Equivalent Beliefs in Dynamic Doxastic Logic

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Abstract Two propositions may be regarded as doxastically equivalent if revision of an agent's beliefs to adopt either has the same effect on the agent's belief state. We enrich the language of dynamic doxastic logic with formulas expressing this notion of equivalence, and provide it with a formal semantics. A finitary proof system is then defined and shown to be sound and complete for this semantics.

1 Introduction

When should two propositions be regarded as equivalent as adopted beliefs? In a theory of belief revision, we will understand this notion of *doxastic equivalence* as follows: ϕ and ψ are equivalent if revision of an agent's set of beliefs to include ϕ has exactly the same effect as revision of that belief set to include ψ. Our interest is in exploring formal logics that represent this notion in their object language, by allowing formation of formulas with syntax $\phi \bowtie \psi$, expressing 'ϕ is doxastically equivalent to ψ'.[1]

But what should we understand by 'has exactly the same effect'? We answer that in the context of the approach to *dynamic doxastic logic* (DDL) for belief revision that has been developed by Krister Segerberg in [11–14] and other papers.[2] This uses multi-modal logics that are designed to formalise reasoning about the beliefs of an agent. These logics have normal modalities of the form $[*\phi]$, generating formulas of type $[*\phi]\theta$ that can be read 'after revision of the agent's beliefs by ϕ, it must be

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[1] $\phi \bowtie \psi$ could conveniently be pronounced 'ϕ tie ψ', from the LaTeX control word `\bowtie` for the symbol $\bowtie$.

[2] An introduction to the modal logic of belief revision appears in [10], and a review of DDL in [9].

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the case that θ'. The dual formula $\langle *\phi\rangle\theta$ can be read, 'after revision by ϕ, it may be that θ'. There are also normal modalities **B** for belief and **K** for 'commitment'. $\mathbf{B}\theta$ expresses that the agent believes θ, while $\mathbf{K}\theta$ asserts that θ is a 'hard-core' belief that the agent is committed to and is not prepared to revise. We follow the syntax of [14] in allowing $\mathbf{B}\theta$ and $\mathbf{K}\theta$ to be well-formed only when θ is a pure Boolean formula, whereas $[*\phi]\theta$ is well-formed when ϕ is pure Boolean and θ is *any* formula, one that may contain (iterations of) modalities.

A typical Kripkean relation semantics for the modalities $[*\phi]$ would assign to each a binary relation $R[\![\phi]\!]$ on a set S. The members of S may be thought of as *belief states* of an agent. Intuitively, a pair (s, t) belongs to $R[\![\phi]\!]$ if the agent may enter belief state t from state s after revising their beliefs to adopt ϕ. There may be more than one such accessible 'result state' compatible with revision by ϕ, so in general $R[\![\phi]\!]$ is a relation that is not a function.

We use such doxastic accessibility relations to interpret the equivalence formulas $\phi \bowtie \psi$, by declaring such a formula to be true in state s precisely when

$$\text{for all belief states t,} \quad (s, t) \in R[\![\phi]\!] \text{ iff } (s, t) \in R[\![\psi]\!]. \tag{1}$$

So $\phi \bowtie \psi$ asserts that revision by ϕ or by ψ leads to exactly the same alternative belief states. This is a *local* notion of equivalence, in that it is tied to a particular initial state s. Global equivalence would mean that $\phi \bowtie \psi$ is true at all states, which amounts to having $R[\![\phi]\!] = R[\![\psi]\!]$.

This chapter shows that there is a finitary axiomatisation of the systems produced by adding $\bowtie$ with the above interpretation to certain dynamic doxastic logics. The postulates for $\bowtie$ that are required are the axiom

$$\phi \bowtie \psi \to ([*\phi]\theta \leftrightarrow [*\psi]\theta);$$

and the inference rule

$$\frac{[*\phi]p \leftrightarrow [*\psi]p}{\phi \bowtie \psi}, \quad \text{if the variable p does not occur in } \phi \text{ or } \psi;$$

along with variants of this rule in which its premiss and conclusion are embedded in other formulas (see Fig. 1 in Sect. 3).

The models of [14] have, in addition to S and R, a set U whose members are thought of as possible worlds, about which the agent may hold beliefs. Certain subsets of U are designated as being *propositions*. Each member of S is a 'selection function', a type of function that assigns to each proposition P a 'theory' representing the set of propositions that the agent comes to believe after revising their beliefs to include P. A selection function can be thought of as embodying the agent's overall disposition to respond to new information. A model has a truth relation $f, u \models \theta$, specifying when θ is true at a pair consisting of a selection function f and a world $u \in U$. A pure Boolean formula ϕ defines a proposition $[\![\phi]\!] \subseteq U$, with $f, u \models \phi$ iff $u \in [\![\phi]\!]$.

Axioms

(TF)	all truth-functional tautologies
($\Box$)	$\Box(\theta \to \omega) \to (\Box\theta \to \Box\omega)$, for each modality $\Box \in \{\mathbf{B}, \mathbf{K}, [*\phi]\}$.
(KB)	$\mathbf{K}\phi \to \mathbf{B}\phi$
($*2$)	$[*\phi]\mathbf{B}\phi$
($*3$)	$[*\top]\mathbf{B}\phi \to \mathbf{B}\phi$
($*4$)	$\mathbf{b}\top \to (\mathbf{B}\phi \to [*\top]\mathbf{B}\phi)$
($*5$)	$\mathbf{b}\top \to (\mathbf{k}\phi \to \langle *\phi\rangle\mathbf{b}\top)$
($*6$)	$\mathbf{K}(\phi \leftrightarrow \psi) \to ([*\phi]\mathbf{B}\chi \leftrightarrow [*\psi]\mathbf{B}\chi)$
($*7$)	$[*(\phi \wedge \psi)]\mathbf{B}\chi \to [*\phi]\mathbf{B}(\psi \to \chi)$
($*8$)	$\langle *\phi\rangle\mathbf{b}\psi \to ([*\phi]\mathbf{B}(\psi \to \chi) \to [*(\phi \wedge \psi)]\mathbf{B}\chi)$
($*$FB)	$\langle *\phi\rangle\mathbf{B}\psi \to [*\phi]\mathbf{B}\psi$
(K$*$)	$\mathbf{K}\psi \to [*\phi]\mathbf{K}\psi$
(BK)	$\mathbf{B}\bot \to \mathbf{K}\bot$
($\bowtie$)	$\phi \bowtie \psi \to ([*\phi]\theta \leftrightarrow [*\psi]\theta)$

Rules

(MP) $\dfrac{\theta, \theta \to \omega}{\omega}$

($\Box$N) $\dfrac{\theta}{\Box\theta}$, for $\Box \in \{\mathbf{B}, \mathbf{K}, [*\phi]\}$ ($\Box$-NECESSITATION)

(CR) $\dfrac{\phi \leftrightarrow \psi}{[*\phi]\theta \leftrightarrow [*\psi]\theta}$ (CONGRUENCE RULE)

($\bowtie$R) $\dfrac{\rho([*\phi]p \leftrightarrow [*\psi]p)}{\rho(\phi \bowtie \psi)}$, if the variable p does not occur in ϕ, ψ or the template ρ.

Fig. 1 Axioms and inference rules

Here we take a more abstract approach in which S can be any set, with each member $s \in S$ being assigned a selection function f^s, allowing the possibility that distinct members of S (belief states) are assigned the same selection function. Thus we may have $s \neq t$ but $f^s = f^t$. This provides flexibility in constructing models, and we will produce (in Sect. 3) a series of small examples that effectively differentiate a number of logics and their properties. Our models also have a function $\mathsf{ws} : S \to U$, with $\mathsf{ws}(s)$ being thought of as the *world state* corresponding to the belief state s. This makes it possible to introduce a simpler truth relation $s \models \theta$, specifying when θ is true at belief state s. For pure Boolean ϕ we have $s \models \phi$ iff $\mathsf{ws}(s) \in [\![\phi]\!]$.

The minimal logic we study, which we call L_{K}, is characterised by models in which ws is surjective, i.e. the image of ws is the whole of U (every possible world is the world state of some belief state). The $\bowtie$-free fragments of logics in general have a canonical model in which ws is surjective. But this condition is stronger than is strictly needed: it suffices that the image of ws is topologically dense in U, under the topology generated by the propositions. To provide every logic having $\bowtie$ with a characteristic canonical model we need to admit models having only this weaker topological condition.

The axiomatisation of the minimal logic L_K is in some respects weaker than that of [14]. We have left out the axioms

$$
\begin{array}{ll}
(*\mathrm{D}) & [*\phi]\theta \to \langle *\phi\rangle\theta \\
(*\mathrm{X}) & \psi \leftrightarrow [*\phi]\psi \\
(*\mathrm{K}) & \mathbf{K}\psi \leftrightarrow [*\phi]\mathbf{K}\psi.
\end{array}
$$

These can be consistently added to L_K (even simultaneously). But each of them is inconsistent with the formula $\neg\mathbf{B}\bot$ which holds of the belief state of a *rational* agent, one who does not believe a contradiction. Moreover, any logic containing (∗K) allows a direct derivation of $\mathbf{B}\bot$, somewhat limiting its interest. Our logic L_K does not include $\neg\mathbf{B}\bot$, but it can be consistently added.

On the other hand, we include the axiom

(K∗) $\mathbf{K}\psi \to [*\phi]\mathbf{K}\psi$

that weakens (∗K), and which appears to capture the essence of a hard-core belief as being one that cannot be revised. We also show that the scheme $\psi \to [*\phi]\psi$ can be consistently added, resulting in logics characterised by models satisfying

$$(s,t) \in R[\![\phi]\!] \quad \text{implies} \quad \mathsf{ws}(s) = \mathsf{ws}(t).$$

Moreover, $\psi \to [*\phi]\psi$ is consistent with $\neg\mathbf{B}\bot$.

The scheme (∗D) is equivalent to $\langle *\phi\rangle\top$. The obstacle to its inclusion is that any logic containing $\neg\mathbf{B}\bot$ must have $\neg\langle *\bot\rangle\top$ as a theorem. But we can use the equivalence connective $\bowtie$ to exclude contradictory formulas, and consider the weaker scheme

$$\neg(\phi \bowtie \bot) \to \langle *\phi\rangle\top. \tag{2}$$

This makes the plausible assertion about rational belief that revision by ϕ is possible provided that ϕ is not equivalent to a contradiction. The logic obtained by adding $\neg\mathbf{B}\bot$ and (2) to L_K is consistent. This is explained in Sect. 7, where we deal with all these issues about axiomatisation.

It is worth noting that the $\bowtie$ concept is not special to DDL. It could be added to any multi-modal logic. Given an indexed set $\{[\alpha] : \alpha \in I\}$ of normal modalities, interpreted by a set $\{R[\![\alpha]\!] : \alpha \in I\}$ of binary relations, we can extend the language by adding formulas $\alpha \bowtie \beta$ for all $\alpha, \beta \in I$, and define $\alpha \bowtie \beta$ to be true at a point s iff

$$\text{for all } t, \quad (s,t) \in R[\![\alpha]\!] \text{ iff } (s,t) \in R[\![\beta]\!].$$

A significant example is *dynamic program logic* [7], where I is a set of programs, and $R[\![\alpha]\!]$ is thought of as the set of input/output pairs of states of program α. Then $\alpha \bowtie \beta$ expresses the natural notion of equivalence of programs as meaning that execution of either program in a given input state induces the same possible output states.

It turns out that in that computational context, $\bowtie$ is a very powerful notion. Elsewhere [6] we have shown that addition of $\bowtie$ to the basic program logic PDL produces a system whose set of valid formulas is not recursively enumerable, and so cannot have a finitary axiomatisation. In fact this holds for any variant of PDL whose class of programs is closed under compositions of programs and formation of WHILE- DO commands. But for DDL, where the modalities $[*\phi]$ are indexed by the rather simpler class of Boolean propositional formulas, a finitary axiomatisation of logics with $\bowtie$ is possible, as we now proceed to show.

2 Syntax and Semantics

This section sets out the formal language and semantics that we use. A good deal of the notation and terminology is adapted from [14].

We take as given some denumerable set of propositional variables, for which the letters p, q are used. From these, *pure Boolean formulas* are constructed by the Boolean connectives, say taking $\to$ and the constant $\bot$ (Falsum) as primitive, and introducing $\wedge$, $\vee$, $\neg$, $\leftrightarrow$ by the usual abbreviations. The letters ϕ, ψ, χ will always denote pure Boolean formulas.

We use θ and ω for arbitrary formulas. These are generated from the propositional variables by the Boolean connectives and the specifications:

- If ϕ is pure Boolean and θ is any formula, then $\mathbf{B}\phi$, $\mathbf{K}\phi$ and $[*\phi]\theta$ are formulas.
- If ϕ and ψ are pure Boolean, then $\phi \bowtie \psi$ is a formula.

Further abbreviations are introduced by writing $\top$ for $\neg\bot$, $\mathbf{b}\phi$ for $\neg\mathbf{B}\neg\phi$, $\mathbf{k}\phi$ for $\neg\mathbf{K}\neg\phi$, and $\langle*\phi\rangle\theta$ for $\neg[*\phi]\neg\theta$.

A *Boolean structure* $(U, Prop)$ comprises a set U and a non-empty collection *Prop* of subsets of U that is closed under binary intersections $X \cap Y$ and complements $-X$, hence under binary unions $X \cup Y$ and Boolean implications $(-X) \cup Y$. So *Prop* is a Boolean subalgebra of the powerset algebra of U. The members of *Prop* are called the *propositions* of the structure. This is in accord with the view of U as a set of possible worlds, with a proposition being identified with the set of worlds in which it is true.[3] Members of U may be thought of as different possible states of the world about which an agent may hold various beliefs.

We make use of the topology on U generated by *Prop*. Since $U \in Prop$ and *Prop* is closed under binary intersections, *Prop* is a *base* for this topology, so every open subset of U is a union of propositions. Since *Prop* is closed under complements, every proposition is also closed, and every closed subset of U is an intersection of propositions. Hence a closed set can be viewed as representing a *theory*, in the sense of a set of propositions, i.e. the theory is identified with the set of worlds in which all of its propositions are true.

