Solid Mechanics and Its Applications
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Seyed Habibollah Hashemi Kachapi
Davood Domairry Ganji

Dynamics and Vibrations

Progress in Nonlinear Analysis

Springer

Solid Mechanics and Its Applications

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The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

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Preface

Introduction

Dynamical and vibratory systems are basically applications of mathematics and science to the solution of real-world problems. In the majority of real-life and applied phenomena in engineering sciences, as well as in a multiplicity of other sciences, solutions of specifically defined problems are the ultimate goal. In order to apply engineering or any other science, it is necessary to fully understand dynamical and vibratory systems and how to solve cases of either linear or nonlinear equations using analytical and numerical methods. It is of particular importance to study nonlinearity in dynamics and vibration, because almost all applied processes act nonlinearly. In addition, nonlinear analysis of complex systems is one of the most important and complicated tasks, especially in engineering and applied science problems.

There are only a handful of books that focus on nonlinear dynamics and vibrations analysis. Some of these books are written at a fundamental level that may not meet ambitious engineering program requirements. Others are specialized in certain fields of oscillatory systems, including modeling and simulation. In this book, we attempt to strike a balance between theory and practice, fundamentals and advanced subjects, and generality and specialization.

None of the books in this area have completely studied and analyzed nonlinear equations in dynamical and vibratory systems using the latest analytical and numerical methods, which, if included, would allow the user to solve problems without needing to study many different references. Therefore, in this book, we have chosen to use the latest analytic and numerical laboratory methods, referring to a bibliography of more than 300 books, papers, and research reports, many of them written by the authors of this book, and to consider almost all possible processes and physical configurations, thereby exploring new theories that have been proposed to solve real-life problems in engineering and applied sciences. In this way, the users (bachelor's, master's, and Ph.D. students, university teachers, and even workers in research centers in different fields of mechanical, civil, aerospace, electrical, chemical, applied mathematics, physics, etc.) can approach such systems with confidence. In the different chapters of the book, not only are linear and nonlinear problems, especially those in an oscillatory form, broadly discussed, but also applied examples are solved in a practical manner by the proposed methodology.

An abundant number of examples and homework problems are provided.

The users of this collection can achieve very strong capabilities in the area, especially of nonlinear phenomena in dynamically and vibratory systems, such as the following:

- A complete understanding of the nonlinearity sources and formulation of dynamical motion equations in different systems using the most general methods (e.g., principle of virtual work, D'Alembert's principle, Newton and Lagrange methods, etc.).
- A complete understanding of the fundamentals of oscillatory systems and their governing nonlinear equations; also analytical and numerical methods in solving applied problems, especially those with nonlinearities.
- A complete study of mathematical problems in engineering, analytic, and numeric methods (e.g., perturbation methods, the homotopy perturbation method, variational methods, energy methods, limit cycles, the parameterized perturbation method, the singular perturbation method, Adomian's decomposition method, the differential transformation method and its modification, He's parameter expansion method, He's amplitude–frequency formulation, the harmonic balance method, the coupled method of homotopy perturbation, the variational method, Floquet theory, etc.).
- Complete familiarity with specialized processes and applications in different areas of the field, studying them, eliminating complexities and controlling them, and also applying them in real-life engineering cases.
- A complete analysis of important engineering systems (e.g., NDOF systems, discs, springs, beams, normal modes, multibody phenomena, shafts, sliders, the human body, nonlinear oscillators in automobile design, rotating rigid frames, flexible beams, rotating rigid hubs, elastic cantilever beams, the human eardrum, etc.).
- A complete analysis of important equations in the field and their generalizations in real-life applications with practical examples (Duffing's oscillation, Van der Pol's oscillation, Mathieu's oscillation, Hamiltonian oscillation, Hill's oscillation, resonances, viscoelasticity, damping, fraction order, cubic nonlinearity, coupled systems, wave equations, etc.).
- The ability to encounter, model, and interpret an engineering process or system and to solve the related complexities engendered by the vibrations property in linear and nonlinear cases.

Audience

This book is a comprehensive and complete text on dynamical and vibratory motions and analytical and numerical methods in applied problems. It is self-contained, and the subject matter is presented in an organized and systematic manner. This book is quite appropriate for several groups of people, including the following:

- Senior undergraduate and graduate students taking courses in the mentioned fields.
- Professionals, for whom the book can be adapted for a short course on the subject matter.
- Design and research engineers, who will be able to draw upon the book in selecting and developing mathematical models for analytical and design purposes in applied conditions.
- Practicing engineers and managers who want to learn about the basic principles and concepts involved in the solving of problems using analytical and numerical methods such as dynamics, vibrations, and systems analysis and how they can be applied at their own workplaces.
- Generally, users who are bachelor's, master's, and Ph.D. students, university teachers, and even researchers at centers in different fields of mechanical, civil, aerospace engineering, applied physics, mathematics, and so forth.

Because the book is aimed at a wide audience, the level of mathematics is kept intentionally low. All the principles presented in the book are illustrated by numerous worked examples. The book draws a balance between theory and practice.

Acknowledgments

We are grateful to all those who have had a direct impact on this work. Many people working in the general areas of nonlinear phenomena, vibrations, oscillations, dynamics, mathematical, and physical problems, and analytical and numerical methods have influenced the format of this book.

The authors are very thankful to Babol Noshirvani University of Technology, Iran, and the National Elite Foundation of Iran (Bonyad Melli Nokhbeghan), Mazandaran Province, Sari, Iran, especially the nonlinear dynamics teams in the Mechanical Engineering Department, and all professors and students of all Iranian universities who helped them develop research skill, editing the electronic text, giving them useful consultation and precious guidance, and providing references for the authors, especially examples of applications that were used in different chapters.

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We are indebted to many colleagues and to numerous authors who have made contributions to the literature in this field. In addition, I am greatly indebted to all the authors of the articles listed in the bibliography of this book, such as Professors Ali Nayfeh, Shaher Momani, Liao, G. Adomian, J. Srinivas, Richard H. Rand, Alexander Fidlin, and other professors for their kind and faithful support and references that were directly used in the writing of this book. I would also like to thank the reviewers for their efforts and for their comments and suggestions, which have well served to compile the best possible book for the intended and targeted audience.

A special acknowledgment is expressed to the chief editor of the publisher for his excellent revision of the English language of the book and for editing the electronic text.

Finally, we would very much like to acknowledge the devotion, encouragement, patience, and support provided by our family members.

I would appreciate being informed of errors or receiving other comments about the book. Please write to the authors at the Babol Noshirvani University of Technology address or send e-mail to: sha.hashemi.kachapi@gmail.com (Seyed H. Hashemi Kachapi) and ddg_davood@ yahoo.com (D. D. Ganji)

We sincerely hope that the final outcome of this book will help students, researchers, and other users in developing an appreciation for the topic of analysis of nonlinear dynamical systems and nonlinear vibration analysis using analytical and numerical methods.

Contents

Notation and Units

Both the SI and the US/English systems of units have been used throughout the book.

Chapter 1
Introduction to Nonlinear Vibrations and Dynamics

1.1 Usual Sources of Nonlinearity in Mechanical and Other Engineering

1.1.1 Introduction

The world around us, and indeed we ourselves, are inherently subject to various nonlinearities. The simplest experiment illustrating this statement is an attempt to bend a wooden beam. As long as the load is small, the deflection of the beam is approximately proportional to the applied force. But at some sufficiently large level of this force, the beam will simply break. This strong and definitely irreversible change is an elementary example of nonlinear behavior that illustrates an important feature forcing us to formulate the first statement more precisely. The world is nonlinear, but in many cases, if we consider only small influences and changes, a linear approximation is often sufficient to understand, predict, and control its behavior.

Nonlinearities and their consequences in the physical and technical world are highly diversified, and the development of the corresponding theoretical framework and mathematical language is still in its infancy. We would like to start with several examples demonstrating the most usual sources of nonlinearity in engineering science, especially mechanical engineering and applied science (see Fidlin 2006).

1.1.2 Geometrical Nonlinearities

The first and the simplest nonlinearity is the geometrical one arising in the form of pure kinematics. The first example shows the pendulum (Fig. 1.1), whose dynamics is governed by the equation

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0. \tag{1.1}$$

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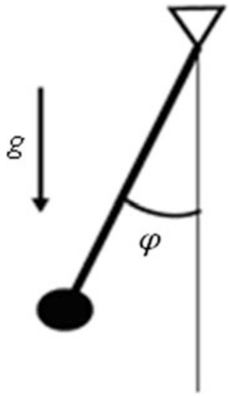


Fig. 1.1 The mathematical pendulum is one of the simplest examples of geometrically nonlinear systems

For small oscillations around the minimum equilibrium point, $\varphi = 0$, and this equation can be linearized. But if one is interested in large oscillations or even in the rotational motions of the pendulum, the system will be significantly nonlinear.

Another example appears in the crank mechanism, which is usual in all kinds of machines (Fig. 1.2). It consists of a rotating rod, which is attached to a fixed point by a spring with stiffness C and free length x_0. We assume that the spring is linear (which means that the deflection of this spring is proportional to the applied force, irrespective of its magnitude). The governing equation for this system has the form

$$\ddot{\varphi} = \frac{CL\left(\sqrt{L^2 + l^2 - 2Ll\cos\varphi} - x_0\right)}{ml\sqrt{L^2 + l^2 - 2Ll\cos\varphi}} \sin\varphi. \tag{1.2}$$

1.1.3 Physical Nonlinearities

The assumption concerning the linearity of the spring is correct only for small deflections. Both rubber (Fig. 1.3) and steel (Fig. 1.4) demonstrate nonlinear relationships between stress and strain if the applied load is sufficiently large.

Figures 1.3 and 1.4 are the most common examples of physical or material nonlinearity in mechanical engineering.

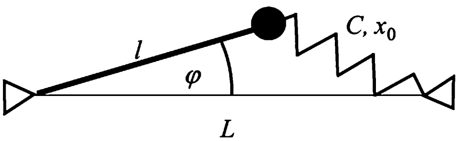


Fig. 1.2 The geometrically nonlinear crank

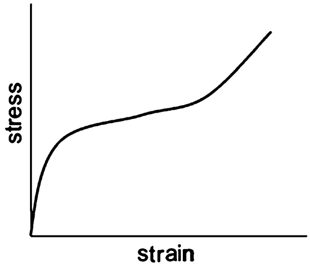


Fig. 1.3 Stress–strain diagram for a rubber-like material

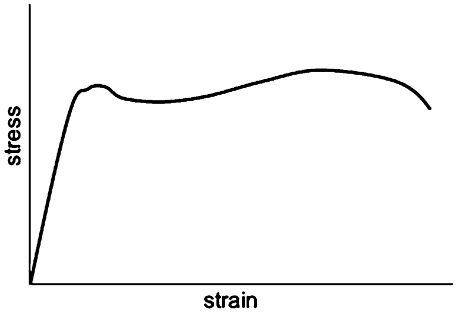


Fig. 1.4 Stress–strain diagram for steel

1.1.4 Structural or Designed Nonlinearities

The nonlinear characteristic of a spring is desired in numerous applications. Two simple examples of the designed nonlinearities are represented in Figs. 1.5 and 1.6.

The stiffness of the spring's system in the first example increases as the deflection exceeds a certain value of x_0, after which the spring on the right-hand side of the mass becomes active. This behavior is usually called *hardening* and is sometimes described as a progressive stiffness characteristic; here, it is achieved through the designed clearance between the two springs.

The stiffness of the spring's system in the second example decreases as the load exceeds a certain value of F_0, after which the spring on the right-hand side of the mass becomes active (since the external force is smaller than the preload F_0, the mass will be pressed against the stop by the spring on the right-hand side of it, and the whole frame moves as a solid body). This behavior is usually called *softening*.

These two examples belong to the group of structural nonlinearities and demonstrate how easily the nonlinear characteristic can be designed through an appropriate combination of linear components.

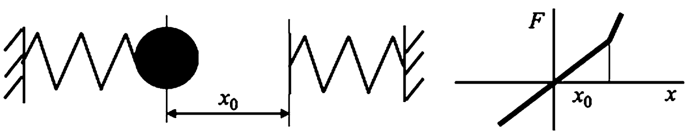


Fig. 1.5 Designed hard nonlinearity

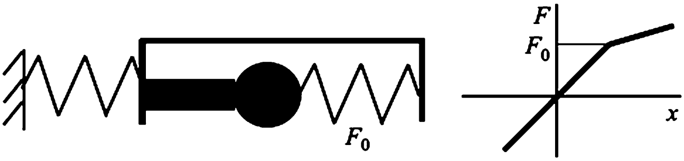


Fig. 1.6 Designed soft nonlinearity

1.1.5 Constraints

Unilateral constraints are another important example of structural nonlinearities. They are important sources of nonlinearities and will be discussed extensively in the following. Here, we give only one example (Fig. 1.7) showing the pendulum suspended near a rigid wall.

This is not a simple nonlinear system; in fact, it is very complex. Some additional assumptions are necessary in order to describe the collisions between the mass and the wall. In any case, this system cannot be linearized, at least as long as the framework of rigid body mechanics is chosen.

Kinematical constraints (not just unilateral ones) are important sources of nonlinearities. Although Newton's equations are linear with respect to coordinates and forces, Lagrange's equations in generalized coordinates (which take constraints implicitly into account and are represented in Sect. 1.1.3) are usually nonlinear. Consider the crank mechanism (Fig. 1.8) as an example.

Its position can be completely characterized by the angle φ between the crank and the horizontal axis. The position of the slider can always be expressed in terms of an angle such as

$$x = r\cos\varphi + l\sqrt{1 - \frac{r^2}{l^2}\sin^2\varphi}. \tag{1.3}$$

This relation is significantly nonlinear and leads to Lagrange's equations governing the mechanism:

Fig. 1.7 The pendulum near the wall and its torque characteristic

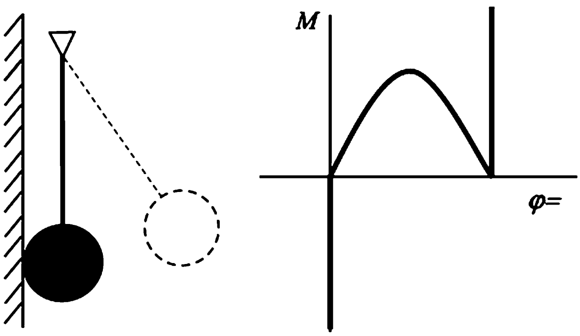
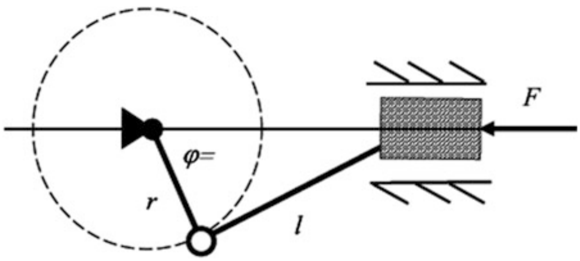


Fig. 1.8 Crank mechanism



$$\begin{aligned}
&T = \frac{1}{2}mr^2\,\dot{\varphi}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}J(\varphi)\,\dot{\varphi}^2\\
&J(\varphi) = mr^2 + Mr^2\sin^2\varphi\left(1 + \frac{r^2}{l^2}\cos^2\varphi \Big/ \left(1 - \frac{r^2}{l^2}\sin^2\varphi\right)\right)\\
&\delta A = F\delta x = Q\delta\varphi \Rightarrow Q = -r\sin\varphi\left(1 + \frac{r}{l}\cos\varphi \Big/ \sqrt{1 - \frac{r^2}{l^2}\sin^2\varphi}\right)\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\,\dot{\varphi}}\right) - \frac{\partial T}{\partial\varphi} = Q \Rightarrow J(\varphi)\,\ddot{\varphi} + \frac{1}{2}\frac{\mathrm{d}J}{\mathrm{d}\varphi}\dot{\varphi}^2 = Q
\end{aligned} \tag{1.4}$$

Here, mr^2 is the crank's inertia, M is the mass of the slider, T is the kinetic energy of the whole mechanism, and Q is the generalized force obtained through the relation for the virtual work.

This equation is totally nonlinear, but considering the second term on the left-hand side, which depends on $\dot{\varphi}^2$, it is obvious that this term is the consequence of the variable effective inertia $J(\varphi)$. Due to kinematical coupling (Eq. 1.3), this nonlinear dependency on the generalized velocities is usual for diverse mechanisms.

1.1.6 Nonlinearity of Friction

The last but not least source of nonlinearity we will mention here is damping. Damping mechanisms are extremely complex and deeply connected with microscopic processes in materials on their surfaces and in thin fluid films. Even the simplest models of viscous damping are nonlinear. Usually, they can be formulated in the form

$$\underline{R} = -f(|\underline{v}|)\frac{\underline{v}}{|\underline{v}|}. \tag{1.5}$$

Here, R is the damping force directed against the velocity. The function $f(|\underline{v}|)$ describes how the friction force depends on the magnitude of the velocity. The usual linear damping corresponds to $f(|\underline{v}|) = b|\underline{v}|$. This damping is extremely rare in applications. The only real case is the stationary flow in a long pipe. Nevertheless, linear damping is usually used if the real damping mechanism is unknown, but some energy dissipation is necessary for the analysis.

More realistic is the power law $f(v) = b|\underline{v}|^{\alpha}$, $1 < \alpha \leq 2$. It describes fluid damping at high Reynolds numbers. The case $\alpha = 2$ corresponds to the fully developed turbulent flow, which is typical for air. Dry friction in the contact between two surfaces depends both on the relative velocity and on the normal force in the contact area, $f(v) = \mu N$. In the one-dimensional case of dry friction, the relation (Eq. 1.5) is usually written as

$$R = -\mu N \operatorname{sgn}(v). \tag{1.6}$$

The friction coefficient μ is, however, not a constant. Even more, the formal relation (1.6) is not a function. It is not defined for $v = 0$. This special case corresponds to the so-called sticking and is usually described by a separate coefficient μ_s. The friction force during sticking cannot be calculated according to Eq. (1.6). It is determined by the condition $v = 0$ as long as the calculated value does not exceed the maximal value:

$$|R| \leq \mu_s N. \tag{1.7}$$

Slipping starts as soon as the inequality (Eq. 1.7) is broken in case of sticking.

The simplest Coulomb's friction law ($\mu = \text{const}$) is shown in Fig. 1.9a. The friction law corresponding with the sticking friction is shown in Fig. 1.9b. Figure 1.9c shows the friction coefficient corresponding with the negative slope of the force–velocity curve at small relative velocities. Decreasing of the friction coefficient was confirmed in numerous experiments for various friction partners and fluids (air, water, and oil) between them. Finally, Fig. 1.9d shows the regularized friction law, which is sometimes used in numerical simulations. The possibility of

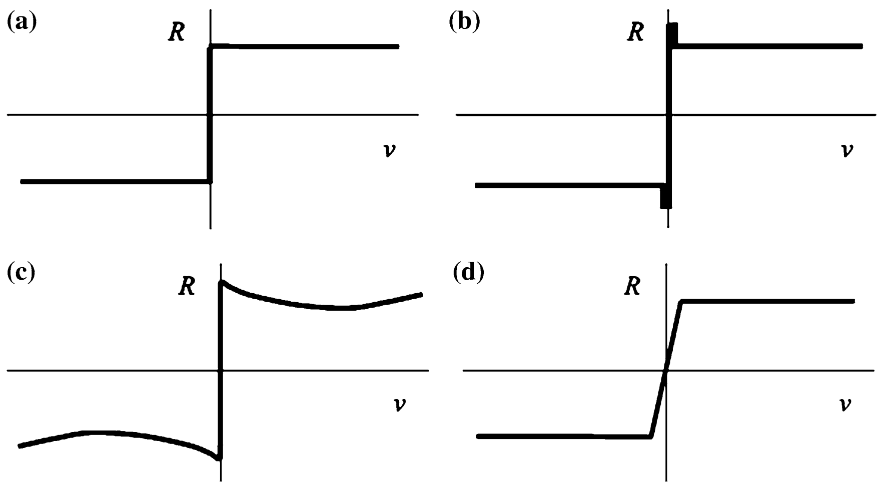


Fig. 1.9 Different idealization for dry friction

sticking (the vertical line for $v = 0$ is replaced through a quasi-viscous damping) is not taken into account and is applicable only to investigations in which sticking does not occur.

For more details in Sect. 1.1, refer to Fidlin (2006).

1.2 Formulation of Equations

1.2.1 Introduction

The most important and fundamental step in analyzing an engineering problem is to derive the equations governing the motion and dynamics of the system, unless investigating the problem proves to be impossible. Since the equations governing the motion of a body or system represent the nature of its analysis, obtaining these equations is of great importance.

These equations lead to the formation of ordinary or partial differential equations and different types of linear and nonlinear equations in general. Therefore, in this chapter, some fundamental methods of obtaining the governing equations are introduced along with applied examples, and in the following chapters, the methods of solving them are explained (see Dukkipati 2004a, b, 2006, 2009, 2010a, b; Dukkipati and Srinivas 2006).

1.2.2 Principle of Virtual Work

The principle of virtual work, due to Johann Bernoulli, is essentially a statement of the static or dynamic equilibrium of such an engineering mechanical system. A virtual displacement, denoted by δr, is an imaginary displacement and occurs without the passage of time. The virtual displacement being infinitesimal, it obeys the rules of differential calculus. It takes place instantaneously—that is, does not necessitate any time to materialize, $\delta t = 0$.

Consider an applied system with N particles in a three-dimensional space where Cartesian coordinates are (x_1, y_1, z_1,..., z_n). Suppose the system is subject to k holonomic constraints (holonomic constraints are those in which the generalized coordinates are related by a constraint equation) $\phi_j(x_1, y_1, z_1,..., z_n, t) = 0$, $j = 1,2,\ldots, k$. The virtual displacements δx_1, δy_1, δz_1, etc. are said to be compatible with the system's constraints if the constraint equations are still satisfied. That is,

$$\phi_j(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1, \ldots, z_N + \delta z_N, t) = 0. \tag{1.8}$$

Note that t is constant during the virtual displacement. Expanding Eq. (1.8) using Taylor's series around the original position, and neglecting the higher order terms in $\delta x_1, \delta y_1, \delta z_1$ etc., we obtain

$$\phi_j(x_1, y_1, z_1, \ldots, z_N, t) + \sum_{i=1}^{N} \left(\frac{\partial \phi_j}{\partial x_i} \delta x_i + \frac{\partial \phi_j}{\partial y_i} \delta y_i + \frac{\partial \phi_j}{\partial z_i} \delta z_i \right) = 0. \tag{1.9}$$

Hence,

$$\sum_{i=1}^{N} \left(\frac{\partial \phi_j}{\partial x_i} \partial x_i + \frac{\partial \phi_j}{\phi y_i} \partial y_i + \frac{\partial \phi_j}{\phi z_i} \partial z_i \right) = 0. \tag{1.10}$$

The condition for the virtual displacements should be compatible with the constraints.

Let $\bar{F}_i$ be the force acting on particle i, which is subject to a virtual displacement, δr_{i}. Then, the virtual work of the system is

$$\delta W = \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i. \tag{1.11}$$

Equation (1.11) represents the virtual work performed by the resultant force vector $\bar{F}_i$ over the virtual displacement vector δr_{i} of particle i.

When the system is in balance, the resultant force acting on each particle is zero. The resultant force is the sum of the applied forces and the reaction forces or the constraint forces. Hence, under balanced conditions, we will have

$$\bar{F}_i + \bar{R}_i = 0. \tag{1.12}$$

Therefore, the virtual work done by all the forces moving through an arbitrary virtual displacement that is compatible with the constraints is zero. Hence,

$$\sum_{i=1}^{N} (\bar{F}_i + \bar{R}_i) \cdot \delta \bar{r}_i = \sum_{i=1}^{N} \bar{F}_i \cdot \delta \bar{r}_i + \sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i. \tag{1.13}$$

In Eq. (1.13), $\sum_{i=1}^{N} \bar{R}_i \cdot \delta \bar{r}_i$ is the work done by constraint forces. Many constraint forces that commonly occur do not do any work during a virtual displacement because either they are perpendicular to the displacement or two equal and opposite reaction forces cancel the work done by each other.

Some examples of workless constraint forces are presented below:

1. A rigid rod connecting two particles (see Fig. 1.10). Internal forces are equal in magnitude and opposite in direction. Hence, the net work done by the internal forces is zero.

$$\begin{aligned} \bar{R}_2 &= -R_1, \\ \bar{e}_r \cdot \delta \bar{r}_1 &= \bar{e}_r \cdot \delta \bar{r}_2, \\ \delta W_c &= \bar{R}_1 \cdot \delta \bar{r}_1 + \bar{R}_2 \cdot \delta \bar{r}_2 = 0. \end{aligned}$$

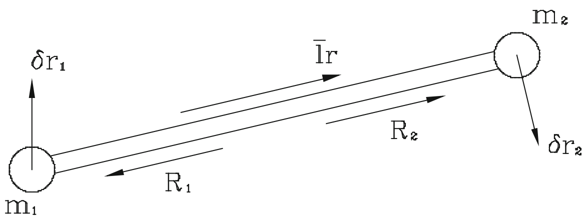


Fig. 1.10 A rigid rod connecting two particles

2. A body sliding without friction on a fixed surface (see Fig. 1.11). The normal reaction is perpendicular to the direction of motion, and hence, the work done by the normal reaction is zero. $\bar{R}$ is normal to $\delta\bar{r}$, and hence, $\bar{R} \cdot \delta\bar{r} = 0$.
3. A circular disk rolling without slipping along a straight horizontal path: (see Fig. 1.12). Note that an instantaneous center does not move during a virtual displacement (see Fig. 1.12). Hence, the work of the friction force, R_{f}, is zero

$$\bar{R}_{\mathrm{f}} \cdot 0 = 0.$$

Such constraints are called workless constraints. If the considered system has workless constraints, regarding Eq. (1.13), we will have

$$\sum_{i=1}^{N} \bar{R}_i \cdot \delta\bar{r}_i = 0. \tag{1.14}$$

Therefore, from Eq. (1.13), we will have

$$\delta W = \sum_{i=1}^{N} \bar{F}_i \cdot \delta\bar{r}_i = 0. \tag{1.15}$$

Equation (1.15) states that the work performed by the applied forces through infinitesimal virtual displacements, compatible with the system of constraints, is zero. This is known as the principle of virtual work.

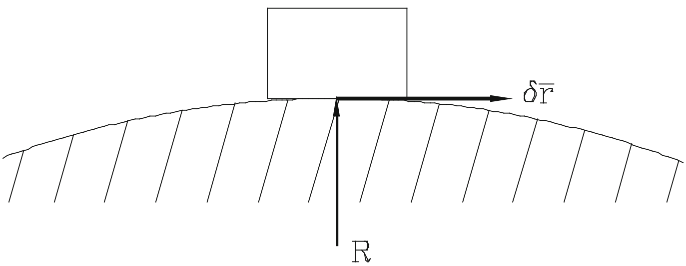


Fig. 1.11 A body sliding without friction on a fixed surface

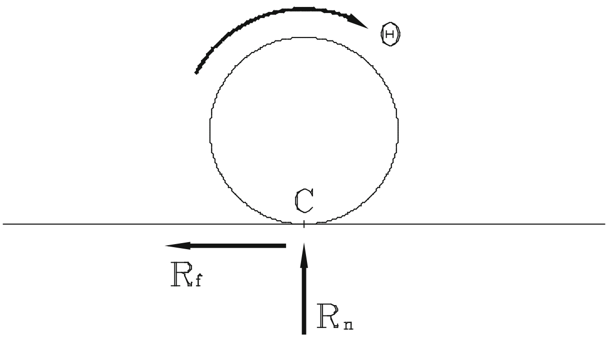


Fig. 1.12 A circular disk rolling without slipping

Now if we assume that the system is not in equilibrium, it will start to move in the direction of the resultant force. Since any motion must be compatible with the constraints, we can always choose a virtual displacement in the direction of the actual motion at each point. In such a case, the virtual work is positive, and therefore,

$$\sum_{i=1}^{N} \bar{F}_i \cdot \delta\bar{r}_i + \sum_{i=1}^{N} \bar{R}_i \cdot \delta\bar{r}_i > 0. \tag{1.16}$$

But the constraints are workless; hence,

$$\sum_{i=1}^{N} \bar{F}_i \cdot \delta\bar{r}_i > 0. \tag{1.17}$$

The above results are summarized in the principle of virtual work: The necessary and sufficient condition for the static equilibrium of the initially motionless scleronomic system, which is subjected to workless constraints, is that the zero virtual work will be done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.

It is sometimes convenient to assume that a set of δx_is, conforming to the instantaneous constraints, occurs during an interval of time δt. The corresponding ratios of the form $\delta x/\delta t$ have the dimensions of velocity and are known as virtual velocities. Accordingly, the principle of virtual power can be formulated: For a system at rest for a finite time, the total power of the system given by the product of the applied forces and the virtual velocity must vanish for all virtual velocities conforming to the constraints.

For the following holonomic and nonholonomic constraints:

$$\phi_j(x_1, y_1, z_1, \ldots, z_N) = 0, \tag{1.18}$$

$$\sum_{i=1}^{3N} a_{\mathrm{ji}} \mathrm{d}q_i + a_{\mathrm{ji}} \mathrm{d}t = 0. \tag{1.19}$$

The conditions for the virtual displacements and virtual velocities to be compatible with the constraints are, respectively,

$$\frac{\partial \phi_j}{\partial t} = 0 \tag{1.20}$$

and

$$a_{\mathrm{jt}} = 0. \tag{1.21}$$

A virtual displacement and virtual velocity can also be a possible real displacement and real velocity described by a set of dxs and assumed to occur during the time increment dt only if conditions (1.20) and (1.21) are satisfied. Since these conditions are not satisfied in a general case, a virtual displacement or virtual velocity is not, in general, a possible real displacement.

1.2.3 d'Alembert's Principle

Here, the principle of virtual work is extended to a form of dynamics, in which it is known as d'Alembert's principle. The Principle of Virtual Work is extended to the dynamic case by considering the inertia forces and considering the systems to be in dynamic equilibrium. Consider a system consisting of N particles. It is assumed that a typical mass particle m_i in a system of particles ($i = 1, 2,\ldots, N$) is acted upon by the applied force $\bar{F}_i$ and the constraint force $\bar{R}_i$. Also, we can apply Newton's second law for particle m_i if any inertial forces are negligibly small.

The equation of motion can be written as

$$\bar{F}_i + \bar{R}_i - m_i\ddot{\bar{r}}_i = 0, \quad i = 1, 2, \ldots, N, \tag{1.22}$$

where $\bar{F}_i$ is the applied force, $\bar{R}_i$ is the constraint force and $-m_i\ddot{r}_i$ is the reversed effective force or the inertia force. Equation 1.22 states that the sum of all the forces, external and inertial, acting on each particle of the system is zero. This is known as d'Alembert's principle. Extending the principle of virtual work to a dynamic equilibrium state results in

$$\delta W = \sum_{i=1}^{N} (\bar{F}_i + \bar{R}_i - m_i\ddot{r})_i \cdot \delta\bar{r}_i = 0. \tag{1.23}$$

Equation 1.23 embodies both the virtual work principle of statics and d'Alembert's principle and is often referred to as the generalized principle of d'Alembert or the Lagrange version of d'Alembert's principle.

If the system has workless constraints and we choose $\delta\bar{r}_i$ to be reversible virtual displacements consistent with the constraints, we will have

$$\sum_{i=1}^{N} (\bar{F}_i - m_i\ddot{r})_i \cdot \delta\bar{r}_i = 0. \quad (1.24)$$

In Eq. 1.24, the sum of the applied force $\bar{F}_i$ and the inertia force $-m_i\ddot{r}_i$ or $(\bar{F}_i - m_i\ddot{r}_i)$ is called the effective force acting on particle m_i.

The generalized principle of d'Alembert states that the virtual work performed by the effective forces through infinitesimal virtual displacements, compatible with the system constraints, is zero.

1.2.4 Lagrange's Equations of Motion

We derive Lagrange's equations of motion for a dynamic system using an application of d'Alembert's principle and the principle of virtual work. First, we treat only holonomic systems and later generalize the results to nonholonomic systems.

1.2.4.1 Holonomic Systems

Consider a dynamic system of N particles. Using d'Alembert's principle and the principle of virtual work, we have seen in Sect. 4.6 that

$$\sum_{k=1}^{N} \left(\vec{F}_k - m_k\ddot{\vec{r}}_k\right)\delta\vec{r}_k = 0, \quad (1.25)$$

where $\vec{F}_K$ is the impressed force on the ith particle of mass m_k. Let the system have n degrees of freedom. For this holonomic system, if we choose q_1, q_2,..., q_n as n generalized coordinates, then we have the transformation equation, between the vector coordinates of the particles and the n generalized coordinates, as

$$\vec{r}_k = \vec{r}_k(q_1, q_2, \ldots, q_n, t). \quad (1.26)$$

Then the velocities of the particles are

$$\dot{\vec{r}}_k = \sum_{i=1}^{n} \frac{\partial r_k}{\partial q_i}\dot{q}_i + \frac{\partial \vec{r}_k}{dt}. \quad (1.27)$$

Expressing the virtual displacements $\delta\bar{r}_k$ in terms of the n generalized coordinates

$$\delta\bar{r}_k = \sum_{i=1}^{n} \frac{\partial \bar{r}_k}{\partial q_i}\delta q_i, \quad (1.28)$$

Equation (1.25) becomes

$$\sum_{k=1}^{N}\sum_{i=1}^{n}\left(\bar{F}_k\frac{\partial\bar{r}_k}{\partial q_i}-m_k\ddot{\bar{r}}_k\frac{\partial\bar{r}_k}{\partial q_i}\right)\delta q_i=0, \tag{1.29}$$

and Eq. (1.29) becomes

$$\sum_{\mathrm{k}=1}^{N}F_k\frac{\partial r_k}{\partial q_i}=Q_i. \tag{1.30}$$

Q_i is called the generalized force in the direction of the ith generalized coordinate.

The other term involving the accelerations—that is, $m_k\ddot{\bar{r}}_k\frac{\partial \mathrm{r_k}}{\partial \mathrm{q_i}}$—may be related to the kinetic energy of the system as follows.

From Eqs. 1.25 and 1.27, we obtain

$$\frac{\partial\dot{\bar{r}}_k}{\partial\dot{q}_i}=\frac{\partial\bar{r}_k}{\partial q_i}. \tag{1.31}$$

The expression for kinetic energy is

$$T=\frac{1}{2}\sum_{k=1}^{N}m_k\dot{\bar{r}}_k{}^2 \tag{1.32}$$

and

$$\frac{\partial T}{\partial\dot{q}_i}=\sum_{k=1}^{N}m_k\dot{\bar{r}}_k\frac{\partial\dot{\bar{r}}_k}{\partial\dot{q}_i}, \tag{1.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_i}\right)=\sum_{k=1}^{N}m_k\ddot{\bar{r}}_k\frac{\partial\dot{\bar{r}}_k}{\partial\dot{q}_i}+\sum_{k=1}^{N}m_k\dot{\bar{r}}_k\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\bar{r}}_k}{\partial\dot{q}_i}\right). \tag{1.34}$$

From Eqs. 1.31 and 1.34, and noting that the order of differentiation can be changed, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_i}\right)=\sum_{k=1}^{N}m_k\ddot{\bar{r}}_k\frac{\partial\bar{r}_k}{\partial q_i}+\sum_{k=1}^{N}m_k\ddot{\bar{r}}_k\frac{\partial\dot{\bar{r}}_k}{\partial q_i}. \tag{1.35}$$

From Eq. 1.32, we have

$$\frac{\partial T}{\partial\dot{q}_i}=\sum_{k=1}^{N}m_k\dot{\bar{r}}_k\frac{\partial\dot{\bar{r}}_k}{\partial\dot{q}_i}. \tag{1.36}$$

Substituting Eq. 1.36 into 1.35, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{q}_i}\right)=\sum_{k=1}^{N}m_k\ddot{\bar{r}}_k\frac{\partial\bar{r}_k}{\partial q_i}+\frac{\partial T}{\partial q_i}. \tag{1.37}$$

Substituting Eqs. 1.37 and 1.28 into 1.27, we get

$$\sum_{i=1}^{n}\left[Q_i - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}\right]\delta q_i = 0. \tag{1.38}$$

Since q_i's are generalized coordinates for a holonomic system, they are independent. Hence, to satisfy Eq. 1.38, the coefficients of δq_i must be zero. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i = 0. \tag{1.39}$$

For conservative systems,

$$Q_i = -\frac{\partial V}{\partial q_i}, \tag{1.40}$$

where V is the potential energy of the system, and hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0. \tag{1.41}$$

Equation 1.41 is Lagrange's equation for a conservative system. Expressing $T - V = L$, known as the Lagrangian correlation, we will have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1.42}$$

If there are forces not derivable from a potential function V—that is, for nonconservative systems

$$Q_i = -\frac{\partial V}{\partial q_i} + Q_i' \tag{1.43}$$

then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'. \tag{1.44}$$

1.2.4.2 Nonholonomic Systems

The derivation of the Lagrange equations for holonomic systems requires the generalized coordinates to be independent. For a nonholonomic system, however, there must be a larger number of generalized coordinates than the number of degrees of freedom. Therefore, the δqs are not independent if we assume a virtual displacement consistent with the constraints. Let n coordinates q_1, q_2,…, q_n be chosen to describe the motion. Then, there are m nonholonomic constraint equations of the form

$$\sum_{i=1}^{n} a_{\mathrm{ji}} \mathrm{d}q_i + a_{\mathrm{jt}} \mathrm{d}t = 0, \quad j = 1, 2, \ldots, m, \tag{1.45}$$

where a_{ji}, $i = 1, 2, \ldots, n$, are functions of q_i. The degrees of freedom are given by $(n - m)$, and the coordinates q_i are not all independent. The δqs must meet the following conditions:

$$\sum_{i=1}^{n} a_{\mathrm{ji}} \delta q_i = 0 \quad j = 1, 2, \ldots, m. \tag{1.46}$$

Let us assume that each generalized applied force Q_i is obtained from a potential function and assume workless constraints. The generalized constraint force C_i must meet the condition

$$\sum_{i=1}^{N} C_i \delta q_i = 0 \tag{1.47}$$

for any virtual displacement consistent with the constraints.

Multiply Eq. 1.46 by factor λ_j, known as the Lagrange multiplier, and obtain the m equations as

$$\lambda_j \sum_{i=1}^{n} a_{\mathrm{ji}} \delta q_i = 0, \quad j = 1, 2, \ldots, m. \tag{1.48}$$

Subtract the sum of these m equations from Eq. 1.47. Then, by interchanging the order of summation, we obtain

$$\sum_{i=1}^{n} \left(C_i - \sum_{j=1}^{m} \lambda_j a_{\mathrm{ji}} \right) \delta q_i = 0. \tag{1.49}$$

Now, if we choose λ_j's in such a way that

$$C_i = \sum_{j=1}^{m} \lambda_j a_{\mathrm{ji}}, \quad i = 1, 2, \ldots, n, \tag{1.50}$$

then the coefficients of δq's are zero, and Eq. (1.49) will apply for any δq's; that is, the δq's are independent.

Now the generalized force C_i can be equated to Q_i' or the force not derivable by a potential function,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{\mathrm{ji}}, \quad i = 1, 2, \ldots, n. \tag{1.50a}$$

In addition to these n equations we have m equations of constraints to solve for $(n + m)$ unknowns—that is, n q's and m λ's. The Lagrange multiplier relates the constraints to constraint forces (see Eq. 1.50).

1.2.4.3 Summary of Lagrange's Method

The following are typical problems with Lagrange formulations (see Mark 2005):

- Be sure to first establish the number of degrees of freedom and then formulate all energy terms in only those variables. Clearly identify which degrees of freedom are relative coordinates versus absolute coordinates. Be careful about rotational/translation problems.
- For kinetic energy terms, be sure to formulate absolute velocities before taking derivatives. Be careful about 2-D and 3-D vector motions.
- For potential energy terms, be sure that the actual deflection, described by relative and/or absolute coordinates, is described as well in terms of spring elements. Be careful about 2-D and 3-D vector motions.
- There should be only one total kinetic energy equation, one total potential energy equation, and one total dissipative energy equation for the system. The kinetic, potential, and dissipative energy equation should be involved only with the N generalized coordinates and the constants (mass, damping, stiffness) of the system.
- Apply the Lagrange equation once for each generalized coordinate. For N degrees of freedom, N generalized coordinates will yield N equations of motion.
- If necessary, linearize the equations of motion by neglecting nonlinear terms in the equations of motion. Note that the linear equations of motion may not adequately describe the original equations of motion.

1.2.5 Newton's Method

The following are typical problems with Newton's (or d'Alembert's) formulations (see Mark 2005):

- Be sure to first establish the number of degrees of freedom and then formulate all terms in only those variables. Clearly identify which degrees of freedom are relative coordinates versus absolute coordinates. Also, clearly identify what will be the positive direction of motion for each coordinate. Be careful about rotational/translation problems. State any constraint relationships related to independent and dependent coordinates.
- Evaluate the static balance for the problem in order to determine whether the orientation of the system in the gravitational field will affect the equation of motion (are the weights of the objects balanced an initial static deflection in the springs?). When in doubt, perform a static force balance to determine the appropriate constraint equation.
- For displacement, velocity, and acceleration terms, be sure to develop absolute or relative displacement, velocity, and acceleration of appropriate points as required. Be careful about 2-D and 3-D vector motions.

- Be sure to draw the appropriate free-body diagram for each mass (or combination of masses) in the system.
- Whenever the system is separated in order to draw a free-body diagram, replace the separation with the appropriate internal force/moments on each side of the separation.
- Do not move force/moments arbitrarily from one mass to another. The internal forces account for the effects of one mass on another.
- Develop one equation of motion for each degree of freedom of the system using Newton's (or d'Alembert's) method. Be sure to watch for moving reference frame issues. Also, check that the units are the same for each term in an equation (forces + moments: NOT!).

If necessary, once the exact equations of motion have been determined, we linearize them by neglecting their nonlinear terms. (Note that this could lead to a not insignificant chance of a description of the motions that will be inadequate for solving the problem.)

1.2.5.1 Newton's Equation

The corresponding Fig. 1.13

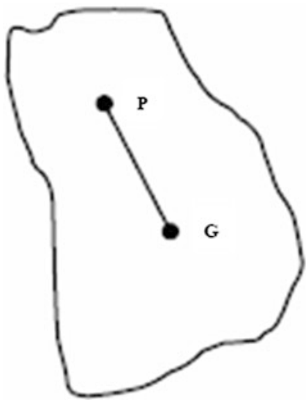
Force balance is

$$\sum \overline{F} = M\ddot{\overline{q}}_G \left(-\text{or} - \sum \overline{F} - M\ddot{\overline{q}}_G = 0\right). \tag{1.51}$$

Moment balance is

$$\sum \overline{M}_p = J_p\ddot{\overline{\theta}} + \overline{r}_{G/p} \times M\ddot{\overline{q}}_p \left(-\text{or} - \sum \overline{M}_p - J\ddot{\overline{\theta}} - \overline{r}_{G/p} \times M\ddot{\overline{q}}_p = 0\right). \tag{1.52}$$

Fig. 1.13



For more details on Sect. 1.2, refer to Dukkipati (2004, 2006), Dukkipati and Srinivas (2006) and Mark (2005).

1.3 Applied Examples

Example 1.1

Consider two rigid bodies connected together and moving in a plane (Fig. 1.14). This may be considered as a typical robot arm. M_1 and M_2 are motor torques.

All the constraints are holonomic, and there are two generalized coordinates. We choose

$$q_1 = \theta_1; \quad q_2 = \theta_2.$$

We could choose ϕ instead of θ_2, but θ_2 is somewhat easier since we need the velocities and angular velocities relative to the inertial frame (x, y) to determine the kinetic energy.

Kinetic Energy. For two rigid bodies, first consider body #1 (Fig. 1.14):

$$v_B = l_1\dot{\theta}_1$$
$$T_1 = \frac{1}{2}m_1 v_B^2 + \frac{1}{2}\bar{I}_1\omega_1^2 = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}\bar{I}_1\dot{\theta}_1^2 = \frac{1}{2}I_1\dot{\theta}_1^2$$

where

$$I_1 = \bar{I}_1 + m_1 l_1^2$$

is the moment of inertia about A by the parallel axis theorem (Fig. 1.15).

Fig. 1.14

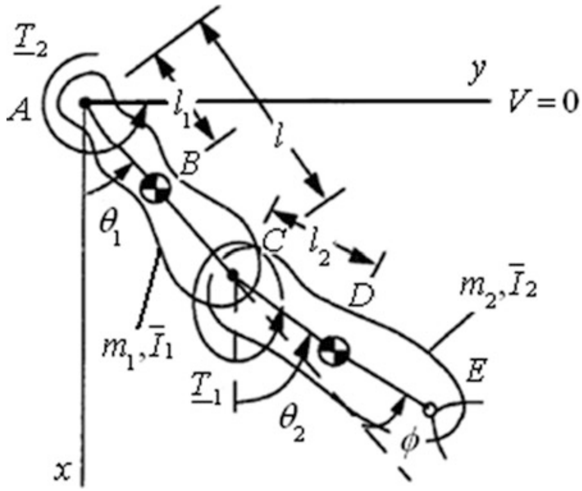
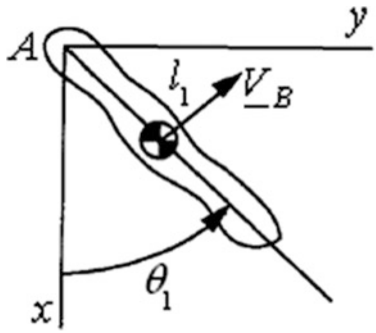


Fig. 1.15



Now consider body #2 (Fig. 1.16). Let $\{\hat{i}_1, \hat{j}_1\}, \{\hat{i}_2, \hat{j}_2\}$ be reference frames fixed in the ground, in link AC and in link CD, respectively. Use kinematic analysis to relate the velocity of D relative to $\{\hat{i}_3, \hat{j}_3\}$ and its velocity relative to $\{\hat{i}_1, \hat{j}_1\}$, and let ω be the angular velocity of $\{\hat{i}_3, \hat{j}_3\}$ relative to $\{\hat{i}_1, \hat{j}_1\}$,

$$\underline{v}_D = \underline{v}_C + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{r}$$

where

$$\underline{v}_D = \frac{\mathrm{d}\underline{r}_D}{\mathrm{d}t}, \quad \underline{v}_C = \frac{\mathrm{d}\underline{r}_C}{\mathrm{d}t}.$$

The terms are

$$\underline{v}_C = l\theta_1\hat{j}_2, \quad \underline{v}_{\text{rel}} = \underline{0}, \quad \underline{\omega} = l\theta_2\hat{k}_3, \quad \underline{r} = l_2\hat{i}_3.$$

The required unit vector transformation is

$$\hat{j}_2 = \sin(\theta_2 - \theta_1)\hat{i}_3 + \cos(\theta_2 - \theta_1)\hat{j}_3.$$

Substituting, we get

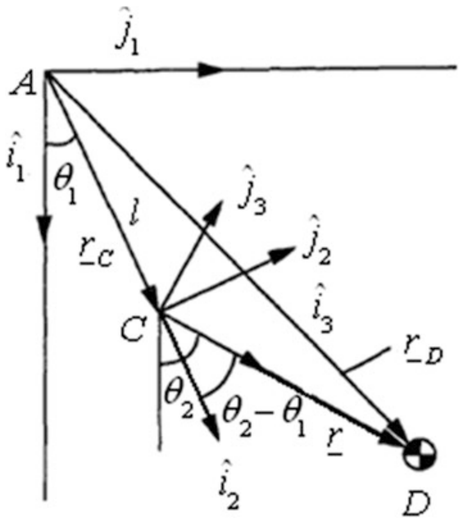
$$\begin{aligned} \underline{v}_D &= l\theta_1\hat{j}_2 + \underline{0} + \theta_2\hat{k}_3 \times l_2\hat{i}_3 \\ &= l\theta_1 \sin(\theta_2 - \theta_1)\hat{i}_3 + \left[l_2\theta_2 + l\theta_1 \cos(\theta_2 - \theta_1)\hat{j}_3\right] \end{aligned}$$

(This also could be obtained geometrically from the law of cosines). T_2 is then

$$\begin{aligned} T_2 &= \frac{1}{2}m_2 v_D^2 + \frac{1}{2}\bar{I}_2\omega_2^2 \\ &= \frac{1}{2}m_2\left[l^2\,\dot{\theta}_1^2 + l_2^2\,\dot{\theta}_2^2 + 2ll_2\,\dot{\theta}_1\,\dot{\underset{2}{\theta}}\cos(\theta_2 - \theta_1)\right] + \frac{1}{2}\bar{I}_2\,\dot{\theta}_2^2. \end{aligned}$$

Note that this equation is quadratic in the velocity components with displacement-dependent coefficients; the constant and linear terms are missing. Consequently,

Fig. 1.16



$$T = T_1 + T_2 = \frac{1}{2}I' \dot{\theta}_1^2 + \frac{1}{2}I_2 \dot{\theta}_2^2 + m_2 l l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1),$$

where

$$I' = I_1 + m_2 l^2; \quad I_2 = \bar{I}_2 + m_2 l_2^2.$$

Potential Energy and Lagrangian. Since the only given force is gravity,

$$\begin{aligned} V &= -m_1 g l_1 \cos\theta_1 - m_2 g(l\cos\theta_1 + l_2\cos\theta_2), \\ L = T - V &= \frac{1}{2}I'\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + m_2 l l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) \\ &\quad + m_1 g l_1 \cos\theta_1 + m_2 g(l\cos\theta_1 + l_2\cos\theta_2) \end{aligned}$$

Lagrange's Equations. Consider the following Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \theta_1}\right) - \frac{\partial L}{\partial \theta_1} = Q_1; \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \theta_2}\right) - \frac{\partial L}{\partial \theta_2} = Q_2.$$

Computing some of the terms,

$$\begin{aligned} \frac{\partial L}{\partial \theta_1} &= I'\theta_1 + m_2 l l_2 \theta_2 \cos(\theta_2 - \theta_1) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \theta_1}\right) &= I'\theta_1 + m_2 l l_2[\theta_2\cos(\theta_2 - \theta_1) - \theta_2\sin(\theta_2 - \theta_1)(\theta_2 - \theta_1)] \\ \frac{\partial L}{\partial \theta_1} &= m_2 l l_2 \theta_1\theta_2 \sin(\theta_2 - \theta_1) - m_1 g l_1 \sin\theta_1 - m_2 g l \sin\theta_1 \end{aligned}$$

Finally, substitution gives the equations of motion:

$$I'\theta_1 + A\cos(\theta_2 - \theta_1)\theta_2 - A\sin(\theta_2 - \theta_1)\theta_2^2 + B\sin\theta_1 = M_1$$
$$I_2\theta_2 + A\cos(\theta_2 - \theta_1)\theta_1 - A\sin(\theta_2 - \theta_1)\theta_1^2 + C\sin\theta_2 = M_2$$

where

$$Q_1 = M_1, \quad Q_2 = M_2$$

and

$$A = m_2 l l_1; \quad B = m_1 g l_1 + m_2 g l; \quad C = m_2 g l_2.$$

Equations of motion are dynamically coupled and highly nonlinear, making their solution difficult. Linearizing for small angles and small angular rates and setting $M_1 = M_2 = 0$ gives the special case of the double physical pendulum:

$$I'\ddot{\theta}_1 + Ac\ddot{\theta}_2 + B\theta_1 = 0$$
$$I_2\ddot{\theta}_2 + A\ddot{\theta}_1 + Cs\theta_2 = 0$$

These equations are linear but still dynamically coupled. They may be easily dynamically uncoupled by a change of variables. Modal analysis of these equations for typical cases reveals two modes, one rapid and one relatively slow.

Example 1.2

Figure 1.17 shows a system of a uniform rigid link of mass m and length ℓ and two linear springs of stiffness k_1 and k_2. The link is in horizontal position when the springs are unstretched. Derive the equilibrium equation for the system using the principle of virtual work.

Solution

Let C be the center of gravity of the link AB.

Referring to Fig. 1.18, the virtual work principle can be written as

$$\delta W = -k_1 x \delta x - k_2 y \delta y + mg\frac{1}{2}\delta y = 0.$$

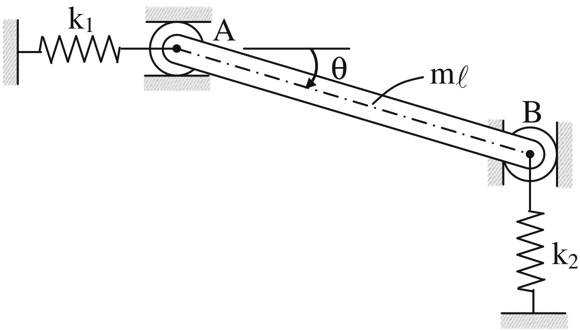
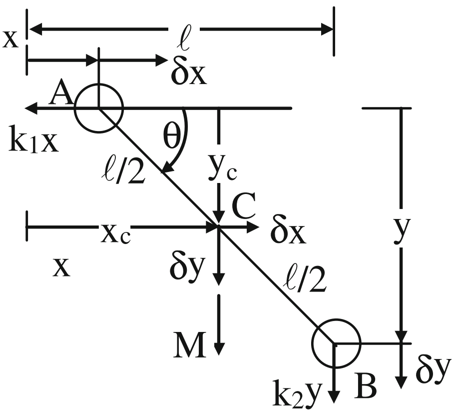


Fig. 1.17 Uniform rigid link

Fig. 1.18



The rectangular coordinates x and y and the rectangular virtual displacements δx and δy in terms of the generalized coordinate θ and the generalized virtual displacement $\delta\theta$ are

$$\begin{aligned} x &= \ell(1 - \cos\theta), \\ y &= \ell\sin\theta, \\ \delta x &= \ell\sin\theta\ \delta\theta, \\ \delta y &= \ell\cos\theta\ \delta\theta \end{aligned}$$

Therefore, the virtual work principle in terms of the generalized displacement is

$$\delta W = \left[-k_1\ell(1 - \cos\theta)\ell\sin\theta - k_2\ell\sin\theta\ \ell\cos\theta + \frac{1}{2}mg\ \ell\cos\theta \right]\delta\theta = 0.$$

Since $\delta\theta$ is arbitrary, the coefficient of $\delta\theta$ must be zero. Hence, the equilibrium equation is

$$\frac{mg}{2\ell} = k_1\ \tan\theta + (k_2 - k_1)\sin\theta.$$

Example 1.3

Derive the equation of motion for the system of Example 1.2 by means of d'Alembert's principle.

Solution

Referring to Fig. 1.18, we have

$$\begin{aligned} x_C &= \frac{1}{2}x, \ddot{x}_C = \frac{1}{2}\ddot{x} \text{ and } \delta x_C = \frac{1}{2}\delta x, \\ y_C &= \frac{1}{2}y, \ddot{y}_C = \frac{1}{2}\ddot{y} \text{ and } \delta y_C = \frac{1}{2}\delta y. \end{aligned}$$

The generalized principle of d'Alembert, for this system can be written as

$$\begin{aligned}&-k_1x\delta x+mg\delta y_C-k_2y\delta y-m\ddot{x}_C\delta x_C-m\ddot{y}_C\delta y_C-I_C\ddot{\theta}\delta\theta\\&=\left(-k_1x-\frac{1}{4}m\ddot{x}\right)\delta x+\left(-k_2y+\frac{1}{2}mg-\frac{1}{4}m\ddot{y}\right)\delta y-I_C\ddot{\theta}\delta\theta=0.\end{aligned}$$

The relations between rectangular coordinates x and y and generalized coordinate θ are given by

$$\begin{aligned}&x=\ell(1-\cos\theta)\Rightarrow\dot{x}=\ell\sin\theta\dot{\theta},\\&\ddot{x}=\ell(\cos\theta\dot{\theta}^2+\sin\ddot{\theta}),\delta x=\ell\sin\theta\delta\theta,\\&y=\ell\sin\theta\Rightarrow\dot{y}=\ell\cos\theta\dot{\theta},\\&\ddot{y}=\ell(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}),\delta y=\ell\cos\theta\delta\theta.\end{aligned}$$

Also,

$$I_c=\frac{1}{12}m\ell^2;$$

therefore,

$$\begin{aligned}&\left\{\left[-k_1\ell(1-\cos\theta)-\frac{1}{4}\ell m(\cos\theta\dot{\theta}^2+\sin\theta\ddot{\theta})\right]\ell\sin\theta\right.\\&\left.+\left[-k_2\ell\sin\theta-\frac{1}{4}\ell m(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta})+\frac{1}{2}mg\right]\ell\cos\theta-\frac{1}{12}m\ell^2\ddot{\theta}\right\}\delta\theta\\&=\left[-k_1\ell^2(1-\cos\theta)\sin\theta-k_2\ell^2\sin\theta\cos\theta+\frac{1}{2}mg\ell\cos\theta-\frac{1}{3}m\ell^2\ddot{\theta}\right]\delta\theta=0.\end{aligned}$$

Since $\delta\theta$ is arbitrary, its coefficient must be zero. Hence, the equation of motion is

$$\frac{1}{3}m\ddot{\theta}+k_1(1-\cos\theta)\sin\theta+k_2\sin\theta\cos\theta-\frac{1}{2}m\frac{g}{L}\cos\theta=0.$$

Example 1.4
Figure 1.19 shows a uniform rigid disk of radius r rolling without slipping inside a circular track of radius R. Obtain the differential equation of motion for the system.

Fig. 1.19

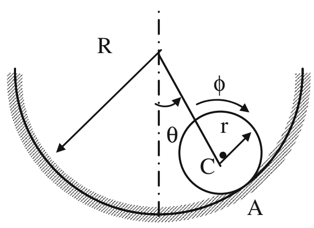
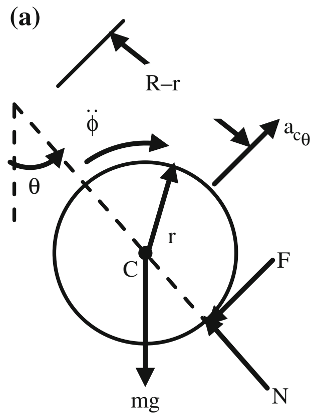


Fig. 1.20



Solution

Referring to the free-body diagram, we can write the force equation in the transverse direction as (Fig. 1.20)

$$F_\theta = -F - mg\sin\theta = ma_{c\theta} = m(R-r)\ddot{\theta}.$$

The moment equation about the mass center is

$$M_C = Fr = I_c\alpha = \frac{mr^2}{2}\ \ddot{\phi}.$$

The velocity v_c of the mass center of the disk is given by

$$v_c = (R-r)\dot{\theta} = r\dot{\phi}.$$

Combining the previous equations, we obtain the equation of motion as

$$\begin{aligned} m(R-r)\ddot{\theta} + F + mg\sin\theta &= m(R-r)\ddot{\theta} + \frac{mr^2}{2}\ \ddot{\phi} + mg\sin\theta \\ &= \frac{3}{2}m(R-r)\ddot{\theta} + mg\sin\theta = 0, \end{aligned}$$

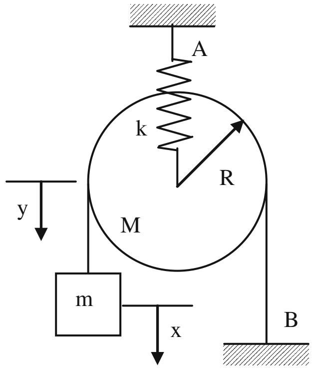
or

$$\ddot{\theta} + \frac{2g}{3(R-r)}\sin\theta = 0.$$

For small angles $\theta \approx 0$, $\sin\theta = \theta$ and the previous equation reduces to

$$\ddot{\theta} + \frac{2g}{3(R-r)}\theta = 0.$$

Fig. 1.21



Example 1.5

Use the energy method to find the natural frequency of vibration of the spring mass pulley system if the mass m is displaced slightly and released. In Fig. 1.21, the cord is inextensible.

Solution

The kinetic energy of the system = kinetic energy of mass + kinetic energy of the pulley:

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left(\frac{\dot{x}^2}{4}\right) + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{2R}\right)^2 \\ &= \frac{1}{2}m\dot{x}^2 + \frac{3}{16}M\dot{x}^2. \end{aligned}$$

The potential energy of the system = the spring's elastic energy:

$$V = \frac{1}{2}ky^2 = \frac{1}{2}k\left(\frac{x}{2}\right)^2 = \frac{1}{8}x^2.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}(T+V) = 0$$

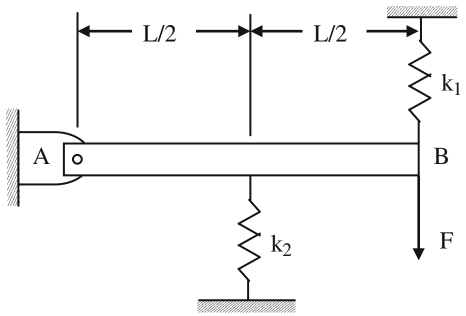
or

$$m\dot{x}\ddot{x} + \frac{3}{8}M\dot{x}\ddot{x} + \frac{1}{4}kx\dot{x} = 0$$

or

$$\dot{x}\left(4m\ddot{x} + \frac{3}{2}M\ddot{x} + kx\right) = 0.$$

Fig. 1.22



The equation of motion is

$$\ddot{x}\left(4m+\frac{3}{2}M\right)+kx=0.$$

The natural frequency of vibration is

$$\omega_n=\sqrt{\frac{k}{\left(4m+\frac{3}{2}M\right)}}\,\text{rad/s}.$$

Example 1.6
For an undamped vibrating system with a single degree of freedom, shown in Fig. 1.22, $k_1 = 10$ lb/in, $k_2 = 20$ lb/in., $F = 30 \sin 5t$, $L = 20$ in.

The weight of the uniform slender bar is 30 lb. Determine the differential equation of motion of the bar for small oscillations.

Solution
For a small oscillation of the bar, and considering that the bar is horizontal at equilibrium, the differential equation of motion can be written as (see Fig. 1.23):

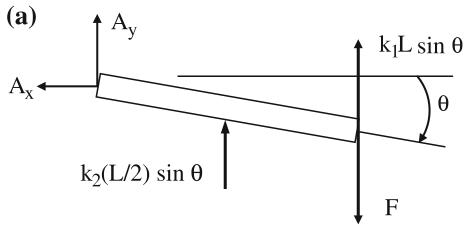
$$\Sigma M_{\mathrm{A}} = I_{\mathrm{A}}\alpha$$

or

$$FL\cos\theta-(k_1L\sin\theta)(\mathrm{L}\cos\theta-\left(k_2\frac{L}{2}\sin\theta\right)\left(\frac{L}{2}\cos\theta\right)=\frac{1}{3}mL^2\ddot{\theta}$$

or

Fig. 1.23



$$\ddot{\theta} + \frac{12k_1 + 3k_2}{4m}\theta = \frac{3F}{mL}.$$

Substituting the given values gives the differential equation of motion as

$$\ddot{\theta} + 579.60\theta = 57.96 \text{ sin } 5t.$$

Example 1.7
The support of a simple pendulum shown in Fig. 1.24 has a specified motion $y = Y_0$ sin ωt. Assuming small oscillations of the system, determine the differential equation of motion.
Solution
For small angular oscillations, the displacement of the mass m in the horizontal direction is given by

$$x = y + \ell\theta = Y_0 \text{ sin } \omega t + \ell\theta.$$

The velocity and acceleration of the mass are

$$\dot{x} = \dot{y} + \ell\dot{\theta} = \omega Y_0 \text{sin } \omega t + \ell\dot{\theta},$$
$$\ddot{x} = \ddot{y} + \ell = -\omega Y_0 \text{ sin } \omega t + \ell\ddot{\theta}.$$

Let R_x and R_y denote the reaction forces, as shown in Fig. 1.25.

Taking the moments of the applied and inertia forces about O, we obtain the dynamic equation

Fig. 1.24

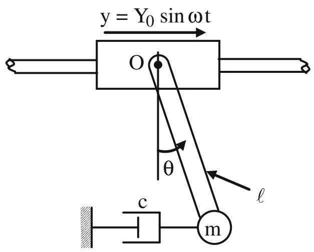
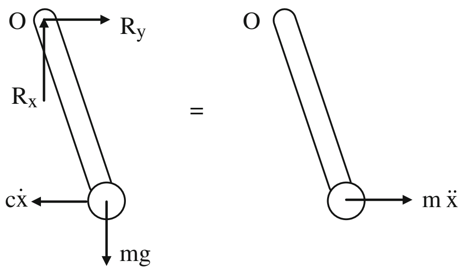


Fig. 1.25



$$-mg\ell\sin\theta - c\dot{x}\ell\cos\theta = m\ddot{x}\ell\cos\theta.$$

For small angular oscillation, sin $\theta \approx \theta$ and cos $\theta \approx 1$. Hence,

$$\mathrm{m}\ddot{x}\ell + \mathrm{c}\ell + \mathrm{mg}\ell\theta = 0.$$

Example 1.8
Figure 1.26 shows a double pendulum. Assuming small amplitudes and $m_1 = m_2 = m$, $\ell_1 = \ell_2 = \ell$ and using the coordinates x_1 and x_2, determine the differential equations of motion.
Solution
Taking moments about O for mass m_1 (see free body diagram, Fig. 1.27), one obtains

Fig. 1.26

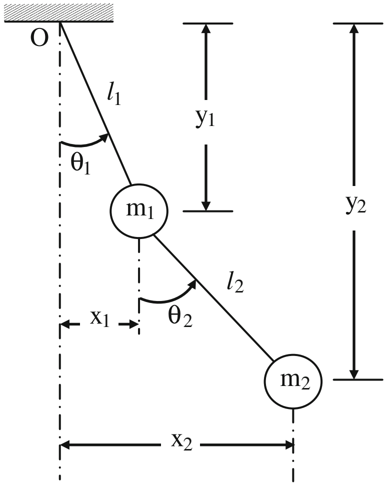
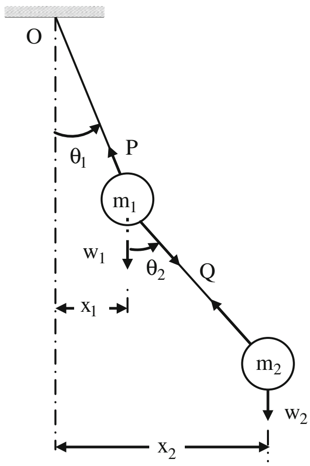


Fig. 1.27



$$m_1\ell_1^2\ddot{\theta}_1 = -w_1(\ell_1 \sin\theta_1) + Q\sin\theta_2(\ell_1\cos\theta_1) - Q\cos\theta_2(\ell_1\sin\theta_1)$$
$$= -w_1\ell_1\theta_1 + w_2\ell_1(\theta_2 - \theta_1),$$

assuming $Q \sim \_w_2$,

$$m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_2\left(\ell_1\ddot{\theta}_1\right) = -w_2(\ell_2\sin\theta_2) = -w_2\ell_2\theta_2.$$

Using the relations $\theta_1 = \frac{x_1}{\ell_1}$ and $\theta_2 = \frac{x_2 - x_1}{\ell_2}$, the previous equations become

$$m_1\ell_1\ddot{x}_1 + \left[\omega_1 + \omega_2\left(\frac{\ell_1 + \ell_2}{\ell_2}\right)\right]x_1 - \frac{\omega_2\ell_1}{\ell_2}x_2 = 0$$
$$m_1\ell_2\ddot{x}_2 - w_2x_1 + w_2x_2 = 0.$$

When $m_1 = m_2 = m,\ \ \ell_1 = \ell_2 = \ell$, and $\omega_1 = \omega_2 = mg$, we get

$$m\ell\ddot{x}_1 + 3mgx_1 - mgx_2 = 0,$$
$$m\ell\ddot{x}_2 - mgx_1 + mgx_2 = 0.$$

Example 1.9
Figure 1.28 shows an undamped vibrating system with two degrees of freedom. Assuming small angles of oscillation, obtain the equation of motion for the system.
Solution
Applying Newton's law $\sum F = ma$, we have $m_1\ddot{x}_1 = -m_2\ddot{x}_2 - 2kx_1$, where $x_2 = x_1 + \ell\sin\theta$ and ℓ is the length of the pendulum. For small angles of oscillation, $\ddot{x}_2 = \ddot{x}_1 + \ell\cos\theta\ \ddot{\theta}$.Hence, the equation of motion can be written as

$$(m_1 + m_2)\ddot{x}_1 + 2kx_1 + m_2\ell\ddot{\theta} = 0$$

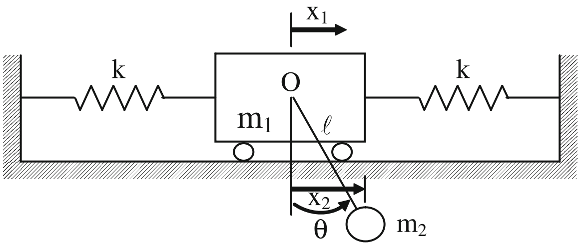
Taking the moments about point O, we get

$$-m_2\ddot{x}_2\ell\cos\theta = (m_2g - m_2\ddot{y})\ell\sin\theta$$

where $y = \ell\cos\ \theta$ and $\ddot{y} = -\ell\sin\theta\ \ddot{\theta}$. Assuming small angles of oscillation, sin $\theta\ \approx\ \theta$ and cos $\theta\ \approx\ 1$, the previous equation becomes

$$\ddot{\theta} + \left(\frac{g}{\ell}\right)\theta + \frac{\ddot{x}_1}{\ell} = 0.$$

Fig. 1.28



If m_1 is held stationary, the system is reduced to a single-degree-of-freedom system having simple pendulum action—that is,

$$\ddot{\theta} + \left(\frac{g}{\ell}\right)\theta = 0.$$

Example 1.10

Determine the two natural frequencies and mode shapes for a thin rod suspended as a pendulum shown in Fig. 1.29.

