

Chapter 3

Equilibrium Calculation

Abstract The modelling of structure mechanics is a matter for finding equilibrium of the structure. The Newton-Raphson method for equilibrium calculation is described. This method is based on the nodes position, the forces on nodes, and the stiffness matrix. Other methods of equilibrium calculation, the methods of Newmark and of the energy minimisation, are described.

Keywords Equilibrium calculation · Newton-Raphson method · Newmark method · Energy minimisation method

3.1 Newton-Raphson Method

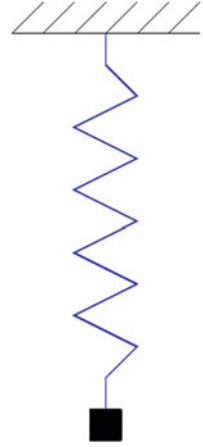
Finite element methods generally use the Newton-Raphson method [4] for the calculation of the equilibrium position of a mechanical structure. The equilibrium position corresponds to that position of the structure in which the sum of forces equals 0. In what follows a few simple examples are given to explain the method under three cases: one dimension, two dimensions and several dimensions.

3.1.1 One Dimension

A spring (Fig. 3.1) equilibrium is reached when the weight is equilibrated by the spring force. At this position the sum of forces equals 0. This position can be calculated using the Newton-Raphson method. In this example there is just one dimension: the vertical position (x) of the mass relatively to the spring fixation which also equals the length of the spring.

The spring equilibrium is calculated by writing the force on the mass: the weight is $-Mg$ (N), and the force of the spring is $+K \frac{x-l_0}{l_0}$ (N).

Fig. 3.1 The equilibrium of the spring is due to the mass weight and the spring force



With

M : mass (kg),

g : acceleration of gravity (m/s^2),

K : spring stiffness (N),

x : position of the mass along the spring axis relative to the fixed point of the spring (m),

l_0 : length of the stretched spring (m).

In this example the stiffness is not constant in order to give a clearer explanation of the Newton-Raphson method. K is equal to Ax . That means that longer the spring is, the stiffer it is.

The sum of forces on the mass (curve on Fig. 3.2) is

$$F(x) = K \frac{x - l_0}{l_0} - Mg \quad (3.1)$$

or, following the previous relations,

$$F(x) = Ax \frac{x - l_0}{l_0} - Mg \quad (3.2)$$

Obviously at the equilibrium $F(x) = 0$. It is clear that this simple equation has an analytical solution, which is

$$x = \frac{\sqrt{l_0 A (4gM + l_0 A)} + l_0 A}{2A} \quad (3.3)$$

The Newton-Raphson method could be used to find the length of the spring (x) at the equilibrium. This method requires knowing the force and the derivative of the force relatively to the position.

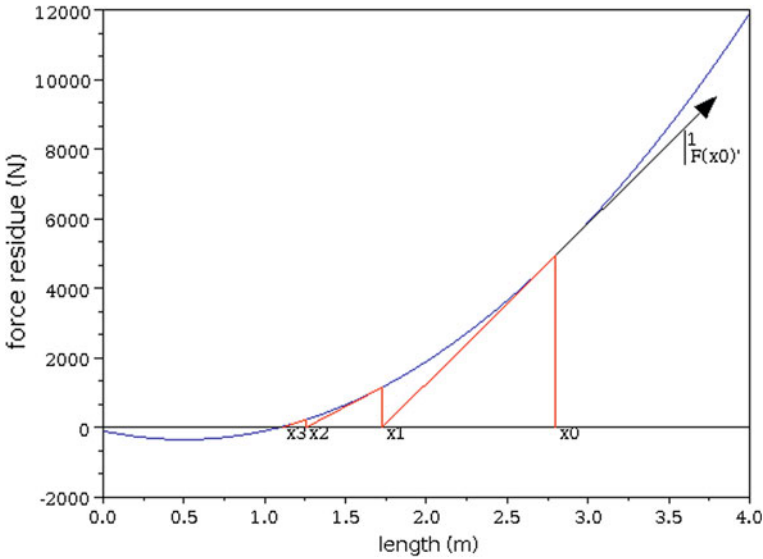


Fig. 3.2 Sum of forces on the mass function of spring length. Three Newton-Raphson iterations starting at $x = 2.8$ m are displayed. The vector tangent at x_0 is shown

The method is iterative and approximates the force curve by its tangent (shown in Fig. 3.2). From a position (x_k), the force ($F(x_k)$) and the derivative of force ($F'(x_k)$) are calculated, and a new position (x_{k+1}) can be found. This new position is generally closer to the equilibrium and is calculated as follows:

$$x_{k+1} = x_k + \frac{F(x_k)}{-F'(x_k)} \tag{3.4}$$

Figure. 3.2 shows three iterations with an initial value x_0 of the spring length of 2.8 m.

