(Mathematics) Curriculum, Teaching and Learning

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Abstract As mathematics curricula around the world have undergone significant reform in recent years, it is time to re-think the role of the curriculum for mathematics learning and teaching. What shape would such a curriculum have (the final product) if it does undergo this kind of reform? Who are the 'end-users' of the curriculum and what are the inter-relationships among the curriculum, the teachers, and the students? This chapter attempts to summarize and comment on the various chapters in this book and to initiate further reflection and discussion on these issues.

Keywords Mathematics curriculum · Mathematics teaching · Mathematics curriculum reform

Prologue

Mathematics curricula around the world have been undergoing reform for more than a decade, so it is now time to re-think the role of the curriculum in the learning and teaching of mathematics. The chapters in this book contribute to our understanding of the role and purpose of a mathematics curriculum, as well as of mathematics teaching and learning in the context of reform in the various educational regions.[1] Rather than setting out a firm position on various issues, the authors have chosen to raise a number of questions on various related aspects, in the hope that these questions would act as fuel to drive further reflection and professional discourse.

[1]The term 'region' is used throughout this chapter for consistency. Some regions (e.g., Japan) are countries, while others (e.g., Hong Kong) are not.

The word 'mathematics' is in parentheses since we believe that many of the issues in this chapter are not confined to mathematics.

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Standardization in the Current Reform of the Mathematics Curriculum

In the West, prior to the modern mathematics movement of the 1960s, there had been virtually no change in the mathematics curriculum and textbooks for a very long time. There was little difference between the pre-second-world-war and post-second-world-war traditional textbooks because both were based on what has since been called the pre-1800 model (Cooper 1985). Following the modern mathematics movement, there were widespread reforms of the mathematics curriculum around the world at the turn of the millennium (Wong et al. 2004). Among other factors, these changes were triggered by several large-scale international comparisons, such as the Second IEA Mathematics Study (SIMS), the 1992 International Assessment of Education Progress (IAEP) mathematics study, and the Third International Mathematics and Science Study (TIMSS).[2] Though Hirabayashi did state that "having a high achievement in international mathematics studies is not the only criterion" for a good curriculum (as quoted in Curriculum and Textbook Workgroup 2002, p. 6; see also Wong et al. 2004), it appears that improving the position of one's country/region in the 'international league table' is still a major goal in the current trend towards mathematics curriculum reform (Anderson 2014; Pang 2014; Stephens 2014).

To avoid falling behind, and to maintain standards, the first thing to do is to establish what the standards are (Reys 2014). Thus, the idea of standardization quietly crept into the mathematics curriculum and teaching. Besides the need to keep up the standard, there is an interest in standardizing (unifying) the mathematics curriculum across the regions and having it benchmarked against the ones used in other countries or regions. Setting up a national curriculum became a trend in Australia, Brazil, Israel, Japan, Korea, Singapore, the UK, the USA (though the US mathematics standard was prepared by a non-governmental body, the National Council of Teachers of Mathematics), and the Chinese regions (Anderson 2014; Even and Olsher 2014; Garnica 2014; Pang 2014; Reys 2014; Stephens 2014; Tam et al. 2014; Wong et al. 2014). All of these cases involved conformation to a single curriculum standard. The first matter to consider is whether such a move is desirable and viable.

It is interesting to note that, although educational autonomy and decentralization were always stressed in the West (van Zanten and van den Heuvel-Panhuizen 2014), there is a long history of a centralized curriculum in the East. Inevitably, if there is too much emphasis on curricular autonomy, there is a possibility that the resulting curriculum will be too laissez-faire. However, too much centralized control has its drawbacks as well. This is particularly an issue in educational regions in the East where there is already a long tradition of a centralized curriculum. The question is whether pushing for strict standardization would further tighten the existing centralized control, which would run counter to the call for school-based curriculum development (Wong and Tang 2012).