Solution

Figure 1.30 shows the free-body diagram of the system.

Applying $\Sigma \underline{M}_G = I_G \ddot{\underline{\theta}}_2$, it results in $-T\frac{\ell}{2}\sin(\theta_2 - \theta_1) = \frac{1}{12}m\ell^2\ddot{\theta}_2$ and $\Sigma \underline{F}_x = m\ddot{\underline{x}}$ gives $-T\sin\theta_1 = m\ddot{x}$.

For small angles, $\sin\theta \simeq \theta$ and $T = W$, $\ddot{x} \approx \frac{\ell}{2}\ddot{\theta}_1 + \frac{\ell}{2}\ddot{\theta}_2$.

Hence, previous equations become

$$-W\frac{\ell}{2}(\theta_2 - \theta_1) = \frac{1}{12}m\ell^2\ddot{\theta}_2,$$
$$-W\theta_1 = m\frac{\ell}{2}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right).$$

Fig. 1.29

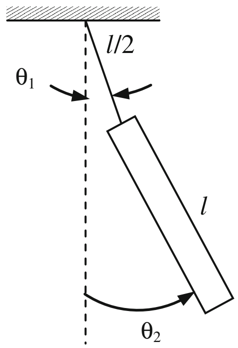


Fig. 1.30

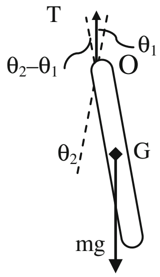
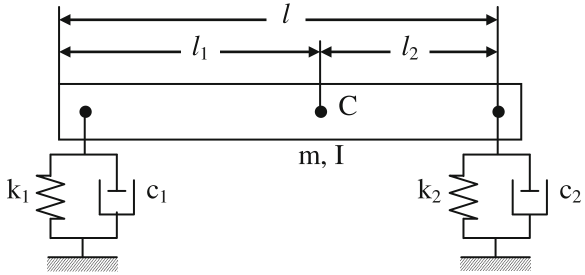


Fig. 1.31



Example 1.11
Determine the matrix differential equations of motion of the damped two-degrees-of-freedom system shown in Fig. 1.31.
Solution
Noting that motion is only in the vertical direction, and applying Newton's laws of motion, the differential equations of motion from the free-body diagrams (Fig. 1.32) can be written as coupled linear and angular oscillations

$$m\ddot{y} = -c_1(\dot{y} - \ell_1\dot{\theta}) - k_1(y - \ell_1\theta) - c_2(\dot{y} + \ell_2\dot{\theta}) - k_2(y + \ell_2\theta),$$
$$I\ddot{\theta} = c_1(\dot{y} - \ell_1\dot{\theta})\ell_1 + k_1(y - \ell_1\theta)\ell_1 - c_2(\dot{y} + \ell_2\dot{\theta})\ell_2 - k_2(y + \ell_2\theta)\ell_2.$$

The previous equations can be written as

$$m\ddot{y} + (c_1 + c_2)\dot{y} - (c_1\ell_1 - c_2\ell_2)\dot{\theta} + (k_1 + k_2)y - (k_1\ell_1 - k_2\ell_2)\theta = 0,$$
$$I\ddot{\theta} + \left(c_1\ell_1^2 + c_2\ell_2^2\right)\dot{\theta} - (c_1\ell_1 - c_2\ell_2)\dot{y} + \left(k_1\ell_1^2 + k_2\ell_2^2\right)\theta - (k_1\ell_1 - k_2\ell_2)y = 0,$$

or in the matrix form,

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -(c_1\ell_1 - c_2\ell_2) \\ -(c_1\ell_1 - c_2\ell_2) & c_1\ell_1^2 + c_2\ell_2^2 \end{bmatrix}\begin{bmatrix} \dot{y} \\ \dot{\theta} \end{bmatrix}$$
$$+ \begin{bmatrix} k_1 + k_2 & k_2\ell_2 - k_1\ell_1 \\ k_2\ell_2 - k_1\ell_1 & k_1\ell_1^2 + k_2\ell_2^2 \end{bmatrix}\begin{bmatrix} y \\ j\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Example 1.12
Determine the general motion of the forced vibration system shown in Fig. 1.33.
Solution
The equations of motion for forced damped vibrations are

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2)x_1 - c_2\dot{x}_2 - k_2x_2 = F_0 \sin \omega t,$$
$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 - c_2\dot{x}_1 - k_2x_1 = 0.$$

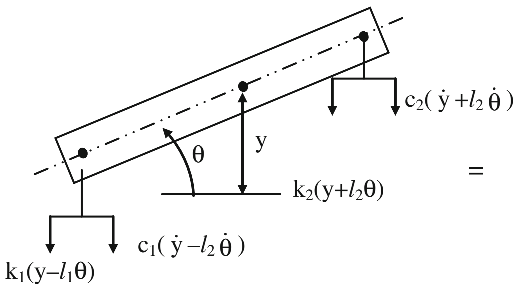
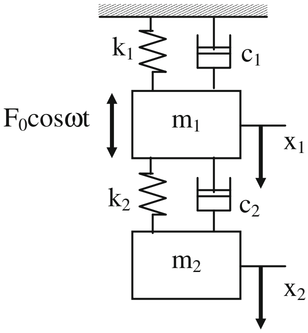


Fig. 1.32 Free body diagrams

Fig. 1.33 Forced vibration system



Example 1.13

For the triple pendulum shown in Fig. 1.34, derive Newton's equations of motion. The angles θ_1, θ_2 and θ_3 can be considered as arbitrarily large.

Solution

Referring to the free-body diagram in Fig. 1.35, we can write

Fig. 1.34 Triple pendulum

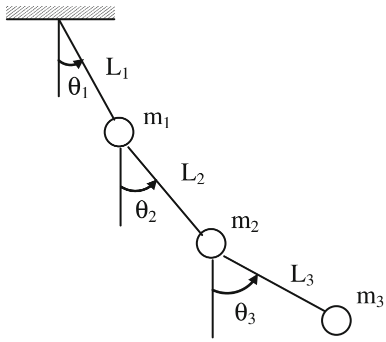
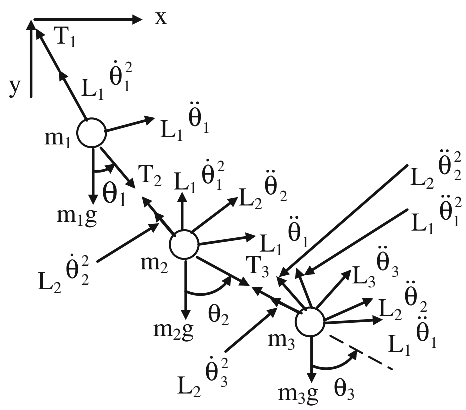


Fig. 1.35 Free-body diagram



$$\sum F_{t1} = T_2\sin(\theta_2-\theta_1) - m_1 g\sin\theta_1 = m_1 L_1\ddot{\theta}_1,$$
$$\sum F_{n1} = T_1 - T_2\cos(\theta_2-\theta_1) - m_1 g\cos\theta_1 = m_1 L_1\dot{\theta}_1^2,$$
$$\sum F_{t2} = T_3\sin(\theta_3-\theta_2) - m_2 g\sin\theta_2$$
$$= m_2\left[L_2\ddot{\theta}_2 + L_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_2-\theta_1)\right],$$
$$\sum F_{n2} = T_2 - T_3\cos(\theta_3-\theta_2) - m_2 g\cos\theta_2$$
$$= m_2\left[L_2\dot{\theta}_2^2 - L_1\ddot{\theta}_1\sin(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\cos(\theta_2-\theta_1)\right],$$
$$\sum F_{t3} = m_3 g\sin\theta_3 = m_3\left[L_3\ddot{\theta}_3 + L_1\ddot{\theta}_1\cos(\theta_3-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_3-\theta_1)\right.$$
$$\left.+L_2\ddot{\theta}_2\cos(\theta_3-\theta_2) + L_2\dot{\theta}_2^2\sin(\theta_3-\theta_2)\right],$$
$$\sum F_{n3} = T_3 - m_3 g\cos\theta_3 = m_3\left[L_3\dot{\theta}_2^3 - L_1\ddot{\theta}_1\sin(\theta_3-\theta_1) + L_1\dot{\theta}_1^2\cos(\theta_3-\theta_1)\right.$$
$$\left.-L_2\ddot{\theta}_2\sin(\theta_3-\theta_2) + L_2\dot{\theta}_2^2\cos(\theta_3-\theta_2)\right].$$

Eliminating T_1, T_2, and T_3 and rearranging, we get the equations of motion as

$$(m_1+m_2+m_3)\left(L_1\ddot{\theta}_1 + g\sin\theta_1\right) + (m_2+m_3)L_2\left[\ddot{\theta}_2\cos(\theta_2-\theta_1)\right.$$
$$\left.-\dot{\theta}_2^2\sin(\theta_2-\theta_1)\right] + m_3 L_3\left[\ddot{\theta}_3\cos(\theta_3-\theta_1) - \dot{\theta}_3^2\sin(\theta_3-\theta_1)\right] = 0,$$
$$(m_2+m_3)\left[L_2\ddot{\theta}_2 + L_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_2-\theta_1) + g\sin\theta_2\right]$$
$$+ m_3 L_3\left[\ddot{\theta}_3\cos(\theta_3-\theta_2) - \dot{\theta}_3^2\sin(\theta_3-\theta_2)\right] = 0,$$
$$m_3\left[L_3\ddot{\theta}_3 + L_1\ddot{\theta}_1\cos(\theta_3-\theta_1) + L_2\ddot{\theta}_2\cos(\theta_3-\theta_2)\right.$$
$$\left.+L_1\dot{\theta}_1^2\sin(\theta_3-\theta_1) + L_2\dot{\theta}_2^2\sin(\theta_3-\theta_2) + g\sin\theta_3\right] = 0.$$

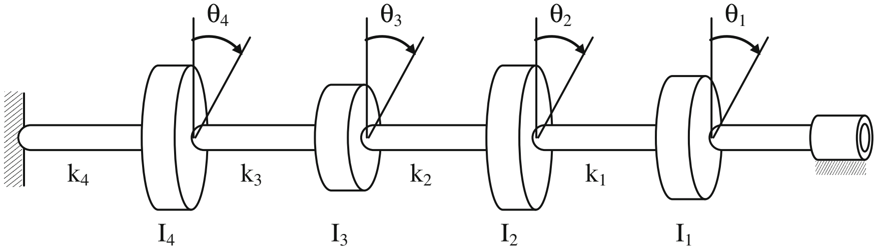


Fig. 1.36 Four-degrees-of-freedom system

Example 1.14

For the four-degrees-of-freedom system shown in Fig. 1.36:

(a) determine the kinetic and potential energy functions for the system,
(b) obtain the equations of motion governing the motion of the system using Lagrange's equation.

Solution

(a)

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}I_4\dot{\theta}_4^2.$$

V is equal to the work done by the shaft as it returns from the dynamic configuration to the reference equilibrium position, or

$$V = \int_{\theta}^{0} (-k\theta)\mathrm{d}\theta = \frac{k\theta^2}{2},$$

(b)

$$V = \frac{1}{2}k_1(\theta_1 - \theta_2)^2 + \frac{1}{2}k_2(\theta_2 - \theta_3)^2 + \frac{1}{2}k_3(\theta_3 - \theta_4)^2 + \frac{1}{2}k_4\theta_4^2.$$

The Lagrange equations for subscript $k = 1$ are

$$\frac{\partial T}{\partial \dot{\theta} 1} = I_1\dot{\theta}_1 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = I_1\ddot{\theta}_1,$$
$$\frac{\partial T}{\partial \theta_1} = 0,$$
$$\frac{\partial V}{\partial \theta_1} = k_1(\theta_1 - \theta_2).$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$$

gives

$$I_1\ddot{\theta}_1 + k_1(\theta_1 - \theta_2) = 0.$$

Similarly, letting subscript $k = 2, 3, 4$ in Lagrange's relation yields

$$\begin{aligned} I_2\ddot{\theta}_2 + k_1(\theta_1 - \theta_2) + k_2(\theta_2 - \theta_3) &= 0, \\ I_3\ddot{\theta}_3 + k_2(\theta_2 - \theta_3) + k_3(\theta_3 - \theta_4) &= 0, \\ I_4\ddot{\theta}_4 + k_3(\theta_3 - \theta_4) + k_4\theta_4 &= 0. \end{aligned}$$

Example 1.15
Derive the differential equations governing the motion of the system shown in Fig. 1.37 using the Lagrange equation and the generalized coordinates θ_1, θ_2, θ_3, and θ_4.
Solution
Writing $k = k_{\mathrm{AB}} = \frac{JG}{\ell}$, we have

$$\begin{aligned} k_{\mathrm{BC}} &= \frac{JG}{0.5\ell} = 2k, \\ k_{\mathrm{CD}} &= \frac{JG}{1.5\ell} = \frac{2}{3}k, \\ k_{\mathrm{DE}} &= \frac{JG}{0.5\ell} = 2k. \end{aligned}$$

The kinetic energy of the system is

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}I_3\dot{\theta}_3^2 + \frac{1}{2}I_4\dot{\theta}_4^2.$$

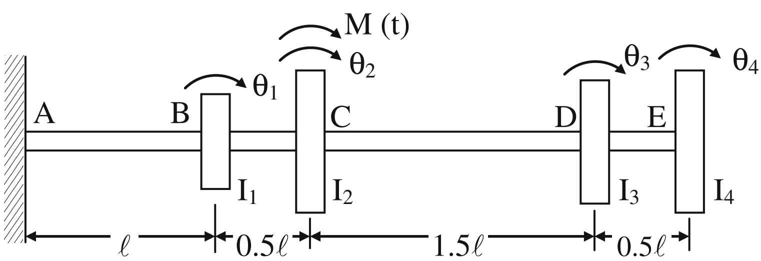


Fig. 1.37 Four-degrees-of-freedom system

The potential energy of the system is

$$V = \frac{1}{2}k\theta_1^2 + \frac{1}{2}(2k)(\theta_2 - \theta_1)^2 + \frac{1}{2}\left(\frac{2}{3}k\right)(\theta_3 - \theta_2)^2 + \frac{1}{2}(2k)\theta_4^2.$$

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\left[I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2 + I_4\dot{\theta}_4^2 - k\theta_1^2 - 2k(\theta_2 - \theta_1)^2 - \frac{2}{3}k(\theta_3 - \theta_2) - 2k\theta_4^2\right]. \end{aligned}$$

Applying Lagrange's equation,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}(I_1\dot{\theta}_1) + k\theta_1 + 2k(\theta_2 - \theta_1)(-1) = 0, \end{aligned}$$

or

$$\begin{aligned} &I_1\ddot{\theta}_1 + 3kq_1 - 2kq_2 = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(I_2\dot{\theta}_2\right) + 2k(\theta_2 - \theta_1) + \frac{2}{3}k(\theta_3 - \theta_2)(-1) = 0, \end{aligned}$$

or

$$\begin{aligned} &I_2\ddot{\theta}_2 2\theta q_1 + \frac{8}{3}k\theta_2 - \frac{2}{3}k\theta_3 = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}(I_3\dot{\theta}_3) + \frac{2}{3}k(\theta_3 - \theta_2) + 2k(\theta_4 - \theta_3)(-1) = 0, \end{aligned}$$

or

$$\begin{aligned} &I_3\ddot{\theta}_3 - \frac{2}{3}k\theta_2 + \frac{8}{3}k\theta_3 - 2k\theta_4 = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_4}\right) - \frac{\partial L}{\partial \theta_4} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}(I_4\dot{\theta}_4) + 2k(\theta_4 - \theta_3) = 0, \end{aligned}$$

or

$$I_4\ddot{\theta}_4 - 2k\theta_3 + 2k\theta_4 = 0.$$

The work done by the external moment when virtual rotations are introduced is given by

$$\delta W = M(t)\delta\theta_2.$$

The components of the force vector are therefore given by

$$\begin{aligned} F_1 &= 0, \\ F_2 &= M(t), \\ F_3 &= 0, \text{ and} \\ F_4 &= 0. \end{aligned}$$

The differential equations of motion can be arranged in the matrix form as

$$\begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_4 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 & 0 \\ -2k & \frac{8}{3}k & \frac{-2}{3}k & 0 \\ 0 & \frac{-2}{3}k & \frac{8}{3}k & -2k \\ 0 & 0 & -2k & 2k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M(t) \\ 0 \\ 0 \end{Bmatrix}$$

Example 1.16
For the simplified model of an automobile shown in Fig. 1.38, derive the equations of motion using Newton's second law of motion.
Solution
Referring to the free-body diagram shown in Fig. 1.39, and applying Newton's second law of motion, we have

$$\begin{aligned} m\dot{x}_3 &= -c_2(\dot{x}_3 - \ell_1\dot{\theta} - \dot{x}_1) - k_2(x_3 - \ell_1\theta - x_1) - c_2(\dot{x}_3 - \ell_1\dot{\theta} - \dot{x}_2) - k_2(x_3 - \ell_2\theta - x_2), \\ J_G\ddot{\theta} &= c_2(\dot{x}_3 - \ell_1\dot{\theta} - \dot{x}_1)\ell_1 + k_2(x_3 - \ell_1\theta - x_1)\ell_1 - c_2(\dot{x}_3 + \ell_2\dot{\theta} - \dot{x}_2)\ell_2 \\ &\quad - k_2(x_3 + \ell_2\theta - x_2)\ell_2, \\ m_1\ddot{x}_1 &= -c_2(\dot{x}_1 - \dot{x}_3 + \ell\dot{\theta}) - k_2(x_1 - x_3 + \ell_1\theta) - c_1\dot{x}_1 - k_1x_1 + F_1, \\ m_2\ddot{x}_2 &= -c_2(\dot{x}_2 - \dot{x}_3 - \ell_2\dot{\theta}) - k_2(x_1 - x_3 - \ell_2\theta) - k_1x_2 - c_1\dot{x}_2 + F_2, \end{aligned}$$

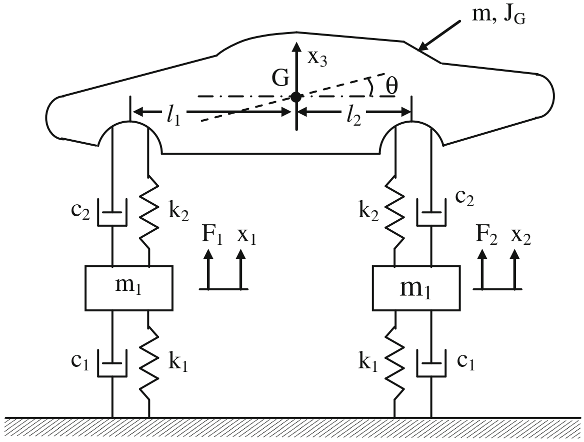


Fig. 1.38 Simplified model of an automobile

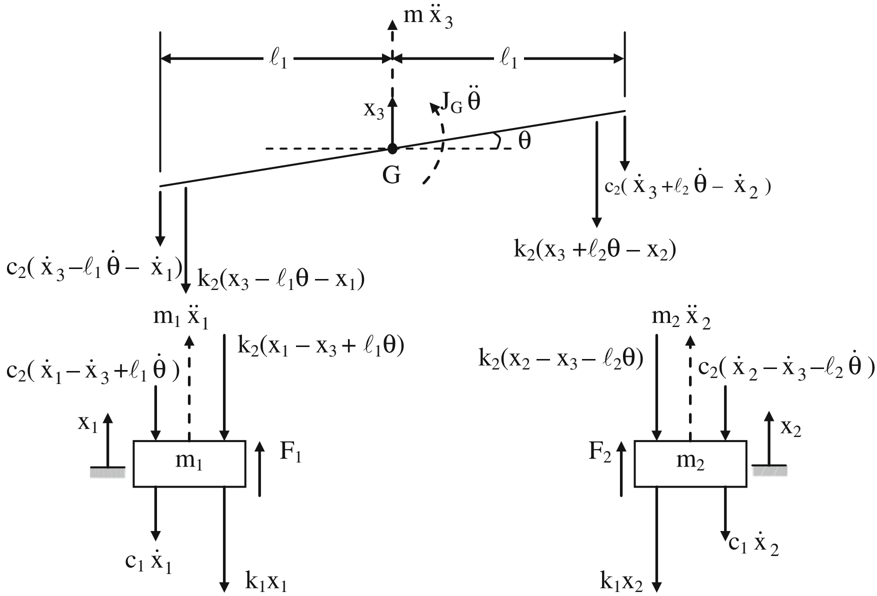


Fig. 1.39 Free-body diagram

which can be rewritten as

$$
\begin{aligned}
&m\dot{x}_3 + 2c_2\dot{x}_3 - c_2\dot{x}_1 - c_2\dot{x}_2 + 2k_2x_3 + \theta(k_2\ell_2 - k_2\ell_1) - k_2x_1 - k_2x_2 = 0,\\
&J_G\ddot{\theta} + (c_2\ell_1^2 + c_2\ell_2^2)\dot{\theta} + (-c_2\ell_1 + c_2\ell_2)\dot{x}_3 + c_2\ell_1\dot{x}_1 - c_2\ell_2\dot{x}_2\\
&\;+ (-k_2\ell_1 + k_2\ell_2)x_3 + \mathrm{k}_2\ell\ell_1x_1 - \mathrm{k}_2\ell_2x_2 + (k_2\ell_1^2 + k_2\ell_2^2)\theta = 0,\\
&m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_3 + c_2\ell_1\dot{\theta} + (k_1 + k_2)x_1 - \mathrm{k}_2x_3 + \mathrm{k}_2\ell_1\theta = F_{1,}\\
&m_2\ddot{x}_2 + (c_1 + c_2)\dot{x}_2 - c_2\dot{x}_3 - c_2\ell_2\dot{\theta} + (k_1 + k_2)\mathrm{x}_2 - k_2x_3 - k_2\ell_2\theta = F_{2.}
\end{aligned}
$$

Example 1.17
Figure 1.40 shows a mass, *m*, attached to the midpoint of a string of length 2ℓ. Assuming the tension in the string as *T*, obtain the governing differential equation of motion for the system.
Solution
Applying Newton's second law of motion to the system in Fig. 1.41 gives

$$\Sigma F_x = -2T\sin\theta = m\ddot{x}$$

and we get

$$\sin\theta = \frac{x}{\ell} = \frac{x}{\sqrt{\ell_0^2 + x^2}} \approx \frac{x}{\ell_0}\left[1 - \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right].$$

Fig. 1.40

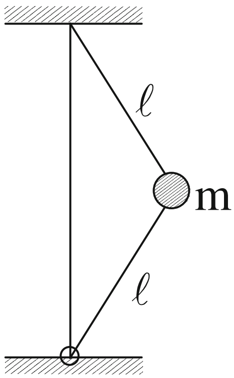
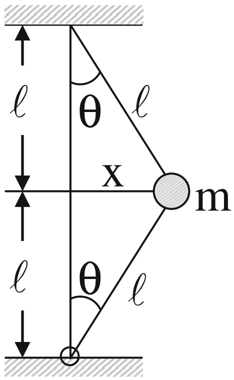


Fig. 1.41



Also,

$$T = T_0 + k(\ell - \ell_0) = T_0 + k\left[\ell_0\left(1 + \frac{x^2}{\ell_0^2}\right)^{1/2} - \ell_0\right]$$

$$\approx T_0 + k\left(\frac{1}{2}\right)\left(\frac{x}{\ell_0}\right)^2.$$

Hence,

$$m\ddot{x} + 2\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\frac{x}{\sqrt{\ell_0^2 + x^2}} = 0$$

or

$$m\ddot{x} + \frac{2}{\ell_0}\left[T_0 + \frac{k}{2}\left(\frac{x}{\ell_0}\right)^2\right]\left[1 + \frac{1}{2}\left(\frac{x}{\ell_0}\right)^2\right]x = 0$$

which is a nonlinear differential equation.For more details in Sect. 1.1.4, refer to Dukkipati (2004, 2006), Dukkipati and Srinivas (2006).

Problems

1.1 Figure 1.42 shows a bead of mass M, free to slide along a smooth circular hoop rotating about a vertical axis with constant angular velocity . Use the principle of virtual work to derive the equilibrium equation for the system.

1.2 Derive the equations of motion for the vibrating system shown in Fig. 1.43 using the principle of virtual work. Use x and θ as generalized coordinates.

1.3 Figure 1.44 shows a pulley of radius r and moment of inertia I supporting two masses m_1 and m_2 connected to each other by a rope. Derive the equation of motion of the system of the system by means of d'Alembert's principle.

1.4 Figure 1.45 shows a governor mechanism for controlling the speed of rotating shafts. A pendulum with mass m and length L is mounted on the rim of the wheel of radius r that rotates with constant angular velocity ω. The pendulum is constrained by two springs as shown. The pendulum remains in a radial position when the wheel is rotating at ω rad/s. However, the pendulum moves to one side or the other of its neutral position when the wheel

Fig. 1.42

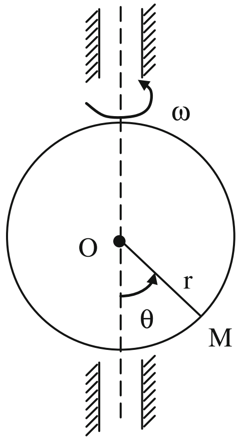


Fig. 1.43

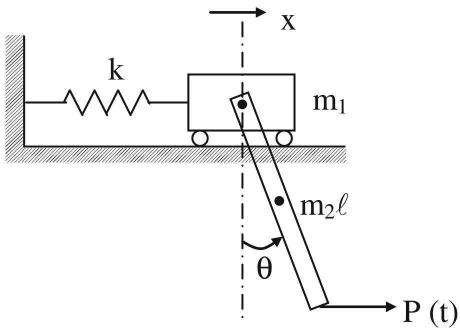


Fig. 1.44

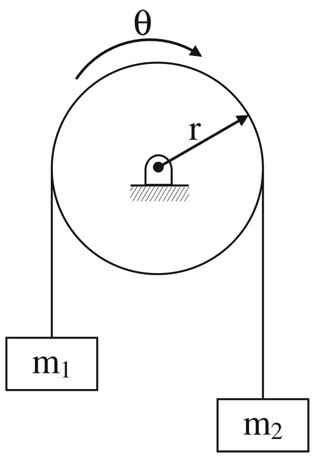
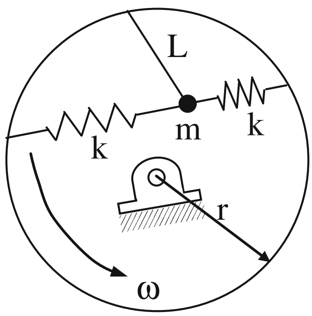


Fig. 1.45



accelerates or decelerates. The angular velocity ω is regulated by allowing the pendulum displacement to control the power input to the system. Determine the equation of motion of the pendulum with respect to the flywheel expressed in terms of the angular velocity and acceleration of the flywheel. Use d'Alembert's principle.

1.5 For the simple pendulum shown in Fig. 1.46,

Fig. 1.46

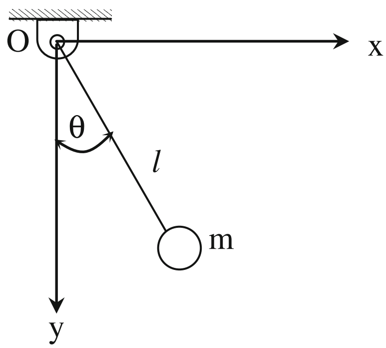


Fig. 1.47

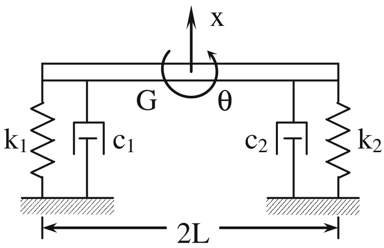
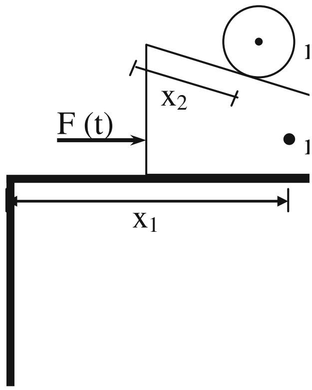


Fig. 1.48



(a) set up the Lagrangian,
(b) obtain the equation describing its motion.

1.6 Derive the equations of motion of the system of Fig. 1.47 using Lagrange's equation.

1.7 A cylinder of mass m_2 and moment of inertia I_2 about its longitudinal axis rolls without slipping on a wedge (Fig. 1.48). The wedge slides on the floor under the action of an applied force $F(t)$. There is friction between the cylinder and wedge, and the coefficient of sliding friction between the wedge and the floor is μ. Choosing x_1 and x_2 as generalized coordinates, obtain Lagrange's equation of motion.

1.8 Using Lagrange's equation, derive the equations of motion for small oscillations of a double pendulum, which consists of two rigid bodies, suspended at O and hinged at A, as shown in Fig. 1.49. The centers of gravity are C_1 and C_2, and the moments of inertia with respect to C_1 and C_2 are I_1 and I_2, respectively. The masses of the upper and lower bodies are m_1 and m_2, respectively.

1.9 Figure 1.50 shows a uniform cylinder rolling without slipping on a horizontal surface and constrained by a linear spring. Determine the differential equation of motion for the system.

Fig. 1.49

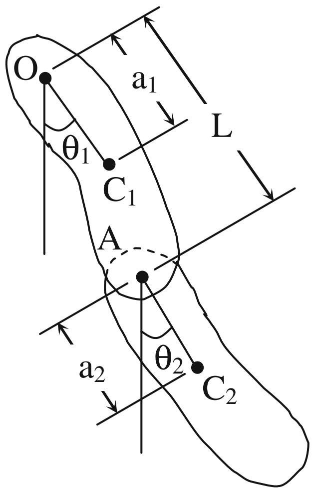


Fig. 1.50

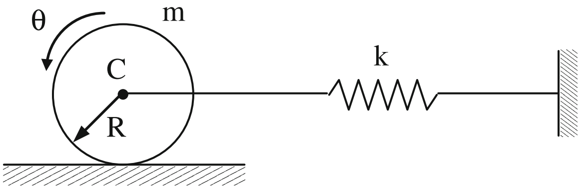


Fig. 1.51

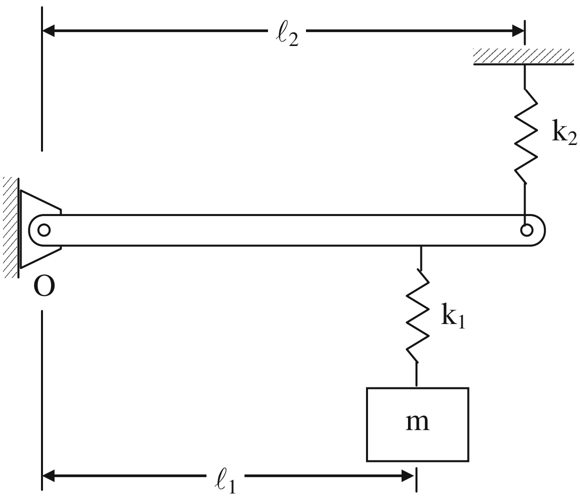


Fig. 1.52

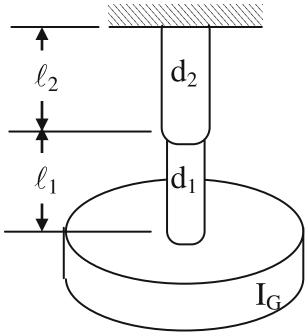


Fig. 1.53

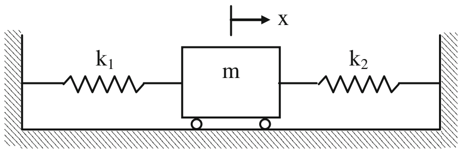
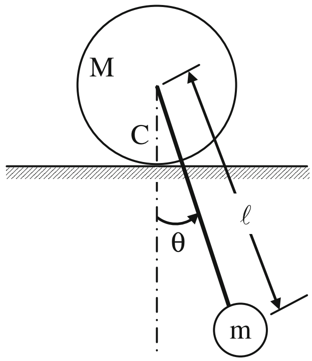


Fig. 1.54



1.10 Determine the differential equation of motion for the system shown in Fig. 1.51, where the massless rigid bar is hinged at *O*.

1.11 Figure 1.52 shows a disk fastened to the stepped shaft. Determine the differential equation of the angular motion of the disk.

1.12 Obtain the equation of motion of the system shown in Fig. 1.53 using the principle of conservation of energy.

1.13 Figure 1.54 shows a mass m attached to one end of a massless rod of length ℓ and the other end of the rod is rigidly attached to the center of a homogeneous cylinder of radius R and mass M. Determine the differential equation of motion for the system if the cylinder rolls without slipping.

1.14 A nonuniform bar is mounted on springs k_1 and k_2 as shown in Fig. 1.55. Determine the equations of motion for the system.
It is given that $k_1 = 3{,}200$ N/mm, $k_2 = 2{,}250$ N/mm, $\ell_1 = 0.48$ m, $\ell_2 = 0.62$ m, $m = 1{,}000$ kg, $I_G = 320$ kgm^2.

Fig. 1.55

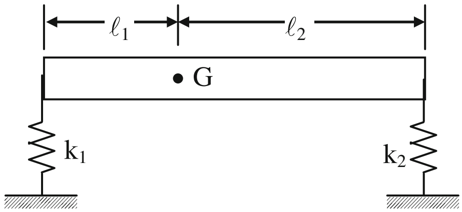


Fig. 1.56

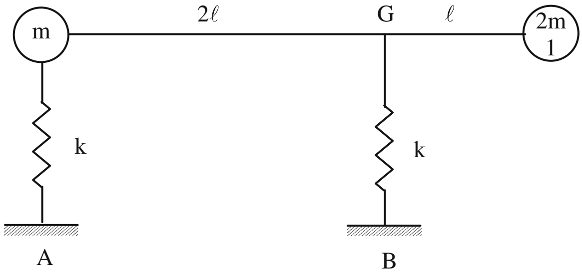
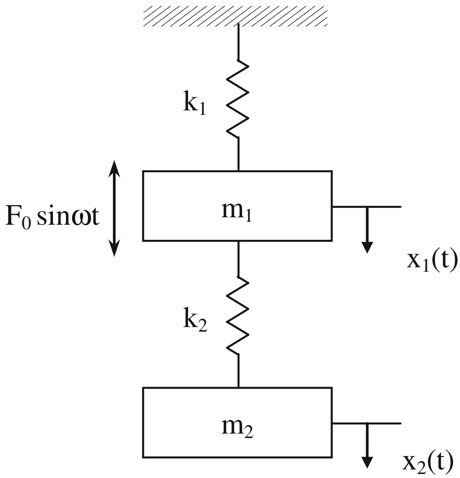


Fig. 1.57



1.15 A massless rigid bar of length 3ℓ and masses *m* and $2m$ is supported by two linear springs of equal stiffness as shown in Fig. 1.56. Determine the differential equations of motion.

1.16 Determine the differential equations of motion of the general two-degrees-of-freedom spring–mass system shown in Fig. 1.57.

1.17 Determine the differential equations of motion of the general two-degrees-of-freedom spring–mass system shown in Fig. 1.58.

1.18 A heavy eccentric disk can rotate about a fixed, smooth, horizontal axis at *O* (Fig. 1.59). Let its mass moment of inertia about the axis of rotation be *I*, and let its mass center *G* be at a distance of *s* from the axis of rotation. A

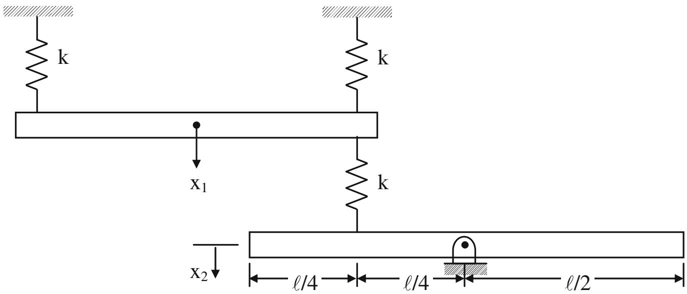
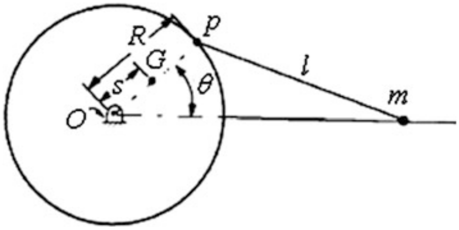


Fig. 1.58

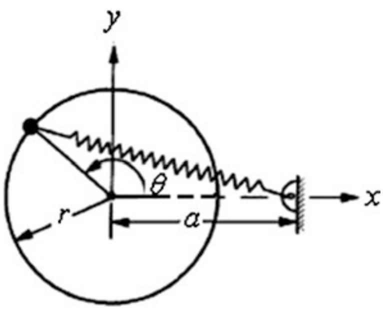
Fig. 1.59



massless connecting rod of length *I* is smoothly hinged to the disk at a point *P* at a distance of *R* from the axis of rotation and connected to a particle of mass *m*, which is constrained to move on a smooth horizontal surface as shown. *O*, *G*, and *P* lie on a straight line. If gravity is the only force acting on the system, define suitable coordinates and construct Lagrange's equations of motion for this system.

1.19 A heavy particle of mass *m* is constrained to move on a circle of radius *r* which lies in the vertical plane, as shown in Fig. 1.60. It is attached to a linear spring of rate *k*, which is anchored at a point on the *x*-axis at a distance of *a* from the origin of the *x*, *y* system, and $a > r$. The free length of the spring is $a - r$. Using the angle θ as generalized coordinate and the equation of motion, calculate the generalized forces arising from the

Fig. 1.60



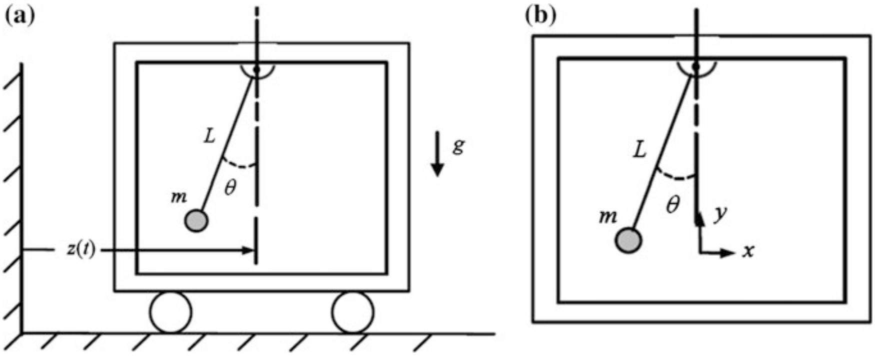
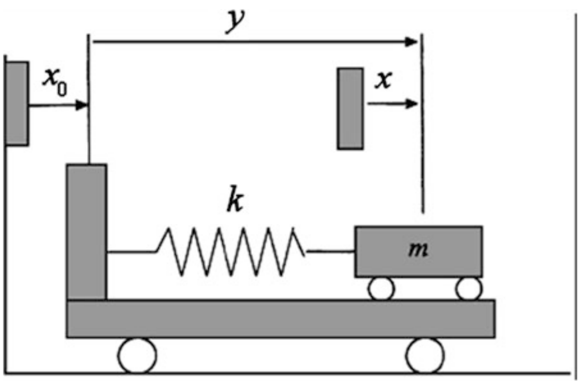


Fig. 1.61

Fig. 1.62



gravitational and the spring forces. Does the answer change if $a < r$ and, if so, how?

1.20 Consider the system in Fig. 1.61a. Define a coordinate system at the pendulum rest position in Fig. 1.61b. Note that the coordinate system origin is fixed within the enclosure. Derive the equation of motion.

1.21 Consider a system with a single degree of freedom with a moving foundation as shown in Fig. 1.62. Derive the equation of motion.

1.22 Figure 1.63 shows a simplified schematic of one half of the door support system. Half of the total door inertia is coupled to one of the overhead

Fig. 1.63 Simplified schematic of garage door support system

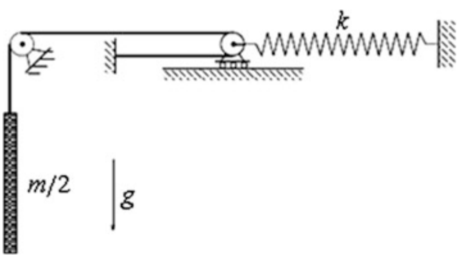
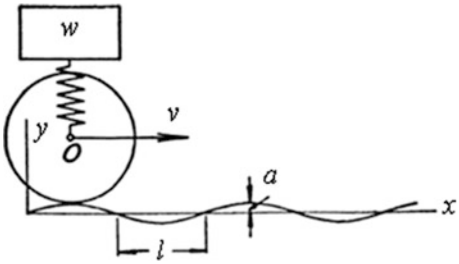


Fig. 1.64



springs by the pulley system. Take the total weight of the door to be 200 lb and the stiffness of each spring to be 5 pounds/in.

(a) Formulate a model to analyze the oscillations of the door.
(b) Estimate the frequency, in hertz, of the oscillations.
(c) List the main assumptions underlying your model.

1.23 A wheel is rolling along a wavy surface with a constant horizontal speed v (Fig. 1.64). Determine the amplitude of the forced vertical vibrations of the load W attached to the axle of the wheel by a spring if the static deflection of the spring under the action of the load W, $\delta_{st} = 3.86$ in., $v = 60$ ft/s, and the wavy surface is given by the equation $y = a \sin \frac{\pi x}{l}$ in which $a = 1$ in. and $l = 38$ in.

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Chapter 2
Perturbation and Variational Methods

2.1 Introduction

In this chapter and Chap. 3, we will use mathematical methods (analytical and numerical methods) for solving strongly nonlinear systems in field dynamics and vibration. More of these methods are mathematics methods that have been introduced by Chinese scientists, especially Professor Ji-Huan He.

In the course of a professional career spanning 2 decades, He has made countless contributions in variational theory, asymptotic techniques, nanotechnology, life science, and high-energy particle physics as well.

In 1997, in his Ph.D. Thesis, He proposed a novel method called the *semi-inverse* method to search for variational formulations directly from field equations and boundary conditions without a Lagrange multiplier (He 1997b, c). In 1998, the well-known variational iteration method (VIM) was suggested by using general Lagrange multipliers and restricted variations (He 1998a, b). Many asymptotic techniques, including the homotopy perturbation method (HPM) (He 1999a), the energy method (He 2006a), modifications of the Lindstedt–Poincaré method (He 2001c), the bookkeeping parameter method (He 2001a), the parametrized perturbation method (He 1999b), the iteration perturbation method (He 2001b), and other methods and their applications were suggested by He during 1999–2006 (He 2000a, b, c, d, 1997a, 2004a, b, c, 2005a, b, c, 2006b, d, 2003a, b, 1998c, 1999c, 2002a, b, c, 2007, 2008; He and Wu 2006a, b; Shou and He 2008; He and Abdou 2007). For a relatively comprehensive survey on the method and its applications, the reader is referred to He's review article (He 2006a) and monograph (He 2006c).

In the following, we introduce some early important methods, with their applications in nonlinear systems and nonlinear equations in dynamics and vibrations, that are today being widely used in engineering and applied sciences.

S. H. H. Kachapi and D. D. Ganji, *Dynamics and Vibrations*,
Solid Mechanics and Its Applications 202, DOI: 10.1007/978-94-007-6775-1_2,

2.2 The Basic Ideas of Perturbation Analysis

Perturbation theory has its roots in early celestial mechanics, where the theory of epicycles was used to make small corrections to the predicted paths of planets. Curiously, it was the need for more and more epicycles that eventually led to the sixteenth century Copernican revolution in the understanding of planetary orbits. The development of basic perturbation theory for differential equations was fairly complete by the middle of the nineteenth century. It was at that time that Charles-Eugène Delaunay was studying the perturbative expansion for the Earth–Moon–Sun system and discovered the so-called "problem of small denominators." Here, the denominator appearing in the *n*th term of the perturbative expansion could become arbitrarily small, causing the *n*th correction to be as large as or larger than the first-order correction. At the turn of the twentieth century, this problem led Henri Poincaré to make one of the first deductions of the existence of chaos, or what is prosaically called the "butterfly effect": that even a very small perturbation can have a very large effect on a system.

Perturbation theory saw a particularly dramatic expansion and evolution with the arrival of quantum mechanics. Although perturbation theory was used in the semiclassical theory of the Bohr atom, the calculations were monstrously complicated and subject to somewhat ambiguous interpretation. The discovery of Heisenberg's matrix mechanics allowed a vast simplification of the application of perturbation theory. Notable examples are the Stark effect and the Zeeman effect, which have a simple enough theory to be included in standard undergraduate textbooks in quantum mechanics. Other early applications include the fine structure and the hyperfine structure in the hydrogen atom.

In modern times, perturbation theory underlies much of quantum chemistry and quantum field theory. In chemistry, perturbation theory was used to obtain the first solutions for the helium atom.

In the middle of the twentieth century, Richard Feynman realized that the perturbative expansion could be given a dramatic and beautiful graphical representation in terms of what are now called Feynman diagrams. Although originally applied only in quantum field theory, such diagrams now find increasing use in any area where perturbative expansions are studied.

A partial resolution of the small-divisor problem was given by the statement of the KAM theorem in 1954. Developed by Andrey Kolmogorov, Vladimir Arnold, and Jürgen Moser, this theorem stated the conditions under which a system of partial differential equations will have only mildly chaotic behavior under small perturbations.

In the late twentieth century, broad dissatisfaction with perturbation theory in the quantum physics community, including not only the difficulty of going beyond second order in the expansion, but also questions about whether the perturbative expansion is even convergent has led to a strong interest in the area of nonperturbative analysis—that is, the study of exactly solvable models. The prototypical model is the Korteweg–de Vries equation, a highly nonlinear equation for which the interesting solutions, the solitons, cannot be reached by perturbation theory, even if the perturbations were carried out to infinite order. Much of the theoretical work in nonperturbative analysis goes under the name of quantum groups and noncommutative geometry.

2.2.1 Variation of Free Constants and Systems in Standard Form

The main idea of perturbation methods is to consider systems being close to an unperturbed one. It is supposed that the solutions of the unperturbed system are easy to find. In other words, it is supposed that the unperturbed system can be integrated in a closed form (see Ref. Fidlin 2006).

Consider a system

$$\dot{z} = Z(z, t, \varepsilon). \tag{2.1}$$

Here, $\varepsilon \ll 1$ is a small parameter. Consider the corresponding unperturbed system as follows:

$$\dot{z}_0 = Z(z_0, t, 0). \tag{2.2}$$

We suppose that its general solution is known as

$$z_0 = f(t, C) \Leftrightarrow \frac{\partial f}{\partial t} = Z(f(t, C), t, 0). \tag{2.3}$$

Here, C is a vector of arbitrary constants. Taking C as a set of new variables—that is, considering Eq. 2.3 as a transformation for the perturbed system Eq. 2.1—the following equation can be easily obtained:

$$\frac{\partial f}{\partial C}\dot{C} + \frac{\partial f}{\partial t} = Z(f(t, C), t, \varepsilon) = Z(f(t, C), t, 0) + \varepsilon \frac{\partial Z}{\partial \varepsilon}|_{\varepsilon=0} + \cdots. \tag{2.4}$$

Here, the symbol $\cdots$ stands for the terms $O(\varepsilon^2)$. Taking Eq. 2.3 into account and supposing the matrix $(\partial f/\partial C)$ not to be degenerated—that is, $\det(\partial f/\partial C) \neq 0$—the following equation for the new variables C can be obtained:

$$\dot{C} = \varepsilon \left(\frac{\partial f}{\partial C}\right)^{-1} + \frac{\partial Z(f(t, C), t, \varepsilon)}{\partial \varepsilon}|_{\varepsilon=0} + \cdots. \tag{2.5}$$

A system in the form of Eq. 2.5—that is, a system in which the right-hand side is multiplied by a small parameter—is called a "system in the standard from" for the averaging method. Usually, it is written as

$$\dot{x} = \varepsilon X(x, t, \varepsilon). \tag{2.6}$$

Here, x is the n-dimensional vector of the state variables, X is the n-dimensional vector function depending on the state variables, time, and, perhaps, the small parameter ε.

Actually, the statement "the system (Eq. 2.6) is in the standard form because the small parameter stays as a factor in front of its right-hand side" is too simplified. The functions at the right-hand side of Eq. 2.6 have to be, in addition, bounded and smooth, and the time average of the right-hand sides must exist (see below). These additional conditions are not always easy to satisfy. Even if the unperturbed system is a linear excited and damped oscillator (Eq. 2.7) and the excitation and damping are not small, it cannot be directly transformed to the standard form:

$$m\ddot{x} + \beta\dot{x} + cx = a\sin\omega t,\ \ a = O(1),\ \ \beta = O(1). \tag{2.7}$$

Nevertheless, there are two large and very important classes of systems suitable for perturbation analysis.

The quasi-conservative, especially quasi-linear, systems belong to the first class. Systems with strong excitation—that is, systems with dominating external and inertial forces—belong to the second class. It is usual to write the governing equations for the problems of this class in a slightly different form:

$$\ddot{x} = F(x, \dot{x}, t) + \omega\Phi(x, t, \tau), \quad \tau = \omega t,\ \omega \gg 1. \tag{2.8}$$

This form expresses better the fact that the term $\omega\Phi(x, t, \tau)$, containing the high-frequency excitation, is dominating here (because ω is the large parameter).

There are many different methods for an asymptotic analysis of perturbed dynamical systems. First of all, these methods differ according to the type of solutions they deal with. There are numerous methods considering only periodic solutions and their stability. Most of them are based on the ideas of Poincaré and Liapunov.

Another group of methods also considers transient solutions of dynamical systems; that is, these methods allow analyzing an infinitesimal vicinity of periodic solutions and their attraction area.

$$\ddot{x} = F(x, \dot{x}, t) + \omega\Phi(x, t, \tau), \quad \tau = \omega t,\ \omega \gg 1. \tag{2.9}$$

Three of these methods are most popular today. We are going to start with the standard averaging.

2.2.2 *Standard Averaging as an Almost Identical Transformation*

Initial value problems in the standard form are investigated by averaging:

$$\dot{x} = \varepsilon X(x, \dot{x}, t), \qquad x(0) = x_0. \tag{2.10}$$

Let us first introduce the time average of the function X:

$$\langle X(x,t,\varepsilon)\rangle_t = \lim_{T\to\infty}\frac{1}{T}\int_0^T X(x,t,\varepsilon)\,\mathrm{d}t. \tag{2.11}$$

The integration here has to be performed with respect to the explicit time (variables x are considered as constant parameters).

If the function X is periodic with respect to the explicit time t, the definition of the time average can be significantly simplified:

$$\begin{aligned} &\forall t : X(x,t+2\pi,\varepsilon) = X(x,t,\varepsilon) \\ &\Rightarrow \langle X(x,t,\varepsilon)\rangle_t = \frac{1}{2\pi}\int_0^{2\pi} X(x,t,\varepsilon)\,\mathrm{d}t \end{aligned} \tag{2.12}$$

The main idea of the averaging method is not to try to solve the system (Eq. 2.10) but to try to find another system, simpler than the original one, in which solutions are close to the solutions of the original system for a sufficiently long time interval. The simplification that can be achieved using averaging is to eliminate the independent variable t from the considered equations—that is, to reduce the effective order of the system by one.

In order to do it formally (without mathematical proof) for the simplest periodic case, the following approximate identical transformation can be applied:

$$x = \xi + \varepsilon u(\xi,t,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.13}$$

It is very important to understand the sense of this transformation in order to comprehend the physical meaning of the method. It splits the solution to Eq. 2.10 into two parts: the large slowly varying part ξ, describing the evolution of the system, and the small fast oscillating part u, which is responsible for the oscillations of the solution around the slow component.

Consider that the new variable ξ is governed by the autonomous equation

$$\dot{\xi} = \varepsilon\,\Xi(\xi,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.14}$$

Both the unknown functions, $u(\xi,t,\varepsilon)$, which have to be periodic functions of time, and $\Xi(\xi,\varepsilon)$ have to be determined by the following procedure. Applying Eqs. 2.13 and 2.14 to Eq. 2.10, the following equation can be obtained:

$$\varepsilon\,\Xi(\xi,\varepsilon) + \varepsilon\frac{\partial u}{\partial t} = \varepsilon X(\xi,t,\varepsilon) + O\left(\varepsilon^2\right). \tag{2.15}$$

Balancing the terms $O(\varepsilon)$ and considering that u has to be a bounded periodic function, we obtain that this condition can be fulfilled only if Ξ is the time average of X:

$$\varepsilon\,\Xi(\xi,\varepsilon) = \langle X(\xi,t,\varepsilon)\rangle_t$$
$$u(\xi,t,\varepsilon) = \int_0^t (X(\xi,\vartheta,\varepsilon) - \Xi(\xi,\varepsilon))\mathrm{d}\vartheta + u_0(\xi,\varepsilon). \tag{2.16}$$

It is usual to choose the free functions $u_0(\xi,\varepsilon)$ in order to guarantee that the time average of the functions u is equal to zero—that is, $\langle u(\xi,t,\varepsilon)\rangle_t = 0$.

It is not the unique possible choice of the functions u_0. Another one is convenient if the Eq. 2.10 have the Hamiltonian form. Then the free functions can be chosen in order to guarantee that the averaged equations also have the Hamiltonian form.

Higher order approximations can be obtained in a similar way. For the second-order approximation, we apply the following transformation:

$$x = \xi + \varepsilon u_1(\xi,t,\varepsilon) + \varepsilon^2 u_2(\xi,t,\varepsilon) + \cdots. \tag{2.17}$$

We require further that the new variable ξ be governed by an autonomous equation:

$$\dot{\xi} = \varepsilon\,\Xi_1(\xi,\varepsilon) + \varepsilon^2\,\Xi_2(\xi,\varepsilon) + \cdots. \tag{2.18}$$

Balancing the terms $O(\varepsilon)$ we obtain, as above,

$$\frac{\partial u_1}{\partial t} = X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon) \Rightarrow$$
$$\Xi_1(\xi,\varepsilon) = \langle X(\xi,t,\varepsilon)\rangle_t, \quad u_1 = \int [X(\xi,t,\varepsilon) - \Xi_1(\xi,\varepsilon)]\mathrm{d}t, \tag{2.19}$$
$$\langle u_1\rangle = 0$$

Balancing the terms $O(\varepsilon^2)$, we obtain, as above,

$$\frac{\partial u_2}{\partial t} = \frac{\partial X}{\partial x}\Big|_{x=\xi}\, u_1 - \Xi_2 - \frac{\partial u_1}{\partial \xi}\,\Xi_1 \Rightarrow$$
$$\Xi_2 = \left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\, u_1 - \frac{\partial u_1}{\partial \xi}\,\Xi_1 \right\rangle_t = \left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\, u_1 \right\rangle_t. \tag{2.20}$$

Finally, the equation of the second-order approximation is

$$\dot{\xi} = \varepsilon\langle X(\xi,t,\varepsilon)\rangle + \varepsilon^2\left\langle \frac{\partial X}{\partial x}\Big|_{x=\xi}\, u_1 \right\rangle_t + O\left(\varepsilon^3\right). \tag{2.21}$$

The above procedure is purely formal, because it does not explain whether one can shorten the equations for ξ neglecting the small terms $O(\varepsilon^2)$ or $O(\varepsilon^3)$ in Eqs. 2.14 and 2.21, respectively. Neither does it explain why and for how long a time the solutions of the original system (Eq. 2.10) and those of the shortened averaged systems (Eqs. 2.14 or 2.21) are close to each other.

Answers to these questions were given by Bogoliubov in his first theorem. Consider the system (2.10) and assume:

1. X is a measurable with respect to t vector function.
2. It is bounded and satisfies the Lipschitz condition with respect to the vector argument x:

$$\begin{aligned} &\|X(x,t,\varepsilon)\| \leq M \\ &\|X(x_1,t,\varepsilon) - X(x_2,t,\varepsilon)\| \leq \lambda\|x_1 - x_2\| \end{aligned} \tag{2.22}$$

3. The time average of the function X exists uniformly with respect to x.

Consider the averaged system satisfying the same initial conditions:

$$\dot{\xi} = \varepsilon\,\Xi(\xi,\varepsilon), \quad \xi(0) = x_0. \tag{2.23}$$

Under these conditions, the mistake made by using the system (Eq. 2.23) with functions Ξ determined by the relationships (Eq. 2.16), instead of the original one, has the magnitude order of the small parameter ε for the asymptotically long time interval $t = O(1/\varepsilon)$.

The proof of this theorem is not very complex. Readers interested in the mathematical background can find it in Fidlin (2006).

If the averaged system (Eq. 2.23) has an asymptotically stable singular point in the linear approximation and the function X is continuously differentiable with respect to it, then the original system (Eq. 2.10) has a solution that remains in the vicinity of this point for infinite time.

For more details in this section refer to Fidlin (2006).

2.2.3 Method of Multiple Scales

In mathematics and physics, the method of multiple scales comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, for both small and large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable and subsequently treating these variables, fast and slow, as if they are independent. In the solution process of the perturbation problem thereafter, the resulting additional freedom—introduced by the new independent variables—is used to remove (unwanted) secular terms. The latter puts constraints on the approximate solution and are called *solvability conditions* (see Ref. de Sterke and Sipe 1988).

The basic logic of the method of multiple scales can be easily illustrated by considering a system in the standard form (Eq. 2.10). The first step of the solution is to convert to two independent variables $\theta = t$ and $\tau = \varepsilon t$, supposing that

$x = \varphi(\theta, \tau)$—that is, to convert from the system of ordinary differential equations (Eq. 2.10) to a system with partial derivatives:

$$\frac{\partial \varphi}{\partial \theta} + \varepsilon \frac{\partial \varphi}{\partial \tau} = \varepsilon X(\varphi(\theta, \tau), \varepsilon). \tag{2.24}$$

The relationship between Eqs. 2.10 and 2.24 is determined by the condition that, if $\varphi(\theta, \tau)$ is a solution to Eq. 2.24, then $x = \varphi(t, \varepsilon t)$ is a solution to Eq. 2.10. In other words, the system (Eq. 2.24) is more general than the original equations (Eq. 2.10) and any solution (Eq. 2.24) taken along the straight line $\theta = t, \tau = \varepsilon t$ satisfies the equation (Eq. 2.10).

We require $\varphi(\theta, \tau)$ to be a 2π—periodic function of τ and try to find φ as a formal asymptotic expansion in terms of ε:

$$\varphi(\theta, \tau) = \psi_0(\theta, \tau) + \varepsilon \psi_1(\theta, \tau) + \cdots. \tag{2.25}$$

All the functions here have to be bounded functions of the fast time θ. Substituting this expression into Eq. 2.24 and balancing the terms with equal powers of ε, the following relationships can be obtained:

$$\begin{aligned} \varepsilon^0 : \quad & \frac{\partial \psi_0}{\partial \theta} = 0 \\ \varepsilon^1 : \quad & \frac{\partial \psi_0}{\partial \tau} + \frac{\partial \psi_1}{\partial \theta} = X(\psi_0(\theta, \tau), \theta, \varepsilon). \end{aligned} \tag{2.26}$$

The first of these equations means that ψ_0 depends only on the slow time τ:

$$\psi_0 = \xi(\tau). \tag{2.27}$$

Substituting Eq. 2.27 into the second relationship from Eq. 2.26, a simple equation for $\psi_1(\theta, \tau)$ can be obtained:

$$\frac{\partial \psi_1}{\partial t} = X(\xi(\tau), t, \varepsilon) - \frac{\partial \xi}{\partial \tau}. \tag{2.28}$$

The function $\psi_1(\theta, \tau)$ has to be a bounded function of θ; that is, its derivative can contain only oscillating components. It is possible if the function $\frac{\partial \xi}{\partial \tau}$ annihilates the constant component of X. It means that

$$\frac{\partial \xi}{\partial \tau} = \langle X(\xi, \theta, \varepsilon) \rangle_\theta. \tag{2.29}$$

Here, the average is calculated with respect to the fast time θ. Returning to the straight line $\theta = t, \tau = \varepsilon t$, we find that the slow component of the solution is governed by the equation

$$\frac{\partial \xi}{\partial \tau} = \langle X(\xi, t, \varepsilon) \rangle_t. \tag{2.30}$$

The fast oscillating small correction ψ_1 can be calculated as

$$\psi_1 = \int_0^t \left(X(\xi, \theta, \varepsilon) - \langle X(\xi, \theta, \varepsilon) \rangle_\theta \right) \mathrm{d}\theta + \psi_1^0(\xi, \varepsilon). \tag{2.31}$$

Comparing Eqs. 2.25 and 2.29–2.31 with the corresponding relationships from the previous subsection describing the averaging method (Eqs. 2.13, and 2.16) Eq. 2.23, it is easy to notice that they are identical ($\psi_1 \equiv u$). Considering the higher order terms in the expansion (Eq. 2.25), the higher order approximations to Eq. 2.10 can be obtained by the multiple scales technique. They are the same as those obtained by the averaging method.

2.2.4 Direct Separation of Motions

Direct separation of motions was formulated originally for systems of second-order differential equations, but it can be easily reformulated as follows.

Consider the system of ordinary differential equations (see Fidlin 2006):

$$\dot{x} = F(x, \varepsilon t) + \Phi(x, \varepsilon t, t). \tag{2.32}$$

The basic idea of the direct separation of motions is to consider only solutions that are a superposition of slow evolution and fast oscillations. The object of main interest is the slow component:

$$\begin{aligned} & x(t) = \xi(\tau) + \psi(\tau, t) \\ & \tau = \varepsilon t; \quad \langle \psi(\tau, t) \rangle_t = \frac{1}{2\pi} \int_0^{2\pi} \psi(\tau, t) \mathrm{d}t = 0 \end{aligned} \tag{2.33}$$

The next step is to go over from the system of n differential equations (Eq. 2.32) to a system of $2n$ integral–differential equations:

$$\begin{aligned} \dot{\xi} &= F(\xi, \tau) + \langle F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) \rangle_t + \langle \Phi(\xi + \psi(\xi, \tau), \tau, t) \rangle_t \\ \dot{\psi} &= F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) - \langle F(\xi + \psi(\xi, \tau), \tau) - F(\xi, \tau) \rangle_t \\ &\quad + \Phi(\xi + \psi(\xi, \tau), \tau, t) - \langle \Phi(\xi + \psi(\xi, \tau), \tau, t) \rangle_t \end{aligned} \tag{2.34}$$

The relationship between systems 2.32 and 2.34 is as follows: If a pair (ξ, ψ) is a solution to Eq. 2.24, then $x(t)$ determined according to Eq. 2.33 is automatically a solution to Eq. 2.32. It means that the system (2.34) is more general than the original one. This system is not more general, but it is at first sight more complex. Nevertheless, in many important cases, it is easy to solve with the assumption that the variable ξ in the second equation (2.34) is constant.

The system in the standard form (Eq. 2.10) can be considered as an example. In this case, we have

$$F(x, \varepsilon t) = 0; \quad \Phi(x, \varepsilon t, t) = \varepsilon X(x, t). \tag{2.35}$$

Substituting Eq. 2.35 into Eq. 2.34, the following equations can easily be obtained:

$$\begin{aligned} \dot{\xi} &= \varepsilon \langle X(\xi + \psi, t) \rangle_t \\ \dot{\psi} &= \varepsilon X(\xi + \psi, t) - \varepsilon \langle X(\xi + \psi, t) \rangle_t \end{aligned} \tag{2.36}$$

Solving the second equation of the system (Eq. 2.36) asymptotically is obvious:

$$\psi = \psi_0 + \varepsilon \psi_1 + \cdots. \tag{2.37}$$

Inserting this expression into the second equation (2.36) and balancing terms with the equal powers of the small parameter, we will have

$$\begin{aligned} \psi_0 &= 0 \\ \dot{\xi} &= \varepsilon \langle X(\xi, t) \rangle \\ \dot{\psi}_1 &= X(\xi, t) - \langle X(\xi, t) \rangle \end{aligned} \tag{2.38}$$

Equation 2.38 does not differ from the equations of the first-order approximation (2.14, 2.16, or 2.30) (Eq. 2.31).

Unfortunately, neither the method of multiple scales nor the method of the direct separation of motions has a mathematical proof differing from that for the standard averaging.

2.2.5 Relationship Between These Methods

All the considered methods are very useful and efficient in the analysis of engineering systems such as oscillating systems. All of them applied to the system in standard form give the same result. (It is actually the necessary condition for such a procedure to be called a method.) So selecting one of them in any special case is mainly a personal preference. From the author's point of view, the multiple scales and, especially, the direct separation of motions are slightly easier for practical use, as compared with the standard averaging method. Their main advantage is the straightforward algorithm used to solve the problem, which does not require an initial transformation of the system to the standard form. This transformation may sometimes be rather difficult.

But this statement is correct only for systems that are in standard form or can be transformed to it. The situation becomes much more interesting if it is impossible to transform a system to the standard form or one of the conditions (2.22 or 2.23) is not fulfilled. In such a case, one can try any of the described methods. The problem is how to make sure that the obtained results are correct. The main advantage of the averaging method becomes clear in these cases. There is a clear way, based on

Gronwall's lemma, to prove the accuracy of the averaging procedure. Thus, the method contains an instrument to generalize itself. This situation enables researchers to move away from pure physical intuition (being the most effective in many cases) and to take the path of rigorous mathematical analysis.

In different chapters, we will use these perturbation techniques.

