Chapter 1
Mathematics & Mathematics Education: Searching for Common Ground

Michael N. Fried

Between these two groups... there is little communication and, instead of fellow-feeling, something like hostility. (C.P. Snow, The Two Cultures, p. 59)

Prologue

If being mathematically educated could be summed up simply as a familiarity with certain key mathematical ideas—integer, algebraic equation, function, proof—their applications, and a facility in working with them, one could state unequivocally what the interests, foundations, and goals of mathematics education as a field should be. Not too long ago, only the conditional form of this statement would strike one as curious and odd. For what else could one mean by being mathematically educated, and what else could one place higher on the agenda of mathematics education research than the teaching and learning of these key mathematical ideas? And, with that, one could hardly imagine challenging the close and natural alignment between mathematics education and mathematics as academic disciplines.

However, over the last quarter century or so, and for better or for worse, this simple notion of where the core of mathematics education lies has been offset by goals and interests allying it, as an academic field, more closely with psychology of learning, cultural differences, and social justice, among others, than with mathematics itself. Thus, while the first two-thirds of the twentieth century could boast of great mathematicians such as Felix Klein, Jacques Hadamard, George Pólya, and Hans Freudenthal making contributions to mathematics education, today, not only are such figures rare in the field, they have also been to an extent alienated by it.

In the spring of 2012 a symposium concerning the relationship between mathematics and mathematics education was held at Ben Gurion University of the Negev. The symposium was in honor of Ted Eisenberg, who over the years has lamented profoundly the growing divide between the mathematics community and the mathematics education community. It has always been his opinion, shared by the editors

M.N. Fried (✉)
Science and Technology Education, Ben-Gurion University of the Negev, Beer Sheva, Israel
e-mail: mfried@bgu.ac.il

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of this volume, that the divide between the two communities is wasteful and unhealthy for both. The work at hand, which grew out of that symposium, confronts this disturbing gap. By examining areas of commonality as well as disagreement we hope to define more clearly the role mathematics as a discipline plays in mathematics education and mathematics education research and will try to establish a basis for fruitful collaboration between these disciplines. We can only hope that in the end readers will be left with a clearer sense of the mutual benefit both communities stand to lose by failing to strengthen the natural bonds between them.

With the exception of the first part, where we have pieces by Ted, Michael N. Fried, and Norma Presmeg set together in a kind of general dialogue, the various parts of the book take up particular subjects, such as proof, history of mathematics, and educational policy, among others, in which mathematicians and mathematics education researchers either both have a stake or a common interest. It is important to remark that in reading the contributions by the mathematicians and mathematics education researchers one should consider not only what is said but also the ways in which the different communities approach their respective tasks. While we have tried to maintain a certain uniformity in format, we have allowed considerable freedom in other regards. This comes out of the recognition that although we stress common interests and shared concerns, there are nevertheless differences between the communities of mathematicians and mathematics education researchers. One must confront these differences and try to understand them. Thus, to introduce the work and frame its theme, we expand a bit more about the distinctions, divisions, and possibility of cooperation between these two communities. Following that, we shall describe the main parts of the book in brief.

Distinctions and Connections

The moment one broaches the possibility of conflict or tension or misunderstanding between the mathematics and mathematics education communities the difficulty immediately arises, how are these to be distinguished, if at all? Not only this, but also a whole set of distinctions that, previously, one could write off as merely academic, become relevant—not only "Mathematics vs Mathematics education," but also "Mathematician vs Mathematics educator" and "Mathematics educator vs Mathematics education researcher" and "Mathematics education vs Mathematics education research." These distinctions are at the heart of the entire problem we are considering in this work. Granted, the distinctions may not be new, but their problematic character is. In the past, the problem of mathematics vs mathematics education, the main distinction we are considering, could only be viewed as a non-problem, a false dichotomy. One could then easily say that mathematics and mathematics education simply belonged to different categories: a whole, "mathematics," and a part, "mathematics education." Asking about the distinction between mathematics and mathematics education would have been like asking about the distinction between mathematics and geometry.

Nor does that view necessarily vanish with mathematics education's becoming a separate academic field (see Kilpatrick 1992 for a very good exposé of how that happened). However, with that change in place, the relationship between mathematics and mathematics education became no longer obvious and necessary: it has now become a question. One must ask, at very least, what justifies the formal, academic distinction between mathematics and mathematics education in the first place? While the separation may be merely bureaucratic and not essential, members of the new field do need to consider their own identity as mathematics educators. At some level, this itself is a bureaucratic necessity, albeit one also requiring genuine introspection—as a separate discipline, a basis has to be established for hiring and promoting mathematics educators: what is it a mathematics educator has to do well, what is that makes a mathematics educator an expert? This runs together with the next distinction, namely, between a mathematics educator and a mathematician. The question of the identity of the discipline thus becomes one of the identity of the practitioner: Is one a mathematician first before one is a mathematics educator? Is a mathematics educator a kind of mathematician?

Of course this begs the further question of what makes one a mathematics educator—in particular, how one should distinguish a mathematics educator from a researcher in mathematics education. Here, since one has the term "mathematics education researcher," one can treat a mathematics educator simply as a mathematics teacher. At the university level, naturally, the difference between a mathematician and mathematics teacher is not nearly so pronounced as it may be at school level since, besides the obvious fact that it is typically mathematics researches teaching mathematics students, university level mathematics already begins to have the feel of mathematics as the mathematician knows it. One could go further and argue that the difference between doing and teaching mathematics is actually never very great in that mathematicians must always communicate their thinking. Consider, in this connection, Andrew Wiles' "graduate seminar" taught principally to fellow mathematician Nick Katz when Wiles was working on Fermat's last theorem. As Simon Singh (1997, p. 242) relates:

> Virtually everything Wiles had done was revolutionary, and Katz gave a great deal of thought as to the best way to examine it thoroughly: "What Andrew had to explain was so big and long that it wouldn't have worked to try and just explain it in his office in informal conversations. For something this big we really needed to have the formal structure of weekly scheduled lectures, otherwise the thing would just degenerate. That's why we decided to set up a lecture course."