There are two more issues to be considered. Firstly, as the notion of the 'curriculum' has expanded to encompass a large number of components, such as attainment

[2] This was later renamed 'Trends in International Mathematics and Science Study.'

targets, teaching approaches, and learning activities, would it still be possible for the central curriculum designers to really know what is happening in each district, each school, and each classroom (even with each student), in order to be able to design an ideal curriculum? Secondly, the above issue will become even more salient when processing abilities, including the so-called generic skills or higher-order thinking skills (e.g., problem solving, creativity and communication), are given greater emphasis in the current reform of the mathematics curriculum. This is particularly true if an attempt is made to use attainment indicators as inputs into the standard mathematics curriculum. How can the mastery of these generic skills be turned into measurable outcomes? In practice, can these generic skills be acquired step by step in line with the progress towards the attainment standards? Or can they only be nurtured holistically? All of these questions deserve reflection on the part of those who care about the mathematics curriculum (Wong et al. 2004).

Curriculum, Textbook and Instruction

After the curriculum is reformed, it needs to be implemented. The model of 'intended, implemented, and attained curricula' has been used since SIMS to analyze the mathematics curriculum (Travers and Westbury 1989). Similar frameworks were also proposed by scholars such as Goodlad (1979) and Marsh and Willis (2007). This type of framework for curriculum analysis is also used in various chapters in this book (for example Reys 2014; Senk et al. 2014). However, the 'intended, implemented, and attained curricula' model might give the impression that action would be taken only in that order. Indeed, it is often emphasized that the word *curriculum* originates in the Latin *currere*, which means 'to race.' In simplistic terms, curriculum designers would set the course (racecourse) for students to follow, leading them to their destination. The first step in this process is to design an intended curriculum. The next step is to guide teachers on how to implement the curriculum. This involves providing them with a set of well-designed curriculum documents, textbooks, and other 'accessories,' as well as a good methodology for instruction. The final step is to cross-check whether the expected curriculum targets have been achieved (e.g., Anderson 2002; Martone and Sireci 2009).

At the present time, the concept of the curriculum can be very broad (see further discussion in later sections). School documents, newspaper articles, committee reports, and many academic textbooks refer to any or all of the subjects offered or prescribed as 'the curriculum of the school' (Marsh 2004, p. 3). Nevertheless, the textbook is still the means most frequently used to actualize the curriculum. In this book, a series of chapters focus on the textbook (Even and Olsher 2014; Li et al. 2014; Senk et al. 2014). Thus textbooks can be seen as a further manifestation of the intended curriculum. Undoubtedly, the design of both the curriculum and the related documents is crucial for effective teaching, especially for novice teachers, giving them confidence that if they follow the curriculum design, they will achieve the desired learning outcomes.

Along with the above line of thought, after the curriculum is designed, the next step could be to design the instruction components, which is the focus of a number of chapters in this book (Huang et al. 2014; Reys 2014; Shafer 2014; Wong et al. 2014). With a carefully designed curriculum and the relevant curriculum documents at hand to guide the teachers on effective instructional methods, it is very likely that the curriculum goals would be achieved, unless there is 'infidelity' in the process, that is, the curriculum is not being implemented strictly in line with its original design (Achinstein and Ogawa 2006; Fullan 2007; Kimpston 1985; O'Donnell 2008; Synder et al. 1992; see also van Zanten and van den Heuvel-Panhuizen 2014).

Once the curriculum standard is laid down, the next step is to design the curriculum material (textbooks included), and then to equip the teachers with the various skills needed to implement the instructional design. If this process is respected, the teachers would then deliver their teaching as prescribed, and arrive at the expected students' learning outcomes. This logic will be considered again towards the end of this chapter. In the following section, one factor—the teacher—that might affect the implementation of the curriculum is examined.

The Curriculum and the Teacher

After the curriculum is finalized, it is the teacher who has to deliver it. So one source of curriculum infidelity is the teacher. Teachers should be professional enough to implement the curriculum as designed, and benefit from the assistance of the underlying instructional design of the curriculum documents. Both the beliefs and knowledge of the teachers (of all kinds, including subject knowledge, pedagogical content knowledge, curriculum knowledge, and knowledge about the students: Bromme 1994; Shulman 1987; Sullivan and Wood 2008) are seen to be of the utmost importance in guaranteeing that the intended curriculum is successfully implemented, and thus yields the expected attainments. However, such an approach still reflects the linear mentality (Fig. 1).

