Commentary on *Probabilistic Thinking: Presenting Plural Perspectives*

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> Martin Gardner's writing is amazingly accurate and reliable. The fact that he made a mistake is simply a testimonial to the difficulty of the [Two Child] problem.
>
> (Khovanova 2011, p. 1)

> But, to my surprise, Erdös said, "No, that is impossible, it should make no difference"... Erdös objected that he still did not understand the reason why, but [after being shown a simulation of the Monty Hall Problem] was reluctantly convinced that I was right.
>
> (Vazsonyi 1999, p. 18)

The above quotations and, more notably, the individuals involved, help cement the popular notion that probability is counterintuitive—just "Ask Marilyn". However, as demonstrated throughout this volume, counterintuitiveness is but one of many different characteristics of probabilistic thinking.

Those of you familiar with research investigating probabilistic thinking in the field of mathematics education, might, at this point in the book, be expecting a "wish list" for future research, which has become customary (e.g., Kapadia and Borovcnik 1991; Jones et al. 2007; Shaughnessy 1992); however, we will not be adding to the list of wish lists. Instead, we have decided to, in this commentary, highlight some of the overarching themes that have emerged from the significant amount of research housed in this volume. Themes emerging from each of the four main perspectives—Mathematics and Philosophy, Psychology, Stochastics and Mathematics Education—are now commented on in turn.

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1 Perspective I: Mathematics and Philosophy

There are three themes emerging from Perspective I: Mathematics and Philosophy that we wish to highlight. The themes are: different philosophical interpretations of probability, modeling, and subjective probability.

The classical, frequentist, and subjective interpretations of probability, which we denote *The Big Three*, are addressed (to varying degrees) in nearly every chapter and in every perspective of this volume. Worthy of note, and not something that we would consider censoring with our editorial hand, this volume continues to see different researchers utilizing different terminology when referring to The Big Three (see, for example, Borovcnik and Kapadia, this volume). Although not censored, issues inherent in the use of different terminology are discussed in this volume (see, for example, Chernoff and Russell). While the terminology associated with The Big Three has not been solidified in this volume, many authors, within their chapters, have clearly defined their philosophical positions relative to The Big Three. Other authors in this volume, (perhaps) heeding the repeated calls for unification found in the mathematics education literature, utilized (all of) The Big Three interpretations of probability in their research (e.g., Borovcnik and Kapadia; Eichler and Vogel; Pfannkuch and Ziedins). By embracing multiple interpretations of probability, this research further reveals the next major theme that emerged from the first perspective.

Beyond, but related to The Big Three (philosophical interpretations of probability), a second theme that emerged from the first perspective was modeling. As stated by Borovcnik and Kapadia (this volume): "Philosophical difficulties have been prevalent in probability since its inception, especially since the idea requires modeling—probability is not an inherent property of an event, but based on the underlying model chosen" (p. 7). Modeling was found to varying degrees in numerous chapters (e.g., Batanero et al.): some with a heavier emphasis on technology and simulation than others (e.g., Eichler and Vogel; Pfannkuch and Ziedins, Prodromou; Lesh); and some making finer distinctions between reality (Borovcnik and Kapadia; Pfannkuch and Ziedins), virtual reality (Eichler and Vogel), or some sanitized version of reality (Greer; Sriraman and Lee). Although the unification of The Big Three may be achieved through modeling, issues associated with subjective probability still persist: the final theme we will comment on from Perspective I.

As evidenced throughout this volume, and unlike the classical and frequentist interpretations of probability, research in mathematics education has yet to adequately define the subjective interpretation of probability. However, Sharma (this volume) has put forth a call, echoed by Watson (this volume), for research investigating subjective probability. We further echo this call, but do so under the caveats presented by Chernoff and Russell (this volume). Addressing the issues surrounding subjective probability may help the field to move beyond The Big Three resulting in new research embracing "new" interpretations of probability (e.g., propensity—mentioned by Borovcnik and Kapadia, this volume). Like Perspective I, different interpretations of probability also underpin themes emerging from Perspective II.

2 Perspective II: Psychology

Numerous themes emerge from the second perspective of this volume. Of these themes, we have selected two that will now be commented on in turn: the research of Daniel Kahneman and Amos Tversky (and colleagues); and the research of Gerd Gigerenzer (and colleagues).

