Three Pillars of First Grade Mathematics, and Beyond

Roger Howe

Abstract An integrated approach to first grade arithmetic is described. It consists of a coordinated development of the three pillars of the title, which are (i) strong conceptual grasp of the operations of addition and subtraction through word problems, (ii) computational skill that embodies place value understanding, and (iii) coordination of counting number with measurement number. The ways in which these three parts interact and reinforce each other is discussed. This approach is highly consistent with CCSSM standards recently released in the United States by the Council of Chief State School Officers.

In a second part, a sketch is given of a further development of these key ideas in later grades. Increasing understanding of the arithmetic operations leads to increasing appreciation of the sophistication and underlying structure of place value notation, eventually making links with polynomials. Linear measurement becomes the basis for developing and exploiting the number line, which later supports coordinatization. Throughout, consistent attention should be given to interpreting and solving increasingly involved word problems. Successful intertwining of these three strands supports the later learning of algebra, and its links to geometry.

Keywords Word problems · Place value · Counting-measurement coordination · Number line

For nearly all students, first grade is the beginning of dedicated intensive instruction in mathematics. Since later mathematics learning builds on earlier learning, getting started right is important. Since arithmetic is the main focus of mathematics education in elementary school, first grade should concentrate on giving students a good start in arithmetic. Some would argue that geometry or data or early algebra should also get attention, and there is probably room time to do something about some of these (and the Common Core State Standards in Mathematics (CCSSM) (CCSSO 2011) calls for some), but starting arithmetic off right is the essential task of first grade.

R. Howe (✉)
Mathematics Department, Yale University, New Haven, USA
e-mail: howe@math.yale.edu

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This is not as simple as it might sound. Getting going in arithmetic involves more than learning how to compute. It entails developing a broad conception of the operations of addition and subtraction, one that includes all the main contexts where these might be used, and one that supports thinking of addition and subtraction as well-defined things with specific properties, about which we can reason. It also entails going beyond situations that are described by counting, to see how arithmetic applies to the arena of measurement. The connection of arithmetic to geometry through measurement both enlarges the conception of arithmetic and provides concrete and conceptual tools to help students think about arithmetic.

In the domain of computation, the overarching idea is that of place value. The standard conception of place value in the U.S. tends to be rather limited: it is frequently treated as a vocabulary issue, that students should know the value of each place in a multi-digit number. However, the principle of place value controls essentially all aspects of arithmetic computation and estimation. Students should eventually come to appreciate and be able to exploit the ubiquitous influence of place value. A good start in first grade can help students reach that goal.

The discussion below of computation and place value is substantially influenced by our reading of East Asian texts and education literature. In particular, we emphasize the value of *addition and subtraction within 20* (Ma 1999) as a context for learning the addition and subtraction facts. This also has been recognized by CCSSM, which has this topic as an explicit standard at grades 1 and 2.

These considerations lead to three main ingredients that are key to starting off right in arithmetic. They are:

(I) A robust understanding of the operations of addition and subtraction.
(II) An approach to arithmetic computation that intertwines place value with the addition/subtraction facts.
(III) Making connections between counting number and measurement number.

Below we enlarge on each of these topics. In two supplemental sections, we will sketch ways in which these basic themes might extend to later grades.

A Robust Understanding of the Operations of Addition and Subtraction

Addition is often described as combining and subtraction as taking away, but the types of situations in which these operations are used are more varied than these brief descriptions would suggest. Mathematics educators have articulated a taxonomy of one-step addition and subtraction word problems.

The CCSSM has adopted a version that recognizes 14 types. The types fall into three main categories: *change*, in which some number changes over time*; comparison*, in which the difference between two quantities plays a role; and *part-part whole*, in which some quantity or collection of objects made up of two parts. These broad classes are similar to those discussed in *Adding It Up* (Kilpatrick et al. 2001), based on *Children's Mathematics* (Carpenter et al. 1999), and also to the discussion in Fuson's paper (2005).

Each of the first and second types can be divided into two subtypes. In problems involving change over time, the initial quantity can either increase or decrease. Similarly, in comparisons of quantities, one quantity can be described either as more or less than the other one. In part-part whole problems, the two parts play equivalent roles, so these form just one family.

Finally, for each of the four subtypes of change or comparison problems, one can pose three different questions, according as to what is unknown. Thus, for change-increase problems, one can ask to find the final total, the amount of change, or the initial amount. For comparison problems, one can ask to find the larger quantity, the smaller quantity, or the difference. For part-part-whole problems, since the two parts play equivalent roles, there are only two questions: what is the size of the whole, or what is the size of an unknown part. In all, this gives $2 \times 2 \times 3 + 2 = 14$ types.

Here are examples of selected types:

Change-increase, total unknown: Shana had three toy trucks. For her birthday, she got four more toy trucks. How many toy trucks did she have then?

Comparison-more, smaller unknown: Shana has seven toy trucks. She has four more toy trucks than her friend Molly. How many toy trucks does Molly have?

Part-part whole, part unknown: Shana has a collection of seven toy trucks. She keeps them on two shelves in her bedroom. There are four trucks on the top shelf. How many trucks are on the lower shelf?

The full taxonomy, with all 14 subtypes (plus a fifteenth, of a different nature), is given as table I on page 88 of the Common Core State Standards (CCSSO 2011).

Although an adult may think of these types of problem as quite similar, mathematics educators have shown that young children find them quite different (Carpenter et al. 1999). For example, consider the problem

Change-increase, original amount unknown: Shana had some toy trucks. For her birthday, she got four more toy trucks, and then she had seven. How many toy trucks did she have before her birthday?

This type of problem turns out to be difficult for many young students to think about, because they are unsure how to model it. To solve the **Change-increase, total unknown** problem, they can count out three tokens, then four more tokens, then count all the tokens to find the answer. To deal with the **Change-increase, change unknown**, they can proceed similarly after some thought. They lay out seven counters to represent the total, and three next to them to represent the original amount. Then they count the unmatched counters in the total. (Effectively, they have converted the change problem to a comparison problem.) However, with the **Change-increase, original amount unknown**, they have trouble getting started. At this stage, the fact that a sum does not depend on the order in which the addends are combined (the *commutative* property of addition), is still to be learned.

The importance of presenting all types of addition and subtraction problems is clear if we take into account that a tremendous amount of learning takes place through examples. Children acquire vocabulary at the rate of several words each day (for passive vocabulary; see http://en.wikipedia.org/wiki/Vocabulary). Mostly, they do not look them up in the dictionary. Rather, they learn them by seeing them used in context, that is, through examples of how a word is used. It is important to obey the maxim of *example sufficiency*, especially in teaching abstract concepts, which are the main content of mathematics. By example sufficiency, I mean giving a broad enough array of examples to provide a well-rounded representation of the concept. A famous example of example insufficiency is the case of triangles. In brief presentations of the concept of triangle, frequently only one example, that of an equilateral triangle with a horizontal base, is given. Perhaps then it should not be surprising that studies have found that many second or third grade students will not identify non equilateral triangles, or even equilateral triangles with non-horizontal bases, as being triangles. With foundational concepts, such as addition and subtraction, which will form the base on which many further ideas are built, it is especially important to present a well-rounded collection of situations where addition or subtraction can be used. Thus, care should be taken in first grade to introduce all types of one-step addition/subtraction word problems, and to use them all repeatedly throughout the year with larger numbers as student technique in symbolic calculation improves.