The advocate of a 'teacher-proof curriculum' (Apple 1993; Priestley 2002), at the peak of behaviorism, further reinforces this line of thought. The curriculum (together with the textbooks) is a 'script' for the teachers to play their parts (Wong 2009). Yet in recent years, teacher ownership (of the curriculum), teacher autonomy, and the community of learning (Cochran-Smith and Lytle 1999; Kirk and MacDonald 2001) among the teachers have been emphasized, which opens up another option for the role of the curriculum (Even and Olsher 2014). Stein et al. (2014) point out that we could guarantee a high-quality instruction even though we use the curricular in a congruent manner. Indeed, in reality, a teacher cannot and should not be only a faithful executor of the intended curriculum (including the textbooks). The roles and inter-relationships among the curriculum, the textbook, and the teacher (not to mention the students, who will be discussed later in this chapter) need to be re-thought (Cohen et al. 2003; Li 2011; McCaffrey et al. 2001).

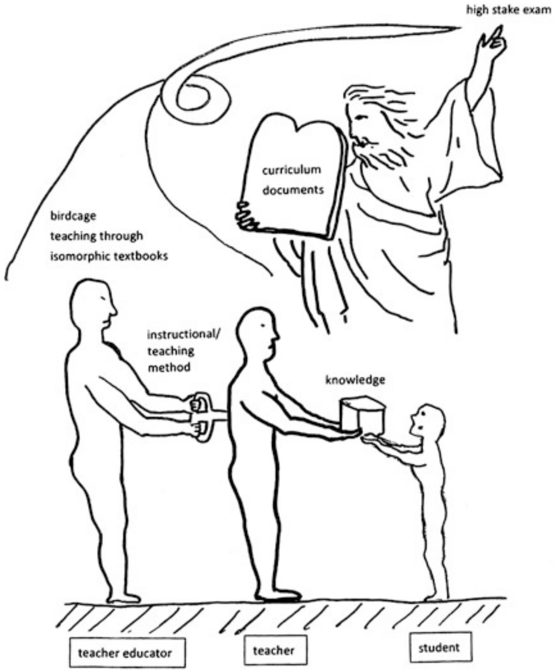

Fig. 1 The curriculum, teacher, and teaching

A number of chapters in this book touch upon the inter-relationships among the curriculum, the teachers, and the teaching process. Some of them focus on the connectedness among the three, while others focus on how curriculum change can facilitate teacher professionalism (Brown and Hodgen 2014; Cai et al. 2014; Takahashi 2014). The notion of teacher ownership of the curriculum is not new, while the concept of the teacher as a reflective practitioner was discussed previously in the 1980s (Schön 1983). From this perspective, the teacher should own the curriculum, and evolve to become an educational researcher, an assessment expert (assessment not just *of* but *for* learning) and a curriculum designer (Clandinin and Connelly 1992). The teaching of each lesson would involve an element of curriculum design and not just the blind respect of a pre-designed instructional practice.

There is yet another aspect of teacher ownership, namely, involving teachers in the curriculum development process. Even and Olsher (2014) describe how teachers became more genuine participants in the process of textbook development, which made them more active participants in curriculum development. Their needs, wishes, and aspirations were also fed back to the professional curriculum developers and the policy makers. Wong et al. (2014) contains an extensive discussion of this idea. Superficial 'town hall' consultations may alienate the teachers and adversely affect the way that they view the curriculum. That chapter then considers the need for curriculum reformers to carefully listen to and to synthesize the views of the various stakeholders holistically (see also Lam et al. in press).

Curriculum, Teaching and Learning

Most, if not all, curriculum design is done out of goodwill and in the hope that it will help students to learn more effectively. In other words, the purpose of all of these efforts is not just to promote teaching but also to facilitate learning. In that sense, the student is the end-user for the curriculum, yet the student voice is often under-represented in curriculum design (Geiger et al. 2014). Most curriculum design is based on a 'hypothetical learning trajectory' (Fuson and Li 2014; Simon 1995), which is essentially an adult perspective (and in particular that of the curriculum designers) rather than a delineation of actual student learning. Furthermore, such a 'hypothetical learning trajectory' often describes the 'shortest learning path.' However, in reality, it is quite natural for students to loop in their learning process. There is a need for students to 'hatch' as they loop around too. Therefore, it seems that the students' opinions, such as their appraisal and/or diagnosis of their learning process, should play a role in curriculum design (Geiger et al. 2014). In the holistic review of the mathematics curriculum in Hong Kong, the opinions of the various stakeholders, including university professors, employers, parents, and students, were solicited. In particular, a questionnaire survey was conducted among 10,000 students (as well as 60 interviews) (Tam et al. 2014). The conclusion was that the student should have a role in the whole curriculum design process.

