \left. + G \left( \frac{K_{p'} + K_p}{2} + i\epsilon \right) V | p \rangle \right\}. \end{aligned} \quad (7.35)$$

Recognizing again that  $VG = TG_0$ , and replacing  $K_0$  by  $K_p$  in the first term, and  $K_{p'}$  in the second, we obtain

$$\begin{aligned} \langle p' | S | p \rangle &= \delta^4(p' - p) \\ &\quad + \lim_{\epsilon \rightarrow 0} \left\{ \frac{1}{K_{p'} - K_p + i\epsilon} + \frac{1}{K_p - K_{p'} + i\epsilon} \langle p' | T \left( \frac{K_{p'} + K_p}{2} + i\epsilon \right) | p \rangle \right\}. \end{aligned} \quad (7.36)$$

With the property of distributions that

$$\lim_{\epsilon \rightarrow 0^+} \left( \frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = -2\pi i \delta(x),$$

we obtain one of the main results of this section, that

$$\langle p' | S | p \rangle = \delta^4(p' - p) - 2\pi i \delta(K_{p'} - K_p) \lim_{\epsilon \rightarrow 0} \langle p' | T(K_p + i\epsilon) | p \rangle \quad (7.37)$$

This result is completely analogous to the result of the nonrelativistic formal scattering theory, where the matrix element corresponding to the scattering amplitude is often defined as

$$t(p' \leftarrow p) = \lim_{\epsilon \rightarrow 0} \langle p' | T(K_p + i\epsilon) | p \rangle. \quad (7.38)$$

As in the nonrelativistic case, it follows from (7.24) that, although the physically relevant value of  $T(z)$  is the so-called ‘‘on-shell’’ value of the limit for  $z \rightarrow K_p + i0$ , the matrix elements  $\langle p' | T(z) | p \rangle$  satisfy the integral Lippmann-Schwinger (Lippman 1950) equation

$$\langle p' | T(z) | p \rangle = \langle p' | V | p \rangle + \int d^4 p'' \frac{\langle p' | V | p'' \rangle}{z - K_{p''}} \langle p'' | T(z) | p \rangle. \quad (7.39)$$

We shall discuss in the next section how the cross section is computed from this amplitude, yielding somewhat different content from the nonrelativistic case, and then study the representation of the scattering amplitude in terms of the analog of the Bessel Legendre expansion of the nonrelativistic theory.

### 7.3 Cross Sections

The computation of the cross sections in the relativistic scattering theory was made by Lavie and Horwitz (1982). The probability for the scattered particle to lie in a region  $d^4 p$  is given by

$$w(d^4 p \leftarrow \phi_{in}) = |\phi_{out}^{scatt}|^2 d^4 p, \quad (7.40)$$

where, from the relation (7.8), determining the out-wave from the in-wave through the  $S$ -matrix, we recognize from (7.37) that the *scattered* part of the out-wave  $\phi_{out}^{sc}$  is constructed by means of the  $T$  matrix. Assuming that the laboratory detectors are sensitive to the direction of the momentum  $\mathbf{p}$  and not its magnitude (momentum filters can be taken into account by restricting the integration), we integrate over all  $\mathbf{p}$  to obtain the probability to find the particle emerging with energy  $dp^0$  around  $p^0$ , and three momentum in the solid angle  $d\Omega$  around  $\mathbf{p}$ ,

$$w(d\Omega dp^0 \leftarrow \phi_{in}) = d\Omega dp^0 \int d|\mathbf{p}| |\mathbf{p}|^2 |\phi_{out}^{scatt}(\mathbf{p}, p^0)|^2. \quad (7.41)$$

Note that the energy  $p^0$  is held fixed for the integration, and, since  $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ , if the particle remains close to “mass shell”, the integration over the magnitude of the momentum may be over a small range.

We have so far taken into account a single incoming wave packet. For a beam of wave packets at impact parameters  $\rho$  distributed over times  $\{x_0\}$  in a pulse in the beam, the total number of scatterings into  $d\Omega dp^0$  would be

$$N_{scatt}(d\Omega dp^0) = \int d^3 \rho \int dx^0 w(d\Omega dp^0 \leftarrow \phi_{in}^{\rho, x^0}) n_{inc}(\rho, x^0), \quad (7.42)$$

where  $n_{inc}(\rho, x^0)$  is the number of packets per unit area and unit time perpendicular to the motion of the beam. Since the beam should cover the potential, the integrals can be extended to infinity without changing the result. For  $n_{inc}$  constant, the cross section can then be defined by

$$\sigma(d\Omega dp^0 \leftarrow \phi_{in}) = \frac{N_{sc}(d\Omega dp^0)}{n_{inc}} = \int d^3 \rho \int dx^0 w(d\Omega dp^0 \leftarrow \phi_{in}^{\rho, x^0}). \quad (7.43)$$

This definition is given in terms of a *number* divided by a *density* (equivalent to a rate divided by a flux on  $\Delta\tau$ ), and has dimension three Cook (1957) has given a similar definition). In four dimensions, we see that, as in three dimensions, the dimension of the cross section is the dimension of the space minus the one dimension of the beam. The extra dimension in the cross section can be understood in terms of the time interval which spans the extension of the potential in the relative time (which may be, for example, of the order of the spatial range divided by  $c$ ).<sup>2</sup> This factor of the time interval emerges explicitly in the nonrelativistic limit, where the scattering amplitude contains a delta function  $\delta(p^0 - \bar{p}^0)$ , where  $\bar{p}^0$  is the average energy of

<sup>2</sup>I am grateful to Gideon Alexander for a discussion of this point, and how the analysis of the actual experiments effectively normalize out this interval.

the packet, a result of integration over  $x^0$  required to sample the potential through its spread in  $t$  (Horwitz 1982).

As in the nonrelativistic theory, an optical theorem can be proven, relating the total cross section to the imaginary part of the forward cross section, as a result of the unitarity of the  $S$  matrix. See (Horwitz 1982), where the Feynman rules, electromagnetic scattering, and many body scattering are discussed, and the Rutherford cross section is obtained.

## 7.4 Two Body Partial Wave Analysis

In this section, we use the framework established in Chap. 5 to describe scattering on the elementary level defined by the partial wave expansions of the nonrelativistic theory. The states of the two body systems are given in terms of the RMS structure applicable to the bound states, since the scattering states lie in the same Hilbert space, in the part of the spectrum above the ionization point of the bound states.

Due to the mass energy equivalence of the relativistic theory, it is possible for the masses of the individual particles to change as a result of the scattering interaction; there is therefore no qualitative difference between “elastic” and “inelastic” scattering. The boost parameter  $\beta$  in the final state controls the distribution of these masses, corresponding to the synchronization of the pair of events generating the particle worldlines.

The partial wave expansion that we shall obtain contains phase shifts labelled by the quantum number  $\ell$  which determines the value of the  $O(3, 1)$  Casimir operator intrinsic to the RMS, and corresponds to the nonrelativistic orbital angular momentum quantum number. As we have seen in the previous section, the hyperarea in four-dimensional spacetime perpendicular to the space direction of the incident beam is three dimensional ( $L^2T$ ). This cross section would include the scattering of an ensemble of events that includes all possible distributions of  $\beta$ , and therefore to possible mass changes of the particles after the scattering. Restricting to a small neighborhood of permissible  $\beta$ 's corresponds to a restriction to a definite mass shift; the result carries a Jacobian  $dx^0/d\beta$ , reducing the dimensionality of the incident flux to that of current per unit area, thus accounting for the factor of time required to cover the action of the potential discussed in the previous section. It is interesting to note that the time dimension of the cross section formula corresponds in this sense to the inclusion of inelastic phenomena (consistent with the  $\Delta t \Delta E$  relation).

It will be recalled that the representations of  $O(3, 1)$  provided by the solution for the bound state problem constituted an induced representation on an orbit over spacelike directions. In this construction, the wave function carries a representation of  $O(2, 1)$  that moves along an orbit labelled by the spacelike vector  $m_\mu$ , accompanied by an  $O(2, 1)$  Wigner “rotation”. The structure of this motion was analyzed (Arshansky 1989) into irreducible representations of  $O(3) \subset O(3, 1)$  with quantum number  $(L, q)$ , thus obtaining the principal series of Gel'fand (1963). The bound states can be described equally well for any choice of  $m_\mu$ ; the mass levels are

completely degenerate with respect to this choice. In the problem of two-body scattering, however, the direction of the beam selects a definite spacelike direction. For the wavefunction with  $m_\mu$  component oriented along this direction, one can argue that the scattered wave will be maximally symmetric about this axis. This maximally symmetric state is the one for which the Gel'fand representation contains only the value of  $L$  corresponding to the lowest weight of the principal series, and we shall assume that the scattering matrix (which is diagonal in  $m_\mu$ ) is described, to a good approximation, by such a state (Arshansky 1989). The result we shall obtain agrees in form with the well-known partial wave expansion. Alternative choices of  $m_\mu$ , resulting in states with less symmetry, evidently do not contribute in an important way to the partial wave expansions which have been useful in describing scattering experiments. While we do not exclude other contributions, we restrict our attention here to this special class of scattering states.

We therefore take the general form of the two body wave function corresponding to a definite value  $c_2$  for the second Casimir operator  $\mathbf{L} \cdot \mathbf{A}$  and a definite direction for  $m_\mu$  the form

$$\begin{aligned} \psi_{n_\mu}^{c_1} &= \frac{1}{\sqrt{\rho \sin \theta \cosh \beta}} \Sigma_{\ell, n, k, L, q} A_{\ell n k}^{Lq} \hat{R}_\ell^k(\rho) \hat{\Theta}_\ell^n(\theta) \\ &\times \Xi_{n, k}^{c_2, L}(u) P_{q-M_k}^L(z) e^{-iq\gamma} \hat{\chi}_{n+l}^{-n}(\beta, \varphi), \end{aligned} \quad (7.44)$$

where  $u = \tanh \alpha$ ,  $z = \sin \omega$ , and  $\alpha, \omega, \gamma$  are the angles and hyperangle representing the orientation of the spacelike vector  $m_\mu$ , and the functions appearing on the the right hand side of (7.44) are defined in Arshansky (1989). The variables  $\rho, \theta, \beta, \varphi$  correspond to the relative coordinates in the RMS defined by  $m_\mu$ . The measure on the Hilbert space  $\mathcal{H}_m$  to which these functions belong is  $d\mu = \rho^2 \sin^2 \theta \cosh \beta d\beta d\theta d\varphi$ . The integer parameter  $n$ , determining the Casimir operator for the  $O(2, 1)$  little group, plays the role of the magnetic quantum number in the corresponding non-relativistic problem; here, it fixes the relation between the value of the first Casimir operator  $\mathbf{L}^2 - \mathbf{A}^2$  and  $c_2$  according to ( $\hat{n} = n + \frac{1}{2}$ )

$$-c_1 = 1 - \hat{n}^2 + c_2^2/\hat{n}^2, \quad (7.45)$$

the consistency relation found in Chap. 5.

The differential equations for the functions  $\hat{R}_\ell^k(\rho)$ ,  $\hat{\Theta}_\ell^n(\theta)$  obtained from separation of variables related to the accompanying coordinates  $y^\mu$ , are identical to the equations satisfied by the corresponding nonrelativistic functions (the ‘‘hats’’ denote the extraction of factors  $1/\sqrt{\rho}$  and  $1/\sqrt{\sin \theta}$  from the functions obtained from the relativistic equations); the extra factors are included explicitly. We have also extracted the factor  $1/\sqrt{\cosh \beta}$  in the function  $\hat{\chi}_{n+l}^{-n}(\beta, \varphi)$ , constituting the irreducible representations of  $O(2, 1)$ .

These functions are of quite a different form from those of the usual partial wave expansion; if, however, we choose  $m_\mu$  to be directed parallel to the incoming beam, which we take to be the  $z$  axis, the parameters  $\alpha, \omega$  and  $\gamma$  are zero, and the accompanying coordinates  $y^\mu$  coincide with the base relative coordinates  $x^\mu$  for this orientation.



Only the lowest weight of the Gel'fand representation,  $L = \frac{1}{2}$  and  $n = k = 0$  contribute. We are therefore left with the simple form

$$\psi(x) = \sum_{\ell=0}^{\infty} \frac{A_{\ell} \hat{R}_{\ell}^{\kappa}(\rho) P_{\ell}(\cos \theta) e^{i\varphi/2}}{\sqrt{\rho \sin \theta \cosh \beta}}, \quad (7.46)$$

very similar to the usual partial wave expansion. The half-integer phase factor, discussed above in connection with the bound state functions, is a particular peculiarity associated with the topology of the RMS, but does not influence the experimental predictions of the scattering theory on this level (as we discuss in the next Chapter, it does not influence the Berry phases of the perturbed oscillator either). One may, however, think of experiments in which there is an interaction for which the particle rotates, and coherent interference would provide evidence for this phase, as for the neutron experiments in a magnetic field (Werner 1975).

The coefficients  $A_{\ell}$  can be determined, as for the usual partial wave expansion for the nonrelativistic problem, by requiring that  $\psi$  take the form of the asymptotic incoming wave ( $\tau \rightarrow -\infty$  for a wave packet on the value  $\kappa$  of the  $z$ -component of momentum),

$$\psi_{inc} \sim \frac{e^{i\kappa\rho \cos \theta} e^{i\varphi/2}}{\sqrt{\rho \cosh \beta \sin \theta}}, \quad (7.47)$$

where we have used the fact that  $\hat{R}_{\ell}^{\kappa}(\rho)$  is a solution of the nonrelativistic radial equation in  $\rho$ , and that  $\rho \rightarrow \infty$  spacelike in this limit. The solution (7.47) is an eigenstate of the four momentum operator  $p_{\mu}$  with eigenvalue  $\kappa$  for  $p_3$ ; asymptotically, the other components vanish, to that  $p_{\mu} \sim (0, 0, 0, \kappa)$ .

The *conserved* current associated with the wave function (7.46) (it is this *relative* current which is associated with the counting of scattering events) is given by

$$j_{\mu} = -\frac{i}{2m} (\psi^{*}(x) \partial_{\mu} \psi - \partial_{\mu} \psi^{*} \psi). \quad (7.48)$$

The  $\tau$  integration that is required to convert the  $\tau$ -dependent current density into a conserved current, as discussed in Chap. 2, in the asymptotic free particle case, serves to link the mass squared values  $\kappa^2$  in the two factors  $\psi$  with a  $\delta$ -function. In an interval  $d\kappa^2/2\pi$ , one obtains (the other components vanish)

$$j_z = \frac{\kappa \sin \theta \cosh \beta}{m \rho}. \quad (7.49)$$

The relation between the parameters  $\beta$  and  $\theta$  determines the synchronization between the pair of events being considered, and this determines the mass change during the scattering.

The function  $\hat{R}_{\ell}^{\kappa}(\rho)$ , as remarked above, is a solution of the nonrelativistic radial equation, and therefore results in an outgoing wave of the form

$$\psi^{(+)} \sim \frac{e^{i\varphi/2}}{\sqrt{\rho \cosh \beta \sin \theta}} \left\{ e^{i\kappa\rho \cos \theta} + \frac{1}{\rho} f(\theta) e^{i\kappa\rho} \right\}, \quad (7.50)$$

where, in the Legendre expansion of  $f(\theta)$  the coefficients, following the usual arguments (Merzbacher 1970) for asymptotic values of  $\rho$ , are related to a set of phase shifts  $\delta_\ell(\kappa)$  according to

$$f(\theta) = \frac{1}{2i\kappa} \sum_{\ell=0}^{\infty} (2\ell + 1)(S_\ell - 1) P_\ell(\cos \theta), \quad (7.51)$$

and

$$S_\ell = e^{2i\delta_\ell}. \quad (7.52)$$

The quantities  $S_\ell$  are the  $\ell$  components of the  $S$  matrix. The set of numbers  $\delta_\ell$  are the same as the nonrelativistic phase shifts (as functions of  $\kappa$ , however) since the radial equation (for  $V(\rho)$  the same form as the nonrelativistic  $V(r)$ ) is identical to that of the nonrelativistic problem.

Using the second part of (7.48) and (7.50) we may compute the outgoing current (derivatives with respect to  $\theta$ ,  $\beta$  and  $\varphi$  contain factors that go like  $1/\rho$  and vanish asymptotically). The contributions of the  $\rho$  derivatives multiplied by the four-volume element divided by  $d\rho$  (the infinitesimal volume element lying on the constant  $\rho$  hypersurface) correspond to the number of particles per unit time scattered through this surface element associated with the currents (7.48) in the outgoing wave. For the part of this flow through a surface element normal to the scattering direction specified by unit vector  $x^\mu/\rho$ , one obtains  $(x^\mu/\rho)j_\mu^{scatt}$  for the number of particles per unit time through this surface element. Dividing by the incident flux (and integrating over the azimuthal angle  $\varphi$ ), we obtain the differential cross section

$$d\sigma(\theta) = 2\pi |f(\theta)|^2 d\Omega(\theta), \quad (7.53)$$

where

$$d\Omega(\theta) = \sin \theta d\theta. \quad (7.54)$$

As we have remarked above, the formulas for scattering we have obtained apply both to elastic and inelastic scattering.

## 7.5 Unitarity and the Levinson Theorem

The total probability for the incoming wave (7.47) is

$$\int \rho^3 \sin^2 \theta \cosh \beta d\rho d\theta d\varphi |\psi_{inc}|^2 = \int_0^R d\rho = R, \quad (7.55)$$

for an interval  $d\beta$ , as for our computation of the current above. In the computation of the norm of the outgoing wave  $\psi^{(+)}(x)$  of (7.50) in this interval  $d\beta$ , with the help of the orthogonality relations for Legendre polynomials, one finds precisely the result (7.54). This demonstrates unitarity of the  $S$  matrix for *each* value of  $\beta$ , i.e., for each

state of inelasticity described by this scattering system. It furthermore follows from the form of  $f(\theta)$  given in (7.51) that the optical theorem follows in the usual form.

The analytic properties of  $S_\ell(\kappa)$  follow from the radial equation and the asymptotic form (7.50) for the outgoing wave. Following Levinson (1949) we identify the part of the wave function with asymptotic behavior  $\sim \exp(+i(\kappa\rho - \pi/2))$ ; the limit of this function for  $\rho \rightarrow 0$  (on the light cone) is called  $D_\ell(\kappa)$ ; then,  $D_\ell^*(\kappa) = (-1)^\ell D_\ell(\kappa)$ . Since

$$S_\ell(\kappa) = D_\ell^*(\kappa)/D_\ell(\kappa), \quad (7.56)$$

integration on  $\kappa$  from  $-\infty$  to  $+\infty$  (noting that  $\delta_\ell(\kappa) = -\delta_\ell(-\kappa)$ ), one finds that

$$\delta_\ell(\infty) - \delta_\ell(0) = -\pi N_b^\ell, \quad (7.57)$$

where  $N_b^\ell$  is the number of bound states for a given  $\ell$ . This connection between the number of bound states and the scattering phase shifts is consistent with our formulation of scattering in the RMS. Moreover, the bound states with support in a given RMS are associated with the scattering for the same direction of  $m_\mu$  of that RMS.

We note from this analysis that when  $\delta_\ell$  goes through the value  $\pi/2$  the cross section goes through a maximum. This fact is associated with the interpretation of a *resonance* for this value of  $\kappa$ , corresponding to a complex pole in the lower half plane of the  $S$  matrix.

## 7.6 Resonances and Semigroup Evolution

We have introduced the Green's function in (7.18) as an important constituent in the development of formal scattering theory. This function also arises in the Laplace transform of the Stueckelberg-Schrödinger equation (and thus becomes associated with the idea of propagation). As for the nonrelativistic Schrödinger equation, for  $\tau$  independent generator  $K$ , the formal integrated solution is

$$\psi_\tau = e^{-iK\tau} \psi_0 \quad (7.58)$$

Thus the unitary evolution generated by  $K$  “propagates” the wave function forward in  $\tau$ , but not necessarily in  $t$ . Thus the spacetime diagrams of Feynman, as he pointed out in his paper (Feynman 1949), can be understood as lines propagating forward and backward in time according to this evolution; in interaction picture, the straight lines correspond to unperturbed propagation, and the vertices, to some interaction carried by the full Hamiltonian  $K$ . The standard interaction picture applied to this evolution can be represented in terms of such diagrams (as pointed out in Chap. 4, for gauge fields, the fifth field induced by gauge invariance of the Stueckelberg-Schrödinger equation must be taken into account for consistency with the continuous evolution picture of the Stueckelberg theory).

In the following, we make a connection between the resonances which occur in scattering theory, recognized as fairly well defined maxima in the cross sections

defined by the scattering amplitude  $t(p' \leftarrow p)$ , corresponding to values of  $\delta_\ell$  defined in (7.51) and (7.52) going through a value of  $\pi/2$  (or multiples of it), and the description of unstable systems (Horwitz (1971)).

In 1926, for the description of the decay of an unstable system in the nonrelativistic theory, Gamow (1928) suggested that the Schrödinger equation be assigned a complex eigenvalue,  $E - i\frac{\Gamma}{2}$  so that the solution would give an exponential decay law. The idea was not theoretically tenable since the momentum is proportional to the square root of the energy in nonrelativistic mechanics, and this would imply a complex momentum and associated instability of the spatial wave function. Wigner and Weisskopf in 1930 (Weisskopf 1930) proposed a model in which the initial state of a quantum mechanical system is considered the “particle”, or the state of the unstable system, and its projection back into that state after time  $t$  would be understood as the amplitude for the particle to remain in its initial state, i.e. the “survival probability amplitude” would be<sup>3</sup>

$$A(t) = (\psi, e^{-iHt}\psi). \quad (7.59)$$

In the relativistic theory we may introduce the analogous definition,

$$A(\tau) = (\psi, e^{-iK\tau}\psi), \quad (7.60)$$

where, for potential type problems,  $K$  may be taken as the reduced motion which, for the two body problem, may be bounded from below.<sup>4</sup>

The Laplace transform of (7.60) provides a useful (Horwitz 1971) interpretation of the Green’s function introduced in (7.18), i.e.

$$\begin{aligned} A(z) &= \int_0^\infty e^{iz\tau} A(\tau) \\ &= i(\psi, \frac{1}{z - K}\psi), \end{aligned} \quad (7.61)$$

is well-defined (and analytic) for  $Im z > 0$ , exhibiting explicitly the Green’s function, now understood in terms of a “propagator”, as the Laplace transform of the unitary evolution.

The inverse Laplace transform is an integral to be carried out on a line in the complex  $z$  plane just above the real axis (where the function is analytic) from  $+\infty$  to  $-\infty$ ,

$$A(\tau) = \frac{1}{2\pi i} \int_{\infty+i\epsilon}^{-\infty+i\epsilon} dz e^{-iz\tau} A(z). \quad (7.62)$$

If the spectrum of  $K$  (or of  $H$  in the nonrelativistic case), for the reduced motion of a two body system runs from zero to infinity (i.e., bounded below), then the integration on the negative real axis can be moved to the lower half plane, running along the

<sup>3</sup>It will be convenient in the remainder of this chapter to use the round bracket for scalar products when using normalized functions, and the angular bracket for non-normalized generalized states.

<sup>4</sup>The discussion which follows can be applied directly to the nonrelativistic case by transposing  $\tau$  to  $t$ . We have maintained the notation  $\tau$  here to be consistent with the relativistic framework.

imaginary axis, where the value is suppressed by the exponential  $\exp(-i z \tau)$ ; the contributions are very small for the interval sufficiently below the branch point, and  $\tau$  sufficiently large. The integral along the positive real axis can be lowered to the second Riemann sheet by considering the difference, with the help of the identity given after (7.36)

$$(\psi, G(\kappa + i\epsilon)\psi) - (\psi, G(\kappa - i\epsilon)\psi) = 2\pi i \chi(\kappa), \quad (7.63)$$

where  $\chi(\kappa)$  is the spectral weight factor of the expectation value of the spectral representation of the operator  $(z - K)^{-1}$ . The term  $(\psi, G(\kappa - i\epsilon)\psi)$  is evaluated by analytic continuation around the real axis to a point just below the positive real axis.<sup>5</sup> Rotating the integral on the positive real axis into the lower half plane (second sheet), one finds that there can be contributions from singularities in the lower half plane. In the case of a pole, which one might understand as the remnant of a bound state, a pole on the real axis, pulled down into the lower half plane by the interaction, the passage of the contour of integration over this pole extracts a residue proportional to  $e^{-i z_p \tau}$ , which may dominate the entire integral (for times not too long and not too short), in agreement with the proposal of Gamow. For very short times this expression will not be dominant, and for very long times, the contribution of the branch cut dominates, for which there may be a polynomial type decay law (Bleistein 1977).

Furthermore, the pole contribution does not correspond to any physical state. It has been shown, on the other hand, that there is a vector in a Banach space (an element of a Gel'fand triple constructed in the dual to a subspace of the original Hilbert space (Horwitz 1978; Baumgartel 1976)) that can be constructed to correspond to this pole with exact exponential decay, but it is difficult to interpret such a construction as a physically meaningful state (expectation values would not be generally defined). Nevertheless, this description provides an experimentally useful definition for resonances in terms of the decay law (Bohm 1989).

The  $T$  matrix, (7.19), contains this function as well, playing an essential role in the construction of the  $S$  matrix and the scattering amplitudes for  $Im z \rightarrow 0_+$ , as in (7.38).

Although the formula (7.60) may provide an exponential behavior for  $|A(\tau)|^2$  for sufficiently long (but not too long) times, as we have seen, for very short times it generally displays a decay law which is not consistent with the exponential form. For short times,

$$A(\tau) = 1 - i \langle K \rangle \tau - \frac{\langle K^2 \rangle}{2} \tau^2 + \dots \quad (7.64)$$

It then follows that

$$|A(\tau)|^2 = 1 - \Delta K^2 \tau^2 + \dots, \quad (7.65)$$

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<sup>5</sup>For the case of a continuous spectrum from  $-\infty$  to  $+\infty$ , two functions, one analytic in the upper half plane, and the other in the lower half plane, according to forward and backward evolution may be defined, and a similar method may be applied by moving the integral along the real line in the upper half plane into the lower half plane.

with  $\Delta K^2 = \langle K^2 \rangle - \langle K \rangle^2$ , not consistent with the semigroup property.

The *semigroup* law of evolution, expected from reasonable arguments to be valid for irreversible processes, such as the decay of an unstable system,<sup>6</sup> is defined by the relation

$$Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2), \quad (7.66)$$

for  $\tau_1, \tau_2 \geq 0$ ;  $Z(\tau)$  has no inverse, unlike a one parameter group, such as the unitary Stueckelberg-Schrödinger evolution  $e^{-iK\tau}$ , which has the property (7.66) for all  $\tau_1, \tau_2$ . The model proposed by Gamow, for which  $|A(t)|^2 \propto e^{-\Gamma t}$ , does have this property, in agreement with experiment, but the derivative of this function at zero is  $\propto -\Gamma$ , so the function approaches unity at zero linearly, not quadratically, as it would for almost any Hamiltonian (with finite dispersion in state  $\psi$ ) in the Wigner-Weisskopf model. One can argue that in many cases the very short time before an approach to exponential (Misra 1977) behavior would not be observable experimentally, and this has justified its use in many cases, but the fact that the evolution is not semigroup has consequences for the application of the idea to two or more dimensions, such as for the neutral  $K$  meson decay, where it has been shown to be quantitatively inapplicable (Cohen 2011).

One finds that the poles of the resolvent for the Wigner-Weisskopf evolution of the two channel system results in non-orthogonal residues that generate interference terms, which make the non-semigroup property evident even for times for which the pole approximation is valid (Cohen 2011), a domain in which exponential decay for the single channel system is very accurately described by the Wigner-Weisskopf model.

The Yang and Wu (1975) parametrization of the  $K^0$  decay processes, based on a Gamow type evolution generated by an effective  $2 \times 2$  non-Hermitian matrix Hamiltonian, on the other hand, results in an evolution that is an exact semigroup. It appears that the phenomenological parametrization of Yang and Wu (1975) is indeed consistent to a high degree of accuracy with the experimental results on  $K$ -meson decay (Olive 2014).

We shall discuss below a theory (based on the work of Lax and Phillips 1967) in which the evolution law is precisely semigroup and identifies the resonance with a quantum state, and discuss how it can be applied to the relativistic evolution of the neutral  $K$  meson system, explaining as well the origin of the phenomenological parametrization of Yang and Wu.

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<sup>6</sup>Based on the argument that one can stop the evolution at any moment and then proceed as if starting from the new initial conditions, with a result equivalent to letting the system develop undisturbed for the entire time, an essentially Markovian hypothesis.

## 7.7 Lax Phillips Theory

The quantum Lax-Phillips theory (Strauss 2000a) which we discuss in the following, constructed by embedding the quantum theory into the original Lax-Phillips scattering theory (Lax 1967) (originally developed for hyperbolic systems, such as acoustic or electromagnetic waves), describes the resonance as a state in a Hilbert space, and therefore it is possible, in principle, to calculate all measurable properties of the system in this state. Moreover, the quantum Lax-Phillips theory provides a framework for understanding the decay of an unstable system as an irreversible process. It appears, in fact, that this framework is categorical for the description of irreversible processes for the evolution of an “isolated” quantum system. There appear to be formal relations to the evolution of a system in a bath with which it interacts (Shikerman 2013) and these relations are presently under investigation.

The scattering theory of Lax and Phillips (1967), originally developed for the description of resonances in classical wave problems such as electromagnetic or acoustic, assumes the existence of a Hilbert space  $\overline{\mathcal{H}}$  of physical states in which there are two distinguished orthogonal subspaces  $\mathcal{D}_+$  and  $\mathcal{D}_-$  with the properties

$$\begin{aligned} U(\tau) \mathcal{D}_+ &\subset \mathcal{D}_+ & \tau > 0 \\ U(\tau) \mathcal{D}_- &\subset \mathcal{D}_- & \tau < 0 \\ \bigcap_{\tau} U(\tau) \mathcal{D}_{\pm} &= \{0\} \\ \overline{\bigcup_{\tau} U(\tau) \mathcal{D}_{\pm}} &= \overline{\mathcal{H}}, \end{aligned} \tag{7.67}$$

i.e., the subspaces  $\mathcal{D}_{\pm}$  are stable under the action of the full unitary dynamical evolution  $U(\tau)$ , a function of the physical laboratory time, which we identify with the universal invariant time  $\tau$  of the Stueckelberg theory discussed above for its application to the relativistic theory. Over all  $\tau$ , the evolution operator generates a dense set in  $\overline{\mathcal{H}}$  from either  $\mathcal{D}_+$  or  $\mathcal{D}_-$ . We shall call  $\mathcal{D}_+$  the *outgoing subspace* and  $\mathcal{D}_-$  the *incoming subspace* with respect to the group  $U(\tau)$ .