2.2.6 Application

Example 2.1

First, consider the forced oscillation of an undamped pendulum,

$$x^2 + \frac{y^2}{\omega^2} = \text{constant}, \tag{2.39}$$

in which, without loss of generality, we can assume that $\omega_0 > 0, \omega > 0$ and $F > 0$ (since $F < 0$ implies a phase difference that can be eliminated by a change of time origin and corresponding modification of initial conditions).

Consider the equation

$$\sin x \approx x - \frac{1}{6}x^3. \tag{2.40}$$

Use Eq. 2.40 to allow for moderately large swings, which is accurate to 1 % for $|x| < 1$ radian (57°). Then Eq. 2.39 becomes approximately

$$\ddot{x} + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = F\cos\omega t. \tag{2.41}$$

Standardize the form of Eq. 2.41 considering the correlation

$$\tau = \omega t, \quad \Omega^2 = \omega_0^2/\omega^2 \ \ (\Omega > 0), \quad \Gamma = F/\omega^2. \tag{2.42}$$

We obtain

$$x'' + \Omega^2 x - \frac{1}{6}\Omega^2 x^3 = \Gamma\cos\tau, \tag{2.43}$$

where dashes represent differentiation with respect to τ. This is a special case of Duffing's equation, which is characterized by a cubic nonlinear term. If Eq. 2.43 actually arises by consideration of a pendulum, the coefficients and variables are all dimensionless.

The methods to be described depend on how small the nonlinear term is. Here, we assume that $\frac{1}{6}\Omega^2$ is small, and then

$$\frac{1}{6}\Omega^2 = \varepsilon_0. \tag{2.44}$$

Then Eq. 2.43 becomes

$$x'' + \Omega^2 x - \varepsilon_0 x^3 = \Gamma \cos \tau. \tag{2.45}$$

Instead of taking Eq. 2.45 as it stands, with $\Omega, \Gamma, \varepsilon_0$ as constants, we consider the family of differential equations

$$x'' + \Omega^2 x - \varepsilon x^3 = \Gamma \cos \tau, \tag{2.46}$$

where ε is a parameter occupying an interval I_ε that includes $\varepsilon = 0$. When $\varepsilon = \varepsilon_0$, we recover Eq. 2.45, and when $\varepsilon = 0$, we obtain the linearized equation corresponding to the family 2.46:

$$x'' + \Omega^2 x = \Gamma \cos \tau. \tag{2.47}$$

The solutions of (2.46) are now thought of as functions of both ε and τ, and we will write $x(\varepsilon, \tau)$.

The most elementary version of the perturbation method is to attempt a representation of the solutions of (2.46) in the form of a power series in ε:

$$x(\varepsilon, \tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots, \tag{2.48}$$

whose coefficients $x_i(\tau)$ are only functions of τ. To form equations for $x_i(\tau), i = 0, 1, 2, \ldots$, substitute the series (Eq. 2.48) into Eq. 2.46:

$$(x_0'' + \varepsilon x_1'' + \cdots) + \Omega^2(x_0 + \varepsilon x_1 + \cdots) - \varepsilon(x_0 + \varepsilon x_1 + \cdots)^3 = \Gamma \cos \tau. \tag{2.49}$$

Since this is assumed to hold for every member of the family Eq. 2.46—that is, for every ε on I_ε—coefficients of the powers of ε must balance, and we obtain

$$x_0'' + \Omega^2 x_0 = \Gamma \cos \tau, \tag{2.50a}$$

$$x_1'' + \Omega^2 x_1 = x_0^3, \tag{2.50b}$$

$$x_2'' + \Omega^2 x_2 = 3x_0^2 x_1, \tag{2.50c}$$

and so on.

We shall be concerned only with periodic solutions having the period, 2π, of the forcing term. Then, for all ε on I_ε and for all τ,

$$x(\varepsilon, \tau + 2\pi) = x(\varepsilon, \tau). \tag{2.51}$$

By Eq. 2.51, it is sufficient that for all τ,

$$x_i(\tau + 2\pi) = x_i(\tau), \quad i = 0, 1, 2, \ldots. \tag{2.52}$$

Equation 2.51, together with the condition 2.52, is sufficient to provide the solutions required. For the present, note that Eq. 2.50a is the same as the "linearized equation" (2.47), necessarily, since putting $\varepsilon_0 = 0$ in Eq. 2.45 implies putting $\varepsilon = 0$ in Eq. 2.48. The major term in Eq. 2.48 is therefore a periodic solution of the linearized equation (2.47). It is therefore clear that this process restricts us to finding the solutions of the nonlinear equations that are close to (or branch from, or bifurcate from) the solution of the linearized equation. The method will not expose any other periodic solutions. The zero-order solution $x_0(\tau)$ is known as a generating solution for the family of Eq. 2.46.

Example 2.2

2.2.7 Introduction

This example is concerned with responses of systems with two degrees of freedom and cubic nonlinearities to multifrequency parametric excitations governed by the following equations:

$$\begin{aligned}
&\ddot{X}_1 + \omega_1^2 X_1 + \varepsilon\Big[2\mu_1 \dot{X}_1 + \alpha_1 X_1^3 + 3\alpha_2 X_1^2 X_2 + \alpha_3 X_1 X_2^2 + \alpha_4 X_2^3 \\
&\qquad\qquad + 2\sum_{m=1}^{M}\{X_1 f_{1m} + X_2 f_{2m}\}\cos(\Omega_m t + \tau_{1m})\Big] = 0, \\
&\ddot{X}_2 + \omega_2^2 X_2 + \varepsilon\Big[2\mu_2 \dot{X}_2 + \alpha_2 X_1^3 + 3\alpha_3 X_1^2 X_2 + \alpha_4 X_1 X_2^2 + \alpha_5 X_2^3 \\
&\qquad\qquad + 2\sum_{n=1}^{M}\{X_1 g_{1n} + X_2 g_{2n}\}\cos(\Omega_n t + \gamma_{1n})\Big] = 0.
\end{aligned} \tag{2.53}$$

where $\omega_n, \mu_n, \alpha_n, f_{mn}, g_{mn}, \Omega_m, \Omega_n, \tau_{1m},$ and γ_{1n} are constants, ε is a small dimensionless parameter, and dots indicate differentiation with respect to the time t. These equations when quadratic terms are included model the responses of ships and bowed structural elements.

2.2.8 The Method of Multiple Scales

To determine a first-order uniform solution of Eq. 2.53, we use the method of multiple scales and let

$$X(t;\varepsilon) = X_{n0}(T_0, T_1) + \varepsilon X_{n1}(T_0, T_1) + \cdots, \tag{2.54}$$

where $T_0 = t$ is a fast scale, which is associated with changes occurring at the frequencies $\omega_n, \Omega_m,$ and Ω_n and $T_1 = \varepsilon t$ is a slow scale, which is associated with modulations in the amplitudes and phases resulting from the nonlinearities and parametric resonances.

In terms of T_0 and T_1, the time derivative becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{2.55}$$

where $D_n = \partial/\partial T_n$. Substituting Eqs. 2.54 and 2.55 into Eq. 2.53 and equating coefficients of power ε, we obtain

$$D_0^2X_{10} + \omega_1^2X_{10} = 0, \qquad D_0X_{20}^2 + \omega_1X_{20} = 0, \tag{2.56}$$

$$\begin{aligned} D_0^2X_{11} + \omega_1^2X_{11} = &- 2D_0D_1X_{10} - 2\mu_1(D_0X_{10}) - \alpha_1X_{10}^3 - 3\alpha_2X_{10}^2X_{20} - 3\alpha_3X_{10}X_{20}^2 \\ &- \alpha_4X_{20}^3 + 2\sum_{m=1}^{M}\{X_{10}f_{1m} + X_{20}f_{2m}\}\cos(\Omega_mt + \tau_{1m}), \end{aligned} \tag{2.57}$$

$$\begin{aligned} D_0^2X_{21} + \omega_2^2X_{21} = &- 2D_0D_1X_{20} - 2\mu_2(D_0X_{20}) - \alpha_2X_{10}^3 - 3\alpha_3X_{10}^2X_{20} - 3\alpha_4X_{10}X_{20}^2 \\ &- \alpha_5X_{20}^3 + 2\sum_{n=1}^{M}\{X_{10}g_{1n} + X_{20}g_{2n}\}\cos(\Omega_nt + \gamma_{1n}). \end{aligned} \tag{2.58}$$

The solution of Eq. 2.56 can be expressed as

$$X_{10} = A_1\exp(i\omega_1T_0) + \mathrm{cc}, \qquad X_{20} = A_2\exp(i\omega_1T_0) + \mathrm{cc}, \tag{2.59}$$

where cc denotes the complex conjugate of the preceding terms. Inserting Eq. 2.59 into Eq. 2.58 yields

$$\begin{aligned} D_0^2X_{11} + \omega_1^2X_{11} = &[-2i\omega_1(A' + \mu_1A_1) + 3\alpha_1A_{10}^2\overline{A}_{10} + 6\alpha_3A_1A_2^2\overline{A}_2]\exp(i\omega_1T_0) \\ &- [6\alpha_2A_1\overline{A}_1A_2 + 3\alpha_4A_2^2\overline{A}_2]\exp(2i\omega_2T_0) \\ &- \alpha_1A_1^3\exp(3i\omega_1T_0) - \alpha_4A_2^3\exp(3i\omega_2T_0) \\ &- 3\alpha_2[A_1^2A_2\exp i(2\omega_1 + \omega_2)T_0 + \overline{A}_1^2A_2\exp i(\omega_2 - 2\omega_1)T_0] \\ &- 3\alpha_3[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0 \\ &\qquad + \overline{A}_1A_2^2\exp i(2\omega_2 - \omega_1)T_0] \\ &- A_1\sum_{m=1}^{M}f_{1m}\exp((\Omega_m{+}\omega_1)T_0 + \tau_{1m}) - \overline{A}_1\sum_{m=1}^{M}f_{1m}\exp((\Omega_m{-}\omega_1)T_0 + \tau_{1m}) \\ &- A_2\sum_{m=1}^{M}f_{2m}\exp((\Omega_m{+}\omega_2)T_0 + \tau_{1m}) - \overline{A}_2\sum_{m=1}^{M}f_{2m}\exp((\Omega_m{-}\omega_2)T_0 + \tau_{1m}), \end{aligned} \tag{2.60}$$

$$\begin{aligned} D_0^2X_{21} + \omega_2^2X_{21} = &- [2i\omega_2(A_2' + \mu_2A_2) + 6\alpha_3A_1\overline{A}_1A_2 + 3\alpha_5\overline{A}_2^2A_2]\exp(i\omega_2T_0) \\ &- [3\alpha_2A_1^2\overline{A}_1 + 6\alpha_4A_2\overline{A}_2]\exp(i\omega_1T_0) \\ &- \alpha_2A_1^3\exp(3i\omega_1T_0) - \alpha_5A_2^3\exp(3i\omega_2T_0) \\ &- 3\alpha_4[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0] \\ &- 3\alpha_3[A_1A_2^2\exp i(\omega_1 + 2\omega_2)T_0 + A_1\overline{A}_2^2\exp i(\omega_1 - 2\omega_2)T_0 \\ &\qquad + \overline{A}_1^2A_2^2\exp i(\omega_2 + 2\omega_1)T_0] \\ &- A_1\sum_{n=1}^{N}g_{1n}\exp((\Omega_n{+}\omega_1)T_0 + \gamma_{1n}) + \overline{A}_1\sum_{n=1}^{N}g_{1n}\exp((\Omega_n{-}\omega_1)T_0 + \gamma_{1n}) \\ &- A_2\sum_{n=1}^{N}g_{2n}\exp((\Omega_n{+}\omega_2)T_0 + \gamma_{1n}) - \overline{A}_2\sum_{n=1}^{N}g_{2n}\exp((\Omega_n{-}\omega_2)T_0 + \gamma_{1n}), \end{aligned}$$

where the over term indicates the complex conjugate and the prime indicates differentiation with respect to T_1. Any particular solution of Eq. 2.60 contains secular or small divisor terms depending on the resonant conditions (1) $\omega_2 \cong 2\omega_1$, internal resonance, and (2) $\Omega_r \cong 2\omega_1$, principal parametric resonance of the first mode. To treat this case, we introduce detuning parameters σ_1 and σ_2 to convert the small divisor terms into secular terms, defined according to the correlation

$$\omega_2 = 2\omega_1 + \varepsilon\sigma_1, \qquad \Omega_r = 2\omega_1 + \varepsilon\sigma_2. \tag{2.61}$$

Substituting Eq. 2.61 into Eq. 2.60 and eliminating the secular terms from X_{11} and X_{21}, we obtain

$$\begin{aligned} &2i\omega_1(A_1' + \mu_1 A_1) + 3\alpha_1 A_1^2\overline{A}_1 + 6\alpha_3 A_1 A_2 \overline{A}_2 \\ &\quad + 3\alpha_2 \overline{A}\,_1^2 A_2 \exp(i\sigma_1 T_1) + \overline{A}_1 f_{1r} \exp i(\sigma_2 T_1 + \tau_{1r}) = 0, \\ &2i\omega_2(A_2' + \mu_2 A_2) + 6\alpha_3 A_1 \overline{A}_1 A_2 + 3\alpha_5 A_2^2 A_2 \overline{A}_2 + \alpha_2 A_1^3 \exp(-i\sigma_2 T_1) = 0. \end{aligned} \tag{2.62}$$

Consequently, the particular solutions of Eq. 2.60 are

$$\begin{aligned} U_{11} = &-\left[\frac{6\alpha_2 A_1 \overline{A}_1 A_2 + 3\alpha_4 A_2^2 \overline{A}_2}{\omega_1^2 - \omega_2^2}\right] \exp(i\omega_2 T_0) \\ &+ \left[\frac{\alpha_1 A_1^3}{8\omega_1^2}\right] \exp(3i\omega_1 T_0) - \left[\frac{\alpha_4 A_2^3}{\omega_1^2 - 9\omega_2^2}\right] \exp(3i\omega_2 T_0) \\ &- \left[\frac{3\alpha_2 A_1^2 A_2}{\{\omega_1^2 - (2\omega_1 + \omega_2)^2\}}\right] \exp i(2\omega_1 + \omega_2)T_0 \\ &- \left[\frac{3\alpha_2 \overline{A}\,_1^2 A_2}{\{\omega_1^2 - (\omega_2 - 2\omega_1)^2\}}\right] \exp i(\omega_2 - 2\omega_1)T_0 \\ &- \left[\frac{3\alpha_3 A_1 A_2^2}{\{\omega_1^2 - (\omega_1 + 2\omega_2)^2\}}\right] \exp i(\omega_1 + 2\omega_2)T_0 \\ &- \left[\frac{3\alpha_3 A_1 \overline{A}\,_2^2}{\{\omega_1^2 - (\omega_1 - 2\omega_2)^2\}}\right] \exp i(\omega_1 - 2\omega_2)T_0 \\ &- \left[\frac{3\alpha_3 \overline{A}_1 A_2^2}{\{\omega_1^2 - (2\omega_2 - \omega_1)^2\}}\right] \exp i(2\omega_2 - \omega_1)T_0 \\ &- \left[\frac{f_{1r} A_1}{\{\omega_1^2 - (\Omega_r + \omega_1 + \tau_{1r})^2\}}\right] \exp i((\Omega_r + \omega_1)T_0 + \tau_{1r}) \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{f_{2r}A_2}{\{\omega_1^2 - (\Omega_r + \omega_2 + \tau_{2r})^2\}}\right] \exp i((\Omega_r + \omega_2)T_0 + \tau_{1r}) \\
&- \left[\frac{f_{2r}\overline{A}_2}{\{\omega_1^2 - (\Omega_r - \omega_2 + \tau_{2r})^2\}}\right] \exp i((\Omega_r - \omega_2)T_0 + \tau_{1r}).
\end{aligned}
$$

$$
\begin{aligned}
U_{21} = &- \left[\frac{3\alpha_2 A_1^2\overline{A}_1 + 6\alpha_4 A_1 A_2 \overline{A}_2}{\omega_2^2 - \omega_1^2}\right] \exp(i\omega_1 T_0) \\
&- \left[\frac{\alpha_2 A_1^3}{\omega_2^2 - 9\omega_1^2}\right] \exp(3i\omega_1 T_0) + \left[\frac{\alpha_5 A_2^3}{8\omega_2^2}\right] \exp(3i\omega_2 T_0) \\
&- \left[\frac{3\alpha_3 A_1^2 A_2}{\{\omega_1^2 - (2\omega_1 - \omega_2)^2\}}\right] \exp i(2\omega_1 + \omega_2)T_0 \\
&- \left[\frac{3\alpha_3 \overline{A}_1^2 A_2}{\{\omega_2^2 - (\omega_2 - 2\omega_1)^2\}}\right] \exp i(\omega_2 - 2\omega_1)T_0 \\
&- \left[\frac{3\alpha_4 A_1 A_2^2}{\{\omega_2^2 - (\omega_1 + 2\omega_2)^2\}}\right] \exp i(\omega_1 + 2\omega_2)T_0 \\
&- \left[\frac{3\alpha_4 A_1 \overline{A}_2^2}{\{\omega_2^2 - (\omega_1 - 2\omega_2)^2\}}\right] \exp i(\omega_1 - 2\omega_2)T_0 \\
&- \left[\frac{3\alpha_4 \overline{A}_1 A_2^2}{\{\omega_2^2 - (2\omega_2 - \omega_1)^2\}}\right] \exp i(2\omega_2 - \omega_1)T_0 \\
&- \left[\frac{g_{1s}A_1}{\{\omega_2^2 - (\Omega_s + \omega_1 + \gamma_{1s})^2\}}\right] \exp i((\Omega_s + \omega_1)T_0 + \gamma_{1s}) \\
&- \left[\frac{g_{1s}\overline{A}_1}{\{\omega_2^2 - (\Omega_s - \omega_1 + \gamma_{1s})^2\}}\right] \exp i((\Omega_s + \omega_1)T_0 + \gamma_{1s}) \\
&- \left[\frac{g_{2r}A_2}{\{\omega_2^2 - (\Omega_s + \omega_2 + \gamma_{2s})^2\}}\right] \exp i((\Omega_s + \omega_2)T_0 + \gamma_{1s}).
\end{aligned} \tag{2.63}
$$

Expressing A_n in the polar notation, we have

$$
A_n = \frac{1}{2} a_n \exp(i\beta_n), \tag{2.64}
$$

and separating the real and imaginary parts of Eq. 2.62, we obtain

$$\begin{aligned}
a_1' &= -\mu_1 a_1 - \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 - \frac{1}{2\omega_1} f_{1r} a_1 \sin\theta_2,\\
a_1\beta_1' &= \frac{3\alpha_1}{8\omega_1} a_1^3 + \frac{3\alpha_3}{4\omega_1} a_1 a_2^2 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 + \frac{1}{2\omega_1} f_{1r} a_1 \cos\theta_2,\\
a_2' &= -\mu_2 a_2 + \frac{\alpha_2}{8\omega_2} a_1^3 \sin\theta_1,\\
a_2\beta_2' &= \frac{3\alpha_3}{4\omega_2} a_1^2 a_2 + \frac{3\alpha_5}{8\omega_2} a_2^3 + \frac{\alpha_2}{8\omega_2} a_1^3 \cos\theta_1.
\end{aligned} \tag{2.65}$$

where

$$\theta_1 = \sigma_1 T_1 + \beta_2 - 3\beta_1, \qquad \theta_2 = \sigma_2 T_1 - 2\beta_1 + \tau_{1r}. \tag{2.66}$$

Inserting Eqs. 2.58 and 2.63 into Eq. 2.54 yields the approximate solutions

$$\begin{aligned}
U_1 = {} & a_1\cos(\omega_1 t + \beta_1) + \varepsilon\Bigg[-\left[\frac{6\alpha_2 a_1^2 a_2 + 3\alpha_4 a_2^2}{4\omega_1^2 - 4\omega_2^2}\right]\cos\psi_2 + \left[\frac{\alpha_1 a_1^3}{32\omega_1^2}\right]\cos 3\psi_1 - \left[\frac{\alpha_4 a_2^3}{4(\omega_1^2 - 9\omega_2^2)}\right]\cos 3\psi_2\\
& - \left[\frac{3\alpha_2 a_1^2 a_2}{\{4\omega_1^2 - 4(2\omega_1+\omega_2)^2\}}\right]\cos(2\psi_1+\psi_2) - \left[\frac{3\alpha_2 a_1^2 a_2}{\{4\omega_1^2 - 4(2\omega_1+\omega_2)^2\}}\right]\cos(\psi_2 - 2\psi_1)\\
& - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_1^2 - 4(\omega_1+2\omega_2)^2\}}\right]\cos(\psi_1+2\psi_2) - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1-2\omega_2)^2\}}\right]\cos(\psi_1 - 2\psi_2)\\
& - \left[\frac{3\alpha_3 a_1 a_2^2}{\{4\omega_2^2 - 4(2\omega_2-\omega_1)^2\}}\right]\cos(2\psi_2-\psi_1) - \left[\frac{f_{1m} a_1}{\{\omega_1^2 - (\Omega_r+\omega_1+\psi_{1r})^2\}}\right]\cos(\psi_3+\psi_1)\\
& - \left[\frac{f_{2r} a_2}{\{\omega_1^2 - (\Omega_r+\omega_2+\psi_{2r})^2\}}\right]\cos(\psi_3+\psi_2) - \left[\frac{f_{2r} a_2}{\{\omega_1^2 - (\Omega_r-\omega_2+\psi_{2r})^2\}}\right]\cos(\psi_4-\psi_2)\Bigg] + o(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
U_2 = {} & a_2\cos(\omega_2 t + \beta_2) + \varepsilon\Bigg[\left[\frac{3\alpha_2 a_1^3 a_2 + 6\alpha_4 a_1 a_2^2}{4\omega_2^2 - 4\omega_1^2}\right]\cos\psi_1 - \left[\frac{\alpha_2 a_1^3}{\omega_2^2 - 9\omega_1^2}\right]\cos 3\psi_2 + \left[\frac{\alpha_5 a_2^3}{32\omega_2^2}\right]\cos 3\psi_2\\
& - \left[\frac{3\alpha_3 a_1^2 a_2}{\{4\omega_2^2 - 4(\omega_2-2\omega_1)^2\}}\right]\cos(\psi_1+2\psi_2) - \left[\frac{3\alpha_3 a_1^2 a_2}{\{4\omega_2^2 - 4(\omega_2-2\omega_1)^2\}}\right]\cos(\psi_2 - 2\psi_1)\\
& - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1+2\omega_2)^2\}}\right]\cos(\psi_1+2\psi_2) - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(\omega_1-2\omega_2)^2\}}\right]\cos(\psi_1 - 2\psi_2)\\
& - \left[\frac{3\alpha_4 a_1 a_2^2}{\{4\omega_2^2 - 4(2\omega_2-\omega_1)^2\}}\right]\cos(2\psi_2-\psi_1) - \left[\frac{g_{1s} a_1}{\{\omega_2^2 - (\Omega_s+\omega_1+\gamma_{1s})^2\}}\right]\cos(\psi_4+\psi_1)\\
& - \left[\frac{g_{1s} a_1}{\{\omega_2^2 - (\Omega_s-\omega_1+\gamma_{1s})^2\}}\right]\cos(\psi_4-\psi_1) - \left[\frac{g_{2r} a_2}{\{\omega_2^2 - (\Omega_s+\omega_2+\gamma_{2s})^2\}}\right]\cos(\psi_4+\psi_2)\Bigg] + o(\varepsilon^2),
\end{aligned} \tag{2.67}$$

where

$$\psi_1 = \omega_1 t + \beta_1, \quad \psi_2 = \omega_2 t + \beta_2, \quad \psi_3 = \Omega_r t + \tau_{1r}, \quad \psi_4 = \Omega_s t + \gamma_{1n}. \tag{2.68}$$

The steady state of Eq. 2.65 is given by

$$\begin{aligned}
&\mu_1 a_1 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 + \frac{1}{2\omega_1} f_{1r} a_1 \sin\theta_2 = 0,\\
&\frac{1}{2}\delta_2 a_1 - \frac{3\alpha_1}{8\omega_1} a_1^3 - \frac{3\alpha_3}{4\omega_1} a_1 a_2^2 - \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 - \frac{1}{2\omega_1} f_{1r} a_1 \cos\theta_2 = 0,\\
&-\mu_2 a_2 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \sin\theta_1 = 0,\\
&\left(\frac{3}{2}\delta_2 - \sigma_1\right) a_2 + \frac{3\alpha_3}{4\omega_2} a_1 a_2^2 + \frac{3\alpha_5}{8\omega_2} a_2^3 + \frac{3\alpha_2}{8\omega_1} a_1^2 a_2 \cos\theta_1 = 0.
\end{aligned} \tag{2.69}$$

There are two possibilities: first, $a_1 = a_2 = 0$ (this is the linear case); second, a_1 and $a_2 \neq 0$ and Eq. 2.69 yield the frequency response equations

$$\begin{aligned}
&\mu_1^2 + \frac{1}{4}\delta_2^2 + \left[\frac{9\alpha_2^2}{64\omega_1^2}\right] a_1^2 a_2^2 + \left[\frac{9\alpha_1^2}{64\omega_1^2}\right] a_1^4 - \frac{1}{4\omega_1^2} f_{1r}^2 - \left[\frac{3\alpha_3\sigma_2}{4\omega_1}\right] a_2^2\\
&- \left[\frac{3\alpha_2\sigma_2}{8\omega_1}\right] a_1 a_2 \cos\theta_1 + \left[\frac{9\alpha_1\alpha_2}{16\omega_1^2}\right] a_1^2 a_2^2 + \left[\frac{9\alpha_1\alpha_2}{32\omega_1^2}\right] a_1^3 a_2 \cos\theta_1 + \left[\frac{9\alpha_2\alpha_3}{16\omega_1^2}\right] a_1 a_2^3 \cos\theta_1 = 0,\\
&\left[\mu_2^2 + \left(\frac{3}{2}\delta_2 - \delta_1\right)^2\right] a_2^2 - \left[\frac{\alpha_2^2}{64\omega_2^2}\right] a_1^6 + \left[\frac{9\alpha_3^2}{16\omega_2^2}\right] a_1^4 a_2^2 + \left[\frac{9\alpha_5^2}{64\omega_2^2}\right] a_2^6\\
&- \left[\frac{3\alpha_3}{2\omega_2}\left(\frac{3}{2}\delta_2 - \delta_1\right)\right] a_1^2 a_2^2 - \left[\frac{3\alpha_5}{4\omega_2}\left(\frac{3}{2}\delta_2 - \delta_1\right)\right] a_2^4 + \left[\frac{3\alpha_3\alpha_5}{32\omega_2}\right] a_1^2 a_2^4 = 0.
\end{aligned} \tag{2.70}$$

Example 2.3

Introduction

In the present example, we consider the following two coupled Duffing–Van der Pol oscillators with a nonlinear coupling:

$$\ddot{x} - \varepsilon d_1 \dot{x}(1 - x^2) + \omega_1^2 x + \varepsilon\alpha_1 x^3 + \varepsilon\delta x y^2 = 0, \tag{2.71a}$$

$$\ddot{y} - \varepsilon d_2 \dot{y}(1 - y^2) + \omega_1^2 y + \varepsilon\alpha_2 y^3 + \varepsilon\delta x^2 y = 0. \tag{2.71b}$$

When $d_1 = d_2 = 0$ the system (Eq. 2.71a, b) consists of the two coupled anharmonic oscillators. If the nonlinear damping is replaced by a linear damping, the system (Eq. 2.71a,b) becomes two coupled Duffing oscillators. When the coupling parameter is set to zero, the system (Eq. 2.71a,b) becomes two uncoupled Duffing–Van der Pol oscillators. The Duffing oscillator corresponds to the choices $\delta = 0, d_1 = d_2 = 0$ and addition of a linear damping.

Analysis with the Method of Multiple Scales

We look for approximate asymptotic solutions of Eq. 2.71a,b by employing the multiple scales method. For small but finite x and y, we consider the solutions in the form of the power series

$$x(t, \varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \cdots, \tag{2.72a}$$

$$y(t,\varepsilon) = y_0(T_0,T_1) + \varepsilon y_1(T_0,T_1) + \cdots, \tag{2.72b}$$

where $T_0 = t$ is a fast scale and $T_1 = \varepsilon t$ is a slow scale. The slow scale T_1 characterizes the modulation in the amplitude and phase caused by the nonlinearity, damping, and coupling. The fast scale T_0 is associated with the relatively fast changes in the response.

The first and second derivatives with respect to time t are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1 + \cdots, \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{2.73}$$

where $D_n = \partial/\partial T_n$. Substituting Eq. 2.72a,b and 2.73 into Eq. 2.71a,b and equating coefficients of equal powers of ε, we obtain $O(\varepsilon^0)$:

$$D_0^2 x_0 + \omega_1^2 x_0 = 0, \tag{2.74a}$$

$$D_0^2 y_0 + \omega_2^2 y_0 = 0, \tag{2.74b}$$

$O(\varepsilon)$:

$$D_0^2 x_1 + \omega_1^2 x_1 = -2D_0 D_1 x_0 + d_1(1 - x_0^2) D_0 x_0 - \alpha_1 x_0^3 - \delta x_0 y_0^2, \tag{2.75a}$$

$$D_0^2 y_1 + \omega_2^2 y_1 = -2D_0 D_1 y_0 + d_2(1 - y_0^2) D_0 y_0 - \alpha_2 y_0^3 - \delta x_0^2 y_0. \tag{2.75b}$$

The general solutions of the linear Eq. 2.74a,b can be written in the form of the correlations

$$x_0 = A_1(T_1) e^{i\omega_1 T_0} + \text{c.c.}, \tag{2.76a}$$

$$y_0 = A_2(T_1) e^{i\omega_2 T_0} + \text{c.c.}, \tag{2.76b}$$

where $c.c.$ represents the complex conjugates of the preceding term. The quantities A_1 and A_2 are arbitrary to this order of approximation and are determined by imposing solvability conditions at the next order of approximation. We substitute Eq. 2.76a,b into Eq. 2.75a,b and obtain

$$\begin{aligned} D_0^2 x_1 + \omega_1^2 x_1 = {} & \left[-2i\omega_1 A_1' + id_1\omega_1 A_1(1 - A_1\overline{A}_1) - 3\alpha_1 A_1^2 \overline{A}_1 - 2\delta A_1 A_2 \overline{A}_2\right] e^{i\omega_1 T_0} \\ & - \left[\alpha_1 A_1^3 + id_1\omega_1 A_1^3\right] e^{3i\omega_1 T_0} - \delta A_1 A_2^2 e^{i(\omega_1 + 2\omega_2)T_0} - \delta \overline{A}_1 A_2^2 e^{i(2\omega_2 - \omega_1)T_0} + \text{ c.c.,} \end{aligned} \tag{2.77a}$$

$$\begin{aligned} D_0^2 y_1 + \omega_2^2 y_1 = {} & \left[-2i\omega_2 A_2' + id_2\omega_2 A_2(1 - A_2\overline{A}_2) - 3\alpha_2 A_2^2 \overline{A}_2 - 2\delta A_1 A_2 \overline{A}_1\right] e^{i\omega_2 T_0} \\ & - \left[\alpha_2 A_2^3 + id_2\omega_2 A_2^3\right] e^{3i\omega_2 T_0} - \delta A_2 A_1^2 e^{i(\omega_2 + 2\omega_1)T_0} - \delta A_1^2 \overline{A}_2 e^{i(2\omega_1 - \omega_2)T_0} + \text{ c.c.,} \end{aligned} \tag{2.77b}$$

where $\overline{A}_n$ denotes the complex conjugate of A_n, the prime denotes differentiation with respect to T_1, and now $c.c.$ represents complex conjugates of each preceding term.

Nonresonant Case

Consider the nonresonant case, $\omega_1 \neq \omega_2$. The arbitrary functions $A_1(T_1)$ and $A_2(T_1)$ are determined from Eq. 2.77a,b by satisfying the solvability conditions for boundedness of the solutions. Particular solutions of Eq. 2.77a,b contain secular terms generated by the first term on the right-hand sides of Eq. 2.77a,b. The conditions for the elimination of secular terms, in Eq. 2.77a,b are

$$2i\omega_1 A_1' - id_1\omega_1 A_1(1 - A_1\overline{A}_1) + 3\alpha_1 A_1^2\overline{A}_1 + 2\delta A_1 A_2\overline{A}_2 = 0, \tag{2.78a}$$

$$2i\omega_2 A_2' - id_2\omega_2 A_2(1 - A_2\overline{A}_2) + 3\alpha_2 A_2^2\overline{A}_2 + 2\delta A_1 A_2\overline{A}_1 = 0. \tag{2.78b}$$

At this step, we introduce the polar forms for the amplitudes A_1 and A_2 as

$$A_1(T_1) = \frac{1}{2}a_1(T_1)e^{i\theta_1(T_1)}, \tag{2.79a}$$

$$A_2(T_1) = \frac{1}{2}a_2(T_1)e^{i\theta_2(T_1)}. \tag{2.79b}$$

Substituting the above expressions for A_1 and A_2 in Eq. 2.78a,b and separating real and imaginary parts, we have

$$\frac{\mathrm{d}a_1}{\mathrm{d}T_1} = \frac{1}{2}d_1 a_1\left(1 - \frac{a_1^2}{4}\right), \tag{2.80a}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}T_1} = \frac{1}{2}d_2 a_2\left(1 - \frac{a_2^2}{4}\right), \tag{2.80b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}T_1} = \frac{1}{8}a_1^2\left(\frac{2\delta}{\omega_2} - \frac{3\alpha_1}{\omega_1}\right) - \frac{1}{8}a_2^2\left(\frac{2\delta}{\omega_1} - \frac{3\alpha_2}{\omega_2}\right), \tag{2.80c}$$

where $\phi = \theta_2 - \theta_1$. Equations 2.80a, b, c are the first-order equations describing the variation of a_1, a_2, and ϕ. The Eqs. 2.80a and 2.80b are uncoupled, and the solutions are

$$a_1(T_1) = 2\left[1 - \left(1 - \frac{4}{a_{10}}\right)e^{-d_1 T_1}\right]^{-1/2}, \tag{2.81a}$$

$$a_2(T_1) = 2\left[1 - \left(1 - \frac{4}{a_{20}}\right)e^{-d_2 T_1}\right]^{-1/2}. \tag{2.81b}$$

The solutions x and y in the zeroth-order approximation are written as

$$x = x_0 = a_1\cos(\theta_1 + \omega_1 T_0), \tag{2.82a}$$

$$y = y_0 = a_2\cos(\theta_2 + \omega_2 T_0). \tag{2.82b}$$

Resonant Case

Next, we consider the resonant case. In order to describe the nearness of ω_2 to ω_1, we introduce a detuning parameter ε through the equation

$$\omega_2 = \omega_1 + \varepsilon\sigma. \tag{2.83}$$

Using the equations $(2\omega_2 - \omega_1)T_0 = \omega_1 T_0 + 2\sigma T_1$ and $(2\omega_1 - \omega_2)T_0 = \omega_2 T_0 - 2\sigma T_1$, the new solvability conditions are

$$2i\omega_1 A_1' - id_1\omega_1 A_1(1 - A_1\overline{A}_1) + 3\alpha_1 A_1^2\overline{A}_1 + 2\delta A_1 A_2\overline{A}_2 + \delta A_2^2\overline{A}_1 e^{i2\sigma T_1} = 0, \tag{2.84a}$$

$$2i\omega_2 A_2' - id_2\omega_2 A_2(1 - A_2\overline{A}_2) + 3\alpha_2 A_2^2\overline{A}_2 + 2\delta A_1 A_2\overline{A}_1 + \delta A_1^2\overline{A}_2 e^{-i2\sigma T_1} = 0. \tag{2.84b}$$

Substituting Eq. 2.80a,b,c into Eq. 2.84a,b and separating the real and imaginary parts of Eq. 2.84a,b, we obtain the following set of equations:

$$\frac{\mathrm{d}a_1}{\mathrm{d}T_1} = \frac{1}{2}d_1 a_1\left(1 - \frac{a_1^2}{4}\right) - \frac{\delta}{8\omega_1}a_1 a_2^2 \sin\phi, \tag{2.85a}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}T_1} = \frac{1}{2}d_2 a_2\left(1 - \frac{a_2^2}{4}\right) + \frac{\delta}{8\omega_2}a_1^2 a_2 \sin\phi, \tag{2.85b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}T_1} = 2a_1^2\left(\frac{\delta}{4\omega_2} - \frac{3\alpha_1}{8\omega_1}\right) - 2a_2^2\left(\frac{\delta}{4\omega_1} - \frac{3\alpha_2}{8\omega_2}\right) + \frac{\delta}{4}\left(\frac{a_1^2}{\omega_2} - \frac{a_2^2}{\omega_1}\right)\cos\phi + 2\sigma, \tag{2.85c}$$

where $\phi = 2\theta_2 - 2\theta_1 + 2\sigma T_1$. For the resonant case, the solutions are given by Eq. 2.83, with the time evolution of the amplitudes and phases as described by Eq. 2.85a,b,c.

2.3 Parameterized Perturbation Method

2.3.1 Introduction

The parameterized perturbation method (PPM) was first proposed by He in 1999b and was further developed in He (2006a).

The method can be applied to nonlinear equations including differential-difference equations (Ding and Zhang 2009; Jalaal et al. 2011).

This approach is an explicit method with high validity for resolution of strong nonlinear systems, which can be used to derive the relationship between period and amplitude in a nonlinear oscillator. In addition, it is more convenient and more efficient, in comparison with traditional methods.

2.3.2 Application

In the following, we consider the method by applied examples.
Example 2.4

Consider the free response of the undamped and single-DOF system that is shown in Fig. 2.1. The restoring forces in the spring are given by

$$F_{\rm sp} = -\left(k\,x(t) + \alpha\,x(t)^3\right), \tag{2.86}$$

with $\alpha > 0$. With this restoring force, the equation of motion of the system is

$$\ddot{x}\,(t) + k\,x(t) + \;\alpha\,x(t)^3 = 0, \tag{2.87}$$

where the equation of motion for this system with a cubic nonlinear stiffness is commonly known as Duffing's equation.

In order to use the parameterized perturbation method, it is necessary to introduce an artificial small parameter β:

$$x(t) = \beta\,v(t). \tag{2.88}$$

Substituting Eq. 2.88 in Eq. 2.84a,b, we obtain

$$\ddot{v}\,(t) + \;k.\,v(t) + \alpha\;\beta^2 v(t)^3 = 0, \qquad v(0) = A/\beta, \qquad \dot{v}\,(0) = 0. \tag{2.89}$$

Suppose that the solution of the Eq. 2.89 and the coefficient, k (or other coefficients), can be expressed in the forms

$$v(t) = v_0(t) + \beta^2 v_1(t) + \beta^4 v_2(t) + \cdots \tag{2.90}$$

$$k = \omega^2 + \beta^2\,\omega_1 + \beta^4\,\omega_2 + \cdots. \tag{2.91}$$

Substituting Eqs. 2.90 and 2.91 into Eq. 2.89 and equating the terms with the identical powers of β yields the following equations:

$$\ddot{v}_0\,(t) + \;\omega^2\,v_0(t) = 0, \qquad v_0(0) = A/\beta, \qquad \dot{v}_0(0) = 0. \tag{2.92}$$

$$\ddot{v}_1\,(t) + \;\omega^2 v_1(t) + \;\omega_1 v_0 + \alpha\;v_0^3 = 0, \qquad v_1(0) = 0, \qquad \dot{v}_1(0) = 0. \tag{2.93}$$

Considering the initial conditions $v_0 = A/\beta$ and $\dot{v}(0) = 0$, the solution of Eq. 2.92 is $v_0(t) = \frac{A}{\beta}\cos\omega t$.

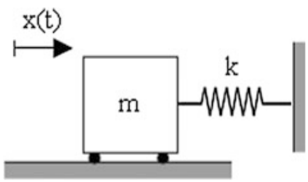


Fig. 2.1 Single-DOF system

Substituting the result into Eq. 2.93, it can be rewritten as

$$\ddot{v}_1(t) + \omega^2 v_1(t) + \frac{A}{\beta}(\omega_1 + \frac{3\,\alpha A^2}{4\beta^2})\cos(\omega t) + \frac{\alpha A^3}{4\beta^3}\cos(3\omega t) = 0. \tag{2.94}$$

Avoiding the presence of a secular term requires

$$\omega_1 = -\frac{3\,\alpha A^2}{4\beta^2}. \tag{2.95}$$

Solving Eq. 2.94, we obtain

$$v_1(t) = \frac{\alpha A^3}{32\omega^2\beta^3}(\cos(3\omega t) - \cos(\omega t)). \tag{2.96}$$

If, for example, its first-order approximation is sufficient, then we will have

$$x(t) = \beta v(t) = \beta\big(v_0(t) + \beta^2 v_1(t)\big) = A\,\cos\omega t + \left(\frac{A^3\alpha}{32\omega^2}\right)(\cos(3\omega t) - \cos(\omega t)). \tag{2.97}$$

Substituting Eq. 2.95 into Eq. 2.91, the first-order angular frequency can be written in the form

$$\omega = \frac{1}{2}\sqrt{4k + 3\,\alpha A^2}. \tag{2.98}$$

The period $T = 2\pi/\omega$ may then be written as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 + \frac{3\,\alpha\,\varepsilon A^2}{4}}}. \tag{2.99}$$

2.4 Singular Perturbation Method

2.4.1 Introduction

The theory of singular perturbations has been with us, in one form or another, for a little over a century (although the term "singular perturbation" dates from the 1940s). The subject and the techniques associated with it have evolved over this period as a response to the need to find approximate solutions (in an analytical form) to complex problems.

Typically, such problems are expressed in terms of differential equations that contain at least one small parameter, and they can arise in many fields: fluid mechanics, particle physics, and combustion processes, to name but three. The essential hallmark of a singular perturbation problem is that a simple and straightforward approximation (based on the smallness of the parameter) does not give an accurate solution throughout the domain of that solution. Perforce, this leads to different approximations being valid in different parts of the domain (usually requiring a "scaling" of the variables with respect to the parameter). This, in turn, has led to the important concepts of breakdown, matching, and so on. The notion of a singular perturbation problem was first evident in the seminal work of L. Prandtl (1874–1953) on the viscous boundary layer.

The singular perturbation method concerns the study of problems featuring a parameter for which solutions of the problem at a limiting value of the parameter are different in character from the limit of solutions of the general problem—namely, the limit is singular. In contrast, for regular perturbation problems, solutions of the general problem converge to solutions of the limit-problem as the parameter approaches the limit-value.

Singular perturbation theory considers systems of the form $\dot{x} = f(x, \varepsilon)$ in which f behaves singularly in the limit $\varepsilon \to 0$. A simple example of such a system is the Van der Pol oscillator in the large damping limit.

2.4.2 Application

Example 2.5

The Van der Pol oscillator is a second-order system with nonlinear damping, of the form

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0. \tag{2.100}$$

The special form of the damping (which can be realized by an electric circuit) has the effect of decreasing the amplitude of large oscillations, while increasing the amplitude of small oscillations.

We are interested in the behavior of the large α. There are several ways to write Eq. 2.100 as a first-order system. For our purpose, a convenient representation is

$$\begin{aligned} \dot{x} &= \alpha\left(y + x - \frac{x^3}{3}\right), \\ \dot{y} &= -\frac{x}{\alpha}. \end{aligned} \tag{2.101}$$

One can easily check that this system is equivalent to Eq. 2.100 by computing $\ddot{x}$. If it is very large, x will move quickly, while y changes slowly. To analyze the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and a slow time $t' = t/\alpha = \sqrt{\varepsilon}t$. This can then be rewritten as

$$\begin{aligned} \varepsilon\frac{dx}{dt'} &= y + x - \frac{x^3}{3}, \\ \frac{dy}{dt'} &= -x. \end{aligned} \tag{2.102}$$

In the limit $\varepsilon \to 0$, we obtain the system

$$\begin{aligned} 0 &= y + x - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -x, \end{aligned} \tag{2.103}$$

which is no longer a system of differential equations. In fact, the solutions are constrained to move on the curve $C: y = \frac{1}{3}x^3 - x$, and eliminating y from the system, we have

$$-x = \frac{\mathrm{d}y}{\mathrm{d}t'} = (x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t'} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t'} = -\frac{x}{x^2 - 1}. \tag{2.104}$$

The dynamics is shown in Fig. 2.2a. Another possibility is to introduce the fast time $t'' = \alpha t = t/\sqrt{\varepsilon}$.

Then Eq. 2.100 becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t''} &= y + x - \frac{x^3}{3}, \\ \frac{\mathrm{d}y}{\mathrm{d}t''} &= -\varepsilon x. \end{aligned} \tag{2.105}$$

In the limit $\varepsilon \to 0$, we get the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t''} &= y + x - \frac{x^3}{3}, \\ \frac{\mathrm{d}y}{\mathrm{d}t''} &= 0. \end{aligned} \tag{2.106}$$

In this case, y is a constant and acts as a parameter in the equation for x. Some orbits are shown in Fig. 2.2b. Of course, the systems (2.101, 2.102, and 2.105) are strictly equivalent for $\varepsilon > 0$. They only differ in the singular limit $\varepsilon \to 0$. The dynamics for small but positive ε can be understood by sketching the vector field. Let us note that $\dot{x}$ is positive if (x, y) lies above the curve C and negative when it

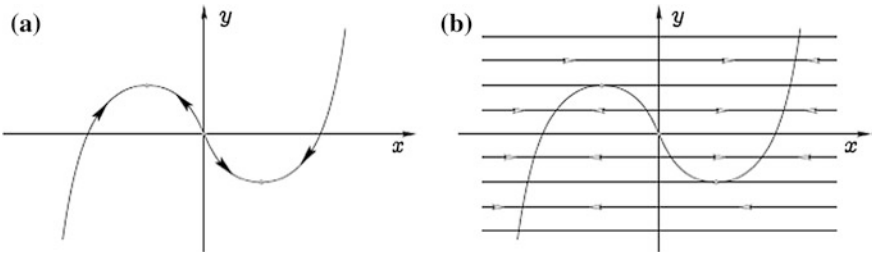


Fig. 2.2 Behavior of the Van der Pol equation in the singular limit $\varepsilon \to 0$, **a** on the slow time scale $t' = \sqrt{\varepsilon}t$, given by Eq. 2.103, and **b** on the fast time scale $t'' = t/\sqrt{\varepsilon}$ (see Eq. 2.105)

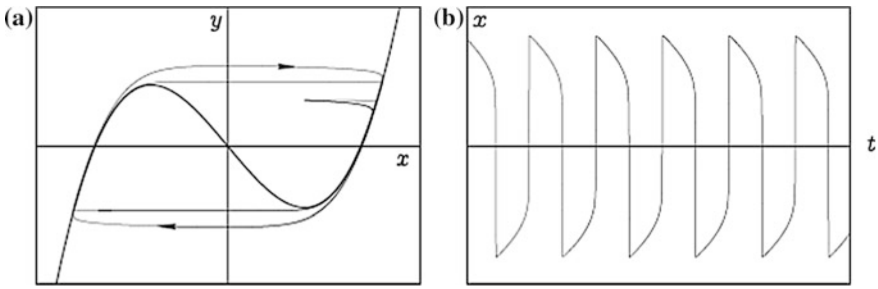


Fig. 2.3 **a** Two solutions of the Van der Pol equations (2.101) (*light curves*) for the same initial condition (1; 0:5), for $\alpha = 5$ and $\alpha = 20$. The heavy curve is the curve $C: y = \frac{1}{3}x^3 - x$. **b** The graph of $x(t)$ $(\alpha = 20)$ displays relaxation oscillations

lies below; this curve separates the plane into regions where x moves to the right or to the left and the orbit must cross the curve vertically; $\mathrm{d}y/\mathrm{d}x$: is very small unless the orbit is close to the curve C, so that the orbits will be almost horizontal except near this curve; Orbits move upward if $x < 0$ and downward if $x > 0$.

The resulting orbits are shown in Fig. 2.3a. An orbit starting somewhere in the plane will first approach the curve C on a nearly horizontal path, in a time t of order $1/\alpha$. Then it will track the curve at a small distance until it reaches a turning point, after a time t of order α. Since the equations forbid it to follow C beyond this point, the orbit will jump to another branch of C, where the behavior repeats. The graph of $x(t)$ thus contains some parts with a small slope and others with a large slope (Fig. 2.3b). This phenomenon is called a *relaxation oscillation*.

2.5 Homotopy Perturbation Method and Its Modifications

2.5.1 A Brief Introduction to the Homotopy Perturbation Method

The HPM was first proposed by He in 1999a (2000d). It has been worked out over a number of years by numerous authors and has matured into a relatively fledged theory thanks to the efforts of many researchers. For a relatively comprehensive survey on the concepts, theory, and applications of the HPM, the reader is referred to the review articles (He 2006a, d, 2008) and (Kachapi and Ganji 2013a, b; Kachapi et al. 2011; Ganji and Kachapi 2011; Hashemi et al. 2007; Tolou et al. 2009; Ganji and Hashemi 2007).

In contrast to the traditional perturbation methods, this technique does not require a small parameter in an equation. In this method, according to the homotopy technique, a homotopy with an imbedding parameter $p \in [0\ ,\ 1]$ is constructed, and the imbedding parameter is considered as a "small parameter," so the method is called the *homotopy perturbation method*, which can take full advantage of the traditional perturbation methods and homotopy techniques. In this method, the solution is considered as the sum of an infinite series, which converges rapidly to accurate solutions.

In this section, three cases of the HPM are applied for solving the governing equation. In the first case, standard HPM is applied. In the second case, time transformation $\theta = \omega t$ is used, and then a homotopy equation is constructed. The parameter expansion technique is used in both cases to expand the square of the unknown angular frequency. And the third case of an HPM is applied for fractional differential equations.

2.5.1.1 First Case of HPM

To explain the basic idea of the HPM for solving nonlinear differential equations, we consider the nonlinear differential equation

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{2.145}$$

subject to boundary condition

$$B(u, \partial u/\partial n) = 0,\ r \in \Gamma, \tag{2.146}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain Ω, and $\partial u/\partial n$ denotes differentiation along the normal drawn outward from Ω.The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Equation 2.145 therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \tag{2.147}$$

In the case in which the nonlinear Eq. 2.145 has no "small parameter," we can construct the homotopy

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p(N(v) - f(r)) = 0, \tag{2.148}$$

where

$$v(r,p){:}\Omega \times [0,1] \to R, \tag{2.149}$$

In Eq. 2.146, $p \in [0\ ,\ 1]$ is an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. 2.146 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \tag{2.150}$$

Also, the homotopy parameter p is used to expand the square of the unknown angular frequency ω as

$$\mu = \omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots, \tag{2.151}$$

or

$$\omega^2 = \mu + p\alpha_1 + p^2\alpha_2 + \ldots, \tag{2.152}$$

where μ is the coefficient of $u(r)$ in Eq. 2.145, the right-hand side of Eq. 2.151 replaces it. Also $\boldsymbol{\alpha}$ ($i = 1, 2, \ldots$) are arbitrary parameters that are to be determined.

The best approximation for solution and angular frequency ω are

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{2.153}$$

$$\omega^2 = 1 + \alpha_1 + \alpha_2 + \cdots, \tag{2.154}$$

when Eq. 2.148 corresponds to Eq. 2.145 and Eq. 2.153 becomes the approximate solution of Eq. 2.145.

2.5.1.2 Second Case of HPM

To explain the basic idea of this case for solving nonlinear differential equations, we consider the nonlinear differential equation

$$\ddot{u} = f(u, \dot{u}), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.155}$$

For the determination of the periodic solution of this equation, we first introduce a linear stiffness term with an unknown (constant) frequency ω into both sides of Eq. 2.155 as

$$\ddot{u} + \omega^2 u = f(u, \dot{u}) + \omega^2 u \equiv g(u, \dot{u}, \omega), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.156}$$

And then an artificial parameter p, is entered into Eq. 2.156, so we have

$$\ddot{u} + \omega^2 u = pg(u, \dot{u}, \omega), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.157}$$

It is obvious that Eq. 2.157 is equal to Eq. 2.156 for $p = 1$.

We now introduce a new independent variable $\theta = \omega t$ so that Eq. 2.157 can be written as

$$u'' + u = pg(u, u', \omega) = p\left(u + \frac{1}{\omega^2} f(u, u', \omega)\right), \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{2.158}$$

The solution of Eq. 2.158 is

$$u(\theta) = u_0(\theta) + pu_1(\theta) + p^2 u_2(\theta) + \cdots, \tag{2.159}$$

with

$$\omega^2 = \omega_0^2(\theta) + p\omega_1^2(\theta) + p^2\omega_2^2(\theta) + \cdots, \tag{2.160}$$

Substituting Eqs. 2.159 and 2.160 into Eq. 2.158, expanding of f or G about $(u_0, u_0,\ \omega_0)$ and setting the coefficients of the monomials p^n, $n \geq 0$ in the resulting series to zero, yields the following sequential equations:

$$u_0'' + u_0 = 0\,, \qquad u_0(0) = A,\ \ u_0'(0) = 0, \tag{2.161}$$

$$u_1'' + u_1 = \left(u_0 + \frac{1}{\omega_0^2} f(u_0, u_0', \omega_0)\right), \qquad u_1(0) = 0,\ \ u_1'(0) = 0, \tag{2.162}$$

$$\begin{aligned} u_2'' + u_2 = {} & \frac{\partial G}{\partial u}(u_0, u_0', \omega_0)u_1 + \frac{\partial G}{\partial u'}(u_0, u_0', \omega_0)u_1' + \frac{\partial G}{\partial \omega}(u_0, u_0', \omega_0)\omega_1\,, \ \ u_2(0) \\ & = 0,\ \ u_2'(0) = 0. \end{aligned} \tag{2.163}$$

The solution of Eq. 2.161 is

$$u_0(\theta) = A\cos(\theta)\,, \tag{2.164}$$

which can be substituted into Eq. 2.162, and the condition that $u_1(\theta)$ is free from secular terms provides an algebraic equation for the determination of ω_0. Similar conditions applied to $u_k(\theta),\ \ k \geq 2$, provide algebraic equations for $\omega_k,\ \ k \geq 1$. Substitution of these values of $u_k(\theta)$ and $\omega_k,\ \ k \geq 0$, into Eqs. 2.159 and 2.160, respectively, and setting $p = 1$ in those equations provide the solution and the frequency of oscillation, respectively.

2.5.1.3 Third Case of HPM

Recently, Shaher Momani applied the HPM to fractional differential equations in 2006 (Momani and Odibat 2006). To illustrate the basic ideas of the modification, we consider the following nonlinear differential equation of fractional order:

$$D_*^\alpha u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, \quad m - 1 < \alpha < m, \tag{2.165}$$

where L is a linear operator that might include other fractional derivatives of order less than a, N is a nonlinear operator that also might include other fractional derivatives of order less than a, f is a known analytic function, and D_*^α is the Caputo fractional derivative of order a, subject to the initial conditions

$$u^k(0) = c_k, \qquad k = 0,\ 1,\ 2,\ \ldots, m - 1 \tag{2.166}$$

In view of the homotopy technique, we can construct the homotopy

$$u^{(m)} - f(t) = p\left[u^{(m)} - L(u) - N(u) - D_*^\alpha u\right], \qquad p \in [0,\ 1]. \tag{2.167}$$

2.5.2 Application

To illustrate its effectiveness and its convenience, several examples with a high order of nonlinearity are used; the result reveals that the first order of approximation obtained by the proposed method is valid uniformly even for a very large parameter and is more accurate than the perturbation solutions.

Example 2.6

Introduction

Consider the generalized Huxley equation

$$u_t - u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma), \quad 0 \le x \le 1, \quad t \ge 0$$

with the initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}},$$

which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. The exact solution of this equation was derived by Wang et al., using nonlinear transformations, and is given by

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma\left(x + \left\{\frac{(1+\delta-\gamma)\rho}{2(1+\delta)}\right\}t\right)\right)\right]^{\frac{1}{\delta}},$$

where $\sigma = \delta\rho/4(1+\delta)$ and $\rho = \sqrt{4\beta(1+\delta)}$.

Applications

After separating the linear and nonlinear parts of the equation, we apply homotopy-perturbation, which can be constructed as (Hashemi et al. 2007)

$$\begin{aligned}
&(1-p)\left(\left(\frac{\partial}{\partial t}v(x,t)\right) - \left(\frac{\partial}{\partial t}u_0(x,t)\right)\right) \\
&+p\begin{pmatrix}\left(\frac{\partial}{\partial t}v(x,t)\right) - \left(\frac{\partial^2}{\partial x^2}v(x,t)\right) - \beta v(x,t) \\ (1-v(x,t)^\delta)(v(x,t)^\delta - \gamma)\end{pmatrix} \\
&= 0.
\end{aligned}$$

Applying HPM into the previous equation and rearranging the resultant equation on the basis of powers of p-terms, we obtain

$$\left(\frac{\partial}{\partial t}v_0(x,t)\right) = 0,$$

$$\begin{pmatrix}\left(\frac{\partial}{\partial t}v_1(x,t)\right) - \beta v_0(x,t)v_0(x,t)^\delta - \left(\frac{\partial^2}{\partial x^2}v_0(x,t)\right) + \beta v_0(x,t)v_0(x,t)^{2\delta} - \\ \beta v_0(x,t)v_0(x,t)^\delta\gamma + \beta v_0(x,t)\gamma\end{pmatrix} = 0.$$

Consider the following conditions:

$$u_0(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}, \quad \frac{\mathrm{d}}{\mathrm{d}t}u_0(x,0) = 0,$$
$$u_i(x,0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}u_i(x,0) = 0, \quad i = 1, 2, \ldots$$

With the effective initial approximation for v_0 from the designated conditions and the solution of previous equations, we will have

$$v_0(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}},$$
$$v_1(x,t) = \frac{1}{\delta^2}\Bigg((1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\Big(2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\tanh(\sigma\gamma x)^2 + 2^{\left(-\frac{1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}$$
$$\sigma^2\tanh(\sigma\gamma x)^2\delta - 2^{\left(\frac{-1+\delta}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\tanh(\sigma\gamma x) + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2 + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta$$
$$\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+\delta}{\delta}\right)}\beta\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1+2\delta}{\delta}\right)}\sigma^2\delta + 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}$$
$$\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{\delta}\beta\delta^2 - 2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}\left(2^{\left(\frac{-1}{\delta}\right)}\gamma^{\left(\frac{1}{\delta}\right)}(1+\tanh(\sigma\gamma x))^{\frac{1}{\delta}}\right)^{2\delta}\delta^2\Big)t\Bigg)$$

In the same manner, the additional components were obtained using the Maple Package.

According to the HPM, we can conclude that

$$u(x,t) = \lim_{p\to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \cdots$$

The numerical results of the exact solution and two-terms approximation of HPM, for $\beta = 1, \ \gamma = 0.001$, and $\delta = 1$, are given in Table 2.1.

Example 2.7

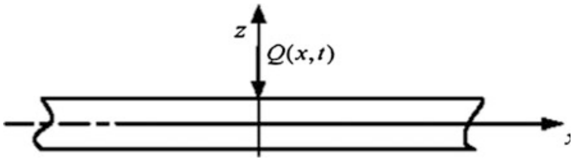
Introduction

Consider a one-dimensional finite beam excited at its center by a suddenly applied transverse force $Q(x,t)$ [variable coefficient fourth-order parabolic partial

Table 2.1 Numerical solutions for $\beta = 1, \ \gamma = 0.001$, and $\delta = 1$

X	t	Exact	HPM
0.1	0.05	5.00030171E-04	5.00005184E-04
	0.1	5.00042665E-04	4.99992690E-04
	1	5.00267553E-04	4.99767803E-04
0.5	0.05	5.00100882E-04	5.00075895E-04
	0.1	5.00113376E-04	5.00063401E-04
	1	5.00338263E-04	4.99838513E-04
0.9	0.05	5.00171593E-04	5.00146605E-04
	0.1	5.00184087E-04	5.00134111E-04
	1	5.00408974E-04	4.99909224E-04

Fig. 2.4 One-dimensional finite beam subject to sudden shear load



differential equations, where $Q(x,t) = 0$]. The applied point load at the origin on a finite beam is shown in Fig. 2.4 (Ganji and Hashemi 2007):

$$\frac{\partial^2}{\partial t^2}u(x,t) + \left(\frac{1}{x} + \frac{1}{120}x^4\right)\left(\frac{\partial^4}{\partial x^4}u(x,t)\right) = 0, \quad \frac{1}{2} < x < 1, \quad t > 0,$$

subject to the initial conditions

$$u(x,0) = 0, \quad \frac{1}{2} < x < 1,$$
$$\frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \quad \frac{1}{2} < x < 1$$

and the boundary conditions

$$u\left(\frac{1}{2},t\right) = \left(1 + \frac{0.5^5}{120}\right)\sin t, \quad u(1,t) = \left(\frac{121}{120}\right)\sin t, \quad t > 0,$$
$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin t, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6}\sin t, \quad t > 0.$$

HPM Method

After separating the linear and nonlinear parts of the equation, we apply HPM as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2 v(x,t)}{\partial t^2} - \frac{\partial^2 v_0(x,t)}{\partial t^2}\right) + p\left(\frac{\partial^2 v(x,t)}{\partial t^2} + \left(\frac{1}{x} + \frac{1}{120}x^4\right)\frac{\partial^4 v(x,t)}{4}\right) = 0,$$
$$p \in [0,1].$$

Applying HPM and rearranging on the basis of powers of p-terms, we have

$$P^0: \frac{\partial^2 v_0(x,t)}{\partial t^2} = 0,$$

$$p^1: \frac{120\left(\frac{\partial^2}{\partial t^2}v_1(x,t)\right)x + x^5\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right) + 120\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right)}{120x} = 0,$$

$$p^2: \frac{120\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right) + 120x\left(\frac{\partial^2}{\partial t^2}v_2(x,t)\right) + x^5\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right)}{120x} = 0,$$

Taking the following conditions into consideration,

$$v_0(x,0)=0,\quad \frac{\partial v_0}{\partial t}(x,0)=1+\frac{x^5}{120},$$
$$v_i(x,0)=0,\quad \frac{\partial v_i}{\partial t}(x,0)|_{t=0}=0,\quad i=1,2,\ldots$$

The solution of previous equations may be written as

$$v_0(x,t)=\left(1+\frac{1}{120}x^5\right)t,$$
$$v_1(x,t)=\left(1+\frac{1}{120}x^5\right)\left(-\frac{1}{6}t^3\right),$$
$$v_2(x,\ t)=\left(1+\frac{1}{120}x^5\right)\left(\frac{1}{120}t^5\right),$$

In the same manner, the remaining components were obtained using the software Package.

According to the HPM, we can conclude that

$$u(x,t)=\lim_{p\to 1}v(x,t)=v_0(x,t)+v_1(x,t)+\cdots$$

Therefore,

$$\begin{aligned} u(x,t)&=\left(1+\frac{1}{120}x^5\right)t+\left(1+\frac{1}{120}x^5\right)\left(-\frac{1}{6}t^3\right)+\left(1+\frac{1}{120}x^5\right)\left(\frac{1}{120}t^5\right)\\ &=\left(1+\frac{1}{120}x^5\right)\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\cdots\right)\end{aligned}$$

The solution $u(x,t)$ in a closed form is

$$u(x,t)=\left(1+\frac{x^5}{120}\right)\sin(t),$$

This is exactly the same as obtained by an exact solution.

Example 2.8

We next consider the parabolic equation

$$\frac{\partial^2}{\partial t^2}u(x,t)+\left(\frac{x}{\sin(x)}-1\right)\left(\frac{\partial^4}{\partial x^4}u(x,t)\right)=0,$$
$$0<x<1,\quad t>0$$

with the initial conditions

$$u(x,0)=x-\sin x,\quad 0<x<1,$$
$$\frac{\partial u}{\partial t}(x,0)=-(x-\sin x),\quad 0<x<1$$

and the boundary conditions

$$u(0,t)=0,\quad u(1,t)=e^{-t}(1-\sin t),\quad t>0,$$
$$\frac{\partial^2 u}{\partial x^2}(0,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(1,t)=e^{-t}\sin t,\quad t>0.$$

A homotopy can be constructed as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{\partial^2 u_0(x,t)}{\partial t^2}\right)+p\left(\frac{\partial^2 v(x,t)}{\partial t^2}+\left(\frac{x}{\sin(x)}-1\right)\frac{\partial^4 v(x,t)}{\partial x^4}\right)=0,$$
$$p\in[0,1].$$

Similar to previous example, after applying the HPM and rearranging based on powers of p-terms, we have

$$p^0\colon\left(\frac{\partial^2}{\partial t^2}v_0(x,t)\right)=0,$$
$$p^1\colon\frac{\left(\frac{\partial^2}{\partial t^2}v_1(x,t)\right)\sin(x)+x\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right)-\sin(x)\left(\frac{\partial^4}{\partial x^4}v_0(x,t)\right)}{\sin(x)}=0,$$
$$p^2\colon\frac{\left(\frac{\partial^2}{\partial t^2}v_2(x,t)\right)\sin(x)+x\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right)-\sin(x)\left(\frac{\partial^4}{\partial x^4}v_1(x,t)\right)}{\sin(x)}=0,$$

with the conditions

$$v_0(x,0)=x-\sin(x),\quad \frac{\partial v_0}{\partial t}(x,0)=-(x-\sin(x)),$$
$$v_i(x,0)=0,\quad \frac{\partial v_i}{\partial t}(x,0)|_{t=0}=0,\quad i=1,2,\ldots$$

The solution of previous equations may be written as

$$v_0(x,t)=(x-\sin(x))(1-t),$$
$$v_1(x,t)=(x-\sin(x))\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right),$$
$$\mathrm{v}_2(\mathrm{x,t})=(x-\sin(x))\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right),$$

In the same manner, the remaining components were obtained using a software Package.

According to the HPM, we can conclude that

$$u(x,t)=\lim_{p\to 1}v(x,t)=v_0(x,t)+v_1(x,t)+\cdots$$

Therefore,

$$u(x,t) = (x - \sin(x))\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right)$$

The solution of $u(x,t)$ in a closed form is

$$u(x,t) = (x - \sin(x))e^{-t},$$

This is exactly the same as that obtained by an exact solution.

Example 2.9

Introduction

In the following, we consider the fourth-order equation in two space variables

$$\frac{\partial^2}{\partial t^2}u(x,y,t) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\left(\frac{\partial^4}{\partial x^4}u(x,y,t)\right) + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\left(\frac{\partial^4}{\partial y^4}u(x,y,t)\right) = 0,$$
$$\frac{1}{2} < x,\ y < 1, \quad t > 0,$$

subject to the initial conditions

$$u(x,y,0) = 0, \quad \frac{1}{2} < x < 1,$$
$$\frac{\partial}{\partial t}u(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^4}{6!}, \quad \frac{1}{2} < x < 1,$$

and the boundary conditions

$$u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{(1/2)^6}{6} + \frac{y^6}{6!}\right)\sin t, \quad u(1,y,t) = \left(2 + \frac{(1/2)^6}{6} + \frac{y^6}{6!}\right)\sin t,$$
$$\frac{\partial^2}{\partial x^2}u\left(\frac{1}{2},y,t\right) = \frac{(1/2)^4}{24}\sin t, \qquad \frac{\partial^2}{\partial x^2}u(1,y,t) = \frac{1}{24}\sin t,$$
$$\frac{\partial^2}{\partial y^2}u\left(x,\frac{1}{2},t\right) = \frac{(1/2)^4}{24}\sin t, \qquad \frac{\partial^2}{\partial y^2}u(x,1,t) = \frac{1}{24}\sin t, \quad t > 0.$$

HPM Method

After separating the linear and nonlinear parts of the equation, a homotopy can be constructed as follows (Ganji and Hashemi 2007):

$$(1-p)\left(\frac{\partial^2}{\partial t^2}v(x,y,t) - \frac{\partial^2}{\partial t^2}u(x,y,0)\right) + p\left(\begin{array}{l}\frac{\partial^2}{\partial t^2}v(x,y,t) + 2\left(\frac{1}{x^2} + \frac{1}{720}x^4\right)\frac{\partial^4}{\partial x^4}v(x,y,t) \\ +\ 2\left(\frac{1}{y^2} + \frac{1}{720}y^4\right)\frac{\partial^4}{\partial y^4}v(x,y,t)\end{array}\right) = 0,$$

$p \in [0,1]$.