With:

The stiffness $A = 1000$ N/m,

The mass $M = 10$ kg,

The acceleration of gravity $g = 9.81$ m/s²,

The unstretched length of the spring $l_0 = 1$ m.

The stretched length at the equilibrium is 1.09 m. That means that the spring stretches 9%.

After five iterations the equilibrium is reached or more exactly $|F(x)| < 0.1$ N. The Fig. 3.2 shows 3 iterations along the curve of force. Figure 3.3 represents the reduction of the force residue ($|F(x)|$) with the five iterations.

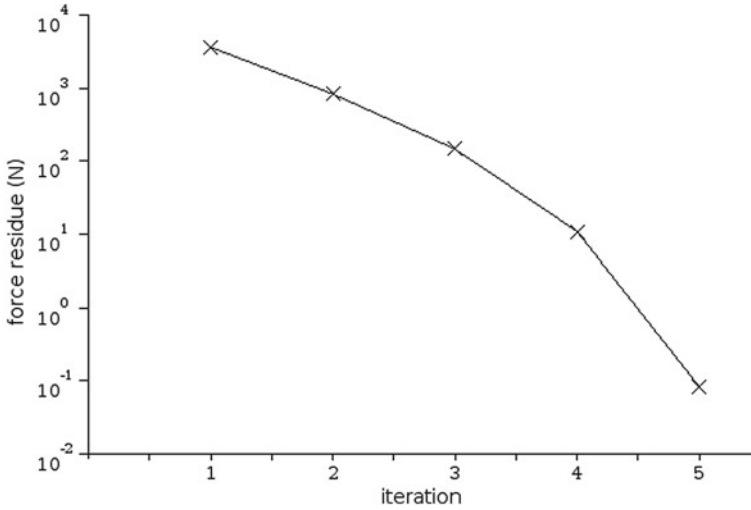


Fig. 3.3 Residue of force for each Newton-Raphson method iteration

3.1.2 Two Dimensions

In this section a simple example in two dimensions is given (Fig. 3.4): a spring with two degrees of freedom, i.e., the horizontal (x) and the vertical (y) positions of the mass relative to the spring fixation. The equilibrium of the system is due to the position of the mass along the vertical and the horizontal. Figure 3.5 shows the variation of the norm of the residue of force ($\sqrt{F_x^2 + F_y^2}$) on the mass due to the positions along x and y of the mass. The equilibrium point is noted by the largest dot.

The stiffness (K) of the spring is not constant: K is equal to Al . That means that the longer the spring is, the stiffer it is. In this condition the horizontal and vertical forces on the mass are due to the spring length and the weight of the mass:

$$F_x = T \frac{x}{l} \quad (3.5)$$

$$F_y = T \frac{y}{l} - Mg \quad (3.6)$$

With:

$$T = Al \frac{l - l_0}{l_0} \quad (3.7)$$

$$l = \sqrt{x^2 + y^2} \quad (3.8)$$

In this case the derivative of the forces is calculated relatively to x and y :

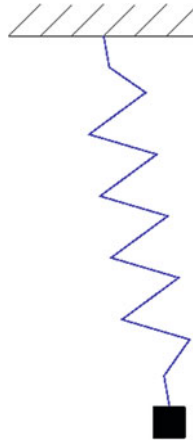


Fig. 3.4 Spring with two degrees of freedom: the *vertical* and *horizontal* positions of the mass. The equilibrium is due to the mass weight and the spring force