[3] The members of *Prop* are sometimes called the *admissible* propositions of the structure, to distinguish them from other subsets of U. See [5].

The topological *closure* of a set $X \subseteq U$ will be denoted $\mathbf{C}X$. This is the intersection of all closed supersets of X, and hence the smallest closed superset.

A *valuation* on a Boolean structure is a map $p \mapsto [\![p]\!]$ assigning to each propositional variable p a proposition $[\![p]\!] \in Prop$. This extends inductively to assign a proposition $[\![\phi]\!]$ to each pure Boolean ϕ in the usual way:

$$[\![\bot]\!] = \emptyset, \ [\![\phi \to \psi]\!] = (U - [\![\phi]\!]) \cup [\![\psi]\!].$$

Hence $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$, $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$, and $[\![\top]\!] = U$.

The belief state of an agent is embodied, not just in their current set of beliefs, but also in their *doxastic dispositions*: how they would respond to new information [11], p. 288. These may be represented by the function assigning to each proposition P the theory representing all propositions that the agent comes to believe after revising their beliefs to include P. Formally, dispositions are modelled by a *selection function* in a Boolean structure $(U, Prop)$, which is a function f assigning to each proposition $P \in Prop$ a closed set (theory) fP, such that for all propositions P and Q:

- $fP \subseteq P$. (INCL)
- if $P \subseteq Q$ and $fP \neq \emptyset$, then $fQ \neq \emptyset$. (MONEYS)
- if $P \subseteq Q$ and $P \cap fQ \neq \emptyset$, then $f(P \cap Q) = P \cap fQ$. (ARROW)[4]

The current belief set of the belief state represented by selection function f may be identified with fU, which corresponds to the set of propositions that the agent believes after revision by the tautologous $\top$.

Every selection function satisfies the stronger condition

- $P \cap fQ \neq \emptyset$ implies $f(P \cap Q) = P \cap fQ$. (STRONG ARROW)

This follows readily by replacing P by $P \cap Q$ in (ARROW) and using (INCL).

A selection function f will be called *null* if $fP = \emptyset$ for all propositions P. By (MONEYS), f is null iff $fU = \emptyset$. Note that every selection function has $f\emptyset = \emptyset$, by (INCL).

The *commitment set* of a selection function f is defined to be

$$\mathscr{C}f = \bigcup\{fP : P \in Prop\}.$$

We now introduce structures of the form

$$\mathfrak{F} = (U, Prop, S, \mathsf{ws}, \mathsf{sf}, R),$$

with $(U, Prop)$ a Boolean structure; S a set; ws a function from S to U; sf a function assigning to each member s of S a selection function $\mathsf{sf}(s)$ on $(U, Prop)$; and R a function assigning to each proposition $P \in Prop$ a binary relation $R(P)$ on S, i.e. $R(P) \subseteq S \times S$.

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[4] INCL stands for 'inclusion', MONEYS for 'monotonicity for nonempty segments', and ARROW is named in honour of Kenneth Arrow. See [14], p. 232.

U may be thought of as a set of possible worlds, as above, and S as the set of possible belief states of an agent. $\mathsf{ws}(s)$ is the world state associated with belief state s. $\mathsf{sf}(s)$ is the selection function representing the agent's doxastic dispositions in belief state s. $R(P)$ is the accessibility relation representing possible changes of belief state resulting from revision to include the belief P.

The selection function $\mathsf{sf}(s)$ will usually be denoted f^s. The structure $\mathfrak{F}$ is called a *(selection) frame* if it satisfies the following conditions:

(F1) if $(s, t) \in R(P)$, then $f^t U = f^s P$.
(F2) if $(s, t) \in R(P)$, then $\mathscr{C} f^t \subseteq \mathbf{C}(\mathscr{C} f^s)$.
(F3) If $f^s P \neq \emptyset$, then there exists $t \in S$ with $(s, t) \in R(P)$.
(F4) The image $\mathsf{ws}(S)$ of the function ws is dense in U, i.e. $\mathbf{C}(\mathsf{ws}(S)) = U$.

Referring to the axioms and inference rules of Fig. 1 in Sect. 3, the frame conditions (F1) and (F3) will play a role in the soundness of several of them, particularly via Lemma 3(3), whose proof uses these conditions. (F2) on the other hand has a specific purpose: the soundness of axiom (K$*$).

(F3) is a weakening of the requirement that $R(P)$ be *serial*, which itself means that for all $s \in S$ there exists $t \in S$ with $(s, t) \in R(P)$.

Note that if ws is *surjective*, i.e. each $u \in U$ is $\mathsf{ws}(s)$ for some s, then (F4) holds. Surjectivity requires that each $u \in U$ belongs to the image-set $\mathsf{ws}(S)$. (F4) is a weakening of this to require only that each u be 'close to' $\mathsf{ws}(S)$, i.e. every open neighbourhood of u intersects $\mathsf{ws}(S)$.

We may call a frame *world-surjective* if its ws-function is surjective. Eventually we will see that the minimal logic we study, and the $\bowtie$-free fragments of all logics, are characterised by models on world-surjective frames. For now we focus on the weaker (F4) itself, and its role in a frame, which is contained in the following result.

Lemma 1 *Let P and Q be any propositions in a frame, such that for all $s \in S$, $\mathsf{ws}(s) \in P$ iff $\mathsf{ws}(s) \in Q$. Then $P = Q$.*

Proof Assume that $\mathsf{ws}(s) \in P$ iff $\mathsf{ws}(s) \in Q$ for all $s \in S$.

Now density of $\mathsf{ws}(S)$ as in (F4) is equivalent to the property that every non-empty open set intersects $\mathsf{ws}(S)$. So if $P - Q \neq \emptyset$, then since $P - Q$ is a proposition and therefore open, it must intersect $\mathsf{ws}(S)$, giving an s such that $\mathsf{ws}(s) \in P$ and $\mathsf{ws}(s) \notin Q$, contrary to assumption. Thus $P - Q = \emptyset$. Likewise $Q - P = \emptyset$, so we must have $P = Q$. $\dashv$

For the use of Lemma 1, and hence the need for (F4), see Lemma 3(2) below and its proof. Ultimately, (F4) is required to ensure the soundness of the Congruence Rule (CR) of Fig. 1.

A *(selection) model* $\mathfrak{M} = (\mathfrak{F}, [\![-]\!])$ on a frame $\mathfrak{F}$ is given by a valuation $[\![-]\!]$ on the Boolean structure of $\mathfrak{F}$. If $s \in S$, the relation 'θ is true at s in $\mathfrak{M}$', written $\mathfrak{M}, s \models \theta$, is defined by induction on the formation of the formula θ, as follows:

$\mathfrak{M}, s \models p$ iff $\mathsf{ws}(s) \in [\![p]\!]$, if p is a propositional variable.
$\mathfrak{M}, s \not\models \bot$, (i.e. not $\mathfrak{M}, s \models \bot$).
$\mathfrak{M}, s \models \theta \to \theta'$ iff $\mathfrak{M}, s \models \theta$ implies $\mathfrak{M}, s \models \theta'$.
$\mathfrak{M}, s \models \mathbf{B}\phi$ iff $f^s(U) \subseteq [\![\phi]\!]$.
$\mathfrak{M}, s \models \mathbf{K}\phi$ iff $\mathscr{C} f^s \subseteq [\![\phi]\!]$.
$\mathfrak{M}, s \models [*\phi]\theta$ iff for all t such that $(s, t) \in R[\![\phi]\!]$, $\mathfrak{M}, t \models \theta$.
$\mathfrak{M}, s \models \phi \bowtie \psi$ iff for all $t \in S$, $(s, t) \in R[\![\phi]\!]$ iff $(s, t) \in R[\![\psi]\!]$.

Writing $R^s[\![\phi]\!]$ for the set $\{t \in S : (s, t) \in R[\![\phi]\!]\}$, the semantics of $\bowtie$ can be given as

$$\mathfrak{M}, s \models \phi \bowtie \psi \quad \text{iff} \quad R^s[\![\phi]\!] = R^s[\![\psi]\!].$$

A formula θ is *true in model* $\mathfrak{M}$, written $\mathfrak{M} \models \theta$, if $\mathfrak{M}, s \models \theta$ for all $s \in S$. θ is *valid at* s *in frame* $\mathfrak{F}$, written $\mathfrak{F}, s \models \theta$, if $\mathfrak{M}, s \models \theta$ for all models $\mathfrak{M}$ on $\mathfrak{F}$. θ is *valid in* $\mathfrak{F}$, written $\mathfrak{F} \models \theta$, if $\mathfrak{F}, s \models \theta$, for all $s \in S$; this is equivalent to requiring that θ is true in all models on $\mathfrak{F}$.

A set Σ of formulas is *satisfied at* s *in* $\mathfrak{M}$, written $\mathfrak{M}, s \models \Sigma$, if $\mathfrak{M}, s \models \theta$ for all $\theta \in \Sigma$. Σ *semantically implies* a formula θ in $\mathfrak{M}$, written $\Sigma \models^{\mathfrak{M}} \theta$, if θ is true at every s satisfying Σ, i.e. $\mathfrak{M}, s \models \Sigma$ implies $\mathfrak{M}, s \models \theta$ for all s in $\mathfrak{M}$. Σ semantically implies θ in frame $\mathfrak{F}$, written $\Sigma \models^{\mathfrak{F}} \theta$, if every model $\mathfrak{M}$ on $\mathfrak{F}$ has $\Sigma \models^{\mathfrak{M}} \theta$.

Satisfaction of a formula is determined by the valuations of the variables that occur in the formula, in the following sense.

Lemma 2 *Let θ be any formula. Then for any models $\mathfrak{M} = (\mathfrak{F}, [\![-]\!])$ and $\mathfrak{M}' = (\mathfrak{F}, [\![-]\!]')$, on the same frame, such that $[\![p]\!] = [\![p]\!]'$ for all variables p that occur in θ, we have*

$$\mathfrak{M}, s \models \theta \quad \textit{iff} \quad \mathfrak{M}', s \models \theta$$

for all $s \in S$.

Proof A straightforward induction on the formation of θ. ⊣

The following facts are useful for proving validity of axioms and soundness of rules.

Lemma 3 *In any selection model $\mathfrak{M}$:*

(1) $\mathfrak{M}, s \models \phi$ iff $\mathsf{ws}(s) \in [\![\phi]\!]$.
(2) $\mathfrak{M} \models \phi \leftrightarrow \psi$ iff $[\![\phi]\!] = [\![\psi]\!]$.
(3) $\mathfrak{M}, s \models [*\phi]\mathbf{B}\psi$ iff $f^s[\![\phi]\!] \subseteq [\![\psi]\!]$.
(4) $\mathfrak{M}, s \models \langle *\phi \rangle \mathbf{b}\psi$ iff $(f^s[\![\phi]\!]) \cap [\![\psi]\!] \neq \emptyset$.
(5) $\mathfrak{M}, s \models \mathbf{K}(\phi \leftrightarrow \psi)$ implies $f^s[\![\phi]\!] = f^s[\![\psi]\!]$.

Proof (1) A straightforward induction on the formation of pure Boolean ϕ.

(2) Using (1), $\mathfrak{M} \models \phi \leftrightarrow \psi$ iff for all $s \in S$, $\mathsf{ws}(s) \in [\![\phi]\!]$ iff $\mathsf{ws}(s) \in [\![\psi]\!]$. But this condition implies $[\![\phi]\!] = [\![\psi]\!]$ by Lemma 1, and is evidently implied by $[\![\phi]\!] = [\![\psi]\!]$.

(3) Let $\mathfrak{M}, s \models [*\phi]\mathbf{B}\psi$. If $f^s[\![\phi]\!] = \emptyset$, then immediately $f^s[\![\phi]\!] \subseteq [\![\psi]\!]$. But if $f^s[\![\phi]\!] \neq \emptyset$, then by (F3) there is a t with$(s, t) \in R[\![\phi]\!]$, hence $\mathfrak{M}, t \models \mathbf{B}\psi$. Then by (F1) and the semantics of $\mathbf{B}$, $f^s[\![\phi]\!] = f^t U \subseteq [\![\psi]\!]$. Conversely, let $f^s[\![\phi]\!] \subseteq [\![\psi]\!]$. Then using (F1) again we argue that $(s, t) \in R[\![\phi]\!]$ implies $f^t U = f^s[\![\phi]\!] \subseteq [\![\psi]\!]$, which implies $\mathfrak{M}, t \models \mathbf{B}\psi$. Hence $\mathfrak{M}, s \models [*\phi]\mathbf{B}\psi$.

(4) By (3) and set algebra, because $\mathfrak{M}, s \models \langle *\phi\rangle\mathbf{b}\psi$ iff $\mathfrak{M}, s \not\models [*\phi]\mathbf{B}\neg\psi$.

(5) Let $\mathfrak{M}, s \models \mathbf{K}(\phi \leftrightarrow \psi)$. Then $\mathscr{C} f^s \subseteq [\![\phi \leftrightarrow \psi]\!]$. Take first the case $f^s[\![\phi]\!] \neq \emptyset$. Now

$$f^s[\![\phi]\!] \subseteq \mathscr{C} f^s \subseteq [\![\phi \leftrightarrow \psi]\!] \subseteq [\![\phi \to \psi]\!],$$

so $f^s[\![\phi]\!] \cap [\![\phi]\!] \subseteq [\![\psi]\!]$, which by (INCL) means that $f^s[\![\phi]\!] \subseteq [\![\psi]\!]$. Then $[\![\psi]\!] \cap f^s[\![\phi]\!] = f^s[\![\phi]\!] \neq \emptyset$, hence by (STRONG ARROW),

$$f^s([\![\psi]\!] \cap [\![\phi]\!]) = [\![\psi]\!] \cap f^s[\![\phi]\!] = f^s[\![\phi]\!] \neq \emptyset.$$

Therefore $f^s[\![\psi]\!] \neq \emptyset$ by (MONEYS).