A theorem of Sinai (Cornfield 1982) assures that  $\overline{\mathcal{H}}$  can be represented as a family of Hilbert spaces obtained by foliating  $\overline{\mathcal{H}}$  along a real line, which we shall call  $\{s\}$ , in the form of a direct integral

$$\overline{\mathcal{H}} = \int_{\oplus} \mathcal{H}_s, \tag{7.68}$$

where the set of auxiliary Hilbert spaces  $\mathcal{H}_s$  are all isomorphic. Representing these spaces in terms of square-integrable functions, we define the norm in the direct integral space (we use Lebesgue measure) as

$$\|f\|^2 = \int_{-\infty}^{\infty} ds \|f_s\|_H^2, \tag{7.69}$$

where  $f \in \overline{\mathcal{H}}$  represents a vector in  $\overline{\mathcal{H}}$  in terms of the  $L^2$  function space  $L^2(-\infty, \infty, H)$ , and  $f_s \in H$ , the  $L^2$  function space representing  $\mathcal{H}_s$  for any  $s$ .

The Sinai theorem furthermore asserts that there are representations for which the action of the full evolution group  $U(\tau)$  on  $L^2(-\infty, \infty, H)$  is translation by  $\tau$  units. Given  $D_{\pm}$  (the  $L^2$  spaces representing  $\mathcal{D}_{\pm}$ ), there is such a representation, called the *incoming translation representation* (Lax 1967), for which functions in  $D_-$  have support in  $L^2(-\infty, 0, H)$ , and another called the *outgoing translation representation*, for which functions in  $D_+$  have support in  $L^2(0, \infty, H)$ . It is clear that  $s$  has the interpretation of an observable; it is conjugate to the “energy”  $\sigma$  defined by Lax and Phillips in the so-called energy representation defined by Fourier transform, as we shall see below.

In the discussion of the bound state problem, we considered the system in the total rest frame, for which this component  $P^0$  is the rest energy of the system (its “rest mass”). The invariant  $P^\mu P_\mu$  is the so-called Mandelstam variable corresponding to the mass of the decaying system (or the total mass of the final state of a scattering system in the  $s$ -channel).<sup>7</sup> We shall find that the resonance described by the Lax Phillips theory corresponds to a pole in the  $S$ -matrix as a function of this variable in the complex plane, as consistent with the usual interpretation of the mass of a resonance. We may therefore identify the foliation variable  $s$  with the time  $T$  of the center of mass of the decaying system (the fourth component of the center of mass position  $X^\mu$  defined in Chap. 5, a variable covariantly dual to the total center of mass energy, the fourth component of the conserved total energy momentum  $P^\mu$ ). The “anti-atom”, or anti-two body state, would be associated with the  $T$ -reversed motion (corresponding to time reversal of the entire system).

Lax and Phillips (Lax 1967) show that there are unitary operators  $W_{\pm}$ , called wave operators, which map elements in  $\overline{\mathcal{H}}$ , respectively, to these representations. They define an  $S$ -matrix,

$$S = W_+ W_-^{-1} \quad (7.70)$$

which connects these representations; it is unitary, commutes with translations, and maps  $L^2(-\infty, 0)$  into itself. The singularities of this  $S$ -matrix, in what is defined as the *spectral representation*, correspond to the spectrum of the generator of the exact semigroup characterizing the evolution of the unstable system.

With the assumptions stated above on the properties of the subspaces  $\mathcal{D}_+$  and  $\mathcal{D}_-$ , Lax and Phillips (1967) prove that the family of operators

$$Z(\tau) \equiv P_+ U(\tau) P_- \quad (\tau \geq 0), \quad (7.71)$$

<sup>7</sup>In analytic  $S$ -matrix theory (Chew 1966; Eden 1967), the “physical” or direct channel, where the total energy momentum of the incoming (or outgoing) particles is given by, for example,  $P^\mu = p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu$ , is called the  $s$  channel. By crossing symmetry, assuming analyticity of the  $S$  matrix, one can consider the related process  $p_1 + \bar{p}_1'$  in what is called the  $t$  channel, or  $p_1 + \bar{p}_2'$  in what is called the  $u$  channel. Although these substitutions can be made directly in the  $S$ -matrix, the corresponding kinematical quantities are off shell until appropriate analytic continuations are made.



where  $P_{\pm}$  are projections into the orthogonal complements of  $\mathcal{D}_{\pm}$ , respectively, is a contractive, continuous, semigroup. This operator annihilates vectors in  $\mathcal{D}_{\pm}$  and carries the space

$$\mathcal{K} = \overline{\mathcal{H}} \ominus \mathcal{D}_+ \ominus \mathcal{D}_- \quad (7.72)$$

into itself, with norm tending to zero for every element in  $\mathcal{K}$ .

We see from this construction that the outgoing subspace  $D_+$  is defined, in the outgoing representation, in terms of support properties (this is also true for the incoming subspace in the incoming representation). One can then easily understand that the fundamental difference between Lax-Phillips theory and the standard quantum theory lies in this property; the projection operators are associated with *subspaces defined by time*. The subspace defining the unstable system in the standard theory is usually defined as the eigenstate of an unperturbed Hamiltonian, and cannot be associated with an interval on time. The subspaces of the Lax-Phillips theory are associated with intervals (e.g., the positive and negative half-lines in the outgoing and incoming free representations). To see this, we remark that the operator  $P_+U(\tau)$  is a semigroup. The product

$$\begin{aligned} P_+U(\tau_1)P_+U(\tau_2) &= P_+U(\tau_1)[1 - (1 - P_+)]U(\tau_2) \\ &= P_+U(\tau_1)U(\tau_2) = P_+U(\tau_1 + \tau_2); \end{aligned} \quad (7.73)$$

this follows from the fact that  $U(\tau_1)$  leaves the subspace  $D_+$  invariant.

We now show that the generator of this semigroup is symmetric but not self-adjoint, and it is therefore not a group. In the outgoing translation representation,

$$(P_+U(\tau)f)(s) = \theta(-s)f(s - \tau), \quad (7.74)$$

and therefore

$$(P_+Kf)(s) = i\theta(-s)\frac{\partial f}{\partial s}(s - \tau)|_{\tau \rightarrow 0_+}, \quad (7.75)$$

where  $f(s)$  is a vector-valued function, and  $K$  is the self-adjoint generator associated with  $U(\tau)$ . If we then compute the scalar product of the vector given in (1.10) with a vector  $g$ , we find that

$$\int_{-\infty}^{\infty} ds g^*(s)(P_+Kf)(s) = i\delta(s)g^*(0)f(0) + \int_{-\infty}^{\infty} ds (P_+Kg)^*(s)f(s). \quad (7.76)$$

The generator is therefore not self-adjoint. It is through this mechanism that the Lax-Phillips theory provides a description that has the semigroup property for the evolution of an unstable system (Horwitz 1973). It has, in fact, a family of complex eigenvalues  $\{\mu\}$  in the upper half-plane; the eigenfunctions are

$$f_{\mu}(s) = \begin{cases} e^{\mu s}n, & s \leq 0; \\ 0, & s > 0, \end{cases} \quad (7.77)$$

where  $n$  is some vector in the auxiliary space.

The semigroup property of the operator  $Z(\tau)$  of (7.71) follows directly from the discussion given above. It clearly vanishes on the subspace  $D_-$ , and by the stability of  $D_+$  under  $U(\tau)$  for  $\tau \geq 0$ , it vanishes on  $D_+$  as well. It is therefore non-zero only on the subspace  $K$ , and on such vectors, the operator  $P_-$  can be omitted; the semigroup property then follows from what we have said above.

If we identify elements in the space  $\overline{\mathcal{H}}$  with *physical states*, and identify the subspace  $\mathcal{K}$  with the unstable system, we see that the quantum Lax Phillips theory provides a framework for the description of an unstable system which decays according to a semigroup law. We remark that, taking a vector  $\psi_0$  in  $\mathcal{K}$ , and evolving it under the action of  $U(\tau)$ , the projection back into the original state is (this follows from (7.71) and the stability of  $\mathcal{D}_\pm$  that  $Z(\tau) = P_{\mathcal{K}}U(\tau)P_{\mathcal{K}}$  as well)

$$\begin{aligned} A(\tau) &= (\psi_0, U(\tau)\psi_0) \\ &= (\psi_0, P_{\mathcal{K}}U(\tau)P_{\mathcal{K}}\psi_0) \\ &= (\psi_0, Z(\tau)\psi_0), \end{aligned} \tag{7.78}$$

so that the survival amplitude (7.59) of the Lax-Phillips theory, analogous to that of the Wigner-Weisskopf formula (7.59), has the exact exponential behavior. The difference between this result and the corresponding expression for the Wigner-Weisskopf theory can be accounted for by the fact that there are translation representations for  $U(\tau)$ , and that the definition of the subspace  $\mathcal{K}$  is related to the support properties along the foliation axis on which these translations are induced.

Functions in the space  $\overline{H}$ , representing the elements of  $\overline{\mathcal{H}}$ , depend on the variable  $s$  as well as the variables of the auxiliary space  $H$ . The measure space of this Hilbert space of states is one dimension larger than that of a quantum theory represented in the auxiliary space alone (the additional dimension may correspond to the center of mass time of the resonance, as pointed out above, a cyclic variable in, for example, the two body problem treated in Chap. 5). With this identification, we may understand this representation of a state as a *virtual history*. The collection of such histories forms a quantum ensemble; the absolute square of the wave function corresponds to the probability that the system would be found, as a result of measurement, at time  $s$  in a particular configuration in the auxiliary space (in the state described by this wave function), i.e., an element of one of the virtual histories (Eisenberg 1997). For example, the expectation value of the position variable  $x$  at a given  $s$  is, in the standard interpretation of the auxiliary space as a space of quantum states,

$$\langle x \rangle_s = \frac{(\psi_s, x\psi_s)}{\|\psi_s\|^2}. \tag{7.79}$$

The full expectation value in the physical Lax-Phillips state, according to (7.69), is then

$$\int ds (\psi_s, x\psi_s) = \int ds \|\psi_s\|^2 \langle x \rangle_s, \tag{7.80}$$

so we see that  $\|\psi_s\|^2$  corresponds to the probability to find a signal which indicates the presence of the system at the time  $s$  (in the same way that  $x$  is interpreted as a dynamical variable in the quantum theory).

One may ask, in this framework, which results in a precise semigroup behavior for an unstable system, whether such a theory can support as well the description of stable systems or a system which makes a transition following the rule of Wigner and Weisskopf (as, for example, the adiabatic rotation of an atom with spin in an electromagnetic field). It is clear that if  $D_{\pm}$  span the whole space, for example, there is no unstable subspace, and one has a scattering theory without the type of resonances that can be associated with unstable systems.

In the following, we give a procedure (Strauss 2000a) for the construction of the subspaces  $D_{\pm}$ , and for defining the representations which realize the Lax-Phillips structure. In this framework, we define the Lax-Phillips  $S$ -matrix.

It follows from the existence of the one-parameter unitary group  $U(\tau)$  which acts on the Hilbert space  $\overline{\mathcal{H}}$  that there is an operator  $K$  which is the generator of dynamical evolution of the physical states in  $\overline{\mathcal{H}}$ ; we assume that there exist *wave operators*  $\Omega_{\pm}$  which intertwine this dynamical operator with an unperturbed dynamical operator  $K_0$ . We shall assume that  $K_0$  has only absolutely continuous spectrum in  $(-\infty, \infty)$  (we discuss below an example in which these assumptions are explicitly valid).

We begin the development of the quantum Lax-Phillips theory (Strauss 2000a) with the construction of the incoming and outgoing translation representations. In this way, we shall construct explicitly the foliations required. The *free spectral representation* of  $K_0$  is defined by

$${}_f\langle\sigma\beta|K_0|g\rangle = \sigma {}_f\langle\sigma\beta|g\rangle, \quad (7.81)$$

where  $|g\rangle$  is an element of  $\overline{\mathcal{H}}$  and  $\beta$  corresponds to the variables (measure space) of the auxiliary space associated to each value of  $\sigma$ , which, with  $\sigma$  (identified with the rest energy of the unstable system, as we have pointed out above), comprise a complete spectral set. The functions  ${}_f\langle\sigma\beta|g\rangle$  may be thought of as a set of functions of the variables  $\beta$  indexed on the variable  $\sigma$  in a continuous sequence of auxiliary Hilbert spaces isomorphic to  $H$ .

We now proceed to define the incoming and outgoing subspaces  $\mathcal{D}_{\pm}$ . To do this, we define the Fourier transform from representations according to the spectrum  $\sigma$  to the foliation variable  $s$  of (7.69), i.e.,

$${}_f\langle s\beta|g\rangle = \int e^{i\sigma s} {}_f\langle\sigma\beta|g\rangle d\sigma. \quad (7.82)$$

Clearly,  $K_0$  acts as the generator of translations in this representation. We shall say that the set of functions  ${}_f\langle s\beta|g\rangle$  are in the *free translation representation*.

Let us consider the sets of functions with support in  $L^2(0, \infty)$  and in  $L^2(-\infty, 0)$ , and call these subspaces  $D_0^{\pm}$ . The Fourier transform back to the free spectral representation provides the two sets of Hardy class functions (Strauss 2000a)

$${}_f\langle\sigma\beta|g_0^{\pm}\rangle = \int e^{-i\sigma s} {}_f\langle s\beta|g_0^{\pm}\rangle ds \in H_{\pm}, \quad (7.83)$$

for  $g_0^{\pm} \in D_0^{\pm}$ .

We may now define the subspaces  $\mathcal{D}_\pm$  in the Hilbert space of states  $\overline{\mathcal{H}}$ . To do this we first map these Hardy class functions in  $\overline{H}$  to  $\overline{\mathcal{H}}$ , i.e., we define the subspaces  $\mathcal{D}_0^\pm$  by

$$\int \sum_{\beta} |\sigma\beta\rangle_f \langle \sigma\beta | g_0^\pm \rangle d\sigma \in \mathcal{D}_0^\pm. \tag{7.84}$$

We shall assume that there are wave operators which intertwine  $K_0$  with the full evolution  $K$ , i.e., that the limits

$$\lim_{\tau \rightarrow \mp\infty} e^{iK\tau} e^{-iK_0\tau} = \Omega_\pm \tag{7.85}$$

exist on a dense set in  $\overline{\mathcal{H}}$ . We emphasize that the operator  $K$  generates evolution of the entire virtual history, i.e., of elements in  $\overline{\mathcal{H}}$ , and that these wave operators are defined in this larger space. These operators are *not*, in general, the usual wave (intertwining) operators for the perturbed and unperturbed Hamiltonians that act in the auxiliary space. The conditions for their existence are, however, closely related to those of the usual wave operators. For the existence of the limit, it is sufficient that for  $\tau \rightarrow \pm\infty$ ,  $\|V e^{-iK_0\tau} \phi\| \rightarrow 0$  for a dense set in  $\overline{\mathcal{H}}$ . As for the usual scattering theory, it is possible to construct examples for which the wave operator exists if the potential falls off sufficiently rapidly, as discussed above in connection with the standard relativistic scattering theory.

The construction of  $\mathcal{D}_\pm$  is then completed with the help of the wave operators. We define these subspaces by

$$\begin{aligned} \mathcal{D}_+ &= \Omega_+ \mathcal{D}_0^+ \\ \mathcal{D}_- &= \Omega_- \mathcal{D}_0^- \end{aligned} \tag{7.86}$$

We remark that these subspaces are not produced by the same unitary map. This procedure is necessary to realize the Lax-Phillips structure non-trivially; if a single unitary map were used, then there would exist a transformation into the space of functions on  $L^2(-\infty, \infty, H)$  which has the property that all functions with support on the positive half-line represent elements of  $\mathcal{D}_+$ , and all functions with support on the negative half-line represent elements of  $\mathcal{D}_-$  in the same representation; the resulting Lax-Phillips  $S$ -matrix would then be trivial. The requirement that  $\mathcal{D}_+$  and  $\mathcal{D}_-$  be orthogonal is not an immediate consequence of our construction; as we shall see, this result is associated with the analyticity of the operator which corresponds to the Lax-Phillips  $S$ -matrix.

In the following, we construct the Lax-Phillips  $S$ -matrix and the Lax-Phillips wave operators.

The wave operators defined by (7.85) intertwine  $K$  and  $K_0$ , i.e.,

$$K \Omega_\pm = \Omega_\pm K_0; \tag{7.87}$$

we may therefore construct the outgoing (incoming) spectral representations from the free spectral representation. Since

$$\begin{aligned} K \Omega_\pm |\sigma\beta\rangle_f &= \Omega_\pm K_0 |\sigma\beta\rangle_f \\ &= \sigma \Omega_\pm |\sigma\beta\rangle_f, \end{aligned} \tag{7.88}$$

we may identify

$$|\sigma\beta\rangle_{in}^{out} = \Omega_{\pm}|\sigma\beta\rangle_f. \quad (7.89)$$

The Lax-Phillips  $S$ -matrix is defined as the operator on  $\overline{H}$  which carries the incoming to outgoing translation representations of the evolution operator  $K$ . Suppose  $g$  is an element of  $\overline{\mathcal{H}}$ ; its incoming spectral representation, according to (7.83), is

$${}_{in}\langle\sigma\beta|g\rangle = {}_f\langle\sigma\beta|\Omega_-^{-1}g\rangle. \quad (7.90)$$

Let us now act on this function with the Lax-Phillips  $S$ -matrix in the free spectral representation, and require the result to be the *outgoing* representer of  $g$ :

$$\begin{aligned} {}_{out}\langle\sigma\beta|g\rangle &= {}_f\langle\sigma\beta|\Omega_+^{-1}g\rangle \\ &= \int d\sigma' \sum_{\beta'} {}_f\langle\sigma\beta|\mathbf{S}|\sigma'\beta'\rangle_f {}_f\langle\sigma'\beta'|\Omega_-^{-1}g\rangle \end{aligned} \quad (7.91)$$

where  $\mathbf{S}$  is the Lax-Phillips  $S$ -operator (defined on  $\overline{\mathcal{H}}$ ). Transforming the kernel to the free translation representation with the help of (7.82), i.e.,

$${}_f\langle s\beta|\mathbf{S}|s'\beta'\rangle_f = \frac{1}{(2\pi)^2} \int d\sigma d\sigma' e^{i\sigma s} e^{-i\sigma's'} {}_f\langle\sigma\beta|\mathbf{S}|\sigma'\beta'\rangle_f, \quad (7.92)$$

we see that the relation (7.91) becomes, after using Fourier transform in a similar way to transform the *in* and *out* spectral representations to the corresponding *in* and *out* translation representations,

$$\begin{aligned} {}_{out}\langle s\beta|g\rangle &= {}_f\langle s\beta|\Omega_+^{-1}g\rangle = \int ds' \sum_{\beta'} {}_f\langle s\beta|\mathbf{S}|s'\beta'\rangle_f {}_f\langle s'\beta'|\Omega_-^{-1}g\rangle \\ &= \int ds' \sum_{\beta'} {}_f\langle s\beta|\mathbf{S}|s'\beta'\rangle_f {}_{in}\langle s'\beta'|g\rangle. \end{aligned} \quad (7.93)$$

Hence the Lax-Phillips  $S$ -matrix is given by

$$S = \{{}_f\langle s\beta|\mathbf{S}|s'\beta'\rangle_f\}, \quad (7.94)$$

in free translation representation. It follows from the intertwining property (7.87) that

$${}_f\langle\sigma\beta|\mathbf{S}|\sigma'\beta'\rangle_f = \delta(\sigma - \sigma') S^{\beta\beta'}(\sigma), \quad (7.95)$$

This result can be expressed in terms of operators on  $\overline{\mathcal{H}}$ . Let

$$w_-^{-1} = \{{}_f\langle s\beta|\Omega_-^{-1}\rangle\} \quad (7.96)$$

be a map from  $\overline{\mathcal{H}}$  to  $\overline{H}$  in the incoming translation representation, and, similarly,

$$w_+^{-1} = \{{}_f\langle s\beta|\Omega_+^{-1}\rangle\} \quad (7.97)$$

a map from  $\overline{\mathcal{H}}$  to  $\overline{H}$  in the outgoing translation representation. It then follows from (7.93) that

$$S = w_+^{-1} w_-, \quad (7.98)$$

is a kernel on the free translation representation. This kernel is understood to operate on the representer of a vector  $g$  in the incoming representation and map it to the representer in the outgoing representation (see Strauss (2000a) for a study of pointwise models corresponding to the nonrelativistic limit of the theory described above, for which the generator acts pointwise on the foliation axis).

## 7.8 Relativistic Lee-Friedrichs Model

We now turn to the relativistic analysis of a resonance in the Lee-Friedrichs model from the point of view of the Lax-Phillips theory. The Lee-Friedrichs model (Friedrichs 1950; Horwitz 1971) is an exactly soluble model for a scattering theory with resonances, originally developed in a nonrelativistic framework. Lee wrote the model in terms of a nonrelativistic quantum field theory, for which the interaction is such that there are sectors that make the problem equivalent to a quantum mechanical model with finite rank potential, corresponding to the formulation of Friedrichs. The *a priori* formulation of the model for the relativistic quantum theory from the point of view of Friedrichs is not so clear, as we shall see, and we therefore follow the procedure of Lee (Horwitz 1995) but for the relativistic quantum field theory (for spin zero bosons) discussed in Chap. 3. We do not indicate here the point  $n_\mu$  on the orbit for the induced representation of these fields, since the entire formulation is at one point on the orbit (scalar fields do not necessarily require an induced representation, but one may construct such boson fields as composite two (or more) fermion systems, as described there, in which case  $n$  is implicit, in the decay modes as well).

Following Lee, let us define the fields  $b(p)$ ,  $a_N(p)$  and  $a_\theta(p)$  as annihilation operators for particles which we shall call the  $V$ ,  $N$  and  $\theta$  particles, and  $M_V$ ,  $M_N$  and  $M_\theta$  the corresponding mass parameters. Writing  $p^2 = p^\mu p_\mu$ , and  $k^2 = k^\mu k_\mu$ , we define

$$K_0 = \int d^4 p \left\{ \frac{p^2}{2M_V} b^\dagger(p) b(p) + \frac{p^2}{2M_N} a_N^\dagger(p) a_N(p) \right\} + \int d^4 k \frac{k^2}{2M_\theta} a_\theta^\dagger(k) a_\theta(k) \quad (7.99)$$

For the interaction, we take

$$V = \int d^4 p \int d^4 k (f(k) b^\dagger(p) a_N^\dagger(p-k) a_\theta(k) + (f^*(k) b(p) a_N(p-k) a_\theta^\dagger(k)), \quad (7.100)$$

describing the process  $V \leftrightarrow N + \theta$ . This interaction is clearly rank one, enabling, as we shall see, one to achieve an exact solution. The coefficient  $f(k)$  is required for the potential to be a bounded operator. As a quantum field theory, if  $f$  is a constant coefficient, the theory becomes poorly defined. The operators

$$\begin{aligned} Q_1 &= \int d^4 p [b^\dagger(p) b(p) + a_N^\dagger(p) a_N(p)] \\ Q_2 &= \int d^4 p [a_N^\dagger(p) a_N(p) - a_\theta^\dagger(p) a_\theta(p)] \end{aligned} \quad (7.101)$$

are strictly conserved, enabling us to decompose the Fock space to sectors. We shall study the problem in the lowest sector  $Q_1 = 1$ ,  $Q_2 = 0$ , for which there is just one  $V$  or one  $N$  and one  $\theta$ . In this sector the generator of evolution can be written in the form

$$K = \int d^4 p K^p = \int d^4 p (K_0^p + V^p), \quad (7.102)$$

where

$$K_0^p = \frac{p^2}{2M_V} b^\dagger(p) b(p) + \int d^4k \left( \frac{(p-k)^2}{2M_N} + \frac{k^2}{2M_\theta} \right) a_N^\dagger(p-k) a_\theta^\dagger(k) a_\theta(k) a_N(p-k) \quad (7.103)$$

and

$$V^p = \int d^4k (f(k) b^\dagger(p) a_N^\dagger(p-k) a_\theta(k) + (f^*(k) b(p) a_N(p-k) a_\theta^\dagger(k)). \quad (7.104)$$

In the corresponding nonrelativistic theory, one can see at this point the essential algebraic content of the Friedrichs model. To understand how this works, and to give an indication of how the resonant structure follows from this picture, we give a brief review of the computation here, essentially based on the results (7.61) and (7.62) for the amplitude. As in the lowest sector of the Lee model, we have a rank one potential  $V$ , and an  $H_0$  for which the eigenstate  $\phi$  satisfies

$$H_0 \phi = E_0 \phi \quad (7.105)$$

The operator  $H_0$  has a continuous spectrum on  $(0, \infty)$ , which we assume is non-degenerate. We furthermore assume that (as by construction in the Lee model),  $\phi$  is orthogonal to these continuum states  $\{|E\rangle\}$ , that  $V$  has only the matrix element  $\langle E|V|\phi\rangle$  (and its conjugate) and that the total  $H = H_0 + V$  has only absolutely continuous spectrum (it can easily be proved that there is no eigenstate (Horwitz 1971)). From the identity (7.20), we may solve for the amplitude (for  $H$  in place of  $K$ ) (7.61). Taking the expectation value of the resolvent equation (7.20) in the state  $\phi$ , one obtains

$$\langle \phi, G(z)\phi \rangle = + \frac{1}{z - E_0} \int dE' \langle \phi|V|E'\rangle \langle E'|G(z)|\phi \rangle \quad (7.106)$$

To obtain an evaluation for  $\langle E'|G|\phi\rangle$ , we now compute

$$\langle E'|G(z)|\phi \rangle = \frac{1}{z - E'} \langle E'|V|\phi \rangle \langle \phi, G(z)\phi \rangle \quad (7.107)$$

This result illustrates the essential point of the Lee-Friedrichs model; since  $V$  does not connect to any other states (rank one), the system is soluble. Substituting (7.107) into (7.106), we obtain the closed solution

$$\langle \phi, G(z)\phi \rangle = \frac{1}{z - E_0} + \frac{1}{z - E_0} \int dE' \frac{|\langle E'|V|\phi \rangle|^2}{z - E'},$$

or

$$(z - E_0 - \int dE' \frac{|\langle E'|V|\phi \rangle|^2}{z - E'}) \langle \phi, G(z)\phi \rangle = 1 \quad (7.108)$$

We see from this result that the complex poles discussed in connection with the weight function Eq. (7.63) have an explicit realization in this model in terms of the zeros of the function

$$h(z) = (z - E_0 - \int dE' \frac{|\langle E'|V|\phi \rangle|^2}{z - E'}), \quad (7.109)$$

It is easy to check that there are no zeros of this function in the first Riemann sheet; continuing through the cut to the second sheet by the method described after (7.63); one finds that there is indeed a possibility for a pole (there is a point where at least the imaginary part can vanish), and the discussion there can be applied explicitly to this model. One sees in this context that the pole does *not* correspond to a physical state in the Hilbert space. The inverse Laplace transform of the wave function contains a contribution due to this pole, but involves an analytic continuation. Sigal and Horwitz (1978) have shown (see also Baumgartel 1976, and Bohm and Gadella 1989) that, as mentioned above, one can construct a function in a Banach space (with explicit use of this model) which is an eigenfunction of the Hamiltonian with complex eigenvalue, but the result is of limited value in embedding the resonance into a quantum theory.

We now return to the development of the covariant relativistic Lax-Phillips theory, following similar procedures.

In the form (7.104) it is clear that both  $K$  and  $K_0$  have a direct integral structure, and therefore the corresponding wave operators  $\Omega_{\pm}^p$  have as well. In this sense, from the expression for  $K_0^p$  we see that  $|V(p)\rangle \equiv b^\dagger(p)|0\rangle$  is a discrete eigenstate of  $K_0^p$ , and is therefore annihilated by  $\Omega_{\pm}^p$ ; it can be shown, in fact, that  $\Omega_{\pm}|V(p)\rangle = 0$  for any  $p$ .

We now construct the Lax-Phillips incoming and outgoing spectral representations for this problem, and discuss the properties of the resonant states. In accordance with the discussion following (7.81), in order to construct the wave operators, we must obtain solutions for (7.81), the unperturbed problem. The complete set of such states may be decomposed into two subsets corresponding to quantum numbers for states containing  $N$  and  $\theta$  particles, which we denote by  $\alpha$  and those containing a  $V$  for which the quantum numbers are denoted by  $\beta$ . Then, the spectral representations are

$$\begin{aligned} |\sigma, \alpha\rangle &= \int d^4p \int d^4k |N(p), \theta(k)\rangle \langle N(p), \theta(k)|\sigma, \alpha\rangle \\ |\sigma, \beta\rangle &= \int d^4p |V(p)\rangle \langle V(p)|\sigma, \beta\rangle, \end{aligned} \quad (7.110)$$

where we define  $|N(p), \theta(k)\rangle \equiv a_N^\dagger(p)a_\theta^\dagger(k)|0\rangle$ , and  $|V(p)\rangle \equiv b^\dagger(p)|0\rangle$ . Therefore, since

$$\begin{aligned} K_0|\sigma, \alpha\rangle &= \omega_{N(p)} + \omega_{\theta(k)}|\sigma, \alpha\rangle = \sigma|\sigma, \alpha\rangle \\ K_0|\sigma, \beta\rangle &= \omega_{V(p)}|\sigma, \beta\rangle = \sigma|\sigma, \beta\rangle, \end{aligned}$$

where  $\omega_{N(p)} = \frac{p^2}{2M_N}$ ,  $\omega_{\theta(k)} = \frac{k^2}{2M_\theta}$  and  $\omega_{V(p)} = \frac{p^2}{2M_V}$ , we must have

$$\begin{aligned} \langle N(p)\theta(k)|\sigma, \alpha\rangle &\propto \delta(\sigma - \omega_{N(p)} - \omega_{\theta(k)}) \\ \langle V(p)|\sigma, \beta\rangle &\propto \delta(\sigma - \omega_{V(p)}) \end{aligned} \quad (7.111)$$

These matrix elements satisfy the requirements of orthogonality and completeness since they are unitary maps.