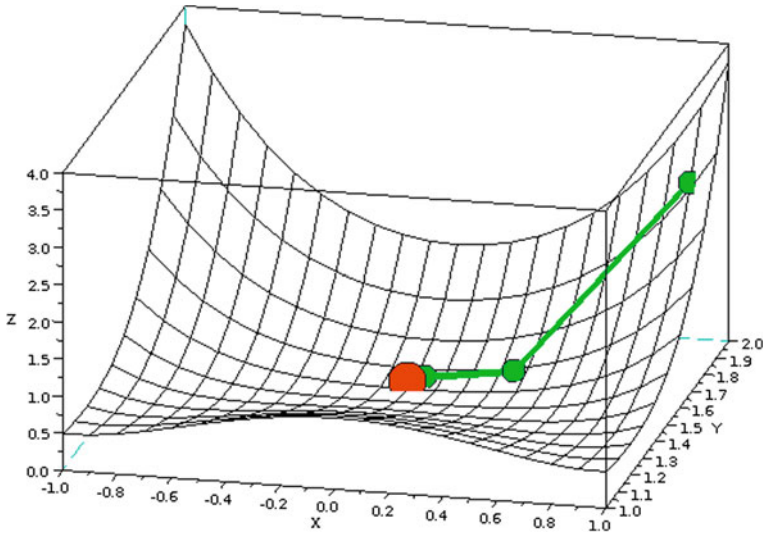


Fig. 3.5 Norm of the force ($Z = \sqrt{F_x^2 + F_y^2}$) function of mass coordinates (X, Y). The largest *dot* is the equilibrium position. The smallest *dots* are the Newton-Raphson iterations starting at $x = 0.9$ m and $y = 1.9$ m

$$\frac{\partial F_x}{\partial x} = A \frac{l - l_0}{l_0} + A \frac{x^2}{ll_0} \quad (3.9)$$

$$\frac{\partial F_x}{\partial y} = A \frac{xy}{ll_0} \quad (3.10)$$

$$\frac{\partial F_y}{\partial x} = A \frac{yx}{ll_0} \quad (3.11)$$

$$\frac{\partial F_y}{\partial y} = A \frac{l - l_0}{l_0} + A \frac{y^2}{ll_0} \quad (3.12)$$

The Newton-Raphson method accesses the equilibrium solution through iterations. At each iteration the new position is calculated by the following relation:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \frac{\mathbf{F}(\mathbf{X}_k)}{-F'(\mathbf{X}_k)} \quad (3.13)$$

With:

$$\mathbf{X}_k = \begin{vmatrix} x_k \\ y_k \end{vmatrix} \quad (3.14)$$

$$\mathbf{F}(\mathbf{X}_k) = \begin{vmatrix} F_x(X_k) \\ F_y(X_k) \end{vmatrix} \quad (3.15)$$

The ratio $\frac{\mathbf{F}(\mathbf{X}_k)}{-F'(\mathbf{X}_k)}$ is the displacement \mathbf{h} , such as $\mathbf{F}(\mathbf{X}_k) = -F'(\mathbf{X}_k)\mathbf{h}$.

With these equations the equilibrium position is assessed (Fig. 3.5). Figure 3.6 represents the reduction of the force residue with the iterations.

3.1.3 Several Dimensions

3.1.3.1 Main Variables

The positions of the nodes are in vector \mathbf{X} , the forces on the nodes are in vector \mathbf{F} , and the stiffness matrix is K ; x_i and F_i refer to the same node along the same axis.

These variables are as follows:

$$\mathbf{X} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{vmatrix} \quad (3.16)$$

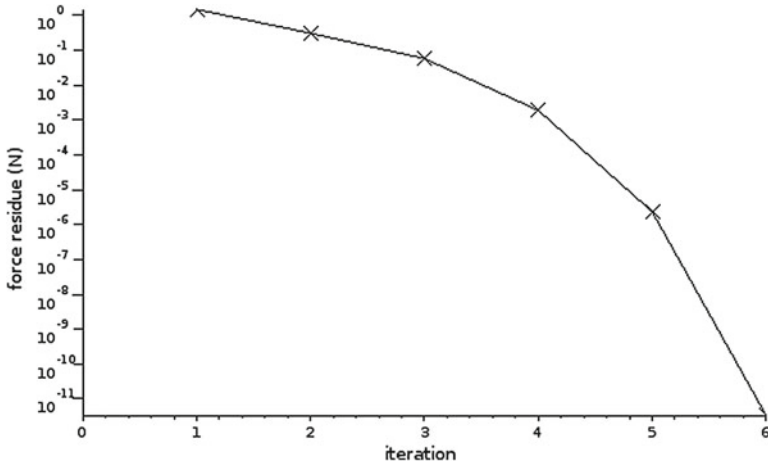


Fig. 3.6 Residue of force ($\sqrt{F_x^2 + F_y^2}$) for each Newton-Raphson method iteration

$$\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_n \end{pmatrix} \tag{3.17}$$

$$K = \begin{pmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & \dots & -\frac{\partial F_1}{\partial x_n} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & \dots & -\frac{\partial F_2}{\partial x_n} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ -\frac{\partial F_n}{\partial x_1} & -\frac{\partial F_n}{\partial x_2} & \dots & -\frac{\partial F_n}{\partial x_n} \end{pmatrix} \tag{3.18}$$