Summing up so far: from $f^s[\![\phi]\!] \neq \emptyset$ we deduced that $f^s[\![\phi]\!] = f^s([\![\psi]\!] \cap [\![\phi]\!])$ and $f^s[\![\psi]\!] \neq \emptyset$. But then from $f^s[\![\psi]\!] \neq \emptyset$, interchanging ϕ and ψ in the above, we can go on to deduce that $f^s[\![\psi]\!] = f^s([\![\phi]\!] \cap [\![\psi]\!])$, which is $f^s([\![\psi]\!] \cap [\![\phi]\!])$, i.e. $f^s[\![\phi]\!]$.

Overall, we showed that if $f^s[\![\phi]\!] \neq \emptyset$, then $f^s[\![\phi]\!] = f^s[\![\psi]\!]$. Likewise, interchanging ϕ and ψ in the overall argument shows that if $f^s[\![\psi]\!] \neq \emptyset$, then $f^s[\![\psi]\!] = f^s[\![\phi]\!]$. That leaves the case that $f^s[\![\phi]\!] = \emptyset$ and $f^s[\![\psi]\!] = \emptyset$, whence of course $f^s[\![\phi]\!] = f^s[\![\psi]\!]$. ⊣

To axiomatise the logic determined by selection frames, we need the notion of a *template*, which can be thought of, approximately, as an expression of the form

$$\theta_0 \to \Box_1(\theta_1 \to \Box_2(\theta_2 \to \cdots \to \Box_n(\theta_{n-1} \to \#)\cdots),$$

where the new symbol # is a place holder for a formula, the θ_i's are formulas, and each $\Box_j$ is a sequence $[*\phi_{j_1}]\cdots[*\phi_{j_{m_j}}]$ of belief revision modalities. Formally, the set of templates is defined inductively by the following stipulations.

- # is a template.
- If ρ is a template, then $\theta \to \rho$ is a template for all formulas θ.
- If ρ is a template, then $[*\phi]\rho$ is a template for all pure Boolean ϕ.

Each template ρ has a single occurrence of the symbol #. We write $\rho(\theta)$ for the formula obtained from ρ by replacing # by the formula θ. Inductively, $\#(\theta) = \theta$, $(\varphi \to \rho)(\theta) = \varphi \to \rho(\theta)$, and $([*\phi]\rho)(\theta) = [*\phi]\rho(\theta)$. The notion of template was introduced in [4], under the name 'admissible form', in order to axiomatise certain dynamic program logics.

In Fig. 1 in Sect. 3 there is an inference rule ($\bowtie$R) involving templates. This rule need not preserve truth in a model. Rather, it preserves validity in a frame, as the following result shows.

Lemma 4 *Let ρ be any template; ϕ, ψ any pure Boolean formulas; and p any variable not occurring in ϕ, ψ or ρ. Then for any $s \in S$ in a frame $\mathfrak{F}$, if $\mathfrak{M}, s \not\models \rho(\phi \bowtie \psi)$ for some model $\mathfrak{M} = (\mathfrak{F}, [\![-]\!])$, then $\mathfrak{M}', s \not\models \rho([*\phi]p \leftrightarrow [*\psi]p)$ for some model $\mathfrak{M}' = (\mathfrak{F}, [\![-]\!]')$ that has $[\![q]\!]' = [\![q]\!]$ for all variables $q \neq p$.*

Proof By induction on the formation of ρ.

For the case $\rho = \#$, suppose p does not occur in ϕ or ψ, and $\mathfrak{M}, s \not\models \phi \bowtie \psi$. Then there exists a $t \in S$ with, say, $(s, t) \in R[\![\phi]\!]$ but $(s, t) \notin R[\![\psi]\!]$. Define $\mathfrak{M}'$ by putting $[\![p]\!]' = \{t' \in S : (s, t') \in R[\![\psi]\!]\}$, and $[\![q]\!]' = [\![q]\!]$ for all variables $q \neq p$.

Now $[\![-]\!]$ and $[\![-]\!]'$ agree on all variables of ϕ, since p is not in ϕ. A simple induction then shows that $[\![\phi]\!] = [\![\phi]\!]'$. Likewise, $[\![\psi]\!] = [\![\psi]\!]'$. Thus $(s, t') \in R[\![\psi]\!]'$ implies $(s, t') \in R[\![\psi]\!]$, which implies $t' \in [\![p]\!]'$. This shows that $\mathfrak{M}', s \models [*\psi]p$. On the other hand, $t \notin [\![p]\!]'$, and so as $(s, t) \in R[\![\phi]\!] = R[\![\phi]\!]'$, this shows $\mathfrak{M}', s \not\models [*\phi]p$. Therefore $\mathfrak{M}', s \not\models [*\phi]p \leftrightarrow [*\psi]p$. The same conclusion follows if, instead, $(s, t) \in R[\![\psi]\!]$ but $(s, t) \notin R[\![\phi]\!]$. That completes the proof that the result holds when $\rho = \#$.

Now assume the result inductively for ρ, and consider a template $\theta \to \rho$ with p not in ϕ, ψ or $\theta \to \rho$. If $\mathfrak{M}, s \not\models \theta \to \rho(\phi \bowtie \psi)$, then $\mathfrak{M}, s \models \theta$ and $\mathfrak{M}, s \not\models \rho(\phi \bowtie \psi)$. But p is not in ϕ, ψ or ρ, so the induction hypothesis gives that $\mathfrak{M}', s \not\models \rho([*\phi]p \leftrightarrow [*\psi]p)$ for some model $\mathfrak{M}'$ on $\mathfrak{F}$ that differs from $\mathfrak{M}$ only on p. Since p is not in θ, we then get $\mathfrak{M}', s \models \theta$ by Lemma 2. It follows that $\mathfrak{M}', s \not\models \theta \to \rho([*\phi]p \leftrightarrow [*\psi]p)$, showing that the result holds for template $\theta \to \rho$.

Finally, again assume the result inductively for ρ, and consider a template $[*\chi]\rho$, with p not in ϕ, ψ or $[*\chi]\rho$. If $\mathfrak{M}, s \not\models [*\chi]\rho(\phi \bowtie \psi)$, then there is a t with $(s, t) \in R[\![\chi]\!]$ and $\mathfrak{M}, t \not\models \rho(\phi \bowtie \psi)$. By the induction hypothesis, $\mathfrak{M}', t \not\models \rho([*\phi]p \leftrightarrow [*\psi]p)$ for some $\mathfrak{M}'$ differing from $\mathfrak{M}$ only on p. Since p is not in χ, we have $[\![\chi]\!] = [\![\chi]\!]'$, so $(s, t) \in R[\![\chi]\!]'$, implying that $\mathfrak{M}', s \not\models [*\chi]\rho([*\phi]p \leftrightarrow [*\psi]p)$. Thus the result holds for template $[*\chi]\rho$. ⊣

Corollary 1 *Let ρ, ϕ, ψ and p be as in the Lemma. Then for any frame $\mathfrak{F}$, and any $s \in S$, if $\mathfrak{F}, s \models \rho([*\phi]p \leftrightarrow [*\psi]p)$, then $\mathfrak{F}, s \models \rho(\phi \bowtie \psi)$.*

Hence if $\rho([*\phi]p \leftrightarrow [*\psi]p)$ is valid in $\mathfrak{F}$, then so is $\rho(\phi \bowtie \psi)$. ⊣

3 Logics

Axioms and rules of inference appear in Fig. 1. There, as usual, ϕ, ψ, χ are pure Boolean formulas, while θ, ω are general formulas.[5] A *selection logic*, or more briefly a *logic*, is defined to be a set L of formulas that contains all instances of these axioms and is closed under these inference rules. The members of L are the *L-theorems*. The *smallest* logic will be denoted L_{K}. This is the intersection of all

[5] Except that in (□) and (□N), θ and ω must be pure Boolean when □ is **B** or **K**.

logics. Since the proof theory is finitary (all rules have finitely many premisses), L_K can also be described as the set of formulas that can be obtained from the axioms by finitely many applications of the inference rules.

The $\bowtie$-free axioms of Fig. 1 are a sub-list of those in [14], except that (K$*$) has replaced ($*$K), as mentioned in the Introduction. The Congruence Rule (CR) is an addition which in fact makes the axiom ($*$6) redundant. In Sect. 7 we discuss this, and explore the consequences of adding a variety of axioms to L_K.

The axiom ($\Box$) and the rule ($\Box$N) define $\Box$ as a *normal* modality, and we use the phrase 'by modal logic' to mean that some conclusion has been obtained by properties of a normal $\Box$ together with tautological reasoning. Note that ($\Box$) and ($\Box$N) hold not just for $\Box \in \{\mathbf{B}, \mathbf{K}, [*\phi]\}$, but also for combinations of these modalities. For instance they hold when $\Box$ denotes the combination $[*\phi]\mathbf{B}$, in the sense that any formula

$$[*\phi]\mathbf{B}(\psi \to \chi) \to ([*\phi]\mathbf{B}\psi \to [*\phi]\mathbf{B}\chi)$$

is an L-theorem; and if ψ is an L-theorem, then so is $[*\phi]\mathbf{B}\psi$.

Axioms ($*$2)–($*$8) are intended to formalise certain postulates of the AGM theory (and preserve their numbering). In the presence of (BK) some of these axioms can be simplified or strengthened. For instance, the consequent of ($*$4) is derivable, and this is the converse of ($*$3). Also the consequent of ($*$5) is derivable, and this can be used to show that the modality $\mathbf{K}$ is definable in L, in the sense that $\mathbf{K}\phi$ is equivalent to $[*\neg\phi]\mathbf{B}\bot$. We record these and other derivability facts now:

Theorem 1 *In any selection logic L:*

(1) $(*4)'$: $\vdash_L \mathbf{B}\phi \to [*\top]\mathbf{B}\phi$.
(2) $(*5)'$: $\vdash_L \mathbf{k}\phi \to \langle *\phi \rangle \mathbf{b}\top$. *Equivalently*, $\vdash_L [*\phi]\mathbf{B}\bot \to \mathbf{K}\neg\phi$.
(3) $\vdash_L \mathbf{K}\phi \leftrightarrow [*\neg\phi]\mathbf{B}\bot$.
(4) If $\vdash_L \phi \leftrightarrow \psi$, then $\vdash_L [*\phi]\mathbf{B}\chi \leftrightarrow [*\psi]\mathbf{B}\chi$.
(5) If $\vdash_L \phi \to \psi$, then $\vdash_L [*\psi]\mathbf{B}\bot \to [*\phi]\mathbf{B}\bot$.
(6) $(*\mathrm{BH})$: $\vdash_L \mathbf{B}\bot \to [*\phi]\mathbf{B}\bot$.
(7) $\vdash_L \rho(\phi \bowtie \psi) \to \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$.

Proof For (1), from (BK) and (K$*$) we obtain $\vdash_L \mathbf{B}\bot \to [*\top]\mathbf{K}\bot$. Since (KB) and modal logic gives $\vdash_L [*\top]\mathbf{K}\bot \to [*\top]\mathbf{B}\bot$ and modal logic gives $\vdash_L [*\top]\mathbf{B}\bot \to [*\top]\mathbf{B}\theta$, this all leads to $\vdash_L \mathbf{B}\bot \to [*\top]\mathbf{B}\phi$, and hence by Boolean logic to

$$\vdash_L \mathbf{B}\bot \to (\mathbf{B}\phi \to [*\top]\mathbf{B}\phi).$$

But ($*$4) is equivalent to $\neg\mathbf{B}\bot \to (\mathbf{B}\phi \to [*\top]\mathbf{B}\phi)$, and these last two L-theorems yield $\vdash_L \mathbf{B}\phi \to [*\top]\mathbf{B}\phi$.

For (2), from (BK) and modal logic we obtain $\vdash_L \mathbf{B}\bot \to \mathbf{K}\neg\phi$, hence by Boolean logic $\vdash_L \mathbf{B}\bot \to ([*\phi]\mathbf{B}\bot \to \mathbf{K}\neg\phi)$. But ($*$5) is equivalent by modal logic to $\neg\mathbf{B}\bot \to ([*\phi]\mathbf{B}\bot \to \mathbf{K}\neg\phi)$, and these last two L-theorems yield $\vdash_L [*\phi]\mathbf{B}\bot \to \mathbf{K}\neg\phi$.

For (3), by (K$*$), (KB) and modal logic we get $\vdash_L \mathbf{K}\phi \to [*\neg\phi]\mathbf{B}\phi$. By this, ($*$2) and modal logic, $\vdash_L \mathbf{K}\phi \to [*\neg\phi](\mathbf{B}\phi \wedge \mathbf{B}\neg\phi)$ and then $\vdash_L \mathbf{K}\phi \to [*\neg\phi]\mathbf{B}\bot$. But from (2) we can derive the converse $\vdash_L [*\neg\phi]\mathbf{B}\bot \to \mathbf{K}\phi$, leading to (3).

(4) is just an instance of the Congruence Rule (CR) (and also follows by axiom ($*$6) and **K**-Necessitation).

For (5), $\vdash_L \phi \to \psi$ implies $\vdash_L \mathbf{K}\neg\psi \to \mathbf{K}\neg\phi$ by modal logic, and this in turn implies $\vdash_L [*\neg\neg\psi]\mathbf{B}\bot \to [*\neg\neg\phi]\mathbf{B}\bot$ by (3). Then $\vdash_L [*\psi]\mathbf{B}\bot \to [*\phi]\mathbf{B}\bot$ follows by (4).

The 'Black Hole' principle ($*$BH) of (6) is derived in [14, Appendix A], using (BK), (K$*$), (KB) and modal logic, similarly to the arguments for (1).