Although the seminar was also a ploy to hide Wiles' secret work on Fermat's last theorem, nevertheless, when all the graduate students had dropped out leaving Wiles and Katz alone, the teacher-student structure remained, as Katz emphasized.

In more ways than one, then, being a mathematician is being a mathematics teacher and communicator, which is a kind of teacher. The converse, however, is far from clear. It is even not entirely clear that a teacher should have a mathematician's training. Surely, mathematics teachers should know what they teach, but saying that begs the question at least and really is a mere platitude. In fact, the question of requisite knowledge for teachers and others is a true question, and an old one. Plato asked a similar question regarding the sophists and teachers of rhetoric. He also asks it most delightfully in a little dialogue centered on a rhapsodist, a reciter of Homer, Ion, after whom the dialogue is named. Socrates claims—and Ion agrees—that an expert rhapsodist must understand what he recites if he is to produce worthy interpretations of the epics, similarly, if he is to distinguish a good rhapsodist from a bad one. To use one of Socrates' examples, if the subject were numbers, one would expect that only an expert in the "arithmetical art," the arithmetical *techne* (*Ion*, 537e), would be able to judge whether the subject was being discussed well. In reciting Homer, Ion speaks about soldiers, generals, and even doctors: is Ion such an expert in these that he can speak so well about them? Ion is no general or doctor. Socrates teases him, saying it must be divine inspiration that he can do so. Yet, as in all Platonic dialogues the issue remains open in the end, for Socrates well knows that rhapsodists *are* successful at what they do, even they are not skilled generals and doctors.

This is true too about mathematics teachers. For this reason, in informal settings and casual conversation, one often hears their success explained by saying that teaching is an art—ironically meaning something closer to Plato's divine inspiration than what the Greeks mean by art, *techne*, a skill informed by knowledge! There may, nevertheless, be some truth to that, though research as to what makes a good mathematics teacher is much more circumspect and far from definitive. As the National Science Foundation report on science and engineering indicators remarks, "No research has conclusively identified the most effective teachers or the factors that contribute to their success, but efforts to improve measures of teaching quality have proliferated in recent years" (National Science Foundation 2012, Teachers of Mathematics and Science, side bar 6).

Be that as it may, the specific relationship between mathematical knowledge and mathematics teaching is equivocal. On the one hand, there is something as powerful as it is inexplicable about simply being in the presence of teachers who have thought deeply about their subjects. The philosopher and literary critic, George Steiner, describes this beautifully reflecting on his own experiences at the University of Chicago:

> Once a young man or woman has been exposed to the virus of the absolute, once she or he has seen, heard, 'smelt' the fever in those who hunt after disinterested truth, something of the afterglow will persist. For the remainder of their, perhaps, quite normal, albeit undistinguished careers and private lives, such men and women will be equipped with some safeguard against emptiness. (Steiner 1997, p. 44)

On the other hand, an early finding of modern mathematics education research showed that a *direct* connection between the depth of teachers' mathematical knowledge and their students' level of achievement cannot be fully maintained. In the course of his work with the School Mathematics Study Group (SMSG), Edward Begle (1972) had shown that empirically there was no significant relationship between teachers' knowledge of advanced algebra and their students' achievement in algebra. This was a result that Ted Eisenberg himself strengthened with a follow up paper in 1977 that controlled for potentially biased factors in Begle's original report.[1]

Although the issue is still open to certain extent, it is hard to doubt that success in mathematics teaching demands *some* combination of subject and non-subject dependent knowledge. The utterly integral character of that combination was driving force of Lee Shulman's (1986) now-standard concept of pedagogical-content-knowledge. But even without the concept, the necessary and simultaneous attention to content and pedagogy is evident in accounts of great mathematics teachers. Thus, Jeremy Kilpatrick says this of Pólya as a teacher:

> One of the things I learned from Pólya was if someone in class had trouble following the presentation, then you slow the class down. Pólya was always willing to slow the class down, but he could still make it interesting. That's one of the remarkable talents he had. He could move at a slower pace so that students could follow his presentation; even the slowest member of the class could get something out of it. Yet, at the same time, what he was presenting was interesting enough and rich enough that the people who understood what was happening could also learn something. He was not interested in getting someplace in the discussion where he felt he should be; he was interested in making what he was doing as illuminating as possible. (Kilpatrick, quoted in Taylor and Taylor 1993, p. 107)

In his writings about mathematics education, the Berkeley mathematician Hung-Hsi Wu agrees that school mathematics teachers need pedagogical-content knowledge (and he uses the term explicitly) and not just content knowledge (Wu 2011). Wu objects to what he calls the "Intellectual Trickle-Down Theory," which holds that extensive mathematical knowledge will effortlessly trickle down into teaching competence; he believes that while elementary school teachers simply need more mathematical knowledge, secondary school teachers need better developed means in order to make solid mathematical knowledge more presentable and understandable for their young, not-yet-mathematically-mature students.

Divisions

If mathematicians like Wu recognize these limits of mathematical knowledge and the concomitant need for insight into teaching and learning, do they also recognize the need for mathematics education research, which is supposed to study the teaching and learning of mathematics? More pointedly, do they recognize the need for mathematics education *researchers*? The answer is yes and no. One sign on the positive side is that there are mathematicians who themselves engage in mathematics education research in the company of other professional mathematics education researchers, Hyman Bass, for example; another is the active field of tertiary mathematics education research which has developed in recent years and is pursued at

[1] Recent work by Ruhama Even (2011), however, has shown that *in their own view*, teachers see advanced mathematical work helpful on three fronts: that it is a knowledge resource; that it improves their understanding of mathematics and what it is; that it provides a model for what learning mathematics feels like.

least with the cooperation of mathematics departments (see, for example, Holton 2002).

For the negative side, we can return to Wu. One could find other examples where a dismissive attitude towards mathematics education researchers is more clearly evident, examples bordering on rancor (the continuing attacks on Jo Boaler by James Milgram and Wayne Bishop, as documented in Boaler 2012, come to mind). But it is more informative to look at Wu (in the context of Wu 2011) in part because he is a mathematician genuinely concerned about mathematics education and somewhat informed about research touching on mathematics education; Wu's case shows the subtle ways one can recognize the need for mathematics education research but not for mathematics education researchers.