Applying HPM and rearranging on the basis of powers of p-terms, we have

$$p^0: \left(\frac{\partial^2}{\partial t^2} v_0(x,y,t)\right) = 0,$$

$$p^1: \frac{1}{360x^2y^2}\left(\begin{array}{l} 360\left(\frac{\partial^2}{\partial t^2} v_1(x,y,t)\right)x^2y^2 + y^2x^6\left(\frac{\partial^4}{\partial x^4} v_0(x,y,t)\right) \\ + 720y^2\left(\frac{\partial^4}{\partial x^4} v_0(x,y,t)\right) + x^2y^6\left(\frac{\partial^4}{\partial y^4} v_0(x,y,t)\right) \\ + 720x^2\left(\frac{\partial^4}{\partial y^4} v_0(x,y,t)\right) \end{array}\right) = 0,$$

$$p^2: \frac{1}{360x^2y^2}\left(\begin{array}{l} 360\left(\frac{\partial^2}{\partial t^2} v_2(x,y,t)\right)x^2y^2 + y^2x^6\left(\frac{\partial^4}{\partial x^4} v_1(x,y,t)\right) \\ + 720y^2\left(\frac{\partial^4}{\partial x^4} v_1(x,y,t)\right) + x^2y^6\left(\frac{\partial^4}{\partial y^4} v_1(x,y,t)\right) \\ + 720x^2\left(\frac{\partial^4}{\partial y^4} v_1(x,y,t)\right) \end{array}\right) = 0,$$

with the conditions

$$\begin{aligned} &v_0(x,y,0) = 0, \quad \frac{\partial}{\partial t} v_0(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^4}{6!}, \\ &v_i(x,y,0) = 0, \quad \frac{\partial}{\partial t} v_i(x,y,0)|_{t=0} = 0, \quad i = 1,2,\ldots \end{aligned}$$

With the effective initial approximation for v_0 from the designated conditions, the solutions of the equations may be written as

$$\begin{aligned} v_0(x,y,t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)t, \\ v_1(x,y,t) &= -\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\frac{t^3}{3!}, \\ v_2(x,y,t) &= \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\frac{t^5}{5!}, \\ &. \\ &. \end{aligned}$$

In the same manner, the remaining components were obtained using a software package.

According to the HPM, we can conclude that

$$u(x,y,t) = \lim_{p\to 1} v(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots$$

Therefore,

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right),$$

The solution $u(x,t)$ in a closed form is

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\ \sin t,$$

This is exactly the same as that obtained by an exact solution.

Example 2.10

This example considers the following nonlinear oscillator with discontinuity (Beléndez et al. 2008):

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x) = 0, \tag{2.168}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0, \tag{2.169}$$

and $\mathrm{sgn}(x)$ is defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x<0, \\ +1, & x \geq 0. \end{cases} \tag{2.170}$$

All the solutions to Eq. 2.168 are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$—that is, the functional behavior of ω as a function of the initial amplitude A.

Equation 2.168 can be rewritten in the form

$$\frac{d^2x}{dt^2} + x = x - \mathrm{sgn}(x). \tag{2.171}$$

Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \cdots, \tag{2.172}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots, \tag{2.173}$$

where $\alpha_i (i = 1, 2, \ldots)$ are to be determined. Substituting Eqs. 2.172 and 2.173 into Eq. 2.171,

$$\begin{aligned}&\left(x_0'' + px_1'' + p^2x_2'' + \cdots\right) + \left(\omega^2 - p\alpha_1 - p^2\alpha_2 - \cdots\right)\left(x_0 + px_1 + p^2x_2 + \cdots\right)\\ &\quad = p\left[\left(x_0 + px_1 + p^2x_2 + \cdots\right) - \mathrm{sgn}\left(x_0 + px_1 + p^2x_2 + \cdots\right)\right]\end{aligned} \tag{2.174}$$

and collecting the terms of the same power of p, we obtain a series of linear equations, of which we write only the first four:

$$x_0'' + \omega^2 x_0 = 0, \qquad x_0(0) = A, \quad x_0'(0) = 0, \tag{2.175}$$

$$x_1'' + \omega^2 x_1 = (1+\alpha_1)x_0 - \operatorname{sgn}(x_0), \qquad x_1(0) = x_1'(0) = 0, \tag{2.176}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1+\alpha_1)x_1, \qquad x_2(0) = x_2'(0) = 0, \tag{2.177}$$

$$x_3'' + \omega^2 x_3 = \alpha_3 x_0 + \alpha_2 x_1 + (1+\alpha_1)x_2, \qquad x_3(0) = x_3'(0) = 0. \tag{2.178}$$

In Eqs. 2.175–2.178, we have taken into account the expression

$$\begin{aligned} f(x) &= f\left(x_0 + px_1 + p^2x_2 + p^3x_3 + \cdots\right) \\ &= f(x_0) + p\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)_{x=x_0} x_1 + p^2\left[\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)_{x=x_0} x_2 + \frac{1}{2}\left(\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}\right)_{x=x_0} x_1^2\right] + O(p^3), \end{aligned} \tag{2.179}$$

where $f(x) = \operatorname{sgn}(x)$ and

$$\frac{\mathrm{d}\operatorname{sgn}(x)}{\mathrm{d}x} = \frac{\mathrm{d}^2\operatorname{sgn}(x)}{\mathrm{d}x^2} = \cdots = 0 \quad \textit{for} \quad x \neq 0,$$

and then

$$\operatorname{sgn}\left(x_0 + px_1 + p^2x_2 + \cdots\right) = \operatorname{sgn}(x_0).$$

The solution of Eq. 2.175 is

$$x_0(t) = A\cos\omega t. \tag{2.180}$$

Substitution of this result into the right-hand side of Eq. 2.176 gives

$$x_1'' + \omega^2 x_1 = (1+\alpha_1)A\cos\omega t - \operatorname{sgn}(A\cos\omega t). \tag{2.181}$$

It is possible to do the following Fourier series expansion:

$$\operatorname{sgn}(\cos\omega t) = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] = a_1\cos\omega t + a_3\cos 3\omega t + \cdots, \tag{2.182}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \operatorname{sgn}(A\cos\tau)\cos[(2n+1)\tau]\mathrm{d}\tau = (-1)^n\frac{4}{(2n+1)\pi}. \tag{2.183}$$

The first term of the expansion in Eq. 2.183 is given by

$$a_1 = \frac{4}{\pi}\int\limits_0^{\pi/2} \operatorname{sgn}(A\cos\tau)\cos\tau d\tau = \frac{4}{\pi}. \quad (2.184)$$

Substituting Eq. 2.182 into Eq. 2.181, we have

$$x_1'' + \omega^2 x_1 = [(1+\alpha_1)A - a_1]\cos\omega t - \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t]. \quad (2.185)$$

No secular terms in $x_1(t)$ require eliminating contributions proportional to $\cos\omega t$ on the right-hand side of Eq. 2.185, and we obtain

$$\alpha_1 = -1 + \frac{a_1}{A} = -1 + \frac{4}{\pi A}. \quad (2.186)$$

From Eqs. 2.173, 2.184, and 2.186, writing $p = 1$, we can easily find that the first-order approximate frequency is

$$\omega_1(A) = \sqrt{\frac{a_1}{A}} = \frac{2}{\sqrt{\pi A}} = \frac{1.128379}{\sqrt{A}} \quad (2.187)$$

and the first-order approximation period can be obtained as

$$T_1(A) = \pi\sqrt{\pi A} = 5.568328\sqrt{A} \quad (2.188)$$

Now, in order to obtain the correction term x_1 for the periodic solution x_0, we consider the following procedure. Taking Eqs. 2.185 and 2.186 into account, we rewrite Eq. 2.185 in the form

$$x_1'' + \omega^2 x_1 = -\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] \quad (2.189)$$

with initial conditions $x_1(0) = 0$ and $x_1'(0) = 0$. The periodic solution of Eq. 2.189 can be written as

$$x_1(t) = \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t]. \quad (2.190)$$

Substituting Eq. 2.190 into Eq. 2.189, we can write the following expression for the coefficients b_{2n+1}:

$$b_{2n+1} = \frac{a_{2n+1}}{4n(n+1)\omega^2} = \frac{(-1)^n}{n(n+1)(2n+1)\pi\omega^2}. \quad (2.191)$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. 2.186 gives

$$b_1 = -\sum_{n=1}^{\infty} b_{2n+1} = \frac{\pi - 3}{\pi\omega^2} = \frac{\sigma}{\omega^2} \quad (2.192)$$

where

$$\sigma = 1 - \frac{3}{\pi}. \tag{2.193}$$

Substituting Eqs. 2.180, 2.186, 2.190, 2.191, and 2.192 into Eq. 2.177 gives the following equation for $x_2(t)$:

$$x_2'' + \omega^2 x_2 = \alpha_2 A \cos\omega t + \frac{\sigma a_1}{A\omega^2}\cos\omega t + \sum_{n=1}^{\infty}\frac{a_{2n+1}}{4n(n+1)A\omega^2}\cos[(2n+1)\omega t]. \tag{2.194}$$

The secular term in the solution for $x_2(t)$ can be eliminated if

$$\alpha_2 = -\frac{\sigma a_1}{A^2\omega^2} = \frac{12-4\pi}{\pi^2 A^2 \omega^2}. \tag{2.195}$$

Similarly for ω_2, and taking $p = 1$, one can easily obtain the following expression for the second-order approximation frequency

$$\omega_2(A) = \frac{1}{\sqrt{2A}}\sqrt{a_1 + \sqrt{a_1^2 - 4\sigma a_1}} = \sqrt{\frac{2+2\sqrt{4-\pi}}{\pi A}} = \frac{1.107452}{\sqrt{A}}, \tag{2.196}$$

and the second-order approximation period is given by

$$T_2(A) = 5.672551\sqrt{A}. \tag{2.197}$$

With the requirement of Eq. 2.198, we can rewrite Eq. 2.194 in the form

$$x_2'' + \omega^2 x_2 = \sum_{n=1}^{\infty}\frac{a_{2n+1}}{4n(n+1)A\omega^2}\cos[(2n+1)\omega t], \tag{2.198}$$

with initial conditions $x_2(0) = 0$ and $x_2'(0) = 0$. The solution of this equation is

$$x_2(t) = \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t]. \tag{2.199}$$

Substituting Eq. 2.199 into Eq. 2.198, we obtain the following expression for the coefficients c_{2n+1}:

$$c_{2n+1} = -\frac{a_1 a_{2n+1}}{16n^2(n+1)^2 A\omega^4} = \frac{(-1)^{n+1}}{n^2(n+1)^2(2n+1)\pi^2 A\omega^4}, \tag{2.200}$$

for $n \geq 1$. Taking into account that $x_2(0) = 0$, Eq. 2.199 gives

$$c_1 = -\sum_{n=1}^{\infty} c_{2n+1} = \frac{\pi^2 + 24\pi - 66}{6\pi^2 A\omega^4} = \frac{\lambda}{A\omega^4}. \tag{2.201}$$

where

$$\lambda = \frac{\pi^2 + 24\pi - 66}{6\pi^2}. \tag{2.202}$$

Substitution of Eqs. 2.180, 2.186, 2.190–2.192, 2.195, and 2.199 into Eq. 2.178 gives the following equation for $x_3(t)$:

$$x_3'' + \omega^2 x_3 = \alpha_3 A \cos \omega t + \frac{\lambda a_1}{A^2\omega^4}\cos \omega t - \frac{\sigma^2 a_1}{A^2\omega^4}\cos \omega t \\ - \sum_{n=1}^{\infty} \frac{\sigma^2 a_{2n+1}}{4n(n+1)A^2\omega^4}\cos[(2n+1)\omega t] - \sum_{n=1}^{\infty} \frac{a_1^2 a_{2n+1}}{16n^2(n+1)^2 A^2\omega^4}\cos[(2n+1)\omega t]. \tag{2.203}$$

The secular term in the solution for $x_3(t)$ can be eliminated if

$$\alpha_3 = \frac{\sigma^2 a_1 - \lambda a_1}{A^3\omega^4} = \frac{240 - 120\pi + 14\pi^2}{3\pi^3 A^3 \omega^4}. \tag{2.204}$$

From Eqs. 2.173, 2.186, 2.195, and 2.204, and taking $p = 1$, one can easily obtain that the following expression for the third-order approximation frequency is

$$\omega_3(A) = \frac{1.111358}{\sqrt{A}}, \tag{2.205}$$

and the third-order approximate period is given by

$$T_3(A) = 5.633609\sqrt{A}. \tag{2.206}$$

Taking Eq. 2.204 into consideration, we can rewrite Eq. 2.203 in the form

$$x_3'' + \omega^2 x_3 = -\sum_{n=1}^{\infty} \frac{\sigma^2 a_{2n+1}}{4n(n+1)A^2\omega^4}\cos[(2n+1)\omega t] \\ - \sum_{n=1}^{\infty} \frac{a_1^2 a_{2n+1}}{16n^2(n+1)^2 A^2\omega^4}\cos[(2n+1)\omega t] \tag{2.207}$$

with initial conditions $x_3(0) = 0$ and $x_3'(0) = 0$. The solution of this equation is

$$x_3(t) = \sum_{n=0}^{\infty} d_{2n+1}\cos[(2n+1)\omega t]. \tag{2.208}$$

Substituting Eq. 2.208 into Eq. 2.207, we obtain the following expression for the coefficients d_{2n+1}:

$$d_{2n+1} = \frac{(-1)^{n+1}[n(n+1)(\pi - 3) + 1]}{n^3(n+1)^3(2n+1)\pi^3 A^2 \omega^6} \tag{2.209}$$

for $n \geq 1$. Taking into account that $x_3(0) = 0$, Eq. 2.207 gives

$$d_1 = -\sum_{n=1}^{\infty} d_{2n+1} = \frac{\pi^3 - 32\pi^2 + 234\pi - 450}{n^3(n+1)^3(2n+1)\pi^3 A^2 \omega^6}. \tag{2.210}$$

For this nonlinear problem, the exact periodic solution and the exact period are given by the equations

$$x_e(t) = \begin{cases} -\dfrac{t^2}{2} + A, & 0 \le t \le \dfrac{T_e}{4}, \\ \dfrac{t^2}{2} - 2\sqrt{2At} + 3A, & \dfrac{T_e}{4} < t \le \dfrac{3T_e}{4}, \\ -\dfrac{t^2}{2} + 4\sqrt{2At} - 15A, & \dfrac{3T_e}{4} < t \le T_e, \end{cases} \tag{2.211}$$

$$T_e(A) = 4\sqrt{2A} = 5.656854\sqrt{A}. \tag{2.212}$$

An easy and direct calculation gives the following series representation for the exact solution $x_e(t)$ (Eq. 2.211):

$$x_e(t) = \frac{32A}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos[(2n+1)\omega_e t], \tag{2.213}$$

where

$$\omega_e(A) = \frac{2\pi}{T_e(A)} = \frac{\pi}{2\sqrt{2A}} = \frac{1.110721}{\sqrt{A}}. \tag{2.214}$$

2.6 Variational Iteration Method

2.6.1 Introduction

The VIM was proposed by the Chinese mathematician He in 1997a and is a modified general Lagrange's multiplier method (Inokuti et al. 1978).

VIM has been favorably applied to various kinds of nonlinear problems. The main property of the method lies in its flexibility and ability to solve nonlinear equations accurately and conveniently, using a linearization assumption as an initial approximation or trial function; then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. The confluence of modem mathematics and symbol computation has posed a challenge to developing technologies capable of handling strongly nonlinear equations, which cannot be successfully dealt with by classical methods. The VIM is uniquely qualified to address this challenge. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solutions. A new iteration formulation is suggested for overcoming the shortcoming. A very useful formulation for determining approximately the period of a nonlinear oscillator is suggested. Examples are given to illustrate the solution procedure.

Consider the following general nonlinear differential equation of an oscillator:

$$u'' + f(u, u', u'') = 0, \tag{2.215}$$

subject to $u(0) = a$ and $u'(0) = b$, where t is time and u is the displacement. The prime denotes differentiation with respect to t.

We rewrite Eq. 2.215 in the form

$$u'' + \Omega^2 u = F(u), \qquad F(u) = \Omega^2 u - f(u). \tag{2.216}$$

We consider that the angular frequency of the oscillator is Ω, and we choose the trial function using an initial condition [such as, for the initial condition $u(0) = A$ and $u'(0) = 0$, the trial function is $u_0(t) = A\cos\Omega t$]. The angular frequency Ω is identified with the physical understanding that no secular terms should appear in $u_1(t)$, which leads to

$$\int_0^T \cos\Omega t\left[\Omega^2 u_0 - f(u_0)\right]\mathrm{d}t = 0, \qquad T = \frac{2\pi}{\Omega}. \tag{2.217}$$

From this equation, Ω can easily be found. It should be especially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to its exact solution, and for this reason, we identify the multiplier from Eq. 2.216 rather than Eq. 2.215.

According to the VIM, we can construct a correction functional as (He 1997a)

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{u_n''(\tau) + \Omega^2 u_n(\tau) - \tilde{F}_n\right\}\mathrm{d}\tau, \tag{2.218}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{F}_n$ is considered as a restricted variation—that is, $\delta\tilde{F}_n = 0$. Under this condition, its stationary conditions of the above correction functional can be written as

$$\begin{aligned} &\lambda''(\tau) + \Omega^2\lambda(\tau) = 0, \\ &\lambda(\tau)|_{\tau=t} = 0, \\ &1 - \lambda'(\tau)|_{\tau=t} = 0. \end{aligned} \tag{2.219}$$

The Lagrange multiplier can, therefore, be readily identified by

$$\lambda = \frac{1}{\Omega}\sin\Omega(\tau - t), \tag{2.220}$$

which leads to the iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_n''(\tau) + f_n\right\} d\tau. \tag{2.221}$$

As we will see in the forthcoming illustrative examples, we usually stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

2.6.2 Application

In this section, several examples are considered for the comparison and usefulness of the method developed.

Example 2.11

Let us consider the following nonlinear oscillators with discontinuities:

$$u'' + u + \varepsilon u|u| = 0,$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

Here the discontinuous function is $f(u) = u + \varepsilon u|u|$. From Eq. 2.217, we can determine the angular frequency (Rafei et al. 2007a) as

$$\int_0^T \cos \Omega t \left[\Omega^2 A \cos \Omega t - (A \cos \Omega t + \varepsilon A \cos \Omega t |A \cos \Omega t|)\right] dt = 0, \qquad T = \frac{2\pi}{\Omega}.$$

Noting that $|\cos \Omega t| = \cos \Omega t$ when $-\pi/2 \le \Omega t \le \pi/2$ and $|\cos \Omega t| = -\cos \Omega t$ when $\pi/2 \le \Omega t \le 3\pi/2$, we can write the previous equation in the form

$$\int_{-\pi/2\Omega}^{\pi/2\Omega} \left[(\Omega^2 - 1)A \cos^2 \Omega t - \varepsilon A^2 \cos^3 \Omega t\right] dt + \int_{\pi/2\Omega}^{3\pi/2\Omega} \left[(\Omega^2 - 1)A \cos^2 \Omega t + \varepsilon A^2 \cos^3 \Omega t\right] dt = 0.$$

From the above equation, one can easily conclude that

$$\Omega = \sqrt{1 + \frac{8}{3\pi} \varepsilon A}.$$

We rewrite Eq. 2.221 in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\Omega} \sin \Omega(\tau - t)\left\{u_n''(\tau) + u_n(\tau) + \varepsilon u_n(\tau)|u_n(\tau)|\right\} d\tau.$$

By the above iteration formula, we can calculate the first-order approximation

$$u_1(t) = \begin{cases} A\cos\Omega t + \int\limits_0^t \frac{1}{\Omega}\sin\Omega(\tau - t)\left\{(1-\Omega^2)A\cos\Omega\tau + \varepsilon A^2\cos^2\Omega\tau\right\}\mathrm{d}\tau, \\ -\frac{\pi}{2} \le \Omega t \le \frac{\pi}{2}, \\ A\cos\Omega t + \int\limits_0^t \frac{1}{\Omega}\sin\Omega(\tau - t)\left\{(1-\Omega^2)A\cos\Omega\tau - \varepsilon A^2\cos^2\Omega\tau\right\}\mathrm{d}\tau, \\ \frac{\pi}{2} \le \Omega t \le \frac{3\pi}{2}, \end{cases}$$

which yields

$$u_1(t) = \begin{cases} A\cos\Omega t + \frac{1}{2\Omega}A(\Omega^2 - 1)t\sin\Omega t + \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2\cos\Omega t) - \frac{\varepsilon A^2}{2\Omega^2}, \\ -\frac{\pi}{2} \le \Omega t \le \frac{\pi}{2}, \\ A\cos\Omega t + \frac{1}{2\Omega}A(\Omega^2 - 1)t\sin\Omega t - \frac{\varepsilon A^2}{6\omega^2}(\cos 2\Omega t + 2\cos\Omega t) + \frac{\varepsilon A^2}{2\Omega^2}, \\ \frac{\pi}{2} \le \Omega t \le \frac{3\pi}{2}, \end{cases}$$

where the angular frequency Ω is defined as $\Omega = \sqrt{1 + \frac{8}{3\pi}\varepsilon A}$.

In order to compare with a traditional perturbation solution, we write Ali Nayfeh's result:

$$u = A\cos\left(1 + \frac{4}{3\pi}\varepsilon A\right)t + \cdots,$$

Example 2.12

Introduction

Two-strand or Sirospun yarns are produced on a conventional ring frame by feeding two roving, drafted simultaneously, into the apron zone at a predetermined separation. Emerging from the nip point of the front rollers, the two strands are twisted together to form a two-ply structure (see Fig. 2.5).

Nonlinear Dynamical Model

Assume that the convergence point (equilibrium position) moves to an instantaneous position (see Fig. 2.6), and the distance x and y are measured from the equilibrium position. Then the motion equations in x and y directions can be expressed (Shou and He 2008) as

$$\begin{aligned} m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + F_1\cos\alpha - F_2\cos\beta = 0, \\ m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + F_1\sin\alpha + F_2\sin\beta - F = 0. \end{aligned} \tag{2.222}$$

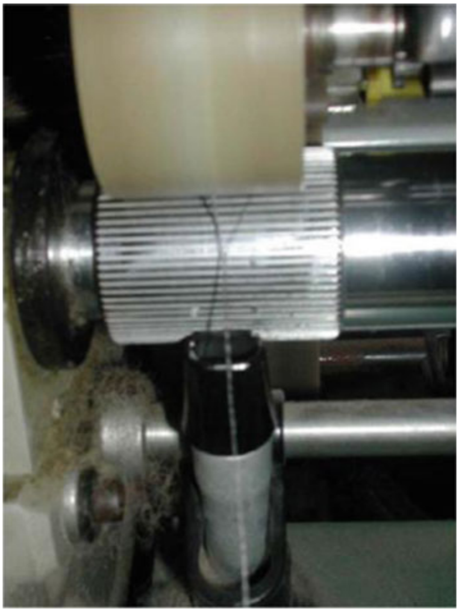


Fig. 2.5 Two-strand yarn spinning

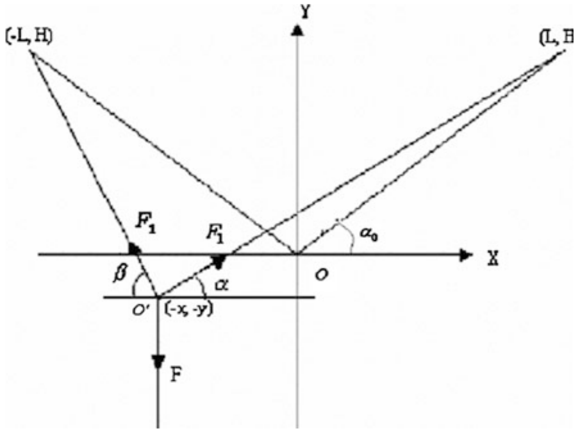


Fig. 2.6 Dynamical illustration of two-strand spun

Here, m is total mass of a fixed control volume, the control volume having been chosen in such a way that the mass center is located on the convergent point (O) of the two strands.

Expanding the trigonometric functions into series of x and y, we can obtain a coupled nonlinear oscillator. In this example, we consider the special

$$\begin{cases} \ddot{x} + \omega_1^2 x + \varepsilon_1 y^2 x = 0 \\ \ddot{y} + \omega_2^2 y + \varepsilon_2 x^2 y = 0 \end{cases}, \tag{2.223}$$

with the initial condition $x(0) = A, \dot{x}(0) = 0, y(0) = B, \dot{y}(0) = 0$.

In our study, ε_1 and ε_2 do not need to be small. We will apply the VIM to solve Eq. 2.222.

Applying the VIM, we can easily construct the following iteration formulations:

$$x_{n+1} = x_n + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\left\{\frac{d^2x}{ds^2} + \omega_1^2 x + \varepsilon_1 y^2 x\right\} ds, \tag{2.224}$$

$$y_{n+1} = y_n + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\left\{\frac{d^2y}{ds^2} + \omega_1^2 y + \varepsilon_1 x^2 y\right\} ds, \tag{2.225}$$

where $\lambda_x = \frac{1}{\omega_1}\sin\omega_1 t,\ \lambda_y = \frac{1}{\omega_2}\sin\omega_2 t.$

We begin with the initial solutions:

$$x_0 = A\cos\Omega_1 t, \tag{2.226}$$

$$y_0 = B\cos\Omega_2 t. \tag{2.227}$$

where Ω_1, Ω_2 are the frequencies in the x and y directions, respectively.

According to the iteration formulations 2.224 and 2.225, we obtain

$$\begin{aligned}
x_1 =& A\cos\Omega_1 t + \frac{1}{\omega_1}\int_0^t \sin\omega_1(s-t)\left\{A\left(\left(\omega_1^2 - \Omega_1^2\right)\cos\Omega_1 s + \varepsilon_1 AB^2\cos\Omega_1 s\cos^2\Omega_2 s\right\} ds \\
=& A\cos\Omega_1 t + A\left(\omega_1^2 - \Omega_1^2\right)\frac{\cos\omega_1 t - \cos\Omega_1 t}{\omega_1^2 - \Omega_1^2} + \frac{\varepsilon_1 AB^2}{2}\frac{\cos\omega_1 t - \cos\Omega_1 t}{\omega_1^2 - \Omega_1^2} \\
&+ \frac{\varepsilon_1 AB^2}{4}\frac{\cos\omega_1 t - \cos(2\Omega_2 + \Omega_1)t}{\omega_1^2 - (2\Omega_2 + \Omega_1)^2} + \frac{\omega_2^2 AB^2}{4}\frac{\cos\omega_1 t - \cos(2\Omega_2 - \Omega_1)t}{\omega_1^2 - (2\Omega_2 - \Omega_1)^2} \\
=& \left\{A + \frac{\varepsilon_1 AB^2}{2(\omega_1^2 - \Omega_1^2)^2} + \frac{\varepsilon_1 AB^2}{4}\left[\frac{1}{\omega_1^2 - (2\Omega_2 + \Omega_1)^2} + \frac{1}{\omega_1^2 - (2\Omega_2 - \Omega_1)^2}\right]\right\}\cos\omega_1 t \\
&- \frac{\varepsilon_1 AB^2}{4}\left[\frac{2\cos\Omega_1 t}{\omega_1^2 - \Omega_1^2} + \frac{\cos(2\Omega_2 + \Omega_1)t}{\omega_1^2 - (2\Omega_2 + \Omega_1)^2} + \frac{\cos(2\Omega_2 - \Omega_1)t}{\omega_1^2 - (2\Omega_2 - \Omega_1)^2}\right]
\end{aligned} \tag{2.228}$$

$$\begin{aligned}
y_1 &= B\cos\Omega_2 t + \frac{1}{\omega_2}\int_0^t \sin\omega_2(s-t)\left\{B(\omega_2^2-\Omega_2^2)\cos\Omega_2 s + \varepsilon_2 BA^2\cos\Omega_2 s\cos^2\Omega_1 s\right\}\mathrm{d}s \\
&= B\cos\Omega_2 t + B\left(\omega_2^2-\Omega_2^2\right)\frac{\cos\omega_2 t-\cos\Omega_2 t}{\omega_2^2-\Omega_2^2} + \frac{\varepsilon_2 BA^2}{2}\frac{\cos\omega_2 t-\cos\Omega_2 t}{\omega_2^2-\Omega_2^2} \\
&\quad + \frac{\varepsilon_2 BA^2}{4}\frac{\cos\omega_2 t-\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_1+\Omega_2)^2} + \frac{\omega_2^2 BA^2}{4}\frac{\cos\omega_2 t-\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2} \\
&= \left\{B + \frac{\varepsilon_2 BA^2}{2\left(\omega_2^2-\Omega_2^2\right)^2} + \frac{\varepsilon_2 BA^2}{4}\left[\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2} + \frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right]\right\}\cos\omega_2 t \\
&\quad - \frac{\varepsilon_2 BA^2}{4}\left[\frac{2\cos\Omega_2 t}{\omega_2^2-\Omega_2^2} + \frac{\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_1+\Omega_2)^2} + \frac{\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right]
\end{aligned} \tag{2.229}$$

Eliminating the secular terms in x_2 and y_2, we require

$$A + \frac{\varepsilon_1 AB^2}{2(\omega_1^2-\Omega_1^2)} + \frac{\varepsilon_1 AB^2}{4}\left[\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2} + \frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\right] = 0, \tag{2.230}$$

$$B + \frac{\varepsilon_2 BA^2}{2(\omega_2^2-\Omega_2^2)} + \frac{\varepsilon_2 BA^2}{4}\left[\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2} + \frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\right] = 0, \tag{2.231}$$

from which the values of Ω_1 and Ω_2 can be determined.

From Eqs. 2.230 and 2.231, we can obtain the resonance condition of the coupled oscillator, which leads to

$$\omega_1^2 - (2\Omega_2-\Omega_1)^2 = 0, \tag{2.232}$$

$$\omega_2^2 - (2\Omega_1-\Omega_2)^2 = 0. \tag{2.233}$$

The condition for resonance can be obtained easily when the parameters are chosen. Resonance occurs when $\omega_1 = 2\Omega_2 - \Omega_1$ or $\omega_2 = 2\Omega_1 - \Omega_2$, where Ω_1 and Ω_2 can be determined from Eqs. 2.232 and 2.233.

2.7 He's Variational Approach

2.7.1 Basic Idea

He's variational approach was proposed by He in 2007. The main property of the method is to solve nonlinear equations accurately and conveniently with a linearization assumption used as an initial approximation or trial function, from which a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution. A very useful formulation for determining approximately the period of a nonlinear oscillator is suggested. Examples are given to illustrate the solution procedure.

Hereby, for a brief introduction of the method, we consider a general nonlinear oscillator in the form

$$\ddot{v}(t) + f(v(t)) = 0. \tag{2.234}$$

Its variational principle can be easily established using the semi-inverse method

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + F(v) \right) \mathrm{d}t, \tag{2.235}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator. Using Eq. 2.235 and $F(v) = \int (\alpha v + \beta v^3)\, \mathrm{d}v$, we obtain

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}\alpha v^2 + \frac{1}{4}\beta v^4 \right) \mathrm{d}t. \tag{2.236}$$

Considering these initial conditions,

$$v(0) = A, \quad \dot{v}(0) = 0. \tag{2.237}$$

Assume that its solution can be expressed as

$$v(t) = A \cos \omega t. \tag{2.238}$$

Substituting Eq. 2.238 into Eq. 2.236 results in

$$\begin{aligned} J(A, \omega) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\alpha A^2 \cos^2 \omega t + \frac{1}{4}\beta A^4 \cos^4 \omega t \right) \mathrm{d}t \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + \frac{1}{2}\alpha A^2 \cos^2 t + \frac{1}{4}\beta A^4 \cos^4 t \right) \mathrm{d}t \end{aligned} \tag{2.239}$$

Applying the Ritz method, we require

$$\partial J/\partial A = 0, \tag{2.240}$$

$$\partial J/\partial \omega = 0. \tag{2.241}$$

But by a careful inspection, for most cases, we find that

$$\partial J/\partial \omega < 0. \tag{2.242}$$

Thus, we modify the conditions 2.240 and 2.241 into the more simple form:

$$\partial J/\partial A = 0 \tag{2.243}$$

2.7.2 Application

Example 2.13

Introduction

The conservative autonomous system of a cubic Duffing equation is represented by the following second-order differential equation that one sees in Kachapi et al. (2010):

$$\ddot{v}(t) + \alpha v(t) + \beta v(t)^3 = 0, \tag{2.244}$$

with initial conditions

$$v(0) = A, \;\; \dot{v}(0) = 0. \tag{2.245}$$

where v and t are generalized dimensionless displacement and time variables, respectively, and α and β are any positive constant parameters.

Case 1

Consider a two-mass system connected with linear and nonlinear stiffnesses (the two-mass system model as shown in Fig. 2.7). The equation of motion is described as

$$m\ddot{x} + k_1(x-y) + k_2(x-y)^3 = 0, \tag{2.246a}$$

$$m\ddot{y} + k_1(y-x) + k_2(y-x)^3 = 0, \tag{2.246b}$$

with initial conditions

$$x(0) = X_0, \;\; \dot{x}(0) = 0. \tag{2.247a}$$

$$y(0) = Y_0, \;\; \dot{y}(0) = 0. \tag{2.247b}$$

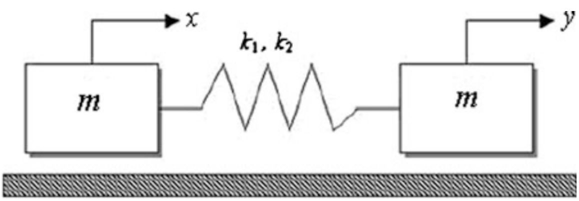


Fig. 2.7 Two masses connected by linear and nonlinear stiffnesses

whose double dots in Eqs. 2.246a and 2.246b denote double differentiation with respect to time t and k_1 and k_2 are linear and nonlinear coefficients of spring stiffness, respectively. Dividing Eqs. 2.246a and 2.246b by mass m yields

$$\ddot{x} + \frac{k_1}{m}(x - y) + \frac{k_2}{m}(x - y)^3 = 0, \tag{2.248a}$$

$$\ddot{y} + \frac{k_1}{m}(y - x) + \frac{k_2}{m}(y - x)^3 = 0. \tag{2.248b}$$

Introducing intermediate variables u and v as follows:

$$x := u, \tag{2.249}$$

$$y - x := v, \tag{2.250}$$

and transforming Eqs. 2.248a and 2.248b yield

$$\ddot{u} - \kappa v - \rho v^3 = 0, \tag{2.251a}$$

$$\ddot{v} + \ddot{u} + \kappa v + \rho v^3 = 0, \tag{2.251b}$$

where $\kappa = k_1/m$ and $\rho = k_2/m$. Equation 2.251a is rearranged as

$$\ddot{u} = \kappa v + \rho v^3 \tag{2.252}$$

Substituting Eq. 2.252 into Eq. 2.251b yields

$$\ddot{v} + 2\kappa v + 2\rho v^3 = 0, \tag{2.253}$$

with initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0. \tag{2.254}$$

Equation 2.253 is equivalent to Duffing's Eq. (a), with $\alpha = 2\kappa$ and $\beta = 2\rho$. For solving Eq. 2.253 using the variational approach, the approximate solutions of $v(\mathrm{t})$ can be back-substituted into Eq. 2.252 to obtain the intermediate variable $u(t)$ by double integration.

Its variational formulation can be readily obtained from Eq. 2.253 as

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \kappa v^2 + \frac{1}{2}\rho v^4 \right) \mathrm{d}t. \tag{2.255}$$

Substituting Eq. 2.239 into Eq. 2.255, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \kappa A^2 \cos^2 \omega t + \frac{1}{2}\rho A^4 \cos^4 \omega\, t \right) \mathrm{d}t. \tag{2.256}$$

The stationary condition with respect to A leads to

$$\begin{aligned}\partial J/\partial A &= \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega\, t + 2\,\kappa A\, \cos^2 \omega\, t + 2\,\rho A^3 \cos^4 \omega\, t\right)\mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + 2\,\kappa A\, \cos^2 t + 2\,\rho A^3 \cos^4 t\right)\mathrm{d}t = 0.\end{aligned} \tag{2.257}$$

This leads to the result

$$\omega = \frac{1}{2}\sqrt{8\,\kappa + 6\,\rho A^2}. \tag{2.258}$$

According to Eqs. 2.239 and 2.258, we can obtain the approximate solution

$$v(t) = A\,\cos\left(\frac{1}{2}t\sqrt{8\,\kappa + 6\,\rho A^2}\right). \tag{2.259}$$

The first-order analytical approximation for $u(t)$ is

$$u(t) = \iint (\kappa\, v + \rho\, v^3)\,\mathrm{d}t\,\mathrm{d}t = \frac{1}{9\,\omega^2} A\cos(\omega\, t)\ (9\kappa + \rho A\, \cos^2(\omega\, t) + 6\rho A^2). \tag{2.260}$$

Therefore, the first-order analytically approximating displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{2.261}$$

$$y(t) = u(t) + A\cos(\omega t). \tag{2.262}$$

Case 2

Consider a two-mass system connected with linear and nonlinear springs and fixed to a body at two ends, as shown in Fig. 2.8. The equation of motion is described as

$$m\ddot{x} + k_1 x + k_2(x - y) + k_3(x - y)^3 = 0, \tag{2.263a}$$

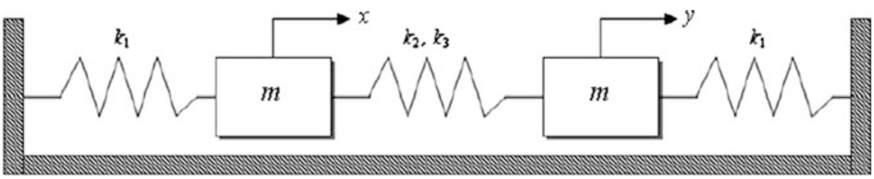


Fig. 2.8 Two-mass system connected with the fixed bodies

$$m\ddot{y} + k_1 y + k_2(y - x) + k_3(y - x)^3 = 0 \tag{2.263b}$$

with initial conditions

$$x(0) = X_0, \;\; \dot{x}(0) = 0. \tag{2.264}$$

$$y(0) = Y_0, \;\; \dot{y}(0) = 0. \tag{2.265}$$

whose double dots in Eqs. 2.263a and 2.263b denote double differentiation with respect to time t and where k_1 and k_2 are linear coefficients of spring stiffness and k_3 is the nonlinear coefficient of spring stiffness. Dividing Eqs. 2.263a and 2.263b by mass m yields

$$\ddot{x} + \frac{k_1}{m}x + \frac{k_2}{m}(x - y) + \frac{k_3}{m}(x - y)^3 = 0, \tag{2.266a}$$

$$\ddot{y} + \frac{k_1}{m}y + \frac{k_2}{m}(y - x) + \frac{k_3}{m}(y - x)^3 = 0. \tag{2.266b}$$

Similar to case 1, transforming the above equations, using intermediate variables, yields

$$\ddot{u} + \gamma\, u - \eta\, v - \lambda\, v^3 = 0, \tag{2.267a}$$

$$\ddot{u} + \ddot{v} + \gamma u + \gamma\, v + \eta v + \lambda\, v^3 = 0 \tag{2.267b}$$

in which $\gamma = k_1/m, \;\; \eta = k_2/m$, and $\lambda = k_3/m$. Rearranging Eq. 2.267a as

$$\ddot{u} = \eta v + \lambda\, v^3 - \gamma u \tag{2.268}$$

and back-substituting into Eq. 2.267b yields

$$\ddot{v} + (\gamma + 2\eta)\, v + 2\lambda\, v^3 = 0, \tag{2.269}$$

with initial conditions

$$v(0) = y(0) - x(0) = Y_0 - X_0 = A, \quad \dot{v}(0) = 0 \tag{2.270}$$

Equation 2.269 is again equivalent to Duffing's Eq. (a) with $\alpha = \gamma + 2\eta$ and $\beta = 2\lambda$. For solving Eq. 2.270 using a coupled variational approach, the approximate solutions of $v(t)$ can be back-substituted into Eq. 2.268 to yield

$$\ddot{u} + \gamma u = \eta v + \lambda\, v^3, \tag{2.271}$$

with initial conditions

$$u(0) = x(0) = X_0, \quad \dot{u}(0) = 0. \tag{2.272}$$

Equation 2.271 is a linear nonhomogeneous second-order ordinary differential equation, and it can be solved readily using a standard method such as a Laplace transformation.

Its variational formulation Eq. 2.273 can be readily obtained as

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}(\gamma + 2\eta)\, v^2 + \frac{1}{2}\lambda\, v^4 \right) \mathrm{d}t. \tag{2.273}$$

Substituting Eq. 2.239 into Eq. 2.273, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2} A^2\omega^2 \sin^2 \omega\, t + \frac{1}{2}(\gamma + 2\eta) A^2 \cos^2 \omega\, t + \frac{1}{2}\lambda A^4 \cos^4 \omega\, t \right) \mathrm{d}t. \tag{2.274}$$

The stationary condition with respect to A reads

$$\begin{aligned} \partial J/\partial A &= \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega\, t + (\gamma + 2\eta)\, A\, \cos^2 \omega\, t + 2\,\lambda A^3 \cos^4 \omega\, t \right) \mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + (\gamma + 2\eta)\, A\, \cos^2 t + 2\,\lambda A^3 \cos^4 t \right) \mathrm{d}t = 0 \end{aligned} \tag{2.275}$$

This leads to the result

$$\omega = \frac{1}{2}\sqrt{8\,\eta + 4\gamma + 6\,\lambda\, A^2}. \tag{2.276}$$

According to Eqs. 2.239 and 2.276, we can obtain the approximate solution

$$v(t) = A\, \cos\left(\frac{1}{2} t \sqrt{8\,\eta + 4\gamma + 6\,\lambda A^2} \right). \tag{2.277}$$

By Eq. 2.275, the first-order analytical approximation for $u(t)$ is

$$\begin{aligned} u(t) = &\frac{\cos\left(\sqrt{\gamma}\, t\right) \left(\lambda A^3\, \gamma - 7\lambda A^3\, \omega^2 - \eta\, \gamma A + 9\, \eta\, \omega^2 A + 5\gamma^2 - 50\, \gamma\, \omega^2 + 45\, \omega^4\right)}{\gamma^2 - 10\, \gamma\, \omega^2 + 9\omega^4} \\ &- \frac{36\left(-\frac{1}{36}(\omega^2 - \gamma)\lambda A^2 \cos(3\, \omega t) + \cos(\omega t) \left(\eta - \frac{3}{4}\, \lambda A^2\right) \left(\omega^2 - \frac{1}{9}\gamma\right)\right) A}{4\gamma^2 - 40\, \gamma\, \omega^2 + 36\omega^4} \end{aligned} \tag{2.278}$$

Therefore, the first-order analytically approximates displacements $x(t)$ and $y(t)$ are

$$x(t) = u(t), \tag{2.279}$$

$$y(t) = u(t) + A \cos(\omega t). \tag{2.280}$$

Discussion of Examples

The exact solution of the dynamical system can be obtained by integrating the governing equation (2.244) and imposing the initial conditions (2.245) as follows:

$$T(A) = \frac{4}{\sqrt{\alpha + \beta A^2}} \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - m \sin^2 t}}, \tag{2.281}$$

for which

$$m = \frac{\beta A^2}{2(\alpha + \beta A^2)}. \tag{2.282}$$

The exact frequency ω_e is also a function of A and can be obtained from the period of the motion as

$$\omega_e(A) = \frac{\pi\sqrt{\alpha + \beta A^2}}{2} \left(\int_0^{\pi/2} \frac{\mathrm{d}t}{1 - m \sin^2 t} \right)^{-1}. \tag{2.283}$$

It should be noted that ω_e contains an integral, which could only be solved numerically in general.

Plotting the exact solution and variational solution, it is clear that the results are in excellent agreement (Figs. 2.9, 2.10).

Example 2.14

As a last example, we consider the following nonlinear Duffing-harmonic oscillation:

$$u'' + u^3/(1 + u^2) = 0, \qquad u(0) = A, \quad u'(0) = 0.$$

in which $f(u) = u^3/(1 + u^2)$.

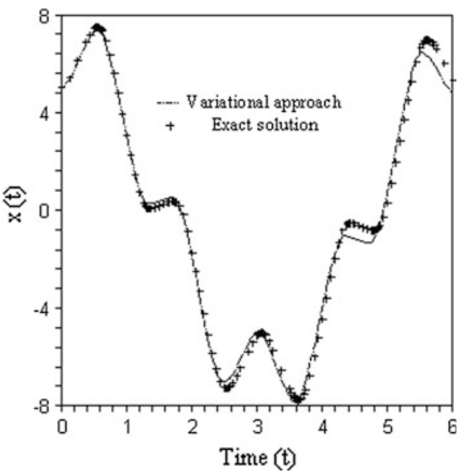
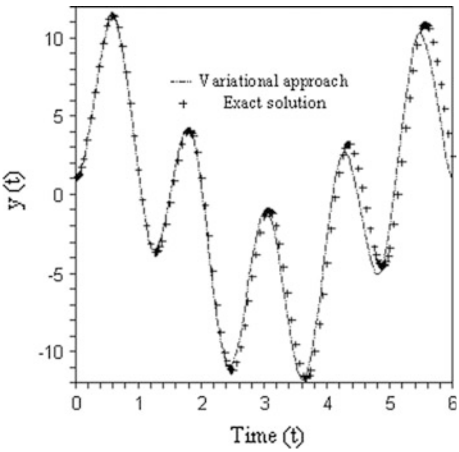


Fig. 2.9 Comparison of the analytical approximates Example 2.13 with the exact solution for $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, with x(0) = 5

Fig. 2.10 Comparison of the analytical approximates Example 2.13 with the exact solution for $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, with $y(0) = 1$



Its variational formulation (Naghipour et al. 2008) is

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \frac{1}{2}\ln(1+u^2) \right) \mathrm{d}t.$$

For similar previous examples, we have

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t - \frac{1}{2}\ln\left(1 + A^2 \cos^2 \omega t\right) \right) \mathrm{d}t,$$

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(-A\,\omega^2 \sin^2 \omega t + A\,\cos^2 \omega t - \left(A\,\cos^2 \omega t\right) / \left(1 + A^2\,\cos^2 \omega t\right) \right) \mathrm{d}t$$

$$= \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + A\,\cos^2 t - \left(A\,\cos^2 t\right) / \left(1 + A^2\,\cos^2 t\right) \right) \mathrm{d}t = 0.$$

From the previous equation, we have

$$\omega = \left((A^2+1)^{1/2} (2\,\mathrm{csgn}\,((A^2+1)^{1/2}) + A^2\,(A^2+1)^{1/2} - 2\,(A^2+1)^{1/2} \right)^{1/2} \Big/ \left(A\,(A^2+1)^{1/2} \right).$$

The csgn is defined in Maple Package Software.

2.8 Couple Variational Method

2.8.1 Introduction

The couple variational method (CVM) is a procedure for studying periodic solutions of strongly nonlinear systems (Kachapi et al. 2009b). The method consists of a combination of variational approaches to determine frequency and amplitude of the system and VIMs to obtain the time response of the system. Some examples are given to illustrate the effectiveness and convenience of the method.

2.8.2 Application

Example 2.15

As a first example, let us consider a family of nonlinear differential equations

$$u'' + \alpha u + \gamma u^{2m+1} = 0\,, \qquad \alpha \geq 0,\quad \gamma > 0,\quad m = 1,\ 2,\ 3,\ \ldots, \tag{2.284}$$

where $\alpha,\ \gamma$, and $m$are constant values. With the initial conditions,

$$u(0) = A, \quad u'(0) = 0. \tag{2.285}$$

For this problem,

$$f(u) = \alpha u + \gamma u^{2m+1} \text{ and } F(u) = \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2}.$$

Its variational formulation can be readily obtained as

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + \frac{1}{2}\alpha u^2 + \frac{\gamma u^{2m+2}}{2m+2} \right) \mathrm{d}t \tag{2.286}$$

Substituting $u_0(t) = A\cos\omega t$ into Eq. 2.286, we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\,\alpha A^2 \cos^2\,\omega t + \frac{\gamma\,(A\,\cos\,\omega t)^{2m+2}}{2m+2} \right) \mathrm{d}t \tag{2.287}$$

The stationary condition with respect to A leads to

$$\begin{aligned}\frac{\partial J}{\partial A} &= \int\limits_0^{T/4}\left(-A\,\omega^2\sin^2\omega\,t+\alpha A\,\cos^2\omega\,t+\frac{\gamma\,(A\,\cos\,\omega\,t)^{2m+2}}{A}\right)\mathrm{d}t\\ &= \int\limits_0^{\pi/2}\left(-A\,\omega^2\,\sin^2 t+\alpha A\,\cos^2 t+\frac{\gamma\,(A\,\cos\,t)^{2m+2}}{A}\right)\mathrm{d}t=0\end{aligned} \tag{2.288}$$

This leads to the result

$$\omega=\frac{\sqrt{A\,\pi\,\Gamma(m+2)\left(\alpha A\,\pi\,\Gamma(m+2)\,+2\,\gamma\,A^{2m+1}\,\Gamma\left(\frac{3}{2}+m\right)\,\sqrt{\pi}\right)}}{A\,\pi\,\Gamma(m+2)}, \tag{2.289}$$

Function Gamma (Γ) is defined in the Mathematical package.
with $T=\frac{2\pi}{\omega}$, yield

$$T=\frac{2A\,\pi^2\,\Gamma(m+2)}{\sqrt{A\,\pi\,\Gamma(m+2)\left(\alpha A\,\pi\,\Gamma(m+2)\,+2\,\gamma\,A^{2m+1}\,\Gamma\left(\frac{3}{2}+m\right)\,\sqrt{\pi}\right)}} \tag{2.290}$$

Thus, we apply VIM and rewrite in the form

$$u_{n+1}(t)=u_n(t)+\int\limits_0^t\frac{1}{\omega}\sin\omega\,(\tau-t)\left(u_n''(\tau)+\alpha\,u_n(\tau)+\gamma\,u_n^{2m+1}(\tau)\right)\mathrm{d}\tau. \tag{2.291}$$

By the above iteration formula, we can calculate the first-order approximation

$$\begin{aligned}u_1(t) &= A\cos\omega\,\mathrm{t}\\ &\quad+\frac{\int_0^t\sin(\tau-t)\left(-\omega^2 A\cos\omega\tau+\alpha A\cos\omega\tau\,+\,\gamma(A\cos\omega\tau)^{2m+1}\right)\mathrm{d}\tau}{\omega}.\end{aligned} \tag{2.292}$$

The angular frequency ω is defined as in Eq. 2.289. For example, for $\alpha=\gamma=A=m=1$, it yields

$$u_1(t)=\,\cos\left(1.3229\,\mathrm{t}\right)-0.012813\cos\mathrm{t}+0.012813\cos\left(3.9687\,\mathrm{t}\right). \tag{2.293}$$

The above results are in good agreement with the results obtained by the exact solutions.

Example 2.16

In dimensionless form, a mass attached to the center of a stretched elastic wire has the equation of motion (Kachapi et al. 2009b)

$$u'' + u - \frac{\lambda\, u}{\sqrt{1+u^2}} = 0, \tag{2.294}$$

This is an example of a conservative nonlinear oscillatory system having an irrational elastic item. All the motions corresponding to Eq. 2.294 are periodic, the system will oscillate between symmetric bounds $[-A, A]$, and its angular frequency and corresponding periodic solution are dependent on the amplitude A.

Its variational formulation can be readily obtained as

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\, u'^2 + \frac{1}{2}\, u^2 - \lambda \sqrt{1+u^2} \right) \mathrm{d}t \tag{2.295}$$

By a similar manipulation, as illustrated in the previous example, we have

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}\, A^2 \omega^2 \sin^2 \omega\, t + \frac{1}{2} A^2 \cos^2 \omega\, t - \lambda \sqrt{1 + A^2\, \cos^2 \omega\, t} \right) \mathrm{d}t. \tag{2.296}$$

The stationary condition with respect to A reads

$$\begin{aligned} \frac{\partial J}{\partial A} &= \int_0^{T/4} \left(-A\, \omega^2 \sin^2 \omega\, t + A \cos^2 \omega\, t - \left(\lambda A\, \cos^2 \omega\, t \middle/ \sqrt{1 + A^2 \cos^2\, \omega\, t} \right) \right) \mathrm{d}t \\ &= \int_0^{\pi/2} \left(-A\, \omega^2 \sin^2 t + A\, \cos^2 t - \left(\lambda A\, \cos^2 t \middle/ \sqrt{1 + A^2 \cos^2 t} \right) \right) \mathrm{d}t = 0 \end{aligned} \tag{2.297}$$

This leads to the result

$$\omega = \frac{\sqrt{\pi \left(4\lambda \left(\mathrm{Elliptic}K\left(\sqrt{-A^2} \right) \right) - 4\lambda \left(\mathrm{Elliptic}E\left(\sqrt{-A^2} \right) \right) + A^2 \pi \right)}}{A\, \pi}, \tag{2.298}$$

where the incomplete elliptic integral Elliptic$E$ and Elliptic$K$ are defined in the Mathematical package. Hence, the approximate period is

$$\mathrm{T} = \frac{2\,\pi}{\omega} = \frac{2\,A\,\pi^2}{\sqrt{\pi \left(4\lambda \left(\mathrm{Elliptic}K\left(\sqrt{-A^2} \right) \right) - \left(4\lambda\, \mathrm{Elliptic}E\left(\sqrt{-A^2} \right) \right) + A^2 \pi \right)}} \tag{2.299}$$

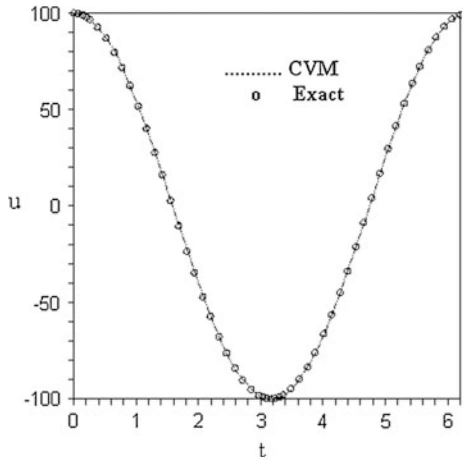


Fig. 2.11 The comparison of the approximate solution (CVM) with the exact solution for $\lambda = 0.1,\ A = 100$

We rewrite Eq. 2.294 in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{\omega} \sin\omega\,(\tau - t)\left(u_n''(\tau) + u_n(\tau) - \frac{\lambda\, u_n(\tau)}{\sqrt{1 + u_n^2(\tau)}}\right) d\tau. \quad (2.300)$$

By the above iteration formula, we can calculate the first-order approximation

$$u_1(t) = A\cos\omega\,\mathrm{t} + \frac{\int_0^t \sin(\tau - t)\left(-\omega^2 A\cos\omega\tau + A\cos\omega\tau - \frac{\lambda A\cos\omega\tau}{\sqrt{1 + A^2\cos^2\omega\tau}}\right) d\tau}{\omega}. \quad (2.301)$$

in which the angular frequency ω is defined as Eq. 2.298. The above results are in good agreement with the results obtained by the exact solution, as illustrated in Fig. 2.11.

2.9 Energy Balance Method

2.9.1 Introduction

In this section, we will introduce a heuristic approach, called the He's energy balance method (EBM), to nonlinear oscillators that were proposed by He in 2002a. In this method, a variational principle for the nonlinear oscillation is established; then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by the collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones.

Some examples reveal that even the lowest order approximations are of high accuracy. They illustrate that the energy balance methodology is very effective and convenient and does not require linearization or small perturbation. It is predicted that the energy balance method will find wide application in engineering problems, as indicated in the following examples.

In order to represent the EBM, we consider a general nonlinear oscillator in the form (He 2002a)

$$u'' + f(u(t)) = 0 \tag{2.302}$$

in which u and t are generalized dimensionless displacement and time variables, respectively.

Its variational principle can be easily obtained as

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 + F(u) \right) \mathrm{d}t \tag{2.303}$$

Its Hamiltonian, therefore, can be written as

$$H = \frac{1}{2}u'^2 + F(u) = F(A) \tag{2.304}$$

or

$$R(t) = \frac{1}{2}u'^2 + F(u) - F(A) = 0 \tag{2.305}$$

Oscillatory systems contain two important physical parameters—that is, the frequency ω and the amplitude of oscillation, A. So let us consider initial conditions such as

$$u(0) = A, \qquad u'(0) = 0 \tag{2.306}$$

Assume that its initial approximation can be expressed as

$$u(t) = A\cos(\omega t) \tag{2.307}$$

Substituting Eq. 2.307 as the u term of Eq. 2.305 yields

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A \cos \omega t) - F(A) = 0 \tag{2.308}$$

If, by any chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. 2.306 is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega = \sqrt{\frac{2(F(A) - F(A \cos \omega t))}{A^2 \sin^2 \omega t}} \tag{2.309}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\frac{2(F(A)-F(A\ \cos\ \omega t))}{A^2 \sin^2\ \omega t}}} \tag{2.310}$$

2.9.2 Application

Example 2.17

We consider the nonlinear oscillator

$$u'' + u^3 + u^{\frac{1}{3}} = 0$$

with the boundary conditions

$$u(0) = A, u'(0) = 0.$$

Its Hamiltonian, therefore, can be written in the form (Ganji et al. 2009b)

$$\Delta H = \frac{1}{2}u'^2 + \frac{1}{4}u^4 + \frac{3}{4}u^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0.$$

Choosing the trial function $u = A\cos(\omega t)$, we obtain the residual equation

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{4}A^4\cos^4(\omega t) + \frac{3}{4}(A\cos(\omega t))^{\frac{4}{3}} - \frac{1}{4}A^4 - \frac{3}{4}A^{\frac{4}{3}} = 0.$$

If we collocate at $\omega t = \pi/4$, we obtain

$$\omega = \sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}},\ T = \frac{\omega}{2\pi}$$

We can obtain the approximate solution

$$u = A\cos\sqrt{\frac{3}{4}A^2 + 1.1101184A^{-\frac{2}{3}}}t$$

Example 2.18

The governing equation of motion and initial conditions of a particle on a rotating parabola can be expressed (Ganji et al. 2009b) as

$$\left(1 + 4q^2u^2\right)\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 + \Delta u = 0,\quad u(0) = A,\ \frac{du}{dt}(0) = 0.$$

where $q > 0$ and $\Delta > 0$ are known positive constants. For this problem,

$$f(u) = 4q^2u^2\frac{d^2u}{dt^2} + 4q^2u\left(\frac{du}{dt}\right)^2 + \Delta u \text{ and } F(u) = -2q^2u^2u'^2 + \frac{1}{2}\Delta u^2.$$

Its variational and Hamiltonian formulations can be readily obtained as

$$J(u) = \int_0^t \left(-\frac{1}{2}u'^2 - 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 \right) \mathrm{d}t,$$

$$H = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 = \frac{1}{2}\Delta A^2,$$

$$R(t) = \frac{1}{2}u'^2 + 2q^2u^2u'^2 + \frac{1}{2}\Delta u^2 - \frac{1}{2}\Delta A^2 = 0,$$

Substituting $u = A\cos(\omega t)$ into $R(t)$, we obtain

$$R(t) = \frac{1}{2}A^2\omega^2\sin^2(\omega t) + 2q^2\omega^2A^4\cos^2(\omega t)\,\sin^2(\omega t) + \frac{1}{2}\Delta A^2\cos^2(\omega t) - \frac{1}{2}\Delta A^2 = 0,$$

which gives us the result

$$\omega = \sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\omega t) + 1)}},$$

with $T = \frac{2\pi}{\omega}$, which yields

$$T = \frac{2\pi}{\sqrt{\frac{\Delta}{(4A^2q^2\cos^2(\omega t)+1)}}},$$

If we collocate at $\omega t = \pi/4$, we obtain

$$\omega_{\mathrm{EBM}} = \sqrt{\frac{\Delta}{(2A^2q^2 + 1)}},$$

with $T = \frac{2\pi}{\omega}$, yielding

$$T_{\mathrm{EBM}} = \frac{2\pi}{\sqrt{\frac{\Delta}{(2A^2q^2+1)}}},$$

The exact period is

$$T_{\mathrm{ex}} = 4\Delta^{-1/2} \int_0^{\pi/2} \left(1 + 4q^2\beta^2\cos^2\varphi\right)^{1/2} \mathrm{d}\varphi.$$

Table 2.2 Comparison between analytical EBM and exact solutions when $\Delta = 1$ and $q = 1$

A	T_{EBM}	T_{ex}	Error percentage
0.1	6.34570	6.34555	0.0024
1.0	10.8827	10.5407	3.2451
10	89.0795	80.4880	10.674
100	888.598	800.071	11.064
1000	8885.76	8000.00	11.071

For comparison, the exact period T_{ex} and the approximate period T_{EBM} are listed in Table 2.2; they can give a good approximate period for values of oscillation amplitude.

Example 2.19

Consider the motion of a rigid rod rocking back and forth on a circular surface without slipping. The governing equation of motion can be expressed as

$$\left(\frac{1}{12}+\frac{1}{16}u^2\right)\frac{d^2u}{dt}+\frac{1}{16}u\left(\frac{du}{dt}\right)^2+\frac{g}{4l}u\cos u=0,\quad u(0)=A,\quad \frac{du}{dt}(0)=0,$$

where $g > 0$ and $l > 0$ are known positive constants.

For the problem, its variational formulation (Ganji et al. 2008b) can be obtained as

$$J(u)=\int_0^t\left(-\frac{1}{2}u'^2-\frac{3}{8}u^2u'^2+\frac{3g(\cos u+\sin u)}{l}\right)dt,$$

By a similar manipulation, as illustrated in the previous example by using Eqs. (2.307) and (2.308) and with $T = \frac{2\pi}{\omega}$, we obtain the result

$$R(t)=\frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{3}{8}A^4\omega^2\cos^2(\omega t)\,\sin^2(\omega t)\\+\frac{3g(\cos(A\cos(\omega t))+A\cos(\omega t)\sin(A\cos(\omega t))-\cos(A)-A\,\sin(A))}{l}=0,$$

$$\omega=\frac{\sqrt{2\left(-6lg(3A^2\cos^2(\omega t)+4)\left(\begin{array}{r}\cos(A\,\cos(\omega t))+A\,\cos(\omega t)\,\sin(A\,\cos(\omega t))\\-\cos(A)-A\,\sin(A)\end{array}\right)\right)}}{(lA(3A^2\cos^2(\omega t)+4)\sin(\omega t))},$$

$$T=\frac{2\pi(lA(3A^2\cos^2(\omega t)+4)\sin(\omega t))}{\sqrt{2\left(-6lg(3A^2\cos^2(\omega t)+4)\left(\begin{array}{r}\cos(A\,\cos(\omega t))+A\,\cos(\omega t)\,\sin(A\,\cos(\omega t))\\-\cos(A)-A\,\sin(A)\end{array}\right)\right)}},$$

Substituting $\omega t = {}^{\pi}/_{4}$ in the previous equations for ω and T, we have

$$\omega_{\text{EBM}} = \frac{4\sqrt{-3\,\lg(8+3A^2)(\eta)}}{l\,A(3A^2+8)},$$
$$T_{\text{EBM}} = \frac{2\,\pi\,l\,A(3A^2+8)}{4\sqrt{-3\,\lg(8+3A^2)(\eta)}}.$$

where η is

$$\eta = 2\cos\left(\frac{A\sqrt{2}}{2}\right) + A\sqrt{2}\sin\left(\frac{A\sqrt{2}}{2}\right) - 2\cos(A) - 2A\sin(A)$$

The exact period of the equation of motion is

$$T_{\text{ex}} = 4\Delta^{-1/2}\int_0^{\pi/2}\left(\frac{(4+3\beta^2\sin^2\varphi)\beta^2\cos^2\varphi}{8[\beta\sin\beta+\cos\beta-\beta\sin\varphi\ \sin(\beta\sin\varphi)-\cos](\beta\sin\varphi)}\right)^{1/2}\mathrm{d}\varphi.$$

The exact period T_{ex} and the approximate period T_{EBM} are shown in Table 2.3. Note that for the problem, the maximum amplitude of oscillation should satisfy $A < \pi/2$.

Example 2.20

Consider a system that undergoes rotary motion with linear and nonlinear stiffness and is fixed to a body at two ends, as shown in Fig. 2.12. Suppose that the two discs of moment of inertia (second moment of mass) are J about their center, the torsional stiffness between the two discs is $f = k_2\theta + k_3\theta^3$, and torsional stiffness at the two ends is $f = k_1\theta$ [].

The equation of motion is given as

$$J\ddot{\theta}_1 + k_1\theta_1 + k_2(\theta_1-\theta_2) + k_3(\theta_1-\theta_2)^3 = 0 \qquad (2.311)$$

$$J\ddot{\theta}_2 + k_1\theta_2 + k_2(\theta_2-\theta_3) + k_3(\theta_2-\theta_3)^3 = 0, \qquad (2.312)$$

with initial conditions

$$\theta_1(0) = \Theta_{10}\,,\ \dot{\theta}_1(0) = 0$$
$$\theta_2(0) = \Theta_{20}\,,\ \dot{\theta}_2(0) = 0.$$

Table 2.3 Comparison of approximate periods with exact solution

A	T_{EBM}	T_{ex}	Error percentage
0.10π	3.76397	3.76397	0.0008
0.15π	3.94064	3.94086	0.0056
0.20π	4.20181	4.20292	0.02642
0.30π	5.05831	5.07728	0.37348
0.40π	6.70586	6.89564	2.7521
0.45π	8.60226	8.94333	3.8136

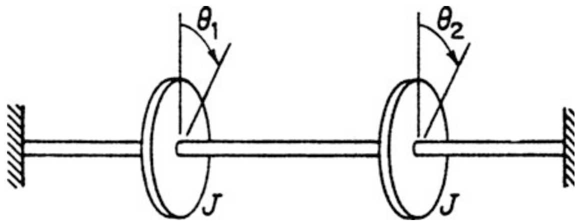


Fig. 2.12 Rotary motion with linear and nonlinear stiffness

Transforming the above equations using intermediate variables in Eqs. 2.311 and 2.312 yields

$$\ddot{u} + \gamma u - \eta v - \lambda v^3 = 0 \tag{2.313}$$

$$\ddot{v} + \ddot{u} + \gamma u + \gamma v + \eta v + \lambda v^3 = 0, \tag{2.314}$$

where $\gamma = \frac{k_1}{J}$, $\eta = \frac{k_2}{J}$ and $\lambda = \frac{k_3}{J}$. Rearranging Eq. 2.313 as follows,

$$\ddot{u} = -\gamma u + \eta v + \lambda v^3 = 0 \tag{2.315}$$

and substituting back into Eq. 2.314 yields

$$\ddot{v} + (\gamma + 2\eta)v + 2\lambda v^3 = 0, \tag{2.316}$$

with initial conditions

$$v(0) = \theta_2(0) - \theta_1(0) = \Theta_{20} - \Theta_{10} = A$$
$$\dot{v}(0) = 0$$

We choose two trial functions $v_1 = A\cos t$ and $v_2 = A\cos\omega t$.
Substituting v_1 and v_2 into Eq. 2.316, we obtain, respectively, the residuals

$$R_1 = -A\cos(t) + (\gamma + 2\eta)A\cos(t) + 2\lambda A^3\cos^3(t) \tag{2.317}$$

and

$$R_2 = -A\omega^2\cos(\omega t) + (\gamma + 2\eta)A\cos(\omega t) + 2\lambda A^3\cos^3(\omega t) \tag{2.318}$$

Also,

$$R_{11} = \frac{2\left(-\frac{1}{4}A\pi + \frac{1}{4}\gamma A\pi + \frac{1}{2}\eta A\pi + \frac{3}{8}\lambda A^3\pi\right)}{\pi} \tag{2.319}$$

and

$$R_{22} = \frac{1}{4}\frac{A(-2\omega^2\pi + 3\lambda A^2\pi + 2\gamma\pi + 4\eta\pi)}{\pi}. \tag{2.320}$$

We therefore obtain

$$\omega = \sqrt{\gamma + 2\eta + \frac{3}{2}A^2\lambda}. \tag{2.321}$$

According to Eq. 2.307, we can obtain the approximate solution

$$v(t) = A\cos\left(\sqrt{\gamma + 2\eta + \frac{3}{2}A^2\lambda}\, t\right). \tag{2.322}$$

By Eq. 2.315, the first-order analytical approximation for $u(t)$ is

$$x(t) = u(t) = \frac{\cos\left(\sqrt{\gamma}t\right)\left(\lambda A^3\gamma - 7\lambda A^3\omega^2 - \eta\gamma A + 9\eta\omega^2 A + 5\gamma^2 - 50\gamma\omega^2 + 45\omega^4\right)}{\gamma^2 - 10\gamma\omega^2 + 9\omega^4} - \frac{36\left(-\frac{1}{36}(\omega^2 - \gamma)\lambda A^2\cos(3\omega t) + \cos(\omega t)\left(\eta - \frac{3}{4}\lambda A^2\right)\left(\omega^2 - \frac{1}{9}\gamma\right)\right)A}{4\gamma^2 - 40\gamma\omega^2 + 36\omega^4} \tag{2.323}$$

2.10 Coupled Method of Homotopy Perturbation and Variational Method

2.10.1 Introduction

The coupled method of homotopy perturbation and variational method has been given much attention recently; it has been proved that this method is very effective in determining the natural frequencies of strongly nonlinear oscillators with high accuracy (He 2006c).

