Chapter 6
History in Mathematics Education. A Hermeneutic Approach

Hans Niels Jahnke

Abstract The paper discusses the possibility of bringing history in the mathematics classroom by studying historical sources with students. A manuscript by Johann Bernoulli about the differential calculus which was brought to a grade 11 classroom serves as an example. Reading a source is fundamentally a hermeneutic activity and can be conceptualised by the term 'horizon merging'. In the so-called hermeneutic circle the horizons of the reader and the author of a text are supposed to merge by a repeated reading. In contrast to common ideas about the genetic principle the hermeneutic approach described in the present paper assumes that students have already some experience with and knowledge of the modern counter-part of the concepts treated in the source. Reading a source is an activity of applying mathematics in a way completely new to students. It provides opportunities for reflecting deeply about their images of the respective mathematical concepts.

Keywords Johann Bernoulli · Concept image · Differential · Genetic principle · Hermeneutics · Hermeneutic circle · Horizon merging · Historical source · Infinitely small quantity

Preliminary Remark

Before I entered the field which usually is called "History and pedagogy of mathematics" (HPM) I had met two rather different notions of what this could mean. One was the idea to consider history of mathematics as a collection of interesting mathematical problems some of which were suitable to be treated at school. This idea was mainly supported by teachers at schools respectively math educators who saw themselves predominantly as teachers. To be sure, some work in this direction is impressive but I asked myself where there was any substantial relation to the history of mathematics. All these problems were meaningful by themselves and could be treated without any reference to history.

H.N. Jahnke (✉)
Faculty of Mathematics, University of Duisburg-Essen, 45117 Essen, Germany
e-mail: njahnke@uni-due.de

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The other idea was in a vague sense related to what could be called the "genetic principle". Prominent mathematicians like Felix Klein and Otto Toeplitz believed that history of mathematics could contribute to the learner's understanding by making visible great lines of development and thereby connecting seemingly unconnected subjects. Klein had exemplified this by his magnificent book "Lectures on the development of mathematics in the 19^{th} century" in which he had reconstructed the immediate pre-history of the mathematics of his time. One can imagine that students who attended these lectures got a sound idea of what was going on in mathematics. But these lectures were definitely intended for an advanced audience.

Otto Toeplitz, on the other hand, was more involved in the teaching of beginning university students of mathematics. He, too, intended to present for the case of analysis the great lines of thought. He justified his "genetic approach" by saying that if we go back to the roots of mathematical concepts the dust of time and the scratches of long use would be removed. Infinitesimal analysis would become attractive for students when they can see that its basic concepts had been objects of an exciting process of research at the time of their invention. But Toeplitz hastened to add the remark that this is completely different from a "historical method". This term, historical method, he says, "... brings to mind the idea, which we, on the contrary, would particularly like to eliminate, of the old and antiquated, the roundabout paths often followed by research, the subjective and haphazard nature of scientific discoveries. It is especially important to me to draw a dividing line in this direction" (Toeplitz 1927, 93, translation by the author and Michael N. Fried, to appear). As a consequence, in his "Genesis of the infinitesimal calculus" he never mentioned Newton's binomial theorem which Newton himself had considered as one of his most important discoveries nor did he made an attempt to discuss indivisibles or infinitesimals. This tension then between a historical and a mathematical perspective on history is a running theme in HPM and an important issue for a book about the common ground between mathematics and mathematics education. I shall return to this at the end of my paper.

A third experience in the beginnings of my involvement with HPM was a talk by Jan van Maanen, later co-editor of the ICMI Study on HPM, which he gave in Toronto in 1992. He reported about a teaching unit on a 17^{th} century Dutch textbook on algebra which he did with pupils of grade 8 (see van Maanen 1997). Obviously, there was a strong contrast to Toeplitz' ideas. There was no great mathematician discovering something new, no exciting solution of new and deep problems. Instead of this, there were questions such as these: How did pupils three hundred years ago solve quadratic equations? Which symbols were used in the textbooks, which procedures, which applications? This means, instead of studying big ideas and great lines of thought pupils were invited to look for the specific and the context. They were invited to compare their own concept image of quadratic equations with that of pupils at their place two hundred years earlier. Listening to this talk it seemed to me obvious that these pupils had learnt a lot of substantial mathematics. Somehow the tension between the historical view of a concept and its modern counterpart itself provided the productive element which was so exciting to students. For similar approaches the reader might consult Arcavi et al. (1982), Laubenbacher and Pengelley (1999) as well as the survey of Tzanakis and Arcavi (2000).