Sometimes, the use of only a limited number of the simplest problem types is justified on the basis that young students have limited reading skills, and that mathematics must be presented in ways that they can understand. This point of view might seem to have increased validity today, when so many students are classified as ELL (English-language learners). However, I would argue that mathematics word problems are as important for their potential to improve reading skills and thinking skills as they are for teaching arithmetic technique. In fact, word problems are the glue that binds mathematics to the real world, and studying them from a language arts point of view, as passages that we want to understand, is as important as solving them. Oral presentation and class discussion can be a vehicle for this, as well as individual reading.

In class discussion, comparative analysis may be an effective tool. Comparing and contrasting pairs of problems, then discussing all three of one of the triples of problems, and ending with comparison of pairs of triples, may give students a sense for the territory of addition and subtraction in a way that just solving problems one at a time could not achieve. A somewhat subtle side benefit of this kind of activity may be that some students come to think of addition and subtraction as having an existence independent of calculation, that is, they may realize that the expression $3 + 8$ is a valid name for a number whether or not we calculate to find that it is 11. This kind of understanding supports algebra.

Comparison problems require a special note of caution. In almost all uses of numbers that occur in everyday life, numbers function as adjectives: two hats, or two dollars, or two train rides all can be interpreted readily; however, “two” by itself does not have a clear meaning. Without a unit to refer to, the meaning of “two” is incomplete. Correspondingly, when we discuss addition, we understand (usually tacitly) that the numbers we are adding all refer to the same unit. The statement “3 dimes and 4 nickels equals 2 quarters” is perfectly intelligible. However, the equation $3 + 4 = 2$ violates our usual understandings of arithmetic. The source of the problem here is that each number is referring to a different unit. To write an equation that expresses the desired relationship, we should make sure that all terms are denominated in the same unit. For example, if we express each coin, nickel, dime and quarter in terms of their value in pennies, we can write a correct equation:

$$3 \times 10 + 4 \times 5 = 2 \times 25.$$

Since ignoring the unit is usually does not cause trouble when dealing with whole numbers, units may often be suppressed in first grade and second grade texts. This can even serve a positive purpose, by emphasizing that arithmetic is independent of the unit: 4 apples and 3 apples make 7 apples, and likewise, 4 trucks and 3 trucks make 7 trucks. However, lack of unit awareness can wreak havoc during the study of fractions.

If they are not formulated carefully, comparison problems may seem to violate the same-unit principle. In such problems, one is often asked to compare the number of birds with the number of worms, or the number of children with the number of cookies. It may then seem that we are subtracting birds from worms, or the other way around, in contravention of the consistent unit principle. What is going on in these problems is more complicated. The problem scenario implicitly sets up a correspondence between the two sorts of things being compared, at some rate (often one-to-one). This implicit correspondence converts (implicitly, of course!) one of the quantities to the other, and subtraction takes place among the quantities of the type that is in abundance. However, this under-the-table correspondence may well be too subtle or confusing for young students to grasp. For this reason, it is advisable to formulate comparison problems so that they are about quantities of essentially the same type. For example, it is easier to assimilate "green apples" and "red apples" under the umbrella unit "apple" than it is to think of "tickets" and "people" as being essentially the same. Note that in the comparison example given above, all numbers referred to toy trucks.

An Approach to Arithmetic Computation that Intertwines Place Value with the Addition/Subtraction Facts

Place value is the central concept of arithmetic computation. It is not simply a vocabulary issue, of knowing the ones place, the tens place, and so on; it is the key organizing principle by which we deal with numbers. Place value, together with the Rules of Arithmetic, specifies the key aspects of how we perform addition/subtraction and multiplication/division (i.e., the algorithms of arithmetic). The vital role of place value is attested to by this quotation from Carl Friedrich Gauss (1777–1855), often named the greatest mathematician since Newton:

> The greatest calamity in the history of science was the failure of Archimedes to invent positional notation. (Eves 2002)

Two-digit numbers, and their addition and subtraction, is the topic where students first engage seriously with place value. The main ingredients in learning two-digit addition and subtraction are:

(a) learning the addition/subtraction facts: knowing the sum of any two digits (that is, the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), and, given the sum and one of the digits, knowing the other digit;
(b) understanding that a two-digit number is made of some tens and some ones; and
(c) in adding or subtracting, you work separately with the tens and the ones, except when regrouping is needed.

Specifically, item (c) comprises two situations:

(i) in adding, when you get more than 10 ones, you convert 10 of them into a ten, and combine that with the other tens; or
(ii) in subtracting, if the ones digit you want to subtract is larger than the ones digit you want to subtract from, you must convert a ten into 10 ones, and subtract from the resulting teen number.

The main US method for teaching this topic has been

(a) learn the addition/subtraction facts by memorization; and
(b) learn the column-wise algorithm for performing the operations.

These are often treated separately, with little or no rationale given for either, and no connections between the two. In recent years, increased use of base ten blocks has probably increased understanding of the regrouping process for some students. However, the learning of the addition facts remains primarily a memorization process, unconnected with the other parts of the package, in particular with regrouping. It is desirable and possible to combine the two key steps in such a way that they support each other, and are both connected to the fundamental principle of place value. We sketch the main steps in this development.

(1) Learn the addition and subtraction facts to 10.

This learning should be fluent, robust and flexible. This means understanding that $3 + 4 = 7$, and $7 - 4 = 3$, and also, that, if you have 4 and want 7, you need 3; and being able to produce any of these statements more or less automatically. (In the U.S., these variants are sometimes referred to as "related facts".) Instruction should be accompanied by many concrete and pictorial illustrations of the relationships involved.

In learning the facts to 10, it is valuable to spend time thinking about all the possible ways to decompose a given number, for example to note that

$$5 = 4 + 1 = 3 + 2 = 2 + 3 = 1 + 4.$$

Besides improving fluency, this work highlights structural facts, such as the commutative rule for addition, which reveals itself here in the symmetry of the possible expansions of 5: each decomposition is paired with another in which the addends are in the opposite order. (Problems that call for all the possible ways to decompose a whole number into two smaller whole numbers are recognized as a fifteenth type of addition and subtraction problem in the Table I of the Common Core Standards, as cited above.)

(2) Learn the teen numbers as a 10 and some ones.

In Chinese this is quite easy, because the number names express this directly: ten and one, ten and two, ten and three, and so on, up to two tens, and onward. It will involve more work in the US, since the number names are not as helpful. There will have to be class discussion about hearing the 10 in "teen", and hearing the 3 in "thir", so that students can think "ten and three" when they hear "thirteen". Similar work will have to be done with the other teen numbers. There will probably have to be some special talk about how "eleven" and "twelve" are pretty dumb names, but you just have to live with them, and think "ten and one" quietly to yourself when you hear "eleven".