From these three variables the displacement vector (\mathbf{h}) can be calculated by solving the following system of linear equations:

$$\mathbf{h}K = \mathbf{F} \tag{3.19}$$

3.1.3.2 Iterations

As mentioned earlier, the Newton-Raphson-method is an iterative one. The steps are as follows:

From the position (\mathbf{X}_k) of the nodes resulting from iteration k:

$$\mathbf{X}_k = \begin{vmatrix} x_{k1} \\ x_{k2} \\ \cdot \\ \cdot \\ x_{kn} \end{vmatrix} \quad (3.20)$$

The force (\mathbf{F}_k) on the nodes and the stiffness (K_k) matrix are calculated:

$$\mathbf{F}_k = \begin{vmatrix} F_{k1} \\ F_{k2} \\ \cdot \\ \cdot \\ F_{kn} \end{vmatrix} \quad (3.21)$$

$$K_k = \begin{vmatrix} K_{k11} & K_{k12} & \cdot & \cdot & K_{k1n} \\ K_{k21} & K_{k22} & \cdot & \cdot & K_{k2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{kn1} & K_{kn2} & \cdot & \cdot & K_{knn} \end{vmatrix} \quad (3.22)$$

The node displacements (\mathbf{h}_k) are calculated:

$$\mathbf{h}_k K_k = \mathbf{F}_k \quad (3.23)$$

The new position of nodes is deduced:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{h}_k \quad (3.24)$$

3.1.4 Singularity of the Stiffness Matrix

In some cases the stiffness matrix (K) could be singular. In this case solving $\mathbf{h}K = \mathbf{F}$ (Sect. 3.1.3, p. 20) could lead to a very large displacement ($h_i \gg 1$) and to divergence of the method.

An example can be shown with the unstretched horizontal bar of Fig. 3.7. This bar has two extremities. If the first extremity (on the left on Fig. 3.7) has the horizontal and vertical coordinates (0, 0), the position vector is:

$$\mathbf{X} = \begin{vmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{vmatrix} \quad (3.25)$$



Fig. 3.7 This bar is articulated around its *left* extremity. A vertical force (F_4) is applied on the *right* extremity. This unstretched bar displays a zero stiffness along the *vertical*

With $x_3 \neq 0$

If the force on the second extremity is vertical, the force vector is:

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_4 \end{pmatrix} \tag{3.26}$$

With $F_4 \neq 0$

As we will see in Sect. 5.2 (p. 71) the stiffness matrix is:

$$K = \begin{pmatrix} K_{11} & 0 & -K_{11} & 0 \\ 0 & 0 & 0 & 0 \\ -K_{11} & 0 & K_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.27}$$

The matrix is singular. This is due to the derivative $\frac{\partial F_4}{\partial x_4}$, which is equal to 0 in this case of an unstretched horizontal bar. (i) If the bar is not horizontal this derivative will not be equal to 0, because the derivative of the bar length will not equal 0. (ii) If the bar is in tension (or compression), even horizontal, the derivative $\frac{\partial F_4}{\partial x_4}$ will not equal 0 because the derivative of the tension direction is not equal to 0.

To avoid problems due to singularity, precautions are available, as described below.

3.1.4.1 Additional Stiffness

A simple way is to add an arbitrary value (α) along the diagonal of the stiffness matrix, such that the previous matrix becomes:

$$K = \begin{pmatrix} K_{11} + \alpha & 0 & -K_{11} & 0 \\ 0 & \alpha & 0 & 0 \\ -K_{11} & 0 & K_{11} + \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \tag{3.28}$$

The added value (α) could decrease along the Newton-Raphson iterations. This added value (α) does not modify the equilibrium position, but only the way to reach this equilibrium.

3.1.4.2 Additional Mechanical Behaviour

Another way to remove singularity is to add further mechanical behaviour. For example, if this bar is in a fluid, air, or water, a vertical displacement will generate a drag in the opposite direction, meaning that the components of the stiffness matrix K_{22} and K_{44} will be not equal to 0.

3.1.4.3 Displacement Limit

A displacement limit could be imposed to avoid too large a value:

$$\mathbf{h}K = \mathbf{F} \quad (3.29)$$

$$\text{if } \mathbf{h}_i > \text{limit } \mathbf{h}_i = \text{limit} \quad (3.30)$$

$$\text{if } \mathbf{h}_i \leq \text{limit } \mathbf{h}_i = \mathbf{h}_i \quad (3.31)$$

3.2 Other Resolution Methods

3.2.1 Newmark Method

The Newmark method is used to find the equilibrium position of a mechanical structure. The following example in one dimension explains the method in a simplified way.