Thus, I decided to continue in this direction. From the outset it was clear that the inclusion of history of mathematics into teaching cannot make things easier to students. Rather the opposite is the case. At all times, scientists have written for a narrow circle of specialists and not for pupils of later generations. Frequently, their ideas were still vague and insufficiently formulated in a clumsy language. Thus as a rule, when it comes to reading texts of eminent mathematicians of the past we have to expect considerable difficulties. This is even a frequent experience of working mathematicians when they consult papers in their field which have been written in a distant past.

In the following I will describe an experience with reading a historical source with students and then discuss general principles and difficulties underlying this enterprise.

Johann Bernoulli's Textbook on the Differential Calculus

First of all I should mention that at the time of this teaching experience I conducted regularly courses for practicing teachers in which we read historical sources. Every course comprised 5 sessions of 2 hours each. The participating teachers were asked to prepare every session by reading a source. Participation was completely voluntary, the teachers did not receive any reward or credit. Some of them intended to use historical material in their classroom, others came out of interest in the history of mathematics. With hindsight, it is still astonishing to me that this rather naïve procedure really worked. But it is a fact that after two years there was a group of ca. 100 interested teachers whom I invited and of which ca. 20 used to participate in these reading courses. It was in the context of this in-service teacher training that I did this teaching experiment, other teachers did similar experiments with their pupils.

Let us now turn to our source. Brothers Jacob Bernoulli (1655–1705) and Johann Bernoulli (1667–1748) were the most important mathematicians of the Leibniz school of analysis. It was due to their substantial work that not Newton's fluxions, but Leibniz' differentials became generally accepted on the European continent. After Leibniz and Newton had passed away Johann Bernoulli was for some 15 years the leading mathematician of Europe. It is an indication of the highly competitive mathematical climate at that time that he was involved in numerous controversies and quarrels with colleagues. He even managed to fall out with his son Daniel because the latter had won a prize of the Paris Academy of the Sciences for which he himself had applied, too. Thus, he earns a place of honour in Ted's list of idiosyncratic mathematicians (Eisenberg 2008).

Johann Bernoulli wrote the manuscript "De calculo differentialium" in 1690/91 when he was 23 years old. At that time, he conducted private lectures about Leibniz' new calculus to the Marquis de L'Hospital (1661–1704). Johann and L'Hospital had agreed on a secret contract saying that Johann would teach L'Hospital and would leave his mathematical discoveries to the latter's exclusive use. In return, L'Hospital payed Johann a considerable salary until his death. For a long time Johann's manuscript "De calculo differentialium" was regarded as lost, some historians even believed that Johann never had written such a paper. Only in 1922 the manuscript was detected by Paul Schafheitlin in the library of the university of Basel. This, of course, was no accident since Johann had been a professor at that university. The more astonishing it is that it took such a long time after Bernoulli's death that historians became aware of this important manuscript.

Schafheitlin published the manuscript and translated it into German (Schafheitlin 1924). A comparison of l'Hospital's textbook "Analyse des infiniment petits" (1696) with Johann's manuscript shows that the latter was sort of a draft to the former. L'Hospital had considerably extended Johann's text, corrected some mathematical mistakes (see below) and dressed it up didactically. In the teaching sequence it was especially stressed that at the time when Bernoulli and L'Hospital wrote their books the meaning of the basic concepts of Leibniz' calculus was not at all clear. Thus, both books were efforts of interpretation and clarification, and not just reproductions of a given body of knowledge.

Bernoulli's "differential calculus" comprises 38 printed pages. It contains:

- 3 postulates,
- calculation rules for differentials,
- 11 problems on the determination of tangents to curves,
- 9 problems of maxima and minima,
- methods for determining points of inflection.

For the teaching unit I produced a montage of texts containing the postulates, the rules for differentials, some problems on determining tangents to curves and problems on minima and maxima.

