Does Classroom Instruction Stick to Textbooks? A Case Study of Fraction Division

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Abstract In this chapter, we examined the consistency between textbook and its implementation in classrooms. By investigating how two selected Chinese teachers taught fraction division over four consecutive lessons, and making use of an existing study on the treatments of the same content unit in textbooks, it was found that the sample teachers essentially adopted their textbooks. The teachers put great effort into developing students' understanding of the meaning of fraction division and justifying why the algorithm of fraction division works by employing a problem-based approach and using multiple representations. They followed the textbooks regarding the conceptualization of concepts and algorithms, the topic coverage, the sequence of content presentation, the approach to developing the concepts and algorithms, and the selection of problems and exercises. Meanwhile, the teachers also demonstrated certain flexibility in constructing their own problems for introducing new knowledge and consolidating the learned knowledge. Finally, the authors argued that the Chinese strategies of adopting textbooks might be attributed to their teaching culture and professional development practice.

Keywords Fraction division · Mathematics curriculum · Mathematics teaching · Mathematical tasks and representations · Curriculum implementation fidelity · Chinese mathematics teaching and learning

Background

Textbooks are seen as an important factor impacting what teachers do and, therefore, what students learn (Tarr et al. 2008). However, most teachers do not teach all

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topics in their textbooks. They may use the same textbooks but teach vastly different lessons; or, even when similar lessons are taught, assignments from the textbooks may be quite different (Huntley and Chval 2010; Kilpatrick 2003; Tarr et al. 2006; Thompson and Senk 2010). Variation in the implementation of a textbook is often cited as a factor that is likely to contribute to mediation of textbooks' impact on student learning (e.g., Remillard 2005). The differences in the implementation of textbooks are linked to various factors, such as state assessment pressures (Huntley and Chval 2010), lack of clarity in textbooks' intents, teachers' beliefs, teachers' prior experiences as students or as pre-service teachers, teachers' knowledge or understanding of the textbook's content and/or the pedagogy called for in the textbooks, the environment in which teachers work, and students' prior knowledge (e.g., Remillard 2005). It is this complexity that calls for studies on "coherence between the textbook and implemented curriculum; that is, consistency between curriculum and instruction is needed in order to actualize student learning in mathematics" (Tarr et al. 2008, p. 275). A written curriculum cannot fully provide guidance for teaching (e.g., Ball and Cohen 1996), and the same textbook could be implemented unevenly within and across schools (e.g., Kilpatrick 2003). Thus, implementation fidelity, the extent to which there is a match between the written curriculum and teachers' practices in the classroom, has become an important issue (National Research Council 2004). A few studies on coherence between reform-oriented or traditional textbooks and their implementation (Tarr et al. 2006; Thompson and Senk 2010) have questioned the appropriateness of textbook adaptation.

Because Chinese students have repeatedly outperformed their Western counterparts in school mathematics in various international comparative assessments (e.g., Mullis et al. 2008; OECD 2009), an examination of the implementation of Chinese textbooks may provide insight into the discussion on implementation fidelity. There are several studies on Chinese mathematics curricula and textbooks (e.g., Li et al. 2009a, 2009b; Liu and Li 2010) and some research on mathematics classroom instruction in China (Huang and Leung 2004; Leung 2005; Li and Huang 2012). Yet, little attention has been devoted to examining the features of textbook implementation. In general, as argued by Park and Leung (2006), "in many East Asian countries, teachers and students regard the textbook as a 'Bible' which contains all the essential knowledge" (p. 230) due to the centralized curriculum and assessment systems (e.g., Leung and Li 2010; Usiskin and Willmore 2008). However, little empirical research has approved or disapproved this statement. The current study is designed to investigate the learning opportunities provided by a sample of teachers and their relationship with the textbooks used. In order to sharpen the research focus, a common topic of fraction division was selected. In particular, this study is sought to address the following research questions:

(1) How is the content of fraction division presented in the selected Chinese classrooms?
(2) How is the content of fraction division enacted in the selected Chinese classrooms?
(3) How are the content focus and organization in the classrooms related to the textbooks?

Research Background

Textbook Use as Following or Subverting

As suggested by Remillard (2005), there are three different ways of examining curriculum use: use as following or subverting, use as interpretation, and use as participation. The stance of textbook use as following or subverting reviews "the written curriculum as embodying discernible and complete images of practice and examine the degree to which teachers follow these guidelines with fidelity" (Stein et al. 2007, p. 343). The view of curriculum use as interpretation holds that teachers bring their own beliefs and experiences to create their own meanings of textbooks, and they implement textbooks based on their interpretation of the authors' intentions. Thus, this notion assumes that it is impossible to examine the fidelity between written teaching materials and classroom action. The third view of curriculum use as participation suggests that use of curriculum materials is a kind of collaboration with the materials. Central to this perspective is the assumption that teachers and curriculum materials are engaged in a dynamic interrelationship.

Given the nature of our research questions, the stance of textbook use as following or subverting was more suitable. The question that now arises is "how far may teachers go in their adaptations without destroying the spirit and meaning of the curriculum they implement in their class?" (Ben-Peretz 1990, p. 31). Tarr et al. (2006) found that their sample of teachers taught 60 to 70 % of the textbooks. Teachers often supplement the textbook, omit problems or sections, and change the order of the lesson presented in textbooks based on different considerations (Huntley and Chval 2010; Tarr et al. 2006). Thus, the key goal is supporting teachers in making well-informed, purposeful decisions that benefit students' learning of mathematics (Huntley and Chval 2010). An examination of the implementation of textbooks in China, where there is a high-achieving education system, may provide some suggestions.