Due to the structure of the dynamics of this model given in (7.103) and (7.104), we may solve explicitly for the matrix elements of the wave operators (as for the



simple nonrelativistic model treated above) (Strauss 2000)

$$\langle V(p+k)|\Omega_+|N(p), \theta(k) \rangle = \langle N(p'), \theta(k')|\Omega_+|N(p), \theta(k) \rangle \quad (7.112)$$

We may apply the integral formula (7.27) for the wave operator to the states  $|N(p)\theta(k) \rangle$  to obtain

$$\Omega_+|N(p)\theta(k) \rangle = |N(p)\theta(k) \rangle - i \lim_{\epsilon \rightarrow 0} \int_0^{-\infty} e^{i(\omega_N(p) + \omega_\theta(k) - i\epsilon)\tau} U(\tau) f(k) b^\dagger(p+k)|0 \rangle . \quad (7.113)$$

To complete the evaluation of this integral, we must now find the evolution of the state  $b^\dagger(p)|0 \rangle$  under the evolution  $U(\tau)$ . The solution for this evolution involves very similar procedures to that outlined above for the nonrelativistic Lee-Friedrichs model, making use of the finite rank property of the interaction. In the sector of the Fock space that we are using, the state  $\psi_\tau$  at any time  $\tau$  can be represented as

$$\psi_\tau = \int d^4q A(q, \tau) b^\dagger(q)|0 \rangle + \int d^4p \int d^4k B(p, k, \tau) a^\dagger(p) a^\dagger(k)|0 \rangle . \quad (7.114)$$

Substituting this into the Stueckelberg-Schrödinger equation with the full Hamiltonian given by (7.102), one obtains

$$\begin{aligned} i \frac{\partial A(q, \tau)}{\partial \tau} &= \frac{q^2}{2M_V} + \int d^4k f(k) B(q-k, k, \tau) \\ i \frac{\partial B(p, k, \tau)}{\partial \tau} &= \left( \frac{p^2}{2M_V} + \frac{k^2}{2M_\theta} \right) B(p, k, \tau) + f^*(k) A(p+k, \tau). \end{aligned} \quad (7.115)$$

These equations reflect the solubility of the model, forming a closed system that can be solved by Laplace transform. Defining

$$\begin{aligned} \bar{A}(q, z) &= \int_0^{-\infty} e^{iz\tau} A(q, \tau) \quad \text{Im } z < 0 \\ \bar{B}(p, k, z) &= \int_0^{-\infty} e^{iz\tau} B(p, k, \tau) \quad \text{Im } z < 0, \end{aligned} \quad (7.116)$$

one obtains

$$\begin{aligned} \left( z - \frac{q^2}{2M_V} \right) \bar{A}(q, z) &= i A(q, 0) + \int d^4k f(k) \bar{B}(q-k, k, z) \\ \left( z - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right) \bar{B}(p, k, z) &= B(p, k, 0) + f^*(k) \bar{A}(p+k, z) \end{aligned} \quad (7.117)$$

the analog of (7.106) and (7.107). Using the initial conditions

$$B(p, k, 0) = 0, \quad A(q, 0) = f(k) \delta^4(q-p-k) \quad (7.118)$$

the Laplace transformed solutions become

$$\begin{aligned} \bar{A}(q, z) &= i \frac{A(q, 0)}{h(q, z)} \\ \bar{B}(p, k, z) &= i \left( z - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta} \right)^{-1} f^*(k) \frac{A(p+k, 0)}{h(p+k, z)}, \end{aligned} \quad (7.119)$$

where

$$h(q, z) = z - \frac{q^2}{2M_V} - \int d^4k \frac{|f(k)|^2}{z - \frac{(q-k)^2}{2M_N} - \frac{k^2}{2M_\theta}}. \quad (7.120)$$

These results are the analog of (7.108). With the Laplace transform of  $\psi_\tau$ , and the formula for the wave operator (7.113) we obtain the matrix elements of the wave operator (Strauss 2000)

$$\langle V(p')|\Omega_+|N(p), \theta(k) \rangle = \lim_{\epsilon \rightarrow 0} \delta^4(p' - p - k) f(k) h^{-1}(p', \omega - i\epsilon) \quad (7.121)$$

and

$$\begin{aligned} \langle N(p'), \theta(k')|\Omega_+|N(p), \theta(k) \rangle &= \delta^4(p' - p) \delta^4(k' - k) \\ &+ i \lim_{\epsilon \rightarrow 0} \left[ -i(\omega - i\epsilon - \frac{p'^2}{2M_N} - \frac{k'^2}{2M_\theta})^{-1} \right. \\ &\times \left. \frac{f^*(k')f(k)}{h(p+k, \omega - i\epsilon)} \right] \delta^4(p' + k' - p - k), \end{aligned} \quad (7.122)$$

where  $\omega = \omega_\theta + \omega_N$ .

According to our previous discussion, we are now in a position to calculate the transformation to the *outgoing* spectral representation (with quantum numbers  $\alpha$ )

$$\langle V(p)|\Omega_+|\sigma, \alpha \rangle_0 = \langle N(p), \theta(k)|\Omega_+|\sigma, \alpha \rangle_0. \quad (7.123)$$

The results can be expressed as

$$\langle V(p)|\Omega_+|\sigma, \alpha \rangle_0 = h^{-1}(p, \sigma - i\epsilon) |n \rangle_{p, \sigma}^\alpha \quad (7.124)$$

and

$$\begin{aligned} \langle N(p), \theta(k)|\Omega_+|\sigma, \alpha \rangle_0 &= \langle N(p), \theta(k)|\sigma, \alpha \rangle_0 + i \lim_{\epsilon \rightarrow 0} \left[ -i(\sigma - i\epsilon - \frac{p^2}{2M_N} - \frac{k^2}{2M_\theta})^{-1} \right. \\ &\times \left. f^*(k') h^{-1}(p' + k', \sigma - i\epsilon) |n \rangle_{p'+k', \sigma}^\alpha \right], \end{aligned} \quad (7.125)$$

where we have defined the vector valued (on  $\alpha$ ) function

$$|n \rangle_{p, \sigma}^\alpha \equiv \int d^4k f(k) \langle N(p-k)\theta(k)|\sigma, \alpha \rangle \quad (7.126)$$

Carrying out a similar calculation for  $\Omega_-$ , we can write an explicit formula for the Lax-Phillips  $S$ -matrix (Strauss 2000)

$${}_0 \langle \sigma', \alpha' | S | \sigma, \alpha \rangle_0 = \delta(\sigma' - \sigma) \delta_{\alpha', \alpha} - 2\pi i \int d^4p \frac{|n \rangle_{p, \sigma'}^{\alpha'} \langle n |_{p, \sigma}^\alpha}{h(p, \sigma + i\epsilon)}, \quad (7.127)$$

or, in matrix notation (on  $\alpha', \alpha$ )

$$S(\sigma) = 1 - 2\pi i \int d^4p \frac{|n \rangle_{p, \sigma}^{\alpha'} \langle n |_{p, \sigma}^\alpha}{h(p, \sigma + i\epsilon)} \quad (7.128)$$

This completes our expression for the Lax-Phillips  $S$ -matrix for the relativistic Lee model (Strauss 2000). The variables of the auxiliary space are treated as a reduced two body system as in Chap. 5, in terms of a direct integral over the total energy momentum  $P^\mu$ ; separating out the variables  $\alpha = \gamma, P$  for the complete set for the auxiliary space, we may write for the corresponding vector valued functions  $|n \rangle_{\sigma, P}^\gamma$

the state corresponding to that of the reduced motion with conservation expressed by  $\delta(\sigma - P^2/2M - p_{rel}^2/2m)$ . The square of the relative momentum is then determined by  $P$  and  $\sigma$ , and we can write the  $S$  matrix entering the direct integral over  $P$  as

$$S_P(\sigma) = 1 - 2\pi i \int d^4 p \frac{|n >_{P,\sigma} < n|_{P,\sigma}}{h(P, \sigma + i\epsilon)} \quad (7.129)$$

It is proved in Strauss (2000) that for  $|n >_P$  a normalized basis vector, we must have the form

$$|n >_{\sigma,P} = g_P(\sigma)|n >_P, \quad (7.130)$$

and that the  $S$ -matrix must commute with the projection operator formed from these states.

Assuming that in the relativistic Lee model there is just one pole in the lower half plane  $\mu_P$ , corresponding to a single resonance, then the  $S$ -matrix takes on the form

$$S_P(\sigma) = S'_{(P)}(\sigma)M_P(\sigma), \quad (7.131)$$

where the resonant poles are carried by the matrix  $S'$ . If the function  $M_P(\sigma)$  is of bounded exponential growth, i.e.  $\ln |M_P(\sigma)|$  is bounded by  $|Im \sigma|$ , then there is an equivalence transformation that can bring the  $S$ -matrix to the form

$$S'_P(\sigma) = 1_{H,\sigma} - |n >_P \langle P < n| + \frac{\sigma - \bar{\mu}_P}{\sigma - \mu_P} |n >_P \langle P < n|. \quad (7.132)$$

The residue of the pole is a projection operator into a state in the auxiliary Hilbert space, identified here as the *state of the resonance*. In terms of the definition of the states of the Lee model (7.126), we see that the resonance becomes a proper state in the auxiliary Hilbert space of the Lax-Phillips theory.<sup>8</sup> Moreover, if there are two or more poles, it is a consequence of the proof that the evolution law is an exact semigroup that the pole residues are orthogonal, corresponding to orthogonal subspaces contained in  $\mathcal{K}$ , the complement of  $\mathcal{D}_\pm$  in the Lax-Phillips Hilbert space. We remark that the resonance pole, associated with the center of mass momentum of the two body system (in general of the final state) may be finitely spread out according to the construction of the normalizable wave packet (Strauss 2000), and the corresponding bilinear form of the residue would then correspond to a mixed state over this small interval.

This result, not achievable in the Wigner-Weisskopf approach to the description of resonances, is made possible by the foliation admitted by the Lax-Phillips theory imbedded in a natural way into the relativistic quantum theory.

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<sup>8</sup>See Strauss and Horwitz (2000) for a detailed discussion of the  $(N, \theta, V)$  particle content of the resonance.

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# Some Applications: The Electron Anomalous Moment, Invariant Berry Phases and the Spacetime Lattice

# 8

In this chapter we describe three important applications of the theory.

In the first section, we discuss the application of the Stueckelberg theory to the calculation of the anomalous moment of the electron (Bennett 2012). The original work of Schwinger (1951), and many later treatments (Itzykson 1980) use the standard formalism of quantum field theory. We show here, following (Bennett 2012), that the results can be obtained, to lowest order, in the framework of the relativistic quantum mechanics that we have developed here, without the necessity of second quantization.

In the second section we discuss the general formulation of Berry phases, the response of a wave function in the quantum theory to a cyclic adiabatic variation of parameters of the Hamiltonian, resulting in a phase when the parameters return to their original value. The basic theory was developed by Berry (1984) using the nonrelativistic quantum theory. Since the Stueckelberg quantum theory has the same structure as the nonrelativistic quantum theory, represented in a well-defined Hilbert space, one can calculate the Berry phases in a similar way (Bachar 2014). We show an example of a perturbed four dimensional harmonic oscillator, of the type considered by Feynman et al. (1971), and Kim and Noz (1977), discussed in some detail in Chap. 5 here, and show that the associated Berry phases are Lorentz invariant, and are therefore an intrinsic property of the relativistic dynamical system.

In the third section, we introduce the idea of a spacetime lattice (Engelberg 2009) and the corresponding Bloch waves for a periodic potential distribution in space and time. The example that we treat is that of an electromagnetic standing wave in a cavity. The corresponding solution of the Schrödinger-Stueckelberg equation is that of Bloch type waves in space and time with associated mass (energy) gaps which appear to be observable in the laboratory.

## 8.1 The Anomalous Moment of the Electron

In Schwinger's original calculation (Schwinger 1951) in the framework of quantum electrodynamics, the interacting Green's function containing the  $J^\mu A_\mu$  interaction in terms of which the anomalous moment is computed is represented by the matrix element,

$$G(x, x') = \langle x | G | x' \rangle, \quad (8.1)$$

where the operator  $G$  is given by

$$G = \frac{1}{\gamma \Pi + m} = i \int_0^\infty ds e^{-i(\gamma \Pi + m)s}. \quad (8.2)$$

In this expression,  $x, x'$  are in the spectrum of the operator valued spacetime variables  $\{x^\mu\}$ ,  $s$  is the so-called "proper time",  $\Pi = p - eA$ , and  $\gamma$  are the Dirac matrices. In this "proper time" formalism, Schwinger calculates the unitary evolution (see remark below on Hermiticity of  $\mathcal{H}$ )

$$U(s) = e^{-i\mathcal{H}s}, \quad (8.3)$$

where

$$\mathcal{H} = -(\gamma \Pi)^2 = \Pi_\mu \Pi^\mu - \frac{1}{2} e \sigma_{\mu\nu} F^{\mu\nu}, \quad (8.4)$$

up to a constant, the squared Dirac operator whose spectrum determines the structure of the Green's function. The unitary kernel

$$\langle x' | U(s) | x'' \rangle = \langle x', s | x'', 0 \rangle \quad (8.5)$$

describes, as stated by Schwinger, the development of a system governed by the "Hamiltonian"  $\mathcal{H}$  in the "time"  $s$ , the matrix element of  $U(s)$  being the transformation function from a state in which  $x$  at  $s = 0$  had the value  $x''$  to a state in which this operator has the value  $x'$  at "time"  $s$ .<sup>1</sup> The structure defined here by Schwinger, for which the "proper time"  $s$  provides a formal structure for computations in quantum field theory which is similar to that of the nonrelativistic quantum theory, and making methods of perturbative analysis used in the nonrelativistic theory available, is clearly recognizable as the content of the Stueckelberg theory that we have discussed in previous chapters (Schwinger refers to the work of Fock (1937), a principle reference of Stueckelberg, and to Nambu (1950) who explains the approach of Feynman (1949) in his paper on spacetime diagrams in these terms as well).

Schwinger then proceeds to compute the quantity of primary interest in achieving the result for the anomalous moment,

$$\langle j_\mu(x) \rangle = i \text{etr} \gamma_\mu G(x, x') |_{x' \rightarrow x}, \quad (8.6)$$

<sup>1</sup>Note that the operator  $\mathcal{H}$  defined in (8.4) is not Hermitian due to the presence of the electric term in the interaction [see Chap. 3]; Schwinger takes the electric field to be zero, avoiding this difficulty. However, the formalism developed in Chap. 3 using the induced representation, with scalar product given by (8.7), as used by Bennett in the calculation we shall describe here, is valid in full generality for the electromagnetic-spin interaction.

where  $tr$  is the spinor trace. We do not describe here the details of this remarkable and rich paper of Schwinger. The structure of the calculation, however, involving the computation of an effective mass shift, strongly suggests that the Stueckelberg theory, in the spin formalism that we have developed in Chap. 5, should be capable of achieving the value of the anomalous moment *without the explicit use of quantum field theory*. Bennett (2012) has, in fact, achieved this result and we explain here the procedure that he applied.<sup>2</sup>

To follow the convention used by Bennett we take the scalar product of the spinor valued wave function in Eq. (3.24) with the minus sign, so that  $-\gamma^0 n^\mu \gamma_\mu$  is a positive Hermitian spinorial operator, clearly seen when in the “rest” frame  $n^\mu$  has the form  $\hat{n}^\mu = (+1, 0, 0, 0)$  (we use the Lorentz metric  $(-1, +1, +1, +1)$  throughout), and define  $\circ \equiv -\gamma^0 n^\mu \gamma_\mu$ . The scalar product then reads

$$\langle \psi | \phi \rangle_n = \int d^4x \psi(x)^\dagger \circ \phi(x) \quad (8.7)$$

The corresponding (positive definite) norm,  $\langle \psi | \psi \rangle_n$ , as for the scalar wave functions of the SHL theory, has the meaning of the probability density for finding an event at world time  $\tau$  in the sector labelled by  $n$ . The wave functions are taken to satisfy the Schrödinger equation

$$i \frac{\partial}{\partial \tau} \psi_n = K_n \psi_n \quad (8.8)$$

where  $K_n$  has the form (3.32).<sup>3</sup> We furthermore take the free particle wave function to have the Dirac type form

$$\phi(x, \tau, n, p, s) = \left(\frac{1}{2\pi}\right)^2 u(n, p, s) \exp(ip \cdot x - \frac{p^2}{2M} \tau), \quad (8.9)$$

where  $s^\mu$  is the spin polarization (the vector  $(0, \hat{s})$  in the “rest” frame). The four-spinor amplitude may be expressed in terms of eigenspinors of the projection operators  $(1 \mp n^\mu \gamma_\nu)/2$  and  $(1 \mp \gamma^5 s^\mu \gamma_\mu)/2$ .

It follows from the results of Chap. 7 that the first order scattering matrix for a spin 1/2 charged particle by an external potential  $A(x)$  is

$$S_{fi}^{(1)} = -i \int d\tau \int d^4x \phi_f^\dagger(x, \tau) \circ \mathcal{V}(x) \phi_i(x, \tau), \quad (8.10)$$

where, with notation from Chap. 3,

$$\mathcal{V} = -\frac{e}{M} A \cdot p - \frac{e}{2M} F \cdot \Sigma_n. \quad (8.11)$$

<sup>2</sup>We thank Cecille DeWitt for her encouragement for the study of this problem.

<sup>3</sup>We do not use here the full generality of the form (4.55) with  $g$  replaced by  $e'$  as in (4.9); the restricted form of  $K$  given here is adequate for gauge invariance under the 5D analog of the Hamilton gauge for which gauge transformations are  $\tau$  independent and the fields may be taken as  $\tau$  independent Maxwell type fields.

It is sufficient for our purposes here to consider only the spin interaction, i.e., take for the interaction

$$\mathcal{V}_s = -\frac{e}{2M} F \cdot \Sigma_n. \quad (8.12)$$

We now insert the free wave functions (8.9) into the formula (8.10) to obtain

$$S_{fi}^{(1)} = -i(2\pi)^{-3} \delta(\Delta\omega) u_f^\dagger \circ \mathcal{V}_s(\Delta p) u_i, \quad (8.13)$$

where  $\Delta\omega = (p_f^2 - p_i^2)/2M$ , and

$$\mathcal{V}_s(p) = \int d^4x \mathcal{V}_s(x) \exp(-ip \cdot x),$$

the 4D Fourier transform. Note that scattering from a  $\tau$ -independent potential does not affect the rest mass of the free particle.

We now wish to calculate the next order correction to the scattering matrix. For this, we consider

$$S_{fi}^{(2)} = -i \int d\tau \int d^4x \int d\sigma \int d^4y \phi_f^\dagger(x, \tau) \circ \mathcal{V}_s(x) G_0(x-y, \tau-\sigma) V_c(y) \phi_i(y, \sigma), \quad (8.14)$$

where

$$V_c = -\frac{e}{m} A \cdot p \quad (8.15)$$

is the convective part of the interaction appearing in (3.32), where the source of  $A$  is the particle itself, and

$$G_0(x' - x, \tau' - \tau) = \frac{1}{\pi^5} \int d\chi \int d^4q \frac{\exp i[q \cdot (x' - x) - \chi(\tau' - \tau)]}{\chi - \frac{q^2}{2m} + i\epsilon} \quad (8.16)$$

is the retarded free particle propagator ( $0 < \epsilon \ll 1$ ). Closing the contour in the complex  $\chi$  plane, below the real axis for  $\tau' > \tau$ , yields

$$G_0(x' - x, \tau' - \tau) = -\frac{i}{(2\pi)^4} \int d^4q \exp i[q \cdot (x' - x) - \frac{q^2}{2M}(\tau' - \tau)] \quad (8.17)$$

The potential  $A$  obeys

$$-\partial_\mu \partial^\mu A^\lambda = e \int (j_c^\lambda(x, \tau) + j_s^\lambda(x, \tau)) d\tau, \quad (8.18)$$

the appropriately concatenated event currents (see (4.24)), where

$$j_c^\lambda(x, \tau) = \phi_f(x, \tau)^\dagger \circ \left(\frac{\bar{p}}{M}\right) \phi_i(x, \tau), \quad (8.19)$$

is the convection current, and  $j_s$  is the spin current (Saad 1989)

$$j_s^\lambda(x, \tau) = -i \left(\frac{\Delta p_\nu}{M}\right) \phi_f(x, \tau)^\dagger \circ \Sigma_n^{\lambda\nu} \phi_i(x, \tau), \quad (8.20)$$

where  $\bar{p} = (p_f + p_i)/2$ , and  $\Delta p = p_f - p_i$ . Neither the spin current  $j_s^\lambda$  nor the spin interaction term in (8.13) need be considered for weak scattering ( $\Delta p \rightarrow 0$ ). Then, (8.18) and (8.19) imply that

$$A^\mu(x) = e \int d\zeta \int d^4z D_F(x-z) \phi_f^\dagger(z, \zeta) \circ \left(\frac{\bar{p}}{M}\right) \phi_i(z, \zeta), \quad (8.21)$$

where  $D_F$  is the forward in time photon propagator Bjorken (1964, Sect. 7.4)

Now, we rearrange (8.14) as in (Bjorken 1964, Sect. 7.5) with the help of

$$\phi_i(z, \zeta) \phi_f^\dagger(x, \tau) \rightarrow i G_0(z-x, \zeta-\tau) \quad (8.22)$$

implied by the form of the free wave and (8.17). Since the incident and final monochromatic plane wave functions are, in practice, normalized beams of finite bandwidth, this expression is justified. Taking into account the orthonormality of the spinor basis, (8.22) becomes exact as  $\Delta p \rightarrow 0$ . Note that the substitution on the quantum mechanical level (8.22) has an analogue in quantum field theory (Weinberg 1995), where the free fermion propagator is a two event correlation. With (8.21) and (8.22), one obtains the *vertex correction*

$$S_{fi}^{(2)} = + \frac{e^2 \bar{p}^2}{M^2} \int d\tau \int d^4x \int d\sigma \int d^4y \int d\zeta \int d^4z \quad (8.23)$$

$$\times \phi_f^\dagger(z, \zeta) G_0(z-x, \zeta-\tau) \circ \mathcal{V}_s(x) G_0(x-y, \tau-\sigma) \phi_i(y, \sigma) D_F(y-z)$$

For the case of the scattering of two electrons, with an interaction that includes both spin and convection terms, there would be a pair of incident and final lines forming a box diagram which converts to a vertex correction by an exchange of vertices for two of the electron lines (Bjorken 1964, Sect. 8.1). Fermi-Dirac statistics then requires a sign change so that the additive contribution of the two diagrams is antisymmetric, resulting in a sign replacement consistent with the sign of (8.23). Bennett points out, at the time of his writing, that the difficulty in making use of this idea in this context emphasizes the absence of a spin-statistics theorem in relativistic quantum mechanics; however, as explained in Chap. 3 here, it indeed follows from the theoretical framework of the relativistic quantum theory, based on induced representations for particles with spin, that there is a spin statistics theorem for relativistic many-body systems consistent with the resulting sign of (8.23).

Fourier transforming the above result reduces (8.23) to

$$S_{fi}^{(2)} = S_{fi}^{(1)} \times (ie^2 \bar{p}^2 / M^2) \times I_{VC} \quad (8.24)$$

as  $\Delta p \rightarrow 0$ , where the vertex correction factor is given by

$$I_{VC}(\bar{p}, M) = \frac{1}{2\pi)^4} \int d^4k \frac{4M^2}{(2M\bar{\omega} - (k + \bar{p})^2 + i\epsilon)^2} \frac{1}{k^2 - i\epsilon}. \quad (8.25)$$

The Fourier transforms of the propagators in (8.24) on  $\tau$  and  $\zeta$  are to be taken only for subluminal  $\bar{\omega}_{i,f} < 0$ . The integrand in (8.25) is  $O(k^3 dk/k^6)$  at high wavenumber, and is therefore convergent, but  $O(k^3 dk/k^4)$ , and is therefore logarithmically



divergent at low wavenumber. Evaluating the integral in the usual way described by, for example, Bjorken and Drell (1964) and Zee (Weinberg 1995), one obtains

$$S_{fi}^{(2)} = S_{fi}^{(1)} \frac{\alpha_R}{2\pi} \quad (8.26)$$

where the renormalized fine structure constant is

$$\alpha_R = \frac{e_R^2}{4\pi} = \frac{e^2}{4\pi} (1 - \ln \kappa^2), \quad (8.27)$$

with  $0 < \kappa \ll 1$ .

Recalling the spin dependent perturbation (8.12) (and the form (8.13)), we infer that the  $g$ -factor of the electron is

$$g = 2\left(1 + \frac{\alpha_R}{2\pi}\right), \quad (8.28)$$

Recalling the spin dependent perturbation (8.12) (and the form (8.13)), we infer that the  $g$ -factor of the electron is

$$g = 2\left(1 + \frac{\alpha_R}{2\pi}\right), \quad (8.29)$$

This remarkable result of quantum electrodynamics (Schwinger 1951) is obtained here by Bennett (2012) entirely in the framework of the relativistic quantum mechanics of SHP.

Note that this result does not depend on the unspecified Galilean target mass  $M$  or on the incident and final squared rest masses  $-p_i^2$ ,  $-p_f^2$ , both close to  $-\bar{p}^2$ . At the time of this writing, studies are being carried out of higher order corrections in this framework, as well as other observable effects that have been previously derived using quantum electrodynamics.

The notion of renormalization in quantum field theory (explained well in Bogliubov 1959) is a procedure for extracting physical meaningful quantities from poorly conditioned mathematical formulas arising from divergent integrals in the theory. Such integrals arise due to “closed loops” in the Feynman diagrams representing higher order effects in the process described. In terms of the Stueckelberg theory, as well as for quantum field theory, these contributions can be understood as due to the occurrence of particle-antiparticle pairs. In the presence of interaction, there is no constraint on world lines to maintain monotonic behavior (in spacetime) even classically, and the quantum theory allows still more freedom for the formation of such virtual configurations. As in the calculation given above, poorly defined integrals are often (such “renormalizability” is characteristic of gauge theories, such as electromagnetism) multiplied by a factor that can cancel the singularity, providing a finite result. However, the result is generally expressed in terms of a mass or coupling constant that differs from the numbers originally entered into the fundamental equations of motion. The physical values of mass and coupling constant that emerge from such calculations then acquire values that are accessible to experiment through these processes; the contribution of the “loops”, which do not contribute to physical processes are then removed. A simple example of such “renormalization” is described in Fetter (1971) where the nonrelativistic many body theory is used to

compute energy shifts and corrected wave functions by eliminating “disconnected” diagrams. The idea is that a theory which admits such fluctuations can result in modified values of the measured coupling and masses; for example, electron-positron pair annihilation and recombination is a process that can be polarized in an electric field and shield the core charge to some extent. It is clear that in the context of the Stueckelberg quantum theory, such virtual processes are admissible and may find a systematic description that has not yet been fully developed.

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## 8.2 Invariant Berry Phases

In 1983, Berry (1984) discovered that for an adiabatically varying Hamiltonian, returning to its initial state after a time  $T$ , the wave function acquires a *geometric phase* in addition to the dynamical phase associated with the eigenvalues of the Hamiltonian. Such a phase (known in the mathematical literature as a Maslov index (Maslov 1976)) plays an important role in the physics of adiabatically perturbed systems. The Aharonov-Bohm effect may be considered as associated with such a phase, along with other applications in quantum optics. It has been pointed out by Sternberg and Kostant (1987), Shapere and Wilczek (1989) and Jackiw (1988) (see also Biedenharn 1987) that the anomaly of quantum field theory can be understood in these terms. The well-known property in Einstein’s theory of gravitation, for which the parallel transport of a vector around a closed path in spacetime, for which it does not recover its original value in the presence of a gravitational field (Weinberg 1972) may be thought of as lying in this category as well. Berry’s result is obtained by assuming that there is a Hamiltonian undergoing an adiabatic variation controlled by a set of parameters. We show in this section that the Stueckelberg theory, providing a Hamiltonian and a symplectic dynamics, admits the existence of Berry phases as well, and that, in particular, for interaction at a distance theories (analogous to potential theories of the nonrelativistic mechanics), investigated in some detail in previous chapters, as well as for gauge interactions in the framework of the induced representations discussed in Chap. 3, the resulting Berry phases are Lorentz invariant.

We begin by reviewing Berry’s original derivation for the nonrelativistic Schrödinger theory, and generalize the procedure to the relativistic case, constructing, as an example, the Berry phases associated with the perturbed four dimensional covariant harmonic oscillator.