In the following I will at first introduce some parts of the source and then describe how the students worked with it. First, the postulates:

Postulates

1. A quantity which is decreased or increased by an infinitely smaller quantity is neither decreased nor increased.
2. Every curved line consists of infinitely many segments which are infinitely small.
3. [omitted here: refers to the integral calculus]

A symbolization of the first postulate might be written as

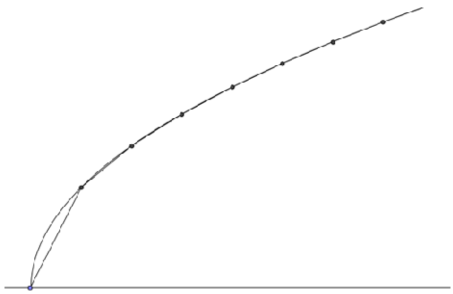
$$x + e = x$$

where e is infinitely smaller than x. According to the ordinary rules of algebra this implies

$$e = 0.$$

But this was an implication not intended by the analysts of the Leibniz school—a fact which does not become clear from the postulate itself but from its later applications to differentials. To be sure, all of his life Leibniz was rather vague about what differentials and the infinitely small were. Thus it was young Johann Bernoulli who in plain language formulated the central rule underlying any calculation with differentials. This was a conscious act of interpretation and a determination of meaning which had not been done before and which later was taken over by the Marquis de L'Hospital.

The second postulate, too, made explicit what the analysts of the Leibniz school had tacitly assumed, namely to consider a curve as a polygon with infinitely many sides of infinitely small lengths.



Thus, the second postulate shows the geometrical meaning of the calculus and determines its application to geometry. In regard to the idea to replace curved lines by straight lines we could anachronistically speak of a geometrical version of what we today call linearization.

What infinitely smaller quantities and infinitely small segments are is nowhere explained. Thus, we have to look at the applications of these concepts to learn more about them.

Thus, let us look at the calculation rules for differentials, e.g. the product rule.

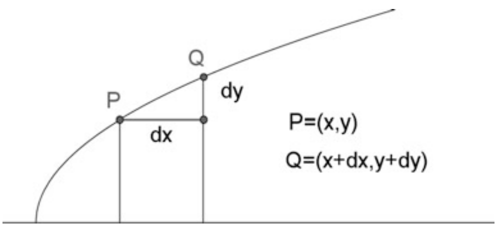
> "The differential of xy is $x\mathrm{d}y + y\mathrm{d}x$. When $x + e$ is multiplied by $x + f$ (where $e = \mathrm{d}x$ and $f = \mathrm{d}y$) then the product is $xy + ey + fx + ef$. After subtracting xy we get $ey + fx + ef$ which is according to postulate 1 equal to $ey + fx = x\mathrm{d}y + y\mathrm{d}x$. q.e.d."

Bernoulli does not explicitly say what a differential is but from his calculation it becomes clear that it is a difference between two states of a quantity which are infinitely near to each other.

$$d(xy) = (x + dx)(y + dy) - xy$$

The calculation gives

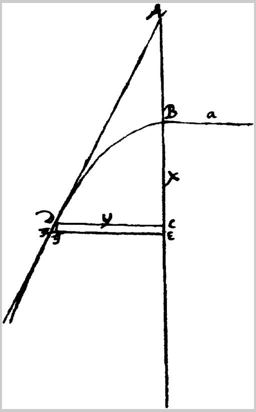
$$d(xy) = xdy + ydx + dx \cdot dy = xdy + ydx$$



Thus $d(xy)$ is infinitely smaller than xy. The critical point in Bernoulli's proof is of course the argument that $dx \cdot dy$ is infinitely smaller than xdy or ydx which are for their part infinitely smaller than the quantities x and y. Thus like Russian dolls we have three nested universes of quantities each one infinitely smaller than its predecessor. I mention in passing that the discussion with the students why $dx \cdot dy$ is infinitely smaller than xdy was far from easy and we escaped to some plausibility arguments. (Please, do not misinterpret the figure. y should not be considered as function of x, rather to each point of the curve the quantities y (ordinate) and x (abscissa) are assigned.)

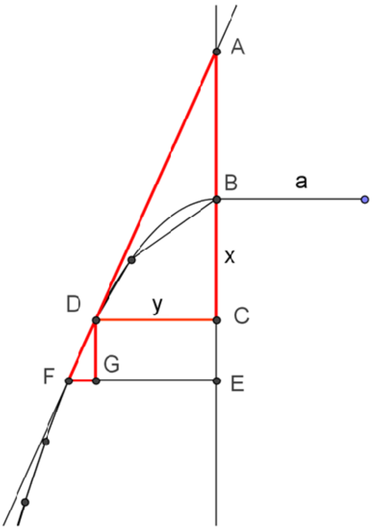
In a further step we consider how the calculus of differentials is applied to the study of curves.