Besides the names, the notation will need explicit attention. The fact that the 1 in 13 stands for ten, and the 3 stands for the three additional ones will probably have to be taken note of repeatedly. We agree not to write the 0 in the ten, to save time and space, but we put the 1 on the left of the three, and this is just a short way of writing $10 + 3$. If our number names reinforced this, learning would probably be quicker and easier, but with sufficient reminders, we can hope that students will retain the idea.

Some amount of work with the teen numbers should be done to help students become comfortable with them. Adding and subtracting a teen number and a single digit, not involving regrouping, asking which number comes just before or just after, asking which of two teen numbers is larger, are examples of exercises to increase familiarity. With regard to ordering, and adding that does not cross decades, students may observe spontaneously that only the ones digit is involved, and that, as far as this digit is concerned, everything is "just like" the parallel single digit behavior. If no student offers this, pointing it out may be helpful.

(3) Learning the higher addition facts.

This is known in East Asia as "addition and subtraction within 20" (Ma 1999). The importance of this topic for providing important connections in the learning of place value is recognized by the adoption of this term in the Common Core Standards in grades 1 and 2 (CCSSO 2011).

Now that the teen numbers are understood in terms of their base 10 structure, the focus returns to single-digit addition and subtraction, and learning the addition and subtraction facts when the total exceeds 10. Here the key point is *not* memorization of the higher addition facts, but understanding how to produce them, and their connection to place value notation. So, for example, to add $6 + 7$, a student should think in terms of making a 10. From stage (1), it is known that starting from 6, one needs 4 more to make 10. One gets the 4 from the 7, and since one also knows that $4 + 3 = 7$, there are 3 left over from the 7, so one gets 10 and 3 more, or 13. The formal expression of this in terms of symbolic manipulation uses the Associative Rule of addition to change the form of the sum:

$$6 + 7 = 6 + (4 + 3) = (6 + 4) + 3 = 10 + 3 = 13.$$

However, at this stage, such niceties can be ignored. Similarly, in subtracting a one-digit number from a two-digit number, one may have to unmake or break apart the 10. There are (at least) two different ways that a student might think about this; either one is valid. These are illustrated in the following computations.

$$13 - 7 = 13 - (3 + 4) = (13 - 3) - 4 = 10 - 4 = 6,$$

or

$$13 - 7 = (10 + 3) - 7 = (10 - 7) + 3 = 3 + 3 = 6.$$

(4) Learn that two-digit numbers are made of some tens and some ones.

When students are fairly fluent in the addition/subtraction facts and making/unmaking 10, attention can then move to larger numbers. The key understanding is that a two-digit number is made of some tens and some ones.

Thus, $43 = 40 + 3$ is 4 tens and 3 ones. The main work is probably in getting students to think of each -ty number as indicating a certain number of tens. Then the general two digit number is gotten by appending some ones, and this is fairly clearly indicated in the name. Again students need to learn to think beyond the names: “twenty” is 2 tens; “thirty” is 3 tens; “forty” is 4 tens; and so forth. The names and what they mean should again be connected with the notation: the 10s digit tells the number of tens, and the 1s digit tells the number of ones.

For many students, a fair amount of counting with verification, that indeed 20 is 2 tens, forty is 4 tens, and so forth, may be required to solidify confidence in the equivalence. As the counting is being done, the benefits of grouping by some manageable amount, which for us is 10, should be promoted. In fact, if counting gets interrupted, the advantage of having made groups of 10 should be evident, in greatly reducing the amount that must be recounted. A hundreds chart can also be useful in this work. In working with a hundreds chart, it may be helpful to point out that a given number tells the number of spaces in the chart up to and including that number. This observation can also be helpful when studying computation (step 5 below), especially in interpreting the effect of adding 1 or adding 10 to a general two-digit number. Some educators advocate having a hundreds chart in which the numbers with a given tens digit run down a column (rather than across a row, which seems to be the more common form).

Manipulatives such as 10-rods and 1-cubes may be helpful in making two-digit numbers tangible and accessible. Often such manipulatives are handled by arranging them in loose groupings, on a mat or other area designated for the work. However, it is probably a good idea to have students do some of this work in the context of linear measurement, with the 10-rods and cubes arranged into a linear train. Among other advantages, this will emphasize that the various rods and cubes are indeed united into a single quantity, with length corresponding to the size of the number. The measurement model for numbers is discussed further below.

Attention should also be paid to ordering two-digit numbers—thinking about which of two numbers is larger. Here the simple principle is, that the 10s digit determines the relative size of two two-digit numbers, except when both numbers have the same 10s digit, in which case, you look at the 1s digit. Since the size difference between the 10-rods and the cubes is starkly apparent when all are assembled into a train, the measurement or length model of numbers, constructed by trains of 10-rods and cubes, can provide a physical and visual way of thinking about the relative sizes of numbers and the order relation.

(5) Add/subtract two-digit numbers by combining tens with tens and ones with ones.

This can be done in stages: add and subtract 1 or 10 from a two-digit number, add/subtract single digit numbers or multiples of ten from a two-digit number, add/subtract two-digit numbers without regrouping, add/subtract single digit numbers to or from two-digit numbers when regrouping is required, and finally, the general case of adding or subtracting two-digit numbers with regrouping. When adding (or subtracting) a single-digit number to (or from) a general two-digit number, if regrouping is required, the corresponding addition fact should be emphasized. Both the reasoning and the mechanics of regrouping have already been learned while learning the addition facts beyond 10.

Manipulatives such as 10-rods and cubes can of course be used to model addition and subtraction. Again, arranging these rods and cubes into trains and working in terms of the length model for numbers can help students think about addition and subtraction. See section "Making Connections Between Counting Number and Measurement Number" for more details.

The ability to work independently with the tens and the ones should enable many students to do two-digit addition and subtraction mentally. To find $53 + 29$, a student could say "$50 + 20$ is 70, and $3 + 9$ is 12, and $70 + 12$ is 82." To compute $64 - 36$, one could subtract 30 from 64 to get 34, reducing to the problem $34 - 6$, which is $20 + 14 - 6$, which one knows is $20 + 8 = 28$, since one has learned how to compute $14 - 6$ as part of addition and subtraction within 20. (We note that this subtraction method, in which the largest place is subtracted first, will work in general. Of course, it may involve more rewriting than the standard algorithm; but for two-digit numbers, it seems quite manageable.)

It should be mentioned that many of the activities in steps 4 and 5 are present in various U.S. curricula, though perhaps without the unifying viewpoint provided by addition and subtraction within 20 (step 3). They are also presented in teacher training courses (Beckmann 2008; Van de Walle 2006).

Making Connections Between Counting Number and Measurement Number

One of the main arenas of application of mathematics is in measurement. Numbers used in measurement, in contrast to counting, may not be whole numbers. They can be rational numbers (meaning quotients of whole numbers), usually represented by fractions (possibly also with a negative sign), or even stranger numbers.[1]

Geometrical measurement is so different from the context of counting, that the classical Greeks did not think of the numbers involved in measurement as numbers, and reserved the term *ratio* for numbers in the context of geometrical figures (Klein 1992). It was only after the invention of symbolic algebra by Francois Viète around 1600 that the notion of number was expanded to include the numbers that arise in measurement. A few decades later, this development led to the invention of the coördinate plane by René Descartes, and to the strong linkage between number and geometry that we take for granted today.