To be fair, Wu does refer to non-mathematician mathematics education researchers, such as Deborah Ball, and not unfavorably, especially when she recognizes the poor mathematical backgrounds of teachers. However, what he sees as the important task of mathematics education is "the *customization* of abstract mathematics for use in schools" (Wu 2011, p. 378, emphasis in the original), and this is a task for mathematicians. Wu describes the paper as "a call for action," namely, a call for mathematicians to recognize the "urgent need of active participation in the education enterprise" (p. 372).

On the face of it, there is nothing wrong with this. In fact, is it not what we ourselves are asking for in this book? The problem is that while Wu bemoans the "communication gap between mathematicians and educators" (p. 382), it is not hard to see that, for him, the gap consists in educators' not taking account of mathematicians rather than mathematician's missing the views and knowledge of educators. It is telling that he chooses to describe the dangers of the communication gap by recalling how Watson and Crick in their work on the DNA molecule benefitted crucially from the visit of a professional crystallographer, Jerry Donohue. And he summarizes the story and its moral as follows:

> ... but for the fortuitous presence of someone truly knowledgeable about physical chemistry, Crick and Watson might not have been able to guess the double helix model, or at least the discovery would have been much delayed.
> The moral one can draw from this story is that, if such misinformation could exist in high-level science, one should expect the same in mathematics education, which is much more freewheeling. This suggests that real progress in teacher education will require both the education and the mathematics communities to collaborate very closely and to be vigilant in separating the wheat from the chaff. In particular, given the long years during which incorrect information about mathematics has been accumulating in the education literature and school textbooks, there should be strong incentive for educators to seek information about the K-12 mathematics curriculum anew and to begin some critical rethinking. (pp. 382–383)

Although Wu speaks about collaboration explicitly, *knowledge* is placed squarely in the mathematicians' camp. He may object to the "intellectual trickle-down theory," but, when it comes down to it, whatever is wrong with the "theory," it is still the mathematicians who must correct it. It is hard to see where mathematics education researchers have a role, other than to sit quietly and listen. Indeed, the paper is addressed to mathematicians, and it appears in the mathematics journal, the *Notices of the AMS* (American Mathematical Society). It is not a call for collaboration: it is, as Wu says, a call for mathematicians to take action, not necessarily for *them* to listen.

But it is not only mathematicians who are to blame for dividing communities that should collaborate. Mathematics education researchers can also be dismissive of what mathematicians might bring to the floor. It can be asked, equally, whether mathematics education researchers recognize a need for mathematicians in mathematics education. Again, one can cite examples of open opposition to mathematicians having a central role in mathematics education research (see the account of the ICMI centenary in the first chapter of *Dialogue on a Dialogue* below). More often, however, what one finds is an agenda that leaves little room for mathematicians, a tendency to give precedence to areas hardly any mathematician would call mathematics and which certainly no mathematics department would include in its program. This is especially manifest when social justice issues are brought into mathematics education. Accordingly, Sriraman, Roscoe and English note that,

> Numerous scholars like Ubiratan D'Ambrosio, Ole Skovsmose, Bill Atweh, Alan Schoenfeld, Rico Gutstein, Brian Greer, Swapna Mukhopadhyay among others have argued that mathematics education has everything to do with today's socio-cultural political and economic scenario. In particular mathematics education has much more to do with politics, in its broad sense, than with mathematics, in its inner sense. (Sriraman et al. 2010, p. 627)

And in her commentary on Sriraman et al. (2010), Keiko Yasukawa confirms this by concluding:

> If we believe that mathematics learning can be a resource to increase democratic participation in society, to increase equity and social justice, then mathematics learning cannot be divorced from learning the politics of the world in which we live. Has the study of politics in mathematics education gone far enough? Evidently not. Can it go further? Yes, through critical mathematics education that will awaken learners to the ways in which mathematics is concealed but active in the dominant discourses that are influencing the ways we think about the fundamental principles of equity and fairness. (p. 643)

If understanding the nature and role of mathematics, not only in science and engineering but also in students' everyday lives, should be considered part of mathematics education, then these kinds of political investigations are not out of place. However, even these authors would have to admit that there is something merely accidental about mathematics' place in the political superstructure. There are other elements of the superstructure, and there *could* be other areas attaining the same prominence as mathematics *if they happened* to be valued in the same way. In other words, this key place of mathematics is not related to mathematics "in its inner sense," to use Sriraman et al. (2010) phrase. In fact, once one puts on the glasses of critical mathematics education, *every* mathematical notion becomes suspect and must be examined for its socio-political function: every mathematical idea has an ulterior meaning. This is almost axiomatic in "critical theory" (which dominates the thought of the authors mentioned by Sriraman, et al.' above), and it may reflect a true state of things. But if so, the mathematical meaning of any given mathematical concept, as the mathematician understands it, becomes not only secondary but also the very thing one must learn to move beyond. Mathematicians, in this way, can be of no help, and critical mathematics education may well see them, if they are not "liberated," as part of the problem.[2]

Distinctions Once Again and the Possibility of Cooperation

The kind of divisive positions we have just described—and there are others—not only distance the possibility of cooperation between mathematicians and mathematics education researchers, they also lead to a lack of coherence in mathematics education, regardless whether it is the mathematicians or the mathematics education researchers who take charge. Mathematicians dismissive of mathematics education researchers and mathematics education researchers dismissive of mathematicians must both find themselves edging towards inconsistency: the first wants to customize advanced mathematics for use in the schools but gives little credence to those who research the conditions and nature of learning; the second wants to teach mathematics and wants it taught while showing that it is only part of a superstructure concealing the non-mathematical political forces.

The truth is these two communities cannot be completely divorced. Even if they feel pushed to declare their loyalty to one camp, they will inevitably have one foot in the other. As argued above, university mathematicians typically and often necessarily take on the role of a teacher, that is, a mathematics educator, and as such must take an interest in how students learn and how best to teach. Mathematics education researchers, on the other hand, still insist that they are interested in *mathematics* education. And there are areas that interest both communities in ways that are quite similar, for example, visualization and problem-solving. Remember Pólya's interests in problem-solving and his work as a mathematician were joined almost seamlessly: consider for example his book with Gábor Szegö on problems and theorems in analysis, a serious book in which the problems were organized according to their solution strategies (see Taylor and Taylor 1993, pp. 24–25).