The original heuristics and biases program of Daniel Kahneman and Amos Tversky (e.g., Kahneman et al. 1982) is seminal to those investigating probabilistic thinking in the field of mathematics education. However, Chernoff (2012), while acknowledging exceptions (e.g., Leron and Hazzan 2006, 2009; Tzur 2011), argues that the mathematics education community has largely ignored more recent developments from the field of cognitive psychology. As a result, developments associated with the original heuristics and biases program (e.g., Gilovich et al. 2002; Kahneman 2011) are not found in mathematics education literature investigating probabilistic thinking. In particular, Chernoff highlights an "arrested development of the representativeness heuristic" (p. 952) in the field of mathematics education, which, if not for certain chapters in this volume (Chiesi and Primi; Ejersbo and Leron; Savard) and the previously mentioned exceptions, may have been extended to heuristics, in general, in mathematics education. Other chapters in this perspective, considering a different notion of heuristics (e.g., Meder and Gigerenzer, this volume), further thwart continuation of this arrested development of heuristics.

The question of whether man is an intuitive statistician, central to the research of Gerd Gigerenzer and colleagues (and, for that matter, Kahneman and Tversky), is found in numerous chapters throughout this volume (e.g., Abrahamson; Brase, Martinie and Castillo-Grouw; Martignon; Meder and Gigerenzer; Saldanha and Liu; Van Dooren). Prior to this volume, the research of Gerd Gigerenzer and colleagues has (inexplicably) been largely ignored by the mathematics education community. We say "inexplicably" because of the two main topics central to the research of Gigerenzer and colleagues: heuristics and risk. Research found in this volume (e.g., Meder and Gigerenzer; Martignon), potentially, signals the dawn of a new era of research for those investigating probabilistic thinking in the field of mathematics education. Such research may not only shape the Assimilation Period, but may, as was the case with Kahneman and Tversky's research during the Post-Piagetian Period, define the Assimilation Period.

Based on the above, we contend that the next period of research, the Assimilation Period, may be defined as a renaissance period for psychological research in mathematics education. If not, then, at the very least, it is essential "for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard et al. 2009, p. 13). As a caveat, we would add that such recognition and incorporation of achievements could be bidirectional.

3 Perspective III: Stochastics

Within Perspective III, we have chosen two themes to highlight: randomness and statistics. However, as discussed in the Preface to this perspective, it would be more accurate to declare the two themes of this perspective to be: perceived randomness and statistics.

Given the close connection between randomness and probability, one might expect a similar volume of research for the two topics. This, however, is not the case. The volume of research into probabilistic thinking, at least in the field of mathematics education, is greater than the volume of research investigating randomness and perceptions of randomness. However, this perspective houses some unique investigations into (perceptions of) randomness (Batanero et al.; Jolfaee et al.; Saldanha and Liu). These studies not only inform the existing research literature related to this topic, but, further, forge new threads of research by combining topics in probability (e.g., randomness) with other areas of research in mathematics education (e.g., gesture). Ultimately, these investigations may stimulate further research on (perceptions of) randomness and, thus, increase the volume of literature on this topic.

Statistics and probability are inextricably linked; yet, their relationship, in the field of mathematics education, is uncertain. In particular instances, major research syntheses combine probability and statistics research (e.g., Shaughnessy 1992) while, in other instances, the two areas are kept separate (e.g., Jones et al. 2007). What is certain, however, is that statistics is a major area of research in mathematics education. Indicators include, for example, conferences and journals solely dedicated to statistics education. However, the same is not true for probability—there is no [*Probability*] *Education Research Journal* or *Journal of* [*Probability*] *Education*. Additional indicators of statistics being a major area of research include the field's influence on other areas of research in mathematics education (e.g., simulation, inference, and modeling). In relation to Perspective III, such an influence upon research into probabilistic thinking is witnessed in this volume (e.g., English; Prodromou; Roth). Statistics education, then, provides an example of the potential for research investigating probabilistic thinking to become, one day, a major area of research in mathematics education (i.e., mainstream), which will influence related areas of research in mathematics education (e.g., statistics and others) and beyond.

4 Perspective IV: Mathematics Education

In considering the themes emerging from the mathematics education perspective, the last of the four main perspectives, we have chosen to comment on three. These themes are: the teaching and learning of topics in probability (and mathematics), areas of research in mathematics education, and methods used in mathematics education research.