Consider a Hamiltonian  $H(\mathbf{R})$ , where

$$\mathbf{R} = R_1, R_2, \dots, R_m$$

are  $m$  parameters depending explicitly on the time  $t$ . These parameters are to follow a closed path in the parameter space as functions of  $t$  such that  $\mathbf{R}(T) = \mathbf{R}(0)$  for  $T$  necessarily large so that the process may be considered to be adiabatic. In the adiabatic approximation, the Schrödinger equation can be studied in terms of the “stationary” eigenstates at each  $t$ , i.e.,

$$H(\mathbf{R}(t))|\psi_n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))|\psi_n(\mathbf{R}(t))\rangle, \quad (8.30)$$

where  $n$  labels the eigenstates, and the parameters  $\mathbf{R}(t)$  are slowly varying (relative to  $\frac{\hbar}{\Delta E}$ ) functions of time. Under these adiabatic conditions, the evolving system remains in the  $n$ th eigenstate for each  $t$ . The solutions (8.30), due to the time dependence in  $H$ , are not precise solutions of the time dependent Schrödinger equation, and it may be expected that there is a deviation from the stationary form with purely dynamical phase factor  $\exp(-iEt)$ . We therefore write the solution as

$$|\psi(t)\rangle = \exp\left\{-\frac{i}{\hbar} \int_0^t dt' E_n(\mathbf{R}(t'))\right\} e^{i\gamma_n(t)} |\psi_n(\mathbf{R}(t))\rangle \quad (8.31)$$

Berry (1984) points out that the phase  $\gamma_n(t)$  cannot be written as a function of  $\mathbf{R}$ , and does not, in general, return to its original value after the closed circuit from  $t = 0$  to  $t = T$ . Substituting the form (8.31) into the Schrödinger equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H(\mathbf{R}(t)) |\psi(t)\rangle, \quad (8.32)$$

one finds that

$$\dot{\gamma}(t) = i \langle \psi_n(\mathbf{R}(t)) | \nabla_{\mathbf{R}} | \psi_n(\mathbf{R}(t)) \rangle \cdot \dot{\mathbf{R}}(t). \quad (8.33)$$

The total phase *change* around the closed path which occurs in

$$|\psi(T)\rangle = \exp\left\{-\frac{i}{\hbar} \int_0^T dt' E_n(\mathbf{R}(t'))\right\} e^{i\gamma_n(C)} |\psi_n(\mathbf{R}(T))\rangle, \quad (8.34)$$

where

$$\gamma_n(C) = i \oint \langle \psi_n(\mathbf{R}) | \nabla_{\mathbf{R}} | \psi_n(\mathbf{R}) \rangle \cdot d\mathbf{R}, \quad (8.35)$$

is independent of how the closed path is traversed.

We can see immediately that  $\gamma_n(C)$  is real, since integrating by parts over the closed path, one finds that

$$\gamma_n(C) = -i \oint \langle \psi_n(\mathbf{R}) | \nabla_{\mathbf{R}} | \psi_n(\mathbf{R}) \rangle^* \cdot d\mathbf{R} = \gamma_n(C)^*. \quad (8.36)$$

This remarkable result, a *geometric* phase, has been found to play an important and fundamental role in many areas of physics, as mentioned above; it is called the *Berry phase*.

The generalization of Berry's argument to the relativistic domain is straightforward in the framework of the SHP quantum theory, and has been done by Bachar et al. (2014), where it was applied to the four dimensional covariant oscillator with  $\tau$  dependent (adiabatic) perturbation. The bound state oscillator wavefunctions are known from the work of Arshansky and Horwitz (1989), and a carefully chosen perturbation is added that breaks the azimuthal symmetry, necessarily with a fractional coefficient for the angular dependence to achieve a nonzero Berry phase. This result is described in the following.

The unperturbed oscillator Hamiltonian for the oscillator in the RMS is discussed in Chap. 5, along with its bound state solutions. We take for the perturbed (reduced) Hamiltonian

$$K = \frac{1}{2m} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda}{\rho^2} \right]$$

$$\begin{aligned}
& + \varepsilon_1 \rho^2 \sin^2 \theta \cos^2 \left( \frac{2}{3} \phi \right) \cosh^2 \beta \\
& + \varepsilon_2 \rho^2 \sin^2 \theta \sin^2 \left( \frac{2}{3} \phi \right) \cosh^2 \beta \\
& + \varepsilon_3 \rho^2 \cos^2 \theta - \varepsilon_0 \rho^2 \sin^2 \theta \sinh^2 \beta,
\end{aligned} \tag{8.37}$$

where we have chosen, in our illustrative example, a fractional coefficient for the  $\phi$ -dependence in order to assure that the computed Berry phase to be nontrivial. This requirement arises technically from the formula involving the scalar product for the relativistic wave functions, as we shall see below.

The unperturbed wave functions are given by Arshansky (1989) and Bachar (2014)

$$\begin{aligned}
\psi_{n_a \ell n m}(\phi, \beta, \theta, \rho) &= \frac{1}{2\pi} e^{i(m + \frac{1}{2}\phi)} \sqrt{n} \sqrt{\frac{\Gamma(1+m+n)}{\Gamma(1+m-n)}} \\
&\times (1 - \tanh^2 \beta)^{\frac{1}{4}} P_m^{-n}(\tanh \beta) (1 - \cos^2 \theta)^{-\frac{1}{4}} \\
&\times P_\ell^n(\cos \theta) \frac{1}{\sqrt{\rho}} \left( \frac{m\omega\rho^2}{\hbar} \right)^{\frac{1}{2}} e^{-\frac{m\omega\rho^2}{2\hbar}} L_{n_a}^{\ell + \frac{1}{2}} \left( \frac{m\omega\rho^2}{\hbar} \right).
\end{aligned} \tag{8.38}$$

Application of standard first order perturbation theory gives the first order correction to the wave function as

$$\psi_{n'_a \ell' n' m'}^{(1)} = \sum_{n_a, \ell \neq n'_a, \ell'} \frac{\langle \psi_{n_a \ell n m} | V | \psi_{n'_a \ell' n' m'} \rangle}{K'_a - K_a} \psi_{n_a \ell n m}, \tag{8.39}$$

where  $V$  is the perturbation given in (8.37) and  $K_a$  is the eigenvalue for the unperturbed Hamiltonian

$$K_a = \hbar\omega \left( 1 + 2n_a + \frac{3}{2} \right). \tag{8.40}$$

Now, suppose that  $\varepsilon_0$  and  $\varepsilon_3$  are taken to be zero; then

$$\psi_{n'_a \ell' n' m'}^{(1)} = \varepsilon_1 \psi'_{n'_a \ell' n' m'} + \varepsilon_2 \psi''_{n'_a \ell' n' m'}, \tag{8.41}$$

where  $\psi'$  and  $\psi''$  contain just the parts of  $V$  proportional to  $\varepsilon_1$  and  $\varepsilon_2$ , i.e.

$$V' = \rho^2 \sin^2 \theta \cos^2 \left( \frac{2}{3} \phi \right) \cosh^2 \beta \tag{8.42}$$

and

$$V'' = \rho^2 \sin^2 \theta \sin^2 \left( \frac{2}{3} \phi \right) \cosh^2 \beta. \tag{8.43}$$

The matrix elements contain the factor

$$\int_0^{2\pi} \pi \exp i(m - m' + \eta)\phi$$

corresponding to the phase factor of the Legendre functions in the scalar product, where  $\eta$  carries the  $\phi$  dependence in the perturbation. For  $\eta$  integer, a Kronecker delta appears; since the remaining integrals are real, there would be no Berry phase. With the choice of perturbation for which  $\eta = \pm 4\pi/3$ , the integral contains a term

with phase  $8\pi/3$ , an angle of  $120^\circ$ , thus contributing an imaginary part which results, according to (8.35), in a nontrivial Berry phase. Any fractional coefficient on  $\phi$  in the perturbation which is not equal to an integer divided by 4 will result in a nontrivial Berry phase (the choice of  $2/3$  was made for simplicity).

We may write the perturbed wave function for the  $j$ th level as (the index  $j$  corresponds to the set of quantum numbers  $\{n_a, \ell, n, m\}$ ) in the form

$$\Psi_j = \psi_j + \psi_j^{(1)} = \psi_j + \varepsilon_1 \psi_j^{(1)} + \varepsilon_2 \psi_j^{(2)}, \quad (8.44)$$

Let us define the parameter vector in the  $\varepsilon_1, \varepsilon_2$  plane as

$$\mathbf{R} = r\{\cos \alpha \hat{\varepsilon}_1 + \sin \alpha \hat{\varepsilon}_2\}, \quad (8.45)$$

where  $\hat{\varepsilon}_1$  and  $\hat{\varepsilon}_2$  are unit vectors along two orthogonal axes. Then,

$$d\mathbf{R} = r\{-\sin \alpha \hat{\varepsilon}_1 + \cos \alpha \hat{\varepsilon}_2\}d\alpha \quad (8.46)$$

We may now compute the scalar product corresponding to (8.35) defining the Berry phase

$$\langle \Psi_j | \nabla_R \Psi_j \rangle = \langle \psi_j + \varepsilon_1 \psi_j^{(1)} + \varepsilon_2 \psi_j^{(2)} | \psi_j' \hat{\varepsilon}_1 + \psi_j'' \hat{\varepsilon}_2 \rangle. \quad (8.47)$$

Only terms in  $\varepsilon_1 d\varepsilon_2$  and  $\varepsilon_2 d\varepsilon_1$  can contribute to the integral, resulting in

$$\gamma_j = -2\pi \text{Im} \langle \psi_j^{(1)} | \psi_j^{(2)} \rangle. \quad (8.48)$$

For small  $r$  and reasonable values of  $\omega$ , and the electron mass, measurable values of the Berry phase for the perturbed oscillator can be achieved.

It is clear that the results we have achieved here are relativistically invariant. The calculation takes place in a given RMS, say, oriented along the  $z$  axis. Under a Lorentz transformation, the frame becomes oriented along a new  $z$ -axis, but all the variables entering into the calculation can be defined in the new frame in the same way, *including the form of the perturbation*. The moving observer sees the physical system in terms of the transformed variables. These variables are related to the variables of the original frame by Lorentz transformation, and under the integration in the scalar product, a change of variables (for which the measure is invariant) leaves the integrals in the form that they had in the original frame (isomorphic for any  $m^\mu$ ), since we are in an induced representation, there is, in addition, an operation corresponding to the Wigner rotation on the wave functions, but this unitary action cancels out as well in the matrix element. Therefore the results are Lorentz *invariant*.

It seems suggestive that the technical necessity for choosing a perturbation with fractional  $\phi$  dependence would have a topological significance, not yet fully understood.

It would be also be of interest to investigate the relation of these phases with the applications to particle physics as envisioned by Leutwyler and Stern (1977).

### 8.3 The Spacetime Lattice

As explained in previous chapters, the Stueckelberg wave function is coherent over a finite range of space and time, accounting for the results, for example, of the Lindner experiment as discussed in Chap. 6. Since the time and space variables enter on a similar level, it is interesting to think of a periodic *spacetime lattice* analogous to a space lattice for which the nonrelativistic theory has solutions in the form of Bloch waves, resulting in energy gaps that form the basis of much of solid state physics and its associated devices. In this section, we study the Bloch waves associated with a lattice constructed of an electromagnetic standing wave in a cavity, containing fields periodic in space as well as time (oscillations at each point). It has been shown by Engelberg and Horwitz (2009) that achievable configurations can lead to potentially observable phenomena associated with the resulting energy gaps, and we discuss this work in the following.

We consider here a Hamiltonian reflecting invariance under the Hamilton gauge alone, with  $\tau$  independent fields and no compensating field for the  $\tau$  derivative in the Stueckelberg-Schrödinger equation, i.e., the usual Maxwell fields, of the form

$$K = \frac{(p_\mu - eA_\mu)(p^\mu - eA^\mu)}{2M}, \quad (8.49)$$

as in Stueckelberg's original paper (Stueckelberg 1941). The numerator actually represents the (negative) mass squared of the particle, but since we shall be working with a system in which there is a particle entering the interaction region (cavity) from an asymptotically free region, for which the (negative) mass squared is represented by  $p_\mu p^\mu$ , we may treat the electromagnetic terms perturbatively. Therefore, we take, for the dynamical problem of the particle in the cavity, the Hamiltonian to be of the approximate form

$$K = \frac{p_\mu p^\mu}{2M} - \frac{e}{2M} \{A^\mu, p_\mu\} + \frac{e^2}{2M} A_\mu A^\mu, \quad (8.50)$$

where we consider the second and third terms as perturbations.

To utilize the periodic behavior of  $A_\mu$ , we now consider first order perturbation theory (e.g., Raimes 1961) for the unperturbed wave functions

$$\psi_k = \frac{1}{\sqrt{v}} e^{ik_\sigma x^\sigma}, \quad (8.51)$$

with *spacetime* vectors  $k^\sigma, x^\sigma$ . For the gauge invariance of our procedure, we must treat both the linear and quadratic terms in  $e$  at each order. The mass (squared) shifts due to the first order perturbation are obtained by calculating the eigenvalues of the perturbation matrix  $V = \{V_{k,k'}\}$ , with, for the term linear in  $e$ ,

$$V_{k,k'} = \int \psi_k^* \frac{ie\hbar}{M} A^\mu \frac{\partial}{\partial x^\mu} \psi_{k'} d^4x. \quad (8.52)$$

Assuming an electromagnetic wave of the form  $A^0 = A^2 = A^3 = 0$ , and  $A^1 = A \sin(\omega t) \cos(\omega t)$ , we obtain for the contribution of the term first order in  $e$

$$\begin{aligned} V_{k,k'} &= \int \psi_k^* \frac{ie\hbar}{M} A^\mu \frac{\partial}{\partial x^\mu} \psi_{k'} d^4x. \\ &= \frac{iek_x A \hbar}{4vM} \int e^{-iK_\sigma x^\sigma} \\ &\quad \cdot (e^{i(\omega_\gamma t + k_\gamma x)} + e^{i(\omega_\gamma t - k_\gamma x)} - e^{i(-\omega_\gamma t + k_\gamma x)} - e^{i(-\omega_\gamma t - k_\gamma x)}) d^4x, \end{aligned} \quad (8.53)$$

where

$$K_\sigma = k'_\sigma - k_\sigma. \quad (8.54)$$

We define the edge of a Brillouin zone as the collection of sets of degenerate states, with the same value of  $m^2 c^4 = E^2 - c^2 p_z^2$ , for which the distance between points in the  $(E, cp_z)$  space can be written as  $\hbar ck_\gamma (n_E, n_p)$  (regarded as a vector), for the lowest possible absolute values of the integers  $n_E$  and  $n_p$ . The occupancy of the edges of the Brillouin zones are given by filling in the possibilities for these lowest possible values. For the nonzero values of the matrix element of the term linear in  $e$ , with  $K_\sigma = (\frac{\omega_\gamma}{c}, 0, 0, \pm k_\gamma)$  or  $(-\frac{\omega_\gamma}{c}, 0, 0, \pm k_\gamma)$ , we can have a mass gap only along the edges of the second Brillouin zone; these points lie, however, entirely on the light cone, where it is not likely to find a massive charged particle.

The contribution of the second order term in  $e$ ,

$$V_{k,k'} = \frac{1}{v} \int \psi_k^* \frac{e^2}{2M} A_\mu A^\mu \psi_{k'} d^4x, \quad (8.55)$$

however, is

$$\begin{aligned} V_{k,k'} &= \frac{e^2 A^2}{8vM} \int e^{iK_\sigma x^\sigma} \\ &\quad \cdot (1 - \frac{1}{2}(e^{2i\omega_\gamma} + e^{-2i\omega_\gamma} - e^{2ik_\gamma z} - e^{-2ik_\gamma z}) - \\ &\quad - \frac{1}{4}(e^{2i(\omega_\gamma t + k_\gamma z)} + e^{2i(\omega_\gamma t - k_\gamma z)} + e^{-2i(\omega_\gamma t - k_\gamma z)} + e^{-2i(\omega_\gamma t + k_\gamma z)})) d^4x \end{aligned} \quad (8.56)$$

In the same way, this gives a nonzero result with mass gaps along the edges of the third and fifth Brillouin zones. The edge of the fifth Brillouin zone lies entirely on the light cone, but there are nontrivial mass gaps at the edge of the third zone, defined by

$$E = \pm \hbar \omega_\gamma \quad (8.57)$$

and

$$cp_z = \pm \hbar ck_\gamma. \quad (8.58)$$

Along these lines, except for four points where the edge of the Brillouin zone crosses the light cone, the eigenstates come in pairs, between which  $K_\sigma$  is parallel either to the energy or momentum axis. Therefore, to find the magnitude of the gap on the

edge of the third zone, we solve for the eigenvalues  $\epsilon_3$  of the two  $2 \times 2$  matrices  $V$  at these values. One obtains

$$\epsilon_{3\pm} = (2 \pm 1) \frac{e^2 A^2}{16M} \quad (8.59)$$

Assuming that the mass of an electron in the spacetime lattice lies in a narrow range around the Galilean target mass  $M_e$  (the on-shell mass), as we have argued in previous chapters, after the splitting the mass at the edge of the Brillouin zone is determined by

$$(M_e c^2)^2 = E^2 - c^2 p_z^2. \quad (8.60)$$

With the value of  $p_z^2$  given by (8.58), it then follows that

$$E = \sqrt{(M_e c^2 + \epsilon_{3\pm})^2 + (\hbar c k_\gamma)^2}, \quad (8.61)$$

which, for small  $\epsilon_{3\pm}$  is well approximated by

$$E = M_e c^2 + \epsilon_{3\pm} + \frac{(\hbar c k_\gamma)^2}{2M_e} \quad (8.62)$$

Finally, we can related the amplitude of the electromagnetic vector potential to the intensity of the beam creating the lattice by

$$A = \frac{\sqrt{2I}}{\omega_\gamma \sqrt{c\epsilon_0}} \quad (8.63)$$

so that all values of the kinetic energy of the electron between

$$E_- = \frac{\hbar^2}{2M_e \lambda^2} + \frac{e^2 I \lambda^2}{16M_e c^3 \epsilon_0} \quad (8.64)$$

and

$$E_+ = \frac{\hbar^2}{2M_e \lambda^2} + 3 \frac{e^2 I \lambda^2}{16M_e c^3 \epsilon_0}, \quad (8.65)$$

where  $\lambda$  is the wavelength of the radiation, are forbidden inside the lattice.

For example, if the beam creating the lattice has a wavelength of 589 nm and an intensity of  $3.13 \times 10^{12}$  W/cm<sup>2</sup>, we find that all kinetic energies of an external beam of electrons between 0.5 and 1.5 eV are forbidden to penetrate the spacetime lattice. A current with monotonically increasing kinetic energy would therefore be seen as conducting with the exception of such a gap. With the reasonable numbers we have chosen, it appears that this could be an observable effect.

Although the spacetime lattice provides a “static” potential distribution in space and time, so that the Bloch wave analysis applies, the corresponding one dimension configuration is time dependent. As we have remarked in our discussion of the Lindner experiment, in principal, Floquet theory could be applied, corresponding in fact to a nonrelativistic limit of the Stueckelberg theory, but would offer quantitatively different results.



# Hamiltonian Map to Conformal Modification of Spacetime Metric: Kaluza-Klein and TeVeS

In this chapter, we discuss the cosmological problem of accounting for the radiation curves of galaxies. It has commonly been assumed that the disagreement of simulations using the Newtonian form for gravitational attraction (with forces proportional to  $1/r^2$  between stellar bodies) with the Tully-Fisher radiation curves (Tully 1977) is due to a matter distribution that is not visible through emitted light (so-called “dark matter”), but it has been difficult to find a viable candidate for what that matter should be. Milgrom (1983) has proposed (MOND) that the Newton law be modified by a law which coincides with Newton’s for large accelerations, but differs from it when the accelerations are small. This suggestion has resulted in models which have been very successful in describing the galaxy radiation curves (e.g. Famaey 2012). However, as emphasized by Bekenstein (2004), it is difficult to change the basic Newton law without changing Einstein’s formulation of gravity in the framework of general relativity (e.g. Weinberg 1972). He proposed that the Einstein metric  $g_{\mu\nu}$  be replaced by a conformal modification  $e^{-2\phi}g_{\mu\nu}$ , where  $\phi$  is a scalar field; in this way the modification proposed by Milgrom can be achieved in the post-Newtonian limit.

Although a suitable choice of  $\phi$  has been shown to account well for the radiation curves, the gravitational distortion of light rays from other stars passing the galaxy is not described properly in this model; it would appear that the unaccounted for “matter” in the galaxy could be responsible. Bekenstein and Sanders (1994, 2004), however, have shown that the introduction of a field which, we shall call  $\mathcal{U}_\mu(x)$ , satisfying the normalization requirement

$$\mathcal{U}_\mu\mathcal{U}^\mu = -1 \quad (9.1)$$

permits the construction of a metric of the form

$$e^{-2\phi}(g_{\mu\nu} + \mathcal{U}_\mu\mathcal{U}_\nu) - e^{-2\phi}\mathcal{U}_\mu\mathcal{U}_\nu \quad (9.2)$$

which does make it possible to describe the bending of light passing by the galaxy as well as the galactic rotation curves without the addition of very much “dark matter” (Bekenstein 2004). It was pointed out by Contaldis et al. (2008) that if the fields  $\mathcal{U}_\mu$  were taken to be gauge fields, they would suffer caustic singularities near large bodies. We show here that in the framework of the SHP theory these fields can be

taken to be gauge fields which are nonabelian, and in the Abelian limit there are residual terms which may cancel the caustic singularities.

We start by discussing the application, originally developed to study the stability of nonrelativistic Hamiltonian dynamical systems (Horwitz 2007), by means of the introduction of a conformal metric, to the relativistic case. Introduction of the conformal modification of the metric in the relativistic framework provides a basis for Bekenstein's model. We remark that this can provide a relationship between the "dark matter" and "dark energy" (presumed responsible for the anomalous expansion of the universe) (Overduin 2008) problems. We discuss, furthermore, how the introduction of gauge fields can be taken into account in this framework and how, in the conformally modified structure, they emerge as (nonabelian in this context) Kaluza-Klein fields (Kaluza 1921). The Lorentz force due to such non-Abelian fields is computed by Hamiltonian methods (see also Land 1995), and it is suggested that small deviations of the orbits of satellites from the Newtonian orbits, such as the Pioneer (Turyshev 2006) (although some thermal effects have been implicated Turyshev 2012) may be accounted for by such nonabelian gauge fields.

## 9.1 Dynamics of a Relativistic Geometric Hamiltonian System

The Hamiltonian (Misner 1970)

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu, \quad (9.3)$$

with Hamilton equations (written in terms of derivatives with respect to the invariant world time  $\tau$ )

$$\dot{x}^\mu = \frac{\partial K}{\partial p_\mu} = \frac{1}{m} g^{\mu\nu} p_\nu \quad (9.4)$$

and

$$\dot{p}_\mu = -\frac{\partial K}{\partial x^\mu} = -\frac{1}{2m} \frac{\partial g^{\lambda\gamma}}{\partial x^\mu} p_\lambda p_\gamma \quad (9.5)$$

lead to the geodesic equation

$$\ddot{x}^\rho = -\Gamma^\rho_{\mu\nu} \dot{x}^\nu \dot{x}^\mu, \quad (9.6)$$

where what has appeared as a compatible connection form  $\Gamma^\rho_{\mu\nu}$  is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right). \quad (9.7)$$

These results can be taken to be tensor relations with respect to diffeomorphisms admitted by the manifold  $\{x^\mu\}$ ; writing the Hamiltonian in terms of (9.3), we see that the square of the invariant interval on an orbit is proportional, through the constant Hamiltonian, to the square of the corresponding interval world time, i.e.,

$$ds^2 = \frac{2}{m} K d\tau^2. \quad (9.8)$$

We shall study, in the following, a generalization of (9.3) consisting of the addition of a scalar field  $\Phi(x)$ . The presence of such a scalar field may be considered as a gauge compensation field for the  $\tau$  derivative in the evolution term of the covariant generalization of (9.3) in the Stückelberg-Schrödinger equation (Saad 1989), an energy distribution not directly associated with electromagnetic radiation in the usual sense. We then follow the method of Horwitz (2007) to show that there is a corresponding Hamiltonian  $\hat{K}$  with a conformally modified metric, and no explicit additive scalar field, which has the form of the construction of Bekenstein and Milgrom (1983), Bekenstein (2004) for the realization of Milgrom's MOND program (modified Newtonian dynamics) (Milgrom 1983) for achieving the observed galactic rotation curves. This simple form of Bekenstein's theory (called RAQUAL), which we discuss in some detail below for the sake of simplicity and clarity in the development of the mathematical method, does not properly account for causality and gravitational lensing; the theory has been further developed to include vector fields (which we shall call Bekenstein-Sanders fields) as well (*TeVes*) (Bekenstein 2004), which has been relatively successful in accounting for these problems. It has been shown Gershon (2009), moreover, that a gauge type Hamiltonian, with Minkowski metric and both vector and scalar fields results, under a conformal map, in an effective Kaluza-Klein theory. We shall indicate here (using a general Einstein metric) how the *TeVes* structure can emerge in terms of a Kaluza-Klein theory in this way, for which the Bekenstein-Sanders fields are considered as gauge fields. As a realization of this possibility, we exhibit a gauge transformation on the underlying quantum theory for which the vector fields, (Bekenstein 1994) which we shall call  $U^\mu(x)$ , emerge as gauge compensation fields, such that, as required by the *TeVes* theory, the property  $U^\mu U_\mu = -1$  is preserved under such gauge transformations. The corresponding quantum theory then has the form of a Hilbert bundle and, in this framework, the gauge fields are of (generalized) Yang-Mills type (Yang 1954). Working in the infinitesimal neighborhood of a gauge in which the fields are Abelian, we show that in the limit the contributions from the nonabelian sector provide nonlinear terms in the field equations which may avoid the caustic singularity found by Contaldi et al. (Contaldi 2008).

For both the RAQUAL and the *TeVes* theories, the correspondence between  $K$  and  $\hat{K}$  implies a relation between the conformal factor in  $\hat{K}$  and the world scalar field  $\Phi$ , and thus a possible connection between the so-called dark matter problem and a dark energy distribution represented by  $\Phi$  (which could be put into correspondence with the fifth gauge field (see Chap. 4) of the general Stueckelberg theory).

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## 9.2 Addition of a Scalar Potential and Conformal Equivalence

The addition of a scalar potential to the Hamiltonian (9.3), in the form

$$K = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu + \Phi(x), \quad (9.9)$$

leads, according to the Hamilton equations, to the geodesic equation<sup>1</sup>

$$\ddot{x}^\rho = -\Gamma^\rho_{\mu\nu} \dot{x}^\nu \dot{x}^\mu - \frac{1}{m} g_{\rho\nu} \frac{\partial \Phi}{\partial x^\nu}. \quad (9.10)$$

Now, consider the Hamiltonian (we carry out the calculations explicitly here since we shall have need of some of the intermediate results)

$$\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu}(y) p_\mu p_\nu. \quad (9.11)$$

It follows from the Hamilton equations that

$$\dot{y}^\mu = \frac{\partial \hat{K}}{\partial p_\mu} = \frac{1}{m} \hat{g}^{\mu\nu} p_\nu,$$

so that

$$p_\nu = m \hat{g}_{\mu\nu} \dot{y}^\mu \quad (9.12)$$

and

$$\dot{p}_\mu = -\frac{\partial \hat{K}}{\partial y^\mu} = -\frac{1}{2m} \frac{\partial \hat{g}^{\lambda\gamma}}{\partial y^\mu} p_\lambda p_\gamma.$$

As in (9.6), it then follows that

$$\ddot{y}^\mu = -\hat{\Gamma}^\mu_{\lambda\sigma} \dot{y}^\lambda \dot{y}^\sigma, \quad (9.13)$$

where, as for (9.6),

$$\hat{\Gamma}^\mu_{\lambda\sigma} = \frac{1}{2} \hat{g}^{\mu\nu} \left\{ \frac{\partial \hat{g}_{\nu\sigma}}{\partial y^\lambda} + \frac{\partial \hat{g}_{\nu\lambda}}{\partial y^\sigma} - \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \right\}. \quad (9.14)$$

We now establish an equivalence between the Hamiltonians (9.9) and (9.11) by assuming the momenta  $p_\mu$  equal at every moment  $\tau$  in the two descriptions. With the constraint

$$\hat{K} = K = k, \quad (9.15)$$

if we assume the conformal form

$$\hat{g}^{\nu\sigma}(y) = \phi(y) g^{\nu\sigma}(x), \quad (9.16)$$

it follows that

$$\phi(y)(k - \Phi(x)) = k. \quad (9.17)$$

The relation (9.17) is not sufficient to construct  $y$  as a function of  $x$ , but if we impose the relation (this relation follows from requiring the momenta in each picture to be equal for all  $\tau$  (Horwitz 2015a))

$$\delta x^\mu = \phi^{-1}(y) \delta y^\mu \quad (9.18)$$

<sup>1</sup>Note that (9.10) does not admit an equivalence principle, but (9.14), arising from (9.11) does.

between variations generated on position in the two coordinate systems, it is sufficient to evaluate derivatives of  $\phi(y)$  in terms of derivatives with respect to  $x$  of the scalar field  $\Phi(x)$  (Horwitz 2015a; see also Calderon 2013). We review this construction briefly below.

We remark that the construction based on Eqs. (9.11) and (9.16) admits the same family of diffeomorphisms as that of (9.9), since  $\phi$  is scalar. Under these diffeomorphisms, both  $g_{\mu\nu}$  and  $\hat{g}_{\mu\nu}$  are second rank tensors, and by construction of the connection forms, (9.64) and (9.13) are covariant relations.

To see how these derivatives are constructed on the constraint hypersurface determined by (9.17), let us, for brevity, define

$$F(x) \equiv \frac{k}{k - \Phi(x)}, \quad (9.19)$$

so that the constraint relation (9.15) reads

$$\phi(y) = F(x). \quad (9.20)$$

Then, since variations in  $x$  and  $y$  are related by (9.18),

$$\phi(y + \delta y) = F(x + \delta x) \cong F(x) + \delta x^\mu \frac{\partial F(x)}{\partial x^\mu}. \quad (9.21)$$

To first order in Taylor's series on the left, we obtain the relation

$$\frac{\partial \phi(y)}{\partial y^\mu} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^\mu}, \quad (9.22)$$

In agreement with the requirement that the momenta are equal for all  $\tau$  (Horwitz 2015a). We may therefore define a derivative, restricted to the constraint hypersurface

$$\frac{\tilde{\partial} F(x)}{\tilde{\partial} y^\mu} = \phi^{-1}(y) \frac{\partial F(x)}{\partial x^\mu} \quad (9.23)$$

The Leibniz relation follows easily for the product of functions, it e.g., for  $\phi(y) g^{\mu\nu}(x)$ .

In a similar way, one finds

$$\frac{\tilde{\partial}^2 F(x)}{\tilde{\partial} y^\mu \tilde{\partial} y^\nu} = \frac{\tilde{\partial}^2 F(x)}{\tilde{\partial} y^\nu \tilde{\partial} y^\mu} \quad (9.24)$$

This implies that the restricted derivative defined by (9.23) behaves as a *bona fide* derivative on the constraint hypersurface, admitting the consistent coexistence of the coordinates  $x$  and  $y$  related by (9.17). It has been shown (Horwitz 2015a) that all derivatives of  $F(y)$  can be expressed in terms of  $\phi(x)$  and its derivatives, and conversely, all derivatives of  $\phi(x)$  can be expressed in terms of  $F(y)$  and its derivatives.