With these common areas of interest in mind, one would expect far more cooperation and collaboration than one typically finds between the communities of mathematicians and mathematics education researchers. True, as remarked above, there are instances of open enmity between these communities that would poison

[2] I am referring to mathematicians and mathematics teachers who, lacking the "critical" outlook, devote themselves to teaching mathematics as if it were a neutral subject. For proponents of critical theory, they, unwittingly, support the power structure rather than reveal it. Thus in his well-known article "Ideology and ideological state apparatuses" (Althusser 1971), Louis Althusser writes: "I ask the pardon of those teachers who, in dreadful conditions, attempt to turn the few weapons they can find in the history and learning they 'teach' against the ideology, the system and the practices in which they are trapped. They are a kind of hero. But they are rare and how many (the majority) do not even begin to suspect the 'work' the system (which is bigger than they are and crushes them) forces them to do, or worse, put all their heart and ingenuity into performing it with the most advanced awareness (the famous new methods!). So little do they suspect it that their own devotion contributes to the maintenance and nourishment of this ideological representation of the School..." (p. 157).

any attempt to work together. However, such *open* enmity is not the rule: most of the time, it is rather only a vague dismissal of one or the other or simply lack of acknowledgement. Moreover, as we also remarked, there is not a total lack of collaboration.[3] So why is there not more?

That question is one of the preoccupations behind this book. Paradoxically, though, having finally arrived at the question of cooperation and collaboration, we must return to the question of how these communities are distinct. For collaboration is a relation between groups that complement one another, and being complementary presupposes difference—difference in focus, in method, in worldview. Without such difference, the communities are thrown into a relation not of collaboration but of competition, as, unfortunately, the relation is all too often perceived.

For our case, a key source of difference between research in mathematics education and in mathematics is the alignment of mathematics education, as part of general education, with the social sciences or even the humanities, and mathematics, with the exact sciences. The emphasis on research is important. For when one considers mathematics education research, one must consider not only its methodology, but also, at a deeper level, what kind of knowledge it generates. Recall how Wu's treatment of the problems of mathematics teaching rested on what kind of knowledge teachers possessed and needed to possess. The social sciences and humanities and the exact sciences have their own sense of knowledge, what it means to know something and what one needs to do to know something. The possibility of cooperation and collaboration, therefore, comes with an appreciation of the more fundamental difference between these two streams of thought: cooperation and collaboration must be premised on coexistence of such different kinds of knowledge and modes of pursuing knowledge.

Of course one can deny this and embrace the tempting assumption that these different kinds of knowledge and modes of pursuing knowledge are, *mutatis mutandis*, the same for the humanities and social sciences on the one side and the exact science on the other. It is the assumption that on both sides there are facts and universal immutable laws which can be verified by methods each side can accept and understand. To be sure, it is not assumed that a law of "learning science" would be a law of physics, but that there *would be* laws; nor is it assumed that, say, a particular experimental technique would be the same in both cases, but that there *would be* experimental techniques whose warrants for accepting or rejecting a claim could be explained each in the other's terms.

[3] One good example of collaboration that does exist is the Klein Project developed and implemented by ICMI. The project was commissioned in 2008 by the International Mathematical Union (IMU) and the International Commission for Mathematical Instruction (ICMI). Its guiding idea was to revisit Felix Klein's book "Elementary Mathematics from an Advanced Standpoint" and produce a book for secondary teachers communicating the breadth and vitality of mathematics as research discipline while connecting it to the secondary school curriculum. An international design team for the project was appointed led by two ICMI presidents: Michèle Artigue and Bill Barton and a book is under preparation. In the meantime, Klein Project has produced a set of "vignettes" for teachers and students. The rationale for this phase of the project and examples of the vignettes already produced can be found at the website: http://blog.kleinproject.org/.

As tempting as this assumption may be, it leads to a whole variety of mutual misunderstandings and false expectations. And more than anything else it is what allows the relationship between the communities to slip into one of competition. It can also obscure self-understanding—particularly in the social sciences (and, there, particularly in educational research)—as can be seen in the common tendency towards "physics envy,"[4] "desiring this [other] man's art… [Ourselves] almost despising," as Shakespeare would say (see *Sonnet xxix*). Yet, the existence of "physics envy" as well as the unreflective use of such terms as "hard science" and "soft science" are only signs that the assumption we are speaking of is adopted widely, even if it be so unconsciously or unacknowledged.

Still, this way of thinking in which the methods and rigor of an intellectual pursuit, indeed, the value of its knowledge, are judged according to its closeness or distance from sciences like physics and chemistry has deep roots. Its greatest expression is in the work of Auguste Comte. Comte's *Cours de Philosophie Positive*, composed between 1830 and 1842, is little read today; yet, despite enormous revisions in how philosophers and historians have come to think about the sciences, including the social sciences, the spirit of this work of Comte haunts the world of research.

Comte invented the word "sociology," and what he meant by that is best seen in the other term he employed, "social physics." He really meant that, as he goes on to describe "social statics" and "social dynamics"! Comte believed that the evolution of society and, therefore, its improvement could be charted by laws comparable to those of physics. In fact, he thought that laws of social phenomenon were incorporated into a greater system of laws including physics. Thus he writes:

> It is the exclusive property of the positive principle to recognize the fundamental law of continuous human development, representing the existing evolution as the necessary result of the gradual series of former transformations, by simply extending to social phenomena the spirit that governs the treatment of all other natural phenomena. This coherence and homogeneousness of the positive principle is further shown by its operation in not only comprehending all the various social ideas in one whole, but in connecting the system with the whole of natural philosophy, and constituting thus the aggregate of human knowledge as a complete scientific hierarchy. (Comte 1975a, p. 211)[5]

[4]This is the lament of a recent opinion piece in the New York Times by political scientists (note the name!) Kevin A. Clarke and David M. Primo ((2012, March 30). Overcoming 'Physics Envy'. Available at http://www.nytimes.com/2012/04/01/opinion/sunday/the-social-sciences-physics-envy.html). Interestingly enough, this phrase, so commonly used regarding the social sciences, was actually coined by Joel E. Cohen with reference to biology. Cohen wrote a book review of a book on dynamical systems in biology (Cohen, J.E. (1971, May 14). Mathematics as Metaphor. *Science 172*, 674–675), which begins, "Everyone likes to discover general and unifying principles in biology" (p. 674) and then goes on to say, creating the famous phrase, "Physics-envy is the curse of biology" (p. 675)! So, even within the natural sciences, one should be careful to recognize that there may not be uniformity in appropriateness of methods and approaches.