In a previous study, Li et al. (2009a, 2009b) examined the textbook treatments of fraction division in China, Japan, and the US. Building on their findings, this study will focus on an examination of fraction division teaching and observe the extent to which the characteristics of fraction division teaching in classrooms are in line with the treatments of fraction division in textbooks.

Teaching and Learning of Fraction Division

Learning of Fraction Division Developing a conceptual understanding of the algorithm of fraction division is a difficult task for both students and teachers (e.g., Carpenter et al. 1989; Li and Kulm 2008). Even though teachers can perform computations of fraction division, it is difficult for them, at least in the United States, to explain the computation of fraction division conceptually and with appropriate representations or connections to their mathematical knowledge (Ma 1999). Researchers have suggested different approaches to help students learn how to divide fractions, including (1) providing mathematical justifications for the fraction division algorithm and (2) using concrete or visual demonstrations to explain how fraction division can be computed through extending the whole-number division to fraction division with the measurement interpretation and the partitive interpretation (Li 2008).

Treatments of Fraction Division in Textbooks In their study, Li et al. (2009a, 2009b) examined the ways of dealing with fraction division in Chinese, Japanese, and US textbooks. The researchers examined three Chinese, three Japanese, and four US textbooks in great detail using a two-level framework. At the macro level, they identified how content topics were placed and organized. At the micro level, they examined how fraction division was conceptualized, which focused on the content topic introduction and potential use of representations and/or examples. In addition, the learning progression—the coverage and sequence of topics presented—was also examined.

Li et al. (2009a, 2009b) found that their sample of Chinese and Japanese textbooks developed fraction division as an inverse operation of fraction multiplication and prominently used examples to illustrate the relationship between the two operations using the "one problem, multiple solutions" approach. In contrast, the focus of US textbooks was on the computational process of fraction division by extending previous understandings of division involving whole numbers. In the US, the concept of division of fractions was either explained directly or through the use of pictorial representations. Thus, the Chinese approach emphasized the mathematical structures of and the relationship between fraction division and multiplication, whereas the US approaches emphasized the computation procedures.

Although both Chinese and US textbooks emphasized multiple representations, the US textbooks generally used pictorial representations to demonstrate the computation process of fraction division while the Chinese textbooks primarily used pictorial representations to develop the concept of fraction division and to explain why the algorithm works. In addition, the Chinese textbooks emphasized the problem-solving approach in the presentation of fraction division content and tended to include larger number and more difficult problems than the US textbooks.

A Framework for Examining Classroom Instruction

A variety of theories and approaches could be used to examine classroom instruction (Richardson 2001). Some studies have focused on investigating the nature of mathematics classroom (Clarke et al. 2006; Cobb and Bauersfeld 1995; Hiebert et al. 2003), while others were aimed at characterizing pedagogical contracts (e.g., Boaler 1998). Due to the purposes of the current study (i.e., examining the nature and characteristics of fraction division teaching and their connections to textbooks used), by reference to the framework used by Li et al. (2009a, 2009b), we will focus our literature review on (1) learning progression of fraction division, (2) mathematics tasks (examples and exercises), and (3) representations.

Learning Trajectory Building on the social constructivist theory, Simon and his collaborators (Simon 1995; Simon and Tzur 2004; Simon et al. 2004) have developed a theory on designing and implementing lessons based on the notion of Learning Trajectory (LT) (Simon and Tzur 2004; Simon et al. 2004). The LT has three components: "the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' understanding will evolve in the context of the learning activities" (Simon 1995, p. 136). In the context of fraction division, different conceptualization approaches project different learning trajectories. In the current study, the instructional objectives stated in lesson plans and learning progressions uncovered in the videotaped lessons were examined to depict learning trajectories constructed in classrooms.

Mathematical Tasks and Student Learning The role mathematical tasks play in engaging students in mathematical thinking and reasoning about substantial concepts and ideas has been realized and investigated for a long time (Doyle 1983, 1988; Hiebert and Wearne 1993; Stein and Lane 1996). Mathematical tasks are fundamental to learning because "tasks convey messages about what mathematics is and what doing mathematics entails" (National Council of Teachers of Mathematics [NCTM] 1991, p. 24). Mathematical tasks can provide a learning environment in which students engage in and develop mathematical concepts and mathematical thinking. Mathematical tasks have potential influences on students' thinking and can broaden, or restrict, their ideas and perspectives on subject matters (Henningsen and Stein 1997). A theory of mathematics teaching, called *teaching with variation*, has been in place for several decades in China (Gu et al. 2004). This theory emphasizes developing knowledge and building essential connections among relevant concepts through working with systematic and interconnected problems that focus on critical features of the objects of learning. Although mathematical tasks generally include projects, questions, constructions, applications, and student exercises, in this study tasks are used to refer to problems (including examples) and exercises. We examined the features of classroom instruction through investigating how teachers developed new knowledge through launching and implementing mathematical tasks.

Pedagogical Representations and Student Learning In addition to use of mathematics tasks, pedagogical representation is a widely used aspect for exploring classroom instruction. When we speak of pedagogical representations, we mean representations used by teachers and students in the classroom. Pedagogical representations are helpful in explaining or illustrating concepts, connections, relationships, or problem solving processes (Cuoco and Curcio 2001). Some representations may be more powerful than others for teaching particular concepts (Leinhardt 2001). Thus, what representations to use and how to use them are important decisions a teacher makes when selecting instructional strategies for a mathematics classroom.

Recently, an attempt to examine how Chinese and US teachers conceptualized and constructed pedagogical representations for mathematics instruction (Cai 2005; Cai and Wang 2006; Huang and Cai 2011) shed insight into understanding of mathematics instruction. Accordingly, in this study we investigated how teachers taught fraction division through examining how they constructed and used representations in the classroom.