The history of mathematics can be a good guide to what is important and what is difficult in learning mathematics. The difficulties evinced by the Greeks, combined with our post-Renaissance understanding that they are joined at the hip, indicate that it is necessary to help students explicitly to bridge the intuitive gap between number and geometry, and that this should start early. There are several benefits to starting in first grade. In particular, this can already help students think geometrically about two-digit numbers. Also, it can help prepare students to appreciate the metric nature of the number line (or ray), the use of which is called for explicitly by CCSSM in second grade.

In the course of civilization, people have learned to measure a huge variety of quantities, and several of the most important (area, volume, weight, time, speed, etc.) are dealt with in school. The most basic and probably simplest type of measurement is *linear measurement*: measurement of length or distance. Most adults probably think of linear measurement in terms of using a ruler. However, one should first lay a foundation by getting students to think of length or distance in terms of the familiar counting numbers, and to model addition by *concatenation of length*—laying rods end to end. (This can be viewed as a case of the part-part-whole aspect of addition.)

This process lends itself well to work with manipulatives in first grade. The basic materials needed are a collection of unit cubes, and rods with the same cross section as the cubes, but of various lengths. All whole number lengths from 1 to 20 would afford exploration of addition and subtraction within 20, in other words, a measurement analog of the addition and subtraction facts. Cuisenaire rods can probably be useful, but they don't have the full range of lengths, and their colors may be a distraction. Besides cubes, a generous supply of rods of length 10 is desirable. Unifix cubes might also be used, although these do not come with the ready-made larger

[1]Irrational numbers, which, with a few exceptions such as some square roots, π and e are not encountered by non-mathematicians, but which can be articulated into an elaborate hierarchy.

lengths. It might be a productive class activity to assemble cubes into rods of various lengths, which could then serve as templates for activities related to addition.

A first activity would be just measuring the length of various rods in terms of the cubes. It might be a good exercise to see if students could learn to recognize various lengths without having to measure. The rods might be marked with their lengths to facilitate later work (or if Cuisenaire rods are being used, many students will probably learn to associate lengths with the colors).

In learning to measure, students should come to appreciate the importance of lining up the cubes carefully, face to face, with no gaps. For some students, this may require a substantial amount of practice. If Unifix cubes are used, it could be instructive to have several groups of students produce bars with the same number of cubes, and to compare their lengths, noting the importance of having the cubes fit tightly for consistent length.

After students have gained familiarity with measuring the rods, and have come to associate a definite length with a given rod, along with associated ideas of order—that longer rods have greater measured lengths—addition and subtraction can be studied. Students should get used to the idea that addition corresponds to putting bars together end-to-end, aka the combination of lengths. Subtraction corresponds to the comparison of lengths: placing two rods side-by-side, and measuring the unmatched part of the longer rod. After a reasonable amount of work like this, the reasons for these correspondences between length measurement and arithmetic should be discussed. Ideally, a student will volunteer the basic reason: we have defined length in terms of measurement by unit lengths, and the collection of units needed to measure a combination of lengths is just the union of the collections that measure each of the individual lengths. Similar reasoning applies to subtraction.

Once addition and subtraction are interpreted in terms of lengths, one can begin to use the length model to bolster understanding of place value. One can introduce 10-rods as a convenient way to simplify the measuring process. The ease of laying down one 10-rod instead of carefully lining up ten unit cubes should be apparent to students. The expression of the teen numbers as a 10 and some 1s is readily modeled with a 10-rod and some cubes, and the modeling of the addition and subtraction facts beyond 10, as well as the making (in addition) and unmaking (in subtraction) of a 10 can be illustrated concretely in terms of length.

At some point, the possibility of measuring other lengths—lengths of pencils, lengths and widths of book covers, various body parts, and anything else that attracts class attention—should be explored. Longer things can be measured as students become accustomed to dealing with larger numbers. (Such activities might also be used as part of introducing larger numbers.) At least some measurement should be done using unit cubes only, so that the huge savings in effort afforded by use of 10-rods instead of only using unit cubes is made evident. Reporting of results of measurement should include units—so many cubes long. If the cube sides are of a standard length, such as a centimeter, this term could be used. Whether it is necessary or advisable at this point to consider different units of length needs study.

Objects in the class environment will typically not be exactly whole numbers of units in length. Often it is advocated to have students say that a given object is "about 14" units long. However, I would favor reporting the length as "between 14 and 15" if it is more than 14, or "between 13 and 14" if less. This kind of language serves to highlight the need for more numbers than whole numbers in the realm of measurement. Indeed, a teacher could tell students that later they will learn about other numbers (fractions, mixed numbers, rational numbers) that can be used to measure more accurately. If the length model for addition and subtraction (and better, its interpretation in terms of the number line, to be introduced later) is well absorbed, it can serve as an anchor for interpreting addition and subtraction of fractions, because although the symbolic representation of addition is substantially more complicated for fractions than for whole numbers, the geometric representation in terms of combination of lengths is uniform.

When students are used to thinking of addition in terms of combining lengths, and are familiar with 10-rods, the length model for addition can be coordinated with base 10 notation. Students can make trains consisting of 10-rods and cubes, to represent two-digit numbers. The convention should be established that the standard way to do this is always to have the 10-rods together on one side of the train (say the left), and the cubes together on the other (the right). This arrangement best displays the base ten structure of the number.

When numbers so represented are added by combining the trains end-to-end, students will probably observe that the resulting train is not in standard form: the 10-rods of the train on the right are to the right of the cubes of the train on the left. To put the train in standard order, these rods and cubes must be rearranged. The resulting train will be seen, perhaps after sufficient teacher direction, to be the result of "combining the tens and combining the ones", just as in the other contexts where two-digit addition is studied. Also, if the sum has more than ten cubes, the regrouping process can be modeled physically by replacing ten of the cubes by one 10-rod. If students fail to do so, it probably should be explicitly noted by the teacher that this process preserves the total length.

The analog of this process for subtraction should also be done carefully. When no regrouping is required, the trains of 10-rods and cubes can be compared to each other, and it should be checked that this yields the same answer as the full train comparison. When regrouping is required, one can convert a 10-rod to cubes to supplement the cubes in the minuend, before comparing with the cubes in the subtrahend. As with addition, the results of the separate comparison processes for the 10-rods and the cubes should be verified to give the same result as the whole train comparison.

This kind of work with lengths can strengthen the learning of arithmetic by reinforcing symbolic work and work with unstructured collections of objects. Equally important, it should get children used to the idea that measurement is a natural domain for application of number ideas. It should prepare them well for introduction of the number line, whose concrete realization is the ruler, as a tool that can be used to measure anything without the need to form trains at all, in second grade.

Beyond First Grade

Above we have argued that coördinated attention to word problems, place value issues in base ten arithmetic, and linear measurement as a domain for number and arithmetic, can form the core of first grade mathematics instruction that gives students a good start. In the remainder of this note, we will sketch how these three topics might continue to develop and support further mathematics learning in later grades.

Second Grade

In many ways, second grade is a continuation and consolidation of first grade, and completes the first stage of mathematics learning. The 3 pillars discussed above remain highly relevant.