To Find the Tangent to the Parabola



According to the definition of the parabola we have $ax = y^2$, thus $adx = 2ydy$ or $a : 2y = \frac{dy}{dx}$ and since according to postulate 2 it is supposed that every curve consists of infinitely many straight lines, the tangent AD and the infinitely small segment DF of the parabola BDF will be a straight line. Therefore, if one draws DG parallel to diameter AE, then triangle $\Delta DGF \sim \Delta ACD$. Thus we have $FG : GD = CD : AC$, and if s designates the subtangent, then $\frac{dy}{dx} = \frac{y}{s} = \frac{a}{2y}$. Consequently, $s = \frac{2y^2}{a} = \frac{2ax}{a} = 2x$. If, therefore, AC is taken twice as large as the abscissa BC of point D and if through A the straight line AD is drawn, then this is the sought tangent.

For the sake of clarity we replace Bernoulli's freehand sketch by a computer drawing. In short: Bernoulli put up the equation of the parabola $ax = y^2$ (in Bernoulli's words its 'nature') from which he derived a differential equation. Then he applied postulate 2 with two remarkable consequences. (1) A tangent line to a curve is simply the prolongation of the infinitely small segment adjacent to that point. (2) This creates two similar triangles, the infinitely small triangle FGD and the finite triangle DCA. This gives $\frac{dy}{dx} = \frac{y}{s} = \frac{a}{2y}$ where s is the subtangent $\overline{AC}$. From this follows



$$s = 2x$$

Three features of this text immediately strike us. (1) There is no coordinate system, instead the variables refer to the symmetry axis of the curve. (2) The equation of the parabola is of a geometrical nature. Any point of the parabola is constructed by transforming geometrically the rectangle ax into the square y^2. See the segment a attached to B. (3) For determining the tangent Bernoulli does not calculate its slope, but he calculates the x-coordinate of a second point of the tangent, namely A, that is the subtangent s. Then the tangent can be constructed as the straight line connecting A and D. (4) We pointed already at the remarkable definition of a tangent as an extension of an infinitely small side of the polygon representing the curve. All in all, we see here a basically geometrical conception of infinitesimal analysis in which the role of the algebraic symbolism is reduced to auxiliary calculations.

Students Read Bernoulli's Text

Sections of Bernoulli's manuscript were read with students of an advanced mathematical course ("Leistungskurs") in grade 11 at a Gymnasium near Bielefeld, Germany (see Jahnke 1995, for details). The students had already been introduced to the fundamentals of the differential calculus and they knew the concepts of limit and derivative and how to apply them in order to determine tangents, extremal values and points of inflection. In Bernoulli's manuscript they found a conceptual framework completely different from their own. The textbook on which their calculus lessons were based consistently avoided differentials, even as a notation. Derivatives were exclusively written as $f'(x)$. However, students had met differentials in their physics classes as very small but finite quantities. Usually, pupils worked in groups on a section of the source to which I had added an assignment with special questions. For example, I asked them to give an intuitive interpretation of Bernoulli's concept of "infinitely small quantity", or I asked them to calculate the differential of x^2 after they had studied Bernoulli's proof of the product rule for differentials or they had to find out from the text what a subtangent is.

The teaching started with a 'map of the history of mathematics.' Students were asked to give names and dates of mathematicians they had heard of. Since they were convinced that they didn't know anything about history of mathematics they were astonished that after half an hour the blackboard was full of names and dates. As it is often the case, as a group they were more successful and brighter than as individuals.

In a second step they read a short sheet of information about the early history of analysis, Johann Bernoulli's biography and the history of the source they were expected to read. Since studying the source took more time than predicted the treatment of extremal values was finally omitted.

Asked for an intuitive interpretation of "infinitely small quantity" students were very inventive. They offered:

- "a quantity is always positive"
- $x + e = x$, but $e \neq 0$?

- they set up the proportion:

  quantity : infinitely small quantity ≡ line : point ≡ area : line ≡ solid : area

- an infinitely small quantity is like the difference between a rational approximation of π and the number π itself
- an infinitely small quantity is like the difference between 1 and 0.999...