[5]The English translation contained in the collection edited by Gertrud Lenzer was produced in Comte's day and, as Lenzer notes, was "enthusiastically approved" by Comte himself. The original French text can be found in Comte (1975b, leçon 46, p. 66).

This "positive," or as we might say, "scientific," knowledge was, for him, the final stage in an evolution of knowledge itself, beginning with what he termed the "theological stage" and then the "metaphysical stage" (see pp. 71–72).[6] Comte notes that social thinking will only bear fruit when it finds its way out of the metaphysical stage and fully enters the positive stage, which, he admits, has not yet been accomplished.

It is Comte's voice, his faith in progress through science, that one hears in the American *No Child Left Behind* policy. There we are told that we must aim for "Scientifically Based Research"[7] in order to bring about true educational improvement. This means research, according to *No Child Left Behind*, that:

(1) Employs systematic, empirical methods that draw on observation or experiment
(2) Involves rigorous data analyses that are adequate to test the stated hypothesis and justify the general conclusion
(3) Relies on measurement or observational methods that provide valid data across evaluators and observers, and across multiple measurements and observations
(4) Is accepted by a peer-reviewed or a panel of independent experts through comparatively rigorous, objective and scientific review (US Department of Education 2002a)

The implication, completely consistent with Comte's doctrine, is that research more philosophical, less empirical and experimental, even if it is "the best one can do now," is ultimately *to be replaced* by this "scientifically based" knowledge.

Interestingly enough, the opposing view, namely, that there are distinct modes of pursuing knowledge dependent on the object, that what might be appropriate for physics is not appropriate for the humanities, or, for that matter, educational studies, was recognized before *and* after Comte. Before Comte, one could point, say, to Aristotle, whose introduction to the *Nichomachean Ethics* begins with a discussion of just this point, saying, for example, that one should not expect probable arguments from a mathematician as one should not expect strict proofs from an rhetorician (Book I, 1095b:25–26). But a better example—one whose cogency remains unabated—is Pascal's distinction between two the different kinds of minds, "l'esprit de géométrie," the geometric mind, which proceeds by drawing conclusions from a few first principles, and "l'esprit de finesse," the intuitive mind, which proceeds with a kind of intuitive understanding of things whose principles are so numerous they cannot be grasped one-by-one but must be seen somehow all at once (tout d'un coup) (Pascal 1962, Lafuma 512). The importance for us is that where

[6]Comte claimed that education was, in his day, motivated by thinking of the theological, metaphysical and literary types. One of his hopes in laying out the positive philosophy was that education would turn in the positive direction: in effect, Comte was, in effect, pressing for education based more on the sciences and mathematics than on the traditional literary curriculum. This theme, now ubiquitous, was taken up often in the 19th century, for example, by the great biologist Thomas Huxley who suggested that liberal education should be science education.

[7]Comte's sense that progress is impeded by less-than-scientific research can be felt the discussion of mathematics education and "Scientifically Based Research" recorded at the US Department of Education Website (US Department of Education 2002b).

we have this complexity of principles (such as with, say, "learning" whose very definition is hard to frame) it is not enough to modify the analytical approach of the "l'esprit de géométrie": an entirely different approach is required. Pascal, makes it clear in this famous pensée, moreover, that one looks ridiculous, as he puts it, when applying the geometric mind to things that demand the intuitive mind, and the contrary; one cannot be reduced to the other.

After Comte, at the end of the 19th century and the beginning of the 20th, in the work of such figures as Wilhelm Dilthey and Wilhelm Windelband one finds an acute awareness of the difference between what was commonly called the "human sciences" and the "exact sciences." Dilthey (1989), for example, made it clear that in the human sciences, one is engaged in an activity of interpretation rather than deduction; one is driven by a kind of "understanding" (Verstehen), as he called it, of one's human subjects and what they produce. Windelband, a figure less known than Dilthey and perhaps less profound, made a pointed distinction between what he called *nomothetic* and the *idiographic* approaches to knowledge (Windelband 1894/1980); the one concerned phenomena that governed by universal law (*nomos* means "law" in Greek), while the other concern phenomena connected with individuals and their own perspectives.

Both Dilthey and Windelband (though their main object was historical inquiry) touch clearly on the kind of inquiry mathematics education research engages in, namely, studying the learning of mathematics and the place of mathematics in a student's life, as opposed to studying mathematics itself; what does it mean for a student to encounter and begin to assimilate a new mathematical idea, for a student to face and overcome a difficulty, to discern a difficulty? These questions involve exploring the understanding of a student from the inside, as it were. Windelband's distinction between *nomothetic* and *idiographic* inquiries is extremely important in this regard, for although mathematics education research often uses statistics and large populations, some of its most enlightening work is the result of case studies involving sometimes two or three students. The way in which one draws insight from an individual student is difficult, if not impossible, to grasp from the nomothetic perspective: how can a universal law be deduced from an individual case?

But perhaps more than figures such as Dilthey, it is Max Weber whom we must take account of in the post-Comtean world. For its in Weber one comes face to face with problem of values in a decisive way. Weber was deeply concerned about the scientific character of his sociological work. This led him to assert forcefully and repeatedly that values, or more precisely, value judgments, must be removed from social science (see the three long essays in Weber 1949).[8] What makes this fascinating and problematic is that work in the human sciences, Weber's work in particular,

[8]This point is also made in Weber's well-known address, "Science as Vocation" (English translation in the collection Gerth and Mills (1958, pp. 129–156) where he also, as in this place, refers to what university professors in science—specifically social science—can see as part of their vocation and what they cannot—and what they cannot includes pronouncements of value among their students.

refers constantly to values and value-systems.[9] There is no overt contradiction in this of course since it is conceivable to speak about a value-laden subject, say religion, in a value-free way. But there are already lines of tension, especially since the inner understanding of such subjects (and Weber has views here that are not inconsistent with Dilthey) presupposes what it is like to be committed to values.[10] These lines are stretched even further by Weber's acceptance of what he calls "value-relevance" in inquiry: the choice of subject matter for investigation may be related to the values of the investigator, even while the investigation itself is value-free (e.g. Weber 1949, pp. 21ff).