The Current Study

In the current study, we examined how selected Chinese teachers taught fraction division regarding: (1) the structure of fraction division in classroom teaching, namely, instructional objectives and the sequence of content knowledge presentation (answering research question #1); (2) the development of the content by examining mathematics tasks and pedagogical representations (answering research question #2). In addition, we examined the connections between the characteristics of fraction division teaching in Chinese classrooms (findings derived from the current data analysis) and the treatment of division of fractions in Chinese textbooks (findings by Li et al. 2009a, 2009b) (answering research question #3).

Method

Data Sources

The data consisted of eight videotaped lessons taught by two Chinese teachers and their corresponding lesson plans, selected from a larger project investigating cross-cultural (Chinese and US teachers') lesson planning and classroom instruction. A total of seven elementary schools from two Chinese provinces participated in the larger research project (Li et al. 2009a, 2009b). With the guidance of Chinese mathematics education experts, the sample schools were selected so that they represented a large range of school qualities based on their reputations. Each selected school received an invitation to the project and an explanation of the objectives, procedures, and instruments used for data collection. For the current study, we selected two teachers based on the reputation of their schools and their teaching experiences so that they represented an average level of teaching.

Each of the two teachers selected for this study taught four consecutive lessons that were videotaped by one of the researchers. These two teachers were from elementary schools located in two different provinces. The first elementary school was located in a suburban area of a medium-sized city; however, the school was in the process of transformation. The school used to serve a student population mainly from the rural areas adjacent to the city, but it has now started to serve some of the urban areas as well. In terms of the school's location, the community it served, students' test scores, and perceived teachers' quality, the school had an average standing in that province. The teacher from the first school (Teacher A) had 8 years of teaching experience. The second school was located in a rural area in another province; its quality was judged as below average for that province. Teacher B, who was from the second school, was a promising teacher with 19 years of teaching experience. There were roughly 45 students in each class.

Data Analysis

We used the videotaped lessons and transcripts as our data sources. One researcher watched the videos and read corresponding transcripts to get an understanding of the Chinese lessons. Then, the researcher identified the main contents of all of the lessons and developed a concept map of teaching division of fractions. We made a detailed examination of all four consecutive lessons from each Chinese teacher with extra attention paid to the content connection and variation across lessons. We examined the use of mathematics tasks for introducing, developing, and consolidating fraction division. Meanwhile, we also examined how teachers constructed pedagogical representations when solving problems. The types of representations included in the lessons were algebraic/symbolic, numeric/tabular, graphic, and verbal/literal (Cuoco and Curcio 2001).

Results

The results are presented in three sections. In the first section, we report the development of a learning trajectory for division of fractions. The second section concerns common features of fraction division teaching in the sample Chinese classrooms. The third section reports an analysis of the relationship between how division of fractions is taught in classrooms and how it is treated in textbooks.

Learning Trajectory Constructed in the Classrooms

Content Coverage and Instructional Objectives The two Chinese teachers, Teacher A and Teacher B, spent four lessons teaching fraction division in a similar manner. The content arrangement and relevant instructional objectives based on lesson plans are displayed in Table 1.

Table 1 shows that Chinese teachers covered essentially the same content and instructional objectives: understanding the meaning of fraction division and the relationship between multiplication and division; understanding and mastering the computational rules for dividing a fraction by a whole number (F/WN); understanding

Table 1 The content arrangement and instructional objectives in the Chinese lessons

Teacher	Lesson 1	Lesson 2	Lesson 3	Lesson 4
A	Understanding the meaning of fraction division and the relationship between multiplication and division; Understanding and mastering the computational rules for dividing fractions by whole numbers (F/WN)	Understanding and mastering the computational rules for dividing whole numbers by fractions (WN/F)	Understanding and mastering the computational rules for dividing fractions by fractions (F/F)	Mastering the methods of solving word problems using fraction division; Understanding comparison of fractions
B	The meaning of fraction division; The relationship between multiplication and division; The algorithm of F/WN and its justification	Understanding the meaning and algorithm of WN/F	Synthesizing the algorithm of F/F; Problem posing and problem solving; Solving word problems	Dividing mixed numbers by mixed numbers; Understanding of comparison of fractions.

and mastering the computational rules for dividing a whole number by a fraction (WN/F); understanding and mastering the computational rules for dividing a fraction by a fraction (F/F); and mastering word problem solving and comparison of fractions before and after division by a fraction. Their developments of these contents were also quite similar, except for the minor differences in emphasis.

Learning Progression for Fraction Division

Both lesson plans and videotaped lessons revealed that the teachers followed a pattern (see Fig. 1) explicitly. The two Chinese teachers made efforts to develop fraction division: (1) developing the concept of fraction division based on students' prior knowledge (meaning of whole number division and the relationship between multiplication and division) (Lesson 1); (2) developing the algorithms coherently and systematically from F/WN (Lesson 1), WN/F (Lesson 2), to F/F (Lesson 3), and (3) applying the algorithms to different contexts such as word problems and comparison of fractions (Lessons 3 and 4). The key of learning fraction division was to understand that the meanings of fraction division and whole number division were the same, and that division is the inverse operation of multiplication. Then, ways of learning about the whole number division were analogized and adapted to fraction division. Second, by effectively using the pictorial representation (segment diagram), the meaning of a fraction and the meaning of division were explicated to help students understand WN/F (lesson 1), WN/F (lesson 2), and F/F (lesson 3). In