(7) is shown by induction on the formation of ρ. When $\rho = \#$, this is just axiom ($\bowtie$). Assuming inductively that (7) holds for ρ, then it holds with $[*\psi]\rho$ in place of ρ by modal logic, and with $\omega \to \rho$ in place of ρ by Boolean logic. $\dashv$

Remark 1 A simpler axiom set could be given by taking the derivable schemes $(*4)'$ and $(*5)'$ in place of ($*$4) and ($*$5), and deleting (BK), which is itself derivable from the simple cases $\mathbf{B}\bot \to [*\top]\mathbf{B}\bot$ of $(*4)'$ and $[*\top]\mathbf{B}\bot \to \mathbf{K}\neg\top$ of $(*5)'$. $\dashv$

A formula θ is *L-derivable* from a set Σ of formulas, in symbols $\Sigma \vdash_L \theta$, if there is some finite subset Σ_0 of Σ such that $(\bigwedge \Sigma_0) \to \theta$ is an L-theorem. The empty conjunction $\bigwedge \emptyset$ is taken to be the formula $\top$. We write $\vdash_L \theta$ when $\emptyset \vdash_L \theta$, which holds iff $\theta \in L$, i.e. iff θ is an L-theorem.

The fundamental derivability fact for a normal modality $\Box$ is the following (see e.g. [1], p. 159).

Lemma 5 ($\Box$-Lemma) *If* $\{\theta : \Box\theta \in \Sigma\} \vdash_L \omega$, *then* $\Sigma \vdash_L \Box\omega$. $\dashv$

A set Σ is *L-consistent* if $\Sigma \nvdash_L \bot$, and is *L-maximal* if it is maximally L-consistent. Familiarity is assumed with the properties of an L-maximal Γ, including that it contains all L-theorems; is closed under tautological consequence; has $\Gamma \vdash_L \theta$ iff $\theta \in \Gamma$; $\neg\theta \in \Gamma$ iff $\theta \notin \Gamma$, etc.

The logic L is called *consistent* if it is L-consistent as a set of formulas. This holds iff $\bot$ is not an L-theorem, or equivalently, iff there is at least one formula that is not an L-theorem.

A set Σ is said to *respect* $\bowtie$ *in L* if, for all templates ρ and all pure Boolean ϕ, ψ,

$$\text{if } \Sigma \vdash_L \rho([*\phi]\theta \leftrightarrow [*\psi]\theta) \text{ for all formulas } \theta, \text{ then } \Sigma \vdash_L \rho(\phi \bowtie \psi). \qquad (3)$$

If Σ is L-maximal, this is equivalent to requiring that for all ρ, ϕ, ψ,

$$\{\rho([*\phi]\theta \leftrightarrow [*\psi]\theta) : \theta \text{ is any formula}\} \subseteq \Sigma \quad \text{implies} \quad \rho(\phi \bowtie \psi) \in \Sigma. \qquad (4)$$

The set Σ is *L-saturated* if it is L-maximal and satsfies (3), or equivalently (4). The set of L-saturated sets will be denoted S_L.

Lemma 6 *Let Σ be a set that respects $\bowtie$ in L. Then*

(1) For each finite set Γ of formulas, $\Sigma \cup \Gamma$ respects $\bowtie$ in L .
(2) $[*\phi]^{-L}\Sigma = \{\theta : \Sigma \vdash_L [*\phi]\theta\}$ respects $\bowtie$ in L.

Proof (1) Let $\Sigma \cup \Gamma \vdash_L \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$ for all formulas θ. If ω is the conjunction of the members of Γ, then $\Sigma \vdash_L \omega \to \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$ for all θ. Applying the fact that Σ respects $\bowtie$ to the template $\omega \to \rho$ then gives $\Sigma \vdash_L \omega \to \rho(\phi \bowtie \psi)$. Hence $\Sigma \cup \Gamma \vdash_L \rho(\phi \bowtie \psi)$.
(2) Let $[*\phi]^{-L}\Sigma \vdash_L \rho([*\chi]\theta \leftrightarrow [*\psi]\theta)$ for all formulas θ. Then by the $\Box$-Lemma 5 with $\Box = [*\phi]$, $\Sigma \vdash_L [*\phi]\rho([*\chi]\theta \leftrightarrow [*\psi]\theta)$ for all θ. Applying the fact that Σ respects $\bowtie$ to the template $[*\phi]\rho$ then gives $\Sigma \vdash_L [*\phi]\rho(\chi \bowtie \psi)$, hence $\rho(\chi \bowtie \psi) \in [*\phi]^{-L}\Sigma$, and so $[*\phi]^{-L}\Sigma \vdash_L \rho(\chi \bowtie \psi)$. $\dashv$

We turn now to the question of the existence of saturated sets, and indeed the existence of 'sufficiently many' of them. The following variant of Lindenbaum's Lemma and related results depend on the fact that our propositional language is countable.

Theorem 2 (1) *Every L-consistent set that L-respects $\bowtie$ has an L-saturated extension.*
(2) *If Σ is L-consistent, and there are infinitely many variables that do not occur in any member of Σ, then Σ has an L-saturated extension.*
(3) *Every finite L-consistent set has an L-saturated extension.*
(4) *$\vdash_L \theta$ iff θ belongs to every L-saturated set.*
(5) *If L is a consistent logic, then the set S_L of L-saturated sets is non-empty.*

Proof (1) Let Σ_0 be L-consistent and respect $\bowtie$ in L. Since there are countably many formulas, there is an enumeration $\{\theta_n : n \geq 0\}$ of the set of all formulas of the form $\rho(\phi \bowtie \psi)$. We define a nested sequence $\Sigma_0 \subseteq \cdots \subseteq \Sigma_n \subseteq \cdots$ of L-consistent sets such that $\Sigma_n - \Sigma_0$ is finite for all $n \geq 0$.
Suppose inductively that we have defined Σ_n that is L-consistent and has $\Sigma_n - \Sigma_0$ finite. Then Σ_n respects $\bowtie$ by part (1) of Lemma 6. If $\Sigma_n \vdash_L \theta_n$, put $\Sigma_{n+1} = \Sigma_n \cup \{\theta_n\}$. If however $\Sigma_n \nvdash_L \theta_n$, with $\theta_n = \rho(\phi \bowtie \psi)$, since Σ_n respects $\bowtie$ there is some formula θ with $\Sigma_n \nvdash_L \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$. Put

$$\Sigma_{n+1} = \Sigma_n \cup \{\neg\rho([*\phi]\theta \leftrightarrow [*\psi]\theta)\}.$$

In both cases we get that Σ_{n+1} is L-consistent, with $\Sigma_{n+1} - \Sigma_0$ finite.
Now put $\Sigma = \bigcup_{n \geq 0} \Sigma_n$. Then Σ is L-consistent, so extends to an L-maximal set Γ in the usual way. It remains to show that Γ respects $\bowtie$. But if $\Gamma \nvdash_L \rho(\phi \bowtie \psi)$, with $\rho(\phi \bowtie \psi) = \theta_n$, then $\Sigma_n \nvdash_L \theta_n$ as $\Sigma_n \subseteq \Gamma$, so by our construction there is a θ with $\neg\rho([*\phi]\theta \leftrightarrow [*\psi]\theta) \in \Sigma_{n+1} \subseteq \Gamma$, so $\Gamma \nvdash_L \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$ as Γ is L-consistent.
(2) Suppose there are infinitely many variables that do not occur in Σ. Then we show that Σ respects $\bowtie$ in L. For, if $\Sigma \vdash_L \rho([*\phi]\theta \leftrightarrow [*\psi]\theta)$ for all θ,

then we choose a variable p that does not occur in Σ or in ρ, ϕ or ψ. Then $\Sigma \vdash_L \rho([*\phi]p \leftrightarrow [*\psi]p)$, so $\vdash_L \omega \to \rho([*\phi]p \leftrightarrow [*\psi]p)$ where ω is the conjunction of some finite subset of Σ. Since p also does not occur in ω, the rule ($\bowtie$R) then applies to the template $\omega \to \rho$ to give $\vdash_L \omega \to \rho(\phi \bowtie \psi)$. It follows that $\Sigma \vdash_L \rho(\phi \bowtie \psi)$.

This confirms that Σ respects $\bowtie$. So if Σ is also L-consistent, by part (1) it has an L-saturated extension.

(3) From part (2), for if Σ is finite, there are infinitely many variables that do not occur in Σ.

(4) If $\vdash_L \theta$, then θ belongs to every L-maximal set, and in particular to the L-saturated ones. But if $\nvdash_L \theta$, then $\{\neg\theta\}$ is L-consistent and finite, hence by (3) there is an L-saturated Γ with $\neg\theta \in \Gamma$, hence $\theta \notin \Gamma$.

(5) If L is consistent, then $\nvdash_L \bot$, so by (4) there is a L-saturated set. ⊣

Theorem 3 *Let Σ be L-saturated, and ϕ a pure Boolean formula. Then*

$$[*\phi]\omega \in \Sigma \quad \textit{iff} \quad \textit{for all } \Delta \in S_L \textit{ such that } \{\theta : [*\phi]\theta \in \Sigma\} \subseteq \Delta,\ \omega \in \Delta.$$

Proof The result from left to right is immediate. For the converse, note first that since Σ is L-maximal,

$$\{\theta : [*\phi]\theta \in \Sigma\} = \{\theta : \Sigma \vdash_L [*\phi]\theta\} = [*\phi]^{-L}\Sigma.$$

Now if $[*\phi]\omega \notin \Sigma$, then $\Sigma \nvdash_L [*\phi]\omega$, so by the $\Box$-Lemma 5 with $\Box = [*\phi]$, we have $[*\phi]^{-L}\Sigma \nvdash_L \omega$. Hence $\Delta_0 = [*\phi]^{-L}\Sigma \cup \{\neg\omega\}$ is L-consistent.

But $[*\phi]^{-L}\Sigma$ respects $\bowtie$ by part (2) of Lemma 6, hence by part (1) of that Lemma, Δ_0 respects $\bowtie$. It follows by Theorem 2(1) that Δ_0 has an L-saturated extension Δ. Then $\neg\omega \in \Delta$, so $\omega \notin \Delta$, and $[*\phi]^{-L} \subseteq \Delta$, as required to complete the proof. ⊣

4 Soundness

First we briefly account for the truth of axioms in models, identifying the model-theoretic properties needed in each case.

Lemma 7 *The axioms in Fig. 1 are true in all models, hence valid in all frames.*

Proof We work in a given model, suppressing its name and writing $s \models \theta$ for its truth relation.

($\Box$): For $\Box = [\![\phi]\!]$, this is true in the model as in standard Kripkean semantics. For $\Box = \mathbf{B}$, observe that if $f^s U \subseteq [\![\phi \to \psi]\!]$ and $f^s U \subseteq [\![\phi]\!]$, then

$$f^s U \subseteq [\![\phi \to \psi]\!] \cap [\![\phi]\!] \subseteq [\![\psi]\!].$$

For $\Box = \mathbf{K}$ the argument is similar, with $\mathscr{C} f^s$ in place of $f^s U$.

(∗2): $f^s[\![\phi]\!] \subseteq [\![\phi]\!]$ by (INCL), so $s \models [\![\phi]\!]\mathbf{B}\phi$ by Lemma 3(3).

(∗3): If $s \models [*\top]\mathbf{B}\phi$, then $f^s[\![\top]\!] \subseteq [\![\phi]\!]$ (Lemma 3(3)), i.e. $f^s U \subseteq [\![\phi]\!]$, and so $s \models \mathbf{B}\phi$.

(∗4): We show that the stronger $(*4)'$ is true. If $s \models \mathbf{B}\phi$, then $f^s[\![\top]\!] = f^s U \subseteq [\![\phi]\!]$, hence $s \models [*\top]\mathbf{B}\phi$. Thus $s \models \mathbf{B}\phi \to [*\top]\mathbf{B}\phi$.

(∗5): We show that the stronger $(*5)'$ is true. If $s \models \mathbf{k}\phi$, then $s \not\models \mathbf{K}\neg\varphi$, so $\mathscr{C} f^s \nsubseteq -[\![\phi]\!]$, so there is a ψ with $[\![\phi]\!] \cap f^s[\![\psi]\!] \neq \emptyset$. Hence $f^s([\![\phi]\!] \cap [\![\psi]\!]) \neq \emptyset$ by (STRONG ARROW), and so $f^s[\![\phi]\!] \neq \emptyset$ by (MONEYS). Thus $(f^s[\![\phi]\!]) \cap [\![\top]\!] \neq \emptyset$, and so $s \models \langle *\phi \rangle \mathbf{b}\top$ by Lemma 3(4). Thus $s \models \mathbf{k}\phi \to \langle *\phi \rangle \mathbf{b}\top$.

(∗6): If $s \models \mathbf{K}(\phi \leftrightarrow \psi)$, then $f^s[\![\phi]\!] = f^s[\![\psi]\!]$ by Lemma 3(5). So in general $f^s[\![\phi]\!] \subseteq [\![\chi]\!]$ iff $f^s[\![\psi]\!] \subseteq [\![\chi]\!]$, hence by Lemma 3(3), $s \models [*\phi]\mathbf{B}\chi$ iff $s \models [*\psi]\mathbf{B}\chi$, showing $s \models [*\phi]\mathbf{B}\chi \leftrightarrow [*\psi]\mathbf{B}\chi$.

(∗7): Let $s \models [*(\phi \wedge \psi)]\mathbf{B}\chi$. Then $f^s[\![\phi \wedge \psi]\!] \subseteq [\![\chi]\!]$.
First, if $(f^s[\![\phi]\!]) \cap [\![\psi]\!] \neq \emptyset$ then, using (STRONG ARROW),

$$(f^s[\![\phi]\!]) \cap [\![\psi]\!] = f^s([\![\phi]\!] \cap [\![\psi]\!]) = f^s[\![\phi \wedge \psi]\!] \subseteq [\![\chi]\!],$$

so $f^s[\![\phi]\!] \subseteq [\![\psi \to \chi]\!]$, and hence $s \models [*\phi]\mathbf{B}(\psi \to \chi)$.
But if $(f^s[\![\phi]\!]) \cap [\![\psi]\!] = \emptyset$, then $(f^s[\![\phi]\!]) \cap [\![\psi]\!] \subseteq [\![\chi]\!]$, anyway, and we get the same conclusion $s \models [*\phi]\mathbf{B}(\psi \to \chi)$.