In the following, we complete our argument of equivalence by reconstructing the equations of motion following from the Hamilton equations applied to (9.9), i.e., Eq. (9.10).

We begin our construction, in analogy with the procedure used in the nonrelativistic problem (Gershon 2009), by defining the new variable  $z_\mu$  such that

$$\dot{z}_\mu = \hat{g}_{\mu\nu}(y) \dot{y}^\nu. \quad (9.25)$$

Substituting  $\dot{y}^\nu = \hat{g}^{\mu\nu}(y)\dot{z}_\mu$  into (9.11), the  $\tau$  derivatives of  $\hat{g}^{\mu\nu}(y)$  generate terms that cancel two of the terms in  $\hat{\Gamma}_{\lambda\sigma}^\mu$ , leaving

$$\ddot{z}_\nu = \frac{1}{2} \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \dot{y}^\lambda \dot{y}^\sigma. \quad (9.26)$$

Now, substituting for  $\dot{y}^\lambda$  from (9.25), and using the identity

$$\hat{g}^{\gamma\lambda} \frac{\partial \hat{g}_{\lambda\sigma}}{\partial y^\nu} \hat{g}^{\sigma\rho} = -\frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu}, \quad (9.27)$$

we find

$$\ddot{z}_\nu = -\frac{1}{2} \frac{\partial \hat{g}^{\gamma\rho}}{\partial y^\nu} \dot{z}_\gamma \dot{z}_\rho. \quad (9.28)$$

Finally, from the variational type argument we used above,

$$\begin{aligned} \hat{g}^{\rho\gamma}(y + \delta y) - \hat{g}^{\rho\gamma}(y) &= \frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \delta y^\nu \\ &= \frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \hat{g}^{\nu\lambda} \delta z_\lambda, \end{aligned} \quad (9.29)$$

so that

$$\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} \hat{g}^{\nu\lambda} = \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda}$$

or

$$\frac{\partial \hat{g}^{\rho\gamma}}{\partial y^\nu} = \hat{g}^{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \quad (9.30)$$

We therefore have the alternative form

$$\ddot{z}_\nu = -\frac{1}{2} \hat{g}_{\nu\lambda} \frac{\partial \hat{g}^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\rho \dot{z}_\gamma. \quad (9.31)$$

This result constitutes a “geometric” embedding of the Hamiltonian motion induced by (9.9) in the same way as for the nonrelativistic case. Substituting the explicit form of  $\hat{g}^{\rho\gamma}$  in terms of the original Einstein metric from (9.16), one obtains

$$\ddot{z}_\nu = -\frac{1}{2} g_{\nu\lambda} \frac{\partial g^{\rho\gamma}}{\partial z_\lambda} \dot{z}_\rho \dot{z}_\gamma - \frac{1}{2} \phi^{-1} g_{\nu\lambda} \frac{\partial \phi}{\partial z_\lambda} g^{\rho\gamma} \dot{z}_\gamma \dot{z}_\rho \quad (9.32)$$

The second term contains the potential field, as in the Hamilton equations, but the first term does not contain the full connection form. We may finally, however, define a “decontraction” of the connection in (9.30) using the Einstein metric. In fact, since according to (9.16),  $\dot{y}^\nu = \phi \dot{x}^\nu$ , and by (9.23),

$$\dot{z}_\mu = \hat{g}_{\mu\nu} \dot{y}^\nu = \phi^{-1} g_{\mu\nu} \dot{y}^\nu, \quad (9.33)$$

it follows that

$$\dot{z}_\mu = g_{\mu\nu} \dot{x}^\nu. \quad (9.34)$$

Making this substitution in (9.32) leads explicitly, taking into account the  $k$  shell constraint (9.15) and the form of (9.9), to the Eq. (9.10). We have thus completed our demonstration of the equivalence between the purely metric form of the Hamiltonian

(9.11) and the Hamilton (9.9), for which the relation (9.31) corresponds to a dynamics generated by a compatible connection form, and constitute a “geometric” embedding of the original Hamiltonian motion.

Our interest in this section has been in relating the Hamiltonian (9.9) to the simplest Bekenstein-Milgrom form of MOND, without concern in the development of this simplified case for lensing or causal effects, for which a *TeV**S* type theory would be required. In the next Section, we indicate how a *TeV**S* theory can be generated in this framework, i.e., as a result of a conformal map.

### 9.3 TeVeS and Kaluza-Klein Theory

In this section, we show that the *TeV**S* theory can be cast into the form of a Kaluza-Klein construction. There has recently been a discussion (Gershon 2009), from the point of view of conformal correspondence, of the equivalence of a relativistic Hamiltonian with an electromagnetic type gauge invariant form (Saad 1989; Oron 2001 and Chap. 4) (here  $\eta^{\mu\nu}$  is the Minkowski metric  $(-1, +1, +1, +1)$ )

$$K = \frac{1}{2m} \eta^{\mu\nu} (p_\mu - ea_\mu)(p_\nu - ea_\nu) - ea_5, \quad (9.35)$$

where the  $\{a_\mu\}$ , as fields, may depend on the affine parameter  $\tau$  as well as  $x^\mu$ , and the  $a^5$  field is necessary for the gauge invariance of the  $\tau$  derivative in the quantum mechanical Stueckelberg-Schrödinger equation, with a Kaluza-Klein theory. As remarked in this work, Wesson (Overduin 2008; Liko 2005), as well as previous work on this structure (Oron 2001), have associated the source of the  $a_5$  field with mass density. A Hamiltonian of the form

$$\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (p_\mu - ea_\mu)(p_\nu - ea_\nu) \quad (9.36)$$

can be put into correspondence, as in Sect. 9.2, with  $K$  by taking  $\hat{g}^{\mu\nu}$  to be

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} \frac{k}{k + ea_5}, \quad (9.37)$$

where  $k$  is the common (constant) value of  $K$  and  $\hat{K}$ . In this correspondence, the equations of motion generated by  $\hat{K}$  through the Hamilton equations, have extra terms, beyond those provided by the connection form associated with  $\hat{g}^{\mu\nu}$ , due to the presence of the gauge fields. These additional terms can be identified as belonging to a connection form associated with a five dimensional metric, that of a Kaluza-Klein theory.

We may apply the same procedure to the Hamiltonian

$$K = \frac{1}{2m} g^{\mu\nu} (p_\mu - e\mathcal{U}_\mu)(p_\nu - e\mathcal{U}_\nu) + \Phi, \quad (9.38)$$

where  $g^{\mu\nu}$  is an Einstein metric,  $\Phi$  is a world scalar field, and  $\mathcal{U}_\mu$  are to be identified with the Bekenstein-Sanders fields for which (Bekenstein 1994)  $\mathcal{U}_\nu \mathcal{U}^\nu = -1$ , with  $\mathcal{U}^\mu = g^{\mu\nu} \mathcal{U}_\nu$ .

We shall discuss in Sect. 9.4 a class of gauge transformations on the wave functions of the underlying quantum theory for which the  $\mathcal{U}_\mu$  arise as gauge compensation fields.

Let us define, as in Eq. (9.37), the conformally modified metric

$$\begin{aligned}\hat{g}^{\mu\nu} &= g^{\mu\nu} \frac{k}{k - \Phi} \\ &\equiv e^{-2\phi} g^{\mu\nu}.\end{aligned}\quad (9.39)$$

The “equivalent” Hamiltonian

$$\hat{K} = \frac{1}{2m} \hat{g}^{\mu\nu} (p_\mu - \epsilon \mathcal{U}_\mu)(p_\nu - \epsilon \mathcal{U}_\nu) \quad (9.40)$$

then generates, through the Hamilton equations, an equation of motion which corresponds to the geodesic equation for an effective Kaluza-Klein metric, as in Gershon (2009).

Now, consider the Hamiltonian

$$K_K = \frac{1}{2m} \tilde{g}^{\mu\nu} p_\mu p_\nu, \quad (9.41)$$

with the Bekenstein-Sanders metric (Bekenstein 1994)

$$\tilde{g}^{\mu\nu} = e^{-2\phi} (g^{\mu\nu} + \mathcal{U}^\mu \mathcal{U}^\nu) - e^{2\phi} \mathcal{U}^\mu \mathcal{U}^\nu \quad (9.42)$$

The Hamiltonian  $K_K$  then has the form

$$K_K = e^{-2\phi} g^{\mu\nu} p_\mu p_\nu - 2 \sinh 2\phi (\mathcal{U}^\mu p_\mu)^2, \quad (9.43)$$

Let us now define a Kaluza-Klein type metric of the form obtained in Gershon (2009), arising from the equations of motion generated by (9.40),

$$g^{AB} = \begin{pmatrix} \hat{g}^{\mu\nu} & \mathcal{U}^\nu \\ \mathcal{U}^\mu & g^{55} \end{pmatrix}. \quad (9.44)$$

Contraction to a bilinear form with the (5D) vectors  $p_A = \{p_\lambda, p_5\}$ , with indices  $\lambda = \nu$  on the right and  $\lambda = \mu$  on the left, one finds

$$g^{AB} p_A p_B = \hat{g}^{\mu\nu} p_\mu p_\nu + 2p_5 (p_\mu \mathcal{U}^\mu) + (p_5)^2 g^{55}. \quad (9.45)$$

If we take

$$p_5 = -\frac{(p_\mu \mathcal{U}^\mu)}{g^{55}} \left(1 \pm \sqrt{1 - 2g^{55} \sinh 2\phi}\right), \quad (9.46)$$

then the Kaluza-Klein theory coincides with (9.41), i.e.,

$$K_K = \frac{1}{2m} g^{AB} p_A p_B. \quad (9.47)$$

As remarked by Wesson (Overduin 2008; Kaluza 1921), one can choose  $g_{55} = \text{const.}$  for consistency with electromagnetism, while Wesson makes the more general choice of a world scalar field. Moreover, the value  $g^{55} = 0$  is well defined (as in Gershon 2009).

Since the fields  $\mathcal{U}^\mu$  are timelike unit vectors (Bekenstein 1994),  $(p^\mu \mathcal{U}_\mu)$  corresponds, in an appropriate local frame, to the energy of the particle, close to its mass shell in the case of a nonrelativistic particle, or to the frequency in the case of on-shell photons. It clearly remains to understand more deeply the apparently *ad hoc* choice of  $p^5$  in (9.46) in terms of a 5D canonical dynamics, along with the structure of the 5D Einstein equations for  $g_{AB}$  that follow from the geometry associated with (9.47).



### 9.4 The Bekenstein-Sanders Vector Field as a Gauge Field

Essential features of the Bekenstein-Sanders field (Bekenstein 1994) of the *TeVeS* theory are that it be a local field, i.e.,  $\mathcal{U}_\mu(x)$ , and there is a normalization constraint

$$\mathcal{U}^\mu \mathcal{U}_\mu = -1, \tag{9.48}$$

so that the vector is timelike. To preserve the normalization condition (9.48) under gauge transformation, we shall study the construction of a class of gauge transformations which essentially moves the  $\mathcal{U}(x)$  field on a hyperbolic surface with a Lorentz type transformation (at the point  $x$ ).

If we think of our underlying quantum structure, which generates the gauge field, as a fiber bundle with base  $x^\mu$ , then we must think of the transformation acting in such a way that the absolute square (norm) of the wave function attached to the base point  $x^\mu$  preserves its value (Yang 1954).

An analogy can be drawn to the usual Yang-Mills gauge (Yang 1954) on  $SU(2)$ , where there is a two-valued index for the wave function  $\psi_\alpha(x)$ . The gauge transformation in this case is a two by two matrix function of  $x$ , and acts only on the indices  $\alpha$ . The condition of invariant absolute square (probability) is

$$\sum_\alpha \left| \sum_\beta U_{\alpha\beta} \psi_\beta \right|^2 = \sum_\alpha |\psi_\alpha|^2 \tag{9.49}$$

Generalizing this structure, one can take the indices  $\alpha$  to be continuous, so that (9.49) becomes

$$\int (d\mathcal{U}) \left| \int (d\mathcal{U}') U(\mathcal{U}, \mathcal{U}') \psi(\mathcal{U}', x) \right|^2 = \int (d\mathcal{U}) |\psi(\mathcal{U}, x)|^2, \tag{9.50}$$

implying that  $U(\mathcal{U}, \mathcal{U}')$  is a unitary operator on a Hilbert space  $L^2(d\mathcal{U})$ . Since we are assuming that  $\mathcal{U}_\mu$  lies on an orbit determined by (9.50), the measure is

$$(d\mathcal{U}) = \frac{d^3\mathcal{U}}{\mathcal{U}^0}, \tag{9.51}$$

i.e., a three dimensional Lorentz invariant integration measure (since  $\mathcal{U}^\mu \mathcal{U}_\mu = -1$ ).

Moreover, the Lorentz transformation on  $\mathcal{U}_\mu$  is generated by a non-commutative operator, and therefore the gauge transformation is non-Abelian. We demonstrate the resulting noncommutativity of the operator valued fields,  $\mathcal{U}'$ , after an infinitesimal gauge transformation of this type, explicitly below.

This construction is somewhat similar to the treatment of the electromagnetic potential vector and its time derivative as oscillator variables in the process of second quantization of the radiation field (the energy density of the field is given by these variables in the form of an oscillator). One can think of such a structure as a *Hilbert bundle* (Dixmeier 1959).

We now examine the gauge condition:

$$(p_\mu - \epsilon \mathcal{U}'_\mu) U \psi = U (p_\mu - \epsilon \mathcal{U}_\mu) \psi \tag{9.52}$$

Identifying  $p_\mu$  with  $-i\partial/\partial x^\mu$ , and cancelling the terms  $Up_\mu\psi$  on both sides, we obtain

$$\mathcal{U}'_\mu = U\mathcal{U}_\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}, \quad (9.53)$$

in the same form as the Yang-Mills theory (Yang 1954). It is evident in the Yang-Mills theory, that due to the matrix nature of the second term, the field will be algebra-valued, resulting in the usual structure of the Yang-Mills nonabelian gauge theory. Here, if the transformation  $U$  is a Lorentz transformation, the numerical valued field  $\mathcal{U}_\mu$  would be carried, in the first term, to a new value on a hyperbolic surface. However, the second term may well be operator valued on  $L^2(d\mathcal{U})$ , and thus, as in the Yang-Mills theory,  $\mathcal{U}'^\mu$  would become nonabelian, implying, in general, that  $\mathcal{U}$  is a nonabelian field.

It follows from (9.51) that the field strengths

$$f_{\mu\nu} = \frac{\partial \mathcal{U}_\mu}{\partial x^\nu} - \frac{\partial \mathcal{U}_\nu}{\partial x^\mu} + i\epsilon[\mathcal{U}_\mu, \mathcal{U}_\nu] \quad (9.54)$$

are related to the the field strengths in the transformed form

$$f'_{\mu\nu} = \frac{\partial \mathcal{U}'_\mu}{\partial x^\nu} - \frac{\partial \mathcal{U}'_\nu}{\partial x^\mu} + i\epsilon[\mathcal{U}'_\mu, \mathcal{U}'_\nu] \quad (9.55)$$

according to

$$f'_{\mu\nu}(x) = Uf_{\mu\nu}(x)U^{-1}, \quad (9.56)$$

just as in the finite dimensional Yang-Mills theories.

This result follows from writing out, from (9.51),

$$\begin{aligned} \frac{\partial \mathcal{U}'_\mu}{\partial x^\nu} &= \frac{\partial U}{\partial x^\nu} \mathcal{U}_\mu U^{-1} + U \frac{\partial \mathcal{U}_\mu}{\partial x^\nu} U^{-1} + U \mathcal{U}_\mu \frac{\partial U^{-1}}{\partial x^\nu} \\ &\quad - \frac{i}{\epsilon} \frac{\partial^2 U}{\partial x^\mu \partial x^\nu} U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} \frac{\partial U^{-1}}{\partial x^\nu}, \end{aligned} \quad (9.57)$$

and subtracting the same expression with  $\mu, \nu$  reversed. Then add the result to

$$\begin{aligned} i\epsilon[\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= i\epsilon U[\mathcal{U}_\mu, \mathcal{U}_\nu]U^{-1} + [U\mathcal{U}_\mu U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}] \\ &\quad + [\frac{\partial U}{\partial x^\mu} U^{-1}, U\mathcal{U}_\nu U^{-1}] - \frac{i}{\epsilon} [\frac{\partial U}{\partial x^\mu} U^{-1}, \frac{\partial U}{\partial x^\nu} U^{-1}] \end{aligned} \quad (9.58)$$

Whenever the combination

$$U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1}$$

appears, it should be replaced by

$$-\frac{\partial U^{-1}}{\partial x^\mu}.$$

The result (9.56) then follows after a little manipulation.

Now, consider the possibility that this finite gauge transformation leaves  $\mathcal{U}_\mu \mathcal{U}^\mu = -1$ .

We write out

$$\begin{aligned}
 (U\mathcal{U}_\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1})(U\mathcal{U}^\mu U^{-1} - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} U^{-1}) &= -1 - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} \mathcal{U}^\mu U^{-1} \\
 &\quad - \frac{i}{\epsilon} U\mathcal{U}_\mu U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1} \\
 &\quad - \frac{1}{\epsilon^2} \frac{\partial U}{\partial x^\mu} U^{-1} \frac{\partial U}{\partial x^\mu} U^{-1} \\
 &= -1 - \frac{i}{\epsilon} \frac{\partial U}{\partial x^\mu} \mathcal{U}^\mu U^{-1} \\
 &\quad + \frac{i}{\epsilon} U\mathcal{U}_\mu \frac{\partial U^{-1}}{\partial x_\mu} \\
 &\quad + \frac{1}{\epsilon^2} \frac{\partial U}{\partial x^\mu} \frac{\partial U^{-1}}{\partial x_\mu}. \quad (9.59)
 \end{aligned}$$

It may be possible that  $U$  can be chosen to make all but the first term in (9.59) vanish, but in the case of finite gauge transformations, it is not so easy to see how to construct examples. For the infinitesimal case, it is, however, straightforward to construct a gauge function with the required properties. For

$$U \cong 1 + iG, \quad (9.60)$$

where  $G$  is infinitesimal, (9.53) becomes

$$\mathcal{U}'_\mu = \mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu} + O(G^2). \quad (9.61)$$

Then,

$$\begin{aligned}
 \mathcal{U}'_\mu \mathcal{U}'^\mu &\cong \mathcal{U}_\mu n_\mu + i(\mathcal{U}_\mu [G, \mathcal{U}^\mu] + [G, \mathcal{U}_\mu] \mathcal{U}^\mu) \\
 &\quad + \frac{1}{\epsilon} \left( \frac{\partial G}{\partial x^\mu} \mathcal{U}^\mu + \mathcal{U}_\mu \frac{\partial G}{\partial x_\mu} \right). \quad (9.62)
 \end{aligned}$$

Let us take

$$\begin{aligned}
 G &= -\frac{i\epsilon}{2} \sum \left\{ \omega_{\lambda\gamma}(\mathcal{U}, x), \left( \mathcal{U}^\lambda \frac{\partial}{\partial \mathcal{U}_\gamma} - \mathcal{U}^\gamma \frac{\partial}{\partial \mathcal{U}_\lambda} \right) \right\} \\
 &\equiv \frac{\epsilon}{2} \sum \left\{ \omega_{\lambda\gamma}(\mathcal{U}, x), N^{\lambda\gamma} \right\} \quad (9.63)
 \end{aligned}$$

where symmetrization is required since  $\omega_{\lambda\gamma}$  is a function of  $\mathcal{U}$  as well as  $x$ , and

$$N^{\lambda\gamma} = -i \left( \mathcal{U}^\lambda \frac{\partial}{\partial \mathcal{U}_\gamma} - \mathcal{U}^\gamma \frac{\partial}{\partial \mathcal{U}_\lambda} \right). \quad (9.64)$$

This construction is valid in the initially special gauge, which we shall call the “special abelian gauge”, in which the components of  $\mathcal{U}^\mu$  commute. The appearance of  $\mathcal{U}^\mu$  in the gauge functions is then admissible since this quantity acts on the wave

functions  $\langle \mathcal{U}, x | \psi \rangle = \psi(\mathcal{U}, x)$  at the point  $x$ , in the representation in which the operator  $\mathcal{U}^\mu$  on  $L^2(d\mathcal{U})$  is diagonal.

Our investigation in the following will be concerned with a study of the infinitesimal gauge neighborhood of this limit, where the components of  $\mathcal{U}^\mu$  do not commute, and therefore constitute a Yang Mills type field. We shall show in the limit that the corresponding field equations acquire nonlinear terms, and may therefore suppress the caustic singularities found by Contaldi et al. (Contaldi 2008). They found that nonlinear terms associated with a non-Maxwellian type action, such as  $(\partial_\mu \mathcal{U}^\mu)^2$ , could avoid this caustic singularity, so that the nonlinear terms we find as a residue of the Yang-Mills structure induced by our gauge transformation might achieve this effect in a natural way.

The second term of (9.62), which is the commutator of  $G$  with  $\mathcal{U}^\mu \mathcal{U}_\mu$  vanishes, since this product is Lorentz invariant (the symmetrization in  $G$  does not affect this result).

We now consider the third term in (9.62).

$$\begin{aligned} \frac{1}{\epsilon} \left( \frac{\partial G}{\partial x^\mu} \mathcal{U}^\mu + \mathcal{U}_\mu \frac{\partial G}{\partial x_\mu} \right) &= \frac{1}{2} \left\{ \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}, N^{\lambda\gamma} \right\} \mathcal{U}^\mu + \mathcal{U}^\mu u \left\{ \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu}, N^{\lambda\gamma} \right\} \\ &= \frac{1}{2} \left\{ N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \mathcal{U}^\mu + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} \mathcal{U}^\mu \right. \\ &\quad \left. + \mathcal{U}^\mu N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \mathcal{U}^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} \right\} \end{aligned} \quad (9.65)$$

There are two terms proportional to

$$\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \mathcal{U}^\mu.$$

If we take (locally)

$$\omega_{\lambda\gamma}(\mathcal{U}, x) = \omega_{\lambda\gamma}(k_\nu x^\nu), \quad (9.66)$$

where  $k_\nu \mathcal{U}^\nu = 0$ , then

$$\frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \mathcal{U}_\mu = k_\mu \mathcal{U}^\mu \omega'_{\lambda\gamma} = 0. \quad (9.67)$$

For the remaining two terms,

$$\begin{aligned} \mathcal{U}^\mu N^{\lambda\gamma} \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} N^{\lambda\gamma} \mathcal{U}^\mu \\ &= N^{\lambda\gamma} \mathcal{U}^\mu \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \\ &\quad + [\mathcal{U}^\mu, N^{\lambda\gamma}] \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} \mathcal{U}^\mu N^{\lambda\gamma} \\ &\quad + \frac{\partial \omega_{\lambda\gamma}}{\partial x^\mu} [N^{\lambda\gamma}, \mathcal{U}^\mu]. \end{aligned} \quad (9.68)$$

Since the commutators contain only terms linear in  $\mathcal{U}_\mu$  and they have opposite sign, and cancel. The remaining terms are zero by the argument (9.67). The condition  $\mathcal{U}_\mu \mathcal{U}^\mu = -1$  is therefore invariant under this gauge transformation, involving the

coefficient  $\omega_{\lambda\gamma}$  which is a function of the projection of  $x^\mu$  onto a hyperplane orthogonal to  $\mathcal{U}_\mu$ , i.e., a function of  $k_\mu x^\mu$ , where  $k_\mu \mathcal{U}^\mu = 0$ . The vector  $k_\mu$ , of course, depends on  $\mathcal{U}_\mu$  (for example,  $k_\mu = \mathcal{U}_\mu(\mathcal{U} \cdot b) + b_\mu$ , for some  $b_\mu \neq 0$ ).

We now demonstrate explicitly the nonabelian nature of the gauge fields after infinitesimal gauge transformation. With (9.61), the commutator term in (9.55) is

$$\begin{aligned} [\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= (\mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu})(\mathcal{U}_\nu + i[G, \mathcal{U}_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu}) \\ &\quad - (\mathcal{U}_\nu + i[G, \mathcal{U}_\nu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\nu})(\mathcal{U}_\mu + i[G, \mathcal{U}_\mu] + \frac{1}{\epsilon} \frac{\partial G}{\partial x^\mu}) \\ &= \frac{1}{\epsilon} \left\{ [\mathcal{U}_\mu, \frac{\partial G}{\partial x^\nu}] - [\mathcal{U}_\nu, \frac{\partial G}{\partial x^\mu}] \right\} \\ &\quad + i[\mathcal{U}_\mu, [G, \mathcal{U}_\nu]] - i[\mathcal{U}_\nu, [G, \mathcal{U}_\mu]], \end{aligned} \quad (9.69)$$

where the remaining terms have identically cancelled. Note that this expression does not contain any noncommutative quantities. Now,

$$[G, \mathcal{U}_\nu] = 2i\epsilon\omega_\nu{}^\gamma \mathcal{U}_\gamma \quad (9.70)$$

and

$$[\mathcal{U}_\mu, \frac{\partial G}{\partial x^\nu}] = 2i\epsilon \mathcal{U}_\lambda \frac{\partial \omega^\lambda{}_\mu}{\partial x^\nu}. \quad (9.71)$$

The terms involving  $[G, \mathcal{U}_\nu]$  and  $[G, \mathcal{U}_\mu]$  therefore cancel, so that

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i\mathcal{U}_\lambda \left( \frac{\partial \omega^\lambda{}_\mu}{\partial x^\nu} - \frac{\partial \omega^\lambda{}_\nu}{\partial x^\mu} \right) \quad (9.72)$$

We have taken  $\omega^\lambda{}_\mu = \omega^\lambda{}_\mu(k_\sigma x^\sigma)$ , so that

$$\frac{\partial \omega^\lambda{}_\mu}{\partial x^\nu} = k_\nu \omega'^\lambda{}_\mu, \quad (9.73)$$

and therefore

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i(k_\nu \omega'^\lambda{}_\mu - k_\mu \omega'^\lambda{}_\nu) \mathcal{U}_\lambda, \quad (9.74)$$

generally not zero. This demonstrates the nonabelian character of the fields. In the Abelian limit, we may take  $\omega' \rightarrow 0$ , but as we shall see, there is a residual nonlinearity, which depends on  $\omega''$  may remain in the field equations.

We now consider the derivation of field equations from a Lagrangian constructed with the  $\psi$ 's and  $f^{\mu\nu} f_{\mu\nu}$ . We take the Lagrangian to be of the form (the indices are raised and lowered with  $g_{\mu\nu}$ )

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_m, \quad (9.75)$$

where

$$\mathcal{L}_f = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} \quad (9.76)$$

and

$$\mathcal{L}_m = \psi^* \left( i \frac{\partial}{\partial \tau} - \frac{1}{2M} (p_\mu - e\mathcal{U}_\mu) g^{\mu\nu} (p_\nu - e\mathcal{U}_\nu) - \Phi \right) \psi + \text{c.c.} \quad (9.77)$$

We shall be working in the infinitesimal neighborhood of the special gauge for Abelian  $\mathcal{U}_\mu$ , for which it has the form given in (9.59) for infinitesimal  $G$ . It is therefore not Abelian to first order, but we take its variation  $\delta\mathcal{U}$  to be a c-number function, carrying the variation, to lowest order, by variation of the first term in (9.61), and not varying the part of  $\mathcal{U}$  introduced by the infinitesimal gauge transformation (evaluated on the original value of  $\mathcal{U}$ ).

In carrying out the variation of  $\mathcal{L}_m$ , the contributions of varying the  $\psi$ 's with respect to  $\mathcal{U}$  vanish due to the field equations (Stueckelberg-Schrödinger equation) obtained by varying  $\psi^*$  (or  $\psi$ ), and therefore in the variation with respect to  $\mathcal{U}$ , only the explicit presence of  $\mathcal{U}$  in (9.77) need be taken into account.

Note that for the general case of  $\mathcal{U}$  operator valued, we can write

$$\psi^*(p_\mu - \epsilon\mathcal{U}_\mu)g^{\mu\nu}(p_\nu - \epsilon\mathcal{U}_\nu)\psi = g^{\mu\nu}((p^\mu - \epsilon\mathcal{U}^\mu)\psi)^*(p_\nu - \epsilon\mathcal{U}_\nu)\psi, \quad (9.78)$$

since the Lagrangian density (9.75) contains an integration over  $(d\mathcal{U}')(d\mathcal{U}'')$  (considered in lowest order) as well as an integration over  $(dx)$  in the action and the operators  $\mathcal{U}$  are Hermitian. In the limit in which  $\mathcal{U}$  is evaluated in the special Abelian gauge (real valued), and noting that  $p_\mu$  is represented by an imaginary differential operator, we can write this as

$$g^{\mu\nu}\psi^*(p_\mu - \epsilon\mathcal{U}_\mu)(p_\nu - \epsilon\mathcal{U}_\nu)\psi = -g^{\mu\nu}(p_\mu + \epsilon\mathcal{U}_\mu)\psi^*(p_\nu - \epsilon\mathcal{U}_\nu)\psi, \quad (9.79)$$

i.e., replacing explicitly  $p_\mu$  by  $-i(\partial/\partial x^\mu) \equiv -i\partial_\mu$ , we have

$$\delta_{\mathcal{U}}\mathcal{L}_m = -i\frac{\epsilon}{2M}\{\psi^*(\partial_\mu - i\epsilon\mathcal{U}_\mu)\psi - ((\partial_\mu + i\epsilon\mathcal{U}_\mu)\psi^*)\psi\}\delta\mathcal{U}^\mu, \quad (9.80)$$

where we have called  $g^{\mu\nu}\delta\mathcal{U}_\nu = \delta\mathcal{U}^\mu$ , or

$$\delta_{\mathcal{U}}\mathcal{L}_m = j_\mu(\mathcal{U}, x)\delta\mathcal{U}^\mu, \quad (9.81)$$

where  $j_\mu$  has the usual form of a gauge invariant current.