The study of addition and subtraction continues, the main advances being progression to more complex problems, and to 3 digit numbers. This is the next stage of a gradual increase in the number of digits students are expected to cope with. CCSSM calls for 4th grade students to deal with numbers up to 1 million, and 5th grade students to also handle decimal fractions to thousandths. CCSSM is superior to many of the state standards that it has replaced, in calling explicitly for students to "Understand the place value system" in fifth grade.[2]

Word Problems Use of the full array of one-step addition and subtraction word problems should continue, amplified by the introduction of some two-step problems. Some problems might ask for addition of three or even four numbers. For example, for her birthday, Shana could get toy trucks from two or even three different people; or she could get some toy trucks for her birthday, and then some more for Christmas; or both.

The reader may convince him/herself by experimentation, that of the 14 types of one-step addition and subtraction problems discussed above, most pairs can be combined to make a two-step problem, so that there are potentially almost 200 ($14 \times 14 = 196$) two-step addition and subtraction problems. This should make obvious the futility of any "key word" approach to dealing with word problems, and also indicate the rich potential, both for mathematics and language arts, that analysis of multistep problems affords.

Place Value In dealing with 3-digit addition and subtraction, one should continue to develop the ideas introduced in first grade:

[2]However, the final stage of understanding, in which the base ten units are written as powers of 10 using exponential notation, linking place value notation with polynomial algebra, can not take place before 6th grade, when exponential notation is first introduced (6.EE 1).

(i) Work with expanded form, adding the 1s, the 10s, the 100s independently, with regrouping at the end, as needed. The point should be made that regrouping from 10s to 100s is strictly parallel to regrouping from 1s to 10s, because 100 is made of ten 10s.
(ii) Work with addition and subtraction in parallel, and observe that regrouping in a subtraction problem just reverses the regrouping in the corresponding addition problem.
(iii) The situations that require regrouping are considerably more varied than in the two-digit case, and probably require some systematic study. There may be no regrouping; regrouping only from ones to tens; addition of multiples of ten requiring regrouping of tens to hundreds; addition of general numbers with no regrouping of ones, but regrouping from tens to hundreds; regrouping of both ones and tens; and the most complicated case, when the tens add to 90, and then a carry from the ones place makes this exactly 100, leaving a zero in the tens place of the sum. This last situation may be called "rollover", by analogy with the change in mechanical odometers when 1 is added to a number with 9 in the 10s place (and perhaps larger places also). This should be studied explicitly, along with the corresponding subtraction situation, which requires "borrowing past a zero". Second grade may be a good time to consolidate addition and subtraction algorithms (although CCSSM waits until 3rd grade to ask for fluency). It probably would be a good idea to delay algorithm development until all these different cases have been considered, and then discuss how the usual right-to-left addition procedure handles all cases in one comprehensive method. Subtraction of course is considerably less comfortable, because of the rollover/borrowing past a zero issue, and more discussion of alternative approaches might be helpful.
(iv) Work with manipulatives should include base ten block work (ones cubes, ten-rods and hundred-flats) for student seat work, but also, in some whole class work, with cubes, ten-rods and hundred-rods (meter sticks can double as these) for forming linear trains representing 3-digit numbers. The same kind of rearranging and trading that was done for two-digit numbers should be continued here, including some of the more difficult symbolic cases, such as borrowing past a zero. One big advantage of forming trains to represent 3-digit numbers is that it emphasizes the size relations between 100s and 10s, as well as 10s and 1, making very visible that the 100s are the dominant part of any such number. This point should be made explicitly. Working with trains also shows that arithmetic can take place wholly in terms of the line—two dimensions (or 3, later used for blocks representing 1000 in the standard base ten block sets) are not a necessity, only a convenience, allowing easy manipulation of the blocks.
(v) In comparing numbers, students should learn that the number of 100s determine which of two numbers is larger, except when both numbers have the same number of 100s, in which case the 10s must be considered, and the 1s only when both numbers also have the same number of 10s.

The Linear Measurement Connection The counting-linear measurement connection should be strengthened and elaborated. We have already mentioned above that trains of 100-rods, 10-rods and 1-cubes should be created to represent three digit numbers, and combined to illustrate addition, and compared for subtraction. However, in second grade, linear measurement should become a major topic (see CCSSM Standards (2.MD.1 through 6)) and the number line (actually, the number *ray*, since at this stage, it will go only in one direction from the zero or base point or origin, which will be on one end of the stick or rod that embodies the line) should be introduced, essentially as a ruler.

The connection of the number ray with measurement should be emphasized. Especially, the idea that a number on the number ray *represents a length*—the distance from the origin (the end), as a multiple of the unit length—should be carefully established in students' minds. To bring home the necessity to choose a unit, number lines based on several different unit lengths should be used at various times, with explicit attention to specifying the unit. Taking the centimeter as unit will probably afford maximum compatibility with base-ten manipulatives. In the linear measurement context, the effect the choice of unit length has on the number obtained by measurement should also get attention—the larger the unit, the smaller the associated number, for a given length. A dramatic example would be that a single digit number of meters is also hundreds of centimeters. In the U.S., taking the inch as unit will afford a good tie-in with commonly encountered measurements, and later on, converting from feet or yards to inches, or from miles to feet or yards, can provide a source of multiplication and division problems.

The number ray should be used in conjunction with addition by lining up trains of base-ten blocks, and it can be observed that, if you position the trains along the ray with so that the end point of one train coincides with the end of the ray, then the other end of the second train will fall on the number that gives the sum—the number line functions as a computer! (This could be the first stage in rediscovering the slide rule, which could make a great manipulative in the later grades.)

Introducing the number ray and relating it to length is a main task of second grade, a key stage in a long learning trajectory that culminates with Cartesian coördinates and infinite decimal expansions. Some later stages in this development are discussed below.

Another key job of second grade that prepares for later work is to raise the consciousness of students concerning units. Linear measurement is a context where this issue clearly needs addressing, but it is relevant in many other contexts also. As we have discussed above in connection with comparison word problems, in everyday life, we don't really encounter naked numbers, but rather, any number we meet has a unit attached, and it expresses quantity in relation to that unit. In the early stages of learning addition, it may be advisable to suppress attention to units, in order to concentrate on the number relationships being established. Also, when dealing with whole numbers, the relevant unit is often clear and does not need to be pointed out.

However, attention to units is essential when learning fractions. Many fallacies, including claims of the sort

$$1/2 + 1/3 = 2/5 \quad \text{(not!)} \tag{1}$$

involve lack of attention to the unit. The error in this statement is analogous to our discussion of nickels, quarters and dimes in section "A Robust Understanding of the Operations of Addition and Subtraction". An equation like (1) is often justified with a picture such as

$$AB + ABB = AABBB \tag{2}$$

The first group is taken to represent $1/2$ (the number of As compared to the total number of symbols), the second is taken to represent $1/3$, and the last collection represents $2/5$. Addition is taken as union of sets.