In the coupled method of homotopy perturbation and variational method, the following homotopy is constructed, and a variational formulation for the nonlinear oscillation is established, from which the natural frequency and approximate solution can be readily obtained.

To illustrate the basic ideas of this method, we consider the following equation (see Sect. 2.5):

$$A(u) - f(r) = 0 \quad r \in \Omega, \tag{2.324}$$

with the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \quad r \in \Gamma, \tag{2.325}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω.

A can be divided into two parts, which are L and N, where L is linear and N is nonlinear. Equation 2.324 therefore can be rewritten as

$$L(u) + N(u) - f(r) = 0 \quad r \in \Omega. \tag{2.326}$$

Homotopy perturbation structure is shown as

$$H(v,p) = (1-p)\ [L(v) - L(u_0)] + p[A(v) - f(r)\ ] = 0, \tag{2.327}$$

where

$$v(r,p) : \Omega \times [0,1] \to R \tag{2.328}$$

In Eq. 2.328, $p \in [0\ ,\ 1]$ is an embedding parameter, and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. 2.328 can be written as a power series in p,

$$v = v_0 + pv_1 + p^2v_2 + \cdots \tag{2.329}$$

And the best approximation for the solution is

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \tag{2.330}$$

Consider the following generalized nonlinear oscillations without forced terms:

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0, \tag{2.331}$$

where f is a nonlinear function of u'', u', u.Its variational functional can be easily obtained as (see Sect. 2.6)

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}\omega_0^2 u_2 + \varepsilon F(u) \right\} \mathrm{d}t, \tag{2.332}$$

where F is the potential,

$$\frac{\mathrm{d}F}{\mathrm{d}u} = f \tag{2.333}$$

2.10.2 Application

As an example of the method, a nonlinear oscillator with discontinuities of a conservative autonomous system can be described by the second-order differential equation

$$u'' + \alpha u' + \beta u^2 \mathrm{sgn}(u) + \gamma u^3 = 0 \text{ or } u'' + \alpha u' + \beta u|\,u| + \gamma u^3 = 0 \tag{2.334}$$

with initial conditions

$$u(0) = A\,,\; u'(0) = 0 \tag{2.335}$$

where sgn(u) is the sign function, equal to +1 if $u > 0$, 0 if $u = 0$ and -1 if $u < 0$.

As we will see in the forthcoming illustrative examples, we always stop at the first-order approximation, and the obtained approximate and accurate solution is valid for the whole solution domain.

In order to assess the advantages and the accuracy of the coupled method of homotopy perturbation and variational, we will consider the following two examples (Akbarzade et al. 2011).

Example 2.21

Let us consider the following nonlinear oscillators with discontinuities:

$$u'' + u + \varepsilon u^2 \mathrm{sgn}(u) = 0 \tag{2.336}$$

with initial conditions

$$u(0) = A\,,\; u'(0) = 0.$$

If $u > 0$,

$$u'' + u + \varepsilon u^2 = 0 \tag{2.337}$$

Suppose that the frequency of Eq. 2.337 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[\varepsilon u^2 + (1 - \omega^2)u] = 0\,,\; p \in [0\;,\;1]. \tag{2.338}$$

We assume that the periodic solution to Eq. 2.338 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.339}$$

Substituting Eq. 2.339 into Eq. 2.338 and collecting terms of the same power of p, gives

$$u_0'' + \omega^2 u_0 = 0\,,\; u_0(0) = A\,,\; u_0'(0) = 0 \tag{2.340}$$

and

$$u_1'' + \omega^2 u_1 + \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{2.341}$$

The solution of Eq. 2.340 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which leads to

$$J(u_1) = \int\limits_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)\,u_0 u_1 + \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\; \mathrm{T} = \frac{2\pi}{\omega} \tag{2.342}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.343}$$

Substituting u_1 into the functional Eq. 2.342 results in

$$J(B,\omega) = \left\{\frac{1}{18}\frac{(18A\pi - 75\omega^2 B\pi - 18A\omega^2\pi + 15A^2\pi)B}{\omega}\right\} \tag{2.344}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \quad \text{and} \quad \frac{\partial J}{\partial \omega} = 0, \tag{2.345}$$

we obtain

$$-0.833\varepsilon A - 1 + \omega^2 = 0, \quad \text{and,} \quad \mathrm{B} = 0. \tag{2.346}$$

A first-order approximate solution is obtained, which leads to

$$\omega_1 = \sqrt{1 + 0.833\varepsilon A}. \tag{2.347}$$

The approximate period is

$$T_1 = \frac{2\pi}{\sqrt{1 + 0.833\varepsilon A}}. \tag{2.348}$$

In order to compare with harmonic balance, we write Hu's result:

$$T_1 = \frac{2\pi}{\sqrt{1 + \frac{8}{3\pi}\varepsilon A}}. \tag{2.349}$$

If $u < 0$,

$$u'' + u - \varepsilon u^2 = 0. \tag{2.350}$$

Suppose that the frequency of Eq. 2.350 is ω.

We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[-\varepsilon u^2 + (1 - \omega^2)u] = 0\,,\, p \in [0\,,\ 1]. \tag{2.351}$$

We assume that the periodic solution to Eq. 2.351 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.352}$$

Substituting Eq. 2.352 into Eq. 2.351, collecting terms of the same power of p, gives

$$u_0'' + \omega^2 u_0 = 0\,,\; u_0(0) = A\,,\; u_0'(0) = 0 \tag{2.353}$$

and

$$u_1'' + \omega^2 u_1 - \varepsilon u_0^2 + (1 - \omega^2) u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{2.354}$$

The solution of Eq. 2.353 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2} u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)\, u_0 u_1 - \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\;\; \mathrm{T} = \frac{2\pi}{\omega}. \tag{2.355}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.356}$$

Substituting u_1 into the functional Eq. 2.355 results in

$$J(B,\omega) = \left\{ \frac{1}{18} \frac{(18A\pi - 75\omega^2 B\pi - 18A\omega^2\pi - 15A^2\pi)B}{\omega} \right\} \tag{2.357}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \quad \text{and} \quad \frac{\partial J}{\partial \omega} = 0, \tag{2.358}$$

we obtain

$$+0.833\varepsilon A - 1 + \omega^2 = 0,\;\text{and, B = 0} \tag{2.359}$$

The first-order approximate solution is obtained, which reads

$$\omega_2 = \sqrt{1 - 0.833\varepsilon A}. \tag{2.360}$$

The approximate period is

$$T_2 = \frac{2\pi}{\sqrt{1 - 0.833\varepsilon A}}. \tag{2.361}$$

In order to compare with harmonic balance, we write He's result

$$T_2 = \frac{2\pi}{\sqrt{1 - \frac{8}{3\pi}\varepsilon A}}. \tag{2.362}$$

We obtain the first approximate period T:

$$T = \frac{T_1 + T_2}{2} \tag{2.363}$$

Example 2.22

Consider the following nonlinear oscillator with discontinuities:

$$u'' + u + \varepsilon u^2 \mathrm{sgn}(u) + u^3 = 0, \tag{2.364}$$

with the initial conditions

$$u(0) = A\,,\; u'(0) = 0.$$

If $u > 0$,

$$u'' + u + \varepsilon u^2 + u^3 = 0 \tag{2.365}$$

Suppose that the frequency of Eq. 2.365 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[u^3 + \varepsilon u^2 + (1 - \omega^2)u] = 0\,,\; p \in [0\;,\;1]. \tag{2.366}$$

We assume that the periodic solution to equation Eq. 2.364 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.367}$$

Substituting Eq. 2.367 into Eq. 2.366, collecting terms of the same power of p gives

$$u_0'' + \omega^2 u_0 = 0\,,\; u_0(0) = A\,,\; u_0'(0) = 0 \tag{2.368}$$

and

$$u_1'' + \omega^2 u_1 + u_0^3 + \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{2.369}$$

The solution of Eq. 2.368 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)\,u_0 u_1 + u_0^3 u_1 + \varepsilon u_0^2 u_1 \right\} \mathrm{d}t,\; \mathrm{T} = \frac{2\pi}{\omega} \tag{2.370}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.371}$$

Substituting u_1 into the functional Eq. 2.370 results in

$$J(B,\omega) = \left\{ -\frac{1}{36}\frac{(-30\varepsilon A^2\pi - 36A\pi + 36\omega^2 A\pi + 150B\omega^2\pi - 27A^3\pi)B}{\omega} \right\}. \tag{2.372}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \text{ and } \frac{\partial J}{\partial \omega} = 0 \tag{2.373}$$

we obtain

$$-0.833\varepsilon A - 075A^2 - 1 + \omega^2 = 0, \text{and, B = 0.} \tag{2.374}$$

The first-order approximate solution is obtained, which reads

$$\omega_1 = \sqrt{1 + 0.75A^2 + 0.833\varepsilon A}. \tag{2.375}$$

The approximate period is

$$T_1 = \frac{2\pi}{\sqrt{1 + 0.75A^2 + 0.833\varepsilon A}}. \tag{2.376}$$

In order to compare with the harmonic balance solution, we write He's result,

$$T_1 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A^2 + \frac{8}{3\pi}\varepsilon A}}. \tag{2.377}$$

If $u < 0$,

$$u'' + u - \varepsilon u^2 + u^3 = 0. \tag{2.378}$$

Suppose that the frequency of Eq. 2.378 is ω. We construct the following homotopy by the same manipulation as the basic idea:

$$u'' + \omega^2 u + p[u^3 - \varepsilon u^2 + (1 - \omega^2)u] = 0, \, p \in [0, 1]. \tag{2.379}$$

We assume that the periodic solution to Eq. 2.379 may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{2.380}$$

Substituting Eq. 2.380 into Eq. 2.379, collecting terms of the same power of p gives

$$u_0'' + \omega^2 u_0 = 0, u_0(0) = A, u_0'(0) = 0 \tag{2.381}$$

and

$$u_1'' + \omega^2 u_1 + u_0^3 + \varepsilon u_0^2 + (1 - \omega^2)u_0 = 0\,,\; u_1(0) = 0\,,\; u_1'(0) = 0. \tag{2.382}$$

The solution of Eq. 2.381 is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)\,u_0 u_1 \; + u_0^3 u_1 - \varepsilon u_0^2 u_1 \right\} \mathrm{d}t, \;\; \mathrm{T} = \frac{2\pi}{\omega}. \tag{2.383}$$

To better illustrate the procedure, we choose the simplest trail function,

$$u_1 = B\left(\cos\omega t - \frac{5}{3}\cos 2\omega t\right) \tag{2.384}$$

Substituting u_1 into the functional Eq. 2.383 results in

$$J(B,\omega) = \left\{ -\frac{1}{36}\frac{(+30\varepsilon A^2\pi - 36A\pi + 36\omega^2 A\pi + 150B\omega^2\pi - 27A^3\pi)B}{\omega} \right\}. \tag{2.385}$$

Setting

$$\frac{\partial J}{\partial B} = 0, \text{ and } \frac{\partial J}{\partial \omega} = 0, \tag{2.386}$$

we obtain

$$+0.833\varepsilon A - 075A^2 - 1 + \omega^2 = 0, \;\; \text{and, B = 0} \tag{2.387}$$

The first-order approximate solution is obtained, which reads

$$\omega_2 = \sqrt{1 + 0.75A^2 - 0.833\varepsilon A}. \tag{2.388}$$

The approximate period is

$$T_2 = \frac{2\pi}{\sqrt{1 + 0.75A^2 - 0.833\varepsilon A}}. \tag{2.389}$$

In order to compare with a harmonic balance solution, we write He's result as

$$T_2 = \frac{2\pi}{\sqrt{1 + \frac{3}{4}A^2 - \frac{8}{3\pi}\varepsilon A}}. \tag{2.390}$$

We obtain the first approximate period:

$$T = \frac{T_1 + T_2}{2}. \tag{2.391}$$

In order to compare with the exact solution,

$$T_e = \int_0^A \frac{2\mathrm{d}x}{\sqrt{A^2 - x^2 + \frac{2}{3}\varepsilon(A^3 - x^3) + \frac{1}{2}(A^4 - x^4)}} + \int_0^A \frac{2\mathrm{d}x}{\sqrt{A^2 - x^2 - \frac{2}{3}\varepsilon(A^3 - x^3) + \frac{1}{2}(A^4 - x^4)}} \quad (2.392)$$

For a relatively comprehensive survey on the concepts, theory, and applications of the methods mentioned in this chapter, see more examples in Hashemi et al. (2009), Kachapi et al. (2009a), Shou and He (2008), Ganji et al. (2007a, b, 2008a, c, d, e, 2009a, c, d, e, f), Varedi et al. (2007), Kimiaeifar et al. (2009), Pashaei et al. (2008), Barari et al. (2010), Ghotbi and Barari (2008), Jamshidi and Ganji (2009), Mehdipour et al. (2009), Ganji and Esmaeilpour (2010), Rafei et al. (2007b, c, d), Babazadeh et al. (2008).

Problems

Solve the following problems using methods presented in this chapter.

2.1 Consider the free response of the undamped, single-DOF system with $\alpha > 0$ so that the equation of motion of the system is

$$\ddot{x}(t) + k\,x(t) + \alpha\,x(t)^3 = 0,$$

where the initial condition is zero.

2.2 We consider the motion of a ring of mass m sliding freely on the wire described by the parabola $y = qu^2$, which rotates with a constant angular velocity λ about the y-axis. The equation describing the motion of the ring is

$$\ddot{u} + \omega^2 u = -4qu(u\ddot{u} + \dot{u}^2),$$

where $\omega^2 = 2gq - \lambda^2$ and the initial conditions are $u(0) = A, \quad \dot{u}(0) = 0$

2.3 The Van der Pol oscillator is a second-order system with nonlinear damping, of the form

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0.$$

2.4 Here, a system consisting of a block of mass m that hangs from a viscous damper with coefficient c and a nonlinear spring of stiffness k_1 and k_3 is considered. Equation of motion is given by the following nonlinear differential equation:

$$\frac{d^2x(t)}{\mathrm{d}t^2} + \frac{k_1}{m}x(t) + \frac{k_3}{m}x^3(t) + \frac{c}{m}\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0,$$

with the following initial conditions:

$$x_0(0) = A, \quad \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0.$$

2.5 In this problem, we shall consider a system of (1 + 1)-dimensional long-wave equations:

$$u_t + uu_x + v_x = 0,$$

$$v_t + (vu)_x + \frac{1}{3}u_{xxx} = 0,$$

with the initial conditions $u(x,0) = f(x)$ and $v(x,0) = g(x)$, where v is the elevation of the water wave and u is the surface velocity of water along the x-direction.

2.6 We consider a uniform cantilever beam in which μ is the constant mass density, EI is bending stiffness, and l is the length of the beam, when its base is given a motion $w_b(t)$ normal to the beam axis. The corresponding fourth-order differential equation of this case is

$$\mu\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = -\mu\,\ddot{w}_b(t),$$

where we can assume

$$w_b(t) = W\sin(\omega t)$$

Thus,

$$\ddot{w}_b(t) = -W\omega^2\sin(\omega t)$$

The boundary conditions are

$$w(0,t) = 0, \quad \frac{\partial w}{\partial x}(0,t) = 0,$$

$$\frac{\partial}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right](l,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(l,t) = 0$$

and the initial displacement and velocity of the beam is assumed to be zero; thus,

$$w(x,0) = g(x) = 0,$$

$$\frac{\partial w}{\partial t}(x,0) = h(x) = 0.$$

2.7 We consider a generalized (scalar) Boussinesq equation of the form

$$u_{tt} - [f(u)]_{xx} - u_{xxtt} = h(x,t) \quad -\infty < x < +\infty, t \geq 0,$$

subject to the initial conditions

$$u(x,0) = a(x), \quad u_t(x,0) = b(x) \quad -\infty < x < +\infty,$$

2.8 We consider the nonlinear oscillator

$$u'' + u^3 = 0$$

with the boundary conditions

$$u(0) = A, u'(0) = 0$$

2.9 In this problem, we consider periodic solutions for sub-harmonic resonances of nonlinear oscillations with parametric excitation. The governing equation is

$$X'' + (1 + \varepsilon\cos(\lambda t))[\alpha X + \beta X^3] = 0,$$

and the boundary conditions for this equation are

$$X(0) = X_0,\ X'(0) = 0.$$

2.10 We consider the coupled Whitham–Broer–Kaup (WBK) equations, which have been studied by Whitham, Broer, and Kaup. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \beta\frac{\partial^2 u}{\partial x^2} = 0,$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \alpha\frac{\partial^3 u}{\partial x^3} - \beta\frac{\partial^2 v}{\partial x^2} = 0,$$

with the initial conditions

$$u(x,0) = \lambda - 2k(\alpha + \beta^2)^{0.5}\coth[k(x + x_0)],$$

$$v(x,0) = -2k^2(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{0.5})\mathrm{csch}^2[k(x + x_0)].$$

2.11 We consider the linear Schrödinger equation, which occurs in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics:

$$u_t + iu_{xx} = 0, \quad u(x,0) = f(x), \quad i^2 = -1$$

and the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \gamma|u|^2 u = 0,\ u(x,0) = f(x), \quad i^2 = -1.$$

2.12 In this problem, we shall consider the Schrödinger equation in the form

$$i\frac{\partial\psi(x,t)}{\partial t} = -{}^1\!/_2\nabla^2\psi + V_d(x)\psi + \beta_d|\psi|^2\psi,\ x \in R^d, \quad t \geq 0$$

where $V_d(x)$ is the trapping potential and β_d is a real constant. With initial data

$$\psi(x,0) = \psi_0(x),\, x \in R^d$$

2.13 We consider nonlinear oscillation systems with parametric excitations, governed by

$$\frac{d^2x(t)}{dt^2} + (1 - \varepsilon\cos(\varphi t))(\lambda x(t) + \beta x(t)^3) = 0 \quad x(0) = A, \dot{x}(0) = 0.$$

where $\varepsilon, \varphi, \beta, \lambda$ are known as physical parameters.

2.14 Consider the Van der Pol equation

$$y''(t) + y(t) = \varepsilon\left[1 - y^2(t)\right]y'(t), \qquad y(0) = 0, \quad y'(0) = 2,$$

2.15 We consider a conservative system with a single degree of freedom for which the equation of motion is

$$(1 + 4r^2x^2(t))x'^2(t) + Ax(t) + 4r^2rx'^2(t)x(t) = 0,$$

where A is:

$$A = 2gr - \Omega^2$$

2.16 The rigid frame is forced to rotate at the fixed rate Ω while the frame rotates and the simple pendulum oscillates. The governing equation can be obtained as

$$\frac{d^2x(t)}{dt^2} + (1 - A\cos(x(t)))\sin(x(t)) = 0.$$

Here, by using the Taylor's series expansion for $\cos(x(t))$ and $\sin(x(t))$, the above equation reduces to

$$\frac{d^2x(t)}{dt^2} + (1 - A)x(t) - \left(\frac{1}{6}x^3(t) + \frac{2}{3}Ax^3(t) - \frac{1}{2}Ax^5(t)\right) = 0,$$

with the boundary conditions

$$x(0) = \lambda,\, x'(0) = 0,$$

2.17 Consider the following nonlinear oscillator governed by

$$u'' + u = \varepsilon u'^2 u$$

with initial conditions of

$$u(0) = A,\, u'(0) = 0$$

2.18 We consider the motion of a particle on a rotating parabola. The governing equation of motion can be expressed as

$$u''(1 + 4q^2u^2) + \alpha^2 u + 4q^2 uu'^2 = 0$$

with initial conditions

$$u(0) = A,\ u'(0) = 0$$

2.19 Consider the following nonlinear oscillator governed by

$$u'' + \Omega^2 u + 4\varepsilon u^2 u'' + 4\varepsilon uu'^2 = 0$$

with initial conditions

$$u(0) = A\ ,\ u'(0) = 0.$$

2.20 In this problem, we have a rigid frame that is forced to rotate at the fixed rate Ω. The governing equation can be simply derived as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (1 - \Lambda\cos\theta)\sin\theta = 0,\ \theta(0) = A,\ \ \frac{\mathrm{d}\theta}{\mathrm{d}t}(0) = 0.$$

2.21 Consider free vibration of a conservative system with a single degree of freedom containing a mass attached to linear and nonlinear springs in series. After transformation, the motion is governed by a nonlinear differential equation of motions:

$$(1 + 3\varepsilon z v^2)v'' + 6\varepsilon z v v'^2 + \omega_e^2 v + \varepsilon\,\omega_e^2 v^3 = 0,$$

with the initial conditions

$$v(0) = A, \quad v'(0) = 0$$

2.22 We consider the RLW equation

$$u_t + u_x + uu_x + u_{xxt} = 0.$$

For the special case of this equation, the solitary wave solution is given in the form

$$u(x,t) = 3B\sec h^2[k(x - (1 + B)t)],$$

where

$$k = \frac{\sqrt{B}}{2\sqrt{1 + B}},$$

and the exact solution is

$$u(x,t) = 3\alpha\sec h^2(\beta(x - (1 + \alpha)t)).$$

2.23 Generalize the KDV equation of two space variables and formulate the well-known Kadomtsev–Petviashvili (KP) equation to provide an explanation of the general weakly dispersive waves. The (2 + 1) KP equation is given in the form

$$(u_t + 6\mu u u_x + u_{xxx})_x + 3u_{yy} = 0,$$

or equivalently,

$$u_{xt} + 6\mu u_x^2 + 6\mu u u_{xx} + u_{xxxx} + 3u_{yy} = 0, \quad \mu = (\pm 1)$$

2.24 We consider the Burger equation

$$u_t + uu_x - u_{xx} = 0,\ x \in R$$

with the exact solution being

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{4}\left(x - \frac{1}{2}t\right)\right).$$

2.25 Consider a nonlinear equation of fourth order:

$$u_{tt} + 3u_{xt}u_{xx} + u_{xxxt} = 0,$$

2.26 The Whitham–Broer–Kaup equation is

$$u_t + uu_x + v_x + \beta\, u_{xx} = 0,$$
$$v_t + (uv)_x + \alpha\, u_{xxx} - \beta\, v_{xx} = 0.$$

2.27 The motivation of this part is to extend the analysis of the sine–cosine method to solve different types of nonlinear equations—namely, sSK and LsKdV and water wave equations, which can be shown in the form

$$u_t + \left(63u^4 + 63\left(2u^2u_{xx} + uu_x^2\right) + 21\left(uu_{xxx} + u_{xx}^2 + u_xu_{xxx}\right) + u_{xxxxxx}\right)_x = 0,$$
$$u_t + \left(35u^4 + 70\left(u^2u_{xx} + uu_x^2\right) + 7\left(2uu_{xxx} + 3u_{xx}^2 + 4u_xu_{xxx}\right) + u_{xxxxxx}\right)_x = 0,$$
$$u_t + 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx} = 0,$$

Respectively, the first equation is known as the seventh-order Sawada–Kotera equation, the second equation is known as Lax's seventh-order KdV equation, and the third equation is known as the water wave equation.

2.28 In this problem, we consider the generalized Zakharov equations, which can be shown in the form

$$i\psi_t + \psi_{xx} - 2\,\alpha\,|\psi|^2\,\psi + 2\,\psi\,v = 0,$$

$$v_{tt} - v_{xx} + \left(|\psi|^2\right)_{xx} = 0.$$

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Chapter 3
Considerable Analytical Methods

3.1 Harmonic Balance Method

3.1.1 Introduction

The harmonic balance method (HBM) is a technique used in systems including both linear and nonlinear parts. The fundamental idea of HBM is to decompose the system into two separate subsystems, a linear part and a nonlinear part. The linear part is treated in the frequency domain, and the nonlinear part in the time domain. The interface between the subsystems consists of the Fourier transform pair. Harmonic balance is said to be reached when a chosen number of harmonics N satisfy some predefined convergence criteria. First, an appropriate unknown is chosen to use in the convergence check, which is performed in the frequency domain. Then the equations are rewritten in a suitable form for a convergence loop. One starts with an initial value of the chosen unknown, applies the different linear and nonlinear equations, and finally reaches a new value of the chosen unknown. If the difference between the initial value and the final value of the first N harmonics satisfies the predefined convergence criteria, harmonic balance is reached. Otherwise, an increment of the initial value is calculated by using a generalized Euler method—namely, the Newton–Raphson method.

It should be mentioned that HBM is similar to other proposed coupling techniques, but one advantage of HBM is the calculation of the increment of the initial value. The method proposed by Gupta and Munjal (1992) also includes an iterative process with a convergence condition. The main difference between their method and the HBM is how the chosen convergence unknown is treated. In HBM, one calculates an increment that depends on the difference of the value at the beginning of the convergence loop and the final value after the loop. This implies a faster and more robust convergence. In the method of Gupta and Munjal, the final value is entered as a new initial value, which easily leads to slower convergence or divergence.

For general dynamical systems, the HBM is widely used, from the simplest Duffing oscillation (Liu et al. 2006), to fluid dynamics (Ragulskis et al. 2006), and

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to complex fluid structural interactions (Liu and Dowell 2005). Wu and Wang (2006) developed Mathematica/Maple programs to approximate the analytical solutions of a nonlinear undamped Duffing oscillation.

For considering this method HBM deals with free vibration of a nonlinear system, having combined the linear and nonlinear springs in series and Nonlinear Normal Modes.

Example 3.1

The conservative oscillation system is formulated as a nonlinear ordinary differential equation having linear and nonlinear stiffness components. The governing equation is linearized and associated with the HBM to establish new and accurate higher-order analytical approximate solutions. Unlike the perturbation method, which is restricted to nonlinear conservative systems with a small perturbed parameter and also unlike the classical HBM which results in a complicated set of algebraic equations, the approach yields simple approximate analytical expressions valid for small as well as large amplitudes of oscillation. Some examples are solved and compared with numerical integration solutions, and the results are published.

3.1.2 Governing Equation of Motion and Formulation

Consider free vibration of a conservative, single-degree-of-freedom system with a mass attached to linear and nonlinear springs in series, as shown in Fig. 3.1. After transformation, the motion is governed by a nonlinear differential equation of motion (see Telli and Kopmaz 2006) as

$$(1 + 3\varepsilon z v^2)v'' + 6\varepsilon z v v'^2 + \omega_e^2 v + \varepsilon \omega_e^2 v^3 = 0, \tag{3.1}$$

where

$$\varepsilon = \frac{\beta}{k_2}, \tag{3.2}$$

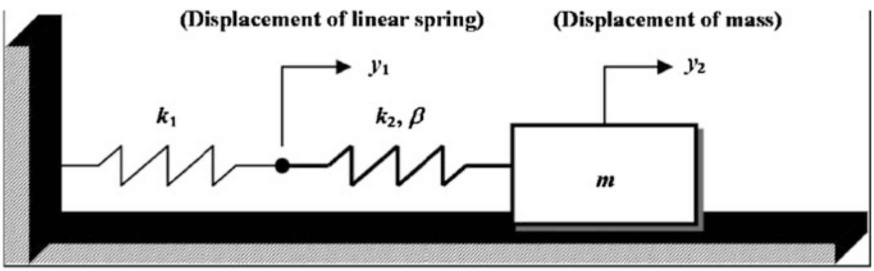

Fig. 3.1 Nonlinear free vibration of a system of mass with serial linear and nonlinear stiffness on a frictionless contact surface

$$\xi = \frac{k_2}{k_1}, \tag{3.3}$$

$$z = \frac{\xi}{1+\xi}, \tag{3.4}$$

$$\omega_e = \sqrt{\frac{k_2}{m(1+\xi)}} \tag{3.5}$$

with the initial conditions

$$v(0) = A, \quad v'(0) = 0, \tag{3.6}$$

in which ε, β, v, ω_e, m, and ξ are perturbation parameter (not restricted to a small parameter), coefficient of nonlinear spring force, deflection of nonlinear spring, natural frequency, mass and the ratio of linear portion k_2 of the nonlinear spring constant to that of the linear spring constant k_1, respectively. Note that the notations in Eqs. (3.1)–(3.5) follow those in Telli and Kopmaz (2006). The deflection of linear spring $y_1(t)$ and the displacement of attached mass $y_2(t)$ can be represented by the deflection of nonlinear spring v in simple relationships as:

$$y_1(t) = \xi\, v(t) + \varepsilon\, \xi\, v(t)^3 \tag{3.7}$$

and

$$y_2(t) = v(t) + y_1(t). \tag{3.8}$$

Introducing a new independent temporal variable, $\tau = \omega t$ (Eqs. 3.1 and 3.6), we have

$$\omega^2 \left[(1 + 3\varepsilon z v^2)\ddot{v} + 6\varepsilon z v \dot{v}^2\right] + \omega_e^2 v + \varepsilon\, \omega_e^2 v^3 = 0 \tag{3.9}$$

and

$$v(0) = A, \quad \dot{v}(0) = 0, \tag{3.10}$$

where a dot denotes differentiation with respect to τ. The deflection of nonlinear spring v is a periodic function of τ with the period of 2π. On the basis of Eq. 3.9, the periodic solution $v(\tau)$ can be expanded in a Fourier series with only odd multiples of τ as follows:

$$v(\tau) = \sum_{n=o}^{\infty} h_{2n+1} \cos(2n+1)\tau. \tag{3.11}$$

To linearize the governing differential equation, we assume $v(\tau)$ as the sum of a principal term and a correction term as

$$v(\tau) = v_1(\tau) + \Delta v_1(\tau). \tag{3.12}$$

Substituting Eq. 3.11 into Eq. 3.9 and neglecting nonlinear terms of $\Delta v_1(\tau)$ yields

$$\begin{aligned}&\omega^2\left[\left(1+3\varepsilon z v_1^2\right)\ddot{v}_1+6\varepsilon z v_1\dot{v}_1^{\,2}\right]+\omega_e^2 v_1+\varepsilon\,\omega_e^2 v_1^3+\left(\omega_e^2+3\varepsilon\,\omega_e^2 v_1^2\right)\Delta v_1\\&+\omega^2\left[\left(1+3\varepsilon z v_1^2\right)\Delta\ddot{v}_1+2(6\varepsilon z v_1\dot{v}_1)\Delta\dot{v}_1+\left(6\varepsilon z v_1\ddot{v}_1+6\varepsilon z\dot{v}_1^{\,2}\right)\Delta v_1\right]=0,\end{aligned}\tag{3.13}$$

and

$$\Delta v_1(0)=0,\quad \Delta\dot{v}_1(0)=0,\tag{3.14}$$

where $v_1(\tau)=A\cos\tau$ is a periodic function of τ period 2π.

Making use of $v_1(\tau)=A\cos\tau$, we have the following Fourier-series expansions:

$$\begin{aligned}\left(1+3\varepsilon z v_1^2\right)\ddot{v}_1+6\varepsilon z v_1\dot{v}_1^{\,2}&=\sum_{i=0}^{\infty}a_{2i+1}\cos(2i+1)\tau\\&=-\frac{A(4+3A^2z\varepsilon)}{4}\cos\tau-\frac{9A^3z\varepsilon}{4}\cos 3\tau,\end{aligned}\tag{3.15}$$

$$\omega_e^2 v_1+\varepsilon\,\omega_e^2 v_1^3=\sum_{i=0}^{\infty}b_{2i+1}\cos(2i+1)\tau=\frac{A\omega_e^2(4+3A^2\varepsilon)}{4}\cos\tau+\frac{A^3\varepsilon\omega_e^2}{4}\cos 3\tau,\tag{3.16}$$

$$1+3\varepsilon z v_1^2=\frac{1}{2}c_0+\sum_{i=1}^{\infty}c_{2i}\cos 2i\tau=\frac{2+3A^2z\varepsilon}{4}+\frac{3A^2z\varepsilon}{2}\cos 2\tau,\tag{3.17}$$

$$2(6\varepsilon z v_1\dot{v}_1)=\sum_{i=0}^{\infty}d_{2(i+1)}\sin 2(i+1)\tau=-6A^2z\varepsilon\sin 2\tau,\tag{3.18}$$

$$6\varepsilon z v_1\ddot{v}_1+6\varepsilon z\dot{v}_1^{\,2}=\frac{1}{2}c_0+\sum_{i=1}^{\infty}e_{2i}\cos 2i\tau=-6A^2\varepsilon\cos 2\tau,\tag{3.19}$$

$$\omega_e^2+3\varepsilon\,\omega_e^2 v_1^2=\frac{1}{2}f_0+\sum_{i=1}^{\infty}f_{2i}\cos 2i\tau=\frac{(2+3A^2\varepsilon)\omega_e^2}{2}+\frac{3A^2\varepsilon\omega_e^2}{2}\cos 2\tau,\tag{3.20}$$

where $a_{2i+1}, b_{2i+1}, c_{2i}, d_{2(i+1)}, e_{2i}$ and f_{2i} for $i=0,1,2,\ldots$ are Fourier-series coefficients.

3.1.3 First-Order Analytical Approximation

For the first-order analytical approximation, we set

$$\Delta v_1(\tau)=0,\tag{3.21}$$

and, therefore,

$$v(\tau)=v_1(\tau)=A\cos\tau.\tag{3.22}$$

Substituting Eqs. 3.15–3.20 into Eq. 3.13, expanding the resulting expression in a trigonometric series and setting the coefficient of $\cos\tau$ to zero yield the solution of the angular frequency ω_1, where subscript ω_1 indicates the first-order analytical approximation. The analytical approximation of ω_1 can be expressed as

$$\omega_1(A) = \omega_e\sqrt{\frac{3\varepsilon A^2 + 4}{3\varepsilon z A^2 + 4}} \tag{3.23}$$

and the periodic solution is

$$v_1(\tau) = A\cos[\omega_1(A)t]. \tag{3.24}$$

3.1.4 Second-Order Analytical Approximation

For the second analytical approximation, we set

$$\Delta v_1(\tau) = x_1(\cos\tau - \cos 3\tau). \tag{3.25}$$

Substituting Eqs. 3.15–3.20 and 3.25 into Eq. 3.13, expanding the resulting expression in a trigonometric series, and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero result in a quadratic equation of ω_2^2, where subscript 2 indicates the second-order analytical approximation. The angular frequency ω_2 can be expressed as

$$\omega_2(A) = \sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}, \tag{3.26}$$

where

$$a = -144A - 252zA^3\varepsilon - 135z^2A^5\varepsilon^2, \tag{3.27}$$

$$b = 160A\omega_e^2 + 124A^3\varepsilon\,\omega_e^2 + 156zA^3\varepsilon\,\omega_e^2 + 150zA^5\varepsilon^2\omega_e^2, \tag{3.28}$$

$$c = -16A\omega_e^4 - 28A^3\varepsilon\,\omega_e^4 - 15A^5\varepsilon^2\omega_e^4 \tag{3.29}$$

where a, b and c are the coefficients of the quadratic equation of ω_2^2. The solution of ω_2 in Eq. 3.26 with respect to $+\sqrt{b^2 - 4ac}$ is omitted, so that, $\omega_2/\omega_1 \approx 1$, and the periodic solution is

$$v_2(\tau) = [A + x_1(A)]\cos[\omega_2(A)t] - x_1(A)\cos[3\omega_2(A)t]. \tag{3.30}$$

where

$$\begin{aligned} x_1(A) = & -[32A\omega_e^2 + 25A^3\varepsilon\,\omega_e^2 + 15A^3z\varepsilon\,\omega_e^2 + 6A^5z\varepsilon^2\omega_e^2 - (1024A^2\omega_e^4 + 1472A^4\varepsilon\,\omega_e^4 \\ & + 2112A^4z\varepsilon\,\omega_e^4 + 421A^6\varepsilon^2\omega_e^4 + 365A^6z\varepsilon^2\omega_e^4 + 981A^6z^2\varepsilon^2\omega_e^4 + 1380A^8z\varepsilon^3\omega_e^4 \\ & + 1980A^8z^2\varepsilon^3\omega_e^4 + 900A^{10}z^2\varepsilon^4\omega_e^4)^{1/2}]/[2\omega_e^2(32 + 51A^2\varepsilon + 21A^2z\varepsilon + 36A^4z\varepsilon^2)]. \end{aligned} \tag{3.31}$$

3.1.5 Third-Order Analytical Approximation

Although the first- and second-order analytical approximations are expected to agree with other solutions, the agreement deteriorates as t progresses during the steady-state response. Therefore, the third-order analytical approximation is derived for a more accurate steady-state response. To construct the third-order analytical approximation, the previous related expressions must be adjusted, due to the interaction between lower-order and higher-order harmonics. Here, $\Delta v_1(\tau)$ and, in Eqs. 3.12, 3.13, and 3.15–3.20, is replaced by $\Delta v_2(\tau)$ and $v_2(\tau)$, respectively, and Eq. 3.13 is modified as

$$\begin{aligned}&\omega^2\left[\left(1+3\varepsilon z v_2^2\right)\ddot{v}_2+6\varepsilon z v_2\dot{v}_2^2\right]+\omega_e^2 v_2+\varepsilon\,\omega_e^2 v_2^3+\left(\omega_e^2+3\varepsilon\,\omega_e^2 v_2^2\right)\Delta v_2\\&+\omega^2\left[\left(1+3\varepsilon z v_2^2\right)\Delta\ddot{v}_2+2(6\varepsilon z v_2\dot{v}_2)\Delta\dot{v}_2+\left(6\varepsilon z v_2\ddot{v}_2+6\varepsilon z\dot{v}_2^2\right)\Delta v_2\right]=0.\end{aligned} \tag{3.32}$$

The right-hand sides of Eqs. 3.15–3.20 in the third-order analytical approximation are completely different from the first- and second-order analytical approximations because $v_1(\tau)$ is replaced by $v_2(\tau)$ of Eq. 3.30. It can be solved directly by substituting the corresponding coefficients of Fourier series in any symbolic software, such as Mathematica.

For the third-order analytical approximation, we set

$$\Delta v_2(\tau)=x_1(\cos\tau-\cos 3\tau)+x_3(\cos 3\tau-\cos 5\tau). \tag{3.33}$$

Substituting the modified Eqs. 3.15–3.20 with $v_1(\tau)$ replaced by $v_2(\tau)$ and Eq. 3.33 into Eq. 3.32, expanding the resulting expression in a trigonometric series, and setting the coefficients of $\cos\tau, \cos 3\tau$, and $\cos 5\tau$ to zero yield ω_3 as a function of A. The corresponding approximate analytical periodic solution can then be solved as

$$\begin{aligned}v_3(\tau)=&[A+x_1(A)+x_2(A)]\cos[\omega_3(A)t]+[x_3(A)-x_2(A)-x_1(A)]\cos[3\omega_3(A)t]\\&-x_3(A)\cos[5\omega_3(A)t].\end{aligned} \tag{3.34}$$

The angular frequency ω_3 is the squared-roots of a quadratic equation of ω_3^2 in the form of

$$a'\left(\omega_3^2\right)^4+b'\left(\omega_3^2\right)^3+c'\left(\omega_3^2\right)^2+d'\left(\omega_3^2\right)\;+\;e'=0, \tag{3.35}$$

where subscript 3 indicates the third-order analytical approximation and a', b', c', d', and e' are coefficients of the quartic equation of ω_3^2. There is a total of eight roots, and the particular root which is closest to ω_2 is identified as the most appropriate solution because ω_3 is a more accurate and of a higher-order approximation to ω_3. Comparison of ω_3 in the following section shows that it is in excellent agreement with the numerical integration solution for small, as well as large, amplitudes of oscillation. The quartic equation can be subsequently solved by any symbolic software, such as MATHEMATICA, for ω_3. The constants x_2 and x_3 in Eq. 3.34 derived in terms of the coefficients of Fourier series are obtained.

3.1.6 Approximate Results and Discussion

The solutions of Eq. 3.1 using the second-order LP perturbation method is briefly derived here. Expanding the frequency $\omega^2 = \omega_{\mathrm{LP}}^2$ and the periodic solution $v(\tau) = v_{\mathrm{LP}}(\tau)$ of Eq. 3.9 into a power series as a function of ε as follows:

$$\omega_{\mathrm{LP}}^2 = \omega_e^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{3.36}$$

$$v_{\mathrm{LP}}(\tau) = v_0(\tau) + \varepsilon v_1(\tau) + \varepsilon^2 v_2(\tau) + \cdots, \quad \tau = \omega_{\mathrm{LP}} t \tag{3.37}$$

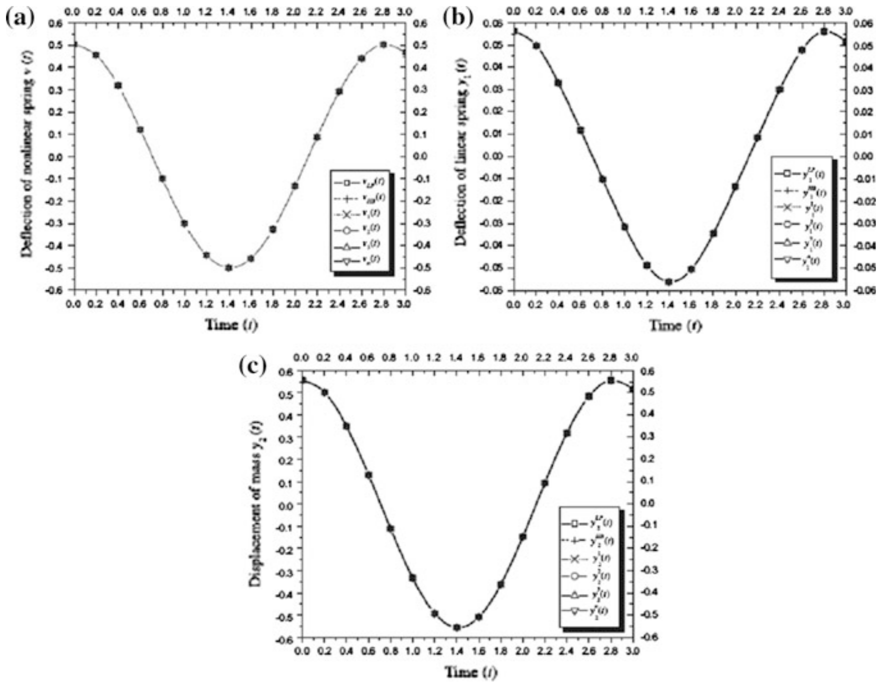

Fig. 3.2 **a** Comparison of deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solution for $m = 1, A = 0.5, \varepsilon = 0.5$, and $\xi = 0.1 (k_1 = 50, k_2 = 5)$. **b** Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 1, \varepsilon = 0.5$. **c** Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 1, \varepsilon = 0.5$, and $\xi = 0.1$ $(k_1 = 50, k_2 = 5)$

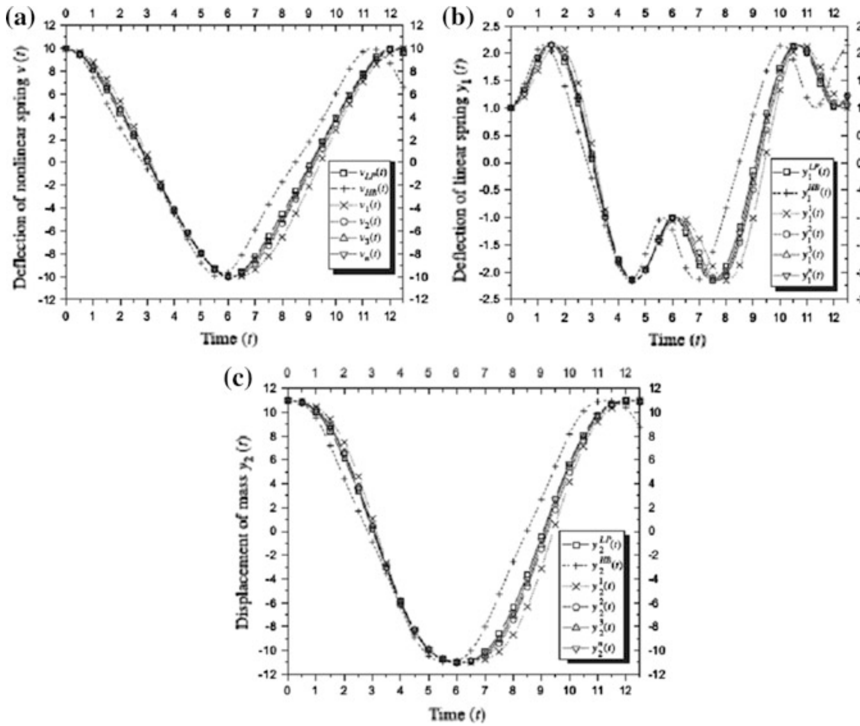

Fig. 3.3 **a** Comparison of the deflection of nonlinear spring $v(t)$ for various analytical approximations and the numerical integration solutions for $m = 4, A = 10, \varepsilon = -0.008$, and $\xi = 0.5(k_1 = 6, k_2 = 3)$. **b** Comparison of the deflection of linear spring $y_1(t)$ for various analytical approximations and the numerical integration solutions for $m = 4, \varepsilon = -0.008$, and $\xi = 0.5(k_1 = 6, k_2 = 3)$. **c** Comparison of the displacement of mass $y_2(t)$ for various analytical approximations and the numerical integration solutions for $m = 4, \varepsilon = -0.008$, and $\xi = 0.5(k_1 = 6, k_2 = 3)$

and setting the coefficients of ε^0, ε^1, and ε^2 as zero yield

$$v_0'' + v_0 = 0, \quad v_0(0) = A, \quad v_0'(0) = 0, \tag{3.38}$$

$$v_1'' + v_1 = -v_0^3 - 6zv_0v_0'^2 - 3zv_0^2v_0'' - \frac{\omega_1 v_0''}{\omega_e^2}, \quad v_1(0) = 0, \quad v_1'(0) = 0, \tag{3.39}$$

$$\begin{aligned} v_2'' + v_2 = & -3v_0^2v_1 - 6zv_1v_0'^2 - \frac{6z\omega_1 v_0 v_0'^2}{\omega_e^2} - 12zv_0v_0'v_1' - 6zv_0v_1v_0'' - \frac{3z\omega_1 v_0^2 v_0''}{\omega_e^2} \\ & - \frac{\omega_2 v_0''}{\omega_e^2} - 3zv_0^2v_1'' - \frac{\omega_1 v_1''}{\omega_e^2}, \quad v_2(0) = 0, \quad v_2''(0) = 0. \end{aligned} \tag{3.40}$$

Solving the linear second-order differential equations 3.38–3.40 with the corresponding initial conditions, we obtain

$$\omega_1 = -\frac{3}{4}A^2\omega_e^2(z-1), \quad \omega_2 = \frac{3}{128}A^4\omega_e^2(15z^2 - 14z - 1), \tag{3.41}$$

$$\begin{aligned} v_0 &= A\cos\omega_{\mathrm{LP}}t, \qquad v_1 = \frac{A^3}{32}(9z-1)(\cos\omega_{\mathrm{LP}}t - \cos 3\omega_{\mathrm{LP}}t), \\ v_2 &= -\frac{A^5(441z^2 - 34z - 32)}{1024}\cos\omega_{\mathrm{LP}}t + \frac{3A^5(9z^2-1)}{128}\cos 3\omega_{\mathrm{LP}}t \\ &\quad + \frac{A^5(225z^2 - 34z + 1)}{1024}\cos 5\omega_{\mathrm{LP}}t. \end{aligned} \tag{3.42}$$

To further illustrate and verify accuracy of this approximate analytical approach, a comparison of the time history response of nonlinear spring deflection $v(t)$, linear spring deflection $y_1(t)$, and mass displacement $y_2(t)$ is presented in Figs. 3.2 and 3.3. Figure 3.2 considers the nonlinear hard-spring cases, while Fig. 3.3 represents the nonlinear soft-spring cases.

3.2 He's Parameter Expansion Method

3.2.1 Introduction

Parameter-expanding methods, including the modified Lindstedt–Poincaré method and the bookkeeping parameter method, can successfully deal with such special cases; however, the classical methods fail. The methods need not have a time transformation like the Lindstedt–Poincaré method; the basic character of the method is to expand the solution and some parameters in the equation.

The parameter expansion method is an easy and straightforward approach to nonlinear oscillators. Anyone can apply the method to find an approximation of the amplitude–frequency relationship of a nonlinear oscillator with only basic knowledge of advance calculus. The basic idea of He's parameter-expanding methods (HPEMs) was provided by Prof. J. H. He in 2002, and the reader is referred to He (2002).

In a case where no parameter exists in an equation, HPEMs can be used (2002). As a general example, the following equation can be considered:

$$m\,u'' + \omega_0^2 u + \varepsilon f(u,\, u',\, u'') = 0,\ u(0) = \lambda,\ u'(0) = 0. \tag{3.43}$$

Various perturbation methods have been applied frequently to analyze Eq. 3.43. The perturbation methods are limited to the case of small ε and $m\,\omega_0^2 > 0$; that is, the associated linear oscillator must be statically stable so that the linear and nonlinear responses are qualitatively similar.

3.2.2 Modified Lindstedt–Poincaré Method

According to the modified Lindstedt–Poincaré method (He 2001b), the solution is expanded into a series of p or ε in the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \tag{3.44}$$

Hereby, the parameter ε (or p) does not require being small ($0 \le \varepsilon \le \infty$ or $0 \le p \le \infty$).

The coefficients m and ω_0^2 are expanded in a similar way

$$\omega_0^2 = \omega^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \quad \text{or} \quad \omega_0^2 = \omega^2 + p\,\omega_1 + p^2\omega_2 + \cdots \tag{3.45}$$

$$m = 1 + \varepsilon m_1 + \varepsilon^2 m_2 + \cdots \quad \text{or} \quad m = 1 + p\,m_1 + p^2 m_2 + \cdots \tag{3.46}$$

ω is assumed to be the frequency of the studied nonlinear oscillator; the values for m and ω_0^2 can be any of these positive, zero, or negative real values.

Here, we are going to solve this problem using HPEM.

3.2.3 Bookkeeping Parameter Method

In this case, no small parameter exists in the equations, so a traditional perturbation method cannot be useful. For this type of problem He introduced a technique in 2001 that provides for a bookkeeping parameter to be entered into the original differential equation (He 2001a).

3.2.4 Application

Example 3.2

This section considers the following nonlinear oscillator with discontinuity (Wang and He 2008):

$$u'' + u|u| = 0, \qquad u(0) = A, \quad u'(0) = 0. \tag{3.47}$$

There exists no small parameter in the equation. Therefore, the traditional perturbation methods cannot be applied directly.

The parameter expansion method entails the bookkeeping parameter method and the modified Lindstedt–Poincaré method.

In order to use the HPEM, we rewrite Eq. 3.47 in the form

$$u'' + 0.u + 1.u|u| = 0. \tag{3.48}$$

According to the parameter expansion method, we may expand the solution, u, the coefficient of u, the zero, and the coefficient of $u|u|$, 1, in series of p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots \tag{3.49}$$

$$0 = \omega^2 + pa_1 + p^2 a_2 + \cdots \tag{3.50}$$

$$1 = pb_1 + p^2 b_2 + \cdots \tag{3.51}$$

Substituting Eqs. 3.49 and 3.50 into Eq. 3.48 and equating the terms with the identical powers of p, we have

$$p^0\!: u_0'' \ + \ \omega^2 u_0 = 0 \tag{3.52}$$

$$p^1\!: u_1'' \ + \ \omega^2 u_1 \ + \ a_1 u_0 \ + \ b_1 u_0 |u_0| = 0 \tag{3.53}$$

$$p^2\!: (1+\omega^2)u_2'' + a_1 u_1'' + a_2 u_0'' + b_1\left(|u_0''|u_1'' + u_0''|u_1''|\right) + b_2 u_0''|u_0''| = 0. \tag{3.54}$$

Considering the initial conditions $u_0(0) = A$ and $u_0'(0) = 0$, the solution of Eq. 3.52 is $u_0 = A\cos\omega t$. Substituting the result into Eq. 3.53, we have

$$u_1'' + \omega^2 u_1 + a_1 A\cos\omega t + b_1 A^2 \cos\omega t|\cos\omega t| = 0. \tag{3.55}$$

It is possible to perform the Fourier series expansion

$$\cos\omega t|\cos\omega t| = \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] = c_1\cos\omega t + c_3\cos\omega t + \cdots, \tag{3.56}$$

where c_i can be determined by the Fourier series, for example

$$c_1 = \frac{2}{\pi}\int_0^{\pi} \cos^2\omega t|\cos\omega t|\ \mathrm{d}(\omega t) = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos^3\tau \mathrm{d}\tau - \int_{\frac{\pi}{2}}^{\pi} \cos^3\tau \mathrm{d}\tau\right) = \frac{8}{3\pi}. \tag{3.57}$$

Substitution of Eq. 3.56 into Eq. 3.55 gives

$$u_1'' + \omega^2 u_1 + \left(a_1 + b_1 A\frac{8}{3\pi}\right)A\cos\omega t + \sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] = 0. \tag{3.58}$$

Not to have a secular term in u_1 requires that

$$a_1 + b_1 A\frac{8}{3\pi} = 0. \tag{3.59}$$

If the first-order approximation is enough, then, setting $p = 1$ in Eqs. 3.50 and 3.51, we have

$$1 = b_1 \tag{3.60}$$

$$0 = \omega^2 + a_1. \tag{3.61}$$

From Eqs. 3.59–3.61, we obtain

$$\omega = \sqrt{\frac{8A}{3\pi}} \approx 2.6667\sqrt{\frac{A}{\pi}}. \tag{3.62}$$

The obtained frequency, Eq. 3.62, is valid for the whole solution domain, $0 < A < \infty$. The accuracy of frequency can be improved if we continue the solution procedure to a higher order; however, the amplitude obtained by this method is an asymptotic series, not a convergent one. For conservative oscillator

$$u'' + f(u)u = 0, \quad f(u) > 0 \tag{3.63}$$

where $f(u)$ is a nonlinear function of u, we always use the zero-order approximate solution. Thus, we have

$$u(t) = A\cos\left(t\sqrt{\frac{8A}{3\pi}}\right). \tag{3.64}$$

Example 3.3

3.2.5 Governing Equation

Considering the mechanical system shown in Fig. 3.4, we determine that there is a mass m grounded by linear and nonlinear springs in series. In this figure, the stiffness coefficient of the first linear spring is k_1; the coefficients associated with the linear and nonlinear portions of spring force in the second spring with cubic nonlinear characteristic are called k_2 and k_3, respectively, by definition ε:

$$\varepsilon = \frac{k_3}{k_2}, \tag{3.65}$$

The case of $k_3 > 0$ corresponds to a hardening spring, while $k_3 < 0$ indicates a softening one. x and y are absolute displacements of the connection point of the two springs and the mass m, respectively. Two new variables have been introduced as follows:

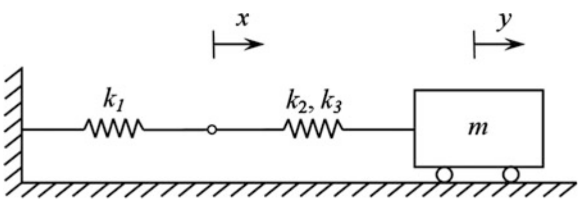

Fig. 3.4 Geometry of the example

$$u = y - x, \quad r = x. \tag{3.66}$$

The following governing equations have been obtained by Telli and Kopmaz (2006):

$$(1 + 3\varepsilon\eta u^2)\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 6\,\varepsilon\eta u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \omega_0^2(u + \varepsilon u^3) = 0, \tag{3.67}$$

$$r = x = \xi(1 + \varepsilon u^2)u, \; y = (1 + \xi + \xi\,\varepsilon u^2)\,u, \tag{3.68}$$

$$\xi = \frac{k_2}{k_1}, \quad \eta = \frac{\xi}{1 + \xi}, \quad \omega_0^2 = \frac{k_2}{m(1 + \xi)}, \tag{3.69}$$

and the initial conditions are

$$u(0) = \lambda, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0. \tag{3.70}$$

3.2.6 HPEM for Solving Problem

According to the HPEM, Eq. 3.67 can be rewritten as (Kimiaeifar et al. 2010):

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \omega_0^2u + \varepsilon\left(3\eta u^2\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 6\,\eta u\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \omega_0^2u^3\right) = 0. \tag{3.71}$$

And initial conditions are

$$u(0) = \lambda, \; \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0. \tag{3.72}$$

The form of solution and the constant one in Eq. 3.71 can be expanded as

$$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2u_2(t) + \cdots \tag{3.73}$$

$$\omega_0^2 = \omega^2 + \varepsilon b_1 + \varepsilon^2b_2 + \cdots \tag{3.74}$$

Substituting Eqs. 3.72–3.74 into Eq. 3.71 and processing as the standard perturbation method, we have

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}t^2} + \omega^2u_0 = 0, \quad u_0(0) = \lambda, \quad \frac{\mathrm{d}u_0}{\mathrm{d}t}(0) = 0. \tag{3.75}$$

The solution of Eq. 3.75 is

$$u_0(t) = \lambda\cos(\omega t). \tag{3.76}$$

Substituting $x_0(t)$ from the above equation into Eq. 3.76 results in

$$\frac{\mathrm{d}^2u_1(t)}{\mathrm{d}t^2}+\omega^2u_1(t)+b_1\lambda\cos(\omega t)-3\eta\lambda^3\cos^3(\omega t)\omega^2+\omega_0^2\lambda^3\cos^3(\omega t)\\+6\eta\lambda^3\cos(\omega t)\sin^2(\omega t)\omega^2=0. \tag{3.77}$$

But considering Eq. 3.74 and assuming two first terms, we have

$$b_1=\frac{\omega_0^2-\omega^2}{\varepsilon}. \tag{3.78}$$

On the basis of trigonometric functions properties, we have

$$\cos^3(\omega t)=1/4\cos(3\omega t)+3/4\cos(\omega t). \tag{3.79}$$

Substituting Eq. 3.79 into Eq. 3.77 and eliminating the secular term leads to

$$b_1\lambda+\frac{3}{4}\omega_0^2\lambda^3k-\frac{3}{4}\eta\lambda^3\omega^2=0. \tag{3.80}$$

Substituting Eq. 3.79 into Eq. 3.80, two roots of this particular equation can be obtained as

$$\omega=\pm\frac{\omega_0\sqrt{(3\eta\lambda^2\varepsilon+4)(4+3\lambda^2\varepsilon)}}{3\eta\lambda^2\varepsilon+4}. \tag{3.81}$$

Replacing ω from Eq. 3.81 into Eq. 3.77 yields

$$u(t)=u_0(t)=\lambda\cos\left(\frac{\omega_0\sqrt{(3\eta\lambda^2\varepsilon+4)(4+3\lambda^2\varepsilon)}}{3\eta\lambda^2\varepsilon+4}t\right). \tag{3.82}$$

3.3 Differential Transformation Method

3.3.1 Introduction

The differential transform method (DTM) is an analytic method for solving differential equations. The concept of a differential transform was first introduced by Zhou in 1986. Its main application therein is to solve both linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The DTM is an alternative procedure for obtaining an analytic Taylor series solution of the differential equations. By using DTM, we get a series solution—in practice, a truncated series solution. The series often coincides with the Taylor expansion of the true solution at point $x = 0$, in the initial value case.

Such a procedure changes the actual problem to make it tractable by conventional methods. In short, the physical problem is transformed into a purely mathematical one for which a solution is readily available. Our concern in this work is the derivation of approximate analytical oscillatory solutions for the nonlinear oscillator equation (Hassan 2002; Momani 2008):

$$y''(t) + cy(t) = \varepsilon f(y(t), y'(t)), \qquad y(0) = a, \quad y'(0) = b, \tag{3.83}$$

where c is a positive real number and ε is a parameter (not necessarily small). We assume that the function $f(y(t), y'(t))$ is an arbitrary nonlinear function of its arguments. The modified DTM will be employed in a straightforward manner without any need for linearization or smallness assumptions.

3.3.2 Differential Transformation Method

This technique, the given differential equation, and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series. This method is useful for obtaining exact and approximate solutions of linear and nonlinear differential equations. There is no need for linearization or perturbations; large computational work and round-off errors are avoided. It has been used to solve effectively, easily, and accurately a large class of linear and nonlinear problems with approximations. The method is well addressed in Ayaz (2004), Hassan (2004) and Liu and Song (2007). The basic definitions of differential transformation are introduced as follows:

Definition 3.1 If $f(t)$ is analytic in the time domain T, then it will be differentiated continuously with respect to time t:

$$\phi(t, k) = \frac{d^k f(t)}{dt^k}, \qquad \forall_t \in T. \tag{3.84}$$

For $t = t_i$, $\phi(t, k) = \phi(t_i, k)$, where k belongs to the set of nonnegative integers, denoted as the K-domain. Therefore, Eq. 3.84 can be rewritten as

$$F(k) = \phi(t_i, k) = \left[\frac{d^k f(t)}{dt^k}\right]\bigg|_{t=t_i}, \qquad \forall_t \in K, \tag{3.85}$$

where $F(k)$ is called the spectrum of $f(t)$ at $t = t_i$ in the K-domain.

Definition 3.2 If $f(t)$ can be represented by the Taylor series, then it can be indicated as

$$f(t) = \sum_{k=0}^{\infty} \left[(t - t_i)^k \Big/ k! \right] F(k). \tag{3.86}$$

Equation 3.86 is called the inverse transform of $F(k)$. With the symbol D denoting the differential transformation process, and upon combining Eqs. 3.85 and 3.86, we obtain

$$f(t) = \sum_{k=0}^{\infty} \left[(t - t_i)^k \Big/ k! \right] F(k) \equiv D^{-1} F(k).$$

Using the differential transformation, a differential equation in the domain of interest can be transformed into an algebraic equation in the K-domain, and $f(t)$ can be obtained by the finite–term Taylor series expansion plus a remainder such as

$$f(t) = \sum_{k=0}^{N} \left[(t - t_i)^k \Big/ k! \right] F(k) + R_{N+1}(t).$$

The fundamental mathematical operations performed by DTM are listed in Table 3.1.

In addition to the above operations, the following theorem, which can be deduced from Eqs. 3.85 and 3.86, is given below:

Theorem 3.1 *If* $f(x) = g_1(x) g_2(x) \cdots g_{m-1}(x) g_m(x)$, *then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1) G_2(k_2 - k_1) \cdots G_{n-1}(k_{n-1} - k_{n-2}) G_n(k - k_{n-1}). \tag{3.87}$$

Table 3.1 The fundamental operations of the differential transform method

Time function	Transformed function
$w(t) = \alpha u(t) \pm \beta v(t)$	$W(k) = \alpha U(k) \pm \beta V(k)$
$w(t) = \mathrm{d}^m u(t)/\mathrm{d}t^m$	$W(k) = \frac{(k+m)!}{k!} U(k+m)$
$w(t) = u(t)v(t)$	$W(k) = \sum_{l=0}^{k} U(l)\, V(k-1)$
$w(t) = t^m$	$W(k) = \delta(k-m) = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if } k \neq m. \end{cases}$
$w(t) = \exp(t)$	$W(k) = 1/k!$
$w(t) = \sin(\omega t + \alpha)$	$W(k) = (\omega^k/k!) \sin(k\pi/2 + \alpha)$
$w(t) = \cos(\omega t + \alpha)$	$W(k) = (\omega^k/k!) \cos(k\pi/2 + \alpha)$

The series solution (3.8) *does not exhibit the periodic behavior that is characteristic of oscillator equations. It converges rapidly only in a small region; in the wide region, they may have very slow convergence rates, and then their truncations yield inaccurate results. In the modified DTM of Shaher Momani, we apply a Laplace transform to the series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants, and then invert the approximant to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution* (Momani 2008).

3.3.3 Padé Approximations

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor's series expansion of a function $y(x)$. The $[L/M]$ Padé approximations of a function $y(x)$ are given by Baker in 1975 as

$$[L/M] = \frac{P_L(x)}{Q_M(x)}. \tag{3.88}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series are

$$y(x) = \sum_{i=1}^{\infty} a_i x^i, \tag{3.89}$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} \ = \ O(x^{L+M+1}), \tag{3.90}$$

which determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation.Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 1.0. \tag{3.91}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have no common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + \cdots P_L x^L \\ Q_M(x) &= q_0 + q_1 x + q_2 x^2 + \cdots q_M x^M \end{aligned}\right\}. \tag{3.92}$$

then, by Eqs. 3.91 and 3.92, we may multiply Eq. 3.90 by $Q_M(x)$, which linearizes the coefficient equations. We can write out Eq. 3.90 in more detail as

$$\left.\begin{aligned} &a_{L+1} + a_L q_1 + \cdots a_{L-M+1} q_M = 0, \\ &a_{L+2} + a_{L+1} q_1 + \cdots a_{L-M+2} q_M = 0, \\ &\vdots \\ &a_{L+M} + a_{L+M-1} q_1 + \cdots a_L q_M = 0, \end{aligned}\right\} \tag{3.93}$$

$$\left.\begin{aligned} &a_o = p_0, \\ &a_0 + a_0 q_1 = p_1, \\ &a_2 + a_1 q_1 + a_0 q_2 = p_1, \\ &\vdots \\ &a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{aligned}\right\}. \tag{3.94}$$

To solve these equations, we start with Eq. 3.93, which is a set of linear equations for all unknown qs. Once the qs are known, Eq. 3.94 gives an explicit formula for the unknown ps, which completes the solution. If Eqs. 3.93 and 3.94 are nonsingular, then we can solve them directly and obtain Eq. 3.95 (Baker 1975), where Eq. 3.95 holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \tag{3.95}$$

To obtain diagonal Padé approximants of different order, such as [2/2], [4/4], or [6/6], we can use the symbolic calculus software, MATHEMATICA.