For the calculation of the variation of  $\mathcal{L}_f$  we note that the commutator term in (9.54) is, in lowest order, a c-number function, as given in (9.74).

Calling

$$\omega'^\lambda{}_\mu\mathcal{U}_\lambda \equiv v_\mu, \quad (9.82)$$

we compute the variation of

$$[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i(k_\nu v_\mu - k_\mu v_\nu) \quad (9.83)$$

Then, for

$$\delta_{\mathcal{U}}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = \delta_{\mathcal{U}_\gamma}\frac{\partial}{\partial\mathcal{U}_\gamma}[\mathcal{U}'_\mu, \mathcal{U}'_\nu], \quad (9.84)$$

we compute

$$\frac{\partial}{\partial\mathcal{U}_\gamma}[\mathcal{U}'_\mu, \mathcal{U}'_\nu] = 2i\left(\frac{\partial k_\nu}{\partial\mathcal{U}_\gamma}v_\mu + k_\nu\frac{\partial v_\mu}{\partial\mathcal{U}_\gamma}\right) - (\mu \leftrightarrow \nu). \quad (9.85)$$

With our choice of  $k_\nu = \mathcal{U}_\nu(\mathcal{U} \cdot b) + b_\nu$ ,

$$\frac{\partial k_\nu}{\partial \mathcal{U}_\gamma} = \delta_\nu^\gamma (\mathcal{U} \cdot b) + \mathcal{U}_\nu b^\gamma, \quad (9.86)$$

so that

$$\begin{aligned} \frac{\partial}{\partial \mathcal{U}_\gamma} [\mathcal{U}'_\mu, \mathcal{U}'_\nu] &= 2i(\delta_\nu^\gamma (\mathcal{U} \cdot b) + \mathcal{U}_\nu b_\gamma) v^\mu \\ &\quad + k_\nu \frac{\partial v_\mu}{\partial \mathcal{U}_\gamma} - (\mu \leftrightarrow \nu) \\ &\equiv \mathcal{O}^\gamma_{\mu\nu}, \end{aligned} \quad (9.87)$$

i.e.

$$\delta_{\mathcal{U}} [\mathcal{U}'_\mu, \mathcal{U}'_\nu] = \mathcal{O}^\gamma_{\mu\nu} \delta \mathcal{U}_\gamma \quad (9.88)$$

The quantity  $v_\mu$  is proportional to the derivative of  $\omega_\mu^\lambda$ . In the limit that  $\omega, \omega' \rightarrow 0$  (cf. (9.83)), the second derivative,  $\omega''$  which appears in  $\mathcal{O}^\gamma_{\mu\nu}$  may not vanish (somewhat analogous to the case in gravitational theory when the connection form vanishes but the curvature does not), so that this term can contribute in limit to the special Abelian gauge.

Returning to the variation of  $\mathcal{L}_f$  in (9.76), we see that

$$\delta \mathcal{L}_f = -\partial^\nu f_{\mu\nu} \delta \mathcal{U}^\mu + 2i f_{\mu\nu} \delta [\mathcal{U}_\mu, \mathcal{U}_\nu], \quad (9.89)$$

where we have taken into account the fact that  $[\mathcal{U}_\mu, \mathcal{U}_\nu]$  is a commuting function, and integrated by parts the derivatives of  $\delta \mathcal{U}$ . With (9.88) we obtain

$$\delta \mathcal{L}_f = -\partial^\nu f_{\mu\nu} \delta \mathcal{U}^\mu + 2i \epsilon f_{\lambda\sigma} \mathcal{O}^{\lambda\sigma}_{\mu} \delta \mathcal{U}^\mu \quad (9.90)$$

Since the coefficient of  $\delta \mathcal{U}^\mu$  must vanish, we obtain, with (9.79), the Yang-Mills equations for the fields given the source currents

$$\partial^\nu f_{\mu\nu} = j_\mu - 2i \epsilon f_{\lambda\sigma} \mathcal{O}^{\lambda\sigma}_{\mu}, \quad (9.91)$$

which is nonlinear in the fields  $\mathcal{U}_\mu$ , as we have seen, even in the Abelian limit, where, from (9.80) and (9.81),

$$j_\mu = -i \frac{\epsilon}{2M} \{ \psi^* (\partial_\mu - i \epsilon \mathcal{U}_\mu) \psi - ((\partial_\mu + i \epsilon \mathcal{U}_\mu) \psi^*) \psi \}. \quad (9.92)$$

We point out that this current corresponds to a flow of the matter field; the absolute square of the wave functions corresponds to an event density. The coupling  $\epsilon$  is not necessarily the electron charge, and the fields  $\mathcal{U}$  are not necessarily electromagnetic even in the Abelian limit. However, the Hamiltonian (9.38) leads directly to a Lorentz type force, similar in form to that generated by the Hilbert-Einstein action (see Chap. 4).

## 9.5 Summary

We have seen in this chapter that a map of the type discussed in Gershon (2009) of a Hamiltonian containing an Einstein metric, generating the connection form of general relativity, and a world scalar field, representing a distribution of energy on the spacetime manifold, into a corresponding Hamiltonian with a conformal metric (and compatible connection form), can account for the structure of the RAQUAL theory of Bekenstein and Milgrom (1983). Furthermore, applying this correspondence to a Hamiltonian with gauge-type structure, we have shown that one obtains a non-compact Kaluza-Klein effective metric which can account for the *TeVes* structure of Bekenstein, Sanders and Milgrom (1989, 1994).

In order to maintain the constraint condition  $\mathcal{U}_\mu \mathcal{U}^\mu = -1$  for the Bekenstein-Sanders fields, under local gauge transformations, we have introduced a class of gauge transformations on the underlying quantum theory which acts on the Hilbert bundle, quite analogous to that arising in the second quantization of the electromagnetic field (where the vector potentials and their time derivatives are considered as quantum oscillator variables) associated with the values of the gauge fields. The action of this class of gauges induces a nonabelian structure on the fields, which therefore satisfy Yang-Mills type field equations with source currents associated with matter flow. In the Abelian limit, these equations contain residual non-linear terms which may avoid the caustic singularities found by Contaldi et al. (2008) for an electromagnetic type gauge field.

The phenomenological constraints placed on the *TeVes* variables in its astrophysical applications and on its MOND limit (Milgrom 1983) would, in principle, place constraints on the vector and scalar fields appearing in the corresponding Hamiltonian model, for which the additive world scalar field corresponds to an energy distribution, not associated with electromagnetic radiation, which could contribute to the anomalous expansion of the universe (Rañada (2003), (2004), Anderson (1998), Rosales (1999)).



In this chapter, we shall discuss the statistical mechanics of a many *event* system, for which the points in space time constitute the fundamental entities for which distribution functions must be constructed to achieve a manifestly covariant theory. Assuming that each event is part of an evolving world line, as in our construction of Chap. 4, the counting of events is essentially equivalent to the counting of world lines corresponding to *particles*. Therefore one should expect that, as we indeed find, the statistical mechanics of events is closely related to the theory of statistical mechanics of particles, as developed, for example, in Synge (1957); see also, de Groot (1980). Hakim (2011), Israel and Kandrup (1984) stress the importance of manifest covariance. We construct a canonical Gibbs ensemble based on a microcanonical ensemble, as is usual in statistical mechanics (e.g. Huang 1967), enabling us to define a temperature and the basic thermodynamic functions (Horwitz 1981).

We obtain the Bose-Einstein and Fermi-Dirac statistics from the relativistic kinetic theory with a form very similar that of the nonrelativistic theory.

Following the work of Weldon et al. (Haber 1982) on the existence of a high temperature Bose-Einstein condensation for boson-antiboson systems, we describe a high temperature Bose-Einstein phase transition that, with appropriate chemical potential, brings the system of particles to mass shell (statistically), providing another explanation, in addition to self-interaction of the fields, that enforces the asymptotic stability of particle masses.

In the quantum case, as we have emphasized in Chap. 3, the particle wave functions, for half-integer spin fermions and for bosons, must lie on corresponding points of their associated induced representation orbit in the corresponding Fock space. In this representation for the boson fields, and with an additional asymptotic gauge condition, we find the correct factor of 2 for the specific heat, even though in principle the 5D gauge fields contain, under canonical quantization (Henneaux 1992), intrinsically three polarization states (Shnerb 1993).

In this framework, we develop a covariant Boltzmann equation. The usual method of derivation of the second law from the Boltzmann equation (Huang 1967) shows that the entropy in the relativistic case increases monotonically in  $\tau$ , but if antiparticles are present, not necessarily in  $t$ .

## 10.1 A Potential Model for the Many Body System

We shall consider a model for the many body system, with particle (event) coordinates and momenta  $\{q_i^\mu\}$  and  $\{p_i^\mu\}$ , for  $i = 1, 2, \dots, N$ , in which the total invariant Hamiltonian is given by

$$K = K_0 + V(\hat{q}_1, \hat{q}_2, \dots, \hat{q}_N), \quad (10.1)$$

where

$$K_0 = \sum_i \frac{\hat{p}_i^\mu \hat{p}_{i\mu}}{2M_i} + \frac{P_\mu P^\mu}{2M}, \quad (10.2)$$

$$\begin{aligned} \hat{p}_i^\mu &= p_i^\mu - (M_i/M)P^\mu \\ \hat{q}_i^\mu &= q_i^\mu - Q^\mu, \end{aligned} \quad (10.3)$$

and

$$M = \sum_i M_i. \quad (10.4)$$

The new variables  $\hat{p}_i^\mu$  and  $\hat{q}_i^\mu$  satisfy the constraints

$$\sum_i \hat{p}_i^\mu = 0, \quad \sum_i M_i \hat{q}_i^\mu = 0. \quad (10.5)$$

Since, by Poincaré invariance,  $V$  is a function of relative coordinates alone, it may be considered as well to be the same function of the  $\{\hat{q}_i^\mu\}$ . As for the Galilean many body problem, the variables  $P^\mu$ ,  $Q^\mu$  are canonical, i.e. the Poisson bracket is

$$\{Q^\mu, P^\nu\}_{PB} = \eta^{\mu\nu}, \quad (10.6)$$

For the relative coordinates, however, it is straightforward to show that

$$\{\hat{q}_i^\mu, \hat{p}_j^\nu\}_{PB} = \eta^{\mu\nu}(\delta_{ij} - \frac{M_j}{M}), \quad (10.7)$$

and therefore the relative coordinates do not satisfy canonical Poisson bracket relations. In the thermodynamic limit, however, these coordinates become canonical. Furthermore, taking into account the constraints (10.4), it follows that along with

$$\frac{dQ^\mu}{d\tau} = \frac{P^\mu}{M}, \quad \frac{dP^\mu}{d\tau} = 0, \quad (10.8)$$

one obtains precisely that (Horwitz 1981)

$$\frac{d\hat{q}_i^\mu}{d\tau} = \frac{\hat{p}_i^\mu}{M_i}, \quad \frac{d\hat{p}_i^\mu}{d\tau} = -\frac{\partial V}{\partial \hat{q}_{i\mu}}, \quad (10.9)$$

as for a canonical system.

In the following, we shall treat the relative variables as a set of canonical coordinates, and simply call them  $q^\mu$ ,  $p^\mu$ . We therefore write for the reduced invariant Hamiltonian

$$K = \sum_i \frac{p_i^\mu p_{i\mu}}{2M_i} + V(q_1, q_2, \dots, q_N). \quad (10.10)$$

## 10.2 The Microcanonical Ensemble

The microcanonical ensemble in classical nonrelativistic statistical mechanics is constructed (Huang 1967) by computing the volume in phase space accessible to a total energy  $E$ . The Hamiltonian (10.10) contains, however, the invariants  $p_i^\mu p_{i\mu} = \mathbf{p}_i^2 - E_i^2/c^2$ , and its value does not bound the phase space (as for the nonrelativistic Hamiltonian containing  $\mathbf{p}_i^2/2M_i$ ). To construct a microcanonical ensemble, we must recognize, however, that the total energy of the system plays a fundamental role in defining equilibrium thermodynamics. A thermometer injected into a system with relative center of mass motion will read a higher temperature than for a system at relative rest; as Eimerl (1975) has pointed out, systems in relative motion cannot be in equilibrium since there is an inevitable exchange of heat. The center of mass of the system is therefore an important element of this definition. Since the entire system is translation invariant in time, the center of mass quantity  $\sum_i E_i$  is a constant of the motion; with this constraint, we can write the microcanonical ensemble as

$$\Gamma(\kappa, E) = \int d^4 p_1 d^4 p_2 \dots d^4 p_N d^4 q_1 d^4 q_2 \dots d^4 q_N \delta(K - \kappa) \delta(\sum_i \frac{E_i}{c} - \frac{E}{c}), \quad (10.11)$$

where  $d^4 p_i = \frac{dE_i}{c} d^3 \mathbf{p}_i$  and  $d_i^q = c dt_i d^3 \mathbf{q}_i$ , now properly bounded by the presence of center of mass energy constraint and the requirement that the particle masses take on values not too far from their Galilean mass shell values (for bounded  $\{E_i\}$ , bounds on the  $\{m_i\}$  enforce bounds on the  $\mathbf{p}_i$ ). For the nonrelativistic microcanonical ensemble, the phase space integral is usually confined to a box with perfect reflecting walls so that the energy is not perturbed by collisions with the walls. If we bound our spacetime volume in time, the reflecting walls will generate particles running backward in time, i.e., antiparticles. Although such states certainly exist, for example, in high temperature plasmas, for the sake of simplicity, we shall assume that the system we are describing constitutes an amorphous cloud of events, which is reasonably bounded in space and time, and retains its structure for a sufficient interval of evolution time  $\tau$  to be able to think of it as a (quasi-)equilibrium state. The integration on  $q_i$  is therefore to be restricted to this domain. The four momentum integration is also to be understood as allowing for variations of the particle masses  $m_i = \sqrt{((\frac{E_i}{c})^2 - \mathbf{p}_i^2)}$  that are not too large, since we are thinking of applications of the theory to systems of particles reasonably close to their Galilean limiting masses.<sup>1</sup> It is in this framework, of systems not too far from having nonrelativistic structure, that we wish to develop the theory here in order to maintain some familiarity with the physical phenomena generally observed. The structure presents, however, a wide range of generalization, holding potentially the prediction of phenomena not at all of familiar type.

<sup>1</sup>As remarked in Chap. 2, in Galilean mechanics, due to the existence of a cohomology in the Lie algebra of the Galilean group, a definite value must be assigned to the value of the mass to achieve an irreducible representation (Sudarshan 1974).

We shall define the microcanonical entropy as

$$S = k_B \ln \Gamma(\kappa, E) \quad (10.12)$$

and the temperature as

$$T^{-1} = \frac{\partial S(\kappa, E)}{\partial E} \quad (10.13)$$

Taking  $c \rightarrow \infty$ , we may obtain the form of the Galilean limit of the relativistic microcanonical ensemble represented by (10.2). As we have pointed out in Chap. 2 and above, the microcanonical ensemble in this limit should contain definite masses for each particle. This requirement can be made precise by requiring that the quantity

$$\begin{aligned} \varepsilon_i &\equiv E_i - M_i c^2 \\ &= c \sqrt{\mathbf{p}_i^2 + m_i^2 c^2} - M_i c^2, \end{aligned} \quad (10.14)$$

constituting a change of variables from energy and momentum to momentum and (variable) mass, as we have done in previous chapters, be bounded in the limit  $c \rightarrow \infty$ . Expanding in powers of  $1/c^2$ , one finds that

$$\varepsilon_i = c^2(m_i - M_i) + \mathbf{p}_i^2/2M + O(1/c^2). \quad (10.15)$$

Hence in this limit,

$$m_i = M_i + \eta_i, \quad (10.16)$$

where

$$\eta_i = c^2(m_i - M_i) \quad (10.17)$$

may have any value in  $(-\infty, \infty)$ , but is bounded as  $c \rightarrow \infty$ . This residual quantity preserves the relativistic Poisson brackets, and in the quantum case where the limits are controlled by the structure of the wave function, the commutation relations. We shall see that is this freedom which permits us to obtain the Galilean microcanonical ensemble.<sup>2</sup>

The kinetic terms of the Hamiltonian (10.10) can then be written

$$\begin{aligned} \sum_i \frac{p_i^\mu p_{i\mu}}{2M_i} &= \sum_i \frac{-E_i^2/c^2 + \mathbf{p}_i^2}{2M_i} \\ &= \sum_i \frac{\mathbf{p}_i^2}{2M_i} - \varepsilon' - \frac{M c^2}{2} - \sum_i \frac{\varepsilon_i^2}{2M_i c^2}, \end{aligned} \quad (10.18)$$

where  $\varepsilon' = \sum_i \varepsilon_i$ , and the last term vanishes as  $O(1/c^2)$  since the  $\varepsilon_i$  are bounded.

Since, according to the equations of motion,

$$\frac{d}{d\tau} c t_i = -\frac{\partial K}{\partial(E_i/c)} = -\frac{\partial K_0}{\partial(\varepsilon_i/c)} = c + \frac{\varepsilon_i}{M_i c}, \quad (10.19)$$

it follows that

$$c t_i = c \tau + \int_0^\tau \frac{\varepsilon_i(\tau')}{M_i c} d\tau' + c t_i(0), \quad (10.20)$$

<sup>2</sup>It was shown by Horwitz and Rotbart (1981a), by examining the scalar product, that the nonrelativistic limit of the relativistic quantum theory is obtained systematically in this way as well.

so that all events become synchronized in  $t$  in the Galilean limit. Hence,  $V$ , which depends only on the differences between the  $\{t_i\}$  becomes independent of time in the Galilean limit, and we obtain (setting all the  $t_i(0) = 0$ ) that

$$K \sim H - \varepsilon' - Mc^2/2, \quad (10.21)$$

where

$$H = \sum_i \frac{\mathbf{p}_i^2}{2M_i} + V(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N), \quad (10.22)$$

the usual  $N$  body nonrelativistic Hamiltonian.

We now turn to the integral measure on the microcanonical ensemble. Noting that

$$\frac{dE_i}{c} = \frac{c^2 m_i dm_i}{\sqrt{\mathbf{p}_i^2 + m_i^2 c^2}} \sim c dm_i, \quad (10.23)$$

we can write the limiting form of the microcanonical ensemble as

$$\begin{aligned} \Gamma(\kappa, E) \sim & \int c dm_1 \dots c dm_N d^3 p_1 \dots d^3 p_N d^4 q_1 \dots d^4 q_N \\ & \cdot \delta(H - \varepsilon' - Mc^2/2 - \kappa) \delta(\varepsilon' - \varepsilon) \cdot c, \end{aligned} \quad (10.24)$$

where  $\varepsilon = E - Mc^2$ , and the integrals are limited by taking the  $m_i$  in a small range  $\mu_i$  around their Galilean limits, and the  $q_i$  in the spacetime volume for the ensemble we have referred to earlier. Now, let  $dm_N = (1/c)d\eta_N$  and integrate over this variable in (10.24). The linear occurrence of  $\eta_N$  in both  $\delta$ -function factors allows us to fold the integration, resulting, for  $\kappa = -Mc^2/2$ , in the standard form of the nonrelativistic microcanonical ensemble

$$\begin{aligned} \Gamma(\kappa, E) \sim & (\Delta mc^2 \Delta t)^{N-1} (c \Delta t) \int d^3 p_1 \dots d^3 p_N d^3 q_1 \dots d^3 q_N \\ & \cdot \delta(H(\mathbf{q}_1 \dots \mathbf{q}_N, \mathbf{p}_1, \dots, \mathbf{p}_N) - \varepsilon). \end{aligned} \quad (10.25)$$

The factors multiplying the integral are not important in the determination of the mean value of any physical observable. We have therefore shown that the relativistic microcanonical ensemble reduces to the usual Galilean microcanonical ensemble in the non-relativistic limit  $c \rightarrow \infty$ . The total energy is identified with  $\sum_i \varepsilon_i$ . We further remark that the result  $\kappa = -Mc^2/2$  implies that the factor  $\delta(K - \kappa)$  in (10.11) restricts the center of mass motion described by  $P^\mu P_\mu$  to mass shell with shift due to  $K_{rel}$ , playing the role of an ‘‘internal energy’’.

We now study some of the properties of systems of free particles (free gases) in the relativistic microcanonical ensemble. From (10.11), with no coordinate dependence in the integral, the integrals over spacetime just yield the total volume of the admissible space, i.e.,

$$\begin{aligned} \Gamma_{free}(\kappa, E) = & V^N (cT)^N \int \frac{dE_1}{c} \dots \frac{dE_N}{c} d^3 p_1 \dots d^3 p_N \cdot \delta\left(\sum_i \frac{\mathbf{p}_i^2}{2M_i} - \frac{E_i^2}{2M_i c^2} - \kappa\right) \\ & \cdot \delta\left(\sum_i \frac{E_i}{c} - \frac{E}{c}\right) \\ = & V^N f(\kappa, E). \end{aligned} \quad (10.26)$$

It then follows that, using the usual thermodynamic relations,

$$S = k_B \ln V^N + k_B \ln f(\kappa, E), \quad (10.27)$$

and

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{k_B}{f(\kappa, E)} \frac{\partial f(\kappa, E)}{\partial E}. \quad (10.28)$$

Differentiating (10.27) with respect to  $V$  at constant  $S$ , we have

$$0 = k_B \frac{N}{V} + k_B \frac{1}{f} \frac{\partial f(\kappa, E)}{\partial E} (-P), \quad (10.29)$$

where

$$P = -\left(\frac{\partial E}{\partial V}\right)_S. \quad (10.30)$$

With (10.28), we then find that the result of Jüttner (1911), i.e.

$$PV = Nk_B T \quad (10.31)$$

is valid for the free relativistic gas.

We now study the ultrarelativistic limit for the free gas by considering the micro-canonical ensemble for  $c \rightarrow 0$ .

With the change of variables

$$\frac{dE_i}{c} = \frac{m_i dm_i c^2}{\sqrt{\mathbf{p}_i^2 + m_i^2 c^2}} \sim c^2 \frac{m_i dm_i}{p_i}, \quad (10.32)$$

where  $p_i = |\mathbf{p}_i|$ , and for  $\kappa = -\frac{1}{2}Mc^2$ , the microcanonical ensemble then reads

$$\begin{aligned} \Gamma(\kappa, E) &\sim (4\pi)^N V^N T^N c^{3N-1} \int m_1 dm_1 \cdots m_N dm_N p_1 dp_1 \cdots p_N dp_N \\ &\cdot \delta\left(\sum_i \frac{m_i^2}{M_i} - M\right) \delta(\sum_i E_i - E) \end{aligned} \quad (10.33)$$

Since  $p_i dp_i = (1/c^2)E_i dE_i$ , we evaluate this integral in the range  $p_i = (0, \infty)$  for which  $E_i = (m_i c^2, \infty) \sim (0, \infty)$ ; taking into account the  $\delta$ -function constraints, the integral becomes

$$\begin{aligned} \Gamma(\kappa, E) &\sim (4\pi)^N V^N c^{N-1} \int m_1 dm_1 \cdots m_N dm_N \delta\left(\sum_i \frac{m_i^2}{M_i} - M\right) \\ &\cdot \int_0^E E_1 dE_1 \cdots \int_0^{E-E_1-\cdots-E_{N-1}} E_{N-2} dE_{N-2} \\ &\cdot \int_0^{E-E_1-\cdots-E_{N-2}} E_{N-1} dE_{N-1} (E - E_1 - E_2 - \cdots - E_{N-1}). \end{aligned} \quad (10.34)$$

By successive differentiation with respect to  $E$ , one finds that the integral has the value  $E^{2N-1}/(2N-1)!$ , so that

$$\Gamma(\kappa, E) \propto E^{2N-1} \sim E^{2N}. \quad (10.35)$$

It then follows from (10.28) that

$$E = 2Nk_B T. \quad (10.36)$$

The same ultrarelativistic limit is obtained, as we shall see below, from the canonical ensemble. For a system with four degrees of freedom (spacetime), and an energy of  $\frac{1}{2}k_B T$  for each particle, this result is consistent with kinetic theory. The canonical ensemble written by Pauli (1921), based on the exponential of  $-c\sqrt{\mathbf{p}^2 + m^2 c^2}/k_B T$  with measure  $d^3 p/p^0$ , although computed in his article for limiting cases only for the nonrelativistic (low temperature) limit, yields  $3Nk_B T$  for the ultrarelativistic (high temperature) case, and does not appear to have a simple kinetic theory interpretation.

Evaluating the relativistic microcanonical phase space integral (10.11) for the rather unrealistic case of a single particle, one obtains

$$\Gamma(\kappa = -\frac{1}{2}Mc^2, E) = 4\pi cVT M \sqrt{E^2/c^2 - M^2 c^2}, \quad (10.37)$$

so that, with (10.28), one obtains

$$k_B T = p \frac{pc}{E/c} = pv, \quad (10.38)$$

in agreement with Pauli; as he remarked, there is no direct connection with equipartition since  $pv$  is not connected in a simple way with the energy of the system as it is in Galilean mechanics.

### 10.3 Canonical Ensemble

To obtain the canonical ensemble, we shall assume that a (small) subsystem  $s$  of the system can exchange only heat (kinetic energy) with the remainder,  $b$ , of the system. The usual assumption of short range forces, justifying the decomposition

$$K \cong K_s + K_b \quad (10.39)$$

must be examined carefully since we are dealing with Lorentz invariant potentials which, in the simplest case, may be considered to be functions of Minkowski squared translation invariant combinations of the  $q_i$ 's. For the case of two body potentials, for example, the quantity  $(q_i - q_j)^\mu (q_i - q_j)_\mu$  may be small, and limited by the shape of the potential as a function of these invariants, but both  $|\mathbf{q}_i - \mathbf{q}_j|$  and  $c(t_i - t_j)$  may be large, thus putting into question the possibility of separating the two subsystems by some boundary in space and time. In this region, however, both events are close to their relative light cone. The surfaces of  $V = \text{const}$  are hyperbolic and asymptotically approach the light cone. A small timelike or spacelike range (in the corresponding invariant measure of distance) implies that the potential is nonvanishing asymptotically only in a thin shell close to the light cone. The existence of a wave operator in scattering theory (discussed in Chap. 7), implying the existence of free asymptotic states in the quantum mechanical problem with potentials of this

type (Horwitz 1980, 1982) provides some corroborative evidence for (10.39). We shall assume its validity.

In our construction of the microcanonical ensemble, it was found that the choice of  $\kappa$  which cancelled the mass shell center of mass motion led to a well-conditioned Galilean limit. The constraint structure of both subsystems must be of the same form as for the entire system. We therefore express the phase space integral for the microcanonical ensemble of the whole system in terms of variable referring to  $s$  and  $b$  as (for  $M = M_s + M_b$ )

$$\begin{aligned} \Gamma(E, M) &= \int d^4 p_1 \cdots d^4 p_{N_s} d^4 q_1 \cdots d^4 q_{N_s} \\ &\quad d^4 p_{N_s+1} \cdots d^4 p_{N_s+N_b} \\ &\quad \cdot d^4 q_{N_s+1} \cdots d^4 q_{N_s+N_b} \delta(K_s + \frac{1}{2} M_s c^2) \delta(K_b + \frac{1}{2} M_b c^2) \\ &\quad \cdot c \delta(E_s + E_b - E), \end{aligned} \quad (10.40)$$

where  $E_s = \sum_{i=1}^{N_s} E_i$ ,  $E_b = \sum_{i=N_s+1}^{N_s+N_b} E_i$ , and we have changed our notation labelling the phase space volume from  $\Gamma(\kappa = -\frac{1}{2} M^2, E)$  to  $\Gamma(E, M)$  for brevity.

Comparing with our original definition (10.11) of the microcanonical ensemble, we can write (10.40) as

$$\begin{aligned} \Gamma(E, M) &= \int d^4 p_1 \cdots d^4 p_{N_s} d^4 q_1 \cdots d^4 q_{N_s} \delta(K_s + \frac{1}{2} M_s c^2) \Gamma_b(E - E_s, M_b) \\ &= \int \frac{dE'}{c} \int d^4 p_1 \cdots d^4 p_{N_s} d^4 q_1 \cdots d^4 q_{N_s} \delta(K_s + \frac{1}{2} M_s c^2) \delta(E_s - E') \cdot c \\ &\quad \Gamma_b(E - E', M_b) \\ &= \int \frac{dE'}{c} \Gamma_s(E', M_s) \Gamma_b(E - E', M_b). \end{aligned} \quad (10.41)$$

Following the usual argument, given *e.g.* by Huang (1967), we assume that there is a maximum in the integrand that dominates the integral at  $E' = \bar{E}$  so that

$$\begin{aligned} S &= k_B \Gamma(E, M) \cong k_B \ln \Gamma_s(\bar{E}, M_s) + k_B \ln \Gamma_b(E - \bar{E}, M_b) \\ &\cong S_s + S_b. \end{aligned} \quad (10.42)$$

From the existence of such a maximum value, implying additivity of the entropy, it follows that (we write the equality assuming that the approximation is very good)

$$\frac{1}{T} = \left. \frac{\partial S_s}{\partial E'} \right|_{E'=\bar{E}} = \left. \frac{\partial S_b}{\partial E'} \right|_{E'=E-\bar{E}}, \quad (10.43)$$

defining  $T$  as a parameter of the equilibrium state. It follows that for any (small) value of  $\bar{E} = E_s$ , which we infer from  $M_b \gg M_s$ ,

$$\begin{aligned} \Gamma_b(E - E_s, M_b) &= e^{S_b(e-E_s, M_b)/k_B} \\ &\cong e^{S_b(E, M_b)/k_B} e^{-\frac{E_s}{k_B T}}. \end{aligned} \quad (10.44)$$



Dropping the subscript  $s$ , the normalized distribution for the canonical ensemble is then

$$D(q, p) = \delta(K + Mc^2)e^{-\beta E} / Q_N(V^{(4)}, T, M), \quad (10.45)$$

where  $\beta = \frac{1}{k_B T}$  and the *relativistic partition function* is then given by

$$\begin{aligned} Q_N(V^{(4)}, T, M) &= \int \frac{d^{4N} p d^{4N} q}{N! h^{4N}} \delta(K + \frac{1}{2} M c^2) e^{-\beta E} \\ &\equiv e^{-\beta A(V^{(4)}, T, M)}, \end{aligned} \quad (10.46)$$

where we have inserted the constant  $h$  of dimension momentum times length, which will occur in numerator and denominator of any expectation value, to make the definition of  $Q_N$  dimensionless. The integration in (10.46) is taken over the constraint  $E = \sum E_i$ ,  $q_i \in \sigma_i$  (subsystem) and  $m_i$  in the small range around the Galilean masses. The formula (10.46) defines the quantity  $A$  that we shall show can be identified with the Helmholtz free energy. Bringing the exponential to the left side, and differentiating

$$\int \frac{d^{4N} p d^{4N} q}{N! h^{4N}} \delta(K + \frac{1}{2} M c^2) e^{\beta(A-E)} = 1 \quad (10.47)$$

with respect to  $\beta$ , one finds

$$\begin{aligned} A = \langle E \rangle - \beta \left( \frac{\partial A}{\partial \beta} \right)_{V^{(4)}} &= \langle E \rangle - T \left( \frac{\partial A}{\partial T} \right)_{V^{(4)}} \\ &\equiv \langle E \rangle - TS, \end{aligned} \quad (10.48)$$

consistent with the interpretation of  $A$  as the Helmholtz free energy. We remark that the relation between the relativistic entropy and the relativistic Helmholtz free energy differs from the nonrelativistic case in that the four-dimensional volume must be held constant.