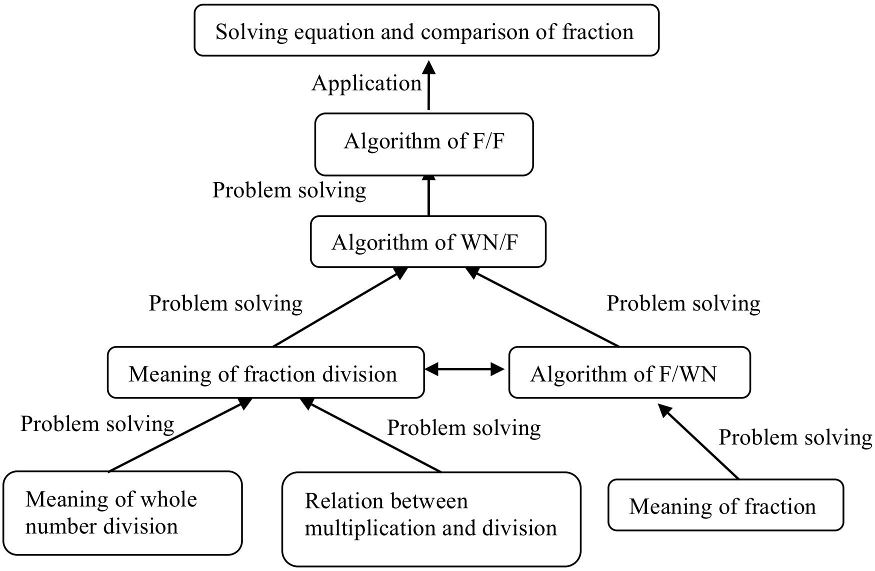


Fig. 1 The concept map of DoF development

this way, the fraction division concept and algorithm were built on and developed from the basic concept of whole number division, the meaning of a fraction, and the relationship between division and multiplication. So, different kinds of knowledge were interconnected. And finally, the new knowledge was linked to problem solving and comparison of fractions. Thus, students' knowledge of fraction division was strengthened and re-structured. This relationship is displayed in Fig. 1.

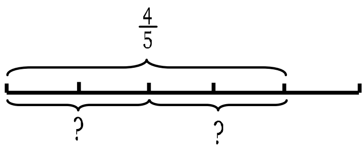
Characteristics of Fraction Division Teaching

The Lesson Structure

By and large, the two teachers shared a similar teaching pattern, which included (1) reviewing previous lesson's content or relevant knowledge for learning the new topic, (2) introducing the new topic through solving mathematical problems related to everyday life, (3) practicing new knowledge with a variety of interconnected problems and summarizing relevant key points or contents in the lesson, and (4) assigning homework. In the sections that follow, we describe the main procedures of the four consecutive lessons of Teacher A.

Lesson 1 After starting with a review of the meaning of whole number division and doing some relevant mental computations, the teacher posted two word problems with pictorial representations: (1) If each of five people eats half a cake, how

Fig. 2 The segment diagram representing $(4/5) \div 2 = (4 \div 2)/5$



much do they eat in total? (2) Can you pose two division problems based on the above information? Students produced three numerical expressions ($5 \times \frac{1}{2} = \frac{5}{2}$; $\frac{5}{2} \div 5 = \frac{1}{2}$; $\frac{5}{2} \div \frac{1}{2} = 5$) as they solved the problems. Students were then led to discover *the meaning of fraction division* and were subsequently asked to read this statement from the textbook in chorus.

The class moved on to explore the algorithm of dividing a fraction by a whole number as they worked on word problems in groups. Four different solutions to the same problem were discussed: (a) $\frac{4}{5} \div 2 = \frac{4}{5} \div \frac{2}{1} = \frac{4 \div 2}{5} = \frac{2}{5}$; (b) using a segment diagram to demonstrate the meaning of $\frac{4}{5}$ and to divide it into two parts (of size $\frac{2}{5}$) (see Fig. 2); (c) using the equivalence, that is "Dividing $\frac{4}{5}$ by 2 is equal to $\frac{1}{2}$ of $\frac{4}{5}$" (i.e., $\frac{4}{5} \div 2 = \frac{4}{5} \times \frac{1}{2}$); and (d) transformation of the equivalence into a decimal operation (i.e., $0.8 \div 2 = 0.4$). This discussion led to the formulation of two common strategies: (1) If the numerator of the fraction is a multiplier of a whole-number divisor, then the quotient equals a fraction with a numerator that is dividing the original numerator by the divisor while the denominator remains the same; and (2) Dividing a fraction by a whole number is equal to the fraction times the reciprocal of the whole number.

Students followed with a variation of the previous word problem so that they legitimized that the second strategy was more convenient and applicable. Moreover, the teacher asked students to read this *computational rule* in chorus (it was emphasized that the divisor cannot be equal to zero). After that, students worked on several exercises from the textbook and some extra problems as they competed in groups or as individual seat-work followed by sharing their work in class. Finally, the teacher summarized the key points of the lesson.

Lesson 2 After reviewing the meaning of fraction division and computational rules for dividing fractions by whole numbers, two word problems from the textbook were discussed. The purpose of the first problem was to review whole number division using the quantitative relationship among velocity (V), time (T), and distance (S) (i.e., $S = VT$). The second problem was designed to explore the new topic, which was dividing whole numbers by fractions ($v = s/t = 12 \div \frac{1}{5}$). Students presented three different ways of computing $12 \div \frac{1}{5}$: (a) $12 \div 0.2 = 60$; (b) $12 \div \frac{1}{5} = 12 \times 5 = 60$; and (c) using a segment diagram including five equal parts, each of them presenting the distance in 1/5 hours (similar to Fig. 2).

Students were also asked to explain different numerical expressions of dividing whole numbers by fractions (e.g., $7 \div \frac{1}{4} = 28$; $24 \div \frac{3}{4}$) using a segment diagram. Based on previous discussions, the computational rule of fraction division was sum-

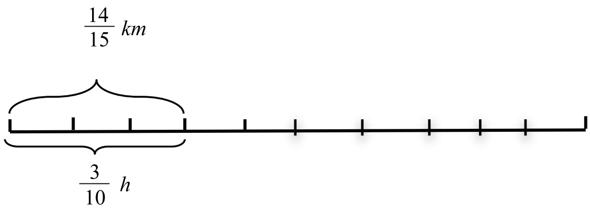


Fig. 3 Diagram representing the relationship between time and distance

marized. Then, the class worked on several exercises on dividing a whole number by a fraction.