It should also be noted that there is a subtle difference between teaching and learning in curriculum design. In the specific context of mathematics, the mathematics curriculum should help students develop their understandings of mathematical **concepts**, in order to **solve** mathematics **problems**, but the words in bold deserve deeper reflection. There is a vast number of meanings of both 'learning' and 'having learned.' Should we allow/encourage students to develop their own concepts (as advocated by constructivism)? Should we impart to them a set of mathematical concepts? Or should we take both of these aspects into account in the design and implementation of a curriculum? Aside from 'ethical' considerations, is it really possible to stop students from conceptualizing their own mathematical experience? In addition, do we have a set of prescribed concepts (conceptual frameworks) for each particular mathematical object? Take division as an example. Is it the inverse of multiplication, the solution to '$bx = a$,' sharing, grouping, dividing a pizza pictorially, or dividing a rectangle pictorially? Can we say that division is any of the above, or that all of them together comprise the notion of division? Is the above list exhaustive, and can these representations help students **understand** division and **solve** problems? What are the grounds for not accepting that students have their own (internal) representations and self-invented problem-solving strategies? And if we value the ability of students to 're-invent' mathematics (Freudenthal 1991; van den Heuvel-Panhuizen 2001), how can we make that re-invention happen, rather than let students just imitate the standard problem-solving strategies (even though these standard strategies are often the 'best and most efficient' ones). All of these questions deserve deeper investigation when we seek to develop a curriculum that enhances **student** learning (Carpenter et al. 1998; Clarke 1997; Fuson et al. 1997; Huang et al. 2014; Threlfall 2000; Tsang 2005).

Attained Curriculum: Student Performance and Problem Solving

Once the curriculum is designed and implemented, it is of undoubted interest to examine whether the desired learning outcomes are attained (Shafer 2014). But first it has to be established what criteria of student performance could be used for this. As mentioned above, there are different facets of 'having learned.' Could these be finding the correct answer, mastering the 'right' procedure to solve problems, stating the definition, drawing a few standard pictorial representations, or all of these? Should attention also be paid to 'deep procedural knowledge,' which is characterized by connectedness and a flexible use of procedures (Star and Rittle-Johnson 2008)? Howe (2014) describes the three pillars of mathematics, namely, conceptual understanding, computational skills, and coordination (which might be closely related to connectedness as mentioned above). This listing suggests that there may be a variety of expected learning outcomes (the attained curriculum), which are at the same time the curriculum objectives. It can also be asked whether or not students' non-cognitive achievements are significant, such as their interest in learning, their self-efficacy, and other affective factors and beliefs. These notions are all emphasized in the current curriculum reform, yet care has to be taken that they do not become parts of the formal, high-stake assessments process (Wong et al. 2004). All of these points provide food for reflection after reading the various chapters in this book.

When considering learning outcomes in mathematics, inevitably, problem solving emerges as a central issue. However, what is the relationship between a generic ability for problem solving and a problem-solving ability in mathematics? This issue is not new and was raised at the beginning of the famous report by Cockcroft:

> It is often suggested that mathematics should be studied in order to develop powers of logical thinking, accuracy and spatial awareness. The study of mathematics can certainly contribute to these ends, but the extent to which it does so depends on the way in which mathematics is taught. Nor is its contribution unique; many other activities and the study of a number of other subjects can develop these powers as well. We therefore believe that the need to develop these powers does not in itself constitute a sufficient reason for studying mathematics rather than other things. (Cockcroft 1982, p. 1)

As students go on to different walks of life after they finish school, and do not restrict themselves to the fields of mathematics or science, it is essential for them to nurture their general (and not mathematics-specific) problem-solving abilities. However, in the context of mathematics learning, this nurturing is done through tackling mathematics problems. How to bridge the gap between these two forms of problem solving, the mathematical and the generic, becomes a task for everyone. It is not confined to learning objectives but also extends to how learning outcomes are assessed.