The notion of value-relevance applies of course even to the purest of sciences and to mathematics: it is at work in deciding whether a mathematical theory or problem is interesting and worth pursuing or whether a particular solution to a problem or proof deserves our praise (see, for example, Corry 1989 and Elkana 1981). In mathematics education research, however, as in other forms of educational research, not only must we speak about value-relevance, but, beyond that, we must speak about a role of values in a more direct way: here, by the very nature of the subject, engaging in "evaluation" is unavoidable (this will be discussed further in *Dialogue on a Dialogue*). For mathematics education research has ultimately the practical aim of *improving* mathematics education, of making it *better*, of saying how we *ought* to teach and how students *ought* to learn. This engagement with values together with its attention to individuals, its idiographic character, and its need to interpret rather than only to describe behavior, sets off mathematics education research from the kinds of research typically pursued in faculties of exact sciences and engineering. One cannot assume, as Comte did, that, in principle, there could be consistency in the general methodologies and general outlooks of these different forms of research.

The first step, therefore, in ameliorating the cooperation between researchers associated with the exact sciences and those in involved in research like mathematics education research is to recognize these radically different ways of pursuing research and to recognize the necessity of those differences. Mathematics education research must be understood as something apart from mathematics and mathematics from mathematics education research: one cannot be subsumed under the other or replaced by the other. It should not be our mission to "convert" mathematicians to what they cannot be as it should not be theirs to determine what mathematics education researchers should research. And yet, to reiterate what has been said in different ways throughout this introduction, this cannot be a formula to go in separate ways: the common focus on mathematics, one way or another, will not allow for that. Cooperation begins when there is at the same time the recognition that each side is looking in the same direction but with very different, complementary eyes.

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[9] Weber's famous 1905 work *Die protestantische Ethik und der Geist des Kapitalismus* (*The Protestant Ethic and the Spirit of Capitalism*) is a case in point.

[10] In his chapter on the fact-value distinction in *Natural Right and History*, Leo Strauss (1953) makes a point along these lines.

The Structure of This Book

Almost as a demonstration of the possibility of cooperation, the authors of this book comprise mathematics education researchers and mathematicians. Some of the authors could wear both hats, and some do. But just putting the two communities together in one room is not enough to begin a dialogue. Indeed, the first part of this book raises the question of dialogue and is centered on a dialogue (Eisenberg and Fried 2009) written by one of the editors, Michael N. Fried, and Ted Eisenberg, to whom this book is dedicated. This dialogue, which concerned the state of mathematics education generally, was in fact a response to paper by Norma Presmeg (Presmeg 2009). In her paper, published in the same issue of *Zentralblatt für Didaktik der Mathematik* (ZDM), Presmeg had argued that since the purview of mathematics education includes more than mathematical content *per se*—that it concerns how students think about mathematics, how mathematics becomes part of students' inner and outer lives, how it is integrated into students' sociocultural world, for example—it is necessarily a multidisciplinary subject. Eisenberg, in particular, felt in the course of broadening mathematics education in this way, mathematical content was in fact becoming lost. The dialogue that he and Fried produced subsequently revolved around the question of mathematics education is truly about as a field, what are its true interests, and has it lost its identity by moving too far away from mathematics.

So *Dialogue on a Dialogue* revisits these two papers[11] and produces a new dialogue with the same players—Eisenberg, Fried, and Presmeg—providing thus three points of view. It sets the stage for the rest of the book by raising questions such as whether mathematics teaching has the same interests as mathematics education research and whether the latter should, as Presmeg originally claimed, be multidisciplinary. It also suggests some of the themes of commonality and difference joining and dividing the communities of mathematics and mathematics education—for example, visualization, proof, policy, problem-solving.

The remaining parts of the book treat eight of these themes. With two exceptions—*Mutual Expectations* and *Problem-Solving*—each part has a similar overall structure: a position paper followed by a chapter containing a series of short responses or reflections on the same subject. In each case, the latter also contains an introduction and synthesis of the main points and problems. To provide the reader with a kind of map for the book, we now summarize these eight parts and set out the players involved.

[11] The papers by Eisenberg and Fried and Presmeg were joined by a third written by David Pimm (Pimm 2009), who also discussed the relationships and provinces of the different disciplines contributing to mathematics education.

Mutual Expectations Between Mathematicians and Mathematics Educators

This part is one of two related to the preconditions for mathematics educators and mathematicians' working together. On the assumption that mathematicians and mathematics education researchers do wish to work together, what do they expect to receive from one another? What kinds of problems do they expect one another to focus upon? This, perhaps more than any other part, addresses the question of how each community is defined in light of the other, for what they expect from one another clearly reflects how they see one another. It is also the part in which one can see the tensions between the two communities, albeit sometimes between the lines.

The introduction and synthesis of the issues involved is written by Tommy Dreyfus; the other contributors include Stephen Lerman, Ioanna Mamona, and Uri Onn. The contributors were chosen carefully so that they would represent a spectrum of views from that of mathematics educator whose work is generally distant from mathematical content to a pure mathematician whose educational interests are closely tied to his university teaching.

History of Mathematics, Mathematics Education, and Mathematics

In a way, this is the oddest of the parts in this book. In contrast to a subject such as "proof," the history of mathematics is neither at the center of mathematics as a discipline nor at the center mathematics education as a discipline. Yet, it is of great interest to both even if it is often misunderstood by both. At the same time it is unavoidable in any effort to see mathematics as a part of general mathematical culture, as Felix Klein put it, and therefore goes far to address the difficulties of mathematical literacy and the meaning of being mathematically educated. A proper understanding of the place of history of mathematics in mathematics and mathematics education may end up being genuine common ground seeming, at present, foreign to both.