Lesson 3 After a review of the previously learned computational rules of fraction division (i.e., $\frac{4}{5} \div 3 = \frac{4}{5} \times \frac{1}{3}$; $4 \div \frac{1}{3} = 4 \times 3$), students identified the commonality among these rules: changing division into multiplication and changing the divisor into its reciprocal. Then, the teacher asked students to read out a word problem from their textbook (i.e., Xiaoming walks 14/15 km in 3/10 hours, how far does he walk in one hour?), and students were asked to express the relation using fraction division (e.g., $\frac{14}{15} \div \frac{3}{10}$). Students were encouraged to make conjectures on how to perform the fraction operation and to justify their conjectures. By using a segment diagram (the teacher drew on the board, see Fig. 3), students were asked to explain the following procedure:

$$\frac{14}{15} \div \frac{3}{10} = \frac{14}{15} \div 3 \times 10 = \frac{14}{15} \times \frac{1}{3} \times 10 = \frac{14}{15} \times \frac{10}{3}$$

The teacher assigned several exercises from the textbook (such as, $\frac{2}{7} \div \frac{5}{6}$; $\frac{1}{12} \div \frac{4}{15}$); several students were invited to write their solutions on the board, and then students' solutions were discussed in the class.

After completing exercises, students were encouraged to summarize the computational rule of fraction division using different representations: Dividing a number by a fraction is equal to the number multiplied by the reciprocal of the divisor (in word); $A \div B(B \neq 0) = A \times \frac{1}{B}$ (in symbol). In particular, students noted the divisor could not be zero. Then, students were asked to read the computational rule in chorus from their textbook.

Subsequently, the students were asked to do more exercises from the textbook, and some students were invited to write their solutions on the board (the teacher explicitly emphasized that it was necessary to increase their computational speed when mastering computational procedures). Finally, the solutions were discussed, and the teacher summarized some key points.

Lesson 4 Starting with a review of the fraction division rule, students were then asked to change four fraction division problems into multiplication problems orally. Subsequently, the teacher presented the topic for the current lesson: word problems. The teacher presented a problem (If 3/8 of a given number is 1/4, what is the given number?) that required students to use two methods (i.e., $\frac{3}{8}x = \frac{1}{4}$; or $\frac{1}{4} \div \frac{3}{8}$) in the solution. The teacher summarized that there were two methods to solve problems such as “Given a proportion of a number, find out the number.”

The teacher then presented four word problems to be solved using equations. Students wrote their solutions on the board, and the teacher commented on the solutions. The teacher then presented four fill in the blank problems for students to solve (e.g., $1/3$ is $5/6$ of ( )?). Students were asked to compare the fraction division expression with the original dividend (e.g., $\frac{6}{7}$ with $\frac{6}{7} \div 3$; 9 with $9 \div \frac{3}{4}$; $\frac{1}{2}$ with $\frac{1}{2} \div \frac{2}{3}$; $\frac{14}{15}$ with $\frac{14}{15} \div \frac{7}{30}$). With this exercise, it was intended to lead students to realize that dividing by a fraction less than 1 would result in the quotient’s increase and dividing by a fraction larger than 1 would result in the quotient’s decrease. After that, the teacher assigned similar exercises from the textbook.

The Common Features of Fraction Division Teaching in China

A detailed description of the four lessons by Teacher B can be found in Appendix. After comparing the lessons by the two teachers, we found that there were more commonalities than differences. The common features included: developing students’ understanding of the algorithm through solving word problems, consolidating the algorithm through systematic and varying exercises, and deepening students’ understanding of the algorithm through purposefully selected representations.

Developing the Algorithm As indicated in the concept map of fraction division (Fig. 1), problem solving is an often-used strategy for introducing, developing, and consolidating knowledge. These two teachers consistently introduced and developed knowledge (concepts and algorithms) by exploring word problems. For example, in lesson 1, in order to explore the meaning of fraction division, both Teacher A and Teacher B used word problems. Both teachers (in lesson 3 by teacher A while in lesson 4 by teacher B) presented word problems that required equations in their solutions, such as “3 times a number is $\frac{2}{5}$. Find the number.”

In order to explore the algorithm of F/WN, Teacher A used two word problems: (1) Divide a rope of $\frac{4}{5}$ meter into two equal parts. How long is each part? and (2) Divide a rope of $\frac{4}{5}$ meter into three equal parts. How long is each part?

Again, in lessons 2 and 3, both teachers used word problems to introduce the algorithm of WN/F and F/F. For example, Teacher A used the problem “A pigeon flies 1.2 km in 1/5 hours. What is the velocity of the pigeon?” ($v = s/t = 1.2 \div \frac{1}{5}$) to introduce WN/F, and another word problem, “A butterfly flies 13/14 km in 3/10 hours. How far can it fly per hour? ” to explore F/F. Similarly, Teacher B used the word problems “If a train travels 60 kilometers per hour, then how far does it travel in 3/4 hour?” and “If a train travels 45 km in 3/4 hours, how far does it travel per hour?” to introduce WN/F, and another word problem, “One red silk belt measures $\frac{9}{10}$ m. If $\frac{3}{10}$ m red silk belt is needed to make a Chinese tie, how many ties can be made from this belt?” to introduce F/F.