In mathematics education, certain research investigating probabilistic thinking focuses on the teaching and learning of particular topics in (and associated with) probability. More specifically, some of the research (in this volume) investigates the teaching and learning of particular topics in probability; for example, the binomial formula (Sanchez and Landin), sample space (Paparistedomou), conditional probability (Huerta), the law of large numbers (Prediger and Schnell), counting methods (Maher and Ahluwalia), and chance (Nilsson). Research into the specific topics mentioned not only contributes to existing research for said particular topics, but, further, to existing research into probabilistic thinking, in general.

The second of the themes emerging from the mathematics education perspective was situating research into probabilistic thinking within other areas of research in mathematics education. Examples commented on and included in this volume are: the affective domain, that is, attitudes and beliefs (Nisbett and Williams), teachers' probabilistic knowledge (Mamolo and Zazkis; Eichler and Vogel; Batanero et al.), representation(s) (Maher and Ahluwalia; Sanchez and Landin), transfer (Watson) and cultural investigations (Sharma). Similar to the first theme discussed, research housed in this volume not only contributes to specific areas of research in mathematics education, but, further, to existing research into probabilistic thinking, in general.

Lastly (in terms of themes commented on for this perspective), methods used in mathematics education research were also represented in this volume. Methods, included and discussed to varying degrees, include: teaching experiments (Nilsson; Nisbett and Williams), teaching/learning arrangements (Prediger and Schnell), longitudinal studies (Maher and Ahluwalia; Prediger and Schnell), frameworks (Sanchez and Landin; Mooney, Langrall and Hertel), and simulation (Nisbett and Williams; Paparistedomou). Similar to the previous two themes, said research not only contributes, specifically, to the methods above, but, generally, to research investigating probabilistic thinking.

5 Final Perspectives

As we have detailed above, a number of different overarching themes have emerged from each of the four main perspectives in this volume. If, however, we had to declare one theme that ran through this entire volume, it would have to be the humanistic tradition in mathematics education (see Sriraman and Lee, this volume). From Borovcnik and Kapadia's historical account of probability and their detailing of puzzles and paradoxes to the historical treatment of randomness (Batanero et al.; Mooney, Langrall and Hertel; Saldanha and Liu) to individual notions and perceptions of randomness (Batanero et al.; Simin, Zazkis and Sinclair) to the question of whether man is an intuitive statistician (see, for example, Perspective II) to cultural influences (Sharma) elements of the humanistic tradition run deep in this volume.

Concluding our commentary on *Probabilistic Thinking: Presenting Plural Perspectives* in the forward looking spirit of the *Advances in Mathematics Education*

Series, we wish to make a few final comments regarding what we referred to as the movement of research investigating probabilistic thinking into mainstream mathematics education, which we have further recognized as a move from the Contemporary Research Period to the Assimilation Period.

Until now, research into probabilistic thinking in mathematics education has drawn heavily on research from other domains. As but one example, such research has relied substantially on the field of psychology and, in many ways, provided the foundation of this field. Consequently, research into probabilistic thinking in mathematics education has not, necessarily, been well situated within mathematics education literature—it has depended on the research from other fields. To illustrate our point, we presented a list of terms that we argued would, one day, be evaluated for their liberal usage, that is, used without the traditional academic scrutiny that would be found in other areas of research in mathematics education. The list of terms was long and included: beliefs, cognition, conceptions (and misconceptions), heuristics, intuition, knowledge, learning, modeling, reasoning, risk, stochastics, subjective probability, teaching, theory (or theories), thinking, and understanding. As evidenced in this volume, we are at the beginning of the end. For example, we consider the chapters from Nisbett and Williams and Sharma as exemplars of this point. Not only are these chapters well situated in the pertinent research literature, but, further, they are well situated in the research on attitudes and beliefs and culture (respectively) found and flourishing in the field of mathematics education.

We await the day when a two-way street is finally opened, that is, research into probabilistic thinking in the field of mathematics education is used to inform other major fields of research (e.g., psychology or philosophy). We consider the chapters of Abrahamson, Bennett, and Simin et al. as exemplars of research that could, potentially, have an impact on other fields of research.

Should the day come, when research investigating probabilistic thinking in the field of mathematics education explicitly influences other areas of research, we wish to note (at this point in time) that we will have moved beyond what we have called the Assimilation Period to whatever the "next" period will be called.

We hope that you enjoyed *Probabilistic Thinking: Presenting Plural Perspectives*. We further hope that each time you delve back into this volume you find another perspective, which will present probabilistic thinking in an ever greater context.

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