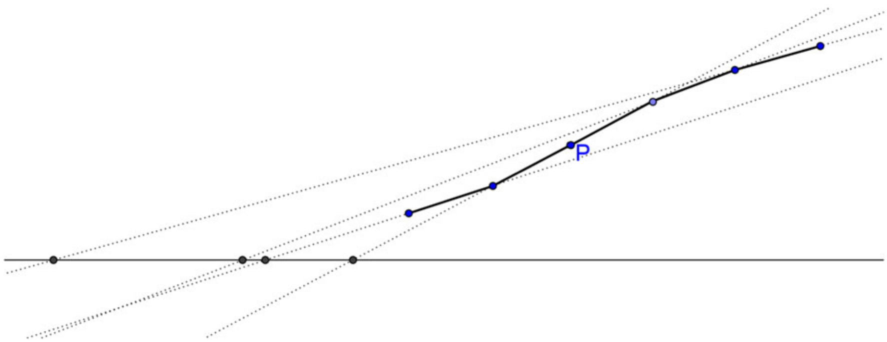
The proportion line : point ≡ area : line suggested that students had heard something about Cavalieri's principle. However, some of them expressed doubts: Can one really say that a line is composed of points?

These questions found a remarkable continuation when the tangent to a parabola was studied. First of all, for the students it was very demanding to handle the unusual form of Bernoulli's equation of a parabola. And it was even more demanding to find out from the involved argument with similar triangles what a subtangent is. Everybody with teaching experience would expect these difficulties. However, after an understanding of the meaning of 'subtangent' had emerged students were excited to realize that in order to determine the tangent Bernoulli did not calculate its slope, but the coordinates of a second point of the tangent. To be sure, the proportion defining the slope was needed to get rid of the infinitely small quantities, but the target quantity was not the slope, but a second point. Two points being known one can construct the tangent by a ruler much easier than using its slope.

Bernoulli's definition of a tangent as a prolongation of an infinitesimal side of the polygon caused lengthy discussions. First of all, if one fixes a point which of the two adjacent sides of the polygon should be chosen? Second, and more important, if the intuition is correct that an infinitely small quantity can be considered as a point, how can such an entity define a direction as is supposed in the definition of a tangent. When it defines a direction an infinitesimal should at least contain two points since two points are needed to determine a direction. The conclusion of this discussion was that an infinitely small quantity must be more than a point, it cannot be 0-dimensional, but has to be 1-dimensional. Of course, this was also the conclusion of the analysts of the Leibniz school. Thus, we have an infinitely small universe in which we can do Euclidean geometry just as we can in the domain of the normally sized quantities.

We stop the analysis of the students' discussion at this point. Of course, the source contains a host of additional interesting problems. Half a year after this teaching unit, students invited me to come again to their class for a further study of Bernoulli's manuscript. We decided to investigate Bernoulli's methods for determining points of inflection. Curve sketching is a routine topic in German calculus classes and the criteria for maxima/minima and points of inflection are dead stuff. They are learnt and mechanically applied. Thus, it was exciting to see that besides the usual $ddy = 0$ Bernoulli hat a second criterion for points of inflection.

When one runs through a curve in the neighbourhood of a point of inflection P then the sketch seems to show that the points of intersection of the successive tangents move along the x-axis in a way that the tangent in P has an extremal position. In his commentary to Bernoulli's manuscript Paul Schafheitlin remarked that this criterion was not adopted by L'Hospital because it is equivalent to the usual



$ddy = 0$. But we found that this is not true. Bernoulli's criterion is neither necessary nor sufficient, it is in general not correct. Of course, the students were impressed to see that such an important mathematician like Bernoulli made such a mistake. To investigate this criterion is a wonderful exercise for good high school and even university students.

The Hermeneutic Approach

In the introduction of my paper I referred to the reconstruction of a great line of thought leading from past roots of a mathematical concept to its modern version. To me this idea (a typically 19th century idea!) is the essence of the genetic principle as it was understood by mathematicians of the late 19th and early 20th century and which is influential even today. For school teaching this is simply unrealistic. This is the case since school teaching is necessarily episodic and progresses in small steps. Thus, we have to be realistic and should not overload the enterprise "History and pedagogy of mathematics" by demands which necessarily must lead to failure. One can call this a *pragmatic categorical imperative*.

Thus, the hermeneutic approach grew first of all out of the idea that you should confine history to a local experience which is quite a modest approach compared to what you have in mind when you imagine a historically guided reconstruction of a mathematical concept. In the hermeneutic approach, students are asked to examine a source in close detail and explore its various contexts of historical, cultural and scientific nature. The hermeneutic approach will not give you an overview. Rather, it is a hope that some pupils will like history and develop a certain interest in it which might motivate them to search for further reading.