In both teachers’ lessons, there was a common effort to encourage students to find multiple solutions to the same problem. Through comparing different solutions, the most reasonable solution was emphasized (usually it was related to an appropriate computational rule). For example, in lesson 1, both teachers explored the computational rule of F/WN through solving word problems. Although students found two strategies, the second strategy (i.e., the rule for F/WN: the fraction times the reciprocal of the whole number) was more convenient and applicable. Thus, the introduction of the rule for F/WN was justified. In lessons 2 and 3, both teachers encouraged students to find different methods to solve the same problem. As a result, new computational rules were discovered, and different concepts and knowledge were applied to develop a deeper understanding. For example, in Teacher B's second lesson, three solutions to the problem "If a train travels 45 km in 3/4 hours, how far does it travel per hour?" were explored:

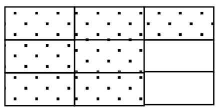
$$45 \div 3 \times 4 = 45 \times \frac{1}{3} \times 4 = 60 \text{ (km)}; \qquad 45 \div 3 + 45 = 60 \text{ (km)}$$

$$45 \div \frac{3}{4} = 45 \times \frac{4}{3} = 60 \text{ (km)}; \qquad x \times \frac{3}{4} = 45 \text{ (km)}.$$

Thus, $45 \div \frac{3}{4} = 45 \times \frac{4}{3}$ (i.e., the rule of WN/F) was discovered and justified. Also, the relationship between the parts and the whole unit was demonstrated by a segment diagram (similar to Fig. 2).

Consolidating the Algorithm In each lesson, there were many classroom exercises for enhancing and applying learned knowledge. The following features were found in common: (1) practice problems were mainly selected from the textbook, though some of them were created by teachers; (2) classroom exercises were conducted in various forms, such as individual work, group work, or competition. Usually, the answers were presented on the board and discussed in class; and (3) types of problems varied, with a focus on the learned content. For example, in lesson 3 of Teacher B, classroom exercises included (1) basic exercises (e.g., $\frac{2}{7} \div () = \frac{2}{7} \times \frac{1}{7}$); (2) computation and reasoning (judging if an equation or inequality is tenable: for example, $\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \div \frac{5}{3}$, $\frac{2}{5} \times \frac{1}{5} < \frac{2}{5} \div \frac{1}{5}$); (3) word problems that required solving equations (e.g., $\frac{1}{3}x = \frac{4}{9}$, $5x = \frac{4}{9}$), and (4) an open-ended problem. The following open-ended problem was presented as group-work:

If the area of the shaded part in the diagram on the right is 28 square meters, what questions can you pose? How can you solve them?



Students, working in groups of four, raised the following questions and solutions:

(1) The area of each shaded block? ($28 \div 7 =?$)
(2) The area of the large rectangle? ($28 \div \frac{7}{9} =?$)
(3) What is the area of the blank part of the large rectangle? ($2 \times (28 \div 7) =?$ or $28 \div \frac{7}{9} - 28 =?$ or $(28 \div \frac{7}{9}) \times \frac{2}{9} =?$)

When arranging and solving problems, we observed characteristics that were common to both teachers. They developed new problems based on a prototype of problems and encouraged students to search for multiple solutions to problems.

Developing the Algorithms through Purposeful Use of Representations The two teachers purposefully used different representations to develop the algorithm of fraction division. In lesson 1, both teachers basically used verbal and numerical representations to review the meaning of whole number division and to develop the meaning of fraction division. The teachers also used pictorial representations and/or a physical model (for example, Teacher A used a half circle representing half a cake, a segment diagram representing the partitioning of ropes, and a physical rope to demonstrate the partitioning of a rope). In lesson 2, both teachers paid great attention to using segment diagrams to develop the algorithm for WN/F. In lesson 3, the teachers used either a segment diagram (Teacher A) or a pictorial representation (Teacher B) to develop the algorithm and solve problems. In lesson 4, symbolic and verbal representations were extensively used to solve equations and to compare fraction values.

It seems that these two teachers used representations selectively and hierarchically: from physical representations to pictorial representations to symbolic and verbal representations. The pictorial and physical representations were only used to develop the algorithm of fraction division. After the algorithm was discovered and justified, they used symbolic/numerical representations for application.

The Relationship Between Textbooks and Classroom Teaching

We presented the relations between textbooks used and classroom teaching from three aspects: (1) the ways of conceptualizing fraction division; (2) the ways of selecting and using mathematical tasks (examples and exercises); and (3) the ways of using representations.

Teacher A used one of the three textbooks examined by Li et al. (2009a, 2009b), and Teacher B used another of the three textbooks. These teachers not only did directly choose workout examples and class exercises from their textbooks but also asked students to read aloud the computation rules stated in the textbooks. Particularly, Teacher B explicitly required students to read the textbook before class. As a result, there was a strong consistency between the textbooks and classroom instruction. Considering findings of Li et al. (2009a, 2009b) and findings of the current study, we identified the following consistencies.

First, over four lessons both teachers put great efforts to progressively develop students' understanding of the meaning of fraction division, the relationship between division and multiplication, and why the algorithm of fraction division worked. This is fairly consistent with the intention of Chinese textbooks (Li et al. 2009a, 2009b).

Second, the Chinese teachers organized word problem solving activities to guide students to discover and justify the algorithm for division of fractions from simple (F/WN) to complex situations (F/F) using different approaches. They both encouraged students to find multiple solutions to the same problem and to recognize the invariant pattern (i.e., computation rule) through comparing different solutions. This feature reflects Chinese textbooks' design that develop division of fractions as an inverse operation of fraction multiplication through solving problems using multiple methods, namely the "one problem, multiple solutions" approach (Li et al. 2009a, p. 824). Meanwhile, these teachers either had students complete some of the textbook exercises in class and/or assigned them as homework. However, teachers paid close attention to some problems in textbooks. For example, both teachers treated "identifying the relationship between quotient and dividend when the divisor increases or decreases" (Li et al. 2009a, p. 823) as an opportunity to develop students' ability to observe and discover. Moreover, they also deliberately designed some problems based on other published teaching materials or their own lesson plans. The open-ended problem posing and solving given by Teacher B (lesson 3) was one example.