The introduction and synthesis of the issues involved is written by Luis Radford; the other contributors include Alain Bernard, Michael N. Fried, Fulvia Furinghetti, and Nathalie Sinclair. Luis Radford was chosen to produce the synthesis, not only because he himself has done some historical work in mathematics and is himself an eminent mathematics educator, but also because of his particular cultural-historical understanding of mathematics. This cultural-historical understanding places mathematics in a grey area between the "two cultures" (using C.P. Snow's famous phrase) and, therefore, shows more clearly the relationship between them.

The part opens with a paper by Hans-Niels Jahnke concerning the hermeneutic approach to history of mathematics, an approach that appreciates the historical character of mathematics of the past while taking into account modern mathematical notions.

Problem-Solving: A Problem for Both Mathematics and Mathematics Education

While history of mathematics may be foreign to both mainstream mathematics and mathematics education, this is certainly not the case with problem-solving. The centrality of problem-solving in mathematicians' own work and in their teaching, is incontrovertible. Problem-solving is also a central topic for mathematics educators, who have developed conceptual frameworks to formulate general ideas about problem-solving (as opposed to the specific ideas needed for solving specific problems). Both mathematics educators and mathematicians have given thought to problems helping students understand ideas, and both have given thought to the process of solving problems: George Pólya, of course, reflected deeply about this, and Pólya's work figures strongly in this chapter. This is, one hastens to add, not only because of the importance of Pólya's work regarding problem-solving, but also because Pólya himself represents a bridge between mathematics and mathematics education: he was an eminent mathematician and also a deep influence on mathematics education.

The introduction and synthesis of the questions raised by problem solving is written by Boris Koichu, who has done extensive work on problem solving especially among talented mathematics students, those most likely later to join the community of mathematicians. The other contributors to this part are Gerald Goldin, Roza Leikin, Shlomo Vinner, and Izzy Weinzweig.

Mathematical Literacy: What Is It and How Is It Determined?

One might say that the guiding question for this part on mathematical literacy is simply what does it mean to say someone is "mathematically educated"? In this light, its subject has a theoretical character. However, it also has a practical side with real consequences for teaching and curriculum development; a notion of literacy is, in this way, also a guide to the design of a mathematics policy, the subject of Part "Policy: What Should We Do, and Who Decides?". Moreover, literacy, precisely because it concerns the ends of policy, is connected to the practical problem of assessing educational policy and achievement. For this reason, operational definitions for literacy have been produced in conjunction with international assessment, notably the PISA program.

The synthesis here is written by Anna Sfard; other contributors include Abraham Arcavi, Iddo Gal, Ron Livné, and Hannah Perl. Sfard's own contribution clarifies the notion of literacy by connecting it to another theme of equal importance to mathematics educators and mathematicians, namely the idea of communication.

The part opens with a paper by Paul Goldenberg, who emphasizes what he calls habits of mind.

Visualization in Mathematics and Mathematics Education

The subject of visualization is important to both mathematics and mathematics education since it characterizes the way both students and mathematicians commonly think about mathematical ideas and solve mathematical problems. For this reason, Hadamard studied visual thinking in his famous work on the psychology of mathematical invention (Hadamard 1945), and mathematicians, such as Stanislav Ulam, writing about their own mathematical thinking attests to the importance of visualization (e.g. Ulam 1976, p. 183). Visualization is also related to the representation of mathematical objects with the aid of computers: the ability of computers to produce and manipulate pictures has allowed new ways for students to study and explore ideas in geometry and analysis. This part, then, takes into account mathematicians use visualization in their teaching, mathematics educators' proposals for employing and developing visual thinking in computer and non-computer environments, as well as research results from mathematics education.

The introduction and synthesis of this topic is written by Elena Nardi; the other contributors include, Rina Hershkowitz, Raz Kupferman, Norma Presmeg, and Michal Yerushalmy.

The part opens with a paper by Ken Clements that discusses, among other things, Clements work with the then young Terence Tao, later Field Prize medalist—a rare view into the ways, often visual ways, a young developing mathematician thinks.

Justification and Proof in Mathematics and Mathematics Education

Common ground here would at first sight seem unproblematic, since "justification," interpreted as "proof," is a subject is crucially important for both mathematics and mathematics education. Yet, there are in fact strong divisions. For in mathematics "proof" and "justification" are identified, whereas in mathematics education much attention is given to forms of justification that fall short of proof but nevertheless are deeply connected with processes of learning. The idea that an incomplete or even incorrect explanation may yield more insight for the mathematics educator than a rigorous proof runs counter to the way of thinking in a discipline that gives little credit to a justification which is not a proof. On the other hand, mathematicians do give weight to proof, even a heuristic argument, that actually persuades them of the truth of mathematical claims. Proofs must have in some sense pedagogical value.

The introduction and synthesis here is written by Keith Weber; the other contributors include, Gila Hanna, Guershon Harel, Ivy Kidron, and Annie and John Selden.

A paper by David Tall concerning research on mathematical reasoning and thinking generally is the opening paper for this part.

Policy: What Should We Do, and Who Decides?

The central concerns of "policy" are the principles and agents of decision-making and the program—the policy—actually decided. To the extent "policy" concerns the agents of decision-making, it is closely related to the subject of collaboration; to the extent it concerns the policy decided, including the curriculum, it must consider the ends the policy tries to achieve and is thus closely related to the subject of literacy. Naturally, beyond the curriculum, the policy decided takes in elements of teaching practice, assessment, and modalities for further decision-making.

The introduction and synthesis of the issues in this part is taken up by Nitsa Movshovitz-Hadar; the other contributors are Jonas Emanuelson, Davida Fischman, Azriel Levy, and Zalman Usiskin.

The part opens with a paper by Mogens Niss who was the architect of the competencies framework used in Denmark. Niss makes it particularly clear how broad the subject of policy is, involving not only decision making but also views about the nature of mathematics teaching and learning and even mathematics itself.

Collaboration Between Mathematics and Mathematics Education

This final part contains accounts of genuine instances of collaboration between mathematicians and mathematics educators or of scholars who have managed to work in both fields. These instances serve as existence proofs for the possibility of collaboration, but not uniqueness proofs. There may be different kinds of models for joint work between mathematicians and mathematics educators.