The method consists first in calculating forces on the structure, then calculating the acceleration on the structure using the dynamic equation ($F = M\gamma$). From this acceleration and using a time step, the speed and the new position of the structure can be calculated [3].

For the example displayed in Fig. 3.1, the equilibrium calculation follows the path shown in Fig. 3.8 with a time step of 0.04 s. Figure 3.9 shows the residue of force. This calculation follows the Newmark explicit method [3].

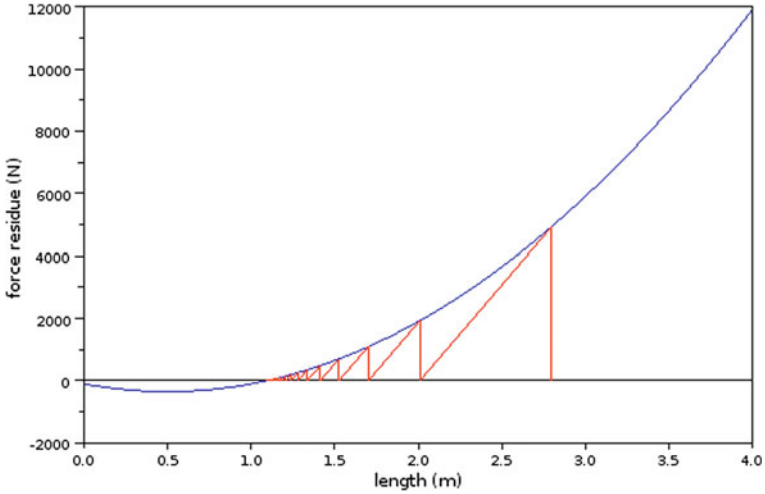


Fig. 3.8 Force on the mass function of spring length and Newmark explicit method iterations

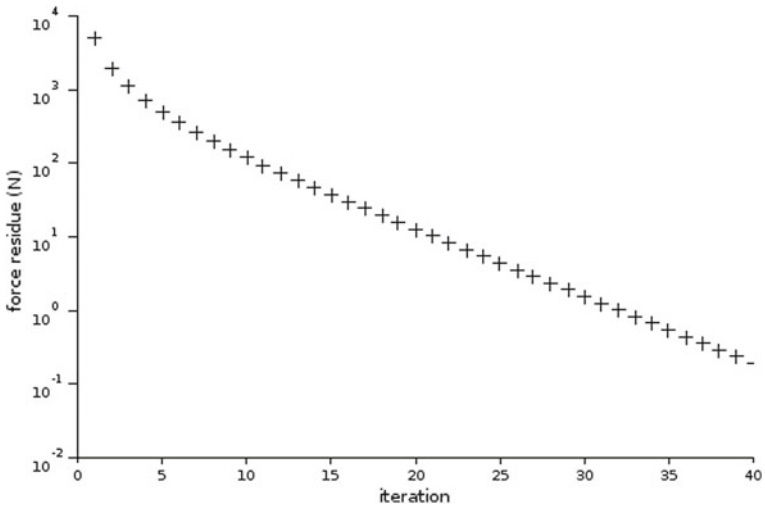


Fig. 3.9 Residue of force for each Newmark explicit method iterations

3.2.2 Energy Minimization

This method consists of finding the position of the structure that leads to the minimum of the energy. The energy involved here is the energy due to the conservative forces only. A conservative force is a force that leads to a variation of energy between two positions independent of the path between these two positions. The main conservative

forces involved in marine structures are weight and tension in elastic cables and netting twines.

In these cases the energy between two positions are quite simple to calculate:

$$E_W = W \Delta h \quad (3.32)$$

$$E_T = \frac{1}{2} K \Delta x^2 \quad (3.33)$$

E_W : energy due to the weight (J),

W : weight (N),

Δh : altitude variation between the two positions (m),

E_T : energy due to the tension (J),

K : constant cable stiffness (N/m),

Δx : cable length variation between the two positions (m).

Some forces are not conservative, as in the case of drag force. In such case the energy consumed by the drag depends on the path followed by the structure between the two positions.

Due to non conservative forces, the method of minimization of energy is not quite adapted to solve the equilibrium of marine structures. In case this method is used, the drag forces could be transformed into constant force.