What Is the Curriculum, in Real Terms?

From the above discussions, it is apparent that the term ‘curriculum,’ mathematics curriculum included, could have very different meanings in different regions. This point should be borne in mind when reading the chapters in this book (and other related articles). In some regions (e.g., China and the UK), the curriculum is mandatory and by law has to be followed. In other countries (e.g., the USA), its use is only recommended. In still other places (e.g., Hong Kong), it is used as a ‘trade off’ to justify a government subsidy, because a school has to conform to the official curriculum if it wants to obtain government funding.

How detailed the curriculum is depends on the level of curriculum control. In some countries, the curriculum document is just a (loose) framework within which different authorities develop their own curriculum. In other countries, where the three (or four) column approach has become popular, not only are the learning targets and contents laid down, but the teaching activities and assessment methods are also suggested (Wong et al. 2004). Thus curriculum documents play different roles and take different forms in different countries and regions.

There are also differences in the end-users of the curriculum. Theoretically, the main audience is the teachers. However, in many cases, when the textbooks are closely aligned to the curriculum, the teachers do not necessarily refer to the curriculum since they believe it is enough to follow the textbooks. In such cases, when drafting a curriculum document, should the textbook developers also be a major target audience? The students are another end-user group. If the students have chosen to study independently (whether they are home-school students, foreign students, or adult students), should the curriculum document also cater to their needs? For instance, could the curriculum be so detailed that it could be followed fully even without a teacher? This question is even more salient for textbooks. Should textbooks be written with the teachers in mind, to guide them in their teaching; or for the students, making it possible to study fully by following them?

Returning to the previous discussion on the ‘intended—implemented—attained’ linear mentality of the curriculum, the curriculum documents are often taken as a starting point for the engineering of a prospective educational reform. In other words, a curriculum document is released to initiate the process of changing the curriculum in subsequent years. However, there are other possibilities too, including the suggestions by Tam et al. (2014). These authors reviewed the historical development of the Hong Kong primary mathematics curriculum in the period 1960–1980. They showed that the curriculum (document) can be seen as a summary and consolidation of a long-term experiment in teaching. It is an ‘end’-product of curriculum reform rather than a starting point. Genuine curriculum reform often originates in day-to-day classroom teaching (Fullan 1999; Stigler and Hiebert 1999). Such teaching experiments could include providing students with more learning opportunities (Anderson 2014) rather than adding specific contents. Again, Geiger et al. (2014) show that offering challenging learning activities and genuine opportunities to students helped them to develop a positive view of mathematics learning and see the connectedness of their learning both within and outside mathematics.

What Mathematics Are We Looking at?

Some of the issues discussed above are general rather than mathematical. When it is boiled down to mathematics, naturally, the aim is for the students to learn some mathematics by following the mathematics curriculum. However, mathematics may be 'just' one means of nurturing a responsible citizen and an 'educated person.' For some time, the school mathematics curriculum has been criticized as being 'de-mathematized' (Zhang 2005). Can this criticism be answered by putting more mathematics back into the school curriculum, or would it be better to explore the *path* of mathematization (Freudenthal 1991; NCTM 1989)?

To this end, students need to be helped to undergo an ontological shift (Chi 1992) from the concrete to the abstract, from the particular to the general, from their own real-life experience to entities in the mathematical world, and from realistic to esoteric mathematics (Cooper and Dunne 1998). For instance, Sinclair and de Freitas (2014) suggest that conceptualizing mathematics as being characteristically *virtual* can bridge the space between the concrete and the abstract. Huang et al. (2014) point out that this shift can occur at several points. The shift occurring at several points is echoed by previous discussions on the design of the *bianshi* curriculum (Wong et al. 2009, 2012a, 2012b). An additional issue is whether or not the ultimate goal of the mathematization process is to achieve a unified, universal form of mathematics. A great deal of discussion has taken place on the subtle differences that might exist between formal/symbolic mathematics, hands-on mathematics, real-life mathematics, mathematics in the ICT environment, etc. (Artigue 2001; Lopez-Real and Leung 2006). When we say that our mathematics curriculum builds a path of mathematization for the students, we need to understand what type of mathematics that path leads to.