Third, the use of different representations (e.g., psychical, pictorial, numerical) in these lessons was intended to help students understand the process of problem solving and why the algorithm worked, which was in line with the intention of the textbooks (Li et al. 2009a, 2009b).

In summary, textbooks had important influences on Chinese teachers' classroom teaching in terms of content coverage, teaching objectives, principles of developing content/learning trajectories, and teaching strategies. Overall, classroom teaching essentially stuck to the textbooks. However, there were some variations and flexibilities in terms of emphasis on certain content points, selection of problems, and assignment of homework. In particular, the teachers adopted some challenging mathematics problems from other resources or adjusted some classroom exercises to meet students' needs. This situation is coined as a Chinese saying, "Teaching should be derived from textbooks, but exceed textbooks."

Conclusion and Discussion

Based on the analysis of selected Chinese teachers' teaching of fraction division, we came to the following conclusions. The Chinese teachers (1) put great effort into developing students' understanding of the meaning of fraction division and their justification of why the algorithm of fraction division works (as inverse operation of fraction multiplication); (2) adopted a problem-based approach to develop the meaning of fraction division, to justify the algorithm, and then to apply the algorithm; and (3) used multiple representations strategically (i.e., visual representations to scaffold the development of algorithms and symbolic representations for extensive applications of algorithms).

A consistency between textbooks and their implementation in classrooms was found regarding the coverage of contents, content development, and selection and use of problems and exercises. The Chinese teachers followed the fundamental principles of their textbooks, such as conceptualizing fraction division as the inverse operation of fraction multiplication and developing the meanings and algorithms of fraction division through word problems. In addition, pictorial representations were used to show why the algorithm of fraction division worked, which also mirrored the textbook treatment. However, the teachers demonstrated flexibility in selecting and constructing examples and exercises.

Given the fact that textbooks in China are official and mandated (Liu and Li 2010), it is not surprising that the sample teachers taught classes by following their textbook seriously. Interestingly, teachers not only followed the sequence of content presentation in the textbook smoothly but also implemented the fundamental principles presented in textbooks essentially. For example, the teachers conceptualized division of fractions as an inverse operation of fraction multiplication as it was presented in textbooks (Li et al. 2009a, 2009b). The teachers adopted problem-based approach to develop the concept and algorithm of fraction division consistently. Surprisingly, the Chinese teachers put much emphasis on the conceptual understanding of such a procedure-oriented content over four lessons. The practice may imply that the Chinese teachers pay great attention to developing students' conceptual understanding and procedural fluency simultaneously.

Use of the Problem-Based Approach Consistently

Emphasis on solving problems and altering problems to promote multiple perspectives are traditional features of Chinese mathematics classrooms (Cai and Nie 2007; Huang et al. 2006), and these approaches were also valued in textbooks (Li et al. 2009a, 2009b; Sun 2011). The core of teaching with variation, a widely adopted teaching method in China, is to vary problems systematically and strategically to promote students' learning (Gu et al. 2004). The problem-based approach is well-recognized as a mathematics learning and teaching method around the world (e.g., Baroody and Dowker 2003; Shimizu 2009); it may make a difference in creating opportunities for students to learn if this approach is valued in both textbooks and classroom instruction intentionally.

Using Representations Flexibly

This study may provide an explanation to why Chinese students prefer using symbolic representation when solving problems (Cai 2005) because Chinese teachers treat concrete representation as scaffolding for developing algorithms, and then they will use symbolic/abstract representations for application of knowledge. Huang and Cai (2011) found that the Chinese teacher in their study tended to use representations selectively based on the nature of problems, while the U.S. teacher in the study tended to use multiple representations simultaneously. Developing students' ability using multiple representations has been called for decades for the development of mathematics knowledge and problem solving (Cuoco and Curcio 2001; Lesh et al. 1987). The use of representations in Chinese textbooks and classrooms suggest that it is crucial to adopt representations purposefully and flexibly, rather than the multiplicity of representations.

Adaptation of Textbooks Strategically

Let us consider "how far may teachers go in their adaptations without destroying the spirit and meaning of the curriculum they implement in their classes?" (Ben-Peretz 1990, p. 31). Although it is difficult, if not impossible, to answer this question precisely, the Chinese practice may shed light on addressing this issue. Firstly, it is crucial to follow the fundamental principles presented in textbooks, such as conceptualizing fraction division as an inverse operation of fraction multiplication, adopting the word problem-based approach, and using pictorial representations to develop the algorithm. Second, introductory problems, examples, and exercises should be used carefully with attention to their purpose and roles and considering students' knowledge readiness and ability. Third, teachers should be encouraged to construct their own examples and exercise problems based on their knowledge of students and pedagogy to individualize their teaching (such as by decreasing or increasing cognitive demands of problems). Considering these factors may help teachers in adapting textbooks in their classes appropriately without destroying the intents of textbooks.

Developing Knowledge and Capacity in Adapting Textbooks

Responding to the call to support teachers in making "well-informed, purposeful decisions (that is, acceptable adaptation) to benefit students' learning of mathematics " (Huntley and Chval 2010, p. 301), it is necessary to realize the importance of studying teaching materials (Ma 1999). In Ma's seminal work, she attributed Chinese elementary teachers' profound understanding of fundamental mathematics to four main factors, including: studying teaching materials intensively, learning mathematics from colleagues, learning mathematics from students, and learning mathematics by doing it. In China, it is fundamentally important to extensively study teaching materials (including textbooks, teaching and learning frameworks, and teachers' manuals) (Ma 1999). Ding et al. (2012) further found that Chinese teachers' knowledge and understanding of mathematics instructional content is mainly attained through intensive studies of textbooks under a supporting professional development system. The sample teachers in their study viewed the "study of textbooks" as an exploration of knowledge beyond textual information, which included (1) identifying the important and difficult points of teaching a lesson, (2) studying the purposes of each worked example and practice problem, (3) exploring the reasons behind certain textbook information, and (4) exploring the best approaches, from the perspectives of students, to present examples. Such a profound understanding of textbooks may help teachers to make appropriate and effective decisions in adapting textbooks to prompt students' learning.