What is wrong with Eq. (1)? A grouping such as AB can provide one reasonable way to represent a fraction, but to avoid confusion, it is essential to see that the $1/2$ refers to the first collection as unit, the $1/3$ refers to the second collection as unit, and the $2/5$ refers to the third collection, the union of the first two, as unit. In order to use a consistent unit, we could choose a single symbol as the unit. Doing this, we see that the above equation of sets translates to the numerical fact

$$(1/2) \times 2 + (1/3) \times 3 = (2/5) \times 5,$$

or

$$1 + 1 = 2,$$

which is indeed a true equation.[3]

In summary, both for purposes of learning the basics of linear measurement, and in preparation for dealing successfully with fractions in third grade, a major duty of second grade is to develop in students an awareness of units, especially, the predilection to ask and the ability to keep track of what the unit is in a given context, and to use units in a consistent fashion.

Third Grade and Later

Third grade, in contrast to second, presents a profusion of new ideas: multiplication and division; fractions; and area measurement. The relations between these new concepts must be presented in carefully orchestrated ways to promote successful learning of each. It is beyond the scope of this essay to detail the key relationships that need exploration, or even the key features of each of the new ideas. We will limit ourselves to sketching how our trio of fundamental constituents of first grade mathematics continue to support learning in this new and richer environment.

[3] Alternatively, if we select the 5-element set as the unit, then the first two sets represent $2/5$ and $3/5$ respectively, and the equation would read

$$(1/2) \times (2/5) + (1/3) \times (3/5) = 2/5,$$

which is also a true equation, representing $2/5$ as a weighted average (*not* a sum!) of $1/2$ and $1/3$.

Word Problems To give students a good perspective on the uses of multiplication and division, a varied collection of one-step multiplication and division problems should be presented, with discussion and analysis, mirroring what was done for addition and subtraction in grades 1 and 2. Page 89 of CCSSM gives a table of common multiplication and division situations, and all should be represented in word problems.

Multiplication and division are subtler operations than addition and subtraction and harder for students to internalize. In particular, although as a numerical operation multiplication is commutative, the two factors typically play different roles, and may well have different units attached.[4] Even in the simplest context, usually used in giving the first definition of multiplication, that of combining equal groups, one factor counts the number of things in one group, and the other factor counts the number of groups. In the standard interpretation of, say 3×5 as the combination of equal groups, the 5 represents the number in each group, and the 3 represents the number of groups. In this interpretation of multiplication, it is far from obvious that 3 groups of 5 have the same number as 5 groups of 3. Thus, on a conceptual level, the commutativity of multiplication is somewhat surprising. The fact that multiplication is indeed commutative should receive explicit attention. A good way to justify it is to use arrays, observing that a, say, 5 by 3 array becomes a 3 by 5 array when rotated by 90°.

Corresponding to the distinct roles of the two factors in multiplication, mathematics educators recognize two types of division. One is *partitive*, or sharing, division, in which a quantity is to be divided into a given number of groups, and the question is, what size will these groups be. The other is *quotative*, or measurement division, in which the size of the groups is specified, and the question is, how many groups can be formed. Parallel problems of the two types with the same numbers should be given, and it should be observed, that the numerical value of answer to both types of question is the same, although what the answer designates will be different. In this work, careful attention to units is especially relevant.

Third grade should also see large numbers of two-step problems, including some that involve any pair of the four operations. There are several hundred different possible types already for two-step problems, so the work required for understanding the problem will increase. Discussions of how to figure out what the problem is asking for, and what needs to be done to answer it, will have to be an important part of instruction, and ongoing as the problems become more complex. The work of Lieven Verschaffel and colleagues (Verschaffel et al. 2000) has documented the worldwide failure of mathematics instruction to enable students to adequately interpret word problems.

Later grades should see problems of increasing complexity, eventually arriving at word problems that require algebra for their solution by 7th or 8th grade. In fact, the boundary between arithmetic and algebra is somewhat fuzzy, and problems that might seem to require algebra can often by solved using only arithmetic supported

[4]The unit attached to the product is then the product of the units attached to the factors.

by a sufficiently insightful analysis (Howe 2010). It may be valuable for students to consider such problems, and to see parallel solutions. The Singapore bar model method (Singapore Ministry of Education 2009) is another approach to solving a broad class of problems that in the U.S. are most commonly handled by algebra. Singapore students start learning how to use this method in 3rd grade, when they are given problems such as

> There are 36 students in a class. There are 8 more boys than girls.
> How many girls are in the class?

Some Singaporean students become so skilled at using bar models that it is difficult to get them to abandon the model method in favor of symbolic algebraic approaches (Singapore Ministry of Education 2006). Something similar could happen with students who become highly skilled at solving word problems using arithmetic methods. Such students should be challenged with problems of increasing difficulty, until they reach a point when the systematic nature of symbolic algebra becomes so advantageous that they use it willingly.

Place Value and Computation Acquisition of multiplication allows students to deepen their understanding of place value, eventually revealing its depth and its connection to polynomial algebra.

In grades 1 and 2, students work with the expanded form, such as

$$243 = 200 + 40 + 3,$$

and learn that, in addition and subtraction, they can combine the parts of like magnitude, using only the single-digit addition and subtraction facts, followed by any necessary regrouping. We will call the numbers like 200 and 40 and 3, with only one non-zero digit, *single place numbers*. Thus, the expanded form of a base ten number expresses it as a sum of single place numbers.

Once students start learning about multiplication, they can begin to appreciate the multiplicative structure of single place numbers. In third grade, they can realize that each single place number is a multiple of a *base ten unit*, which is a single place number whose non-zero digit is 1. Thus, $200 = 2 \times 100$ and $40 = 4 \times 10$ and $3 = 3 \times 1$. This allows students to refine the expanded form to

$$243 = 200 + 40 + 3 = 2 \times 100 + 4 \times 10 + 3 \times 1.$$

Thus they would now think of 243 as being made of two 100s, and four 10s, and three 1s. They were in effect using this structure in adding and subtracting, but now they have a language to express what they were doing.

In fourth grade, students should refine their understanding of the base ten units, seeing the ones larger than 10 as repeated products of 10s. Thus,

$$100 = 10 \times 10, \qquad 1{,}000 = 10 \times 10 \times 10, \qquad 10{,}000 = 10 \times 10 \times 10 \times 10,$$

and so forth.[5]

Understanding the structure of base ten units supports the appreciation of the quantity aspect of place value: each base ten unit is ten times as large as the next smaller unit (the place to the right), and only $1/10$ as large as the next larger one (the place to the left). In particular, as one moves to the right in the places, the value of the unit shrinks by 10 at each step. This can support the idea of continuing places to the right of the 1s place, and making

$$1/10 = (1/10) \times 1, \qquad 1/100 = (1/10\} \times (1/10), \qquad 1/1000 = (1/10) \times (1/100),$$

and so forth; thus it prepares for thinking about and dealing with decimal fractions.