3.3.4 Application

Example 3.4

In this example, the DTM is used to solve subharmonic resonances of nonlinear oscillation systems with parametric excitations, governed by Hassan (2002)

$$\frac{d^2 x(t)}{dt^2} + (1 - \varepsilon \cos(\phi t))(\lambda x(t) + \beta x(t)^3) = 0 \quad x(0) = A,\ \dot{x}(0) = 0, \tag{3.96}$$

where $\varepsilon, \phi, \beta, \lambda$ are known as physical parameters.

A comparison of the present results with those yielded by the established Runge–Kutta method confirms the accuracy of the proposed method.

Applying the DTM to Eq. 3.96 with respect to t gives (Hassan 2002)

$$(k+2)(k+1)X_{k+2} + \lambda X_k - \lambda\varepsilon\left(\sum_{l=0}^{k}\frac{X_{k-l}\phi^l\cos\left(\frac{1}{2}\pi l\right)}{l!}\right) + \beta\left(\sum_{l=0}^{k}X_{k-l}\left(\sum_{p=0}^{l}X_{l-p}X_p\right)\right)$$
$$-\beta\varepsilon\left(\sum_{l=0}^{k}\frac{1}{(k-l)!}\left(\phi^{(k-l)}\cos\left(\frac{1}{2}(k-l)\pi\right)\right)\left(\sum_{p=0}^{l}X_{l-p}\left(\sum_{q=0}^{p}X_{p-q}X_q\right)\right)\right) = 0. \tag{3.97}$$

Suppose that X_0 and X_1 are apparent from boundary conditions. By solving Eq. 3.97 with respect to X_{k+2}, we will have

$$X_2 = \frac{1}{2}\beta\varepsilon X_0^3 - \frac{1}{2}\lambda X_0 + \frac{1}{2}\lambda\varepsilon X_0 - \frac{1}{2}\beta X_0^3, \tag{3.98}$$

$$\begin{aligned} X_3 = &-\frac{1}{6}\lambda X_1 + \frac{1}{6}\lambda\varepsilon X_1 + \frac{1}{6}\lambda\varepsilon X_0\phi\cos\left(\frac{1}{2}\pi\right), \\ &-\frac{1}{2}\beta X_1 X_0^2 + \frac{1}{6}\beta\varepsilon\phi\cos\left(\frac{1}{2}\pi\right)X_0^3 + \frac{1}{2}\beta\varepsilon X_1 X_0^2, \end{aligned} \tag{3.99}$$

$$\begin{aligned} X_4 = &-\frac{1}{3}\lambda\beta\varepsilon X_0^3 + \frac{1}{24}\lambda^2 X_0 - \frac{1}{12}\lambda^2\varepsilon X_0 + \frac{1}{6}\lambda\beta X_0^3 \\ &+\frac{1}{6}\lambda\beta\varepsilon^2 X_0^3 + \frac{1}{24}\lambda^2\varepsilon^2 X_0 + \frac{1}{12}\lambda\varepsilon X_1\phi\cos\left(\frac{1}{2}\pi\right) + \cdots \end{aligned} \tag{3.100}$$

$$\begin{aligned} X_5 = &\frac{1}{20}\beta\varepsilon X_1^3 + \frac{9}{40}X_0^4\beta^2 X_1 + \frac{1}{120}\lambda^2\varepsilon^2 X_1 - \frac{1}{60}\lambda^2\varepsilon X_1 - \frac{1}{30}\lambda^2\varepsilon X_0\phi\cos\left(\frac{1}{2}\pi\right) \\ &+\frac{1}{120}\lambda^2 X_1 + \frac{1}{120}\lambda\varepsilon X_0\phi^3\cos\left(\frac{3}{2}\pi\right) + \frac{1}{40}\lambda\varepsilon X_1\phi^2\cos(\pi) + \cdots \end{aligned} \tag{3.101}$$

The above process is continuous. Substituting Eqs. 3.98–3.101 into the main equation on the basis of DTM, the closed form of the solutions can be obtained:

$$\begin{aligned} x(t) = &X_0 + tX_1 + \frac{t^2}{2!}\left(\frac{1}{2}\beta\varepsilon X_0^3 - \frac{1}{2}\lambda X_0 + \frac{1}{2}\lambda\varepsilon X_0 - \frac{1}{2}\beta X_0^3\right) \\ &+\frac{t^3}{3!}\left(-\frac{1}{6}\lambda X_1 + \frac{1}{6}\lambda\varepsilon X_1 + \frac{1}{6}\lambda\varepsilon X_0\phi\cos\left(\frac{1}{2}\pi\right) - \frac{1}{2}\beta X_1 X_0^2\right. \\ &\left.+\frac{1}{6}\beta\varepsilon\phi\cos(\frac{1}{2}\pi)X_0^3 + \frac{1}{2}\beta\varepsilon X_1 X_0^2\right) + \cdots \end{aligned} \tag{3.102}$$

In this stage, in order to achieve higher accuracy, we use a subdomain technique; that is, the domain of t should be divided into some adequate intervals. The values at the end of each interval will be the initial values of the next one. For example, at the first subdomain, it is assumed that the distance of each interval is 0.2. For the first interval, $0 \to 0.2$, boundary conditions are the ones given in Eq. 3.96 at point $t = 0$. By exerting a transformation, we will have

$$X_0 = A. \tag{3.103}$$

And the other boundary conditions are considered as

$$X_1 = 0. \tag{3.104}$$

As was mentioned above, for the next interval, $0.2 \to 0.4$, new boundary conditions are

$$X_0 = x(0.2). \tag{3.105}$$

The next boundary condition is considered as

$$X_1 = \frac{dx}{dt}(0.2). \tag{3.106}$$

For this interval function, $x(t)$ is represented by power series whose center is located at 0.2, which means that in this power series t converts to$(t - 0.2)$.

In order to verify the effectiveness of the proposed DTM, by using the Maple 10, package, the fourth-order Runge–Kutta as a numerical method is used to compute the displacement response of the nonlinear oscillator for a set of initial amplitudes and different physical parameters. These results are then compared with the DTM corresponding to the same set of amplitudes.

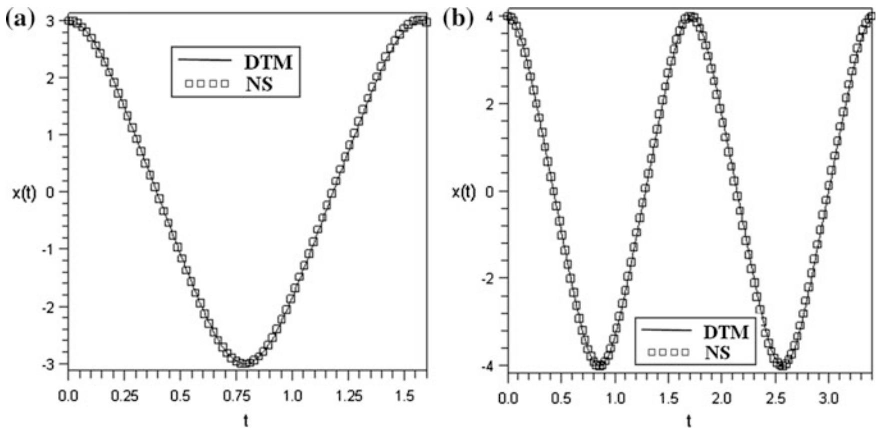

Fig. 3.5 The comparison between the differential transformation method (DTM) and numerical solutions (NS) of $x[m]$ to $t[s]$, for **a** $A = 3$, $\lambda = 3$, $\beta = 2$, $\varepsilon = 0.01$, $\phi = 10$ and **b** $A = 4$, $\lambda = 2$, $\beta = 2$, $\varepsilon = 0.01$, $\phi = 10$

The results for the different methods of DTM and Runge–Kutta are compared in Fig. 3.5.

Example 3.5

In order to assess the advantages and the accuracy of the modified DTM for solving nonlinear oscillatory systems, we have applied the method to a variety of initial-value problems arising in nonlinear dynamics. All the results are calculated by using Mathematica.

Consider the Van der Pol equation,

$$y''(t) + y(t) = \varepsilon\left[1 - y^2(t)\right]y'(t), \qquad y(0) = 0, \quad y'(0) = 2, \tag{3.107}$$

With respect to the initial conditions, we have

$$y(0) = 0, \quad y'(0) = 2. \tag{3.108}$$

Taking the differential transform of both sides of Eq. 3.107, we obtain the recurrence relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)} \times \left[\varepsilon\left((k+1)Y(k+2) - \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}(k-k_2+1)Y(k_1)Y(k_2-k_1)Y(k-k_2+1)\right) - Y(k)\right]. \tag{3.109}$$

The initial conditions given in Eq. 3.109 can be transformed at $t_0 = 0$ as

$$Y(0) = 0, \quad Y(1) = 2. \tag{3.110}$$

By using Eqs. 3.109, 3.110, and 3.107, the solution of the following series is obtained:

$$y(t) = 2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right) + \varepsilon\left(t^2 - \frac{5t^4}{6} + \frac{91t^6}{360} - \frac{41t^8}{1008}\right) + \cdots \tag{3.111}$$

This series does not exhibit the periodic behavior that is characteristic of the oscillatory system (3.107 and 3.108). Comparison of the approximate solution (3.111) for $\varepsilon = 0.3$ and the solution obtained by the fourth-order Runge–Kutta method in Fig. 3.6 shows that it converges in a small region but yields a wrong solution in a wider region. In order to improve the accuracy of the differential transform solution (3.111), we implement the modified DTM as follows.

Applying the Laplace transform to the series solution (3.111), yields

$$L[y(t)] = 2\left(\frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8}\right) + \varepsilon\left(\frac{2}{s^3} - \frac{20}{s^5} + \frac{182}{s^7} - \frac{1640}{s^9}\right) + \cdots \tag{3.112}$$

For simplicity, let $s = 1/t$; then

$$L[y(t)] = 2\left(t^2 - t^4 + t^6 - t^8\right) + \varepsilon\left(2t^3 - 20t^5 + 182t^7 - 1640t^9\right) + \cdots \tag{3.113}$$

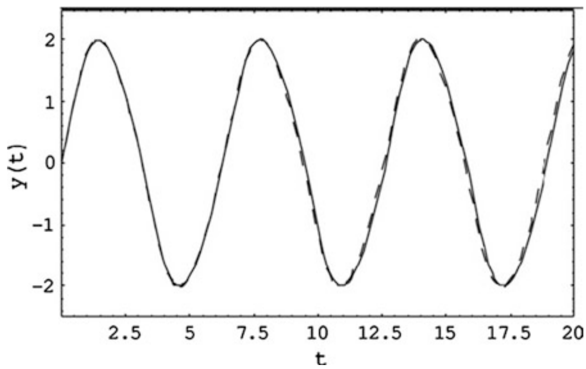

Fig. 3.6 Plots of displacement y versus time t: Runge–Kutta method, (—); Eq. 3.8, (– –)

The [4/4] Padé approximation for the terms containing ε^0, $\varepsilon^1, \ldots$ separately gives

$$\left[\frac{4}{4}\right] = 2\left(\frac{t^2}{1+t^2}\right) + \varepsilon\left(\frac{2t^3}{1+10t^2+9t^4}\right).$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s as

$$\left[\frac{4}{4}\right] = 2\left(\frac{1}{s^2+1}\right) + \varepsilon\left(\frac{2s}{s^4+10s^2+9}\right).$$

By using the inverse Laplace transform to the [4/4] Padé approximation, we obtain the modified approximate solution

$$y(t) = 2\sin(t) + \varepsilon\cos(t)\sin^2(t). \tag{3.114}$$

3.4 Adomian's Decomposition Method

3.4.1 Basic Idea of Adomian's Decomposition Method

The Adomian decomposition method (ADM) is a nonnumerical method for solving nonlinear differential equations, both ordinary and partial. The general direction of this work is toward a unified theory for partial differential equations (PDEs). The method was developed by George Adomian, chair of the Center for Applied Mathematics at the University of Georgia, in 1984. This method is a semianalytical method.

The ADM had been represented by Adomian (1994a, b, 1992). This method is a semianalytical method and has been modified by Wazwaz (1999a, b, 2001) and, more recently, by Luo (2005) and Zhang et al. (2006). This method is useful for obtaining closed form or numerical approximation for a wide class of stochastic and deterministic problems in science and engineering. These problems involve algebraic, linear or nonlinear ordinary or partial differential equations, and integro-differential, integral, and differential delay equations.

Let us discuss a brief outline of the ADM. For this, we consider a general nonlinear equation in the form

$$Lu + Ru + Nu = g \tag{3.115}$$

where L is the highest order derivative that is assumed to be easily invertible, R is the linear differential operator of less order than L, Nu presents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq. 3.115 and using the given conditions, we obtain

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu) \tag{3.116}$$

where the function $f(x)$ represents the terms arising from integration of the source term $g(x)$, using given conditions. For nonlinear differential equations, the nonlinear operator $Nu = F(u)$ is represented by an infinite series of the so-called Adomian polynomials as

$$F(u) = \sum_{m=0}^{\infty} A_m. \tag{3.117}$$

The polynomials A_m are generated for all kinds of nonlinearity so that A_0 depends only on u_0, A_1depends on u_0 and u_1, and so on. The Adomian polynomials introduced above show that the sum of subscripts of the components of u for each term of A_m is equal to n.

The Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{m=0}^{\infty} u_m. \tag{3.118}$$

In the case of $F(u)$, the infinite series is a Taylor expansion about u_0,

$$F(u) = F(u_0) + F'(u_0)(u - u_0) + F''(u_0)\frac{(u - u_0)}{2!} + F'''(u_0)\frac{(u - u_0)^2}{3!} + \cdots \tag{3.119}$$

Rewriting Eq. 3.118 as $u - u_0 = u_1 + u_2 + u_3 + \cdots$, substituting it into Eq. 3.119, and then equating two expressions for $F(u)$ found in Eqs. 3.119 and 3.117 define formulas for the Adomian polynomials in the form of

$$\begin{aligned} F(u) &= A_1 + A_2 + \cdots \\ &= F(u_0) + F'(u_0)(u_1 + u_2 + \cdots) + F''(u_0)\frac{(u_1 + u_2 + \cdots)^2}{2!} + \cdots \end{aligned} \tag{3.120}$$

By equating terms in Eq. 3.120, the first few Adomian's polynomials A_0, A_1, A_2, A_3, and A_4 are given by:

$$A_0 = F(u_0), \tag{3.121}$$

$$A_1 = u_1 F'(u_0), \tag{3.122}$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \tag{3.123}$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \tag{3.124}$$

$$A_4 = u_4 F'(u_0) + \left(\frac{1}{2!} u_2^2 + u_1 u_3\right) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} u_1^4 F^{(iv)}(u_0). \tag{3.125}$$

$\vdots$

Since A_m is known now, Eq. 3.117 can be substituted into Eq. 3.116 to specify the terms in the expansion for the solution of Eq. 3.125.

3.4.2 Application

Example 3.6

3.4.2.1 Introduction

The aim of this example is to employ ADM to obtain the exact solutions for linear and nonlinear Schrödinger equations, which occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity, and quantum mechanics (Sadighi and Ganji 2008a).

We consider the linear Schrödinger equation:

$$u_t + iu_{xx} = 0, \; u(x,0) = f(x), \; i^2 = -1$$

and the nonlinear Schrödinger equation

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \; u(x,0) = f(x), \; i^2 = -1$$

where γ is a constant and $u(x,t)$ is a complex function.

3.4.2.2 Analysis of the ADM

To illustrate the basic concepts of ADM for solving the linear Schrödinger equation, first we rewrite it in the following operator form (Sadighi and Ganji 2008a):

$$L_t u(x,t) + iL_{xx}u(x,t) = 0$$

where the notations are

$$L_t = \frac{\partial}{\partial t} \text{ and } L_{xx} = \frac{\partial^2}{\partial x^2}.$$

Assuming that L_t is invertible, then the inverse operator L_t^{-1} is given by

$$L_t^{-1} = \int_0^t (.)\mathrm{d}t.$$

Operating with the inverse operator on both sides of equation $L_t u(x,t) + iL_{xx}u(x,t) = 0$, we obtain

$$u(x,t) = u(x,0) - iL_t^{-1}(L_{xx}u(x,t)).$$

The Adomian method defines the solution $u(x,t)$ by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Substituting the previous decomposition series into $u(x,t)$ yields

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) - iL_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right).$$

To determine the components of $u_n(x,t)$, the Adomian decomposition method uses the recursive relation

$$u_0(x,t) = u(x,0),$$
$$u_{n+1}(x,t) = -iL_t^{-1}(L_{xx}u_n(x,t)).$$

With this relation, the components of $u_n(x,t)$ are easily obtained. This leads to the solution in a series form. The solution in a closed form follows immediately if an exact solution exists.

Proceeding as before, for solving the nonlinear Schrödinger equation by using ADM, we rewrite it in the operator form

$$iL_t u(x,t) + L_{xx}u(x,t) + \gamma u(x,t)^2 \bar{u}(x,t) = 0.$$

By using the inverse operators, we can write

$$iu(x,t) = iu(x,0) - L_t^{-1}(L_{xx}u(x,t)) - L_t^{-1}\left(\gamma u(x,t)^2\bar{u}(x,t)\right) = 0.$$

Substituting $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ into the previous equation yields

$$i\sum_{n=0}^{\infty} u_n(x,t) = iu(x,0) - L_t^{-1}\left(L_{xx}\sum_{n=0}^{\infty} u_n(x,t)\right) - \gamma L_t^{-1}(A_n) = 0.$$

Using the recursive relation to determine the components of $u_n(x,t)$, we obtain

$$u_0(x,t) = u(x,0),\ iu_{n+1}(x,t) = -L_t^{-1}(L_{xx}u_n(x,t)) - \gamma L_t^{-1}A_n,$$

where A_n are Adomian's polynomials and can be obtained as

$$\begin{aligned}
A_0 &= u_0^2\bar{u}_0,\\
A_1 &= 2u_0u_1\bar{u}_0 + u_0^2\bar{u}_1,\\
A_2 &= 2u_0u_2\bar{u}_0 + u_1^2\bar{u}_0 + 2u_0u_1\bar{u}_2 + u_0^2\bar{u}_2,\\
A_3 &= 2u_0u_3\bar{u}_0 + u_1^2\bar{u}_1 + 2u_1u_2\bar{u}_0 + u_0^2\bar{u}_3 + 2u_0u_2\bar{u}_1 + 2u_0u_1\bar{u}_2.
\end{aligned}$$

$\vdots$

3.4.2.3 Case 1

Consider the linear Schrödinger equation

$$u_t + iu_{xx} = 0$$

subjected to the initial condition

$$u(x,0) = 1 + 2\cosh(2x).$$

Considering the given initial condition, we can assume $u_0(x,y) = 1 + \cosh(2x)$ as an initial approximation. Next, we use the recursive relation to obtain the rest of the components of $u_n(x,y)$:

$$\begin{aligned}
u_1(x,t) &= -iL_t^{-1}(L_{xx}u_0(x,t)) = -4it\cosh(2x),\\
u_2(x,t) &= -iL_t^{-1}(L_{xx}u_1(x,t)) = \frac{(4it)^2}{2!}\cosh(2x),\\
u_3(x,t) &= -iL_t^{-1}(L_{xx}u_2(x,t)) = -\frac{(4it)^3}{3!}\cosh(2x).
\end{aligned}$$

Similarly, the remaining components can be found. The solution in a series form is given by

$$u(x,t) = 1 + \cosh(2x)\left(1 - 4it + \frac{(4it)^2}{2!} - \frac{(4it)^3}{3!} + \vdots\right) = 1 + \cosh(2x)e^{-4it}.$$

So the exact solution is

$$u(x,t) = 1 + \cosh(2x)e^{-4it}.$$

3.4.2.4 Case 2

We then consider the linear Schrödinger equation

$$u_t + iu_{xx} = 0$$

subjected to the initial condition

$$u(x,0) = e^{3ix}.$$

Considering $u(x,0) = e^{3ix}$, we can assume $u_0(x,y) = e^{3ix}$ as an initial approximation. Next, we use the recursive relation to obtain the rest of the components of $u_n(x,y)$.

$$u_1(x,t) = -iL_t^{-1}(L_{xx}u_0(x,t)) = 9ite^{3ix},$$
$$u_2(x,t) = -iL_t^{-1}(L_{xx}u_1(x,t)) = \frac{(9it)^2}{2!}e^{3ix},$$
$$u_3(x,t) = -iL_t^{-1}(L_{xx}u_2(x,t)) = \frac{(9it)^3}{3!}e^{3ix}.$$

Similarly, the remaining components can be found. The solution in a series form is given by

$$u(x,t) = e^{3ix}\left(1 + 9it + \frac{(9it)^2}{2!} + \frac{(9it)^3}{3!} + \cdots\right) = e^{3i(x+3t)}.$$

So the exact solution is

$$u(x,y) = e^{3i(x+3t)}.$$

This solution is the same as that of ADM.

3.4.2.5 Case 3

We now consider the nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2u = 0$$

subjected to the initial condition

$$u(x,0) = e^{ix}.$$

Considering the given initial condition, we can assume $u_0(x,y) = e^{ix}$ as an initial approximation. We next use the recursive relation to obtain the rest of the components of $u_n(x,y)$.

$$u_1(x,t) = iL_t^{-1}(L_{xx}u_0(x,t)) + i\gamma L_t^{-1}A_0 = ite^{ix},$$
$$u_2(x,t) = iL_t^{-1}(L_{xx}u_1(x,t)) + i\gamma L_t^{-1}A_1 = \frac{(it)^2}{2!}e^{ix},$$
$$u_3(x,t) = iL_t^{-1}(L_{xx}u_2(x,t)) + i\gamma L_t^{-1}A_2 = \frac{(it)^3}{3!}e^{ix}.$$

Similarly, the remaining components can be found. The solution in a series form is given by

$$u(x,t) = e^{ix}\left(1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right) = e^{i(x+t)}.$$

Therefore, the exact solution in closed form will be

$$u(x,t) = e^{i(x+t)},$$

which is the same as that obtained by ADM.

3.4.2.6 Case 4

Finally, we consider the nonlinear Schrödinger equation

$$iu_t + u_{xx} - 2|u|^2u = 0,$$

subjected to the initial condition

$$u(x,0) = e^{ix}.$$

Proceeding as before with the initial conditions, in the upper equation, gives

$$u(x,t) = e^{ix}\left(1 - 3it + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \cdots\right) = e^{i(x-3t)}.$$

Therefore, the exact solution in closed form will be

$$u(x,t) = e^{i(x-3t)},$$

which is the same as that of ADM.

3.5 He's Amplitude–Frequency Formulation

3.5.1 Introduction

He's amplitude–frequency formulation (HAFF), derived on the basis of an ancient Chinese mathematical method, is an effective method for treating nonlinear oscillators and is applied to obtain the amplitude–frequency relationship. This method was used by He in 2004.

This method considers the general nonlinear oscillators

$$u''(t) + f(u(t), u'(t), u''(t)) = 0. \tag{3.126}$$

Oscillation systems contain two important physical parameters—that is, the frequency ω and the amplitude of oscillation, A. Therefore, let us consider initial conditions

$$u(0) = A\,,\ u'(0) = 0.$$

According to HAFF, we choose two trial functions, $u_1 = A\cos t$ and $u_2 = A\cos\omega t$.

Substituting u_1 and $u_2$into Eq. 3.126, we obtain the following residuals, respectively:

$$R_1 = u_1''(t) + f\big(u_1(t), u_1'(t), u_1''(t)\big) \tag{3.127}$$

and

$$R_2 = u_2''(t) + f\big(u_2(t), u_2'(t), u_2''(t)\big). \tag{3.128}$$

If, by chance, u_1 or u_2 is chosen to be the exact solution, then the residual, Eqs. 3.127 or 3.128, vanishes completely. In order to use HAFF, we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\ \mathrm{d}t, T_1 = 2\pi \tag{3.129}$$

and,

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\ \mathrm{d}t\,,\ T_2 = \frac{2\pi}{\omega}. \tag{3.130}$$

Applying HAFF, we have

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}}, \tag{3.131}$$

where

$$\omega_1 = 1,\, \omega_2 = \omega. \tag{3.132}$$

Finally, we solve this integral to determine both k and ω:

$$u(t) = A\cos\omega t, \tag{3.133}$$

$$\int_0^{T/4} (\omega^2 u(t) + f(u(t))) \times \cos\omega t \mathrm{d}t = 0, \tag{3.134}$$

$$T = \frac{2\pi}{\omega}. \tag{3.135}$$

3.5.2 Applications

In order to assess the advantages and the accuracy of HAFF, we will consider the following examples:

Example 3.7

Consider a nonlinear oscillator governed by

$$u'' + u = \varepsilon u'^2 u$$

with initial condition

$$u(0) = A\,,\; u'(0) = 0,$$

where

$$f(u(t), u'(t), u''(t)) = u(t) - \varepsilon u'^2(t)u(t).$$

According to HAFF, we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$,where ω is assumed to be the frequency of the nonlinear oscillator upper equation. Substituting u_1 and u_2 into the previous equation, we obtain the following residuals, respectively (Ganji 2010):

$$R_1 = -\varepsilon A^3 \sin^2 t \cos t$$

and

$$R_2 = -A\cos(\omega t)\omega^2 + A\cos(\omega t) - \varepsilon A^3 \sin^2(\omega t)\omega^2 \cos(\omega t).$$

In order to use HAFF, we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1 \cos(t)\ \mathrm{d}t = -\frac{1}{8}\varepsilon \mathrm{A}^3\ ,\ T_1 = 2\pi$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega t)\ \mathrm{d}t = -\frac{1}{8}\frac{A(A^2\varepsilon\omega^2\pi + 4\omega^2\pi - 4\pi)}{\pi}, \quad T_2 = \frac{2\pi}{\omega}.$$

Applying HAFF, we have

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}},$$

where

$$\omega_1 = 1,\ \omega_2 = \omega.$$

We therefore obtain

$$\omega^2 = \frac{4}{\varepsilon A^2 + 4}.$$

The first-order approximate solution is obtained, which leads to

$$\omega = \sqrt{\frac{1}{1 + \frac{1}{4}\varepsilon A^2}}.$$

For small ε, it follows that

$$\omega = \left(1 - \frac{1}{8}\varepsilon A^2\right).$$

This agrees with Nayfeh's (2000) perturbation result.

In order to compare this argument with the homotopy perturbation method, we write He's result:

$$\omega = \sqrt{\frac{1}{1 + \frac{1}{4}\varepsilon A^2}}.$$

Therefore, it may be concluded that the perturbation method is not reliable for large amplitudes, whereas the method presented in this study yields reasonable results.

Example 3.8

The next example considered here is the motion of a particle on a rotating parabola. The governing equation of motion and initial conditions can be expressed as (Ganji et al. 2009b)

$$u''(1+4q^2u^2)+\alpha^2 u+4q^2uu'^2=0$$

with the initial condition

$$u(0)=A\,,\;u'(0)=0.$$

We consider the motion of a ring of mass m sliding freely on a wire described by the parabola $z=qx^2$, which rotates with a constant angular velocity about the z-axis.

According to HAFF, we choose two trial functions $u_1=A\cos t$ and $u_2=A\cos\omega t$, where ω is assumed to be the frequency of the nonlinear oscillator of the upper equation. Substituting u_1 and u_2 into the equation of motion, we obtain the following residuals, respectively:

$$R_1=-A\cos t(1+4q^2A^2\cos^2 t)+\alpha^2A\cos t+4q^2A^3\cos t\sin^2 t$$

and

$$\begin{aligned}R_2=&-A\cos(\omega t)\omega^2(1+4q^2A^2\cos^2(\omega t))\\&+\alpha^2A\cos(\omega t)+4q^2A^3\cos(\omega t)\sin^2(\omega t)\omega^2.\end{aligned}$$

In order to use HAFF, we set

$$R_{11}=\frac{4}{T_1}\int_0^{\frac{T_1}{4}}R_1\cos(t)\;\mathrm{d}t=\frac{2}{\pi}\left(-\frac{1}{4}A\pi-\frac{1}{2}A^3q^2\pi+\frac{1}{4}\alpha^2A\pi\right),\;T_1=2\pi$$

and

$$R_{22}=\frac{4}{T_2}\int_0^{\frac{T_2}{4}}R_2\cos(\omega t)\;\mathrm{d}t=-\frac{1}{2}\frac{A(2q^2A^2\omega^2\pi+\omega^2\pi-\alpha^2\pi)}{\pi},\;T_2=\frac{2\pi}{\omega}.$$

Applying HAFF, we have

$$\omega^2=\frac{\omega_1^2R_{22}-\omega_2^2R_{11}}{R_{22}-R_{11}},$$

where

$$\omega_1=1,\;\omega_2=\omega,$$

from which we therefore obtain

$$\omega^2 = \frac{\alpha^2}{2A^2q^2 + 1}.$$

The first-order approximate solution is obtained, which leads to

$$\omega = \frac{\alpha}{\sqrt{2(Aq)^2 + 1}}.$$

In order to compare with the Parameterized perturbation method, we write He's results:

$$\omega = \frac{\alpha}{\sqrt{2(Aq)^2 + 1}}.$$

Its approximate period can be written in the form

$$T = \frac{2\pi}{\alpha}\sqrt{2(Aq)^2 + 1}.$$

In the case where qA is sufficiently small—that is, $0 < qA << 1$—it follows that

$$T_{\text{perturbation}} = \frac{2\pi}{\alpha}(1 + q^2A^2).$$

In our present study, qA needs not be small, and even in the case of $qA \to \infty$, the present results still show high accuracy;

$$\lim_{qA\to\infty} \frac{T_{ex}}{T} = \frac{\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sqrt{1 + 4q^2A^2\cos^2 t}}{\sqrt{2(Aq)^2 + 1}}\mathrm{d}t = \frac{2\sqrt{2}}{\pi} = 0.900.$$

Therefore, for any value of $qA \to \infty$, it can be easily proved that the maximal relative error is less than 10 % on the whole solution domain.

Example 3.9

Considering the following nonlinear oscillator (Ganji et al. 2009b) governed by

$$u'' + \Omega^2 u + 4\varepsilon u^2 u'' + 4\varepsilon u u'^2 = 0$$

with initial condition

$$u(0) = A\,,\; u'(0) = 0.$$

According to HAFF, we choose two trial functions $u_1 = A\cos t$ and $u_2 = A\cos\omega t$, where ω is assumed to be the frequency of the nonlinear oscillator of the upper equation, and then substituting u_1 and u_2 in this equation, we obtain the following residuals, respectively:

$$R_1 = -A\cos t + \Omega^2 A\cos t - 4\varepsilon A^3\cos^3 t + 4\varepsilon A^3\cos t\sin^2 t$$

and

$$R_2 = -A\cos(\omega t)\omega^2 + \Omega^2 A\cos\omega t - 4\varepsilon A^3\cos^3(\omega t)\omega^2 + 4\varepsilon A^3\cos(\omega t)\sin^2(\omega t)\omega^2.$$

In order to use HAFF, we set

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1\cos(t)\ \mathrm{d}t = \frac{2}{\pi}\left(-\frac{1}{4}A\pi + \frac{1}{4}\Omega^2 A\pi - \frac{1}{2}A^3\varepsilon\pi\right), \quad T_1 = 2\pi$$

and

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2\cos(\omega t)\ \mathrm{d}t = \frac{1}{2\pi}\left(-2\varepsilon A^2\omega^2\pi - \omega^2\pi + \Omega^2\pi\right),\ T_2 = \frac{2\pi}{\omega}.$$

Applying HAFF, we have

$$\omega^2 = \frac{\omega_1^2 R_{22} - \omega_2^2 R_{11}}{R_{22} - R_{11}},$$

where

$$\omega_1 = 1,\ \omega_2 = \omega.$$

We therefore obtain

$$\omega^2 = \frac{\Omega^2}{2\varepsilon A^2 + 1}.$$

The first-order approximate solution is obtained, which leads to

$$\omega = \frac{\Omega}{\sqrt{1 + 2\varepsilon A^2}},$$

where the period is

$$T = \frac{2\pi}{\Omega}\sqrt{1 + 2\varepsilon A^2},$$

while the exact period reads

$$T_{ex} = \frac{4}{\Omega}\sqrt{1 + 4\varepsilon A^2}\int_0^{\frac{\pi}{2}}\sqrt{1 - k\sin^2 t}\mathrm{d}t,$$

where

$$k = \frac{4\varepsilon A^2}{1 + 4\varepsilon A^2}.$$

It is evident that our result is valid for all $\varepsilon > 0$. Even in case $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon \to \infty} \frac{T_{ex}}{T} = 0.9003.$$

For a relatively comprehensive survey on the concepts, theory, and applications of the methods cited in this chapter, see more examples in Ganjia and Seyed (2013a, b, 2011a, b), Momeni et al. (2011a, b), Ganji and Esmaeilpour (2010), Fereidoon et al. (2010), Safari et al. (2009), Ganji et al. (2009a, 2010a, b, 2007), Sadighi et al. (2008), Sadighi and Ganji (2008b, 2007), Kimiaeifar et al. (2009a).

3.5.3 Problems

Solve the following problems using presented methods in this chapter.

3.1 We consider the free oscillation of a nonlinear oscillator with quadratic and cubic nonlinearities:

$$\ddot{x} + \omega^2 x + ax^2 + bx^3 = 0, \qquad x(0) = A, \quad \dot{x}(0) = 0$$

where a and b are constants.

3.2 Consider a family of nonlinear differential equations

$$\ddot{x} + \alpha x + \gamma x^{2n+1} = 0, \qquad \alpha \geq 0, \gamma > 0, n = 1, 2, 3, \ldots$$

with the initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0.$$

The corresponding exact period T is

$$T_{ex} = 4\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{\alpha + \frac{\gamma}{n+1} A^{2n}(1 + \sin^2\theta + \sin^4\theta + \cdots \sin^{2n}\theta)}}.$$

3.3 In this problem the vibration of a mass–spring oscillator with strong quadratic nonlinearity and one degree of freedom is analyzed. Both hard and soft springs are considered.

The vibration of a one-degree-of-freedom mass–spring system is described by differential equation

$$\ddot{x} + cx + (\pm)\, a^2 \mathrm{sign}|x|(x^2) = 0,$$

subject to the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0.$$

3.3.1 **Hard Spring**
For the case of the hard spring, there exists only one fixed point $(x_1, y_1) = (0, 0)$, which is denoted by the vanishing of the vector field $-cx - a^2 \text{sign}|x|(x^2)$.

3.3.2 **Soft Spring**
For the soft spring and vector field $-cx + a^2 sign|x|(x^2)$, the following fixed points exist:

$$(x_1, y_1) = (0, 0), \qquad (|x_2|, y_2) = \left(\frac{c}{a^2}, 0\right).$$

3.4 Consider a more complex example in the form

$$u'' + au + bu^3 + cu^{1/3} = 0$$

with the exact one

$$T_{ex} = \frac{4}{\sqrt{1 + bA^2}} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - k \sin^2 x}},$$

where $k = 0.5bA^2 / (1 + bA^2)$.

3.5 When damping is neglected, the differential equation governing the free oscillation of the mathematical pendulum is given by

$$ml\ddot{\theta} + mg \sin \theta = 0$$

or

$$\ddot{\theta} + a \sin \theta = 0.$$

Here m is the mass, l the length of the pendulum, g the gravitational acceleration, and $a = g/l$. The angle θ designates the deviation from the vertical equilibrium position.
We rewrite the equation in the form

$$\ddot{\theta} + \Omega^2 \theta = \theta \left(\Omega^2 - a \frac{\sin \theta}{\theta} \right),$$

where Ω is an unknown frequency of the periodic solution. Here, the initial conditions are $\theta(0) = A$, $\dot{\theta}(0) = 0$, the inputs of the starting function are $\theta_{-1}(t) = \theta_0(t) = A \cos \Omega t$ and $g(t, \theta, \dot{\theta}, \ddot{\theta}) = \Omega^2 - a \frac{\sin \theta}{\theta}$, while the exact period reads

$$T_{ex} = \frac{4}{\sqrt{a}} \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}}; \quad k = \sin\frac{A}{2}.$$

3.6 We consider the structure shown in Fig. 3.7. The mass m moves in the horizontal direction only. Neglecting the weight of all but the mass, the governing equation for the motion of m is

$$m\ddot{u} + \left(k_1 - \frac{2p}{l}\right)u + \left(k_3 - \frac{p}{l^3}\right)u^3 + \cdots = 0,$$

The previous equation can be put in the general form

$$\ddot{u} + \alpha_1 u + \alpha_3 u^3 + \cdots = 0.$$

where the spring force is given by

$$F_{\text{spring}} = k_1 u + k_3 u^3 + \cdots.$$

3.7 In this problem, we consider a particle of mass m moving under the influence of the central force field of magnitude k/r^{2n+3}. The equation of the orbit in the polar coordinates (r, θ) is

$$\frac{d^2 u}{\mathrm{d}\theta^2} + u = -cu^{2n+1},$$

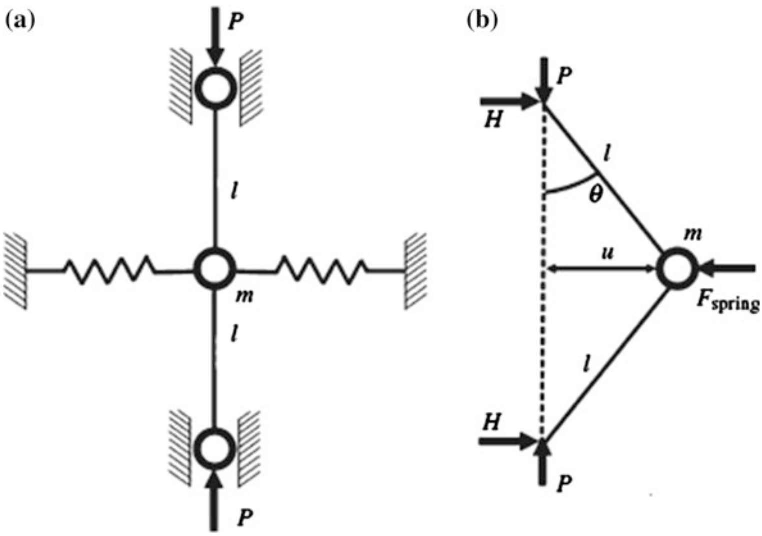

Fig. 3.7 Model for the buckling of a column

where k and c are constants and $u = 1/r$ In this case, let us consider a family of nonlinear differential equations:

$$u'' + \alpha u + \gamma u^{2n+1} = 0, \qquad \alpha > 0, \quad \gamma > 0, \quad n = 1, 2, 3, \ldots,$$
$$u(0) = A, \quad u'(0) = 0.$$

The corresponding exact period T is

$$T_{\text{ex}} = 4 \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{\alpha + \frac{\gamma}{n+1} A^{2n} (1 + \sin^2\theta + \sin^4\theta + \cdots \sin^{2n}\theta)}}.$$

3.8 Consider the nonlinear equation

$$y''(t) + y(t) = -\varepsilon y^2(t) y'(t),$$

subject to the initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

This equation can be appropriately called the "unplugged" Van der Pol equation, and all of its solutions are expected to oscillate with decreasing amplitude to zero.

3.9 Consider the following Duffing equation:

$$y''(t) + y(t) + 0.3y^3(t) = 0,$$

3.10 The example of a nonlinear vibrating system is a nonlinear periodic system. It can be describe by its governing motion equation as

$$\begin{cases} x_2(t) - \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = 0 \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} + 2.25x_1(t) + [x_1(t) - 1.5\sin(t)]^3 - 2\sin(t) = 0 \end{cases},$$

for which the boundary conditions are in the form

$$x_1(0) = 0, \qquad x_2(0) = 1.59929.$$

Guidance: With the effective initial approximation for x_{10}, x_{20} from the boundary conditions to the previous equation, we construct $x_{10}(t), x_{20}(t)$ as

$$x_{10}(t) = \sin(t), \qquad x_{20}(t) = 1.59929\cos(t).$$

3.11 The example is the initial-value problem of an ordinary nonlinear dynamic equation. The nonlinear motion equation of this system can described as

$$\frac{d^2x(t)}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x(t) - \ln t = 0,$$

whose boundary conditions are in the form

$$x(1) = 0, \quad \dot{x}(1) = 1.$$

Hint: With the effective initial approximation for $x(0)$ from the boundary conditions to the previous equation, we construct $x_0(t)$ as $x_0(t) = t - 1$.

3.12 A particle of mass m_1 is attached to a light rigid rod of length l, which is free to rotate in the vertical plane as shown below (see Fig. 3.8). A bead of mass m_2 is free to slide along the smooth rod under the action of the spring. Show that the governing equations are

$$\ddot{u} + \omega_1^2 u - u\dot{\theta}^2 + \omega_2^2(1 - \cos\theta) = \omega_1^2 u_e,$$
$$(1 + mu^2)\ddot{\theta} + (1 + mu)\omega_2^2 \sin\theta + 2mu\dot{u}\dot{\theta} = 0,$$

where $\omega_1^2 = k/\,m$, $\omega_2^2 = g/l$, $m = m_2/m_1$, $u = x/l$, and u_e is the equilibrium position, and then solve it.

3.13 The nonlinear parametric pendulum is described by

$$\frac{d^2\theta}{dt^2} + 2\gamma\frac{d\theta}{dt} + \omega_0^2[1 + h\cos 2(\omega_0 + \varepsilon)t]\sin\theta = 0.$$

For this problem, choose $\omega_0 = 1$. Unless otherwise specified, use $\gamma = 0$ and $\varepsilon = 0$.
The initial conditions are

$$i)\ \dot{\theta}(0) = 0, \quad \theta(0) = 0.01.$$
$$ii)\ \dot{\theta}(0) = 0, \quad \theta(0) = 3.0.$$

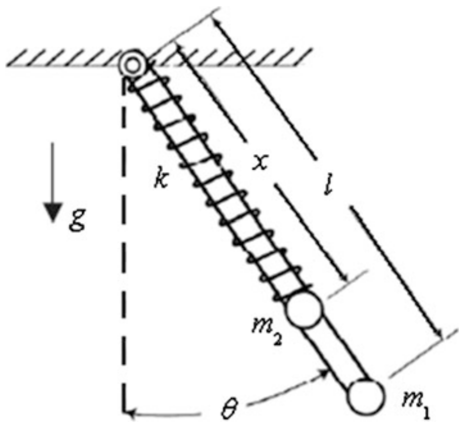

Fig. 3.8 A particle of mass m_1 is attached to a light rigid rod of length l, which is free to rotate in the vertical plane

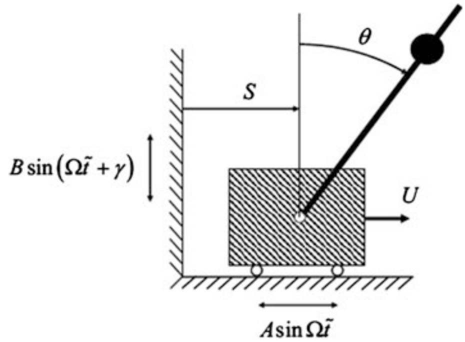

Fig. 3.9 Inverted pendulum balanced by a moving cart

3.14 The motion of a damped pendulum can be described by

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2 \sin\theta = 0,$$

where θ is the angle the pendulum makes with the vertical ($\theta = 0$ is down), γ is a damping factor, and $\omega = \sqrt{g/l}$ is the natural frequency of the pendulum.

3.15 Figure 3.9 shows the standard system normally used to test control algorithms. It contains a cart used to balance a pendulum in the up-pointing position against the gravitation force. The system state can be described through two degrees of freedom—the position of the cart S and pendulum angle θ as observed from the rigid platform. The cart has the mass M and the linear damping coefficient d. The pendulum has the mass m and the torsion inertia J about its center of gravity at distance L from the loss free hinge. The system's reaction to perturbations is governed by a feedback control force $U = U(S, \dot{S}, \theta, \dot{\theta})$. The rigid platform can be excited kinematically relative to the fixed inertial frame.

The system's motion is governed by the equations

$$\ddot{s} + 2\beta\,\dot{s} + \alpha\left(\ddot{\theta}\cos\theta - \ddot{\theta}^2\sin\theta\right) = u + a\omega^2\sin\omega t$$
$$\ddot{\theta} - \left(1 - b\omega^2\sin(\omega t + \gamma)\right)\sin\theta + \ddot{s}\cos\theta = a\omega^2\sin\omega t\cos\theta$$

3.16 For the damped pendulum equation with a forcing term,

$$x + kx + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = F\cos\omega t.$$

3.17 The equation of motion in the Van der Pol plane for the forced, damped pendulum equation is

$$x + kx + x - \frac{1}{6}x^3 = \Gamma\cos\omega t,\ k > 0$$

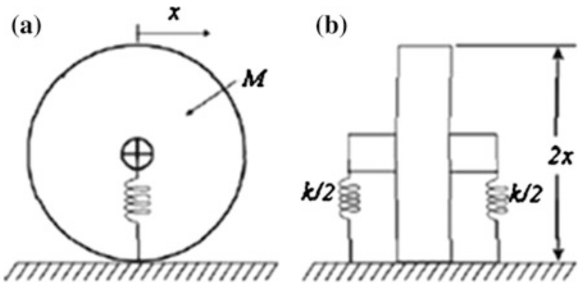

Fig. 3.10 The cylinder rolls back and forth without slip

3.18 For the modal equation in a rotating coordinate system, if the x-, y-coordinate system is rotating relative to a Newtonian frame with angular speed ω, the presence of Coriolis and centripetal accelerations produces the differential equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\omega\frac{\mathrm{d}y}{\mathrm{d}t} - \omega^2 x = -\frac{\partial V}{\partial x}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\omega\frac{\mathrm{d}x}{\mathrm{d}t} - \omega^2 y = -\frac{\partial V}{\partial y}.$$

3.19 The cylinder rolls back and forth without slip, as shown in Fig. 3.10. Fig. 3.10a show that the equation of motion can be written in the form

$$\dot{x} + \omega^2[1 - l(1+x^2)^{-1/2}]x = 0,$$

where $\omega^2 = 2k/3M$ and l is the free length of the spring. All lengths were made dimensionless with respect to the radius r. Fig. 3.10b solve this problem.

3.20 The motion of a particle restrained by a linear Coulomb and square damping is governed by

$$\ddot{u} + \omega_0^2 u + \varepsilon(\mu_0 \mathrm{sgn}\dot{u} + \mu_2 \dot{u}|\dot{u}|) = 0,$$

where $(\mu_0, \mu_2) > 0$ and $\varepsilon \ll 1$.

Show that

$$u = a\cos(\omega_0 t + \beta) + O(\varepsilon),$$

where

$$\dot{a} = -\varepsilon\left(\frac{2\mu_0}{\pi\,\omega_0} + \frac{4}{3\pi}\mu_2\omega_0 a^2\right)$$

and

$$\dot{\beta} = 0.$$

3.21 Consider a two-degree-of-freedom system consisting of two concentrated masses and two springs with a linear damper, under a harmonic excitation

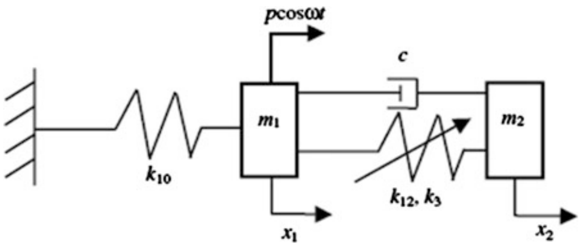

Fig. 3.11 Mechanical model of a two-degrees-of-freedom oscillatory system with cubic nonlinearity

as shown in Fig. 3.11. One of the springs is linear with the stiffness coefficient k_{10}, and another one is a cubic nonlinear spring. The restoring force is defined as

$$f = k_{12}(x_1 - x_2) + k_3(x_1 - x_2)^3.$$

The governing equations of the system can be expressed in the following matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{10}+k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p\cos\omega t - k_3(x_1 - x_2)^3 \\ k_3(x_1 - x_2)^3 \end{bmatrix}.$$

In the above equation, x_1 and x_2 are the displacements of the concentrated masses m_1 and m_2, and $k_{10}, k_{12}, k_3, c, p, \omega, t$ designate the coefficients of linear stiffness, coefficient of nonlinear stiffness, coefficient of damping, excitation amplitude, excitation frequency, and time, respectively. Solve this problem by the present methods.

3.22 In this problem, consider large time behavior of the solutions of the linear PDE problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) -_{txx}(x,t) = 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon[u_x(1,t) + \alpha u_{tx}(1,t) + r\,u_t(1,t)] \end{cases}$$

for $x \in (0,1)$, $t > 0$ α, $\varepsilon \geq 0$ and $r > 0$. In this model, $u(x,t)$ represents the longitudinal displacement at time t of the x particle of a viscoelastic spring. This spring is attached at one end $(x = 0)$ to a fixed wall, and it is attached to a rigid moving body of mass $1/\varepsilon$ at the other end $(x = 1)$. The possible spring inner viscosity or damping is represented by the parameter $\alpha \geq 0$.

3.23 Consider the free oscillation of a suspension system, which is represented schematically in Fig. 3.12 by two bodies of mass m_1 and m_2 linked with each other by a nonlinear spring (k_3), a linear one (k_1), and a shock damper with viscous damping (d_1). Mass m_2 is contacting with the ground through

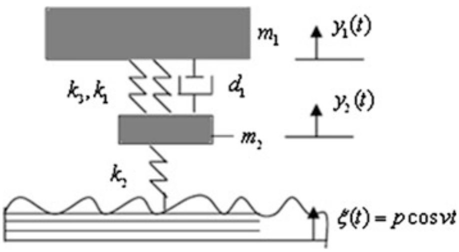

Fig. 3.12 The free oscillation of a suspension system

a linear spring (k_2). The free vibration without damping $d_1 = 0$ is governed by the nonlinear equations

$$\ddot{z}_1 = b_{11}z_1 + b_{12}z_2 + b_{13}z_1^3,$$
$$\ddot{z}_2 = b_{21}z_1 + b_{22}z_2 + b_{23}z_1^3,$$

where it is written as

$$z_1 = y_1 - y_2,\ z_2 = y_2,\ \omega_1^2 = k_1/m_1,\ \omega_2^2 = k_2/m_2,\ \beta = k_3/m_1,\ b_{11} = -\omega_1^2(1+\mu)$$
$$b_{12} = \omega_2^2,\ b_{13} = -\beta(1+\mu),\ b_{21} = \omega_1^2\mu,\ b_{22} = -\omega_2^2,\ b_{23} = \beta\mu,\ \mu = m_1/m_2.$$

3.24 Consider the forced periodic vibration of the suspension system shown in Fig. 3.12, which is governed by the differential equation system

$$\ddot{z}_1 = -\omega_1^2(1+\mu)z_1 + \omega_2^2 z_2 - \beta(1+\mu)z_1^3 - \zeta(1+\mu)\dot{z}_1 - p\cos\nu t,$$
$$\ddot{z}_2 = \omega_1^2\mu z_1 - \omega_2^2 z_2 + \beta\mu z_1^3 + \zeta\mu\dot{z}_1 + p\cos\nu t,$$

where

$$\zeta = d_1/m_1,\ p = k_2(m_1+m_2)/2m_2.$$

3.25 Consider an MEMS translational gyroscope. Focusing attention on the drive direction and considering only rigid modes, the gyroscope's dynamic behavior is equivalent to that of the lumped parameter model shown in Fig. 3.13. The equation of motion of the modal system is

$$m^*\ddot{x} + r^*\dot{x} + k^*x + 4k_3x^3 = F^*.$$

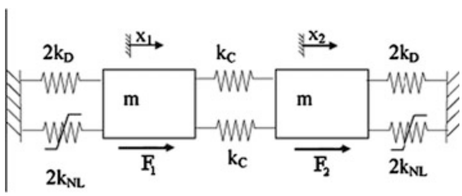

Fig. 3.13 Equivalent lumped-parameter model of the designed gyroscope while moving along drive direction

Since the actuation forces are in counterphase, only one vibration mode is excited. Therefore, using the modal superposition approach, it is possible to further simplify the two-degrees-of-freedom lumped-parameter model to a one-degree-of-freedom modal system having the following mass and stiffness parameter values:

$$m^* = 2m, \quad k^* = 4k_d + 4k_c + 4k_1, \quad k^*_{NL} = 4k_3, \quad F^* = F_1 - F_2.$$

This property is useful to easily synchronize sense and drive resonances, thus increasing the sensibility of the MEMS gyroscope.

3.26 The equation of motion is given by

$$\begin{aligned} &M\ddot{x} + kx(1 + g\,\mathrm{sgn}(x\dot{x})) = 0,\\ &x(0) = a, \quad \dot{x}(0) = 0, \end{aligned}$$

where k is the spring constant and g is the "nonlinearity parameter." The "signum" function is defined as

$$\mathrm{sgn}(\theta) = \begin{cases} +1 & \text{for} \quad \theta > 0 \\ 0 & \qquad\ \ \theta = 0\,. \\ -1 & \text{for} \quad \theta < 0 \end{cases}$$

3.27 We consider the system depicted in Fig. 3.14, composed of a chain of 10 strongly coupled linear oscillators (designated as the "primary system") with a strongly nonlinear (nonlinearizable) end attachment [designated as the nonlinear energy sink (NES)]. The system possesses weak viscous damping, and the mass of the NES is assumed to be small, as compared with the overall mass of the chain. The governing equations of motion of the system are given by:

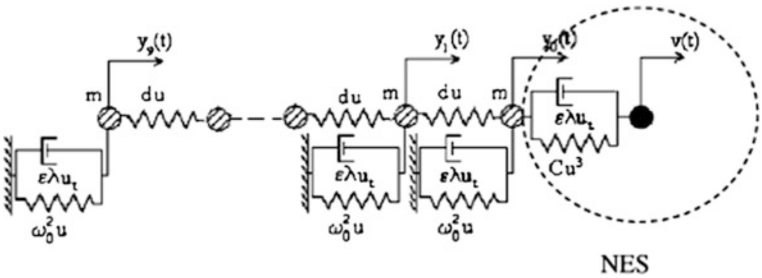

Fig. 3.14 The chain of linear coupled oscillations (the primary system) with strongly nonlinear end attachment (the NES)

$$\begin{aligned}
&\varepsilon\ddot{v} + \varepsilon\lambda(\dot{v} - \dot{y}_0) + C(v - y_0)^3 = 0,\\
&\ddot{y}_0 + \varepsilon\lambda\dot{y}_0 + \omega_0^2 y_0 - \varepsilon\lambda(\dot{v} - \dot{y}_0) - C(v - y_0)^3 + \mathrm{d}(y_0 - y_1) = 0,\\
&\ddot{y}_j + \varepsilon\lambda\dot{y}_j + \omega_0^2 y_j + \mathrm{d}(2y_j - y_{j-1} - y_{j+1}) = 0, \quad j = 1,\ldots 8,\\
&\ddot{y}_9 + \varepsilon\lambda\dot{y}_9 + \omega_0^2 y_9 + \mathrm{d}(y_9 - y_8) = 0,
\end{aligned}$$

where we introduce the small parameter ε, $0 < \varepsilon << 1$ and all other parameters are assumed to be quantities of $O(1)$. In addition, we assume that the system is initially at rest and that an impulse of magnitude F is applied at $t = 0$ at the left boundary of the linear chain, corresponding to the following initial conditions for the system:

$$\begin{aligned}
&v(0) = \dot{v}(0) = 0, \quad y_p(0) = 0, \quad p = 0,\ldots,9,\\
&\dot{y}_k(0) = 0, \quad k = 0,\ldots,8, \quad \dot{y}_9(0+) = F.
\end{aligned}$$

3.28 We consider a nonlinear damping term with a fractional exponent covering the gap between viscous, dry friction, and turbulent damping phenomena. The equation of motion has the form

$$\ddot{x} + \alpha\dot{x}|\dot{x}|^{p-1} + \delta x + \gamma \mathrm{sgn}(x)|x|^{q-1} = \mu\cos\omega t,$$

where x is displacement and $\dot{x}$ velocity, respectively, while the external force is

$$F_x = -\delta x - \gamma \mathrm{sgn}(x)|x|^{q-1},$$

3.29 We consider the stochastic dynamical system

$$\ddot{x} + \left(r + \alpha x^2 - \xi(t)\right)\dot{x} + \alpha x = -bx^3,$$

where $\xi(t)$ is a white noise with intensity D and the parameters a and b are taken to be positive in order to have a stabilizing effect.

3.30 The quadratically-damped Mathieu equation is

$$\ddot{x} + (\delta + \varepsilon\cos t)x + \mu\dot{x}|\dot{x}| = 0,$$

where the parameter μ is assumed to be small.
Guidance: We further expand δ and x as follows:

$$\begin{aligned}
&x = x_0 + \mu x_1 + \mu^2 x_2 + \mu^3 x_3 + \mu^4 x_4 + \mu^5 x_5 + \cdots\\
&\delta = \delta_0 + \mu\delta_1 + \mu^2\delta_2 + \mu^3\delta_3 + \mu^4\delta_4 + \mu^5\delta_5 + \cdots
\end{aligned}$$

And we further introduce the parameter ε_1 defined by

$$\varepsilon = \varepsilon_0 + \mu\varepsilon_1.$$

3.31 The response of a nonlinear system to harmonic excitation is governed by the equation

$$\ddot{x} + 2\varsigma\dot{x}|\dot{x}| + x + \beta\varepsilon x^3 = \cos\frac{\Omega}{\omega_0}t,$$

where $\Omega/\omega_0 \approx 1$. Assume light damping $(\varsigma \ll 1)$ and weak nonlinearity $(0 < \varepsilon \ll 1)$ with

$$\beta = O(1).$$

3.32 The system considered in the present problem consists of a harmonically excited 2d system of linear coupled oscillators (with identical masses) and NES attached to it. By the term NES, we mean a small mass (relative to the linear oscillator mass) attached via essentially a nonlinear spring (pure cubic nonlinearity) and linear viscous damper to the linear subsystem, as illustrated in Fig. 3.15.
As was mentioned above, masses of linear oscillators are identical and, therefore, may be taken as unity without loss of generality $(M = 1)$. The system is described by the following equations:

$$\ddot{y}_2 + k_2y_2 + k_1(y_2 - y_1) = \varepsilon F_2\cos(\omega t),$$
$$\ddot{y}_1 + k_2y_1 + k_1(y_1 - y_2) + \varepsilon k_v(y_1 - v)^3 + \varepsilon\lambda(\dot{y}_1 - \dot{v}) = \varepsilon F_1\cos(\omega t),$$
$$\varepsilon\ddot{v} + \varepsilon k_v(v - y_1)^3 + \varepsilon\lambda(\dot{v} - \dot{y}_1) = 0,$$

where y_1, y_2, v are the displacements of the linear oscillators and NES, respectively, $\varepsilon\lambda$ is the damping coefficient, and $\varepsilon F_i (i = 1, 2)$ are the amplitudes of excitation of each linear oscillator.

3.33 To show the response of a nonlinear oscillator under a harmonic excitation, we consider the weakly nonlinear system

$$\ddot{u} + \mu\dot{u} + \omega^2u + \mu_3\dot{u}^3 + \alpha_2u^2 + \alpha_3u^3 + \alpha_4u^4 + \alpha_5u^5 = F\cos(\Omega t + \gamma),$$

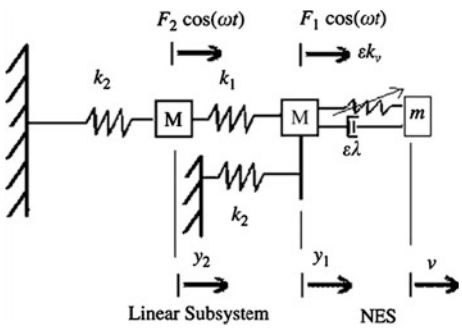

Fig. 3.15 Mechanical model of the system

where $\dot{u} = \mathrm{d}u/\mathrm{d}t$, t is the time, α_i are constants, μ and μ_3 are damping coefficients, F is the excitation amplitude, ω is the linear natural frequency, $\Omega(\approx \omega)$ is the excitation frequency, and γ is the phase angle of the excitation w.r.t. the response.

3.34 Consider a nonlinear oscillator in the form

$$u'' + \omega_n^2 u + \mu u^3 = F_0 \cos(\omega t)$$

with the initial condition

$$u(0) = A,\ u'(0) = 0$$

3.35 Consider the nonlinear cubic-quintic Duffing equation, which reads

$$u'' + f(u) = 0\,, f(u) = \alpha u + \beta u^3 + \gamma u^5$$

with the initial conditions

$$u(0) = A,\ \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0.$$

3.36 We assume that the anchor spring is nonlinear with a force–displacement relation (see Fig. 3.16):

$$f = \delta + \delta^3.$$

The second spring is assumed to be linear with characteristics $f = \delta$. The equations of motion are given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2x - y + x^3 = 0,\quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y - x = F\cos\omega t$$

3.37 Consider the nonlinear oscillator in Fig. 3.17.
This oscillator is very applicable in automobile design where a horizontal motion is converted into a vertical once or vice versa.
The equation of motion and appropriate initial conditions for this case can be given as

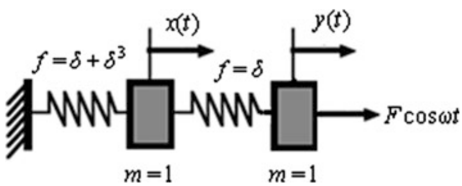

Fig. 3.16 Forced mass–spring system with nonlinear spring

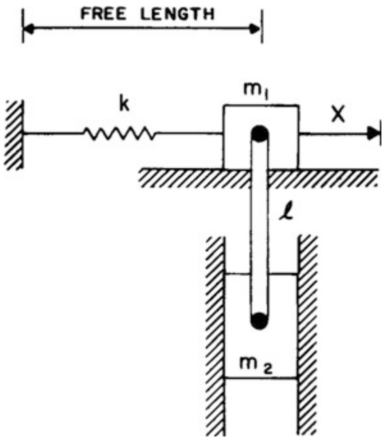

Fig. 3.17 Geometry of the problem

$$(1 + Ru(t)^2)\left(\frac{d^2}{dt^2}u(t)\right) + Ru(t)\left(\frac{d}{dt}u(t)\right)^2 + \omega_0^2 u(t) + \frac{1}{2}\frac{Rgu(t)^3}{l} = 0$$

$$u(0) = A, \frac{du}{dt}(0) = 0,$$

where

$$\omega_0^2 = \frac{k}{m_1} + \frac{Rg}{l}, \quad R = \frac{m_2}{m_1}.$$

3.38 We consider the motion of a ring of mass m sliding freely on the wire described by the parabola $y = qu^2$, which rotates with a constant angular velocity λ about the y-axis. The equation describing the motion of the ring is

$$\ddot{u} + \omega^2 u = -4qu(u\ddot{u} + \dot{u}^2),$$

where $\omega^2 = 2gq - \lambda^2$ and the initial conditions are $u(0) = A, \ \dot{u}(0) = 0$.

3.39 The generalized Huxley equation

$$u_t - u_{xx} = \beta u(1 - u^\delta)(u^\delta - \gamma), \quad 0 \le x \le 1, \quad t \ge 0$$

with the initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}$$

.

3.40 This problem considers a nonlinear oscillator with discontinuity,

$$\frac{d^2x}{dt^2} + \mathrm{sgn}(x) = 0$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0$$

and $\operatorname{sgn}(x)$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ +1, & x \geq 0. \end{cases}$$

3.41 Here, a system consisting of a block of mass m that hangs from a viscous damper with coefficient c and a nonlinear spring of stiffness k_1 and k_3 is considered. The equation of motion is given by the nonlinear differential equation

$$\frac{d^2x(t)}{dt^2} + \frac{k_1}{m}x(t) + \frac{k_3}{m}x^3(t) + \frac{c}{m}\frac{dx(t)}{dt} = 0,$$

with the initial conditions

$$x_0(0) = A, \quad \frac{dx_0}{dt}(0) = 0.$$

3.42 In this problem, we shall consider a system consisting of a (1+1)-dimensional long-wave equation:

$$u_t + uu_x + v_x = 0,$$
$$v_t + (vu)_x + \frac{1}{3}u_{xxx} = 0$$

with the initial conditions of $u(x,0) = f(x)$ and $v(x,0) = g(x)$, where v is the elevation of the water wave and u is the surface velocity of water along the x-direction.

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Chapter 4
Introduction of Considerable Oscillatory Systems

In this chapter, we introduce some considerable oscillatory systems, including Duffing's oscillation systems, Van der Pol oscillator systems, Mathieu's equation, and Ince's equation, with their applications, that are the most important ones for analysis of dynamical and vibratory systems.

4.1 Duffing's Oscillation Systems

4.1.1 Introduction to Duffing's Oscillation

4.1.1.1 Introduction

The differential equation

$$\ddot{x}(t) + x(t) + \varepsilon\,\alpha\,x(t)^3 = 0, \quad \varepsilon > 0 \tag{4.1}$$

is called the *Duffing* oscillator, in which x and t are generalized dimensionless displacement and time variables, respectively, and α and ε are constant parameters in the nonlinear *Duffing* oscillator.

This oscillator is a model of a structural system that includes nonlinear restoring forces (for example, springs). It is sometimes used as an approximation for the pendulum shown in Fig. 4.1:

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta = 0. \tag{4.2}$$

Expanding $\sin\theta = \theta - \frac{\theta^3}{6} + O(\theta^6)$ and then setting $\theta(t) = \sqrt{\varepsilon}\,x(t)$, we get

$$\ddot{x}(t) + \frac{g}{L}\left(x(t) - \varepsilon\,\frac{x(t)^3}{6}\right) = O(\varepsilon^2). \tag{4.3}$$

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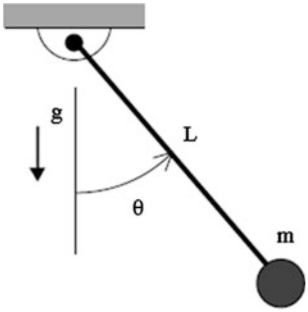
Solid Mechanics and Its Applications 202, DOI: 10.1007/978-94-007-6775-1_4,

Fig. 4.1 Simple pendulum



Now we stretch time with $z = \sqrt{(g/L)}\,t$ and get

$$\ddot{x}(t) + x(t) - \varepsilon\,\frac{x(t)^3}{6} = O(\varepsilon^2), \tag{4.4}$$

which is Eq. 4.1 considering $\alpha = -1/6$.

In order to understand the dynamics of Duffing's Eq. 4.1, we begin by writing it as a first-order system:

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = -x(t) - \varepsilon\alpha\,x(t)^3. \tag{4.5}$$

For a given initial condition $(x(0), y(0))$, Eq. 4.5 specifies a trajectory in the x–y phase plane—that is, the motion of a point in time. The integral curve along which the point moves satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{-x(t) - \varepsilon\alpha\,x(t)^3}{y(t)}. \tag{4.6}$$

Equation 4.6 may be easily integrated to give

$$\frac{y(t)^2}{2} + \frac{x(t)^2}{2} + \varepsilon\,\alpha\frac{x(t)^4}{4} = \text{constant}. \tag{4.7}$$

Equation 4.7 corresponds to the physical principle of conservation of energy. In the case of a positive α, Eq. 4.7 represents a continuum of closed curves surrounding the origin, each of which represents a motion of Eq. 4.1 and is periodic in time. In the case in which α is negative, all motions that start sufficiently close to the origin are periodic. However, in this case, Eq. 4.5 has two additional equilibrium points besides the origin—namely, $x(t) = \pm 1/\sqrt{-\alpha\,\varepsilon},\ \ y(t) = 0$. The integral curves that go through these points separate motions that are periodic from motions that grow unbounded and are called *separatrices* (singular: *separatrix*).