The introduction and synthesis is written by Pat Thompson; the other contributions are by Michèle Artigue, Ehud de Shalit, and Günter Törner.

The part opens with an account of collaboration written by Hyman Bass and Deborah Ball. They themselves are a superb example of the kind of collaboration that is possible.

One Final Word

Since the symposium from which this book emerged was held in the Negev, it is fitting keep in mind a Bedouin custom. When one comes to a Bedouin tent, the host offers coffee. It is always very strong, almost bitter. Then there is talk and food and talk. Finally, tea is served. It is always very sweet. The meaning of this, at least according to one account, is that a visit begins with a little unease, a little uncertainty—thus the bitter coffee. But after conversation, turning things over and exchanging thoughts, the visit ends sweet.

In a way, this is an image of how we hope readers of this book move from the beginning to the end. It is, we think, a hopeful book. So while the expectations discussed in the second part focus some of the uneasiness and friction existing between mathematics education and mathematics, the final part, *Collaboration*, shows signs that cooperation is possible. In between, many issues and questions are raised. These are not resolved completely, even at the end. However, the instances of cooperation and collaboration described in *Collaboration* and, perhaps more trenchantly, the very fact mentioned above that mathematicians and mathematics educators participated in the writing of this book show there is no inevitability in the growing distance between our two communities and that together we can work out these questions which are of mutual concern.

References

Althusser, L. (1971). *Lenin and philosophy and other essays*. New York: Monthly Review Press (trans: B. Brewster).

Begle, E. G. (1972). *Teacher knowledge and student achievement in algebra*. School Mathematics study group reports, number 9.

Boaler, J. (2012). When academic disagreement becomes harassment and persecution. http://www.stanford.edu/~joboaler/. Accessed 10 Dec 2012.

Comte, A. (1975a). In *Auguste Comte and positivism: the essential writings*. G. Lenzer (Ed.). New York: Harper & Row.

Comte, A. (1975b). *Physique social: cours de philosophie positive, leçons 46 à 60*. Paris: Hermann. Presentation and notes by Jean-Paul Enthoven.

Corry, L. (1989). Linearity and reflexivity in the growth of mathematical knowledge. *Science in Context*, *3*, 409–440.

Dilthey, W. (1989). In R. A. Makkreel & F. Rodi (Eds.). *Selected works: Vol. I. Introduction to the human sciences*. Princeton: Princeton University Press.

Eisenberg, T. (1977). Begle revisited: teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education*, *8*, 216–222.

Eisenberg, T., & Fried, M. N. (2009). Dialogue on mathematics education: two points of views on the state of the art. *ZDM. The International Journal on Mathematics Education*, *41*(1&2), 143–150.

Elkana, Y. (1981). A programmatic attempt at an anthropology of knowledge. In E. Mendelsohn & Y. Elkana (Eds.), *Sciences and cultures: anthropological and historical studies of the sciences* (Vol. 5, pp. 1–68). Dordrecht: Reidel.

Even, R. (2011). The relevance of advanced mathematics to teaching secondary school mathematics. *ZDM. The International Journal on Mathematics Education*, *43*(6&7), 941–950.

Gerth, H. H. & Mills, C. W. (Eds.) (1958). *From Max Weber: essays in sociology*. New York: Oxford University Press.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Princeton: Princeton University Press.

Holton, D. (Ed.) (2002). *The teaching and learning of mathematics at university level: an ICMI study*. New York: Kluwer Academic.

Howe, R. (1999). Knowing and teaching elementary mathematics. *Notices of the American Mathematical Society*, *46*(8), 881–887.

Kilpatrick, J. (1992). History of research in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 3–37). New York: Macmillan Co.

National Science Foundation (2012). Science and engineering indicators. http://www.nsf.gov/statistics/seind12/start.htm. Accessed Dec 2012.

Pascal, B. (1962). *Pensées*. Paris: Seuil.

Pimm, D. (2009). Method, certainty and trust across disciplinary boundaries. *ZDM. The International Journal on Mathematics Education*, *41*(1&2), 155–159.

Presmeg, N. (2009). Mathematics education research embracing arts and sciences. *ZDM. The International Journal on Mathematics Education*, *41*(1&2), 131–142.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–31.

Singh, S. (1997). *Fermat's enigma*. New York: Walker and Company.

Snow, C. P. (1963). *The two cultures: and a second look*. New York: Mentor.

Sriraman, B., Roscoe, M., & English, L. (2010). Politicizing mathematics education: has politics gone too far? Or not far enough? In B. Srirarman & L. English (Eds.), *Theories of mathematics education: seeking new frontiers* (pp. 621–638). New York: Springer.

Steiner, G. (1997). *Errata*. London: Weidenfeld & Nicolson.

Strauss, L. (1953). *Natural right and history*. Chicago: University of Chicago Press.

Taylor, H., & Taylor, L. (1993). *George Pólya: master of discovery*. Palo Alto: Dale Seymour Publications.

Ulam, S. (1976). *Adventures of a mathematician*. New York: Scribner's.

US Department of Education (2002a). Comprehensive school reform program overview. http://www2.ed.gov/programs/compreform/csrdoverview/edlite-sld011.html. Accessed 24 Dec 2012.

US Department of Education (2002b). Scientifically based research. http://www2.ed.gov/nclb/methods/whatworks/research/page_pg6.html. Accessed 24 Dec 2012.

Weber, M. (1949). In E. A. Shils & H. A. Finch (Eds.), *The methodology of the social sciences*, Glencoe: Free Press (trans: Shils, E. A., & Finch, H. A.).

Windelband, W. (1894/1980). *Rectorial address, Strasbourg*. Reprinted and translated in *History and Theory*, *19*(2), 169–185.

Wu, H. (2011). The mis-education of mathematics teachers. *Notices of the American Mathematical Society*, *58*(3), 372–384.

Yasukawa, K. (2010). Commentary on politicizing mathematics education: has politics gone too far? Or not far enough? In B. Srirarman & L. English (Eds.), *Theories of mathematics education: seeking new frontiers* (pp. 639–643). New York: Springer.