The final stage of understanding the place value system can be presented in sixth grade, when whole number exponents are introduced. This allows the shorthand notation

$$1 = 10^0, \qquad 10 = 10^1, \qquad 100 = 10^2, \qquad 1000 = 10^3,$$

and so forth. In combination with the earlier work on the structure of single place numbers, this permits the last stage in the progression

$$\begin{aligned} 243 &= 200 + 40 + 3 \\ &= 2 \times 100 \quad +4 \times 10 \quad +3 \times 1 \\ &= 2 \times (10 \times 10) +4 \times 10 \quad +3 \times 1 \\ &= 2 \times 10^2 \quad +4 \times 10^1 +3. \end{aligned}$$

The last stage in this progression shows that a base 10 number can be regarded as a "polynomial in 10". In 6th grade, it probably would serve mainly as an application or example of the use of exponential notation. However, it also highlights the sophistication involved in base ten place value notation, which implicitly uses all the operations of algebra (addition, multiplication, exponentiation), just to write numbers. The full implications of the final expression can be profitably investigated in 8th grade when the algebra of polynomial expressions is discussed. Students can verify that, if a base ten number is turned into a polynomial, by the recipe

$$243 \rightarrow 2x^2 + 4x + 3,$$

and if calculations (addition, subtraction, multiplication) are done with the resulting polynomials, and then 10 is substituted for x, the usual numerical answer will be

[5] At this point, it might be a good idea explicitly to discuss the issue of associativity of multiplication, that it does not matter how we group the factors in these (or any) repeated multiplications, the result will not depend on the grouping. Thus, $10{,}000 = 10 \times 1000$, but just as well, $10{,}000 = 100 \times 100$. In fact, associativity of multiplication is a somewhat subtle property, and its justification using geometric models involves volumes of 3 dimensional bricks. See for example (Epp and Howe 2008) for a fuller discussion.

obtained. For some students, this observation can provide an "Aha!" moment that will tie together eight years of study of mathematics.

The discussion above of course is quite standard mathematics, and in earlier years this author tended to treat the five-stage progression above as common knowledge. However, there is evidence that many students arrive in college without even stage 2, the basic expanded form, as part of their intellectual toolkit (Thanheiser 2009), and the value of making this progression explicit, and to give it emphasis in the curriculum is supported by Teachers of India (2012). Also, the lack of understanding even of the basic meanings of the places by mid-elementary students was documented by Kamii (1986) long ago.

Linear Measurement and the Number Line The connections of arithmetic with linear measurement developed in grades 1 and 2 are the beginnings of a long development of the intimate relationship between number and geometry. In third grade, the basic understanding of the number ray established in second grade would allow studying the nature of fractions from a geometric viewpoint. Although the array and area models will play an important role in helping students understand and work with fractions, the number line can also contribute.

The understanding that the numbers on the number ray tell distances from the endpoint/origin provides a sound basis for placing fractions on the line. The CCSSM advocates understanding fractions as multiples of unit fractions. Thus,

$$2/3 = 2 \times (1/3), \qquad 5/3 = 5 \times (1/3),$$

and so forth. To locate $1/3$ on the number line, one should divide the unit interval into 3 equal parts. Then the other end of the part with one end at 0 is $1/3$ of the way from 0 to 1, and so should be labeled as $1/3$. Then $2/3$ is the point that is two $1/3$ intervals from 0, and $5/3$ is the point that is five $1/3$ intervals away from 0. Repeating this process for all multiples of $1/3$, one finds that they form a system of equally spaced points, very much like the whole numbers, except three $1/3$s fit inside each unit interval—we could say they are three times closer together, or three times as dense, or only $1/3$ as far apart. It is of course the same for whole number multiples $n/d = n \times (1/d)$ of any fixed unit fraction $1/d$. They form a system of equally spaced points on the number line, each one at distance $1/d$ from its neighbors, with d intervals inside the unit interval. Thus, the number line affords a compelling visualization of the systematic nature of the multiples of a fixed unit fraction.

Two ideas crucial to understanding and working with fractions are

(i) repeated subdivision, and
(ii) reconstitution.

Repeated subdivision involves understanding that a unit fraction such as $1/5$, which resulted from subdividing the original unit into 5 equal pieces, constitutes a new unit that can itself be subdivided. The result of the subdivision will then be a unit fraction, with denominator equal to the product of the two denominators. Thus, if we divide $1/5$ into fourths, the result will consist of $(1/4) \times (1/5) = 1/20$. The general relationship is

$$(1/e) \times (1/d) = 1/ed$$

Reconstitution is the reverse process to repeated subdivision. Just as 4 copies of $1/4$ make 1, the unit, so also 4 copies of $1/20 = (1/4) \times (1/5)$ make $1/5$. In symbols, we would write $4 \times (1/20) = 1/5$. The general relationship is

$$e \times (1/ed) = 1/d, \quad \text{or} \quad e/ed = 1/d.$$

The second form of the relationship shows that reconstitution is the justification for the symbolic move of "canceling the same factor from numerator and denominator".

These relationships should be illustrated in a variety of contexts so that students can see how they work and get used to working with them. The number line can be one of those contexts, and the regular subdivisions of the line provided by the whole number multiples of a unit fraction can be used to show many examples of repeated subdivision and reconstitution, by considering the relationship between the subdivision given by the multiples n/d of a given unit fraction $1/d$, and the multiples m/ed of a unit fraction whose denominator is a multiple of d. This study can also contribute to the understanding of how to add fractions. For example, if one works with $1/6$, then since $6 = 3 \times 2$, reconstitution would tell us that $1/2 = 3/6$. Since also $6 = 2 \times 3$, reconstitution would also tell us that $1/3 = 2/6$. Thus, we could conclude that

$$1/2 + 1/3 = 3/6 + 2/6 = 5/6.$$

This kind of formula can also be shown explicitly on the number line. An important pedagogical consideration here is that the linear measurement interpretation of fraction addition is exactly the same as whole number addition: it is combination of lengths. Similarly, subtraction of fractions amounts to comparing lengths. This consistency over different types of numbers, when the symbolic representations and the necessary manipulations may seem dissimilar, can provide a firm basis for understanding and reasoning.

With the introduction of signed numbers, the number ray must become the number line, that is, it must be (in principle) infinite in both directions. Currently, the typical practice is not to distinguish between the number line and the number ray, and to use the term "number line" for both, but if the distinction were made, the change in terminology could provide a signal that something new is going on.

On the doubly infinite line, the origin loses its distinguished position as the endpoint, because there is no endpoint. Thus, the origin must now be specified explicitly. That done, we see that distance from the origin no longer specifies a unique point—for each distance, there are two possibilities, on either side of the origin. To distinguish between them, we must introduce the idea of *orientation*: left, right, or positive, negative. The need for specifying orientation should be given a lot of emphasis, including the use of 'trick' problems such as

James, Randolph and Rebecca live on Elm Street.

If James lives 2 blocks from Randolph, and Rebecca lives 3 blocks from James, how many blocks does Rebecca live from Randolph?

To deal successfully with signed numbers, several conceptual changes in student thinking about numbers are necessary. The most obvious, of course, is the understanding that a number no longer simply gives information about magnitude, but also about direction (which in one dimension reduces to a dichotomy: left, right; plus, minus). This necessary revision gives rise to another surprise: addition and subtraction become merged into a single operation, with subtraction of a given number amounting to addition of its *additive inverse* (aka *negative* or *opposite*). Thus, we think of $2-6$ as $2+(-6)$. Also, for the first time, subtraction can be performed with *any* two numbers: $1-2$ now makes as much sense as $2-1$.