The basic guidelines of the hermeneutic procedure can be summarized in 6 principles.

(1) Students study a historical source *after* they have acquired a good understanding of the respective mathematical topic in a *modern* form and a *modern perspective*. The source is studied in a phase of teaching when the new subject-matter

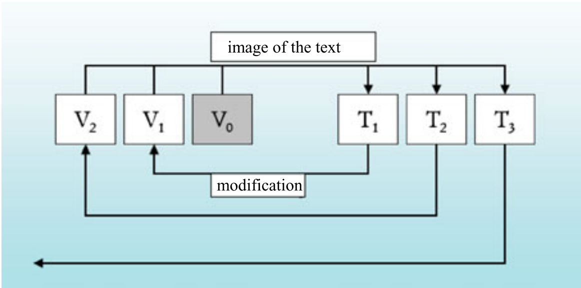
is applied and technical competencies are trained. Reading a source in this context is another manner of applying new concepts, quite different from usual exercises.

(2) Students gather and study information about *context* and *biography* of the author.
(3) The historical *peculiarity* of the source is kept as far as possible.
(4) Students are encouraged to produce *free associations*.
(5) The teacher insists on *reasoned arguments*, but not on accepting an interpretation which has to be shared by everybody.
(6) The historical understanding of a concept is contrasted with the modern view, that is the source should encourage processes of reflection.

What then is hermeneutics? For the following the reader should compare Glaubitz (2010, 2011) and Jahnke (1994, 1996). Simply said it is the "art or the science of interpreting texts." It distinguishes systematically between the author and the reader of a text and their different perspectives. Thus, the strong tension between the historical perspective and the modern view on a mathematical topic is not something which should be smoothed out or eliminated, but is considered as the essential achievement a historical text might contribute to the intellectual development of a person. Thus, the whole enterprise of reading a source rests on experiences of "dépaysement" as the French say or "Verfremdung" ("alienation") as the German writer Bertold Brecht would have said. Sources introduce into teaching an unwieldy element. But how comes it that such unwieldy elements do not lead to failure? This is so only when they have anchor points. The student who deals with something that he already knows but that is presented in a radically different, unfamiliar way or an unknown guise, should be able to make connections to these anchor points. In hermeneutics you would say: His horizon merges with the horizon of the past. *Horizon merging* is a term that was coined by Hans Georg Gadamer (1900–2002). In the horizon merging the student may begin to wonder and to reflect upon what he possibly had never thought about before. In essence he begins to develop deeper awareness. This is in fact an instance of broadening one's horizon. And it does so by utilizing a strategy of dissonance. It is well known that this kind of incompatible information ensures greater retention and ease of retrieval from memory. But to do so, there must be a familiar reference frame. It is therefore applied only to subject-matters that students are already familiar with.

In hermeneutics the process by which the merging of horizons occurs is described by a spiral, the so-called 'hermeneutic circle' which points to the necessity of already possessing an interpretation of a text in order to gain a new interpretation. For us as mathematics educators this appears not so strange as it might be for other people since we are used to reflect about spiral processes, the most prominent being the process of modelling. I take a picture from the dissertation of my former Ph.D. student Michael Glaubitz (2011, 61).

You start with a certain image of the text reflecting your expectations about what it might be about. Then you read the text and realize that some aspects of your image do not agree with what is said in the source. Thus, you have to modify your image,



read again, modify and so on until you are satisfied with the result or simply do not like to continue.

In our case students started with the expectation that Bernoulli's text might be about determining tangents to and extremal values of curves, that the idea of a limit of an infinite process might be central to the subject, that concepts like derivative and slope of a tangent (a quotient) will frequently appear and that all is sort of algebra, with new rules, but symbolic in nature. After some windings of the hermeneutic spiral they had realized that the source was in fact about tangents to and extremal values of curves, but that there was no limit concept, instead there was the difficult concept of infinitely small quantity. Also, the slope of a tangent was less important than expected. It was used in the source to get rid of the infinitely small quantities, but the target object was a second point of the tangent. Thus, the status of a slope changed to that of an auxiliary object. And so on.

On a more basic level the hermeneutic circle can be considered as a process in which a *hypothesis* is put up, tested against the source, modified, tested again and so on until the reader arrives at a satisfying result. For example, the students were asked to infer from the source what a differential is. From their knowledge of calculus some students formed the idea that a differential is something similar to a derivative. With this hypothesis they studied Bernoulli's derivation of the product rule and realized that this cannot be true, since Bernoulli did not calculate a quotient. After some further attempts they saw that a differential is in fact a difference. In a similar manner they found out what a subtangent is.