Fluctuations in energy may be estimated in the usual way (Huang 1967) by differentiating

$$0 = \int \frac{d^{4N} p d^{4N} q}{N! h^{4N}} \delta(K + \frac{1}{2} M c^2) (E - \langle E \rangle) e^{\beta(A-E)} \quad (10.49)$$

with respect to  $\beta$  at constant four volume  $V^{(4)}$  to obtain

$$\langle (E - \langle E \rangle)^2 \rangle = - \frac{\partial \langle E \rangle}{\partial \beta} = k_B T^2 \frac{\partial \langle E \rangle}{\partial T}. \quad (10.50)$$

Assuming that the absorption of heat does not change the time interval  $\Delta t$  of the ensemble, the definition

$$\left( \frac{\partial \langle E \rangle}{\partial T} \right)_{V^{(4)}} = C_V \quad (10.51)$$

coincides with the usual definition of specific heat in the nonrelativistic limit.

The partition function (10.46) can be written as an integral over microcanonical ensembles (10.11) as (absorbing the factors  $N!h^{4N}$  into  $\Gamma$  for now)

$$Q_N(V^{(4)}, T, M) = \int \frac{dE}{c} e^{-\beta E} \Gamma(-\frac{1}{2}Mc^2, E) \quad (10.52)$$

Since, in the microcanonical ensemble,  $S^{mic} = k_B \ln \Gamma$ ,

$$Q_N(V^{(4)}, T, M) = \int \frac{dE}{c} e^{\beta(TS^{mic}(E)-E)}. \quad (10.53)$$

The principal contribution to the integral is at a maximum of the exponent, i.e., where

$$T \frac{\partial S^{mic}(E)}{\partial E} - 1 = 0, \quad \frac{\partial^2 S^{mic}(E)}{\partial E^2} < 0. \quad (10.54)$$

The first of (10.54), coinciding with the definition of temperature in the microcanonical ensemble, implies that the stationary point occurs at  $E = \langle E \rangle$ , and the second corresponds to

$$\frac{\partial}{\partial E} \frac{1}{T(E)} = -\frac{1}{T^2} \frac{1}{C_V} < 0, \quad (10.55)$$

an expected property for physical systems.

Evaluating the integral (10.53) after expanding to second order, one finds, as in the nonrelativistic theory, that the definitions of entropy for the canonical and microcanonical ensemble differ only by terms of the order  $\langle N \rangle$ . It is important to remember that in this analysis we are discussing the statistical mechanics of *events* rather than *particles*. However, we understand that on the condition that every event is a point along a trajectory in spacetime that corresponds to a particle (Weyl 1952), in the sense we have explained in Chap. 4, i.e., the object constituting a conserved four vector current is represented in a covariant way as an integral over the world line (Jackson 1974). Therefore, the counting of events in covariant statistical mechanics can be understood as a counting of particles. Covariant statistical mechanics should, as we have shown above, have a close relationship, with essentially the same meaning for the thermodynamic potentials, to the statistical mechanics of particles associated with world lines.

Following the arguments for the nonrelativistic limit of the microcanonical ensemble given above, one finds that

$$Q_N(V^{(4)}, T, M) \sim \frac{(c\Delta t)^N (c\Delta m)^{N-1}}{N!h^{4N}} \frac{1}{c} e^{-\beta M c^2} \int d^3 p_1 \cdots d^3 p_N d^3 q_1 \cdots d^3 q_N e^{-\beta H}, \quad (10.56)$$

The relativistic canonical ensemble therefore reduces to the usual nonrelativistic one in the limit  $c \rightarrow \infty$  (the factor  $e^{-\beta M c^2}$  corresponds to a shift in  $A$ ).

For the relativistic free gas in the canonical ensemble, let us take all the  $M_i = M_0$ . To compare with the usual formulation, for example, of (Jüttner 1911; Pauli 1921),

we shall restrict the range of masses to be close to the common value  $M_0$ . It then follows that

$$Q_N(V^{(4)}, T, M) \cong \frac{c^{2N-2} (c\Delta t)^N (\delta m)^{N-1}}{N! h^{4N}} 2M_0 \left( V \int d^3 p \frac{e^{-\beta c \sqrt{\mathbf{p}^2 + M_0^2 c^2}}}{\sqrt{\mathbf{p}^2 + M_0^2 c^2}} \right)^N. \quad (10.57)$$

The last factor in (10.57) may be compared to Pauli's formula (Pauli 1921)

$$Q^{Pauli} = \left( V \int d^3 p e^{-\beta c \sqrt{\mathbf{p}^2 + M_0^2 c^2}} \right)^N, \quad (10.58)$$

which differs from our approximate expression (10.57), most significantly at large  $\mathbf{p}^2$ .

Using the relation

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Q_N, \quad (10.59)$$

the result of the integral (Pauli 1921) for (10.58) is

$$\langle E \rangle = N k_B T \left\{ 1 - \sigma \frac{i H_2^{(1)}(i\sigma)}{i H_2^{(1)}(i\sigma)} \right\}, \quad (10.60)$$

where  $H_2^{(1)}$  is the Hankel function of the second kind, and  $\sigma = M_0 c^2 / k_B T$ . For  $\sigma$  large ( $c$  large or  $T$  small, the nonrelativistic limit), one finds (Pauli 1921)

$$\langle E \rangle \sim \frac{3}{2} N k_B T + N M_0 c^2, \quad (10.61)$$

and for  $\sigma$  small ( $c$  small or  $T$  large, the relativistic limit),

$$\langle E \rangle \sim 3 N k_B T. \quad (10.62)$$

The first result, obtained explicitly by Pauli shows consistency with the physics of the nonrelativistic limit of his form of the canonical ensemble, but the last result, not reported in Pauli's work, indicates that in the relativistic limit there is no simple interpretation using the rule of  $\frac{1}{2} k_B T$  for each degree of freedom. However, for the covariant ensemble of (10.57), using the result

$$\int d^3 p \frac{e^{-\beta c \sqrt{\mathbf{p}^2 + M_0^2 c^2}}}{\sqrt{\mathbf{p}^2 + M_0^2 c^2}} = -2\pi^2 \frac{M_0}{\beta} H_1^{(1)}(i\sigma), \quad (10.63)$$

where  $H_1^{(1)}$  is the first type Hankel function, is that for the nonrelativistic limit,

$$\langle E \rangle \sim \frac{3}{2} N k_B T, \quad (10.64)$$

and in the relativistic limit

$$\langle E \rangle \sim 2 N k_B T. \quad (10.65)$$

The latter result is consistent with  $\frac{1}{2} k_B T$  for each degree of freedom (there are four, for spacetime). This regime should be valid for  $T > 10^{12}$  degrees Kelvin.

## 10.4 Grand Canonical Ensemble

We shall consider the grand canonical ensemble as composed of a set of subsystems corresponding to canonical ensembles generated by the exchange of both heat energy and particles with the bath. We cannot specify in advance the quantity  $M = \sum_i M_i$  because the number of particles is indefinite. There are  $N!/N_s!(N - N_s)!$  ways of selecting a subsystem of  $N_s$  particles, and therefore the canonical partition function for this set may be written

$$\begin{aligned} Q_N(V^{(4)}, T, M) &= \sum_{N_s+N_b=N} \int \frac{d^{4N_s} p d^{4N_s} q d^{4N_b} p d^{4N_b} q}{N! h^{4N}} \frac{N!}{N_s!(N - N_s)!} \\ &\cdot e^{-\beta(E_s+E_b)} \delta(K_s + K_b + \frac{1}{2} M c^2) \\ &= \sum_{N_s=0}^N \int \frac{d^{4N_s} p d^{4N_s} q}{N_s! h^{4N_s}} e^{-\beta E_s} Q_{N-N_s}(V^{(4)} - V_s^{(4)}, T, M + \frac{2K_s}{c^2}), \end{aligned} \quad (10.66)$$

where we have written the integral measures in a compact form. Let us choose a temperature  $T$  so that the principal contributions to the phase space integral come from  $E_s \ll E$  and therefore  $N_s \ll N$ . Then,  $K_s \ll K \cong -\frac{1}{2} M c^2$ , and we may approximate

$$\begin{aligned} Q_{N-N_s}(V^{(4)} - V_s^{(4)}, T, M + \frac{2K_s}{c^2}) \\ \cong \exp \left\{ \beta V_s^{(4)} \frac{\partial A}{\partial V^{(4)'}} \Big|_{V^{(4)'}=V^{(4)}} - \beta \frac{2K_s}{c^2} \frac{\partial A}{\partial M'} \Big|_{M'=M} + \beta N_s \frac{\partial A}{\partial N'} \Big|_{N'=N} \right\} \end{aligned} \quad (10.67)$$

if the Helmholtz free energy is sufficiently slowly varying at the endpoint values. The first term corresponds to the change in the free energy due to a change in spacetime volume due to the exchange of particles between the subsystem and the bath. We shall assume that the class of subsystems that we are selecting are typical in the sense that they have the same *a priori* time interval associated with them as the bath; therefore the relevant derivative of  $A$  holds the time interval fixed, and its associated  $\Delta t$  cancels out. Hence,

$$\frac{\partial A}{\partial V^{(4)'}} \Big|_{V^{(4)'}=V^{(4)}} = V_s \frac{\partial A}{\partial V'} \Big|_{V'=V} = -V_s P, \quad (10.68)$$

where  $V$  is the spatial volume of the canonical ensemble, and  $V_s$  is the spatial volume of the subsystem; the pressure is defined in terms of its usual relation to the Helmholtz free energy.

The second term is an essentially relativistic effect, associated with the mass degree of freedom. We define a corresponding chemical potential

$$\frac{\partial A}{\partial M'} \Big|_{M'=M} = \frac{1}{2} \mu_K c^2, \quad (10.69)$$

and refer to  $\mu_K$  as the *mass chemical potential* for the subsystem.

The last term in (10.67) can be immediately identified with the usual chemical potential for the number of particles in the subsystem, so that

$$\frac{\partial A}{\partial N'} \Big|_{N'=N} = \mu. \quad (10.70)$$

Taking our sum to infinity, we can now define the grand canonical ensemble partition function providing  $P, V, T$  relations, as

$$Q(V^{(4)}, \zeta, z, T) = \sum_N z^N \hat{Q}_N(V^{(4)}, \zeta, T) \equiv e^{\beta PV}, \quad (10.71)$$

where

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \int \frac{d^{4N} p d^{4N} q}{N! h^{4N}} e^{-\beta E} \zeta^K, \quad (10.72)$$

and we have dropped the subscript  $s$  for the subsystem.

As in the nonrelativistic theory, it follows immediately that

$$\langle N \rangle = z \frac{\partial}{\partial z} \ln Q = k_B T \frac{\partial}{\partial \mu} \ln Q \quad (10.73)$$

and that

$$\frac{PV}{k_B T} = \ln Q(V^{(4)}, \zeta, z, T). \quad (10.74)$$

The average value of  $K$  is given by

$$\langle K \rangle = \zeta \frac{\partial}{\partial \zeta} \ln Q = -k_B T \frac{\partial}{\partial \mu_K} \ln Q, \quad (10.75)$$

so that it follows from (10.73) and (10.75) that

$$\frac{\partial \langle N \rangle}{\partial \mu_K} = - \frac{\partial \langle K \rangle}{\partial \mu}, \quad (10.76)$$

associating the number dependence on mass and the mass dependence on number in a symmetrical way.

We define the Helmholtz free energy for the grand canonical ensemble as

$$A = \langle N \rangle k_B T \ln z + \langle K \rangle k_B T \ln \zeta - k_B T \ln Q, \quad (10.77)$$

from which it follows that

$$Q = z^{\langle N \rangle} \zeta^{\langle K \rangle} e^{-\beta A} \quad (10.78)$$

$$1 = \sum_N z^{N - \langle N \rangle} e^{\beta A} \int \frac{d^{4n} p d^{4N} d}{N! h^{4n}} e^{-\beta E} \zeta^{K - \langle K \rangle} \quad (10.79)$$

Taking the derivative with respect to  $\beta$ , holding  $z, \zeta, V^{(4)}$  fixed, we obtain

$$0 = - \frac{\partial \langle N \rangle}{\partial \beta} \ln z - \frac{\partial \langle K \rangle}{\partial \beta} \ln \zeta + \frac{\partial}{\partial \beta} (\beta A) - \langle E \rangle. \quad (10.80)$$

It follows from (10.77) that

$$\frac{\partial}{\partial \beta}(\beta A) = \frac{\partial \langle N \rangle}{\partial \beta} \ln z + \frac{\partial \langle K \rangle}{\partial \beta} \ln \zeta - \frac{\partial}{\partial \beta} \ln \mathcal{Q}, \quad (10.81)$$

so that

$$U \equiv \langle E \rangle = -\frac{\partial}{\partial \beta} \ln \mathcal{Q}, \quad (10.82)$$

and hence, from the definitions of  $A$  and  $U$ ,

$$U = A - \mu \langle N \rangle + \mu_K \langle K \rangle + k_B T \ln \mathcal{Q} + k_B T^2 \frac{\partial}{\partial T} \ln \mathcal{Q}. \quad (10.83)$$

From the relation  $U = A + TS$ , we identify the entropy

$$S = \frac{\partial}{\partial T} (k_B T \ln \mathcal{Q}) + \frac{\mu_K \langle K \rangle}{T} - \frac{\mu \langle N \rangle}{T}. \quad (10.84)$$

With these results, it is easy to verify that the Maxwell relations

$$\begin{aligned} S &= -\left(\frac{\partial A}{\partial T}\right)_{V, \langle N \rangle, \langle K \rangle}, \\ P &= -\left(\frac{\partial A}{\partial V}\right)_{T, \langle N \rangle, \langle K \rangle}, \end{aligned} \quad (10.85)$$

and to verify, in terms of our original definitions, that

$$\begin{aligned} \mu &= \left(\frac{\partial A}{\partial N}\right)_{\langle K \rangle, T, V} \\ \mu_K &= -\left(\frac{\partial A}{\partial \langle K \rangle}\right)_{\langle N \rangle, T, V} \end{aligned} \quad (10.86)$$

If the grand canonical ensemble peaks strongly at a particular value of  $N$ , the canonical ensemble is recovered for that  $N$ , and similarly for a particular value  $\bar{K}$  of  $K$ , which we could call  $-\frac{1}{2}\bar{M}c^2$ .

Using the asymptotic form for  $E$  previously studied, the Galilean limit of the grand canonical ensemble can be obtained for each  $N$  using the relation

$$\begin{aligned} e^{-\beta E} e^{-\beta \mu_K K} &\sim e^{-\beta(\epsilon' + Mc^2)} e^{-\beta \mu_K (H - \epsilon' - Mc^2/2)} \\ &= e^{-\beta M c^2} e^{\beta \mu_K M c^2/2} e^{\beta \epsilon' (1 - \mu_K)} e^{-\beta \mu_K H}, \end{aligned} \quad (10.87)$$

from which we see that  $\mu_K \rightarrow 1$  as  $c \rightarrow \infty$ . Then the  $\epsilon'$  dependence cancels, and the first two factors combine to  $e^{-\beta M c^2/2}$ . The exponential damping factor is compensated if

$$\mu \sim \mu_k + \frac{1}{2} \frac{M}{N} c^2, \quad (10.88)$$

where the quantity  $M/N$ , the average mass per particle, should not depend on  $N$ . With these assumptions, and the approximation  $dE_i/c \sim c dm_i$ , one finds that

$$\mathcal{Q}(V^{(4)}, \zeta, z, T) \sim \Sigma_0^\infty z'^N \hat{\mathcal{Q}}_N(V, T) \quad (10.89)$$

where  $z' = e^{\mu_N R \beta}$ , and

$$\hat{\mathcal{Q}}_N(V, T) = \int \frac{d^{3N} p d^{3N} q}{N! h^{3N}} e^{-\beta H}, \quad (10.90)$$

and we have taken the limit in  $\Delta m$ ,  $\Delta t$  integrations so that

$$c\Delta mc\Delta t/h = \Delta\eta\Delta t/h = 1. \quad (10.91)$$

Using the general formulas for the dispersions from the grand canonical ensemble, by differentiating with respect to  $\mu_K$  and  $\mu$ ,

$$\begin{aligned} \langle (K - \langle K \rangle)^2 \rangle &= -k_B T \frac{\partial \langle K \rangle}{\partial \mu_K}, \\ \langle (N - \langle N \rangle)^2 \rangle &= k_B T \frac{\partial \langle N \rangle}{\partial \mu} \end{aligned} \quad (10.92)$$

For the free particle gas, one finds that (Horwitz 1981)

$$\frac{\Delta K_0}{\langle K_0 \rangle} = \frac{1}{\sqrt{\langle N \rangle}} + O\left(\frac{1}{c^4}\right), \quad (10.93)$$

just as for the fluctuations in  $N$ , thus verifying the close relationship between  $K$  and the number of particles. In a similar way, one can verify that (Horwitz 1981)

$$\frac{\langle E \rangle - M_0 c^2 \langle N \rangle}{\langle N \rangle} = \frac{3}{2} k_B T + O\left(\frac{1}{c^2}\right) \quad (10.94)$$

and the the Dulong-Petit law for the specific heat becomes

$$\frac{\partial \langle E \rangle - M_0 c^2 \langle N \rangle}{\partial T \langle N \rangle} = \frac{3}{2} k_B + O\left(\frac{1}{c^2}\right). \quad (10.95)$$

Therefore, the relativistic grand canonical ensemble that we have developed here contains the standard results of the nonrelativistic theory, and can serve as a useful generalization with applications from classical microscopic systems to astronomical calculations (Hakim 2011).

## 10.5 Relativistic Quantum Quantum Statistical Mechanics

In this section we show how the results of the previous sections on classical statistical ensembles can be extended to the quantum case. In the quantum theory, the density operator for a quantum state composed of a maximal mixture of energies, which corresponds to a microcanonical ensemble, is represented by

$$\rho = \sum_{k, E \in \Delta} \psi_{k, E} \psi_{k, E}^*, \quad (10.96)$$

where  $\psi_{k, E}$  are eigenfunctions (or generalized eigenfunctions in the case of continuous spectrum) of the total  $K$  operator, and  $E$  is the value of the total energy operator (well-defined by translation invariance of the whole system), as discussed for the classical microcanonical ensemble. The masses  $m_i$  in this representation may be specified to lie in small intervals  $\mu_i$  as for the classical case discussed above. Then, the total number of states with  $k$ ,  $E \in \Delta$  and  $m_i \in \mu_i$  is

$$\Gamma(k, E) = Tr \rho = \sum_{k, E \in \Delta} \|\psi_{k, E}\|^2, \quad (10.97)$$

and the entropy is defined as

$$S(k, E) = k_B \ln \Gamma(k, E). \quad (10.98)$$

The canonical ensemble is defined as for the classical case discussed above, and is given by

$$\rho = e^{-\beta E} \delta_{K, -Mc^2/2}, \quad (10.99)$$

where  $\beta = 1/k_B T$ , and the partition function is

$$Q_N(V^{(4)}, T, M) = \text{Tr}(e^{\beta E} \delta_{K, -Mc^2/2}), \quad (10.100)$$

so that the expectation of any operator  $\mathcal{O}$  in this ensemble is given by

$$\langle \mathcal{O} \rangle_N = \frac{\text{Tr}(\mathcal{O} e^{\beta E} \delta_{K, -Mc^2/2})}{Q_N}. \quad (10.101)$$

To obtain the grand canonical ensemble, we select a subsystem  $s$  of the system described by a canonical ensemble of  $N$  particles, and write the partition function, as done for the classical case, as in (10.46), as

$$\begin{aligned} Q_N(V^{(4)}, T, M) &= \sum_{N_s+N_b=N} \text{Tr}_s \text{Tr}_b (e^{-\beta E_s} e^{-\beta E_b} \delta_{K_s+K_b, -Mc^2/2}) \\ &= \sum_{N_s=0}^N \text{Tr}_s (V^{(4)} - V_s^{(4)}, T, M + 2K/c^2) \end{aligned} \quad (10.102)$$

Following the procedure of the classical case, we obtain (10.71), i.e.,

$$\mathcal{Q}(V^{(4)}, \zeta, z, T) = \sum_N z^N \hat{Q}_N(V^{(4)}, \zeta, T) \equiv e^{\beta PV}, \quad (10.103)$$

where now

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \text{Tr}_N (e^{-\beta E} \zeta^K), \quad (10.104)$$

and, and as we shall see,  $\zeta = e^{-\mu_K \beta}$ .

For the ideal free quantum gas in a spacetime box of dimension

$$-L/2 \leq x, y, z \leq L/2, \quad \Delta t/2 \leq t \leq \Delta t/2,$$

the microcanonical distribution is characterized by the spectrum

$$2MK = \hbar^2(k_1^2 + k_2^2 + k_3^2 - k_0^2) \quad (10.105)$$

where

$$k_0 = \frac{2\pi}{\Delta t} \nu_0, \quad k_j = \frac{2\pi \nu_j}{L} \quad j = 1, 2, 3, \quad (10.106)$$

and  $\nu_0, \nu_j = 0, \pm 1, \pm 2, \dots$ . Then,  $\mathbf{p} = (2\pi\hbar/L)^\circ$  and  $\varepsilon = (2\pi\hbar/\Delta t)\nu_0$ . The integral measure is given by

$$d^3 p d\varepsilon \sim \frac{(2\pi\hbar)^4}{V^{(4)}}, \quad V^{(4)} = L^3 \Delta t. \quad (10.107)$$

We now compute the Bose-Einstein, Fermi-Dirac and Boltzmann distributions in terms of the discrete sums characteristic of kinetic theory. Consistently with our remarks in Chap. 3, we recall that the wavefunctions of all constituents are associated



with the same value of  $n^\mu$  on the inducing orbit, and therefore the BE and FD cases obey the spin-statistics theorem. Let

$$\begin{aligned} i &= \text{cell around } \mathbf{p}, \varepsilon, m \in \mu, \\ g_i &= \text{number of mass and momentum states in each cell} \\ n_i &= \sum_{\mathbf{p}, \varepsilon} n_{\mathbf{p}, \varepsilon}, \end{aligned} \quad (10.108)$$

where  $n_{\mathbf{p}, \varepsilon}$  is the number of particles with energy momentum  $\mathbf{p}, \varepsilon$ . Let  $W(\{n_i\})$  be the number of states associated with the distribution  $\{n_i\}$ . Then, the total number of states in phase space is

$$\Gamma(E, K_0) = \sum_{\{n_i\}} W(\{n_i\}) \quad (10.109)$$

with the constraints

$$E = \sum_i \varepsilon_i n_i \quad K_0 = \sum_i K_i n_i \quad N = \sum_i n_i, \quad (10.110)$$

where  $K_i$  is the average value of  $K(\mathbf{p}, \varepsilon)$  in the  $i$ th cell. Taking into account the constraints (10.110) we wish to find  $\{n_i\}$  such that

$$\delta \{ \ln W(\{n_i\}) - \alpha \sum_i n_i - \beta \sum_i \varepsilon_i n_i + \gamma \sum_i K_i n_i \} = 0 \quad (10.111)$$

where  $\alpha, \beta, \gamma$  are Lagrange parameters implementing the constraints. Permitting up to  $g_i$  states in each cell for Fermi-Dirac statistics, and all integer values for Bose-Einstein statistics, we find the distributions

$$\begin{aligned} W(\{n_i\}) &= \Pi_i \frac{(n_i + g_i - 1)!}{N_i! (g_i - 1)!} \quad (\text{Bose-Einstein}) \\ &= \Pi_i \frac{g_i!}{n_i! (g_i - n_i)!} \quad (\text{Fermi-Dirac}) \\ &= \Pi_i \frac{g_i!}{n_i!} \quad (\text{Boltzmann}) \end{aligned} \quad (10.112)$$

and obtain the average occupation number distributions (the sign of  $K_i$  is important in establishing the sign of the second variation)

$$\begin{aligned} \bar{n}_i &= \frac{g_i}{z^{-1} \zeta^{-K_i} e^{\beta \varepsilon_i} - 1} \quad (\text{Bose-Einstein}) \\ &= \frac{g_i}{z^{-1} \zeta^{-K_i} e^{\beta \varepsilon_i} + 1} \quad (\text{Fermi-Dirac}) \\ &= g_i z \zeta^{K_i} e^{-\beta \varepsilon_i} \quad (\text{Boltzmann}), \end{aligned} \quad (10.113)$$

where  $z = e^\alpha$  and  $\zeta = e^\gamma$ . Using the maximal distributions in (10.109), the entropy is given by

$$S = k_B \ln W(\{\bar{n}_i\}) \quad (10.114)$$

Pinching down the size of the cells to obtain continuum distributions, we can write (taking  $g_i = 1$ )

$$\begin{aligned} \bar{n}_{\mathbf{p}, \varepsilon} &= \frac{1}{z^{-1} \zeta^{-K(\mathbf{p}, \varepsilon)} e^{\beta \varepsilon} - 1} \quad (\text{Bose - Einstein}) \\ &= \bar{n}_{\mathbf{p}, \varepsilon} \\ &= \frac{1}{z^{-1} \zeta^{-K(\mathbf{p}, \varepsilon)} e^{\beta \varepsilon} + 1} \quad (\text{Fermi - Dirac}) \\ &= z \zeta^{K(\mathbf{p}, \varepsilon)} e^{-\beta \varepsilon} \quad (\text{Boltzmann}). \end{aligned} \quad (10.115)$$

The parameters  $z, \beta, \zeta$  are to be determined from

$$\begin{aligned}\Sigma_{\mathbf{p},\varepsilon}\varepsilon\bar{n}_{\mathbf{p},\varepsilon} &= E, \\ \Sigma_{\mathbf{p},\varepsilon}\bar{n}_{\mathbf{p},\varepsilon} &= N \\ \Sigma_{\mathbf{p},\varepsilon}K(\mathbf{p},\varepsilon)\bar{n}_{\mathbf{p},\varepsilon} &= K_0,\end{aligned}\tag{10.116}$$

where the sums are to be taken over a narrow range of masses  $\Delta m$ . Comparing, in the Boltzmann case, with the classical grand canonical distributions, we identify  $\beta = \frac{1}{k_B T}$ ,  $z = e^{\mu\beta}$ ,  $\zeta = e^{-\mu_K\beta}$  (we show below that these results can be derived directly from the quantum grand canonical ensemble).

Note that in the Fermi-Dirac distribution we have counted as distinct states the several values of  $\varepsilon$  for each  $\mathbf{p}$  which lie within the admissible width  $\Delta m$ . Although the distributions we have obtained are formally very similar (except for the factor  $\zeta^K$ ), the usual notion of the Fermi-Dirac statistics treats all of these states as the same; the role of the mass potential  $\mu_K$  is to control this mass distribution within the small interval  $\Delta m$ , and thus it is expected that the results of any expectation value remain, in general, closely the same as in the mass shell theories.

Using Stirling's approximation for the factorials, one finds that for the Boltzmann gas,

$$S/k_B = \beta E - K \ln \zeta - N \ln z.\tag{10.117}$$

We now turn to a study of the ideal gas from the point of view of the grand canonical ensemble.

For Boltzmann statistics, (10.72) can be written as

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \Sigma_{n_{\mathbf{p},\varepsilon}} \frac{1}{N!} \left( \frac{N!}{\prod_{\mathbf{p},\varepsilon} n_{\mathbf{p},\varepsilon}} \right) e^{-\beta E(\mathbf{p},\varepsilon)} \zeta^{K(\mathbf{p},\varepsilon)},\tag{10.118}$$

where

$$\begin{aligned}E(\{n_{\mathbf{p},\varepsilon}\}) &= \Sigma_{\mathbf{p},\varepsilon} \varepsilon n_{\mathbf{p},\varepsilon}, \\ K(\{n_{\mathbf{p},\varepsilon}\}) &= \Sigma_{\mathbf{p},\varepsilon} K(\mathbf{p},\varepsilon)\end{aligned}\tag{10.119}$$

and

$$N = \Sigma_{\mathbf{p},\varepsilon} n_{\mathbf{p},\varepsilon}\tag{10.120}$$

as a constraint.