Signed numbers are introduced in CCSSM in 6th grade, which is also the grade in which simple algebraic expressions are introduced. Thus, in 6th grade, students are asked to understand expressions such as

$$2+x, \quad \text{and} \quad 2x,$$

as meaning "Pick a number x and add 2 to it", and "Pick a number x and multiply it by 2"; or somewhat more colloquially, "Add 2 to any number," and "Multiply any number by 2". The change in point of view is perhaps somewhat subtle, but it is highly significant, and it must be given enough attention to ensure that students grasp it. Instead of thinking of addition, or multiplication, as a binary operation, something we do with two numbers, we are asked to think of "adding 2"or "multiplying by 2" as a *unary operation*, something we do to *any* single number.

Thinking of "adding 2" as an operation on any number allows us to think of it as a *transformation of the number line*, a recipe that takes each point, corresponding to some number x, and moves it to the point corresponding to $2+x$. If students study what this operation does to many points, they may be able to formulate themselves what this transformation does: it moves each point 2 units in the positive direction (to the right, in the usual orientation of the number line). In other words, it is a *translation* of the number line through 2 units to the right. Similar work with adding -2 should reveal it as a translation of the number line through 2 units to the left. This provides a graphic understanding of the fact that adding -2 undoes adding 2, so that it is the same as subtracting 2.

This transformational view of addition can be reinforced by use of a slide rule to add and subtract numbers, by sliding one copy of a number line along a parallel copy. Care should be taken to correlate this new perspective on addition and subtraction with the original understandings of combining and comparing lengths. For adding or subtracting any given pair of numbers, they amount to essentially the same thing. The difference is, when thinking of "adding 2" as a transformation, we are fixing one addend, and letting the other vary.

The operation of "multiplying by 2" likewise can be visualized as a transformation of the number line. Again, by looking at many examples, we can see that it takes any number and moves it to a number that is twice as far away from the origin.

Thus, "times 2" is a stretching of the number line by a factor of 2, from the origin (which does not move). Also, it preserves direction: positive numbers go to positive numbers, and negatives go to negatives. Students should be made to notice that by this transformation, the length of every interval is doubled, not just the intervals with one end at the origin. This is the geometric embodiment of the Distributive Rule.

When this transformational interpretation is extended to fractions, it provides a way of seeing that multiplying by $1/d$ is the same as dividing by d, so that, in the rational numbers, multiplication and division are two aspects of the same operation. More precisely, division by a given number is the same as multiplication by its reciprocal.[6] This relationship is built out of two more basic ones:

(i) For a whole number d, division by d is the same as multiplication by $1/d$; and
(ii) Multiplication by a fraction n/d amounts to multiplication by n, and multiplication by $1/d$, and it does not matter which is done first.

Combining statements (i) and (ii) produces: multiplying by n/d amounts to multiplying by n and dividing by d, in either order.

The ideas that

(i) division by a given number is the inverse of multiplication by that number, and
(ii) division by a given number may be accomplished by multiplication by the reciprocal, which combine to
(iii) division by a given number is the same as multiplication by the reciprocal,

are the key ingredients in the "invert and multiply" rule for division by fractions.

Multiplication by negative numbers, a well-known trouble spot, fits easily and elegantly into the transformational viewpoint (Friedberg and Howe 2008). The main observation is that multiplication by -1 is reflection across the origin. Every number goes to its negative. Then multiplication by -2 would be multiplication by 2, followed by reflection across the origin (or the other way around—it doesn't matter, since multiplication is commutative). In this picture, it is clear why the product of two negative numbers is positive: reflecting twice across the origin leaves orientation unchanged. For many students, this geometric insight into the nature of multiplication by negative numbers may be more convincing than a formal symbolic argument.

These geometric/transformational interpretations of the operations, and their connections with the basic algebraic expressions are not explicitly emphasized in the CCSSM, and the ability of students to grasp the transformational viewpoint is not well documented. However, the picture afforded by these ideas is quite compelling, and the connections to more advanced mathematics are also strong. In particular, this viewpoint fits very well with the CCSSM emphasis on transformations in geometry. It seems possible that some students could benefit from the "multiplication is stretching" idea from the time multiplication is introduced, in 3rd grade,

[6]Unfortunately, this basic principle is not explicitly enunciated in the CCSSM. One hopes that this defect will be remedied in the next revision.

and that it could form a useful supplement to the "repeated addition" and array/area interpretations that are explicitly recommended by CCSSM.

A final place where the number line can provide a useful interpretation of a numerical construction is in decimal expansions. In fact, it is hard to imagine developing a firm grasp of decimal expansions without invoking the number line. Essentially, decimal expansions provide an address system on the number line. We should think of successive digits in a decimal expansion as providing successively finer information about the location of a point on the line. As an example, consider the decimal

$$3.14159265358979323\ldots.$$

The whole number part of this number, namely 3, locates the point somewhere in the interval [3, 4] from 3 to 4 (including the endpoints, 3 and 4). The digits to the right of the decimal place are instructions about how to locate this number more precisely. To interpret the 1 just to the right of the decimal place, we should picture the interval between 3 and 4 as being divided into 10 equal subintervals, namely

$$[3.0,\ 3.1],\quad [3.1,\ 3.2],\quad [3.2,\ 3.3],\quad [3.3,\ 3.4],\quad [3.4,\ 3.5],$$
$$[3.5,\ 3.6],\quad [3.6,\ 3.7],\quad [3.7,\ 3.8],\quad [3.8,\ 3.9],\quad [3.9,\ 4.0].$$

The .1 in this decimal expansion tells us that the number belongs in the second interval, [3.1, 3.2], from 3.1 to 3.2. To use the next digit, we should further subdivide the interval [3.1, 3.2] into ten equal subintervals, namely

$$[3.10,\ 3.11],\quad [3.11,\ 3.12],\quad [3.12,\ 3.13],\quad [3.13,\ 3.14],\quad [3.14,\ 3.15],$$
$$[3.15,\ 3.16],\quad [3.16,\ 3.17],\quad [3.17,\ 3.18],\quad [3.18,\ 3.19],\quad [3.19,\ 3.20].$$

Then the 4 in the second place to the right of the decimal point tells us that the number is somewhere in the fifth of these intervals, namely in the interval [3.14, 3.15] from 3.14 to 3.15. Each succeeding decimal digit has an analogous interpretation. Any initial segment of the decimal expansion locates the number in a certain interval. We then should break up this interval into 10 equal subintervals, and the next digit in the decimal expansion tells us in which of these 10 subintervals the number lies. If students carry out this process for some examples, they should come to appreciate that the first few decimal places locate the number to sufficient accuracy for most simple purposes, and indeed, that it is rather difficult to resolve an interval into ten subintervals after only a few steps of this procedure, and essentially impossible after only a few more. Our most powerful microscopes allow us to continue the process for several more places, but after 10 to 20 places, depending on how large the starting size was, the remaining decimal places lose physical meaning. A conclusion that should be made explicitly is that it is quite remarkable that our symbolic computational system is capable of producing arbitrarily long decimal expansions for many of the numbers that arise in the course of computation, including π, e, $\sqrt{2}$, $1/3$, etc.

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