Can one say then that students behave like historians of mathematics when they read a source? In principle, this is the case. When they entered the source they had questions similar to those a professional historian of mathematics would ask. Roughly spoken, these questions refer to the different meanings of concepts and the different conceptual structures at the time of Bernoulli and today. There are other natural questions they did not explicitly pose but which were obviously in their minds. These refer to what math educators are used to call concept images. Another question they asked was whether Bernoulli really believed that infinitely small quantities exist or whether he considered them as useful but meaningless tools. Of course, a professional historian would ask this question, too. There are other questions a historian would routinely study and which our students didn't ask, namely to compare Bernoulli's text with other writings of the Leibniz school.

The most important difference concerns the previous knowledge a historian and a student have at their disposal. For example, consider the segment in Bernoulli's sketch of the parabola representing the parameter a. A professional historian knows of course that the segment hints at the ancient ruler-and-compass construction of the parabola. Of course, the students do not know this. For the teacher it is a difficult question whether he should tell this to his students. I decided not to do that and preferred to stop with what they could find out by themselves. Thus, to the students the segment remained one of the peculiarities of the source they couldn't explain.

Discussion

In a recent paper Uffe Jankvist (2009) has distinguished between the use of *history as a tool* and that of *history as a goal*. The first concerns the use of history as an assisting means in the teaching and learning of mathematics (mathematical concepts, theories, methods, motivation and so on). In contrast to this, a use of history as a goal does not serve the primary purpose of being an aid, but rather that of being an aim in itself. For instance, it is considered a goal to show students that mathematics exists and evolves in time and space, that it is a discipline which has undergone an evolution over millennia, that human beings have taken part in the evolution. The distinction between tool and goal is quite useful for becoming conceptually clear about the "whys" and "hows" of history of mathematics in teaching. However, as Jankvist himself remarked both dimensions are intertwined, and I would say inseparably intertwined. In the hermeneutic approach, mathematics enters at least in two ways. First, there is the experience of dissonance or alienation. Students learn something about their own mathematics by experiencing and *reflecting on the contrast between modern concepts and their historical counterparts*. And the point of the "hermeneutic circle" as understood here is that the reflection is in both directions, so that the students deepen both their understanding of history and of their own set of modern conceptualizations. Second, and equally important, is the fact that in reading a source (modern) mathematics itself is *applied as a tool*. The task to think oneself into the situation of persons living at a time long ago requires to be able to argue from the assumptions of these persons, to use their symbols and methods of calculation. This poses completely new demands on the students' abilities to argue and to prove mathematically. The teaching unit I have described in this paper showed clearly that it operated on the upper limit of the students. It was a real stress test to the mathematics they had learnt. Thus, reading a source deepens the mathematical understanding on both levels, on that of *doing mathematics* and on that of *reflecting about mathematics*.

Michael Fried nicely distinguishes between different attitudes (of mathematicians) in regard to the mathematics of the past (2011), that of "colleagues", "treasure-hunters", "conquerors", "privileged observers" and "historical historians of mathematics". The latter "view the past as fundamentally different from the present and see the treatment of the past demanding more than present mathematical knowledge". Of course, this tacitly says that modern mathematical knowledge *is* necessary for understanding the past.

How then can we describe this "more", this extra component which goes beyond modern mathematics? This brings us back to the theme of this conference "the search for a common ground between mathematics and mathematics education". I would like to describe this "more" by the term "respect". At first sight "respect" is a category of human relations. When we communicate honestly with other people mutual respect is a necessary condition. In the present context I use this concept with an additional connotation. This is an *epistemological* one and says that we have to accept that there are different legitimate perspectives on the history of mathematics and none of them has the right to claim exclusive truth for itself. According to hermeneutics understanding a text consists in the merging of different horizons, the horizon of the reader and the horizon of the text/author. Different readers with their different backgrounds arrive at different interpretations. Thus, a history of ideas (produced by a leading mathematician) which might neglect many historical details is as legitimate as a social or cultural history of mathematics or the history produced by our students. However, respect as an epistemological category is injured when somebody neglects the difference between his reconstruction of the past and the past itself and takes the mathematical tools he applies as the matter itself.

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