(∗8): Let $s \models \langle *\phi \rangle \mathbf{b}\psi$. Then by Lemma 3(4), $(f^s[\![\phi]\!]) \cap [\![\psi]\!] \neq \emptyset$, so by (STRONG ARROW), $f^s[\![\phi \wedge \psi]\!] = (f^s[\![\phi]\!]) \cap [\![\psi]\!]$.
Now if $s \models [*\phi]\mathbf{B}(\psi \to \chi)$, then $f^s[\![\phi]\!] \subseteq [\![\psi \to \chi]\!]$, and hence $(f^s[\![\phi]\!]) \cap [\![\psi]\!] \subseteq [\![\chi]\!]$. Thus $f^s[\![\phi \wedge \psi]\!] \subseteq [\![\chi]\!]$, implying $s \models [*(\phi \wedge \psi)]\mathbf{B}\chi$. Altogether this shows that $s \models [*\phi]\mathbf{B}(\psi \to \chi) \to [*(\phi \wedge \psi)]\mathbf{B}\chi$.

(∗FB): If $s \models \langle *\phi \rangle \mathbf{B}\psi$, there exists t with $(s, t) \in R[\![\phi]\!]$ and $t \models \mathbf{B}\psi$. Then by (F1), $f^s[\![\phi]\!] = f^t U \subseteq [\![\psi]\!]$, implying $s \models [*\phi]\mathbf{B}\psi$.

(K∗): Let $s \models \mathbf{K}\psi$. Then $\mathscr{C} f^s \subseteq [\![\psi]\!]$, hence $\mathbf{C}(\mathscr{C} f^s) \subseteq [\![\psi]\!]$ as $[\![\psi]\!]$ is closed. If $(s, t) \in R[\![\phi]\!]$, then by (F2), $\mathscr{C} f^t \subseteq \mathbf{C}(\mathscr{C} f^s) \subseteq [\![\psi]\!]$, hence $t \models \mathbf{K}\psi$. This shows that $s \models [*\phi]\mathbf{K}\psi$.

(KB): If $s \models \mathbf{K}\phi$, then $f^s U \subseteq \mathscr{C} f^s \subseteq [\![\phi]\!]$, hence $s \models \mathbf{B}\phi$.

(BK): If $s \models \mathbf{B}\bot$, then $f^s U = \emptyset$, so for all propositions $P \subseteq U$, $f^s P = \emptyset$ by (MONEYS), hence $\mathscr{C} f^s = \emptyset = [\![\bot]\!]$, implying $s \models \mathbf{K}\bot$.

(⋈): If $s \models \phi \bowtie \psi$, then for all t, $(s, t) \in R[\![\phi]\!]$ iff $(s, t) \in R[\![\psi]\!]$. Hence $s \models [*\phi]\theta$ iff $s \models [*\psi]\theta$, and so $s \models [*\phi]\theta \leftrightarrow [*\psi]\theta$. ⊣

Theorem 4 *For any frame $\mathfrak{F}$, the set $L_{\mathfrak{F}} = \{\theta : \mathfrak{F} \models \theta\}$ of formulas valid in $\mathfrak{F}$ is a logic. If $S \neq \emptyset$ in this frame, then $L_{\mathfrak{F}}$ is consistent.*

Proof By the Lemma just proved, all axioms belong to $L_{\mathfrak{F}}$. So we need to check that $L_{\mathfrak{F}}$ is closed under the inference rules.

The rules (MP) and (□N) are readily seen to preserve truth in each model on $\mathfrak{F}$, hence preserve validity in $\mathfrak{F}$, so $L_{\mathfrak{F}}$ is closed under these rules.

For closure of $L_{\mathfrak{F}}$ under the Congruence Rule (CR), suppose that $\mathfrak{M} \models \phi \leftrightarrow \psi$ where $\mathfrak{M}$ is any model on $\mathfrak{F}$. Then $[\![\phi]\!] = [\![\psi]\!]$ by Lemma 3(2), and therefore $R[\![\phi]\!] = R[\![\psi]\!]$. Hence for any θ we get $\mathfrak{M}, s \models [*\phi]\theta$ iff $\mathfrak{M}, s, \models [*\psi]\theta$ for all s in $\mathfrak{M}$, and so $\mathfrak{M} \models [*\phi]\theta \leftrightarrow [*\psi]\theta$. Thus (CR) preserves truth in each model on $\mathfrak{F}$, hence preserves validity in $\mathfrak{F}$.

Finally, Corollary 1 states that $L_{\mathfrak{F}}$ is closed under the rule ($\bowtie$R), completing the proof that $L_{\mathfrak{F}}$ is a logic.

Now suppose there exists some $s \in S$ (so also $U \neq \emptyset$ as then $\mathsf{ws}(s) \in S$). Then $\bot$ is falsified at s, showing that $\bot \notin L_{\mathfrak{F}}$, as need for consistency of this logic. $\dashv$

We can now demonstrate the soundness of the minimal logic L_{K} with respect to our semantics. From the Theorem just proved we infer that

$$If \vdash_{L_{\mathrm{K}}} \theta, \textit{then } \theta \textit{is valid in all frames.}$$

For if $\vdash_{L_{\mathrm{K}}} \theta$, then θ belongs to every logic, and hence belongs to the logic $L_{\mathfrak{F}}$ of any frame $\mathfrak{F}$. We can then extend this to Strong Soundness results:

Theorem 5 *Let $\mathfrak{F}$ be any selection frame.*

(1) *If $\Sigma \vdash_{L_{\mathfrak{F}}} \theta$, then $\Sigma \models^{\mathfrak{F}} \theta$.*
(2) *If Σ is satisfiable in $\mathfrak{F}$, then it is $L_{\mathfrak{F}}$-consistent.*

Proof For (1): Let $\Sigma \vdash_{L_{\mathfrak{F}}} \theta$. Then $\vdash_{L_{\mathfrak{F}}} \omega \to \theta$, where ω is the conjunction of some finite subset of Σ. Thus $\omega \to \theta$ is valid in $\mathfrak{F}$.

To show that $\Sigma \models^{\mathfrak{F}} \theta$, suppose that $\mathfrak{M}, s \models \Sigma$ in some model $\mathfrak{M}$ on $\mathfrak{F}$. Then $\mathfrak{M}, s \models \omega$. But as $\mathfrak{F} \models \omega \to \theta$, it follows that $\mathfrak{M}, s \models \theta$. This shows that $\Sigma \models^{\mathfrak{M}} \theta$ for all models $\mathfrak{M}$ on $\mathfrak{F}$, as required.

For (2): If Σ is satisfiable at some point s in some model on $\mathfrak{F}$, then since $s \not\models \bot$ we get $\Sigma \not\models^{\mathfrak{F}} \bot$, hence $\Sigma \nvdash_{L_{\mathfrak{F}}} \bot$ by (1). $\dashv$

Corollary 2 (1) *If $\Sigma \vdash_{L_{\mathrm{K}}} \theta$, then $\Sigma \models^{\mathfrak{F}} \theta$ for all frames $\mathfrak{F}$.*
(2) *If Σ is satisfiable, then it is L_{K}-consistent.* $\dashv$

Next we give a series of examples of frames and models, designed to demonstrate various properties of logics and their maximal sets.

Example 1 This is a frame with one world and two belief states.

Put $U = \{u\}$, $Prop = \{\emptyset, U\}$, $S = \{0, 1\}$, $R(U) = \{(0, 0), (1, 1)\}$ and $R(\emptyset) = \{(0, 1), (1, 1)\}$. ws is the unique function from S onto U, while f^0 and f^1 are defined by the following table.

P	f^0P	f^1P
$\emptyset$	$\emptyset$	$\emptyset$
U	U	$\emptyset$

In other words, f^0 is the identity function and f^1 the null function. They are both selection functions, and this structure is a frame.

Now $\mathscr{C} f^1 = \emptyset$ while $\mathscr{C} f^0 = U \neq \emptyset$, so $1 \models \mathbf{K}\bot$ but $0 \not\models \mathbf{K}\bot$. By definition of $R[\![\bot]\!]$, this gives $0 \models [*\bot]\mathbf{K}\bot$ (actually $[*\bot]\mathbf{K}\bot$ is valid in all frames). Hence

$$0 \not\models [*\bot]\mathbf{K}\bot \to \mathbf{K}\bot.$$

Also, as $f^0 U \neq \emptyset$ we get $0 \not\models \mathbf{B}\bot$, while $f^1 U = \emptyset$ and so $1 \not\models \neg\mathbf{B}\bot$.

Thus the logic $L_{\mathfrak{F}}$ determined by this frame contains none of the formulas $\mathbf{B}\bot$, $\neg\mathbf{B}\bot$, $\mathbf{K}\bot$, $[*\bot]\mathbf{K}\bot \to \mathbf{K}\bot$. Therefore L_{K} contains none of them. ⊣

In Example 1, the relations $R(P)$ are serial, so the frame validates the (∗D)-scheme $\langle *\phi \rangle \top$. We now show that this can fail. To do so requires only one world and one belief state, in which case we may as well identify them. Such a structure will be called a *singleton frame*.

The main purpose of the example is to show that $\neg\mathbf{B}\bot$ can be consistently added to L_{K}. As already mentioned, we will see in Sect. 7 that a logic cannot contain both $\neg\mathbf{B}\bot$ and scheme (∗D).

Example 2 The Rational Singleton Frame

Define a frame $\mathfrak{F}_r$ by putting $U = S = \{r\}$, $Prop = \{\emptyset, U\}$, $\mathsf{ws}(r) = r$, f^r = the identity function on *Prop*, $R(\emptyset) = \emptyset$, and $R(U) = \{(r, r)\}$.

Since $f^r U \neq \emptyset$, $\mathbf{B}\bot$ is false at r in any model on the frame. Thus the frame validates $\neg\mathbf{B}\bot$. The logic of the frame is consistent and contains $\neg\mathbf{B}\bot$, and hence the smallest logic containing $\neg\mathbf{B}\bot$ is consistent.

The formula $\langle *\phi \rangle \top$ is false in any model on this frame that has $R[\![\phi]\!] = \emptyset$. In particular, $\mathfrak{F}_r$ validates $\neg\langle *\bot \rangle \top$.

$\mathfrak{F}_r$ is the only singleton frame (up to isomorphism) that validates $\neg\mathbf{B}\bot$. For, in any singleton frame based on $\{r\}$, from $r \models \neg\mathbf{B}\bot$ we infer that $f^r U \neq \emptyset$, so $f^r U = U$ and f^r is the identity function on $\{\emptyset, U\}$. Moreover, if we had $(r, r) \in R(\emptyset)$, then by (F1) we would get the contradictory $f^r U = f^r \emptyset = \emptyset$. Hence we must have $R(\emptyset) = \emptyset$. Also from $f^r U \neq \emptyset$, by (F3) we infer $R(U) \neq \emptyset$, so we must have $R(U) = \{(r, r)\}$.

In summary, a singleton frame validates $\neg\mathbf{B}\bot$ iff its single selection function is the identity function, and if this condition holds, then the structure of the frame is uniquely determined as being that of $\mathfrak{F}_r$. Therefore, any different kind of singleton frame must have a null selection function. There are four such “null frames”, which we describe in Example 5 below. ⊣

The next example validates every formula of the form $[*\phi]\mathbf{B}\psi$. Nonetheless its points can be distinguished by other kinds of formulas. The construction will serve a signficant purpose at at the end of the chapter, where we use it to show that in the canonical models we construct in the next section, distinct belief states may have the same selection function and the same associated world state.

Example 3 As in Example 1, put $U = \{u\}$, $Prop = \{\emptyset, U\}$, $S = \{0, 1\}$, and ws = the unique function $S \to U$. But now, for both $s \in S$, let f^s be the null function, i.e. $f^s(\emptyset) = f^s(U) = \emptyset$. Thus $\mathscr{C} f^s = \emptyset$. Let $R(\emptyset) = R(U) = \{(1, 1)\}$. It is readily checked that this is a frame. In particular, (F3) holds vacuously, as there is no case of $f^s P \neq \emptyset$.

By definition of R, for every pure Boolean ϕ, the formula $\langle *\phi \rangle \top$ is true at 1 in every model on the frame, but false at 0 in every model. Also, in any such model, since $f^s P = \emptyset \subseteq [\![\psi]\!]$ for all s and P, we have $[*\phi]\mathbf{B}\psi$ true in the model for all ϕ and ψ by Lemma 3(3).

Now fix a model $\mathfrak{M}$ on this frame, and let $\Gamma_s = \{\theta : \mathfrak{M}, s \models \theta\}$. Then by the Strong Soundness Theorem 5(2), Γ_0 and Γ_1 are both L-consistent, where L is the logic of this frame. Since in general $\neg\theta \in \Gamma_s$ iff $\theta \notin \Gamma_s$, both are L-maximal. Moreover, both are closed under the rule ($\bowtie$R). This is because the conclusion $\rho(\phi \bowtie \psi)$ of a such a rule is true in $\mathfrak{M}$, hence belongs to Γ_s, since $R[\![\phi]\!] = R[\![\psi]\!]$. Thus Γ_0 and Γ_1 are both L-saturated. Hence they are L_{K}-saturated.

Since $\mathsf{ws}(0) = \mathsf{ws}(1)$, Γ_0 and Γ_1 contain exactly the same pure Boolean formulas (Lemma 3(1)). They both contain all formulas of the form $[*\phi]\mathbf{B}\psi$, since these are all valid in the frame.

On the other hand, Γ_1 contains all formulas $\langle *\phi \rangle \top$, while Γ_0 contains none of them. $\dashv$

The next example in this series shows that there are maximal sets that are not saturated.