Concluding Remarks

As was said at the beginning of this introduction, it is time to re-think the mathematics curriculum, and the learning and teaching of mathematics, as well as to re-think the textbooks, teachers, and students. A number of questions arise from the above discussion. Could and should the curriculum encompass all the aspects of teaching and learning? Should the curriculum be a guideline for teaching or a means to enhance the professionalism of teachers? What is the primary concern or goal of curriculum reform, for example a means to improve the position in the 'international league table,' or a contribution to the whole-person development? Some of these questions have already been raised in articles such as Wong et al. (2004) (though it was published a decade ago), but the authors hope that these questions can continue to provide food for further reflections and investigation as the chapters of this book are read.

As Albert Einstein (1879–1955) said, "Education is that which remains, if one has forgotten everything he learned in school" (Einstein 1950, p. 36). A well-designed curriculum, together with effective delivery, is a necessity and lays the foundation for the emergence of wisdom. However, this may be just the first half of the 'story' (Wong et al. 2012a, 2012b). One has to 'transcend'[3] the *way* after 'entering,' and going along with the *way* (Wong 2006). As a conclusion, two little Chan stories can illustrate this point:

> There was a group of learned monks visiting Master Big Pearl (a great Chan master in the Tang dynasty). One of them asked, "Can Master take a question?" Master replied, "Just like a big pond reflecting the moon, feel free to search on it (implying that one can only touch the shadow rather than the moon!)." A monk asked, "Who is the Buddha?" Master answered, "Sitting on the other side of the pond (i.e. Master Big Pearl himself), other than the Buddha, who can it be?" Everyone was stunned. After a while, another monk asked, "What teaching method do you use to enlighten the others?" Master said, "I did not use any method." The monk murmured, "This is the style of Chan masters." Master asked back, "Then, what method do you use?" The monk replied, "I teach the *Diamond sutra*." Master asked, "How many times have you taught?" The monk replied, "More than 20." Master asked, "Who spoke the *sutra*?" The monk responded in a loud voice, "Are you kidding, isn't it spoken by the Buddha?" Master said, "[But isn't it precisely said in the *sutra* that], if someone said the Buddha has anything to teach, it is a blasphemy, and that person doesn't understand the meaning. But if someone said the *sutra* is not spoken by the Buddha, it is blasphemy against the *sutra* itself. What do you think then?" ... The monk said, "I am getting confused here." Master said, "You never had understood, so how can you say you get confused ... You taught the *sutra* over 20 times, but you have not yet attained Buddhahood (the essence of the teaching)."

> One day a company of several monks came to visit Zhauzhou (778–897) (another great Chan master in the Tang Dynasty). The first one asked, "I am just a beginner, Master, please reveal to me the teaching." Zhauzhou asked, "Have you taken breakfast today?" The monk responded "Yes sir." Master spoke with a loud voice, "Why then are you stand idling there, go now and wash the bowl!" The monk attained realization upon hearing this. The second monk asked, "I am also a novice, could Master please teach me?" Zhauzhou asked, "When did you arrive?" Reply, "Just today." Question, "Have you drunk the tea?" Reply, "Yes sir." Master then said, "You should then report to the reception immediately!" Again, this monk attained realization. At this moment, a third monk who had been studying in the monastery for a long time said, "Sir, I have been here for more than 10 years and never heard your teachings. I wish to take leave from here and learn from others." Zhauzhou was very angry upon hearing this, "Young man, why have you wrongly accused me? Starting from the first day you arrived, whenever you presented me with tea, I drank for you! You presented me with rice, I ate for you. When you

[3] In Wong (2006), originally the word 'exiting' was used, but 'transcending' is a more appropriate term.

bowed, I lowered my eyebrows, and when you prostrated yourself, I nodded my head. I have been teaching you in each of these instances!"

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