If we were to numerically integrate Eq. 4.1, we would see that the period of the periodic motions depended on which closed curve in the phase plane we were on. This effect is typical of nonlinear vibrations and is referred to as the dependence of period on amplitude. In the next section, we will use a perturbation method to investigate this. Figure 4.2 shows a phase plan for Duffing's equation at different αs.

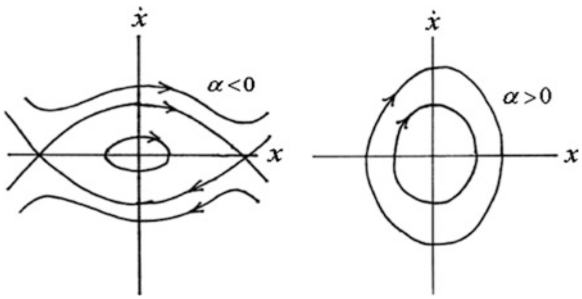


Fig. 4.2 Phase plan for Duffing's equation

4.1.1.2 Solution Procedures Using Analytical Approaches

In this section, we solve Duffing's equation using some analytical approaches.

Parameterized Perturbation Method

Parameterized perturbation method is a perturbation technique, where the coefficients in an equation are also expressed in power of the artificial parameter, which can be used to derive the relationship between period and amplitude in Duffing's oscillator (Eq. 4.1).

In order to use the traditional perturbation methods, it is necessary to introduce an artificial small parameter β. We let

$$x(t) = \beta\, v(t) \tag{4.8}$$

in Eq. 4.1 and obtain

$$\ddot{v}(t) + 1.v(t) + \alpha\,\varepsilon\,\beta^2 v(t)^3 = 0, \quad v(0) = A/\beta \quad \dot{v}(0) = 0. \tag{4.9}$$

Suppose that the solution of Eq. 4.8 and the coefficient, 1, can be expressed in the forms

$$v(t) = v_0(t) + \beta^2 v_1(t) + \beta^4 v_2(t) + \cdots, \tag{4.10}$$

$$1 = \omega^2 + \beta^2\,\omega_1 + \beta^4\,\omega_2 + \cdots. \tag{4.11}$$

Substituting Eqs. 4.10 and 4.11 into Eq. 4.9 and equating the terms with the identical powers of β yields the equations

$$\ddot{v}_0(t) + \omega^2\, v_0(t) = 0, \quad v_0(0) = A/\beta, \quad \dot{v}_0(0) = 0, \tag{4.12}$$

$$\ddot{v}_1(t) + \omega^2 v_1(t) + \omega_1 v_0 + \alpha\,\varepsilon\, v_0^3 = 0, \quad v_1(0) = 0, \quad \dot{v}_1(0) = 0. \tag{4.13}$$

Considering the initial conditions $v(0) = A/\beta$ and $\dot{v}(0) = 0$, the solution of Eq. 4.12 is

$$v_0(t) = \frac{A}{\beta}\cos\omega t. \tag{4.14}$$

Substituting the result into Eq. 4.13 can therefore be rewritten as

$$\ddot{v}_1(t) + \omega^2 v_1(t) + \frac{A}{\beta}\left(\omega_1 + \frac{3\,\alpha\,\varepsilon A^2}{4\beta^2}\right)\cos(\omega t) + \frac{\alpha\,\varepsilon A^3}{4\beta^3}\cos(3\omega t) = 0. \tag{4.15}$$

Avoiding the presence of secular terms requires

$$\omega_1 = -\frac{3\,\alpha\,\varepsilon A^2}{4\beta^2}. \tag{4.16}$$

Solving Eq. 4.15, we obtain

$$v_1(t) = \frac{\alpha\,\varepsilon A^3}{32\omega^2\beta^3}(\cos(3\omega t) - \cos(\omega t)). \tag{4.17}$$

If, for instance, its first-order approximation is sufficient, then we have

$$x(t) = \beta v(t) = \beta\big(v_0(t) + \beta^2 v_1(t)\big) = A\ \cos\omega t + \left(\frac{A^3\alpha\,\varepsilon}{32\omega^2}\right)(\cos(3\omega t) - \cos(\omega t)). \tag{4.18}$$

Substituting Eq. 4.16 into Eq. 4.11, the angular frequency can be written in the form

$$\omega = \frac{1}{2}\sqrt{4 + 3\,\alpha\,\varepsilon A^2}. \tag{4.19}$$

The period $T = 2\pi/\omega$ may then be written as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 + \frac{3\,\alpha\,\varepsilon A^2}{4}}}. \tag{4.20}$$

Variational Approach

In the present part, we repeat the variational approach of Chap. 2, which considers a general nonlinear oscillator in the form

$$\ddot{v}(t) + f(v(t)) = 0. \tag{4.21}$$

Its variational principle can be easily established using the semi-inverse method

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + F(v) \right) \mathrm{d}t, \tag{4.22}$$

where $T = 2\pi/\omega$ is the period of the nonlinear oscillator. Using Eq. 4.22 and $F(v) = \int (\alpha v + \beta v^3)\,\mathrm{d}v$, we get

$$J(v) = \int_0^{T/4} \left(-\frac{1}{2}\dot{v}^2 + \frac{1}{2}\alpha v^2 + \frac{1}{4}\beta v^4 \right) \mathrm{d}t. \tag{4.23}$$

Consider such initial conditions:

$$v(0) = A, \quad \dot{v}(0) = 0. \tag{4.24}$$

Assume that its solution can be expressed as

$$v(t) = A \cos \omega t. \tag{4.25}$$

Substituting Eq. 4.23 into Eq. 4.25 results in

$$\begin{aligned} J(A, \omega) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\alpha A^2 \cos^2 \omega t + \frac{1}{4}\beta A^4 \cos^4 \omega t \right) \mathrm{d}t \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + \frac{1}{2}\alpha A^2 \cos^2 t + \frac{1}{4}\beta A^4 \cos^4 t \right) \mathrm{d}t \end{aligned} \tag{4.26}$$

Applying the Ritz method, we require

$$\begin{aligned} \partial J/\partial A &= 0, \\ \partial J/\partial \omega &= 0. \end{aligned} \tag{4.27}$$

But by a careful inspection, for most cases, we find that

$$\partial J/\partial \omega < 0. \tag{4.28}$$

Thus, we modify the conditions 4.25 and 4.26 into the more simple form

$$\partial J/\partial A = 0. \tag{4.29}$$

Its variational principle can be easily established using the semi-inverse method for the nonlinear Duffing equation (4.1), Using Eq. 4.22 and $F(x) = \int \left(x(t) + \varepsilon\,\alpha\, x(t)^3 \right) \mathrm{d}x$ yields

$$J(x)=\int_0^{T/4}\left(-\frac{1}{2}\dot{x}^2+\frac{1}{2}x^2+\frac{1}{4}\alpha\,\varepsilon\,x^4\right)\mathrm{d}t. \tag{4.30}$$

Substituting Eq. 4.25 into Eq. 4.30 results in

$$\begin{aligned} J(A,\omega)&=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+\frac{1}{2}A^2\cos^2\omega t+\frac{1}{4}\alpha\,\varepsilon A^4\cos^4\omega t\right)\mathrm{d}t\\ &=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2 t+\frac{1}{2}A^2\cos^2 t+\frac{1}{4}\alpha\,\varepsilon A^4\cos^4 t\right)\mathrm{d}t \end{aligned}. \tag{4.31}$$

Then substituting Eq. 4.31 into Eq. 4.29 results in

$$\partial J/\partial A=\frac{A}{\omega}\int_0^{\pi/2}\left(-\omega^2\sin^2 t+\cos^2 t+\alpha\varepsilon A^2\cos^4 t\right)\mathrm{d}t=0. \tag{4.32}$$

This leads to the result

$$\omega=\frac{1}{2}\sqrt{4+3\alpha\,\varepsilon A^2} \tag{4.33}$$

Corresponding to Eq. 4.33, Eq. 4.19 showed absolutely the same result. These results are shown where the variational approach yields an extended rather than a lesser order than the parameterized perturbation method.

Hence, the approximate period is

$$\mathrm{T}=2\,\pi/\omega=4\pi\Big/\sqrt{4+3\alpha\,\varepsilon A^2}. \tag{4.34}$$

Therefore, the analytically approximated displacement $x(t)$ is gained by substituting Eq. 4.33 into Eq. 4.24, from which we can obtain the approximate solution

$$x(t)=A\cos\left(\frac{1}{2}\sqrt{4+3\alpha\,\varepsilon A^2}t\right). \tag{4.35}$$

Discussion and Results

To illustrate and verify accuracy of the variational approach and parameterized perturbation method, a comparison with an exact solution is presented. The exact frequency ω_e for a dynamic system governed by Eq. 4.1 can be derived as shown in Eq. 4.36:

$$\omega_e(A) = 2\pi \left(4 \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - \frac{\alpha\varepsilon}{2} A^2 (1 + \cos^2 t)}} \right)^{-1}. \tag{4.36}$$

We first present a derivation of ω_e.

The exact solution of the dynamical system can be obtained by integrating the governing Eq. 4.1 and imposing the initial conditions $x(0) = A, \quad \dot{x}(0) = 0$, as follows. Equation 4.1 can be expressed as

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{4}\alpha\,\varepsilon\,x^4 = C. \tag{4.37}$$

in which C is a constant. Imposing the previous initial conditions yields

$$C = \frac{1}{2}A^2 + \frac{1}{4}\alpha\,\varepsilon\,A^4. \tag{4.38}$$

Equating Eqs. 4.37 and 4.38 yields

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{(A^2 - x^2) + \frac{\alpha\varepsilon}{2}(A^4 - x^4)}}. \tag{4.39}$$

Integrating Eq. 4.39, the period of oscillation is

$$T(A) = 4 \int_0^{A} \frac{\mathrm{d}x}{\sqrt{(A^2 - x^2) + \frac{\alpha\varepsilon}{2}(A^4 - x^4)}}. \tag{4.40}$$

Substituting $x = -A \cos t$ into Eq. 4.40 and integrating,

$$T(A) = 4 \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - \frac{\alpha\varepsilon}{2} A^2 (1 + \cos^2 t)}}. \tag{4.41}$$

The exact frequency ω_e is also a function of A and can be obtained from the period of the motion as

$$\omega_e(A) = 2\pi \left(4 \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1 - \frac{\alpha\varepsilon}{2} A^2 (1 + \cos^2 t)}} \right)^{-1}. \tag{4.42}$$

It should be noted that ω_e contains an integral that can only be solved numerically, in general.

The corresponding analytical approximation results between the zero-order variational approach and the first-order parameterized perturbation method with exact solution are tabulated in Table 4.1 for different values of ε and a constant value $A = 1$ and also $\alpha = -1/6$ for the Duffing oscillator, where, corresponding to

Table 4.1 Comparison of angular frequencies in Eq. 4.1 from various approximations of analytical methods with the exact solution

Constants			Results			
A	α	ε	Exact solution ω_e	Variational approach ω	Parameterized perturbation ω	Percentage error
1	−1/6	0.1	0.98766	0.99373	0.99373	0.61
1	−1/6	0.5	0.94119	0.96825	0.96825	2.80
1	−1/6	1	0.88889	0.93542	0.93542	4.98
1	−1/6	3	0.72727	0.79058	0.79058	8.00
1	−1/6	5	0.66667	0.70710	0.70710	5.72

Eq. 4.34, these results are absolutely the same as those reported in Eq. 4.19. From the table, the percentages of errors of the variational approach and parameterized perturbation method are 0.61 and 2.8 % for $\varepsilon = 0.01$ and $\varepsilon = 0.05$, respectively. Of course, the accuracy can be improved upon using higher-order approximate solutions for approximation methods. The lower-order approximated solutions are of a high accuracy, and the percentage of error improves significantly from lower-order to higher-order analytical approximations for different parameters and initial amplitudes. Hence, we conclude that we can provide excellent agreement with the exact solutions for the nonlinear Duffing equation.

To further illustrate and verify the accuracy of these approximate analytical approaches, a comparison of the time history oscillatory displacement responses for a nonlinear Duffing's oscillator with the exact solution is presented in Figs. 4.3, 4.4. The figures represent the displacement $x(t)$ for the undamped nonlinear Duffing's oscillator including nonlinear restoring forces. Apparently, it is confirmed that the lower-order analytical approximations show excellent agreement with the exact solution, using a Jacobin elliptic function.

4.1.2 The Forced Duffing Oscillator

4.1.2.1 Introduction

The differential equation (see Rand 2005)

$$\frac{d^2x}{dt^2} + x + \varepsilon c \frac{dx}{dt} + \varepsilon \alpha x^3 = \varepsilon F \cos \beta\omega t \tag{4.43}$$

is called the forced Duffing equation. It is used to model the forcing of a damped elastic structure when the displacements are sufficiently large to make nonlinear elastic effects significant. In contrast to the unforced Duffing equation (4.35), Eq. 4.43 is nonautonomous; that is, time t explicitly appears in the equation in the $\cos \omega t$ term. The phase plane is no longer a suitable arena in which to investigate this equation, since the vector field at a given point changes in time, allowing a

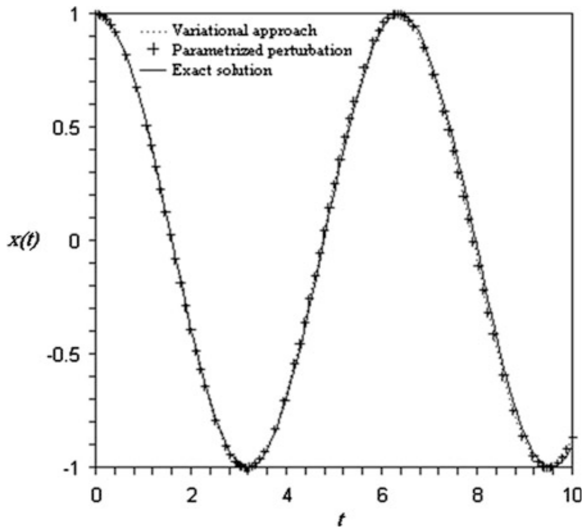


Fig. 4.3 Comparison of the analytical approximates with the exact solution for $A = 1, \alpha = -1/6, \varepsilon = 0.1$

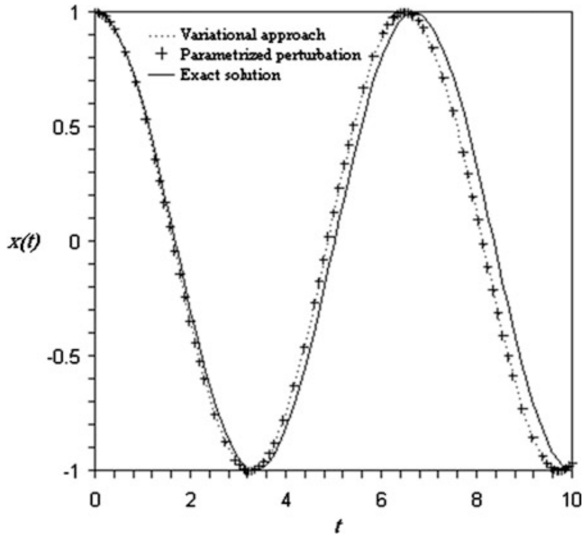


Fig. 4.4 Comparison of the analytical approximates with the exact solution for $A = 1, \alpha = -1/6, \varepsilon = 0.5$

trajectory to return to that point and intersect itself. The system may be made autonomous, however, by increasing its dimension by 1:

$$\frac{dx}{dt} = y, \tag{4.44}$$

$$\frac{dy}{dt} = -x - \varepsilon c y - \varepsilon \alpha x^3 + \varepsilon F \cos z. \tag{4.45}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \omega. \tag{4.46}$$

This system of three first-order ODEs is defined on a phase space with topology $R^2 \times S$, where the circle S comes from the fact that the vector field of Eqs. 4.44–4.46 is 2π—periodic in z.

A convenient scheme for viewing this three-dimensional flow in two dimensions is by way of a Poincaré map M. This map is generated by the flow's intersection with a surface of section $\sum$, which may be taken as $\sum : z = 0 \pmod{2\pi}$. The Poincaré map $M : \sum \to \sum$ is defined as follows: Let p be a point on $\sum$ and, using it as an initial condition for the flow Eqs. 4.44–4.46, let the resulting trajectory evolve in time until $z = 2\pi$—that is, until it once again intersects $\sum$, this time at some point q. Then M maps p to q. Note that a fixed point of the Poincaré map corresponds to a 2π periodic motion of the flow. In the cases of Eqs. 4.44–4.46 when $F = 0$, we could still use this setup, even though, in that case, the system would be autonomous and the phase plane would be more appropriate. We use the three-dimensional space instead, in order to draw conclusions about the $F > 0$ case from the structure of the $F = 0$ case. Thus, when $F = 0$, the equilibria that would normally lie in the x–y phase plane now become closed loops in the $R^2 \times S$ phase space—that is, "periodic" orbits of period 2π. If we now allow F to be nonzero, a continuity argument may be expected to yield the result that each of these periodic orbits continues to persist, giving rise to the conclusion that for each equilibrium point of the $F = 0$ system, there is a 2π periodic motion of the $F > 0$ system, at least for small enough Fs. Such a periodic motion would be a limit cycle in the $R^2 \times S$ phase space and a fixed point in the Poincaré map. The *continuity argument* is called structural stability and offers conditions under which this story holds true. The equilibria in the autonomous system must be hyperbolic; that is, the linearized constant coefficient system valid in the neighborhood of a given equilibrium point must have no eigenvalues with zero real part.

4.1.2.2 Two-Variable Expansion Method

In this section, we use a perturbation method to investigate the dynamics of Eq. 4.43 for small values of ε. We could use averaging for this purpose, but instead we use another method that is equivalent to first-order averaging. The idea of the method is that the expected form of solution of many nonlinear vibration problems involves two time scales: the time scale of the periodic motion itself and a slower time scale that represents the approach to the periodic motion. The method proposes to distinguish between these two time scales by associating a separate independent (time-like) variable with each one. We will use the notation that ξ represents stretched time ωt and η represents slow time εt:

$$\xi = \omega t, \quad \eta = \varepsilon t. \tag{4.47}$$

In order to substitute these definitions into the forced Duffing equation (4.43), we need expressions for the first and second derivatives of x with respect to t. We obtain these by using the chain rule:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial \xi}\frac{\mathrm{d}\xi}{\mathrm{d}t} + \frac{\partial x}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \omega\frac{\partial x}{\partial \xi} + \varepsilon\frac{\partial x}{\partial \eta}, \tag{4.48}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \omega^2\frac{\partial^2 x}{\partial \xi^2} + 2\omega\varepsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \varepsilon^2\frac{\partial^2 x}{\partial \eta^2}. \tag{4.49}$$

Substituting Eqs. 4.48 and 4.49 into Eq. 4.43 gives the partial differential equation

$$\omega^2\frac{\partial^2 x}{\partial \xi^2} + 2\omega\varepsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \varepsilon^2\frac{\partial^2 x}{\partial \eta^2} + x + \varepsilon c\left(\omega\frac{\partial x}{\partial \xi} + \varepsilon\frac{\partial x}{\partial \eta}\right) + \varepsilon\alpha x^3 = \varepsilon F\cos\xi. \tag{4.50}$$

Next, we expand x and ω in power series:

$$x(\xi,\eta) = x_0(\xi,\eta) + \varepsilon x_1(\xi,\eta) + \cdots, \quad \omega = 1 + k_1\varepsilon + \cdots. \tag{4.51}$$

Substituting Eq. 4.50 into Eq. 4.51 and neglecting terms of $O(\varepsilon^2)$ gives, after collecting terms,

$$\frac{\partial^2 x_0}{\partial \xi^2} + x_0 = 0, \tag{4.52}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = -2\frac{\partial^2 x_0}{\partial\xi\partial\eta} - 2k_1\frac{\partial^2 x_0}{\partial \xi^2} - c\frac{\partial x_0}{\partial \xi} - \alpha x_0^3 + F\cos\xi. \tag{4.53}$$

We take the general solution to Eq. 4.52 in the form

$$x_0(\xi,\eta) = A(\eta)\cos\xi + B(\eta)\sin\xi. \tag{4.54}$$

Note here that the *constants* of integration A, B are, in fact, arbitrarily a function of slow time η, since Eq. 4.52 is a PDE. Substituting Eq. 4.54 into Eq. 4.53 and simplifying the resulting trig terms, we obtain an equation of the form

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = (\cdots)\sin\xi + (\cdots)\cos\xi + \text{nonresonant terms}. \tag{4.55}$$

For no resonant terms, we require the coefficients of sin ξ and cos ξ to vanish, giving the following slow flow:

$$2\frac{\mathrm{d}A}{\mathrm{d}\eta} + cA + 2k_1B - \frac{3}{4}\alpha B(A^2 + B^2) = 0, \tag{4.56}$$

$$2\frac{\mathrm{d}B}{\mathrm{d}\eta} + cB + 2k_1A - \frac{3}{4}\alpha A(A^2 + B^2) = F. \tag{4.57}$$

Equilibrium points of the slow flow Eqs. 4.56 and 4.57 correspond to periodic motions of the forced Duffing equation (4.43). To determine them, set $\mathrm{d}A/\mathrm{d}\eta$ and $\mathrm{d}B/\mathrm{d}\eta$ to zero. Multiplying Eq. 4.56 by A and adding it to Eq. 4.57 multiplied by B gives

$$R^2c = BF, \quad \text{where} \quad \mathrm{R}^2 = \mathrm{A}^2 + \mathrm{B}^2. \tag{4.58}$$

Similarly, multiplying Eq. 4.56 by B and subtracting it from Eq. 4.57 multiplied by A gives

$$-2k_1R^2 + \frac{3}{4}\alpha R^4 = AF. \tag{4.59}$$

Squaring Eq. 4.58 and adding it to the square of Eq. 4.59 gives

$$R^2\left(c^2 + \left(-2k_1 + \frac{3}{4}\alpha R^2\right)^2\right) = F^2. \tag{4.60}$$

Equation 4.60 may be solved for k_1, which, with Eq. 4.51, gives the following relation between the response amplitude R and the frequency ω of the periodic motion:

$$\omega = 1 + \frac{3}{8}\varepsilon\alpha R^2 \pm \varepsilon\frac{1}{2}\sqrt{\frac{F^2}{R^2} - c^2}. \tag{4.61}$$

Note that if both the forcing F and the damping c are zero, then Eq. 4.61 gives ω to be a single-valued function of R. The resulting curve, when plotted in the $\omega - R$ plane, is called a *backbone curve*. If $c = 0$ but $F > 0$, then Eq. 4.61 gives ω to be a double-valued function of R that is valid for every R. On the other hand, if both $F > 0$ and $c > 0$, then Eq. 4.61 gives ω to be a double-valued function of R, which, however, is valid only for $R < F/c$.

The slow flow Eqs. 4.56 and 4.57 may also be used to determine the stability of these periodic motions (which correspond to slow flow equilibria). We do so in the special case of zero damping. Setting $c = 0$ in Eqs. 4.56 and 4.57, we obtain

$$\frac{\mathrm{d}A}{\mathrm{d}\eta} = -k_1B + \frac{3}{8}\alpha B(A^2 + B^2), \tag{4.62}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\eta} = k_1A - \frac{3}{8}\alpha A(A^2 + B^2) + \frac{F}{2}. \tag{4.63}$$

Equations 4.62 and 4.63 have equilibria at

$$B = 0, \quad A = \pm R, \quad \text{where} \quad \mathrm{k}_1 = \frac{3}{8}\alpha\mathrm{R}^2 \mp \frac{\mathrm{F}}{2\mathrm{R}}, \tag{4.64}$$

where we use the convention that $R > 0$. In order to determine the stability of these equilibriums, we set $B = u$ and $A = \pm R + v$ and linearize the resulting equations in u, v, giving:

$$\frac{\mathrm{d}v}{\mathrm{d}\eta} = \left(\frac{3}{8}\alpha R^2 - k_1\right)u, \quad \frac{\mathrm{d}u}{\mathrm{d}\eta} = \left(-\frac{9}{8}\alpha R^2 + k_1\right)v. \tag{4.65}$$

From Eq. 4.65, we see that the equilibrium is a center if

$$\left(\frac{3}{8}\alpha R^2 - k_1\right)\left(\frac{9}{8}\alpha R^2 - k_1\right) > 0. \tag{4.66}$$

If this same quantity is negative, the equilibrium is a saddle. Equation 4.66 can be simplified by using Eq. 4.64 to eliminate k_1, giving that the equilibrium is a center if

$$\pm\frac{F}{2R}\left(\frac{3}{4}\alpha R^2 \pm \frac{F}{2R}\right) > 0. \tag{4.67}$$

Now let's consider each branch separately. For the upper sign, $A = +R > 0$, and condition 4.67 is satisfied so that the equilibrium is a center. For the lower sign, $A = -R < 0$, and condition 4.67 states that the equilibrium is a center if

$$\frac{3}{4}\alpha R^2 - \frac{F}{2R} < 0. \tag{4.68}$$

Equation 4.68 can be simplified by using Eq. 4.61, which, in this case, may be written as

$$\omega = 1 + k_1\varepsilon = 1 + \frac{3}{8}\varepsilon\alpha R^2 + \frac{F\varepsilon}{2R}. \tag{4.69}$$

Differentiating Eq. 4.69 with respect to R, we obtain

$$\frac{\mathrm{d}\omega}{\mathrm{d}R} = \varepsilon\left(\frac{3}{4}\alpha R - \frac{F}{2R^2}\right). \tag{4.70}$$

The comparison of Eq. 4.70 with Eq. 4.68 shows that the slow flow equilibrium point corresponding to the lower sign in Eq. 4.64 will be a center if $\frac{\mathrm{d}\omega}{\mathrm{d}R} < 0$ and a saddle if $\frac{\mathrm{d}\omega}{\mathrm{d}R} > 0$.

If we imagine the forcing frequency ω to be varied quasistatically, then as it attains the value at which $\frac{\mathrm{d}\omega}{\mathrm{d}R} = 0$, a saddle-node bifurcation occurs in which the saddle and center (which have been shown to occur for parameters that satisfy Eq. 4.69 merge and disappear. The number of slow flow equilibria will have changed from three to one, and a motion that was circulating around the bifurcating center would now find itself circulating around the other center. If the system included some damping, $c > 0$, the centers would become stable spirals, and a motion that had been close to the bifurcating spiral would, after the bifurcation, find itself approaching the remaining spiral. This motion is known as a jump phenomenon. Before the bifurcation, each of the stable spirals had its own basin of attraction—that is, its own set of initial conditions, which would approach it as $t \to \infty$. As the bifurcation occurs, the basin of attraction of the bifurcating spiral disappears along with the spiral itself, and a motion originally in that basin of attraction now finds itself in the basin of attraction of the remaining spiral. If the forcing frequency were now to reverse its course (again quasistatically), the bifurcation would occur in reverse, and the saddle and spiral pair would be reborn, and, with them, the basin of attraction of the spiral would reappear. However, now the motion that was originally in the basin of attraction of the bifurcating spiral has been relocated into the basin of attraction of the other spiral, where it remains. When the value of ω has returned to its original value, the motion in question will have moved from one basin of attraction to the other. This process is called *hysteresis*.

4.1.3 Universalization and Superposition in Duffing's Oscillator

Nonlinear systems do not obey the same superposition law that linear systems do. One of the main violations of superposition in nonlinear systems is the interaction between the free (complementary) and forced (particular) responses. Moreover, the free and forced responses of nonlinear systems interact to varying degrees depending on the system and operating parameters. For instance, consider the single-degree-of-freedom (SDOF) Duffing oscillator in Fig. 4.5. The output equation of motion that describes this system is given by (see Philips 2006)

$$m\ddot{x} + c\dot{x} + kx + k_n x^3 = f(t). \tag{4.71}$$

Assume that a sinusoidal excitation is used. For a particular forcing frequency and given mass, damping, and stiffness parameters, the steady state response of this system is sensitive to the initial conditions. Figure 4.6 shows the steady state response of the system in Eq. 4.71 to a sinusoidal excitation with a frequency that is slightly more than the undamped natural frequency of the underlying linear system. Note that for different initial conditions, the steady state response amplitude of the underlying system is invariant, whereas the response amplitude of the nonlinear system can be different. This phenomenon is called a *pitchfork* bifurcation or *jump catastrophe* and will be more thoroughly discussed later in the

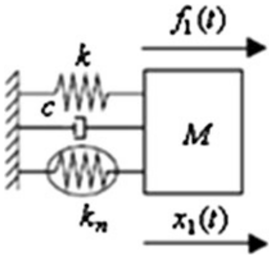


Fig. 4.5 Single-degree-of-freedom Duffing oscillator

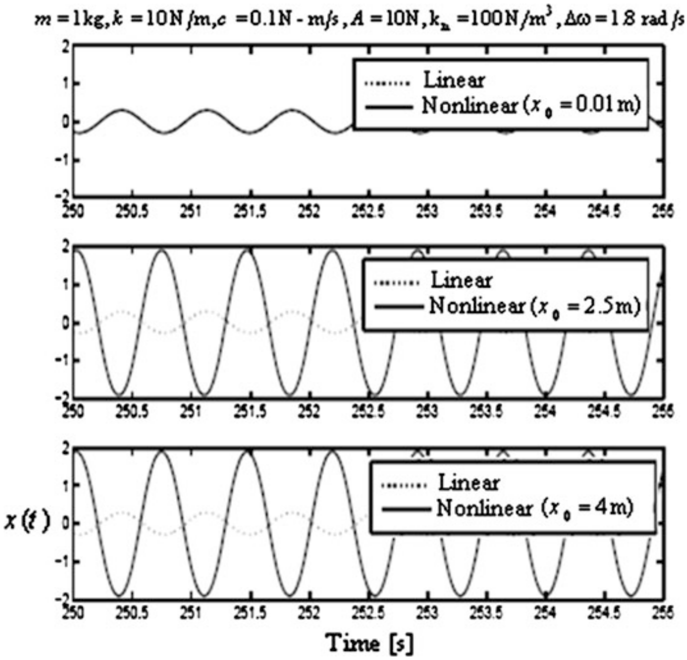


Fig. 4.6 Steady state response of the single-degree-of-freedom Duffing oscillator to a sinusoidal excitation for different sets of initial conditions. Underlying linear system response, $(\cdots)$; nonlinear system response, $(-)$

course. The important point is that the steady state response of the nonlinear system depends on the initial conditions and other parameters as well.

Although nonlinear systems do not obey the principle of superposition in the traditional sense, they do obey their own kind of nonlinear superposition principle. In fact, for a given simulated or measured set of inputs and outputs, "superposition" holds so long as the external inputs and the internal feedback forces due to the nonlinearities are combined to form the total external input to the underlying linear system:

$$\text{Total force} = \text{External force} + \text{Nonlinear internal force}\,. \tag{4.72}$$

This nonlinear superposition principle is tantamount to a simple feedback loop between the external inputs and the system outputs (Fig. 4.7). For example, the equation for the SDOF Duffing oscillator in Eq. 4.71 can be rewritten as follows to directly account for the superposition of the external and nonlinear internal forces:

$$m\ddot{x} + c\dot{x} + kx = f(t) - k_n x^3 = f(t) - f_n(t). \tag{4.73}$$

The output equation of motion for the Duffing oscillator was given in Eq. 4.71, which is rewritten below for reference:

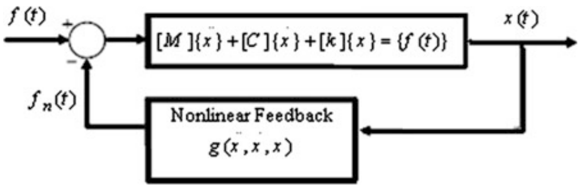


Fig. 4.7 Superposition of external inputs and internal forces due to nonlinearities

$$
\begin{aligned}
m\ddot{x} + c\dot{x} + kx - f_n(x) &= f(t)\\
m\ddot{x} + c\dot{x} + kx + \alpha k_n x^3 &= f(t).
\end{aligned}
\tag{4.74}
$$

The parameters associated with the underlying linear system are the mass m, viscous damping c, and stiffness k, α and k_n are associated with the nonlinear stiffness characteristic. When α is positive ($\alpha = 1$), the system is said to exhibit hardening stiffness. For negative $\alpha(\alpha = -1)$, the system exhibits softening stiffness. This terminology can be better understood by rewriting Eq. 4.74 in a slightly different form as follows:

$$
\begin{aligned}
m\ddot{x} + c\dot{x} + kx + k_n x^3 &= f(t)\\
m\ddot{x} + c\dot{x} + (k + \alpha k_n x^2)x &= f(t)\\
m\ddot{x}(t) + c\dot{x}(t) + (k + n(x))x(t) &= f(t),
\end{aligned}
\tag{4.75}
$$

in which $k + n(x)$ is the nonlinear stiffness associated with a unit displacement in $x(t)$. Plots of both $kx + f_n(x)$ and $k + n(x)$ are shown in the top and bottom of Fig. 4.8 for positive, negative, and zero α with $k = 2\,\text{N/m}$ and $k_n = 1\,\text{N/m}$. Note that the stiffness for the linear system $(-)$ is constant, whereas the hardening and softening stiffnesses for nonlinear systems vary with amplitude.

4.1.3.1 Unforced (Homogeneous)

The free vibration characteristics of the Duffing oscillator in Eq. 4.74 can be very different from those of the underlying linear system ($\alpha = 0$). One feature of the free vibration response of this system is illustrated in Fig. 4.9. It was demonstrated there that higher harmonics are created in the response due to internal feedback, as in Fig. 4.7.

A second feature of the free vibration response of nonlinear systems, frequency modulation, was illustrated in conjunction with Fig. 4.8. There, it was demonstrated that the restoring force in a simple pendulum is tantamount to a softening stiffness and that the natural frequency of oscillation decreased with increasing amplitude. In contrast, consider the system in Eq. 4.74 with $\alpha = 1$ (hardening stiffness) and $f(t) = 0\,N$:

$$
m\ddot{x} + c\dot{x} + kx + k_n x^3 = 0. \tag{4.76}
$$

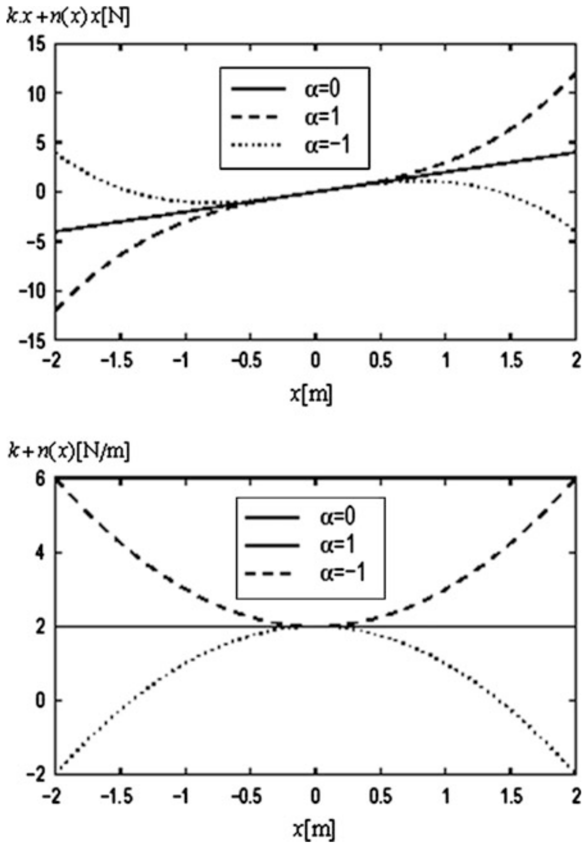


Fig. 4.8 *Top* total internal force due to the linear and nonlinear stiffness $kx - f_n(x)$. *Bottom* linear and nonlinear stiffness for Duffing oscillators with hardening and softening stiffness, k and $k + n(x)$

Figure 4.9 shows the free vibration response to the initial conditions, $x(0) = 1\,\text{m}$ and $\dot{x}(0) = 0\,\text{m/s}$. The time history indicates that the frequency of oscillation is large for large amplitudes and small for small amplitudes, which reflects the hardening stiffness nonlinearity in the system. Note that the frequency spectrum of the response history (bottom) extends across a frequency range from 1 to 12 Hz. It is also clear from the spectrum that larger frequencies correspond to larger amplitude oscillations. This is consistent with the free decay and frequency variations in the time history.

4.1.3.2 Forced (Nonhomogeneous)

The forced Duffing oscillator with hardening stiffness, $m\ddot{x} + c\dot{x} + kx + k_n x^3 = f_0 \cos(\omega_0 t)$, can respond in many different ways depending on the size of the nonlinear internal force, $k_n x^3$, relative to sizes of the internal linear forces and the amplitudes of the excitation and the response. This idea will be discussed later in

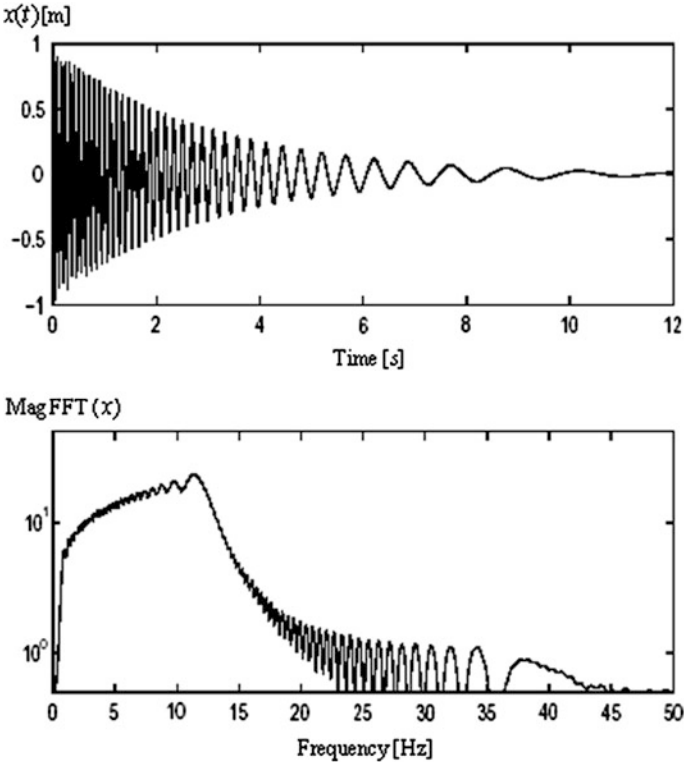


Fig. 4.9 *Top* free (complementary) response of system in Eq. 4.76 to $x(0) = 1\,\text{m}$ and $\dot{x}(0) = 0\,\text{m/s}$ initial condition. *Bottom* magnitude of FFT spectrum (fourier series coefficient)

the course in conjunction with perturbation and other methods of solving for the responses of nonlinear systems.

For now, consider the variation in the steady state response of this system with $m = 1\,\text{kg}$, $c = 0.5\,(\text{N.s})/\text{m}$, $k = 10\,\text{N/m}$, $k_n = 100\,\text{N/m}^3$, and $f_0 = 1\,\text{N}$ to a slowly swept sinusoidal excitation (chirp). The time history and frequency spectrum of the excitation are shown in the top left and right of Fig. 4.10. The slow chirp varies in frequency from $2\,\text{rad/s}$ to $6\,\text{rad/s}$. The responses of the underlying linear system $\left(k_n = 0\,\text{N/m}^3\right)$ and the nonlinear system $\left(k_n = 100\,\text{N/m}^3\right)$ are shown in the middle-left and bottom-left plots. Note that the linear system response passes through the resonance condition smoothly, as expected. The frequency content of the linear response is shown in the middle-right plot, which also indicates that the transition is smooth through resonance.

In contrast, the nonlinear system steady state response amplitude increases until the excitation frequency is approximately 5 rad/s and then becomes unstable and suddenly transitions to a lower steady state amplitude. When the higher amplitude response becomes unstable, the system jumps to the lower response amplitude and resumes a stable steady response.

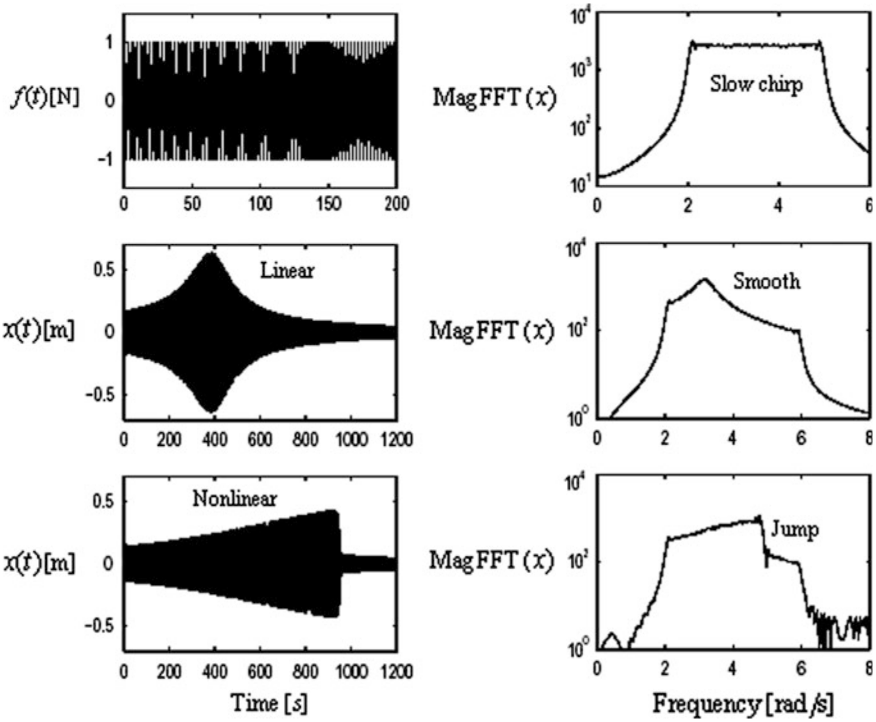


Fig. 4.10 *Top–left*, *right* slow swept sine (chirp) excitation and spectrum. *Middle–left*, *right* linear response and spectrum. *Bottom–left*, *right* nonlinear response and spectrum for Duffing oscillator with hardening stiffness showing jump bifurcation near 5 rad/s

4.2 The Van der Pol Oscillator Systems

4.2.1 The Unforced Van der Pol Oscillator

The following differential equation,

$$\frac{d^2x}{dt^2} + x - \varepsilon(1 - x^2)\frac{dx}{dt} = 0, \quad \varepsilon > 0, \tag{4.77}$$

is called the *Van der Pol oscillator*. It is a model of a nonconservative system in which energy is added to and subtracted from the system in an autonomous fashion, resulting in a periodic motion called a *limit cycle*. Here, we can see that the sign of the damping term, $-\varepsilon(1 - x^2)\frac{dx}{dt}$, changes, depending upon whether $|x|$ is larger or smaller than unity. Van der Pol's equation has been used as a model for stick–slip oscillations, aero-elastic flutter, and numerous biological oscillators, to name but a few of its applications (Rand 2005).

Numerical integration of Eq. 4.77 shows that every initial condition (except $x = \frac{dx}{dt} = 0$) approaches a unique periodic motion. The nature of this limit cycle is dependent on the value of ε For small values of ε, the motion is nearly sinusoidal, whereas for large values of ε, it is a relaxation oscillation, meaning that it tends to resemble a series of step functions, jumping between positive and negative values twice per cycle. If we write Eq. 4.78 as a first-order system,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \varepsilon(1 - x^2)y. \tag{4.78}$$

We find that there is no exact closed form solution. Numerical integration shows that the limit cycle is a closed curve enclosing the origin in the x–y phase plane. From the fact that Eq. 4.78 are invariant under the transformation $x \mapsto -x,\ y \mapsto -y$, we may conclude that the curve representing the limit cycle is point symmetric about the origin.

4.2.1.1 Hopf Bifurcations

Before proceeding to further examine the properties of the Van der Pol equation, we pause to consider how a limit cycle may get born. In particular, we consider the following equation, which is a generalization of Van der Pol's equation:

$$\begin{aligned}\frac{d^2z}{dt^2} + z = c\frac{dz}{dt} + \alpha_1 z^2 + \alpha_2 z\frac{dz}{dt} + \alpha_3\left(\frac{dz}{dt}\right)^2 + \beta_1 z^3 + \beta_2 z^2\frac{dz}{dt}\\ + \beta_3 z\left(\frac{dz}{dt}\right)^2 + \beta_4\left(\frac{dz}{dt}\right)^3,\end{aligned} \tag{4.79}$$

where c is the coefficient of linear damping, where the α_is are coefficients of quadratic nonlinear terms and where the β_is are coefficients of cubic nonlinear terms. For some values of these parameters, Eq. 4.79 may exhibit a limit cycle, whereas for other values, it may not. We are interested in understanding how such a periodic solution can be born as the parameters are varied.

We shall investigate this question by using Lindstedt's method. We begin by introducing a small parameter ε into Eq. 4.79 by the scaling $z = \varepsilon x$, which gives

$$\begin{aligned}\frac{d^2x}{dt^2} + x = c\frac{dx}{dt} + \varepsilon\left[\alpha_1 x^2 + \alpha_2 x\frac{dx}{dt} + \alpha_3\left(\frac{dx}{dt}\right)^2\right]\\ + \varepsilon^2\left[\beta_1 x^3 + \beta_2 x^2\frac{dx}{dt} + \beta_3 x\left(\frac{dx}{dt}\right)^2 + \beta_4\left(\frac{dx}{dt}\right)^3\right].\end{aligned} \tag{4.80}$$

There remains the question of how to scale the coefficient of linear damping c. Let us expand c in a power series in ε:

$$c = c_0 + c_1\varepsilon + c_2\varepsilon^2 + \cdots. \tag{4.81}$$

In order to perturb the simple harmonic oscillator, we must take $c_0 = 0$ Next, consider c_1. As we shall see, although the quadratic terms are of $O(\varepsilon)$, their first contribution to secular terms in Lindstedt's method occurs at $O(\varepsilon^2)$. Thus, if c_1 were not zero, the perturbation method would fail to obtain a limit cycle regardless of the values of the α_i and β_i coefficients. Physically speaking, the damping would be too strong, relative to the nonlinearities, for a limit cycle to exist.

Thus, we scale the coefficient c to be $O(\varepsilon^2)$, and we set $c = \varepsilon^2\mu$:

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = \varepsilon\left[\alpha_1x^2 + \alpha_2x\frac{\mathrm{d}x}{\mathrm{d}t} + \alpha_3\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right]& \\ + \varepsilon^2\left[\mu\frac{\mathrm{d}x}{\mathrm{d}t} + \beta_1x^3 + \beta_2x^2\frac{\mathrm{d}x}{\mathrm{d}t} + \beta_3x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \beta_4\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3.\right]&\end{aligned} \tag{4.82}$$

In order to apply Lindstedt' s method to Eq. 4.82, we first set $\tau = \omega t$, and then we expand

$$\omega = 1 + k_1\varepsilon + k_2\varepsilon^2 + \cdots, \quad x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2x_2(\tau) + \cdots. \tag{4.83}$$

Substituting Eq. 4.83 into Eq. 4.82 and collecting terms gives

$$\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + x_0 = 0, \tag{4.84}$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2k_1\frac{d^2x_0}{d\tau^2} + \alpha_1x_0^2 + \alpha_2x_0\frac{dx_0}{d\tau} + \alpha_3\left(\frac{dx_0}{d\tau}\right)^2, \tag{4.85}$$

$$\frac{\mathrm{d}^2x_2}{\mathrm{d}\tau^2} + x_2 = 12\,\text{terms which are not listed for brevity.} \tag{4.86}$$

We take the solution to Eq. 4.84 as

$$x_0(\tau) = A\cos\tau. \tag{4.87}$$

Substituting Eq. 4.87 into Eq. 4.85 and simplifying the trig terms requires us to take $k_1 = 0$ for no secular terms, and we obtain the following expression for $x_1(\tau)$:

$$x_1(\tau) = \frac{A^2}{6}(3(\alpha_1 + \alpha_3) + (\alpha_3 - \alpha_1)A\cos 2\tau + \alpha_2\sin 2\tau). \tag{4.88}$$

Substituting these results into the x_2 equation (4.86) and requiring the coefficients of both the $\sin\tau$ and $\cos\tau$ to vanish (for no secular terms), we obtain

$$A = 2\sqrt{\frac{-\mu}{\alpha_2(\alpha_1 + \alpha_3) + \beta_2 + 3\beta_4}}, \tag{4.89}$$

as well as the following expression for k_2:

$$k_2 = \frac{\mu(10\alpha_1^2 + \alpha_2^2 + 10\alpha_1\alpha_3 + 4\alpha_3^2 + 9\beta_1 + 3\beta_3)}{6\alpha_1\alpha_2 + 6\alpha_2\alpha_3 + 6\beta_2 + 18\beta_4}. \tag{4.90}$$

According to this approximate analysis, a limit cycle will exist if the expression (4.89) for the amplitude A is real. This requires that the linear damping coefficient μ have the opposite sign to the quantity S defined by

$$S = \alpha_2(\alpha_1 + \alpha_3) + \beta_2 + 3\beta_4. \tag{4.91}$$

If we imagine a situation in which S is fixed and μ is allowed to vary (quasi-statically), then as μ goes through the value zero, a limit cycle is either created or destroyed. This situation is called a *Hopf bifurcation*. There are two cases, $S > 0$ and $S < 0$. In either case, the stability of the equilibrium point at the origin of the phase plane is influenced only by the sign of μ, and not by the value of the α_is or β_is. This may be seen by rewriting Eq. 4.82 in the form

$$\frac{d^2x}{dt^2} + x - \varepsilon^2\mu\frac{dx}{dt} = \text{linear terms}, \tag{4.92}$$

from which we see that the origin is stable for $\mu < 0$ and unstable for $\mu > 0$. Moreover, the stability of the limit cycle is opposite to the stability of the origin, since motions that leave the neighborhood of the origin must accumulate on the limit cycle because of the two-dimensional nature of the phase plane. Thus, in the case $S < 0$, the limit cycle exists only when $\mu > 0$, in which case the origin is unstable and the limit cycle is stable. This case is called a *supercritical Hopf*. The other case, in which $S > 0$ and which involves the limit cycle being unstable, is called a *subcritical Hopf*. In both cases, the amplitude of the newly born limit cycle grows like $\sqrt{|\mu|}$, a function that has infinite slope at $\mu = 0$, so that the size of the limit cycle grows dramatically for parameters close to the bifurcation value of $\mu = 0$.

4.2.1.2 Relaxation Oscillations

We have seen that for small values of ε, the limit cycle in Van der Pol's equation (4.77) is nearly a circle of radius 2 in the phase plane and its frequency is approximately equal to unity. The character of the limit cycle gradually changes as ε is increased until, for very large values of ε, it becomes a relaxation oscillation. In this section, we obtain an approximation for the limit cycle for large ε, by using a perturbation technique called *matched asymptotic expansions*. We begin by defining a new small parameter, $\varepsilon_0 = \frac{1}{\varepsilon} \ll 1$. Substituting this into Eq. 4.77 gives

$$\varepsilon_0\frac{d^2x}{dt^2} + \varepsilon_0 x - (1 - x^2)\frac{dx}{dt} = 0. \tag{4.93}$$

Next, we scale time by setting $t = \varepsilon_0^v t_1$; here, v is to be determined:

$$\varepsilon_0^{1-2v}\frac{d^2x}{dt_1^2}+\varepsilon_0 x-\varepsilon_0^{-v}(1-x^2)\frac{dx}{dt_1}=0. \tag{4.94}$$

The idea of the method is to select v so that we get a distinguished limit—that is, so that two of the three terms in Eq. 4.94 are of the same order of ε_0 and are larger than the other term. The first and third terms will balance if $1-2v=-v$—that is, if $v=1$. Another distinguished limit is $v=-1$, for which value the second and third terms will balance. We consider each of these limits separately.

First, we set $v=-1$ in Eq. 4.94, which gives

$$\varepsilon_0^2\frac{\mathrm{d}^2x}{\mathrm{d}t_1^2}+x-(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t_1}=0, \quad t_1=\varepsilon_0 t. \tag{4.95}$$

Note that t_1 is slow time. Neglecting terms of $O(\varepsilon_0^2)$, we get a first-order differential equation, which can be solved by separation of variables:

$$\frac{(1-x^2)}{x}\mathrm{d}x=\mathrm{d}t_1 \Rightarrow \ln|x|-\frac{x^2}{2}=t_1+\text{constant}. \tag{4.96}$$

The motion proceeds according to Eq. 4.96 until it reaches $x=\pm l$, where the speed $\mathrm{d}x/\mathrm{d}t_1$ is infinite. At this point, the motion undergoes a jump, the dynamics of which are given by the other distinguished limit, as follows. We set $v=1$ in Eq. 4.94, and to avoid confusion of notation, we use (y,t_2) here in place of (x,t_1):

$$\frac{\mathrm{d}^2y}{\mathrm{d}t_2^2}-(1-y^2)\frac{\mathrm{d}y}{\mathrm{d}t_2}+\varepsilon_0^2 y=0, \quad t_2=\frac{t}{\varepsilon_0}. \tag{4.97}$$

Note that t_2 is fast time. Neglecting terms of $O(\varepsilon_0^2)$, we get a second-order differential equation that has the first integral

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t_2}-y+\frac{y^3}{3}\right)=0, \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t_2}-y+\frac{y^3}{3}=\text{constant}. \tag{4.98}$$

The second equation of (4.98) gives a flow along the y-line, which represents a jump in the relaxation oscillation. We wish to choose the constant of integration so that $y=1$ is an equilibrium point of this flow, in which case the motion will proceed from $y=1$ to some as yet unknown second equilibrium point, which will determine the size of the jump. (The value $y=1$ is obtained from the other distinguished limit, Eq. 4.96, as described above.) For equilibrium at $y=1$, we find

$$\frac{\mathrm{d}y}{\mathrm{d}t_2}=0=y-\frac{y^3}{3}+\text{constant}=1-\frac{1}{3}+\text{constant} \Rightarrow \text{constant}=-\frac{2}{3}. \tag{4.99}$$

Using this value of the integration constant in Eq. 4.98, we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t_2} = y - \frac{y^3}{3} - \frac{2}{3} = -\frac{1}{3}(y-1)^2(y+2). \tag{4.100}$$

From Eq. 4.100, we see that the second equilibrium point lies at $y = -2$. Thus, the jump goes from $y = 1$ to $y = -2$. In a similar fashion, we would find that a jump starting at $y = -1$ ends up at $y = 2$.

It remains to compute the period of the relaxation oscillation. Since t_2 is fast time and t_1 is slow time, the time spent in making the jump is negligible, as compared with the time spent moving according to the second equation in Eq. 4.96. That is, half the period is spent in going from $x = 2$ to $x = 1$ via Eq. 4.96, then a nearly instantaneous jump occurs from $x = 1$ to $x = -2$, then the other half of the period is spent in going from $x = -2$ to $x = -1$, again via Eq. 4.96, and, finally, another nearly instantaneous jump occurs from $x = -1$ to $x = 2$.

A half-period on t_1 time scale is

$$= \left[\ln|x| - \frac{x^2}{2}\right]_{x=2}^{x=1} = \frac{3}{2} - \ln 2. \tag{4.101}$$

If we let T represent the period of the limit cycle on the original time scale t, we find

$$T = (3 - 2\ln 2)\varepsilon \approx 1.614\varepsilon. \tag{4.102}$$

4.2.2 *The Forced Van der Pol Oscillator*

4.2.2.1 Introduction

The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x - \varepsilon(1 - x^2)\frac{\mathrm{d}x}{\mathrm{d}t} = \varepsilon F \cos \omega t \tag{4.103}$$

is called the *forced Van der Pol equation* (Rand 2005). It is a model for situations in which a system that is capable of self-oscillation is acted upon by another oscillator—in this case, represented by the $\varepsilon F \cos \omega t$ term. When a damped Duffing-type oscillator is driven with a periodic forcing function, we have seen that the result may be a periodic response at the same frequency as the forcing function. Since the unforced oscillation is dissipated due to the damping, we are not surprised to find that it is absent from the steady state forced behavior. In the case of a periodically forced limit cycle oscillator, however, we may expect that the steady state forced response might include both the unforced limit cycle oscillation and a response at the forcing frequency. If, however, the forcing is strong enough, and the frequency difference between the unforced limit cycle oscillation and the forcing function is small enough, the response may occur only at the forcing frequency. In this case, the unforced oscillation is said to have been *quenched,* the forcing function is said to have *entrained* or *enslaved* the limit cycle oscillator, and the system is said to be *phase-locked* or *frequency-locked,* or just simply *locked.*

A biological application involves the human sleep–wake cycle, in which a person's biological clock is modeled by a Van der Pol oscillator and the daily night–day cycle caused by the earth's rotation is modeled as a periodic forcing term. Experiments have shown that the limit cycle of a person's biological clock typically has a period that is slightly different than 24 h. Normal sleep patterns correspond to the entrainment of a person's biological clock by the 24-h night–day forcing cycle. Insomnia and other sleep disorders may result if the limit cycle of the biological clock is not quenched, in which case we may expect a quasiperiodic response composed of both the limit cycle and forcing frequencies.

4.2.2.2 Entrainment

In this section, we will use the two-variable expansion method to derive a slow flow system that describes the dynamics of Eq. 4.103 for small ε. We replace time t by $\xi = \omega t$ and $\eta = \omega t$, giving

$$\omega^2 \frac{\partial^2 x}{\partial \xi^2} + 2\omega\varepsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial^2 x}{\partial \eta^2} + x - \varepsilon(1 - x^2)\left(\omega \frac{\partial x}{\partial \xi} + \varepsilon \frac{\partial x}{\partial \eta}\right) = \varepsilon F \cos \xi. \quad (4.104)$$

Next, we expand x and ω in power series:

$$x(\xi, \eta) = x_0(\xi, \eta) + \varepsilon x_1(\xi, \eta) + \cdots, \quad \omega = 1 + k_1 \varepsilon + \cdots. \quad (4.105)$$

Note that the second of Eq. 4.105 means that we are restricting the following discussion to cases where the forcing frequency is nearly equal to the unforced limit cycle frequency, which is called *1:1 resonance*. Substituting Eq. 4.105 into Eq. 4.104 and neglecting terms of $O(\varepsilon^2)$ gives, after collecting terms,

$$\frac{\partial^2 x_0}{\partial \xi^2} + x_0 = 0, \quad (4.106)$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - 2k_1 \frac{\partial^2 x_0}{\partial \xi^2} + \left(1 - x_0^2\right)\frac{\partial x_0}{\partial \xi} + F \cos \xi. \quad (4.107)$$

We take the general solution to Eq. 4.106 in the form

$$x_0(\xi, \eta) = A(\eta) \cos \xi + B(\xi) \sin \xi. \quad (4.108)$$

Removing resonant terms, we obtain the slow flow

$$2\frac{dA}{d\eta} = -2k_1B + A - \frac{A}{4}(A^2 + B^2), \tag{4.109}$$

$$2\frac{dB}{d\eta} = 2k_1A + B - \frac{B}{4}(A^2 + B^2) + F. \tag{4.110}$$

Eqsuations 4.109 and 4.110 can be simplified by using polar coordinates ρ and θ in the A–B slow flow phase plane:

$$A = \rho\cos\theta, \quad B = \rho\sin\theta, \tag{4.111}$$

which produces the following expression for x_0, from Eq. 4.108:

$$x_0(\xi, \eta) = \rho(\eta)\cos(\xi - \theta(\eta)). \tag{4.112}$$

Substituting Eq. 4.111 into Eqs. 4.109 and 4.110 gives

$$\frac{d\rho}{d\eta} = \frac{\rho}{8}(4 - \rho^2) + \frac{F}{2}\sin\theta, \tag{4.113}$$

$$\frac{d\theta}{d\eta} = k_1 + \frac{F}{2\rho}\cos\theta. \tag{4.114}$$

We seek equilibrium points of the slow flow Eqs. 4.113 and 4.114. These represent locked periodic motions of Eq. 4.103. Setting $\frac{d\rho}{d\eta} = \frac{d\theta}{d\eta} = 0$, solving for $\sin\theta$ and $\cos\theta$, and using $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$F^2 = \rho^2\left(1 - \frac{\rho^2}{4}\right)^2 + 4k_1^2\rho^2. \tag{4.115}$$

Expanding Eq. 4.115,

$$\frac{u^3}{16} - \frac{u^2}{2} + (4k_1^2 + 1)u - F^2 = 0, \tag{4.116}$$

where we have set $u = \rho^2$ in order to simplify the algebraic expressions. Eq. 4.116 is a cubic polynomial in u, and application of Descartes's rule of signs gives, in view of its three sign changes, that it has either three positive roots or one positive and two complex roots. The transition between these two cases occurs when there is a repeated root, and the condition for this transition is that the partial derivative of Eq. 4.116 should vanish, which gives

$$\frac{3u^3}{16} - u + 1 + 4k_1^2 = 0. \tag{4.117}$$

Eliminating u between Eqs. 4.117 and 4.116, we obtain

$$\frac{F^4}{16} - \frac{F^2}{27}(1 + 36k_1^2) + \frac{16}{27}k_1^2(1 + 4k_1^2)^2 = 0. \tag{4.118}$$

Equation 4.118 plots as two curves meeting at a cusp in the $k_1 - F$ plane. As one of these curves is traversed quasistatically, a saddle-node bifurcation occurs. At the cusp, a further degeneracy occurs, and there is a triply repeated root. The condition for this is that the partial derivative of Eq. 4.117 should vanish, which gives

$$\frac{3u}{8} - 1 = 0. \tag{4.119}$$

Substituting $u = 8/3$ into Eqs. 4.117 and 4.116 gives the location of the cusp as

$$k_1 = \frac{1}{\sqrt{12}} \approx 0.288, \quad F = \sqrt{\frac{32}{27}} \approx 1.088. \tag{4.120}$$

Before we can conclude that the perturbation analysis predicts that the forced Van der Pol equation (4.103) supports entrainment, we must investigate the *stability* of the slow flow equilibria. Let (ρ_0, θ_0) be an equilibrium solution of Eqs. 4.113 and 4.114. To determine its stability, we set

$$\rho = \rho_0 + v, \quad \theta = \theta_0 + w, \tag{4.121}$$

where v and w are small deviations from equilibrium. Substituting Eq. 4.121 into Eqs. 4.113 and 4.114 and linearizing in v and w gives the constant coefficient system:

$$\frac{\mathrm{d}v}{\mathrm{d}\eta} = \frac{v}{2} - \frac{3}{8}\rho_0^2 v + \frac{F}{2}\cos\theta_0 w, \tag{4.122}$$

$$\frac{dw}{d\eta} = -\frac{F}{2\rho_0^2}\cos\theta_0 v - \frac{F}{2\rho_0}\sin\theta_0 w. \tag{4.123}$$

Equations 4.122 and 4.123 may be simplified by using the following expressions from Eqs. 4.113 and 4.114 at equilibrium:

$$\frac{F}{2}\sin\theta_0 = -\frac{\rho_0}{2} + \frac{\rho_0^3}{8}, \quad \frac{F}{2}\cos\theta_0 = -k_1\rho_0. \tag{4.124}$$

Thus, stability is determined by the eigenvalues of the matrix

$$M = \begin{bmatrix} \frac{1}{2} - \frac{3}{8}\rho_0^2 & -k_1\rho_0 \\ \frac{k_1}{\rho_0} & \frac{1}{2} - \frac{1}{8}\rho_0^2 \end{bmatrix}. \tag{4.125}$$

The trace and determinant of M are given by

$$\mathrm{tr(M)} = 1 - \frac{\rho_0^2}{2}, \qquad \mathrm{det(M)} = \left(-\frac{1}{2} + \frac{3}{8}\rho_0^2\right)\left(-\frac{1}{2} + \frac{1}{8}\rho_0^2\right) + \mathrm{k}_1^2 \tag{4.126}$$

The eigenvalues λ of M satisfy the characteristic equation

$$\lambda^2 - \mathrm{tr(M)}\lambda + \det(\mathrm{M}) = 0. \tag{4.127}$$

For stability, the eigenvalues of M must have negative real parts. This requires that $\mathrm{tr(M)} < 0$ and $\det(M) > 0$, which become, using the notation $u = \rho_0^2$,

$$\mathrm{tr(M)} = 1 - \frac{\mathrm{u}}{2} < 0, \quad \det(\mathrm{M}) = \frac{1}{4}\left(\frac{3\mathrm{u}^2}{16} - \mathrm{u} + 1 + \mathrm{k}_1^2\right) > 0. \tag{4.128}$$

Comparison of this expression for $\det(M)$ and Eq. 4.117 shows that $\det(M)$ vanishes on the curves (Eq. 4.118) along which there are saddle-node bifurcations. This illustrates a very typical phenomenon that characterizes nonlinear vibrations—namely, that *a change in stability is accompanied by a bifurcation.* (This is not true of linear systems, in which a change in stability *cannot* be accompanied by a bifurcation.) The condition (Eq. 4.128) on the tr(M) requires that $u > 2$ for stability. Substituting $u = 2$ into Eq. 4.116, we obtain

$$F^2 = \frac{1}{2} + 8k_1^2. \tag{4.129}$$

Hopf bifurcations occur along the curve represented by Eq. 4.129 (assuming $\det(M) > 0$). This curve (Eq. 4.129) intersects the lower curve of saddle-node bifurcations, Eq. 4.118, at a point we shall refer to as point P, and it intersects and is tangent to the upper curve of saddle-node bifurcations at a point we shall refer to as point Q:

$$P : k_1 = \frac{\sqrt{5}}{8} \approx 0.279, \quad F = \frac{3}{\sqrt{8}} \approx 1.060, \quad Q : k_1 = \frac{1}{4} = 0.25, \quad F = 1. \tag{4.130}$$

It turns out that the perturbation analysis predicts that the forced Van der Pol equation (4.103) exhibits stable entrainment solutions everywhere in the first quadrant of the $k_1 - F$ parameter plane *except* in that region bounded by (1) the lower curve of saddle-node bifurcations (Eq. 4.118), from the origin to the point P; (2) the curve of Hopf bifurcations (Eq. 4.129), from point P to infinity; and (3) the k_1 axis. In physical terms, this means that *for a given detuning k_1, there is a minimum value of forcing F required in order for entrainment to occur.* Moreover, as the detuning k_1 gets larger, entrainment requires larger forcing amplitude F. Also note that since k_1 always appears in the form k_1^2 in the equations of the bifurcation and stability curves, the above conclusions are invariant under a change of sign of k_1; that is, they are independent of whether we are above or below the 1:1 resonance.