With the constraint (10.120), the sum in (10.118) becomes

$$\hat{Q}_N(V^{(4)}, \zeta, T) = \frac{1}{N!} (\Sigma_{\mathbf{p},\varepsilon} e^{-\beta\varepsilon} \zeta^{K(\mathbf{p},\varepsilon)})^N,\tag{10.121}$$

and therefore

$$Q(V^{(4)}, \zeta, z, T) = \exp \{ z \Sigma_{\mathbf{p},\varepsilon} e^{-\beta\varepsilon} \zeta^{K(\mathbf{p},\varepsilon)} \}.\tag{10.122}$$

The equation of state can then be obtained explicitly by noting that, as in the classical case,

$$\langle N \rangle = z \frac{\partial}{\partial z} \ln Q = \ln Q = \frac{PV}{k_B T} \quad (10.123)$$

Evaluating the distributions for Bose-Einstein and Fermi-Dirac statistics one obtains (Horwitz 1981)

$$\begin{aligned} Q(V^{(4)}, \zeta, z, T) &= \prod_{\mathbf{p}, \varepsilon} \frac{1}{1 - ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}} \quad (BE) \\ &= \prod_{\mathbf{p}, \varepsilon} (1 + ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}) \quad (FD). \end{aligned} \quad (10.124)$$

The equations of state for the relativistic free quantum gas are

$$\begin{aligned} \frac{PV}{k_B T} &= \ln Q = -\sum_{\mathbf{p}, \varepsilon} \ln(1 - ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}) \quad (BE) \\ &= \sum_{\mathbf{p}, \varepsilon} \ln(1 + ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}) \quad (FD). \end{aligned} \quad (10.125)$$

The total number of particles is

$$\begin{aligned} N &= z \frac{\partial}{\partial z} \ln Q = \sum_{\mathbf{p}, \varepsilon} \frac{ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}}{1 - ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}} \quad (BE) \\ &= \sum_{\mathbf{p}, \varepsilon} \frac{ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}}{1 + ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}} \quad (FD) \end{aligned} \quad (10.126)$$

Similarly, by differentiating with respect to  $\beta\varepsilon - \ln K(\mathbf{p}, \varepsilon)$ , we find that the average occupation numbers are given by

$$\langle n_{\mathbf{p}, \varepsilon} \rangle = \frac{ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}}{1 \mp ze^{-\beta\varepsilon} \zeta^{K(\mathbf{p}, \varepsilon)}}. \quad (10.127)$$

Equation (10.126) then correspond to

$$N = \sum_{\mathbf{p}, \varepsilon} \langle n_{\mathbf{p}, \varepsilon} \rangle. \quad (10.128)$$

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## 10.6 Relativistic High Temperature Boson Phase Transition

Haber and Weldon (1982) showed that in the usual (mass shell) form of relativistic quantum mechanics, that taking into account both the particle and antiparticle distribution functions, a system of bosons can undergo a high temperature phase transition. The introduction of antiparticles in the theory, by application of the arguments of Haber and Weldon, imply the addition of another term in the total number expectation with a negative sign, carrying an opposite sign for the energy chemical

potential, i.e., formula (10.126) (for the boson case) is written as (Burakovsky 1996) (dividing numerator and denominator by the numerator factor).<sup>3</sup>

$$N = V^{(4)} \sum_{k^\mu} \left[ \frac{1}{e^{(E-\mu-\mu_K m^2/2M)/T} - 1} - \frac{1}{e^{(E+\mu-\mu_K m^2/2M)/T} - 1} \right] \quad (10.129)$$

As assumed by Haber and Weldon, the total particle number remains unchanged in the equilibrium state, but the presence of antiparticles implies annihilation and creation processes. Thus, in counting the total number of particles, the antiparticle distribution must carry a negative sign, consistent with the interpretation of Stueckelberg as given in the early chapters of the book. On the other hand, both the terms in the sum in Eq. (10.129) must separately be positive, implying the inequalities

$$\begin{aligned} m - \mu - \mu_K \frac{m^2}{2M} &\geq 0, \\ m + \mu - \mu_K \frac{m^2}{2M} &\geq 0, \end{aligned} \quad (10.130)$$

resulting in the inequalities representing the nonnegativeness of the discriminants in the mass quadratic formulas,

$$-\frac{M}{2\mu_K} \leq \mu \leq \frac{M}{2\mu_K}. \quad (10.131)$$

The bounds of the intersection of the regions satisfying the inequalities (10.130) are given by

$$\frac{M}{\mu_K} \left(1 - \sqrt{1 - \frac{2|\mu|\mu_K}{M}}\right) \leq m \leq \frac{M}{\mu_K} \left(1 + \sqrt{1 - \frac{2|\mu|\mu_K}{M}}\right), \quad (10.132)$$

which for small  $\frac{|\mu|\mu_K}{M}$  reduces, as in the no-antiparticle case to

$$|\mu| \leq m \leq \frac{2M}{\mu_K} \quad (10.133)$$

Replacing the summation in (10.129) by integration, one obtains the formula for the number density

$$\begin{aligned} n &= \frac{1}{4\pi^3} \int_{m_1}^{m_2} m^3 dm \int_{-\infty}^{\infty} \sinh^2 \beta d\beta \\ &\times \left[ \frac{1}{e^{(m \cosh \beta - \mu - \mu_K m^2/2M)/T} - 1} - \frac{1}{e^{(m \cosh \beta + \mu - \mu_K m^2/2M)/T} - 1} \right], \end{aligned} \quad (10.134)$$

<sup>3</sup>Since the sign of the energy of the antiparticle is opposite to that of the particle, the chemical potential  $\mu$  must change sign for the antiparticle, but the mass squared of both particle and antiparticle are positive, and therefore the sign of  $\mu_K$  does not change.

where  $m_1$  and  $m_2$  are defined by the bounds (10.132). Integrating out the  $\beta$  variable, one finds for high temperature  $\mu/T \ll 1$ , one finds

$$n \cong \frac{1}{\pi^3} \left( \frac{M}{\mu_K} \right)^2 \mu T \sqrt{1 - \frac{2|\mu|\mu_K}{M}}. \quad (10.135)$$

For  $T$  above a critical value, the range of admissible masses become pinched down to zero, corresponding to a phase transition, where the dispersion

$$\delta m = \sqrt{\langle m^2 \rangle - \langle m \rangle^2}$$

vanishes as  $\sqrt{T - T_c}$ , a second order transition, corresponding to a ground state with  $p_\mu p^\mu = -(M/\mu_k)^2$ . States with temperature  $T > T_c$  correspond to off-shell excitations of such a ground state.

The phase transition that we have described selects a definite mass for the particles, but this result is statistical. Although the mean fluctuations vanish, there is nevertheless sufficient freedom in the phase space for each particle to fulfil the off-shell requirements for the formulation of the Stueckelberg theory.

This mechanism provides an insight into a possibly more general formulation which would explain the stability of the asymptotic mass of a particle in the Stueckelberg theory in the presence of arbitrary number of collisions; the existence of several solutions could give rise to what appears phenomenologically as mass spectra of observed particles (Kirsten 1991).

## 10.7 Black Body Radiation

As we have discussed in previous chapters, the Stueckelberg-Schrödinger equation implies that the electromagnetic gauge fields are five dimensional, including an  $a_5$  field which compensates for the  $\tau$  derivative of the evolving wave function. The usual argument for two polarization states of the four dimensional Maxwell field is that, of the four degrees of freedom, there is a gauge condition and the constraint of the Gauss law, leaving two polarization states. The factor of two on the Bose-Einstein distribution is essential for the computation of the specific heat of a black body, but the argument of the existence of two constraints leaves the possibility of three polarization states. In the following, we show that the observable radiation field of a black body indeed carries just two polarization states (Horwitz 2015).

The canonical quantization of the 5D radiation field was carried out by Shnerb and Horwitz (1993) following the basic ideas of Teitelboim and Henneaux (1992) and Haller (1972) using algebraic methods. Taking for this discussion, as in Shnerb (1993), the signature of the 5D manifold to be  $[\sigma, +, -, -, -]$ , we write the action for the interacting fields (in this section we work in the framework of both quantized gauge fields and quantized wave functions  $\psi$ ) as

$$\begin{aligned}
S = & \int_{-\infty}^{\infty} d^5x \left\{ -\frac{\lambda}{4} f^{\alpha\beta} f_{\alpha\beta} - G(x) [\partial_\alpha a^\alpha(x)] + \frac{1}{2\lambda} G^2(x) \right. \\
& i\psi^\dagger(x) \frac{\partial\psi(x)}{\partial\tau} \\
& - \frac{1}{2M} \psi^\dagger(x) \frac{\partial\psi(x)}{\partial\tau} - \frac{1}{2M} \psi^\dagger [\partial^\mu - ie' a^\mu(x)] [\partial_\mu - ie' a_\mu(x)] \psi(x) \\
& \left. + e' \psi^\dagger(x) a_\tau(x) \psi(x) \right\}, \tag{10.136}
\end{aligned}$$

where  $\lambda$  is a quantity with dimensions of length (it will play the role of the  $\tau$ -correlation length of the wave function in the Maxwell limit). As discussed in earlier chapters,  $e'$  is the coupling constant of the covariant theory, which also has dimension of length, and  $G$  plays the role of an auxiliary field (Haller 1972) (somewhat analogous to the Fadeev-Popov ghosts (Fadeev 1967) of the path integral approach). The canonically conjugate momenta are given by

$$\begin{aligned}
\pi^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\tau a_\mu)} = -\lambda f^{\tau\mu}, \\
\pi^\tau &= \frac{\delta\mathcal{L}}{\delta(\partial_\tau a_\tau)} = -\sigma G, \\
\pi_\psi &= \frac{\delta\mathcal{L}}{\delta(\partial_\tau \psi)} = i\psi^\dagger. \tag{10.137}
\end{aligned}$$

We now impose equal time commutation relations

$$[\pi^\alpha(x), a_\beta(y)] = -i\delta_\beta^\alpha \delta(x-y) \tag{10.138}$$

and (we are assuming  $\psi$  a boson field for our present purposes)

$$[i\psi^\dagger(x), \psi(y)] = -i\delta(x-y). \tag{10.139}$$

The Hamiltonian (generating unitary evolution in  $\psi$  and  $a^\alpha$  then takes the form

$$\begin{aligned}
K &= \sigma \int d^4x [\pi^\mu (\partial_\tau a_\mu) + \pi^\tau (\partial_\tau a_\tau) + i\psi^\dagger \partial_\tau \psi - \mathcal{L}] \\
&= K_\gamma + K_m + K_{\gamma m}, \tag{10.140}
\end{aligned}$$

where

$$\begin{aligned}
K_\gamma &= \int d^4x \left\{ -\frac{1}{2\lambda} \pi^\mu \pi_\mu - \frac{\lambda\sigma}{4} f^{\mu\nu} f_{\mu\nu} \right. \\
&\quad \left. + \pi^\mu (\partial_\mu a^\tau) - \pi^\tau (\partial_\mu a^\mu) - \frac{1}{2\lambda} \pi^\tau \pi_\tau \right\} \tag{10.141}
\end{aligned}$$

and

$$\begin{aligned}
K_m &= \frac{\sigma}{2M} \int d^4x \psi^\dagger \partial_\mu \partial^\mu \psi, \\
K_{\tau m} &= d^4x \left\{ -e' \psi^\dagger a_\tau \psi - \frac{ie'}{2M} \psi^\dagger [a^\mu \partial_\mu + (\partial_\mu a^\mu)] \right. \\
&\quad \left. - \frac{e'^2}{2M} \psi^\dagger \psi a^\mu a_\mu \right\}. \tag{10.142}
\end{aligned}$$

The stability condition on the states for the restriction to the Gauss law

$$\langle \partial_\mu \pi^\mu + j^\tau \rangle = 0 \quad (10.143)$$

implies that  $\langle \pi^\tau \rangle = 0$ ; one can then eliminate the longitudinal part of the field  $a^\mu$ .

In case the four vector  $k^\mu$  in the Fourier decomposition of the  $a^\mu$  field is timelike, for which the  $O(4, 1)$  theory is stable, one can eliminate, by a unitary transformation (as in the Maxwell case), the time component of  $a^\mu$ . There remains, except for the Coulomb term, three spacelike polarization components  $a^i$ , and the Hilbert space has positive norm.

We have argued in Chap. 3 that vector bosons must lie in a representation of angular momentum with spin 1; as discussed in Jauch and Rohrlich (Jauch 1955, p. 41), these components, with canonical commutation relations, provide a representation, in any choice of gauge, that meets this requirement. For the asymptotic photons of the black body radiation, the components for  $k^\mu$  spacelike, for which the stable solutions are representations of  $O(2, 1)$  do not meet this requirement. Furthermore, in the case that  $k^\mu$  is lightlike, the elimination of longitudinal modes corresponds exactly to the removal of both  $a_0$  and  $a_\parallel$ , leaving just two polarization states. This limiting case is realized for the asymptotic photons of the black body when  $\tau \rightarrow \infty$ , leaving, by application of the Riemann-Lebesgue lemma, the “massless” zero mode.<sup>4</sup> We make this argument explicit in the following.

The analog of the radiation gauge (e.g. Bjorken 1964) for the 5D fields would correspond to setting the  $a_5$  field equal to zero; this corresponds to subtracting the 5-gradient of the indefinite integral of the  $a_5$  field from the  $a_\alpha$  fields, i.e., for

$$a^{5'} = a^5 + \partial^5 \Lambda, \quad (10.144)$$

we can take

$$\Lambda(x, \tau) = - \int^\tau a^5(x, \tau') + \tilde{\Lambda}(x). \quad (10.145)$$

Then, since the second term is independent of  $\tau$ ,  $a^{5'} = 0$ . Furthermore, since

$$a^{\mu'} = a^\mu + \partial^\mu \Lambda(x), \quad (10.146)$$

it follows that

$$a^{0'} = a^0(x, \tau) - \partial^0 \int^\tau a^5(x, \tau') d\tau' + \partial^0 \tilde{\Lambda}(x). \quad (10.147)$$

Under the assumption that the asymptotic fields are independent of  $\tau$ , assuming as well convergence of the indefinite integral in (10.147) for large  $\tau$ , we can make  $a^{0'} = 0$  asymptotically with the choice

$$\tilde{\Lambda}(x) = - \int^t a^0(\mathbf{x}, t', \tau) dt' + \int^t \int^\tau a^5(\mathbf{x}, t', \tau') d\tau' dt'. \quad (10.148)$$

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<sup>4</sup>It was suggested by Andrew Bennett (private communication) that the concatenated field equations, corresponding to an integral over  $\tau$  would equivalently lead to this result.

The remaining term of the generalized Lorentz gauge  $\partial_\alpha a^\alpha = 0$  is just the condition  $\nabla \cdot \mathbf{a} = 0$ , exhibiting the required rotational invariance on the orbit of the induced representation for the  $a^\mu$  field. The longitudinal component along the  $\mathbf{k}$  vector must therefore vanish, and we are left with two effective polarization states.

Therefore, with the Gauss law and the additional gauge condition on the  $5D$  fields, there are three constraints on the  $5D$  fields, leaving (asymptotically) two degrees of freedom.

The remaining degrees of freedom correspond, in the induced representation, to two polarization states that are directly interpretable as angular momentum states of the photon in  $SU(2)$  on the orbit.

The boson distribution function obtained above with the remaining two degrees of freedom, then gives the usual result for the specific heat for black body radiation (Horwitz 2015).

We remark that the relativistic Gibbs ensembles worked out above (Sects. 10.3 and 10.4) were assumed, for simplicity, that there were no antiparticles (the Boltzmann counting construction did not make this assumption). The existence of the  $a_5$  field makes possible, as we have seen in Chap. 4, the (classical) particle-antiparticle transition on particle world lines. The analog of the radiation gauge requirement that we have imposed above as a second gauge condition, resulting in residually two degrees of freedom for the radiation field, would not admit this mechanism in the detectors. The presence of pair production (expected to be very small (Schwinger 1951)) in the detector would therefore suggest that there may be this additional degree of freedom in the boson gas, with a concomitantly small correction in the black body radiation formula.

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## 10.8 Manifestly Covariant Relativistic Boltzmann Equation

In this section, we shall derive a covariant Boltzmann equation with collision terms obtained from the binary scattering of events as described by relativistic scattering theory. We give here the basic ideas, and refer the reader to the work of Horwitz et al. (1981) for details.

We study the case of  $N$  identical particles, and use, for convenience, the formalism of second quantization. The field which annihilates an event at the point  $q = (\mathbf{q}, t)$  is related to the operator which annihilates an event of energy momentum  $p = (\mathbf{p}, E/c)$  by the Fourier transform ( $\hbar = 1$ )

$$\psi(q) = \frac{1}{(2\pi)^2} \int d^4 p \psi(p) e^{ip \cdot q}. \quad (10.149)$$

An arbitrary operator  $A$  on the Hilbert space of events can be represented as

$$A = \sum_{s=1}^N \frac{1}{s!} \int d^4 q_1 \dots d^4 q_s \psi^\dagger(q_1) \dots \psi^\dagger(q_s) \hat{A}_s \psi(q_1) \dots \psi(q_s), \quad (10.150)$$

where  $\hat{A}$  are operators acting on the space associated with every  $s$ -event subspace of the  $N$  event system. The expectation value of such an operator can be expressed in



terms of the trace with the density matrix  $\rho$  as

$$\langle A \rangle = \text{Tr}(\rho A). \quad (10.151)$$

The Weyl correspondence (Weyl 1952) applies, as in the nonrelativistic theory, to every  $s$ -event operator represented as (Balescu 1975)

$$\hat{A}_s = \int d^4 k_1 d^4 j_1 \dots d^4 k_s d^4 j_s A_s(k_1 j_1 \dots k_s j_s) \exp\{i \sum_{n=1}^s k_n \cdot \hat{q}_n + j_n \cdot \hat{p}_n\}, \quad (10.152)$$

where the operators  $\hat{q}_n, \hat{p}_n$  satisfy the canonical commutation relations

$$[\hat{q}_n^\mu, \hat{p}_{n'}^\nu] = i g^{\mu\nu} \delta_{n,n'} \quad (10.153)$$

There is a corresponding function  $A_s(q_1, p_1, \dots, q_s, p_s)$  of the classical variables containing the same coefficients  $A_s(k_1 j_1 \dots k_s j_s)$  which is its classical limit. Consider, in particular, the case  $s = 1$ . Then, the quantity  $\langle A_1 \rangle$  is given by

$$\langle A_1 \rangle = \int d^4 q \int d^4 k d^4 j A_1(k, j) \text{Tr}(\rho \psi^\dagger(q) e^{i(k \cdot \hat{q} + j \cdot \hat{p})} \psi(q)) \quad (10.154)$$

The exponential can be factorized to

$$\exp(ik \cdot q + j \cdot \hat{p}) = \exp(ik \cdot q) \exp(j \cdot \hat{p}) \exp(ik \cdot j/2). \quad (10.155)$$

Then (10.148) becomes

$$\langle A_1 \rangle = \int d^4 q d^4 p A_1(q, p) f_1^W(q, p), \quad (10.156)$$

where  $A_1(q, p)$  is the classical function corresponding to the operator  $\hat{A}_1$  through the Weyl correspondence, and we have defined the *one particle relativistic Wigner function*

$$\begin{aligned} f_1^W(q, p) &= \frac{1}{(2\pi)^4} \int d^4 j e^{-ij \cdot p} \text{Tr}(\rho \psi^\dagger(q - \frac{j}{2}) \psi(q + \frac{j}{2})) \\ &= \frac{1}{(2\pi)^4} \int d^4 k e^{ik \cdot q} \text{Tr}(\rho \psi^\dagger(p - \frac{k}{2}) \psi(p + \frac{k}{2})). \end{aligned} \quad (10.157)$$

As for the nonrelativistic analog of this procedure,  $f_1^W(q, p)$  is not necessarily positive, and cannot be interpreted as a pointwise probability density. It has the advantage, as we shall see, that the equations of motion are very analogous to the classical equations in phase space, and the results are immediately applicable to classical transport theory. Furthermore, note that

$$\int d^4 q f_1^W(q, p) = \text{Tr}(\rho \psi^\dagger(p) \psi(p)) \geq 0, \quad (10.158)$$

and that

$$\int d^4 q d^4 p f_1^W(q, p) = \int d^4 q \text{Tr}(\rho \psi^\dagger(q) \psi(q)). \quad (10.159)$$

Since

$$\int d^4q \psi^\dagger(q) \psi(q) = \int d^4q \psi^\dagger(p) \psi(p) = N, \quad (10.160)$$

the number operator for the total absolutely conserved number of the set of events, a superselection rule (Wick 1952) for this system, and therefore just a simple classical number,

$$\int d^4q d^4p f_1^W(q, p) = N \text{Tr} \rho = N, \quad (10.161)$$

i.e. a “normalization” for the Wigner function.

We now consider the  $\tau$  evolution of the one particle distribution function. To do this in a convenient way, we study the Fourier transform

$$f_1^W(k, p) = \int d^4q e^{ik \cdot q} f_1^W(q, p) = \text{Tr}(\rho \psi^\dagger(p - \frac{k}{2}) \psi(p + \frac{k}{2})). \quad (10.162)$$

Using the cyclic properties of operators under a trace with the density matrix, it then follows from the Stueckelberg-Schrödinger evolution that

$$\partial_\tau f_1^W(k, p) = -i \text{Tr}[(\rho \psi^\dagger(p - \frac{k}{2}) \psi(p + \frac{k}{2})), K] \quad (10.163)$$

We assume that  $K$  has the form

$$K = K_0 + V \quad (10.164)$$

where

$$K_0 = - \int d^4q \psi^\dagger(q) \frac{\partial^\mu \partial_\mu}{2M} \psi(q), \quad (10.165)$$

and

$$V = \frac{1}{2} \int d^4q' d^4q'' \psi^\dagger(q') \psi^\dagger(q'') V(q' - q'') \psi(q'') \psi(q') \quad (10.166)$$

is the two body operator (Poincaré invariant) corresponding to a two-event interaction potential. Carrying out the commutator with this model, one finds that the time dependence of the one particle Wigner function depends on the two particle Wigner function, defined by

$$\begin{aligned} f_2^W(k_1 p_1, k_2 p_2) &= \int d^4q_1 d^4q_2 e^{-ik_1 \cdot q_1 - ik_2 \cdot q_2} f_2^W(q_1 p_1, q_2 p_2) \\ &= \text{Tr}(\rho \psi^\dagger(p_1 - k_1/2) \psi^\dagger(p_2 - k_2/2) \\ &\quad \times \psi(p_1 + k_1/2) \psi(p_2 + k_2/2)), \end{aligned} \quad (10.167)$$

according to

$$\begin{aligned} \partial_\tau f_1^W(k_1, p_1) &= L_0 f_1^W(k_1, p_1) \\ &\quad + \int d^4p_2 d^4k_2 \delta^4(k_2) L_{12} f_2^W(k_1 p_1, k_2 p_2), \end{aligned} \quad (10.168)$$

where  $L_0$  and  $L_{12}$  are differential operators induced by the commutator with  $K_0$ . This procedure may be applied again to every  $f_s^W$  for  $s = 1, 2 \dots N$ , and results in a set of equations of precisely the same form as the well-known BBGKY hierarchy (Balescu 1975) for the nonrelativistic case. One obtains in this way a relativistically covariant generalization of the BBGKY hierarchy derived from basic dynamical principles.

The higher order relations invoke higher order correlations, and for a dilute gas of events, we may assume that truncation at the level of two body correlations will suffice. Furthermore, the two body correlation terms can be represented to fairly good accuracy, as in the non-relativistic case, by two body scattering amplitudes, consisting of two basic terms, one scattering events into the quasi-equilibrium ensemble, and the other, scattering events out. The basic ingredients needed are derived in Chap. 7 on scattering theory. The scattering, as for the nonrelativistic case, induces changes in the distribution function, i.e., the rate of change of  $f$  due to collisions is

$$D_c f(q, p) = D_c^+ f(q, p) - D_c^- f(p, q) \quad (10.169)$$

where  $D_c^- f(p, q)d^4q d^4p \delta\tau$  is the number of collisions in the interval  $\delta\tau$  in which one of the events is in  $d^4q d^4p$ , and  $D_c^+ f(p, q)d^4q d^4p \delta\tau$  is the number of collisions in  $\delta\tau$  in which one of the final events is in  $d^4q d^4p$ . Denoting by  $\dot{P}$  the transition rate derived from the two body scattering theory for this potential, we have

$$\begin{aligned} D_c^+ f(q, p) &= \int d^4p_1 d^4p'_1 d^4p' \dot{P}(p'_1 p' \rightarrow p_1 p) f(q, p') f(q, p'_1), \\ D_c^- f(q, p) &= \int d^4p_1 d^4p'_1 d^4p' \dot{P}(p_1 p \rightarrow p'_1 p') f(q, p) f(q, p_1), \end{aligned} \quad (10.170)$$

Furthermore these results can be put into terms of the experimentally measured scattering cross sections (Horwitz 1982) in the form (we denote  $q_1 - q_2$  by  $q_r$ , and  $\frac{1}{2}(p_1 - p_2)$  by  $p_r$ ,  $P = p_1 + p_2$ , and assume a narrow distribution over the mass shifts)

$$\begin{aligned} \partial_\tau f(q, p) + \frac{p^\mu}{M} \frac{\partial}{\partial q^\mu} f(q, p) &= 4\pi \int d^3p_r d^3p'_r \frac{|\mathbf{p}'_r|}{M} \frac{d\sigma^{exp}}{d^3p_r}(p'_r \rightarrow p_r; P) \\ &\times \{f(q, p') f(q, p'_1) - f(q, p) f(q, p_1)\}. \end{aligned} \quad (10.171)$$

With this final form of the Boltzmann equation, we can discuss the relativistic  $H$ -theorem. Defining the functional (Huang 1967)

$$H(\tau) = \int d^4q d^4p f(q, p, \tau) \ln f(q, p, \tau) \equiv -S(\tau)/k_B, \quad (10.172)$$

where  $S(\tau)$  is the entropy. Then, the derivative of  $H(\tau)$  is

$$\begin{aligned} \frac{dH}{d\tau} &= \frac{1}{64} \int d^4q d^4p d^4p_1 d^4p'_1 [\ln f(q, p_1) f(q, p) - \ln f(q, p'_1) f(q, p')] \\ &\times \{f(q, p') f(q, p) f(q, p_1)\} \dot{P}\left(\left(\frac{p_1 - p}{2} \rightarrow \frac{p'_1 - p'}{2}\right); P\right). \end{aligned} \quad (10.173)$$

Since  $\dot{P}(p_r \rightarrow p'_r : P) \geq 0$ , and the remaining factor in the integrand is non-positive, we obtain

$$\frac{dH(\tau)}{d\tau} \leq 0, \quad (10.174)$$

the relativistic  $H$ -theorem.

This result implies that the entropy  $S(\tau)$  is monotonically increasing as a function of  $\tau$ , but *not necessarily in  $t$* , since the directions of  $t$  and  $\tau$  for the antiparticle are opposite. In the nonrelativistic limit, the standard  $H$  theorem is recovered, since  $t$  and  $\tau$  become identical.

The fundamental basis for the formulation of a relativistically manifestly covariant quantum theory was given in the introductory chapter of this book. The thought experiment of Einstein, constituting two frames of reference, in relative inertial motion, for the generation and reception of light signals forms the basis for the construction of the special theory of relativity. Calling these frames  $F$  and  $F'$ , according to this experiment, two signals emitted successively from  $F$  at, say,  $\tau_1$  and  $\tau_2$  are received in the frame  $F'$  at, respectively,  $\tau'_1$  and  $\tau'_2$ , with the relation between them

$$\tau'_2 - \tau'_1 = \frac{\tau_2 - \tau_1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (11.1)$$

where  $v$  is the relative velocity of the two frames, and  $c$  is the velocity of light. The relation (11.1) follows, according to Lorentz, from the null result of the Michelson-Morley experiment. Einstein defines the *observed* difference as the time interval

$$\Delta t = \tau'_2 - \tau'_1. \quad (11.2)$$

Therefore we see that the *Einstein time*, which transforms, along with the observed interval  $\Delta x$  between the places in the frame  $F$ . As observed in  $F'$ , to provide the observed spacetime point  $(t', x')$  (understanding these variables as intervals), transforming according to the Lorentz transformation. These coordinates must be considered as *observables*, the outcome of a measurement. As our discussion of the gravitational redshift shows, this is true as well for the spacetime manifold of general relativity, for which it is remarkable that an assumed local diffeomorphism invariance of the physical laws provides a set of equations (the Einstein equations) which relate the *geometry* of such observable quantities to the energy momentum of the system.

In the Galilean (Newtonian) description of dynamics, the universal time  $t$  postulated by Newton provides a parameter for the description of the evolution of the state of a particle in phase space,  $\mathbf{x}(t)$ ,  $\mathbf{p}(t)$ . Since, in the relativistic world, as we have argued,  $t$  is an observable on the same level as  $\mathbf{x}$ , the description of the evolution of the system requires the introduction of a parameter  $\tau$ , admitting the description of

a phase space  $t(\tau), \mathbf{x}(\tau)$  with  $E(\tau), \mathbf{p}(\tau)$ . For this motion, Stueckelberg postulated the existence of a Hamiltonian for which this evolution follows a generalized set of Hamilton equations. In order to be able to treat the  $N$  body problem, Horwitz and Piron asserted that the parameter  $\tau$  is universal, and therefore plays the role of the universal parameter of time postulated by Newton.

As for the nonrelativistic theory, one can then write a Schrödinger equation for a quantum wavefunction  $\psi_\tau(x)$ , a covariant function on spacetime in a Hilbert space  $L^2(R^4)$ , satisfying all the requirements of a full quantum theory.

Many of the properties of such a theory are described in these chapters, including bound state spectra, scattering theory and diffraction experiments constituting interference in time, such as the remarkable experiment performed by Lindner et al. The very high frequencies involved in such phenomena form an entrance into the developing field of attosecond physics.

As for the nonrelativistic quantum theory, the construction of the tensor product spaces leading to Fock space and second quantization is straightforward, and some of the properties of the resulting quantum fields are discussed. Although it has been shown by Andrew Bennett (and described here) that the lowest order correction to the electron magnetic moment can be computed without recourse to second quantization, since the first quantized theory can describe what appears to be particle creation and annihilation in the laboratory (as in Stueckelberg's original paper), there are clearly phenomena that can be associated with the creation and annihilation of *events*. The development of quantum statistical mechanics given here illustrates the utility of this concept in the description of relativistic many body systems.

Although we have discussed some applications of this framework to the geometrical approach to the dark matter problem of the galaxies of Milgrom, Bekenstein and Sanders, a general discussion of the application of the theory presented here to general relativity remains to be formulated.

The several phenomenological consequences of this theory, making contact with experiment in some important areas, as described here, with potential applications to general relativity and the emerging field of attosecond physics, provide a strong motivation for a continued effort to develop the theory.

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