Example 4 Let $U = \{u\}$, $Prop = \{\emptyset, U\}$, $S = \{0, 1, 2\}$, ws = the unique function $S \to U$, f^s = the null function for all $s \in S$, $R(\emptyset) = \{(0, 1)\}$ and $R(U) = \{(0, 2)\}$. Again we have a frame.

In any model $\mathfrak{M}$ on this frame, the points 1 and 2 are semantically indistinguishable, i.e.

$$\mathfrak{M}, 1 \models \theta \quad \text{iff} \quad \mathfrak{M}, 2 \models \theta \tag{5}$$

for all formulas θ. This is shown by induction on the formation of θ. The fact that $\mathsf{ws}(1) = \mathsf{ws}(2)$ ensures that (5) holds when θ is a variable, and the inductive cases for the Boolean connectives are routine. The fact that f^1 and f^2 are null ensures that every formula of the form $\mathbf{B}\chi$ or $\mathbf{K}\chi$ is true at both 1 and 2. The fact that there are no pairs $(1, t)$ or $(2, t)$ in any $R[\![\phi]\!]$ ensures that every formula of the form $[*\phi]\chi$ or $\phi \bowtie \psi$ is true at both 1 and 2. Thus (5) holds in all cases.

Since $(0, 1)$ is the only member of $R[\![\bot]\!]$, in $\mathfrak{M}$ we have $0 \models [*\bot]\theta$ iff $1 \models \theta$, for all θ. Similarly, $0 \models [*\top]\theta$ iff $2 \models \theta$. Hence by (5), $0 \models [*\bot]\theta$ iff $0 \models [*\top]\theta$, and therefore $0 \models ([*\bot]\theta \leftrightarrow [*\top]\theta)$ for all θ. But since $(0, 1)$ is in $R[\![\bot]\!] - R[\![\top]\!]$, we have $0 \not\models \bot \bowtie \top$.

Now let Γ be the L_{K}-maximal set $\{\theta : \mathfrak{M}, 0 \models \theta\}$. What we have just shown is that

$$\{[*\bot]\theta \leftrightarrow [*\top]\theta : \theta \text{ is any formula}\} \subseteq \Gamma,$$

while $\bot \bowtie \top \notin \Gamma$. So Γ does not respect $\bowtie$ (i.e. (4) fails with $\rho = \#$), and therefore Γ is not L_{K}-saturated. $\dashv$

Besides the frame $\mathfrak{F}_r$ of Example 2, there are four other singleton frames. They all validate $\mathbf{B}\bot$:

Example 5 The Null Singleton Frames.

Let $U = S = \{\nu\}$, $Prop = \{\emptyset, U\}$, $\mathsf{ws}(\nu) = \nu$, and f^ν = the null function on *Prop*. Since $f^\nu U = \emptyset$, any frame on this structure is going to have $\nu \models \mathbf{B}\bot$. There are four such frames, according to their definitions of the relations $R(P)$:

Name	R∅	RU	Validates
$\mathfrak{F}_\nu$	$\emptyset$	$\emptyset$	$\neg\langle *\phi\rangle\top$
$\mathfrak{F}_\top$	$\emptyset$	$\{(\nu,\nu)\}$	$\langle *\phi\rangle\top \leftrightarrow (\phi \bowtie \top)$
$\mathfrak{F}_\bot$	$\{(\nu,\nu)\}$	$\emptyset$	$\langle *\phi\rangle\top \leftrightarrow (\phi \bowtie \bot)$
$\mathfrak{F}_\mathrm{D}$	$\{(\nu,\nu)\}$	$\{(\nu,\nu)\}$	$\langle *\phi\rangle\top$

The frame $\mathfrak{F}_\mathrm{D}$ validates all three of the schemes (∗D), (∗X) and (∗K) mentioned in the Introduction. ⊣

The following fact about models on singleton frames will be used in Theorem 7 in the next section.

Theorem 6 *If $\mathfrak{M}$ is any model on a singleton frame, then the set $\{\theta : \mathfrak{M} \models \theta\}$ is closed under the rule* ($\bowtie$ R).

Proof We need to show of any template ρ that for all ϕ, ψ:

$$\text{if } \mathfrak{M} \models \rho([*\phi]\theta \leftrightarrow [*\psi]\theta) \text{for all formulas } \theta, \text{then} \mathfrak{M} \models \rho(\phi \bowtie \psi).$$

For this it suffices that for any ρ,

$$\mathfrak{M} \models \rho([*\phi]\bot \leftrightarrow [*\psi]\bot) \quad \text{implies} \quad \mathfrak{M} \models \rho(\phi \bowtie \psi). \tag{6}$$

We show this by induction on ρ. (Note that the converse of (6) holds in any model, by soundness—see Theorem 1(7)).

Now if s is the single element of $\mathfrak{M}$, then in general $\mathfrak{M} \models \theta$ iff $\mathfrak{M}, s \models \theta$, and $R[\![\phi]\!]$ is either $\emptyset$ or $\{(s,s)\}$. From these facts we see that

$$\begin{array}{lll} R[\![\phi]\!] = \emptyset & \text{iff} & \mathfrak{M} \models [*\phi]\bot. \\ R[\![\psi]\!] = \emptyset & \text{iff} & \mathfrak{M} \models [*\psi]\bot. \\ R[\![\phi]\!] = R[\![\psi]\!] & \text{iff} & \mathfrak{M} \models [*\phi]\bot \leftrightarrow [*\psi]\bot. \end{array}$$

Since $\mathfrak{M} \models \phi \bowtie \psi$ iff $R[\![\phi]\!] = R[\![\psi]\!]$ (in any model), this confirms that (6) holds when $\rho = \#$.

Now assume inductively that (6) holds for a template ρ. Then for any χ, if $\mathfrak{M} \not\models [*\chi]\rho(\phi \bowtie \psi)$, then $R[\![\chi]\!] \neq \emptyset$ and $\mathfrak{M} \not\models \rho(\phi \bowtie \psi)$. By induction hypothesis on ρ, $\mathfrak{M} \not\models \rho([*\phi]\bot \leftrightarrow [*\psi]\bot)$. This implies $\mathfrak{M} \not\models [*\chi]\rho([*\phi]\bot \leftrightarrow [*\psi]\bot)$, since $R[\![\chi]\!] = \{(s,s)\}$. Hence (6) holds with $[*\chi]\rho$ in place of ρ.

Also, if $\mathfrak{M} \not\models \omega \to \rho(\phi \bowtie \psi)$, then $\mathfrak{M} \models \omega$ and $\mathfrak{M} \not\models \rho(\phi \bowtie \psi)$, so by induction hypothesis, $\mathfrak{M} \not\models \rho([*\phi]\bot \leftrightarrow [*\psi]\bot)$, and thus $\mathfrak{M} \not\models \omega \to \rho([*\phi]\bot \leftrightarrow [*\psi]\bot)$. Hence (6) holds with $\omega \to \rho$ in place of ρ. That completes the proof of (6).

$\dashv$

5 Canonical Model for L

Fix a logic L. We will construct a model $\mathfrak{M}_L$, based on the set S_L of L-saturated sets, such that the formulas true in $\mathfrak{M}_L$ are precisely the L-theorems.

A *Boolean L-maximal set* is a set u of pure Boolean formulas that is maximally L-consistent within the set of all pure Boolean formulas. Equivalently, u is L-consistent and *negation complete* in the sense that for all pure Boolean ϕ, either $\phi \in u$ or $\neg\phi \in u$. Let U_L be the set of all Boolean L-maximal sets. Any L-consistent set of pure Boolean formulas can be extended to a member of U_L.

For each pure Boolean ϕ, define $[\![\phi]\!]_L = \{u \in U_L : \phi \in u\}$. Put

$$Prop_L = \{[\![\phi]\!]_L : \phi \text{ is pure Boolean}\}.$$

Then $Prop_L$ is a Boolean set algebra, since $U_L - [\![\phi]\!]_L = [\![\neg\phi]\!]_L$; $[\![\phi]\!]_L \cap [\![\psi]\!]_L = [\![\phi \wedge \psi]\!]_L$; $[\![\phi]\!]_L \cup [\![\psi]\!]_L = [\![\phi \vee \psi]\!]_L$; $U_L = [\![\top]\!]_L$, $\emptyset = [\![\bot]\!]_L$ etc. Thus $(U_L, Prop_L)$ is a Boolean structure.[6]

Moreover, $\vdash_L \phi \to \psi$ iff $[\![\phi]\!]_L \subseteq [\![\psi]\!]_L$, and hence $\vdash_L \phi \leftrightarrow \psi$ iff $[\![\phi]\!]_L = [\![\psi]\!]_L$. The only part of that which is not routine is to observe that if $\nvdash_L \phi \to \psi$, then $\{\phi, \neg\psi\}$ is L-consistent and so extends to some $u \in U_L$ with $u \in [\![\phi]\!]_L - [\![\psi]\!]_L$.

Each L-maximal set Γ gives rise to a function f^Γ on *Prop* by putting

$$f^\Gamma[\![\phi]\!]_L = \{u \in U_L : \{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\} \subseteq u\}.$$

f^Γ is well-defined, in the sense that the definition of $f^\Gamma[\![\phi]\!]_L$ does not depend on how the proposition $[\![\phi]\!]_L$ is named. For if $[\![\phi]\!]_L = [\![\phi']\!]_L$, then $\vdash_L \phi \leftrightarrow \phi'$, so by the rule (CR), $\vdash_L [*\phi]\mathbf{B}\psi \leftrightarrow [*\phi']\mathbf{B}\psi$ for any ψ; hence as Γ is an L-maximal set, $\{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\} = \{\psi : [*\phi']\mathbf{B}\psi \in \Gamma\}$.

We also use the fact that, by normal modal logic,

$$u \in f^\Gamma[\![\phi]\!]_L \quad \text{iff} \quad \{\langle *\phi\rangle\mathbf{b}\psi : \psi \in u\} \subseteq \Gamma. \tag{7}$$

Lemma 8 *If Γ is L-maximal, then for any ϕ the following are equivalent.*

(1) $f^\Gamma[\![\phi]\!]_L = \emptyset$.
(2) $\{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\}$ *is L-inconsistent.*

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[6] In the models of [14], *Prop* is taken to be the set of clopen subsets of a topology on U that makes it a *Stone space*, i.e. compact and totally separated. It can be shown that $Prop_L$ generates a Stone topology on U_L, for which the clopen sets are precisely the members of $Prop_L$. But we do not make any use of those additional properties.

(3) $[*\phi]\mathbf{B}\bot \in \Gamma$.

Proof (1) implies (2): If $\{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\}$ is L-consistent, then it is included in a Boolean L-maximal set u, which then belongs to $f^\Gamma[\![\phi]\!]_L$, so $f^\Gamma[\![\phi]\!]_L \neq \emptyset$.

(2) implies (3): If $\{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\} \vdash_L \bot$, then by the $\Box$-Lemma 5 with $\Box = [*\phi]\mathbf{B}$ we have $\Gamma \vdash_L [*\phi]\mathbf{B}\bot$, hence $[*\phi]\mathbf{B}\bot \in \Gamma$.

(3) implies (1): If $[*\phi]\mathbf{B}\bot \in \Gamma$, then any $u \in f^\Gamma[\![\phi]\!]_L$ would have $\bot \in u$, contrary to L-consistency, so in fact $f^\Gamma[\![\phi]\!]_L = \emptyset$. ⊣

Lemma 9 *f^Γ is a selection function on $(U_L, Prop_L)$.*

Proof (INCL): By (∗2), $[*\phi]\mathbf{B}\phi \in \Gamma$. Hence if $u \in f^\Gamma[\![\phi]\!]_L$, then $\phi \in u$, so $u \in [\![\phi]\!]_L$. This confirms that $f^\Gamma[\![\phi]\!]_L \subseteq [\![\phi]\!]_L$.

(MONEYS): Let $[\![\phi]\!]_L \subseteq [\![\psi]\!]_L$. Then $\vdash_L \phi \to \psi$, so by Theorem 1(5), $[*\psi]\mathbf{B}\bot \to [*\phi]\mathbf{B}\bot$ belongs to Γ. But now if $f^\Gamma[\![\phi]\!]_L \neq \emptyset$, then by Lemma 8, $[*\phi]\mathbf{B}\bot \notin \Gamma$, hence $[*\psi]\mathbf{B}\bot \notin \Gamma$, and so $f^\Gamma[\![\psi]\!]_L \neq \emptyset$.

(STRONG ARROW): Suppose that $[\![\psi]\!]_L \cap f^\Gamma[\![\phi]\!]_L \neq \emptyset$. Then we have to show that

$$f^\Gamma([\![\psi]\!]_L \cap [\![\phi]\!]_L) = [\![\psi]\!]_L \cap f^\Gamma[\![\phi]\!]_L. \tag{8}$$

Note that $f^\Gamma([\![\psi]\!]_L \cap [\![\phi]\!]_L) = f^\Gamma[\![\psi \wedge \phi]\!]_L = f^\Gamma[\![\phi \wedge \psi]\!]_L$.

By assumption, there is some element of $f^\Gamma[\![\phi]\!]_L$ that contains ψ, which by (7) implies

$$\langle *\phi\rangle\mathbf{b}\psi \in \Gamma. \tag{9}$$

Now take $u \in f^\Gamma([\![\psi]\!]_L \cap [\![\phi]\!]_L)$. Then as (INCL) holds, $u \in f^\Gamma([\![\psi]\!]_L)$. Also if $[*\phi]\mathbf{B}\chi \in \Gamma$, then by modal logic $[*\phi]\mathbf{B}(\psi \to \chi) \in \Gamma$, which by (∗8) and (9) gives $[*(\phi \wedge \psi)]\mathbf{B}\chi \in \Gamma$. Hence $\chi \in u$ as $u \in f^\Gamma[\![\phi \wedge \psi]\!]_L$. This shows that $u \in f^\Gamma[\![\phi]\!]_L$, and altogether that the left-right inclusion of (8) holds.