Methodology of Studying Implementation Fidelity

Previous studies mainly conducted surveys, interviews, and classroom observations (Huntley and Chval 2010; Tarr et al. 2006) as their methodologies to study textbook implementation but paid less attention to the teaching of specific contents. In contrast, this study extends efforts to examine *implementation fidelity* through investigating what really happens in the classroom. Focusing on a specific topic over consecutive lessons may provide an additional way of researching implementation of textbooks.

In conclusion, the sample teachers essentially adapted their textbooks. They followed the textbooks regarding the conceptualization of concepts and algorithms, the topic coverage, the sequence of content presentation, the approach to developing the concepts and algorithms, and the selection of problems and exercises. The teachers also demonstrated certain flexibility in constructing their own problems for introducing and consolidating new knowledge. The strategies of adapting textbooks may be related to their teaching culture and professional development practice. Extensively studying teaching materials may be an effective way to develop teachers' knowledge of and capability in adapting textbooks in their classrooms.

Appendix: Brief Description of Teacher B's Lessons

Lesson 1 Two methods of fraction division were discussed via a word problem. The teacher asked students to state the meaning of fraction division and the relationship between multiplication and division. After explicitly expressing that the meaning of fraction division was the same as the meaning of whole number division, and fraction division was the inverse operation of fraction multiplication, the teacher led the class to discuss the algorithm of dividing a fraction by a whole number.

Then, the teacher asked students to express the algorithm for dividing a fraction by a whole number. To practice this algorithm, students posed several problems related to dividing a fraction by a whole number (e.g., $\frac{2}{7} \div 3$, $\frac{4}{9} \div 2$) and discussed their solutions and justification in terms of two classifications (i.e., when the numerator is divisible by the divisor and when it is not). For example, students explained why the following procedure worked: $\frac{4}{9} \div 2 = \frac{4 \div 2}{9} = \frac{2}{9}$. Students explained the procedure according to the meaning of fraction and whole number division. In order to help students understand why dividing a fraction by a whole number is equal to the fraction times the reciprocal of the whole number, the teacher organized a hands-on demonstration activity: one student was asked to classify 12 magnetic pads into 3 equal groups, and another student was asked to take away one third of the 12 magnetic pads.

Through comparing the two methods of arranging magnetic blocks, students realized that dividing a fraction by a whole number was equal to the fraction times the reciprocal of the whole number. Then, three types of exercise problems were organized: questions for oral answers, word application problems, and competition problems.

Lesson 2 Beginning with a word problem, the class explored the meaning and algorithm of dividing a fraction by a fraction. The problem was used to recall the method of using a diagram to represent the quantitative relationship between a standard (unit) quantity, partial rate, and partial quantity (similar to Fig. 3). The teacher presented another word problem as follows: If a train runs 45 km in 3/4 hours, how far does it run per hour? By using a similar diagram, students found three solutions to the problem and justified $45 \div \frac{3}{4} = 45 \times \frac{4}{3}$.

Based on this discussion, students discovered the algorithm of dividing a whole number by a fraction. Immediately, the teacher assigned a similar word problem for students to solve, and students presented their three solutions on a small board.

Lesson 3 The lesson began with a review of dividing fractions by whole numbers and dividing whole numbers by fractions. The teacher presented one word problem (i.e., There is a red silk strip measuring 9 over 10 meter in length. If making one Chinese tie requires 3 over 10 of a red silk strip, how many Chinese ties can be made using the strip? How can this problem be expressed numerically? The answer to the question resulted in the following numerical expression: $\frac{9}{10} \div \frac{3}{10} = \frac{9}{10} \times \frac{10}{3} = 3$. Then, the teacher asked students to generalize this rule by providing another concrete example. Finally, the rule of fraction division was synthesized in general:

> Dividing a number A by a number B is equal to the number A times the reciprocal of the number B $(B \neq 0)$.

After that, students worked on several different types of exercises: basic exercises, comparing sizes of two expressions (e.g., $\frac{1}{2} \div \frac{3}{5} = \frac{1}{2} \div \frac{5}{3}$, $\frac{2}{5} \times \frac{1}{5} < \frac{2}{5} \div \frac{1}{5}$), open-ended problems, and word problem solutions (e.g., $\frac{1}{3}x = \frac{4}{9}$, $5x = \frac{4}{9}$).

Lesson 4 After reviewing the rules of fraction division, the teacher presented several fraction division expressions that included at least one mixed number (e.g., $\frac{7}{8} \div 1\frac{5}{6}$; $4\frac{2}{7} \div 1\frac{11}{14}$). Students worked on these problems individually and shared their solutions (some corrections were made). Then, the rule for division of mixed numbers was summarized: first transforming the mixed number to an improper fraction, then using the rule of fraction division.

Then, some exercises from the textbook were assigned to four student groups to be solved, and the results were checked in class. After that, the class discussed two sets of computation problems to make the following observations: (1) When dividing by a fraction less than 1, the quotients will increase, and when dividing by a fraction larger than 1, the quotients will decrease; (2) When the denominators are the same, the larger the numerator is, the larger the fraction is. On the other hand, when the numerators are the same, the larger the denominator is, the smaller the fraction is.

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