Conversely, let $u \in [\![\psi]\!]_L \cap f^\Gamma[\![\phi]\!]_L$. If $[*(\phi \wedge \psi)]\mathbf{B}\chi \in \Gamma$, then $[*\phi]\mathbf{B}(\psi \to \chi) \in \Gamma$ by (∗7), so $\psi \to \chi \in u$ as $u \in f^\Gamma[\![\phi]\!]_L$. But $\psi \in u$ as $u \in [\![\psi]\!]_L$, so then $\chi \in u$. This shows that $u \in f^\Gamma[\![\phi \wedge \psi]\!]_L$, completing the proof of the right-left inclusion of (8). ⊣

If Γ is any L-maximal set, let $\mathsf{ws}_L(\Gamma) = \{\psi : \psi \in \Gamma\}$, the set of all pure Boolean formulas that belong to Γ. This is the *world state* of Γ, and is evidently a Boolean-maximal set, i.e. $\mathsf{ws}_L(\Gamma) \in U_L$. Restricting this to L-saturated sets Γ gives a function $\mathsf{ws}_L : S_L \to U_L$.

The *canonical frame* of L is the structure

$$\mathfrak{F}_L = (U_L, Prop_L, S_L, \mathsf{sf}_L, R_L),$$

based on the Boolean structure $(U_L, Prop_L)$, such that S_L is the set of all L-saturated sets; $\mathsf{ws}_L : S_L \to U_L$ is the function just defined; $\mathsf{sf}_L(\Gamma) = f^\Gamma$ for all $\Gamma \in S_L$; and for any $[\![\phi]\!]_L \in Prop_L$,

$$(\Gamma, \Delta) \in R_L[\![\phi]\!]_L \quad \text{iff} \quad \{\theta : [*\phi]\theta \in \Gamma\} \subseteq \Delta.$$

The definition of $R_L[\![\phi]\!]_L$ does not depend on how the proposition $[\![\phi]\!]_L$ is named. For if $[\![\phi]\!]_L = [\![\phi']\!]_L$, then $\vdash_L \phi \leftrightarrow \phi'$, hence by the rule (CR), $\vdash_L [*\phi]\theta \leftrightarrow [*\phi']\theta$ for all formulas θ, so $\{\theta : [*\phi]\theta \in \Gamma\} = \{\theta : [*\phi']\theta \in \Gamma\}$.

By standard modal logic,

$$(\Gamma, \Delta) \in R_L[\![\phi]\!]_L \quad \text{iff} \quad \{\langle *\phi\rangle\theta : \theta \in \Delta\} \subseteq \Gamma. \tag{10}$$

Lemma 10 $\mathfrak{F}_L$ *is a selection frame.*

Proof We verify the four defining frame conditions.

(F1): Let $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$. We have to show that $f^\Delta U_L = f^\Gamma[\![\phi]\!]_L$. Note that $U_L = [\![\top]\!]_L$. Suppose that $u \in f^\Delta[\![\top]\!]_L$. Then if $[*\phi]\mathbf{B}\psi \in \Gamma$, since $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$ we have $\mathbf{B}\psi \in \Delta$, hence $[*\top]\mathbf{B}\psi \in \Delta$ by $(*4)'$ (see Theorem 1(1)), so $\psi \in u$ as $u \in f^\Delta[\![\top]\!]_L$. This shows that $\{\psi : [*\phi]\mathbf{B}\psi \in \Gamma\} \subseteq u$, i.e. $u \in f^\Gamma[\![\phi]\!]_L$.

Conversely, let $u \in f^\Gamma[\![\phi]\!]_L$. Then if $[*\top]\mathbf{B}\psi \in \Delta$, by axiom $(*3)$ we have $\mathbf{B}\psi \in \Delta$, hence $\langle *\phi\rangle\mathbf{B}\psi \in \Gamma$ by (10), so $[*\phi]\mathbf{B}\psi \in \Gamma$ by $(*\text{FB})$, and therefore $\psi \in u$ as $u \in f^\Gamma[\![\phi]\!]_L$. This shows that $u \in f^\Delta[\![\top]\!]_L$ as required.

(F2): Let $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$. We have to show that $\mathscr{C} f^\Delta \subseteq \mathbf{C}(\mathscr{C} f^\Gamma)$. So suppose that $u \in U_L$ has $u \notin \mathbf{C}(\mathscr{C} f^\Gamma)$. Since $\mathbf{C}(\mathscr{C} f^\Gamma)$ is topologically closed, there must then be some basic open set $P \in Prop$ that contains u and is disjoint from $\mathbf{C}(\mathscr{C} f^\Gamma)$. Then the complement of P is also a basic open set, hence of the form $[\![\psi]\!]_L$, that includes $\mathbf{C}(\mathscr{C} f^\Gamma)$ and does not contain u. Now

$$f^\Gamma[\![\neg\psi]\!]_L \subseteq \mathscr{C} f^\Gamma \subseteq \mathbf{C}(\mathscr{C} f^\Gamma) \subseteq [\![\psi]\!]_L.$$

But $f^\Gamma[\![\neg\psi]\!]_L \subseteq [\![\neg\psi]\!]_L = -[\![\psi]\!]_L$ by (INCL) (Lemma 9), so we conclude that $f^\Gamma[\![\neg\psi]\!]_L = \emptyset$. Hence by Lemma 8, $[*\neg\psi]\mathbf{B}\bot \in \Gamma$. Thus by Theorem 1(3), $\mathbf{K}\psi \in \Gamma$. It follows by axiom $(\text{K}*)$ that $[*\phi]\mathbf{K}\psi \in \Gamma$. So $\mathbf{K}\psi \in \Delta$ as $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$.

Then for every χ we get $[*\chi]\mathbf{K}\psi \in \Delta$ by $(\text{K}*)$, while $\psi \notin u$ as $u \notin [\![\psi]\!]_L$, showing that $u \notin f^\Delta[\![\chi]\!]_L$. Hence $u \notin \bigcup_\chi f^\Delta[\![\chi]\!]_L = \mathscr{C} f^\Delta$, completing the proof that $\mathscr{C} f^\Delta \subseteq \mathbf{C}(\mathscr{C} f^\Gamma)$.

(F3): Suppose $f^\Gamma[\![\phi]\!]_L \neq \emptyset$. Then by Lemma 8, $[*\phi]\mathbf{B}\bot \notin \Gamma$. Therefore by Theorem 3, there is some $\Delta \in S_L$ (with $\mathbf{B}\bot \notin \Delta$) such that $\{\theta : [*\phi]\theta \in \Gamma\} \subseteq \Delta$ and hence $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$.

(F4): To show that $\mathsf{ws}_L(S_L)$ is dense in U_L, it is enough to show that it is intersected by every non-empty basic open set. Since *Prop* is a base for the topology, a basic open set has the form $[\![\phi]\!]_L$, and if this is non-empty, then $\{\phi\}$ is L-consistent, and obviously finite. So by Theorem 2(3), there is a $\Gamma \in S_L$ with $\{\phi\} \subseteq \Gamma$. Then $\phi \in \mathsf{ws}_L(\Gamma)$, so $[\![\phi]\!]_L \cap \mathsf{ws}_L(S_L)$ contains $\mathsf{ws}_L(\Gamma)$ and is therefore non-empty as required. ⊣

Concerning (F4), we now give a sufficient criterion for $\mathfrak{F}_L$ to be world-surjective, a criterion that holds when $L = L_{\text{K}}$.

Theorem 7 *If a logic L is validated by some singleton frame, then in the canonical frame $\mathfrak{F}_L$, the function $\mathsf{ws}_L : S_L \to U_L$ is surjective.*

Proof Let L be validated by some singleton frame $\mathfrak{F}$ having $S = U = \{s\}$. Given any Boolean L-maximal set $u \in U_L$, define a valuation $[\![-]\!]_u$ on $\{s\}$ by declaring that $s \in [\![p]\!]_u$ iff $p \in u$, for all variables p. This gives a model $\mathfrak{M}_u = (\mathfrak{F}, [\![-]\!]_u)$ on $\mathfrak{F}$, for which $\mathfrak{M}_u, s \models p$ iff $p \in u$.

Let $\Gamma_u = \{\theta : \mathfrak{M}_u \models \theta\} = \{\theta : \mathfrak{M}_u, s \models \theta\}$. Since $\mathfrak{F} \models L$ we have $L \subseteq \Gamma_u$, and so Γ_u is L-maximal. A straightforward induction shows that for all pure Boolean ψ, $\psi \in \Gamma_u$ iff $\psi \in u$. Hence $\mathsf{ws}_L(\Gamma_u) = u$.

But by Theorem 6, Γ_u is closed under the rule ($\bowtie$R), and so is L-saturated by (4). Hence $\Gamma_u \in S_L$ as required to conclude that ws_L maps S_L onto U_L. ⊣

The *canonical L-model* is $\mathfrak{M}_L = (\mathfrak{F}_L, [\![-]\!]_L)$, with $[\![p]\!]_L = \{u \in U_L : p \in u\}$, as above, for all variables p.

Theorem 8 (The 'Truth Lemma')

Let θ be any formula. Then for all $\Gamma \in S_L$,

$$\mathfrak{M}_L, \Gamma \models \theta \quad \textit{iff} \quad \theta \in \Gamma.$$

Proof By induction on the formation of θ. In considering each case, we suppress the symbol $\mathfrak{M}_L$, writing $\Gamma \models \theta$ etc.

- For the case of a variable p we have $\Gamma \models p$ iff $\mathsf{ws}(\Gamma) \in [\![p]\!]_L$ iff $p \in \mathsf{ws}(\Gamma)$ iff $p \in \Gamma$ as p is pure Boolean.
- For the case of a formula $\phi \bowtie \psi$, suppose that $\Gamma \models \phi \bowtie \psi$. Take a formula $[*\psi]\omega$ in Γ. If $\Delta \in S_L$ has $\{\theta : [*\phi]\theta \in \Gamma\} \subseteq \Delta$, then $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$ by definition, hence $(\Gamma, \Delta) \in R_L[\![\psi]\!]_L$ as $\Gamma \models \phi \bowtie \psi$, so $\{\theta : [*\psi]\theta \in \Gamma\} \subseteq \Delta$, and thus $\omega \in \Delta$. This shows that $\{\theta : [*\phi]\theta \in \Gamma\} \subseteq \Delta$ implies $\omega \in \Delta$, which by Theorem 3 means that $[*\phi]\omega \in \Gamma$.
  Altogether we showed that $[*\psi]\omega \in \Gamma$ implies $[*\phi]\omega \in \Gamma$. Similarly $[*\phi]\omega \in \Gamma$ implies $[*\psi]\omega \in \Gamma$. Hence $([*\phi]\omega \leftrightarrow [*\psi]\omega) \in \Gamma$ for all formulas ω. By the case $\rho = \#$ of (4), this ensures that $\phi \bowtie \psi \in \Gamma$, since Γ is L-saturated.
  Conversely, suppose $\phi \bowtie \psi \in \Gamma$. Let $(\Gamma, \Delta) \in R_L[\![\psi]\!]_L$. Then if $[*\phi]\theta \in \Gamma$, since $([*\phi]\theta \leftrightarrow [*\psi]\theta) \in \Gamma$ by axiom ($\bowtie$), we get $[*\psi]\theta \in \Gamma$, and hence $\theta \in \Delta$ as $(\Gamma, \Delta) \in R_L[\![\psi]\!]_L$. This shows that $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$. Similarly, $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$ implies $(\Gamma, \Delta) \in R_L[\![\psi]\!]_L$. Thus in general, $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$ iff $(\Gamma, \Delta) \in R_L[\![\psi]\!]_L$, which means that $\Gamma \models \phi \bowtie \psi \in \Gamma$.
- For the case of a formula $\mathbf{B}\phi$, assume first that $\Gamma \models \mathbf{B}\phi$, and so $f^\Gamma[\![\top]\!]_L \subseteq [\![\phi]\!]_L$. Suppose then, for the sake of contradiction, that $\{\psi : \mathbf{B}\psi \in \Gamma\} \nvdash_L \phi$. Then $\{\psi : \mathbf{B}\psi \in \Gamma\} \cup \{\neg\phi\}$ is L-consistent, so extends to a Boolean-maximal set u. But now $u \notin [\![\phi]\!]_L$ as $\phi \notin u$, while for any formula $[*\top]\mathbf{B}\psi \in \Gamma$ we have $\mathbf{B}\psi \in \Gamma$ by (∗3), hence $\psi \in u$ by construction. But this shows that $u \in f^\Gamma[\![\top]\!]_L$, contradicting $f^\Gamma[\![\top]\!]_L \subseteq [\![\phi]\!]_L$. So we must conclude that $\{\psi : \mathbf{B}\psi \in \Gamma\} \vdash_L \phi$. Hence by the $\Box$-Lemma 5 with $\Box = \mathbf{B}$ we get $\Gamma \vdash_L \mathbf{B}\phi$, and therefore $\mathbf{B}\phi \in \Gamma$.

Conversely, suppose $\mathbf{B}\phi \in \Gamma$. Then by $(*4')$, $[*\top]\mathbf{B}\phi \in \Gamma$, so if $u \in f^{\Gamma}[\![\top]\!]_L$ we have $\phi \in u$ and hence $u \in [\![\phi]\!]_L$. This shows that $f^{\Gamma}[\![\top]\!]_L \subseteq [\![\phi]\!]_L$, implying that $\Gamma \models \mathbf{B}\phi$.

- The case of $\bot$ and the inductive case of an implicational formula $\theta \to \omega$ are standard, given the semantics for $\bot$ and $\to$ and the properties of Γ as a maximal set.
- For the inductive case of a formula $[*\phi]\omega$, make the induction hypothesis that the Theorem holds for ω. Then by this hypothesis and Theorem 3, we have $[*\phi]\omega \in \Gamma$ iff

$$\text{for all } \Delta \in S_L \text{ such that } (\Gamma, \Delta) \in R_L[\![\phi]\!]_L,\ \ \Delta \models \omega,$$

which is precisely the condition for $\Gamma \models [*\phi]\omega$. Hence the Theorem holds for $[*\phi]\omega$.

- Finally, we can apply the proof thus far to deal with the case of a formula $\mathbf{K}\phi$, using its equivalence to $[*\neg\phi]\mathbf{B}\bot$. The formula $\mathbf{K}\phi \leftrightarrow [*\neg\phi]\mathbf{B}\bot$ is a theorem of every selection logic (Theorem 1(3)), so is valid in every frame, and in particular is true at every point in $\mathfrak{M}_L$. Moreover, as this formula is an L-theorem, it belongs to every member of S_L. Since the present Theorem holds for $[*\neg\phi]\mathbf{B}\bot$ from the above, we can then argue that
$\Gamma \models \mathbf{K}\phi$
iff $\Gamma \models [*\neg\phi]\mathbf{B}\bot$ as $\Gamma \models \mathbf{K}\phi \leftrightarrow [*\neg\phi]\mathbf{B}\bot$,
iff $[*\neg\phi]\mathbf{B}\bot \in \Gamma$ as the Theorem holds for $[*\neg\phi]\mathbf{B}\bot$,
iff $\mathbf{K}\phi\bot \in \Gamma$ as $(\mathbf{K}\phi \leftrightarrow [*\neg\phi]\mathbf{B}\bot) \in \Gamma$.

That completes the proof of all cases. ⊣

Corollary 3 *For any formula θ, $\mathfrak{M}_L \models \theta$ iff $\vdash_L \theta$.*

Proof $\mathfrak{M}_L \models \theta$ iff θ belongs to every L-saturated set, iff $\vdash_L \theta$ by Theorem 2(4). ⊣

6 Strong Completeness for L_{K}

The Corollary just proven leads to the completeness of L_{K} with respect to frame validity:

Theorem 9 *For any formula θ:*

(1) *If θ is valid in all frames, then $\vdash_{L_{\mathrm{K}}} \theta$.*
(2) *If θ is L_{K}-consistent, then it is true at some point of some model.*

Proof (1) If θ is valid in all frames, then in particular it is true in the model $\mathfrak{M}_{L_{\mathrm{K}}}$, hence $\vdash_{L_{\mathrm{K}}} \theta$ by Corollary 3.

(2) If θ is L_{K}-consistent, then $\nvdash_{L_{\mathrm{K}}} \neg\theta$, so by part (1), $\neg\theta$ is false at some point of some model. ⊣

Now *Strong Completeness* of L_{K} would state that

$$\textit{If } \Sigma \models^{\mathfrak{F}} \theta \textit{ for all frames } \mathfrak{F}, \textit{ then } \Sigma \vdash_{L_{\mathrm{K}}} \theta.$$

This is equivalent to:

Every L_{K}*-consistent set of formulas is satisfiable in some model.*

We can prove that by carrying out the canonical model construction for an expanded language, along the following lines. Suppose that Σ is an L_{K}-consistent set of formulas of the present language. Add a countably infinite set of new variables, and let L'_{K} be the smallest set of formulas of the enlarged language that constitutes a logic.

Then Σ is L'_{K}-consistent, by a well known argument. For, if Σ were not L'_{K}-consistent, we would have $\vdash_{L'_{\mathrm{K}}} \neg\theta$, where θ is the conjunction of some finite subset of Σ. Since our proof theory is finitary, this means that there is some finite sequence of formulas that is an L'_{K}-derivation of $\neg\theta$ by axioms and rules of L'_{K}. This sequence involves only finitely many of the new variables, so we can uniformly replace them by variables from the old language that do not occur in the sequence (there are infinitely many such old variables). This replacement does not alter $\neg\theta$, and it provides a new sequence demonstrating that $\vdash_{L_{\mathrm{K}}} \neg\theta$, contradicting the L_{K}-consistency of Σ.

Thus Σ is L'_{K}-consistent, and there are infinitely many variables in the new language that do not occur in Σ (all the new variables at least). Hence by Theorem 2(2), Σ has an L'_{K}-saturated extension Γ. Then in the model $\mathfrak{M}_{L'_{\mathrm{K}}}$, since $\Sigma \subseteq \Gamma$, the Truth Lemma implies that $\mathfrak{M}_{L'_{\mathrm{K}}}, \Gamma \models \Sigma$, showing that Σ is satisfiable, as required.

In conclusion, we note that the minimal logic L_{K} is strongly complete with respect to the world-surjective frames. The singleton frames validates L_{K} (since every frame does), so by Theorem 7, the canonical frame of L_{K} is world-surjective. Thus

Every L_{K}*-consistent set of formulas is satisfiable in a model on a world-surjective frame.*

7 Commentary

The main objective of this chapter has been to show how the equivalence construct $\bowtie$ can be incorporated into a multi-modal logic. But our work has consequences for the non-$\bowtie$ part of this kind of doxastic logic, and we provide here some observations about additions and adjustments to its axioms, simplification of its semantics, and properties of its models.

Avoiding Inconsistency

The scheme $[*\phi]\mathbf{K}\psi \to \mathbf{K}\psi$, converse to to axiom (K$*$), can be consistently added to L_{K}, as shown by any of the frames of Examples 3, 4 and 5, which validate the scheme since they validate $\mathbf{K}\psi$.

But this scheme is inconsistent with the rational-agent formula $\neg\mathbf{B}\bot$. Even the instance $\psi = \bot$ of the converse is incompatible, as shown by the following derivation.

1. $[*\bot]\mathbf{K}\bot \to \mathbf{K}\bot$	converse to (K*)
2. $\mathbf{B}\bot \to \mathbf{K}\bot$	axiom (BK)
3. $[*\bot]\mathbf{B}\bot \to [*\bot]\mathbf{K}\bot$	from 2 by modal logic
4. $[*\bot]\mathbf{B}\bot$	axiom (*2)
5. $[*\bot]\mathbf{K}\bot$	3, 4, modus ponens
6. $\mathbf{K}\bot$	1, 5, modus ponens
7. $\mathbf{K}\bot \to \mathbf{B}\bot$	axiom (KB)
8. $\mathbf{B}\bot$	6, 7, modus ponens.

This shows that any logic containing $[*\bot]\mathbf{K}\bot \to \mathbf{K}\bot$ must contain $\mathbf{K}\bot$ and $\mathbf{B}\bot$, and be inconsistent with $\neg\mathbf{B}\bot$.

Now axiom (BK) is a tautological consequence of $\neg\mathbf{B}\bot$, so even without assuming (BK), we see from the derivation that:

if a modal logic contains the axioms $(*2)$ *and* (KB), *as well as the formula* $[*\bot]\mathbf{K}\bot \to \mathbf{K}\bot$, *then adding* $\neg\mathbf{B}\bot$ *to it would allow derivation of* $\mathbf{B}\bot$, *hence yield an inconsistency.*

Example 1 showed that none of $\mathbf{B}\bot$, $\neg\mathbf{B}\bot$ and $[*\bot]\mathbf{K}\bot \to \mathbf{K}\bot$ is a theorem of L_{K}.

A similar situation applies to the seriality scheme $\langle *\phi\rangle\top$, equivalent as an axiom to the $(*\mathrm{D})$-scheme $[*\phi]\theta \to \langle *\phi\rangle\theta$. The logic of the frame of Example 1 contains $(*\mathrm{D})$ (as does the logic of the frame $\mathfrak{F}_{\mathrm{D}}$ of Example 5). This shows that $(*\mathrm{D})$ can be consistently added to L_{K}. But consider the derivation

1. $\neg\mathbf{B}\bot$	
2. $[*\bot]\neg\mathbf{B}\bot$	from 1 by $[*\bot]$-Necessitation
3. $[*\bot]\mathbf{B}\bot$	axiom (*2)
4. $[*\bot]\bot$	from 2, 3 by modal logic
5. $\neg\langle *\bot\rangle\top$	from 4 by modal logic.

This shows that a logic cannot consistently contain both $\langle *\bot\rangle\top$ and $\neg\mathbf{B}\bot$.

It is also revealing to look at this semantically. Suppose that $\neg\mathbf{B}\bot$ is true in a model $\mathfrak{M}$. Then at each point $t \in S$ we have $f^t U \neq \emptyset$. Now if there were a pair (s, t) in $R(\emptyset)$, by (F1) and (INCL) we would have $f^t U = f^s \emptyset = \emptyset$, contradicting $f^t U \neq \emptyset$. Therefore the relation $R(\emptyset)$ is empty, so $\langle *\bot\rangle\top$ is false at every point.

In the Introduction we proposed the scheme (2), i.e.

$$\neg(\phi \bowtie \bot) \to \langle *\phi\rangle\top,$$

as a suitable weakening of $(*\mathrm{D})$. The rational singleton frame $\mathfrak{F}_r$ of Example 2 validates this scheme as well as $\neg\mathbf{B}\bot$, showing that the two can be jointly added to L_{K} to produce a consistent logic. (2) itself is not a theorem of L_{K}, as it is not valid in the null frame $\mathfrak{F}_\bot$ of Example 2, and indeed is false in any model on that frame that has $[\![\phi]\!] = U$.

Status of Axiom(∗6)
Axiom (∗6) was not used in our completeness proof. It *could have* been used to prove that $f^\Gamma[\![\phi]\!]_L$ is well defined, since this requires the result

$$\vdash_L \phi \leftrightarrow \psi \quad \text{implies} \quad \vdash_L [*\phi]\mathbf{B}\chi \leftrightarrow [*\psi]\mathbf{B}\chi$$

of Theorem 1(4), which, as we noted, follows by (∗6) and **K**-Necessitation. But the result itself is just an instance of the Congruence Rule (CR).

Thus (CR) supersedes (∗6), which can be dropped from the axiomatisation of L_{K}. But (∗6) is valid (Lemma 7), so it must then be derivable from the rest of the axiomatisation. It would be an interesting exercise to formulate such a derivation.
Adding $\psi \rightarrow [*\phi]\psi$
It is readily checked that the axiom

$$\psi \rightarrow [*\phi]\psi \tag{11}$$

is valid in all frames that satisfy

$$(s, t) \in R[\![\phi]\!] \quad \text{implies} \quad \mathsf{ws}(s) = \mathsf{ws}(t), \tag{12}$$

a condition expressing that 'belief revision does not affect the world' [14],p. 231. Moreover, the presence of (11) in a logic forces its canonical frame to satisfy (12):

Lemma 11 *Let L be any logic that contains the scheme (11). Then in* $\mathfrak{F}_L$, *if* $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$, *then* $\mathsf{ws}_L(\Gamma) = \mathsf{ws}_L(\Delta)$.

Proof Let $(\Gamma, \Delta) \in R_L[\![\phi]\!]_L$. If $\psi \in \Gamma$, then $[*\phi]\psi \in \Gamma$ by axiom (11), so $\psi \in \Delta$. But if $\psi \notin \Gamma$, then $\neg\psi \in \Gamma$, hence $[*\phi]\neg\psi \in \Gamma$ by (11) again, so $\neg\psi \in \Delta$ and thus $\psi \notin \Delta$. This shows that Γ and Δ contain the same pure Boolean formulas. ⊣

It follows from these observations that the smallest logic containing (11) is strongly sound and complete for validity in all frames satisfying (12). Moreover, this logic is valid in all singleton frames, which satisfy (12). Hence this logic has a world-surjective canonical frame, and is characterised by validity in the world-surjective frames satisfying (12). Note that (11) is valid in the frame of Example 2, and so is consistent with $\neg\mathbf{B}\bot$.

The converse of (11) is $[*\phi]\psi \rightarrow \psi$. This is consistent with L_{K}, since it is validated by the frame of Example 1. But any logic containing the scheme $[*\phi]\psi \rightarrow \psi$ is inconsistent with $\neg\mathbf{B}\bot$, since when $\psi = \bot$ the scheme becomes $[*\phi]\bot \rightarrow \bot$, equivalent to $\langle *\phi\rangle\top$. We saw above that even $\langle *\bot\rangle\top$ is inconsistent with $\neg\mathbf{B}\bot$.

Simpler ⋈-Free Semantics
For the ⋈-free fragment of L_{K}, we can replace frame condition (F4) in general by the stronger, and simpler, condition that the function $\mathsf{ws} : S \rightarrow U$ is surjective.

For the canonical model construction, we take S_L to be the set of all L-maximal sets (and not L-saturated ones, as ⋈ is no longer present). U_L remains as the set of

Boolean L-maximal sets, and we define $\mathsf{ws}_L : S_L \to U_L$, as before, by

$$\mathsf{ws}_L(\Gamma) = \{\psi : \psi \in \Gamma\}.$$

But now ws_L is surjective, because every $u \in U_L$ is L-consistent hence extends to an L-maximal $\Gamma \in S_L$ with $\mathsf{ws}_L(\Gamma) = u$.

This construction can be used to show that the class of $\bowtie$-free formulas that are valid in all world-surjective frames is axiomatised by the $\bowtie$-free fragment of L_K.

Γ is not determined by $\mathsf{sf}_L(\Gamma)$ and $\mathsf{ws}_L(\Gamma)$

In a canonical model $\mathfrak{M}_L$, there is more to a belief state $\Gamma \in S_L$ than its associated selection function f^Γ and and world state $\mathsf{ws}_L(\Gamma)$. There may be other L-saturated (or L-maximal in the $\bowtie$-free case) sets with the same selection function and world state.

This is illustrated by the two sets Γ_0 and Γ_1 defined in Example 3 of Sect. 4. These belong to S_L when L is the logic of the frame of that Example, and also when $L = L_K$. Γ_0 and Γ_1 both contain all formulas of the form $[*\phi]\mathbf{B}\psi$. In particular they contain all formulas $[*\phi]\mathbf{B}\bot$, which ensures that $f^{\Gamma_0} = f^{\Gamma_1} =$ the null function (Lemma 8). Also Γ_0 and Γ_1 contain the same Boolean formulas, so $\mathsf{ws}_L(\Gamma_0) = \mathsf{ws}_L(\Gamma_1)$.

But as was shown in Example 3, $\Gamma_0 \neq \Gamma_1$.

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