4.2.2.3 Secondary Resonances in the System with Cubic Nonlinearity and Strong Excitation

The considered resonance at $\omega \approx 1$ is the strongest but not the only one in the system with cubic nonlinearity. Let us assume that the amplitude of the excitation is not small and investigate whether additional resonances are possible in that system (see Fidlin 2006):

$$\ddot{x} + 2\beta\dot{x} + x + \varepsilon x^3 = a\cos\omega t, \quad \varepsilon \ll 1, \ a = O(1), \ \beta = O(\varepsilon). \tag{4.131}$$

The unperturbed system corresponding to Eq. 4.131 is $(\varepsilon = 0, \ \beta = 0)$:

$$\ddot{x}_0 + x_0 = a\cos\omega t. \tag{4.132}$$

Its general solution can be easily obtained if we exclude the main resonance from the further analysis:

$$\begin{aligned} x_0 &= A\sin(t+\alpha) + \frac{a}{1-\omega^2}\cos\omega t \\ \dot{x}_0 &= A\cos(t+\alpha) - \frac{a\omega}{1-\omega^2}\sin\omega t. \end{aligned} \tag{4.133}$$

A and α are the free constants here. We apply the modified Van der Pol's transformation based on the solution (Eq. 4.133) in order to investigate the perturbed system:

$$x = A\sin\varphi + 2\lambda\cos\omega t; \quad \dot{x} = A\cos\varphi - 2\lambda\sin\omega t, \quad \lambda = \frac{q}{2(1-\omega^2)} = O(1). \tag{4.134}$$

The new variables A and φ are governed by the equations

$$\begin{aligned} \dot{A} &= -2\beta\cos\varphi(A\cos\varphi - 2\lambda\omega\sin\psi) - \varepsilon\cos\varphi(A\sin\varphi + 2\lambda\cos\psi)^3 \\ \dot{\varphi} &= 1 + \frac{2\beta}{A}\sin\varphi(A\cos\varphi - 2\lambda\omega\sin\psi) + \frac{\varepsilon}{A}\sin\varphi(A\sin\varphi + 2\lambda\cos\psi)^3 \\ \dot{\psi} &= \omega. \end{aligned} \tag{4.135}$$

What is a resonance in such a system? Resonance is such a combination of parameters that the time average of the right-hand sides becomes discontinuous. The basic idea behind this definition is as follows. Consider a product of two trigonometric functions $\sin\omega_1 t \, \sin\omega_2 t$. Its time average is always equal to zero, except one parameter combination $\omega_1 = \omega_2$. Then the time average is equal to 1/2. Thus, this parameter combination corresponds to the resonance.

Let us investigate how this definition works for the system (Eq. 4.135). These equations can be transformed to more convenient form if we use trigonometric identities:

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$
$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x \qquad (4.136)$$
$$\cos x \sin^2 x = \frac{1}{4}\cos x - \frac{1}{4}\cos 3x$$

Applying Eq. 4.136 to Eq. 4.135, one obtains

$$\begin{aligned}\dot{A} =& -2\beta A\cos^2\varphi + 4\beta\lambda\omega\cos\varphi\sin\psi - \varepsilon A^3\cos\varphi\sin^3\varphi - \frac{3}{2}\varepsilon A^2\lambda(\cos\varphi - \cos 3\varphi)\cos\psi \\ & - 3\varepsilon A\lambda^2\sin 2\varphi(1+\cos 2\psi) - 2\varepsilon\lambda^3\cos\varphi(3\cos\psi + \cos 3\psi) \\ \dot{\varphi} =& 1 + \beta\sin 2\varphi - \frac{4\beta\lambda\omega}{A}\sin\varphi\sin\psi + \varepsilon A^2\sin^4\varphi + \frac{3}{2}\varepsilon A\lambda(3\sin\varphi - \sin 3\varphi)\cos\psi \\ & + 3\varepsilon\lambda^2(1-\cos 2\varphi)(1+\cos 2\psi) + 2\varepsilon\frac{\lambda^3}{A}\sin\varphi(3\cos\psi + \cos 3\psi) \\ \dot{\psi} =& \omega.\end{aligned} \qquad (4.137)$$

Which terms in these equations can produce a discontinuous average? In order to see that, we can replace φ through t and ψ through ωt. Then it becomes obvious that the main resonance corresponds to the parameter constellation $\omega = 1$. This case was investigated in the previous subsection. There are, however, two further parameter constellations producing discontinuous terms:

$$\begin{aligned}\omega &= 3 \rightarrow \cos 3\varphi\cos\psi \\ \omega &= \frac{1}{3} \rightarrow \cos\varphi\cos 3\psi.\end{aligned} \qquad (4.138)$$

These frequencies correspond to the secondary resonances in our system.

Let us consider the first case, $\omega = 3$. Here, the natural frequency of the linearized system is smaller than the frequency of the external excitation. The corresponding resonance is called *subharmonic*.

In the second case, $\omega = \frac{1}{3}$, and the natural frequency is larger than the frequency of the external excitation. The corresponding resonance is called *superharmonic*.

Approximate predictions for the stationary amplitudes in these cases can be obtained similarly as in the previous subsection.

4.2.3 Two Coupled Limit Cycle Oscillators

A limit cycle oscillator, such as the Van der Pol oscillator, is capable of autonomously generating an attractive periodic motion. This section concerns what happens if we couple two such oscillators together (Rand et al. 2005). A contemporary example involves the interaction of two lasers. A laser is an oscillator that produces a coherent beam of light. If two lasers operate physically near one another, the light from either one of them can influence the behavior of the other. Although both oscillators will, in general, have different frequencies, the effect of the coupling may be to produce a motion that is phase and frequency locked.

We will distinguish between three states of a system of two coupled limit cycle oscillators: strongly locked, weakly locked, and unlocked. The motion will be said to be strongly locked if it is both frequency locked and phase locked. If the motion is frequency locked (on the average) but the relative phase of the oscillators is not constant, we will say that the system is weakly locked. If the frequencies are different (on the average), we will say that the system is unlocked or drifting.

4.2.3.1 Two Coupled Van der Pol Oscillators

In this section, we investigate the dynamics of a pair of coupled Van der Pol oscillators in the small ε limit:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x - \varepsilon(1-x^2)\frac{\mathrm{d}x}{\mathrm{d}t} = \varepsilon\alpha(y-x), \tag{4.139}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + (1+\varepsilon\Delta)y - \varepsilon(1-y^2)\frac{\mathrm{d}y}{\mathrm{d}t} = \varepsilon\alpha(x-y), \tag{4.140}$$

where ε is small, where Δ is a parameter relating to the difference in uncoupled frequencies, and where α is a coupling constant.

We use the two variable expansion methods to obtain a slow flow. Working to $O(\varepsilon)$, we set $\xi = (1+k_1\varepsilon)t, \eta = \varepsilon t$, and we expand $x = x_0 + \varepsilon x_1$ and $y = y_0 + \varepsilon y_1$, giving

$$\frac{\partial^2 x_0}{\partial \xi^2} + x_0 = 0, \tag{4.141}$$

$$\frac{\partial^2 y_0}{\partial \xi^2} + y_0 = 0, \tag{4.142}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - 2k_1\frac{\partial^2 x_0}{\partial \xi^2} + (1-x_0^2)\frac{\partial x_0}{\partial \xi} + \alpha(y_0 - x_0), \tag{4.143}$$

$$\frac{\partial^2 y_1}{\partial \xi^2} + y_1 = -2\frac{\partial^2 y_0}{\partial \xi \partial \eta} - 2k_1\frac{\partial^2 y_0}{\partial \xi^2} - \Delta y_0 + (1-y_0^2)\frac{\partial y_0}{\partial \xi} + \alpha(x_0 - y_0). \tag{4.144}$$

We take the general solution to Eqs. 4.141 and 4.142 in the form

$$x_0(\xi,\eta) = A(\eta)\cos\xi + B(\eta)\sin\xi, \quad y_0(\xi,\eta) = C(\eta)\cos\xi + D(\eta)\sin\xi. \tag{4.145}$$

Removing resonant terms in Eqs. 4.143 and 4.144, we obtain the slow flow

$$2\frac{dA}{d\eta} = -2k_1 B + A - \frac{A}{4}(A^2 + B^2) + \alpha(B - D), \tag{4.146}$$

$$2\frac{dB}{d\eta} = 2k_1 A + B - \frac{B}{4}(A^2 + B^2) + \alpha(C - A), \tag{4.147}$$

$$2\frac{dC}{d\eta} = -2k_1 D + \Delta D + C - \frac{C}{4}(C^2 + D^2) + \alpha(D - B), \tag{4.148}$$

$$2\frac{dD}{d\eta} = 2k_1 C - \Delta C + D - \frac{D}{4}(C^2 + D^2) + \alpha(A - C). \tag{4.149}$$

Equations 4.146–4.149 can be simplified by using polar coordinates R_i and θ_i:

$$A = R_1 \cos\theta_1, \quad B = R_1 \sin\theta_1, \quad C = R_2 \cos\theta_2, \quad D = R_2 \sin\theta_2, \tag{4.150}$$

which gives the following expressions for x_0 and y_0, from Eq. 4.172:

$$x_0(\xi,\eta) = R_1(\eta)\cos(\xi - \theta_1(\eta)), \quad y_0(\xi,\eta) = R_2(\eta)\cos(\xi - \theta_2(\eta)). \tag{4.151}$$

Substituting Eq. 4.151 into Eqs. 4.146–4.149 gives

$$2\frac{dR_1}{d\eta} = R_1\left(1 - \frac{R_1^2}{4}\right) + \alpha R_2 \sin(\theta_1 - \theta_2), \tag{4.152}$$

$$2\frac{dR_2}{d\eta} = R_2\left(1 - \frac{R_2^2}{4}\right) - \alpha R_1 \sin(\theta_1 - \theta_2), \tag{4.153}$$

$$2\frac{d\theta_1}{d\eta} = 2k_1 - \alpha + \frac{\alpha R_2 \cos(\theta_1 - \theta_2)}{R_1}, \tag{4.154}$$

$$2\frac{d\theta_2}{d\eta} = 2k_1 - \Delta - \alpha + \frac{\alpha R_1 \cos(\theta_1 - \theta_2)}{R_2}. \tag{4.155}$$

This system of four slow flow ODEs can be reduced to a system of three ODEs by defining ϕ to be the phase difference between the x and y oscillators, $\phi = \theta_1 - \theta_2$:

$$2\frac{dR_1}{d\eta} = R_1\left(1 - \frac{R_1^2}{4}\right) + \alpha R_2 \sin\phi, \tag{4.156}$$

$$2\frac{dR_2}{d\eta} = R_2\left(1 - \frac{R_2^2}{4}\right) - \alpha R_1 \sin\phi, \tag{4.157}$$

$$2\frac{d\phi}{d\eta} = \Delta + \alpha\cos\phi\left(\frac{R_2}{R_1} - \frac{R_1}{R_2}\right). \tag{4.158}$$

We seek equilibrium points of the slow flow Eqs. 4.156–4.158. These represent strongly locked periodic motions of the original system (Eqs. 4.139 and 4.140). We multiply Eq. 4.156 by R_1 and Eq. 4.157 by R_2 and add to get

$$R_1^2 + R_2^2 - \left(\frac{R_1^4 + R_2^4}{4}\right) = 0. \tag{4.159}$$

Next, we multiply Eq. 4.156 by R_2 and Eq. 4.184 by R_1 and subtract to get

$$\sin\phi = \frac{R_1 R_2 \left(R_1^2 - R_2^2\right)}{4\alpha\left(R_1^2 + R_2^2\right)}. \tag{4.160}$$

Now we use Eq. 4.158 to solve for $\cos\phi$:

$$\cos\phi = \frac{R_1 R_2 \Delta}{\alpha\left(R_1^2 - R_2^2\right)}. \tag{4.161}$$

Using the identity $\sin^2\phi + \cos^2\phi = 1$ in Eqs. 4.160 and 4.161 and setting

$$P = R_1^2 + R_2^2, \quad \text{and} \quad \mathrm{Q} = \mathrm{R}_1^2 - \mathrm{R}_2^2, \tag{4.162}$$

we get

$$Q^6 - P^2 Q^4 + \left(16\Delta^2 + 64\alpha^2\right) P^2 Q^2 - 16\Delta^2 P^4 = 0. \tag{4.163}$$

Using the P and Q notation of Eq. 4.162, Eq. 4.159 becomes

$$Q^2 = 8P - P^2. \tag{4.164}$$

Substituting Eq. 4.164 into Eq. 4.163, we get

$$P^3 - 20P^2 + P\left(16\Delta^2 + 32\alpha^2 + 128\right) - \left(64\Delta^2 + 256\alpha^2 + 256\right) = 0. \tag{4.165}$$

Using Descartes's rule of signs, we see that Eq. 4.165 has either one or three positive roots for P. At bifurcation, there will be a double root that corresponds to requiring the derivative of Eq. 4.165 to vanish:

$$3P^3 - 40P + 16\Delta^2 + 32\alpha^2 + 128 = 0. \tag{4.166}$$

Eliminating P from Eqs. 4.165 and 4.166 gives the condition for saddle-node bifurcations as

$$\Delta^6 + \left(6\alpha^2 + 2\right)\Delta^4 + \left(12\alpha^4 - 10\alpha^2 + 1\right)\Delta^2 + 8\alpha^6 - \alpha^4 = 0. \tag{4.167}$$

Equation 4.167 plots as two curves intersecting at a cusp in the $\Delta - a$ plane. At the cusp, a further degeneracy occurs, and there is a triple root in Eq. 4.165. Requiring the derivative of Eq. 4.166 to vanish gives $P = 20/3$ at the cusp, which gives the location of the cusp as

$$\Delta = \frac{1}{\sqrt{27}} \approx 0.1924, \quad \alpha = \frac{2}{\sqrt{27}} \approx 0.3849. \tag{4.168}$$

Next we look for Hopf bifurcations in the slow flow system (Eqs. 4.156–4.158). The presence of a stable limit cycle in the slow flow surrounding an unstable equilibrium point, as occurs in a supercritical Hopf, represents a weakly locked quasiperiodic motion in the original system (Eqs. 4.139 and 4.140). Let (R_{10}, R_{20}, ϕ_0) be an equilibrium point. The behavior of the system linearized in the neighborhood of this point is determined by the eigenvalues of the Jacobian matrix

$$\frac{1}{2}\begin{pmatrix} -\frac{3R_{10}^2-4}{4} & \alpha \sin\phi_0 & \alpha \sin\phi_0 R_{20} \\ -\alpha \sin\phi_0 & -\frac{3R_{20}^2-4}{4} & -\alpha\cos\phi_0 R_{10} \\ -\frac{\alpha\cos\phi_0\left(R_{20}^2+R_{10}^2\right)}{R_{10}^2 R_{20}} & \frac{\alpha\cos\phi_0\left(R_{20}^2+R_{10}^2\right)}{R_{20}^2 R_{10}} & -\frac{\alpha\sin\phi_0\left(R_{20}^2-R_{10}^2\right)}{R_{10}R_{20}} \end{pmatrix}. \tag{4.169}$$

This matrix may be simplified by using Eqs. 4.160 and 4.161 to replace $\sin\phi_0$ and $\cos\phi_0$, then using Eqs. 4.162 to replace R_{10} and R_{20} by P and Q, and then using Eq. 4.164 to replace Q. This turns out to give the following cubic equation on the eigenvalues λ of the matrix (Eq. 4.169):

$$\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0, \tag{4.170}$$

where

$$C_2 = \frac{P-4}{2}, \tag{4.171}$$

$$C_1 = \frac{7P^3 - 112P^2 + (-16\Delta^2 + 512)P - 512}{64P - 512}, \tag{4.172}$$

$$C_0 = \frac{P^4 - 22P^3 + 160P^2 - (32\Delta^2 + 384)P}{128P - 1024}. \tag{4.173}$$

For a Hopf bifurcation, the eigenvalues γ will include a pure imaginary pair, $\pm i\beta$, and a real eigenvalue, γ. This requires the characteristic equation to have the form

$$\lambda^3 - \gamma\lambda^2 + \beta^2\lambda - \beta^2\gamma = 0. \tag{4.174}$$

Comparing Eqs. 4.169 and 4.174, we see that a necessary condition for a Hopf is

$$C_0 = C_1C_2 \Rightarrow 3P^4 - 59P^3 + (-8\Delta^2 + 400)P^2 + (48\Delta^2 - 1088)P + 1024 = 0. \tag{4.175}$$

Eliminating P between Eqs. 4.175 and 4.167 gives the condition for a Hopf as

$$\begin{aligned}&49\Delta^{s} + (266\alpha^2 + 238)\Delta^6 + (88\alpha^4 + 758\alpha^2 + 345)\Delta^4 \\ &+ (-1056\alpha^6 + 1099\alpha^4 + 892\alpha^2 + 172)\Delta^2 - 1152\alpha^8 - 2740\alpha^6 - 876\alpha^4 + 16 = 0.\end{aligned} \tag{4.176}$$

This curve (Eq. 4.176) intersects the lower curve of saddle-node bifurcations (Eq. 4.167) at a point we shall refer to as point P, and it intersects and is tangent to the upper curve of saddle-node bifurcations at a point we shall refer to as point Q:

$$P : \Delta \approx 0.1918, \ \ \alpha \approx 0.3846, \ \ Q : \Delta \approx 0.1899, \ \ \alpha \approx 0.3837. \tag{4.177}$$

We may obtain the asymptotic behavior of the curve for large Δ and large α by keeping only the highest order terms in Eq. 4.176:

$$49\Delta^{s} + 266\alpha^2\Delta^6 + 88\alpha^4\Delta^4 - 1056\alpha^6\Delta^2 - 1152\alpha^8 = 0, \tag{4.178}$$

which may be factored to give

$$\left(\Delta^2 - 2\alpha^2\right)\left(\Delta^2 + 4\alpha^2\right)\left(-7\Delta^2 + 12\alpha^2\right)^2 = 0, \tag{4.179}$$

which gives the asymptotic behavior

$$\Delta \sim \sqrt{2}\alpha. \tag{4.180}$$

However, there is an additional bifurcation here which did not occur in the forced problem. There is a *homoclinic bifurcation*, which occurs along a curve emanating from point Q. This involves the destruction of the limit cycle that was born in the Hopf. The limit cycle grows in size until it gets so large that it hits a saddle and disappears in a saddle connection. For points on this curve far from point Q, we find that the limit cycle changes its topology into a closed curve in which φ changes by 2π each time around. Unfortunately, we do not have an analytic expression for the curve of homoclinic bifurcations. In summary, we see that the transition from strongly locked behavior to unlocked behavior involves an intermediate state in which the system is weakly locked. In the three-dimensional slow flow space, we go from a stable equilibrium point (strongly locked) to a stable limit cycle (weakly locked) and, finally, to a periodic motion that is topologically distinct from the original limit cycle (unlocked). As in the case of the forced Van der Pol oscillator, in order for strongly locked behavior to occur, we need either a small difference in uncoupled frequencies (small Δ) or a large coupling constant α.

4.3 Mathieu's Equation

4.3.1 Introduction

The differential equation

$$\frac{d^2x}{dt^2} + (\delta + \varepsilon \cos t)x = 0 \tag{4.181}$$

is called *Mathieu's equation*. It is a linear differential equation with variable (periodic) coefficients. It commonly occurs in nonlinear vibration problems in two different ways: (1) in systems in which there is periodic forcing and (2) in stability studies of periodic motions in nonlinear *autonomous* systems (see Rand 2005).

As an example of (1), take the case of a pendulum whose support is periodically forced in a vertical direction. The governing differential equation is

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L} + \frac{A\omega^2}{L}\cos \omega t\right) \sin x = 0, \tag{4.182}$$

where the vertical motion of the support is $A \cos \omega t$, g is the acceleration of gravity, L is the pendulum's length, and x is its angle of deflection. In order to investigate the stability of one of the equilibrium solutions $x = 0$ or $x = \pi$, we would linearize Eq. 4.182 about the desired equilibrium, giving, after suitable rescaling of time, an equation of the form of Eq. 4.181.

As an example of (2), we consider a system known as "the particle in the plane." This consists of a particle of unit mass that is constrained to move in the x–y plane and is restrained by two linear springs, each with spring constant of 1/2. The anchor points of the two springs are located on the x-axis at $x = 1$ and $x = -1$. Each of the two springs has unstretched length L. This autonomous two-degrees-of-freedom system exhibits an exact solution corresponding to a mode of vibration in which the particle moves along the x-axis:

$$x = A \cos t, \quad y = 0. \tag{4.183}$$

In order to determine the stability of this motion, one must first derive the equations of motion, then substitute $x = A \cos t + u$, $y = 0 + v$, where u and v are small deviations from the motion (Eq. 4.183), and then linearize in u and v. The result is two linear differential equations on u and v.

The u equation turns out to be the simple harmonic oscillator and cannot produce instability.

The v equation is

$$\frac{d^2v}{dt^2} + \left(\frac{1 - L - A^2 \cos^2 t}{1 - A^2 \cos^2 t}\right) v = 0. \tag{4.184}$$

Expanding Eq. 4.184 for small A and setting $\tau = 2t$, we obtain

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \left(\frac{2 - 2L - A^2 L}{8} - \frac{A^2 L}{8}\cos\tau + O(A^4)\right) v = 0, \tag{4.185}$$

which is, with respect to $O(A^4)$, in the form of Mathieu's Eq. 4.181 with $\delta = \frac{2-2L-A^2L}{8}$ and $\varepsilon = -\frac{A^2L}{8}$.

The chief concern with regard to Mathieu's equation is whether or not all solutions are bounded for given values of the parameters δ and ε. If all solutions are bounded, the corresponding point in the $\delta - \varepsilon$ parameter plane is said to be stable. A point is called unstable if an unbounded solution exists.

4.3.1.1 Perturbations

In this section, we will use the two-variable expansion method to look for a general solution to Mathieu's Eq. 4.181 for small ε. Since Eq. 4.181 is linear, there is no need to stretch time, and we set $\xi = t$ and $\eta = \varepsilon t$, giving

$$\frac{\partial^2 x}{\partial \xi^2} + 2\varepsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial^2 x}{\partial \eta^2} + (\delta + \varepsilon \cos\xi)x = 0 \tag{4.186}$$

Next, we expand x in a power series

$$x(\xi,\eta) = x_0(\xi,\eta) + \varepsilon x_1(\xi,\eta) + \cdots. \tag{4.187}$$

Substituting Eq. 4.187 into Eq. 4.181 and neglecting terms of $O(\varepsilon^2)$ gives, after collecting terms,

$$\frac{\partial^2 x_0}{\partial \xi^2} + \delta x_0 = 0, \tag{4.188}$$

$$\frac{\partial^2 x_1}{\partial \xi^2} + \delta x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos\xi. \tag{4.189}$$

We take the general solution to Eq. 4.188 in the form

$$x_0(\xi,\eta) = A(\eta)\cos\sqrt{\delta}\xi + B(\eta)\sin\sqrt{\delta}\xi. \tag{4.190}$$

Substituting Eq. 4.190 into Eq. 4.189, we obtain

$$\begin{aligned}\frac{\partial^2 x_1}{\partial \xi^2} + \delta x_1 = {} & 2\sqrt{\delta}\frac{\mathrm{d}A}{\mathrm{d}\eta}\sin\sqrt{\delta}\xi - 2\sqrt{\delta}\frac{\mathrm{d}B}{\mathrm{d}\eta}\cos\sqrt{\delta}\xi - A\cos\sqrt{\delta}\xi\cos\xi \\ & - B\sin\sqrt{\delta}\xi\sin\xi.\end{aligned} \tag{4.191}$$

Using some trig identities, this becomes

$$\begin{aligned}\frac{\partial^2 x_1}{\partial \xi^2} + \delta x_1 = & 2\sqrt{\delta}\frac{dA}{d\eta}\sin\sqrt{\delta}\xi - 2\sqrt{\delta}\frac{dB}{d\eta}\cos\sqrt{\delta}\xi \\ & -\frac{A}{2}\left(\cos(\sqrt{\delta}+1)\xi + \cos(\sqrt{\delta}-1)\xi\right) \\ & -\frac{B}{2}\left(\sin(\sqrt{\delta}+1)\xi + \sin(\sqrt{\delta}-1)\xi\right).\end{aligned} \tag{4.192}$$

For a general value of δ, removal of resonance terms gives the trivial slow flow

$$\frac{dA}{d\eta} = 0, \frac{dB}{d\eta} = 0. \tag{4.193}$$

This means that for general δ, the $\cos t$ driving term in Mathieu's Eq. 4.181 has no effect. However, if we choose $\delta = 1/4$, Eq. 4.192 becomes

$$\begin{aligned}\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}x_1 = & \frac{dA}{d\eta}\sin\frac{\xi}{2} - \frac{dB}{d\eta}\cos\frac{\xi}{2} \\ & -\frac{A}{2}\left(\cos\frac{3\xi}{2} + \cos\frac{\xi}{2}\right) \\ & -\frac{B}{2}\left(\sin\frac{3\xi}{2} + \sin\frac{\xi}{2}\right).\end{aligned} \tag{4.194}$$

Now removal of resonance terms gives the slow flow

$$\frac{dA}{d\eta} = -\frac{B}{2}, \frac{dB}{d\eta} = -\frac{A}{2} \Rightarrow \frac{d^2A}{d\eta^2} = \frac{A}{4}. \tag{4.195}$$

Thus, $A(\eta)$ and $B(\eta)$ involve exponential growth, and the parameter value $\delta = 1/4$ causes instability. This corresponds to a 2:1 subharmonic resonance in which the driving frequency is twice the natural frequency.

This discussion may be generalized by "detuning" the resonance—that is, by expanding δ in a power series in ε:

$$\delta = \frac{1}{4} + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots. \tag{4.196}$$

Now Eq. 4.189 gets an additional term,

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0\cos\xi - \delta_1 x_0, \tag{4.197}$$

which results in the following additional terms in the slow flow Eq. 4.195:

$$\frac{dA}{d\eta} = \left(\delta_1 - \frac{1}{2}\right)B, \frac{dB}{d\eta} = -\left(\delta_1 - \frac{1}{2}\right)A \Rightarrow \frac{d^2A}{d\eta^2} = \left(\delta_1^2 - \frac{1}{4}\right)A = 0. \tag{4.198}$$

Here, we see that $A(\eta)$ and $B(\eta)$ will be sine and cosine functions of slow time η if $\delta_1^2 - \frac{1}{4} > 0$—that is, if either $\delta_1 > \frac{1}{2}$ or $\delta_1 < -\frac{1}{2}$. Thus, the following two curves in the $\delta - \varepsilon$ plane represent stability changes and are called *transition curves*:

$$\delta = \frac{1}{4} \pm \frac{\varepsilon}{2} + O(\varepsilon^2). \tag{4.199}$$

These two curves emanate from the point $\delta = 1/4$ on the δ axis and define a region of instability called a *tongue*. Inside the tongue, for small ε, x grows exponentially in time. Outside the tongue, from Eqs. 4.190 and 4.197, x is the sum of terms each of which is the product of two periodic (sinusoidal) functions with generally incommensurate frequencies; that is, x is a quasiperiodic function of t.

4.3.1.2 Floquet Theory

In this section, we present Floquet theory—that is, the general theory of linear differential equations with periodic coefficients. Our goal is to apply this theory to Mathieu's equation (4.181) (Rand et al. 2005).

Let x be an $n \times 1$ column vector, and let A be an $n \times n$ matrix with time-varying coefficients that have period T. Floquet theory is concerned with the following system of first-order differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad A(t+T) = A(t). \tag{4.200}$$

Note that if the independent variable t is replaced by $t + T$, the system (Eq. 4.200) remains invariant. This means that if $x(t)$ is a solution (vector) of Eq. 4.200 and if, in the vector function, $x(t), t$ is replaced everywhere by $t + T$, then the new vector, $x(t+T)$, which in general will be completely different from $x(t)$, is also a solution of Eq. 4.200. This observation may be stated conveniently in terms of fundamental solution matrices.

Let $X(t)$ be a fundamental solution matrix of Eq. 4.200. $X(t)$ is then an $n \times n$ matrix, with each of its columns consisting of a linearly independent solution vector of Eq. 4.200. In particular, we choose the ith column vector to satisfy an initial condition for which each of the scalar components of $x(0)$ is zero, except for the ith scalar component of $x(0)$, which is unity. This gives $X(0) = I$, where I is the $n \times n$ identity matrix. Since the columns of $X(t)$ are linearly independent, they form a basis for the n-dimensional solution space of Eq. 4.200, and thus any other fundamental solution matrix $Z(t)$ may be written in the form $Z(t) = X(t)C$, where C is a nonsingular $n \times n$ matrix. This means that each of the columns of $Z(t)$ may be written as a linear combination of the columns of $X(t)$.

From our previous observations, replacing t by $t + T$ in $X(t)$ produces a new fundamental solution matrix $X(t + T)$. Each of the columns of $X(t + T)$ may be written as a linear combination of the columns of $X(t)$, so that

$$X(t+T) = X(t)C. \tag{4.201}$$

Note that at $t = 0$, Eq. 4.201 becomes $X(T) = X(0)C = IC = C$; that is,

$$C = X(T). \tag{4.202}$$

Equation 4.202 says that the matrix C (about which we know nothing up to now) is, in fact, equal to the value of the fundamental solution matrix $X(t)$ evaluated at time T—that is, after one forcing period. Thus, C could be obtained by numerically integrating Eq. 4.200 from $t = 0$ to $t = T$, n times, once for each of the n initial conditions satisfied by the ith column of $X(0)$.

Equation 4.201 is a key equation here. It has replaced the original system of ODEs with an iterative equation. For example, if we were to consider Eq. 4.201 for the set of t values $t = 0, T, 2T, 3T, \ldots$, we would be generating the successive iterates of a Poincaré map corresponding to the surface of section $\sum : t = 0 \pmod{2\pi}$. This immediately gives the result that $X(nT) = C^n$, which shows that the question of the boundedness of solutions is intimately connected to the matrix C.

In order to solve Eq. 4.201, we transform to normal coordinates. Let $Y(t)$ be another fundamental solution matrix, as yet unknown. Each of the columns of $Y(t)$ may be written as a linear combination of the columns of $X(t)$:

$$Y(t) = X(t)R, \tag{4.203}$$

where R is an as yet unknown $n \times n$ nonsingular matrix. Combining Eqs. 4.201 and 4.203, we obtain

$$Y(t+T) = Y(t)R^{-1}CR. \tag{4.204}$$

Now let us suppose that the matrix C has n linearly independent eigenvectors. If we choose the columns of R as these n eigenvectors, then the matrix product $R^{-1}CR$ will be a diagonal matrix with the eigenvalues λ_i of C on its main diagonal. With $R^{-1}CR$ diagonal, the element $y_{ji}(t)$ of the matrix $Y(t)$ will satisfy

$$y_{ji}(t+T) = \lambda_i y_{ji}(t). \tag{4.205}$$

Equation 4.205 is a linear functional equation. Let us look for a solution to it in the form

$$y_{ji}(t) = \lambda_i^{kt} p_{ji}(t), \tag{4.206}$$

where k is an unknown constant and $p_{ji}(t)$ is an unknown function. Substituting Eq. 4.206 into Eq. 4.205 gives

$$y_{ji}(t+T) = \lambda_i^{k(t+T)} p_{ji}(t+T) = \lambda_i(\lambda_i^{kt} p_{ji}(t)) = \lambda_i y_{ji}(t). \tag{4.207}$$

Equation 4.207 is satisfied if we take $k = 1/T$ and $p_{ji}(t)$ a periodic function of period T,

$$y_{ji}(t) = \lambda_i^{t/T} p_{ji}(t), \quad p_{ji}(t+T) = p_{ji}(t). \tag{4.208}$$

Here, Eq. 4.208 is the general solution to Eq. 4.208. The arbitrary periodic function $p_{ji}(t)$ plays the same role here that an arbitrary constant plays in the case of a linear first-order ODE.

Since we are interested in the question of boundedness of solutions, we can see from Eq. 4.208 that if $|\lambda_i| > 1$, then $y_{ji} \to \infty$ as $t \to \infty$, whereas if $|\lambda_i| < 1$, then $y_{ji} \to 0$ as $t \to \infty$. Thus, we see that the original system (Eq. 4.200) will be stable (all solutions bounded) if every eigenvalue λ_i of $C = X(T)$ has modulus less than unity. If any one eigenvalue λ_i has modulus greater than unity, then Eq. 4.200 will be unstable (an unbounded solution exists).

Note that our assumption that C has n linearly independent eigenvectors could be relaxed, in which case we would have to deal with a Jordan canonical form. The reader is referred to Cesari (1963), Sect. 4.1 for a complete discussion of this case.

4.3.1.3 Hill's Equation

In this section, we apply Floquet theory to a generalization of Mathieu's equation (Eq. 4.181), called *Hill's equation* (see Rand et al. 2005):

$$\frac{d^2x}{dt^2} + f(t)x = 0, \quad f(t+T) = f(t). \tag{4.209}$$

Here, x and f are scalars, and $f(t)$ represents a general periodic function with period T. Equation 4.209 includes examples such as Eq. 4.184.

We begin by defining $x_1 = x$ and $x_2 = \frac{dx}{dt}$ so that Eq. 4.209 can be written as a system of two first-order ODEs:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -f(t) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4.210}$$

Next, we construct a fundamental solution matrix out of two solution vectors, $\begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$ and $\begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}$, which satisfy the initial conditions

$$\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_{21}(0) \\ x_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.211}$$

As we saw in the previous section, the matrix C is the evaluation of the fundamental solution matrix at time T:

$$C = \begin{bmatrix} x_{11}(T) & x_{21}(T) \\ x_{12}(T) & x_{22}(T) \end{bmatrix}. \tag{4.212}$$

From Floquet theory, we know that stability is determined by the eigenvalues of C:

$$\lambda^2 - (\mathrm{trC})\lambda + \det C = 0, \tag{4.213}$$

where trC and $\det C$ are the trace and determinant of C. Now Hill's Eq. 4.209 has the special property that $\det C = 1$. This may be shown by defining W (the Wronskian) as

$$W(t) = \det C = x_{11}(t)x_{22}(t) - x_{12}(t)x_{21}(t). \tag{4.214}$$

Taking the time derivative of W and using Eq. 4.210 gives $\frac{\mathrm{d}W}{\mathrm{d}t} = 0$, which implies that $W(t) = \text{constant} = \mathrm{W}(0) = 1$. Thus Eq. 4.213 can be written as

$$\lambda^2 - (\mathrm{trC})\lambda + 1 = 0, \tag{4.215}$$

which has the solution

$$\lambda = \frac{\mathrm{trC} \pm \sqrt{\mathrm{trC}^2 - 4}}{2}. \tag{4.216}$$

Floquet theory showed that instability results if either eigenvalue has modulus larger than unity. Thus, if $|\mathrm{trC}| > 2$, then Eq. 4.216 gives real roots. But the product of the roots is unity, so if one root has modulus less than unity, the other has modulus greater than unity, with the result that this case is unstable and corresponds to exponential growth in time.

On the other hand, if $|\mathrm{trC}| < 2$, then Eq. 4.216 gives a pair of complex conjugate roots. But since their product must be unity, they must both lie on the unit circle, with the result that this case is stable. Note that the stability here is neutral stability, not asymptotic stability, since Hill's Eq. 4.209 has no damping. This case corresponds to quasiperiodic behavior in time.

Thus, the transition from stable to unstable corresponds to those parameter values that give $|\mathrm{trC}| = 2$. From Eq. 4.216, if $\mathrm{trC} = 2$, then $\lambda = 1, 1$, and from Eq. 4.208, this corresponds to a periodic solution with period T. On the other hand, if $\mathrm{trC} = -2$, then $\lambda = -1, -1$, and from Eq. 4.208, this corresponds to a periodic solution with period $2T$. This gives the important result that *on the transition curves in parameter space between stable and unstable, there exist periodic motions of period T or $2T$.*

The theory presented in this section can be used as a practical numerical procedure for determining stability of a Hill's equation. Begin by numerically integrating the ODE for the two initial conditions (Eq. 4.211). Carry each numerical integration out to time $t = T$ and so obtain $\mathrm{trC} = \mathrm{x}_{11}(\mathrm{T}) + \mathrm{x}_{22}(\mathrm{T})$. Then $|\mathrm{trC}| > 2$ is unstable, while $|\mathrm{trC}| < 2$ is stable. Note that this approach allows you to draw conclusions about large time behavior after numerically integrating for only one forcing period. Without Floquet theory, you would have to numerically integrate out to large time in order to determine whether a solution was growing unbounded, especially for systems that are close to a transition curve, in which case the asymptotic growth is very slow.

4.3.1.4 Harmonic Balance

In this section, we apply Floquet theory to Mathieu's equation (4.191). Since the period of the forcing function in Eq. 4.191 is $T = 2\pi$, we may apply the result obtained in the previous section to conclude that on the transition curves in the δ–ε parameter plane, there exist solutions of period 2π or 4π. This motivates us to look for such a solution in the form of a Fourier series (see Rand et al. 2005):

$$x(t) = \sum_{n=0}^{\infty} a_n \cos\frac{nt}{2} + b_n \sin\frac{nt}{2}. \tag{4.217}$$

This series represents a general periodic function with period 4π, and includes functions with period 2π as a special case (when a_{odd} and b_{odd} are zero). Substituting Eq. 4.217 into Mathieu's equation (4.191), simplifying the trig, and collecting terms (a procedure called *harmonic balance)* gives four sets of algebraic equations on the coefficients a_n and b_n. Each set deals exclusively with $a_{\text{even}}, b_{\text{even}}, a_{\text{odd}}$, and b_{odd}, respectively. Each set is homogeneous and of infinite order, so for a nontrivial solution, the determinants must vanish. This gives four infinite determinants (called *Hill's determinants*):

$$a_{\text{even}}: \begin{vmatrix} \delta & \varepsilon/2 & 0 & 0 & \\ \varepsilon & \delta-1 & \varepsilon/2 & 0 & \cdots \\ 0 & \varepsilon/2 & \delta-4 & \varepsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0, \tag{4.218}$$

$$b_{\text{even}}: \begin{vmatrix} \delta-1 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \delta-4 & \varepsilon/2 & 0 & \cdots \\ 0 & \varepsilon/2 & \delta-9 & \varepsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0, \tag{4.219}$$

$$a_{\text{odd}}: \begin{vmatrix} \delta-1/4+\varepsilon/2 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \delta-9/4 & \varepsilon/2 & 0 & \cdots \\ 0 & \varepsilon/2 & \delta-25/4 & \varepsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0, \tag{4.220}$$

$$b_{\text{odd}}: \begin{vmatrix} \delta-1/4-\varepsilon/2 & \varepsilon/2 & 0 & 0 & \\ \varepsilon/2 & \delta-9/4 & \varepsilon/2 & 0 & \cdots \\ 0 & \varepsilon/2 & \delta-25/4 & \varepsilon/2 & \\ & & \cdots & & \end{vmatrix} = 0. \tag{4.221}$$

In all four determinants, the typical row is of the form

$$\cdots \quad 0 \quad \varepsilon/2 \quad \delta - n^2/4 \quad \varepsilon/2 \quad 0 \quad \cdots$$

(except for the first one or two rows).

Each of these four determinants represents a functional relationship between δ and ε, which plots as a set of transition curves in the δ–ε plane. By setting $\varepsilon = 0$ in these determinants, it is easy to see where the associated curves intersect the δ axis. The transition curves obtained from the a_{even} and b_{even} determinants intersect the δ axis at $\delta = n^2, n = 0,\ 1,\ 2, \ldots$, while those obtained from the a_{odd} and b_{odd} determinants intersect the δ axis at $\delta = \frac{(2n+1)^2}{4}$, $n = 0,\ 1,\ 2, \ldots$ For $\varepsilon > 0$, each of these points on the δ axis gives rise to two transition curves, one coming from the associated a determinant, and the other from the b determinant. Thus, there is a tongue of instability emanating from each of the following points on the δ axis:

$$\delta = \frac{n^2}{4}, \quad n = 0, 1, 2, 3, \ldots \tag{4.222}$$

The $n = 0$ case is an exception, since only one transition curve emanates from it, as a comparison of Eq. 4.218 with Eq. 4.219 will show.

Note that the transition curves (Eq. 4.199) found earlier in this chapter by using the two variable expansion methods correspond to $n = 1$ in Eq. 4.222. Why did the perturbation method miss the other tongues of instability? It was because we truncated the perturbation method, neglecting terms of $O(\varepsilon^2)$. The other tongues of instability turn out to emerge at higher order truncations in the various perturbation methods (two-variable expansion, averaging, Lie transforms, normal forms, even regular perturbations). In all cases, these methods deliver an expression for a particular transition curve in the form of a power series expansion:

$$\delta = \frac{n^2}{4} + \delta_1 \varepsilon + \delta_2 \varepsilon^2 + \cdots. \tag{4.223}$$

As an alternative method of obtaining such an expansion, we can simply substitute Eq. 4.223 into any of the determinants (Eqs. 4.218–4.221) and collect terms, in order to obtain values for the coefficients δ_i. As an example, let us substitute Eq. 4.223 for $n = 1$ into the a_{odd} determinant (Eq. 4.220). Expanding a 3×3 truncation of Eq. 4.220, we get (using computer algebra)

$$-\frac{\varepsilon^3}{8} - \frac{\delta\varepsilon^2}{2} + \frac{13\varepsilon^2}{8} + \frac{\delta^2\varepsilon}{2} - \frac{17\delta\varepsilon}{4} + \frac{225\varepsilon}{32} + \delta^3 - \frac{35\delta^2}{4} + \frac{259\delta}{16} - \frac{225}{64}. \tag{4.224}$$

Substituting Eq. 4.223 with $n = 1$ into Eq. 4.224 and collecting terms gives

$$(12\delta_1 + 6)\varepsilon + \frac{\left(24\delta_2 - 16\delta_1^2 - 8\delta_1 + 3\right)\varepsilon^2}{2} + \cdots. \tag{4.225}$$

Requiring the coefficients of ε and ε^2 in Eq. 4.225 to vanish gives

$$\delta_1 = -\frac{1}{2}, \quad \delta_2 = -\frac{1}{8}. \tag{4.226}$$

This process can be continued to any order of truncation. Here are the expansions of the first few transition curves:

$$\delta = -\frac{\varepsilon^2}{2} + \frac{7\varepsilon^4}{32} - \frac{29\varepsilon^6}{144} + \frac{68687\varepsilon^8}{294912} - \frac{123707\varepsilon^{10}}{409600} + \frac{8022167579\varepsilon^{12}}{19110297600} + \cdots, \tag{4.227}$$

$$\begin{aligned}\delta = \frac{1}{4} - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \frac{\varepsilon^3}{32} - \frac{\varepsilon^4}{384} - \frac{11\varepsilon^5}{4608} + \frac{49\varepsilon^6}{36864} - \frac{55\varepsilon^7}{294912} - \frac{83\varepsilon^8}{552960} \\ + \frac{12121\varepsilon^9}{117964800} - \frac{114299\varepsilon^{10}}{6370099200} - \frac{192151\varepsilon^{11}}{15288238080} + \frac{83513957\varepsilon^{12}}{8561413324800} + \cdots,\end{aligned} \tag{4.228}$$

$$\begin{aligned}\delta = \frac{1}{4} + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} - \frac{\varepsilon^3}{32} - \frac{\varepsilon^4}{384} + \frac{11\varepsilon^5}{4608} + \frac{49\varepsilon^6}{36864} + \frac{55\varepsilon^7}{294912} - \frac{83\varepsilon^8}{552960} \\ - \frac{12121\varepsilon^9}{117964800} - \frac{114299\varepsilon^{10}}{6370099200} + \frac{192151\varepsilon^{11}}{15288238080} + \frac{83513957\varepsilon^{12}}{8561413324800} + \cdots,\end{aligned} \tag{4.229}$$

$$\begin{aligned}\delta = 1 - \frac{\varepsilon^2}{12} + \frac{5\varepsilon^4}{3456} - \frac{289\varepsilon^6}{4976640} + \frac{21391\varepsilon^8}{7166361600} \\ - \frac{2499767\varepsilon^{10}}{14447384985600} + \frac{1046070973\varepsilon^{12}}{97086427103232000} + \cdots,\end{aligned} \tag{4.230}$$

$$\begin{aligned}\delta = 1 + \frac{5\varepsilon^2}{12} - \frac{763\varepsilon^4}{3456} + \frac{1002401\varepsilon^6}{4976640} - \frac{1669068401\varepsilon^8}{7166361600} \\ + \frac{4363384401463\varepsilon^{10}}{14447384985600} - \frac{40755179450909507\varepsilon^{12}}{97086427103232000} + \cdots.\end{aligned} \tag{4.231}$$

4.3.2 Effect of Damping

In this section, we investigate the effect that damping has on the transition curves of Mathieu's equation by applying the two-variable expansion method to the following equation, known as the *damped Mathieu equation* (Rand et al. 2005):

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (\delta + \varepsilon\cos t)x = 0. \tag{4.232}$$

In order to facilitate the perturbation method, we scale the damping coefficient c to be $O(\varepsilon)$:

$$c = \varepsilon\mu. \tag{4.233}$$

We can use the same setup from earlier in this section, where upon Eq. 4.186 becomes

$$\frac{\partial^2 x}{\partial \xi^2} + 2\varepsilon \frac{\partial^2 x}{\partial \xi \partial \eta} + \varepsilon^2 \frac{\partial^2 x}{\partial \eta^2} + \varepsilon\mu\left(\frac{\partial x}{\partial \xi} + \varepsilon \frac{\partial x}{\partial \eta}\right) + (\delta + \varepsilon \cos \xi)x = 0. \tag{4.234}$$

Now we expand x as in Eq. 4.187 and δ as in Eq. 4.196, and we find that Eq. 4.197 gets an additional term:

$$\frac{\partial^2 x_1}{\partial \xi^2} + \frac{1}{4}x_1 = -2\frac{\partial^2 x_0}{\partial \xi \partial \eta} - x_0 \cos \xi - \delta_1 x_0 - \mu \frac{\partial x_0}{\partial \xi}, \tag{4.235}$$

which results in two additional terms appearing in the slow flow Eqs. 4.198:

$$\frac{dA}{d\eta} = -\frac{\mu}{2}A + \left(\delta_1 - \frac{1}{2}\right)B, \quad \frac{dB}{d\eta} = -\left(\delta_1 - \frac{1}{2}\right)A - \frac{\mu}{2}B. \tag{4.236}$$

Equations 4.236 are a linear constant coefficient system that may be solved by assuming a solution in the form $A(\eta) = A_0 \exp(\lambda\eta)$, $B(\eta) = B_0 \exp(\lambda\eta)$. For nontrivial constants A_0 and B_0, the following determinant must vanish:

$$\begin{vmatrix} -\frac{\mu}{2} - \lambda & -\frac{1}{2} + \delta_1 \\ -\frac{1}{2} - \delta_1 & -\frac{\mu}{2} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = -\frac{\mu}{2} \pm \sqrt{-\delta_1^2 + \frac{1}{4}}. \tag{4.237}$$

For the transition between stable and unstable, we set $\lambda = 0$, giving the value

$$\delta_1 = \pm\frac{\sqrt{1-\mu^2}}{2}. \tag{4.238}$$

This gives the following expressions for the $n = 1$ transition curves:

$$\delta = \frac{1}{4} \pm \varepsilon\frac{\sqrt{1-\mu^2}}{2} + O(\varepsilon^2) = \frac{1}{4} \pm \frac{\sqrt{\varepsilon^2 - c^2}}{2} + O(\varepsilon^2). \tag{4.239}$$

Equation 4.239 predicts that for a given value of c, there is a minimum value of ε that is required for instability to occur. The $n = 1$ tongue, which for $c = 0$ emanates from the δ-axis, becomes detached from the δ axis for $c > 0$. This prediction is verified by numerically integrating Eq. 4.232 for fixed c, while δ and ε are permitted to vary.

4.3.3 *Effect of Nonlinearity*

In the previous sections of this chapter, we have seen how unbounded solutions to Mathieu's equation (4.181) can result from resonances between the forcing frequency and the oscillator's unforced natural frequency. However, real physical systems do not exhibit unbounded behavior.

The difference lies in the fact that the Mathieu equation is linear. The effects of nonlinearity can be explained as follows: As the resonance causes the amplitude of the motion to increase, the relation between period and amplitude (which is a characteristic effect of nonlinearity) causes the resonance to detune, decreasing its tendency to produce large motions.

A more realistic model can be obtained by including nonlinear terms in the Mathieu equation. For example, in the case of the vertically driven pendulum (Eq. 4.182), if we expand $\sin x$ in a Taylor series, we get

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{g}{L} - \frac{A\omega^2}{L}\cos\omega t\right)\left(x - \frac{x^3}{6} + \cdots\right) = 0. \tag{4.240}$$

Now if we rescale time by $\tau = \omega t$ and set $\delta = \frac{g}{\omega^2 L}$ and $\varepsilon = \frac{A}{L}$, we get

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (\delta - \varepsilon\cos\tau)\left(x - \frac{x^3}{6} + \cdots\right) = 0. \tag{4.241}$$

Next, if we scale x by $x = \sqrt{\varepsilon}y$ and neglect terms of $O(\varepsilon^2)$, we get

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} + (\delta - \varepsilon\cos\tau)y - \varepsilon\frac{\delta}{6}y^3 + O(\varepsilon^2) = 0. \tag{4.242}$$

Motivated by this example, in this section, we study the nonlinear Mathieu equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (\delta + \varepsilon\cos t)x + \varepsilon\alpha x^3 = 0. \tag{4.243}$$

We once again use the two-variable expansion method to treat this equation. Using the same setup that we did earlier in this chapter, Eq. 4.186 becomes

$$\frac{\partial^2 x}{\partial\xi^2} + 2\varepsilon\frac{\partial^2 x}{\partial\xi\partial\eta} + \varepsilon^2\frac{\partial^2 x}{\partial\eta^2} + (\delta + \varepsilon\cos\xi)x + \varepsilon\alpha x^3 = 0. \tag{4.244}$$

We expand x as in Eq. 4.187 and δ as in Eq. 4.196, and we find that Eq. 4.197 gets an additional term

$$\frac{\partial^2 x_1}{\partial\xi^2} + \frac{1}{4}x_1 = -2\frac{\partial^2 x_0}{\partial\xi\partial\eta} - x_0\cos\xi - \delta_1 x_0 - \alpha x_0^3, \tag{4.245}$$

where x_0 is of the form

$$x_0(\xi,\eta) = A(\eta)\cos\frac{\xi}{2} + B(\eta)\sin\frac{\xi}{2}. \tag{4.246}$$

Removal of resonant terms in Eq. 4.245 results in the appearance of some additional cubic terms in the slow flow Eq. 4.198:

$$\frac{\mathrm{d}A}{\mathrm{d}\eta} = \left(\delta_1 - \frac{1}{2}\right)B + \frac{3\alpha}{4}B(A^2 + B^2), \quad \frac{\mathrm{d}B}{\mathrm{d}\eta} = -\left(\delta_1 + \frac{1}{2}\right)A - \frac{3\alpha}{4}A(A^2 + B^2). \tag{4.247}$$

In order to more easily work with the slow flow (Eq. 4.247), we transform to polar coordinates in the A–B phase plane:

$$A = R\cos\theta, \quad B = R\sin\theta. \tag{4.248}$$

Note that Eqs. 4.248 and 4.246 give the following alternate expression for x_0:

$$x_0(\xi, \eta) = R(\eta)\cos\left(\frac{\xi}{2} - \theta(\eta)\right). \tag{4.249}$$

Substitution of Eq. 4.248 into the slow flow Eq. 4.274 gives

$$\frac{\mathrm{d}R}{\mathrm{d}\eta} = -\frac{R}{2}\sin 2\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\eta} = -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4}R^2. \tag{4.250}$$

We seek equilibria of the slow flow (Eq. 4.250). From Eq. 4.249, a solution in which R and θ are constant in slow time η represents a periodic motion of the nonlinear Mathieu Eq. 4.243, which has one half the frequency of the forcing function—that is, such a motion is a 2:1 subharmonic. Such slow flow equilibria satisfy the equations

$$-\frac{R}{2}\sin 2\theta = 0, \quad -\delta_1 - \frac{\cos 2\theta}{2} - \frac{3\alpha}{4}R^2 = 0. \tag{4.251}$$

Ignoring the trivial solution $R = 0$, the first equation of Eq. 4.251 requires $\sin 2\theta = 0$ or $\theta = 0, \frac{\pi}{2}, \pi$, or $\frac{3\pi}{2}$.

Solving the second equation of Eq. 4.251 for R^2, we get

$$R^2 = -\frac{4}{3\alpha}\left(\frac{\cos 2\theta}{2} + \delta_1\right). \tag{4.252}$$

For a nontrivial real solution, $R^2 > 0$. Let us assume that the nonlinearity parameter $a > 0$.

Then, in the case of $\theta = 0$ or $\pi, \cos 2\theta = 1$ and nontrivial equilibria exist only for $\delta_1 < -\frac{1}{2}$. On the other hand, for $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}, \cos 2\theta = -1$, and nontrivial equilibria require $\delta_1 < \frac{1}{2}$.

Since $\delta_1 = \pm\frac{1}{2}$ corresponds to transition curves for the stability of the trivial solution, the analysis predicts that bifurcations occur as we cross the transition curves in the δ–ε plane. That is, imagine quasistatically decreasing the parameter δ while ε is kept fixed and moving through the $n = 1$ tongue emanating from the point $\delta = \frac{1}{4}$ on the δ axis. As δ decreases across the right transition curve, the trivial solution $x = 0$ becomes unstable, and simultaneously, a stable 2:1 subharmonic motion is born. This motion grows in amplitude as δ continues to decrease.

When the left transition curve is crossed, the trivial solution becomes stable again, and an unstable 2:1 subharmonic is born. This scenario can be pictured as involving two pitchfork bifurcations.

If the nonlinearity parameter $a < 0$, a similar sequence of bifurcations occurs, except that, in this case, the subharmonic motions are born as δ *increases* quasi-statically through the $n = 1$ tongue.

4.4 Ince's Equation

4.4.1 Introduction

The equation

$$(1 + a\cos 2t)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\sin 2t\frac{\mathrm{d}x}{\mathrm{d}t} + (c + d\cos 2t)x = 0, \tag{4.253}$$

which is called *Ince's equation*, occurs in a variety of mechanics problems. It includes Mathieu's equation as a special case (for which $a = b = 0$). However, because Ince's equation contains four parameters instead of only two for Mathieu's equation, a certain phenomenon called *coexistence* can occur in Ince's equation, but not in Mathieu's equation. The phenomenon of coexistence involves the disappearance of tongues of instability that would ordinarily be there (see Rand et al. 2005).

As an example, consider the equation

$$\left(1 + \frac{\varepsilon}{2}\cos 2t\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\varepsilon}{2}\sin 2t\frac{\mathrm{d}x}{\mathrm{d}t} + cx = 0, \tag{4.254}$$

which is Ince's equation with $a = b = \varepsilon/2$ and $d = 0$. We are interested in the location of the transition curves of Eq. 4.254 in the c–ε plane, which separate regions of stability (all solutions bounded) from regions of instability (an unbounded solution exists). A straightforward line of reasoning leads us to expect tongues of instability to emanate from the points $c = n^2, n = 1, 2, 3, \ldots$ on the c-axis. Let us examine this reasoning. We have seen that Floquet theory tells us that equations of the form of Hill's equation,

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + f(t)z = 0, \quad f(t + T) = f(t), \tag{4.255}$$

have periodic solutions of period T or $2T$ on their transition curves. However, Eq. 4.254 is not of the form of Hill's equation (4.255). Nevertheless, if we set

$$x = \left(1 + \frac{\varepsilon}{2}\cos 2t\right)^{\frac{1}{4}}z, \tag{4.256}$$

then it turns out that Eq. 4.254 becomes a Hill's equation (4.255) on $z(t)$, with the coefficient

$$f(t) = \frac{\varepsilon^2 \cos 4t + 16\varepsilon(c-1)\cos 2t + 32c - 9\varepsilon^2}{4(\varepsilon^2 \cos 4t + 8\varepsilon \cos 2t + 8 + \varepsilon^2)}. \tag{4.257}$$

Here, $f(t)$ is periodic with period π. Thus, Floquet theory tells us that the resulting Hill's equation on $z(t)$ will have solutions of period π or 2π on its transition curves. Now from Eq. 4.256, the boundedness of $z(t)$ is equivalent to the boundedness of $x(t)$, so transition curves for the z equation occur for the same parameters as do those for the x equation (4.254). Also, since the coefficient $\left(1 + \frac{\varepsilon}{2}\cos 2t\right)^{\frac{1}{4}}$ in Eq. 4.256 has period π, we may conclude that Eq. 4.254 has solutions of period π or 2π on its transition curves. Now when $\varepsilon = 0$, Eq. 4.254 is of the form $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + cx = 0$ and has solutions of period $\frac{2\pi}{\sqrt{c}}$. These will correspond to solutions of period π or 2π when $\frac{2\pi}{\sqrt{c}} = \frac{2\pi}{n}$, since a solution with period $\frac{2\pi}{n}$ may also be thought of as having period π (n even) or 2π (n odd), which gives $c = n^2, n = 1, 2, 3, \ldots$, as claimed above.

To reiterate, the purpose of the preceding long-winded paragraph was to show that we can expect Eq. 4.254 to have tongues of instability emanating from the points $c = n^2, n = 1, 2, 3, \ldots$ on the c-axis. While this would be true in general for an equation of the type 4.253, the coefficients in Eq. 4.254 have been especially chosen to illustrate the phenomenon of coexistence. In fact, Eq. 4.254 has only one tongue of instability, which emanates from the point $c = 1$ on the c-axis.

4.4.2 *Coexistence*

In order to understand what happened to all the tongues of instability that we expected to occur in Eq. 4.254, we use the method of harmonic balance. Since the transition curves are characterized by the occurrence of a periodic solution of period π or 2π, we expand the solution x in a Fourier series:

$$x(t) = \sum_{n=0}^{\infty} a_n \cos nt + b_n \sin nt. \tag{4.258}$$

This series represents a general periodic function of period 2π and includes functions of period π as a special case (when a_{odd} and b_{odd} are zero). Substituting Eq. 4.258 into Eq. 4.254, simplifying the trig, and collecting terms, we obtain four sets of algebraic equations on the coefficients a_n and b_n. Each set deals exclusively with $a_{\text{even}}, b_{\text{even}}, a_{\text{odd}}$, and b_{odd}, respectively. Each set is homogeneous and of infinite order, so for a nontrivial solution, the determinants must vanish. This gives four infinite determinants:

$$a_{\text{even}}: \begin{vmatrix} c & -\frac{3\varepsilon}{2} & 0 & 0 & 0 & \\ 0 & c-4 & -5\varepsilon & 0 & 0 & \cdots \\ 0 & -\frac{\varepsilon}{2} & c-16 & -\frac{21\varepsilon}{2} & 0 & \\ 0 & 0 & -3\varepsilon & c-36 & -18\varepsilon & \\ & & & \cdots & & \end{vmatrix} = 0 \tag{4.259}$$

$$b_{\text{even}}: \begin{vmatrix} c-4 & -5\varepsilon & 0 & 0 & \\ -\frac{\varepsilon}{2} & c-16 & -\frac{21\varepsilon}{2} & 0 & \cdots \\ 0 & -3\varepsilon & c-36 & -18\varepsilon & \\ & & \cdots & & \end{vmatrix} = 0 \tag{4.260}$$

$$a_{\text{odd}}: \begin{vmatrix} c-1-\frac{\varepsilon}{2} & -3\varepsilon & 0 & 0 & \\ 0 & c-9 & -\frac{15\varepsilon}{2} & 0 & \cdots \\ 0 & -\frac{3}{2}\varepsilon & c-25 & -14\varepsilon & \\ & & \cdots & & \end{vmatrix} = 0 \tag{4.261}$$

$$b_{\text{odd}}: \begin{vmatrix} c-1+\frac{\varepsilon}{2} & -3\varepsilon & 0 & 0 & \\ 0 & c-9 & -\frac{15\varepsilon}{2} & 0 & \cdots \\ 0 & -\frac{3}{2}\varepsilon & c-25 & -14\varepsilon & \\ & & \cdots & & \end{vmatrix} = 0. \tag{4.262}$$

If we represent by Δ_0 the determinant (Eq. 4.260) of the b_{even} coefficients, then the determinant (4.259) of the a_{even} coefficients may be written $c\Delta_0$, a result obtainable by doing a Laplace expansion down the first column. This gives us the result that $c = 0$ is the exact equation of a transition curve.

Examination of Eq. 4.259 shows that on $c = 0$, we have the exact solution $x(t) = a_0$, the other a_{even} coefficients vanishing on $c = 0$. Note that $x(t) = a_0$ (= 1, say) may be considered a π-periodic solution.

On the other hand, we may also satisfy Eq. 4.259 by taking $\Delta_0 = 0$, which corresponds to taking $a_0 = 0$, while the other a_{even} coefficients do not, in general, vanish. Note that this same condition $\Delta_0 = 0$ gives a nontrivial solution for the b_{even} coefficients. Thus, on the transition curves corresponding to $\Delta_0 = 0$, we have the *coexistence* of two linearly independent π-periodic solutions, one even and the other odd. Now a region of instability usually lies between two such transition curves, one of which has an even π-periodic solution, and the other of which has an odd π-periodic solution. In the case where two such periodic functions coexist, the instability region disappears (or rather, has zero width). In the case of Eq. 4.281, all of the even coefficient (π-periodic) tongues disappear.

Let us turn now to Eqs. 4.261 and 4.262 on the coefficients a_{odd} and b_{odd}, respectively. The determinant 4.261 may be written $(c - 1 - \varepsilon/2)\Delta_1$, and the determinant 4.262 may be written $(c - 1 + \varepsilon/2)\Delta_1$, where Δ_1 is the infinite determinant:

$$\Delta_1 = \begin{vmatrix} c-9 & -\frac{15\varepsilon}{2} & 0 & \\ -\frac{3\varepsilon}{2} & c-25 & -14\varepsilon & \cdots \\ & \cdots & & \end{vmatrix}. \tag{4.263}$$

We may satisfy Eq. 4.261 by taking $c = 1 + \varepsilon/2$. This corresponds to taking a_1 nonzero, and all the other $a_{\text{odd}} = 0$. Similarly, we may satisfy Eq. 4.262 by taking $c = 1 - \varepsilon/2$, which corresponds to taking b_1 nonzero, and all the other $b_{\text{odd}} = 0$. Thus, we have obtained the following exact expressions for two transition curves emanating from $c = 1$ on the c-axis:

$$c = 1 + \frac{\varepsilon}{2} \quad \text{on which} \quad \mathrm{x(t) = cos\,t}, \tag{4.264}$$

$$c = 1 - \frac{\varepsilon}{2} \quad \text{on which} \quad \mathrm{x(t) = cos\,t}. \tag{4.265}$$

All the other transition curves correspond to the vanishing of Δ_1. This condition guarantees a nontrivial solution for both the a_{odd} and the b_{odd} coefficients, respectively. Since the same relation between c and ε produces two linearly independent 2π-periodic solutions, we have another instance of coexistence, and the associated tongues of instability do not occur.

4.4.3 Ince's Equation

Let us now apply the foregoing approach to the general version of Ince's equation (4.253). We substitute the Fourier series (Eq. 4.258) into Eq. 4.253, perform the usual trig simplifications, and collect terms, thereby obtaining four sets of algebraic equations on the coefficients a_n and b_n. For a nontrivial solution, these require that the following four infinite determinants vanish (Rand 2005):

$$a_{\text{even}}: \begin{vmatrix} c & \frac{d}{2}-b-2a & 0 & 0 & 0 & \\ d & c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ 0 & \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \cdots \\ 0 & 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \\ 0 & 0 & & \frac{d}{2}+3b-18a & c-64 & \\ & & & \cdots & & \end{vmatrix} = 0 \tag{4.266}$$

$$b_{\text{even}}: \begin{vmatrix} c-4 & \frac{d}{2}-2b-8a & 0 & 0 & \\ \frac{d}{2}+b-2a & c-16 & \frac{d}{2}-3b-18a & 0 & \\ 0 & \frac{d}{2}+2b-8a & c-36 & \frac{d}{2}-4b-32a & \cdots \\ 0 & 0 & \frac{d}{2}+3b-18a & c-64 & \\ & & \cdots & & \end{vmatrix} = 0 \tag{4.267}$$

$$a_{\text{odd}}:\begin{vmatrix} c-1+\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \cdots \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{vmatrix}=0 \tag{4.268}$$

$$b_{\text{odd}}:\begin{vmatrix} c-1-\frac{d-b-a}{2} & \frac{d-3b-9a}{2} & 0 & 0 & \\ \frac{d+b-a}{2} & c-9 & \frac{d-5b-25a}{2} & 0 & \\ 0 & \frac{d+3b-9a}{2} & c-25 & \frac{d-7b-49a}{2} & \cdots \\ 0 & 0 & \frac{d+5b-25a}{2} & c-49 & \\ & & \cdots & & \end{vmatrix}=0. \tag{4.269}$$

The notation in these determinants may be simplified by setting

$$Q(m)=\frac{d}{2}+bm-2am^2, \tag{4.270}$$

$$P(m)=Q\left(m-\frac{1}{2}\right)=\frac{d+2b\left(m-\frac{1}{2}\right)-4a\left(m-\frac{1}{2}\right)^2}{2}$$
$$=\frac{d+2b(2m-1)-a(2m-1)^2}{2}. \tag{4.271}$$

Using the notation of Eqs. 4.270 and 4.271, the determinants (Eqs. 4.266–4.269) become

$$a_{\text{even}}:\begin{vmatrix} c & Q(-1) & 0 & 0 & 0 & \\ 2Q(0) & c-4 & Q(-2) & 0 & 0 & \\ 0 & Q(1) & c-16 & Q(-3) & 0 & \cdots \\ 0 & 0 & Q(2) & c-36 & Q(-4) & \\ 0 & 0 & 0 & Q(3) & c-64 & \\ & & & \cdots & & \end{vmatrix}=0 \tag{4.272}$$

$$b_{\text{even}}:\begin{vmatrix} c-4 & Q(-2) & 0 & 0 & \\ Q(1) & c-16 & Q(-3) & 0 & \\ 0 & Q(2) & c-36 & Q(-4) & \cdots \\ 0 & 0 & Q(3) & c-64 & \\ & & \cdots & & \end{vmatrix}=0 \tag{4.273}$$

$$a_{\text{odd}}:\begin{vmatrix} c-1+P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \\ 0 & P(2) & c-25 & P(-3) & \cdots \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{vmatrix}=0 \tag{4.274}$$

$$b_{\text{odd}}: \begin{vmatrix} c-1-P(0) & P(-1) & 0 & 0 & \\ P(1) & c-9 & P(-2) & 0 & \\ 0 & P(2) & c-25 & P(-3) & \cdots \\ 0 & 0 & P(3) & c-49 & \\ & & \cdots & & \end{vmatrix} = 0. \tag{4.275}$$

Comparison of determinants 4.272 and 4.273 shows that if the first row and first column of Eq. 4.299 are removed, then the remainder of Eq. 4.272 is identical to Eq. 4.273. The significance of this observation is that if any one of the off-diagonal terms vanishes—that is, if $Q(m) = 0$ for some integer m (positive, negative, or zero)—then coexistence can occur, and an infinite number of possible tongues of instability will not occur.

In order to understand how this works, suppose that $Q(2) = 0$. Then we may represent Eqs. 4.272 and 4.273 symbolically as follows:

$$a_{\text{even}}: \begin{vmatrix} X & X & 0 & 0 & 0 & \\ X & X & X & 0 & 0 & \\ 0 & X & X & X & 0 & \cdots \\ 0 & 0 & Q(2) & X & X & \\ 0 & 0 & 0 & X & X & \\ & & & \cdots & & \end{vmatrix} = 0 \tag{4.276}$$

$$b_{\text{even}}: \begin{vmatrix} X & X & 0 & 0 & \\ X & X & X & 0 & \\ 0 & Q(2) & X & X & \cdots \\ 0 & 0 & X & X & \\ & & \cdots & & \end{vmatrix} = 0, \tag{4.277}$$

where we have used the symbol X to represent a term which is nonzero. The vanishing of $Q(2)$ "disconnects" the lower (infinite) portion of these equations from the upper (finite) portion.

There are now two possible ways in which to satisfy these equations with $Q(2) = 0$.

1. For a nontrivial solution to the lower (infinite) portion, the (disconnected, infinite) determinant must vanish. Since this determinant is identical for both the as and the bs, coexistence is present, and the associated tongues emanating from $c = 36, 64, \ldots$ do not occur. The coefficients $a_6, a_8, a_{10}, \ldots$ and $b_6, b_8, b_{10}, \ldots$ will not, in general, vanish. In this case, the upper portion of the determinant will not vanish, in general, and the coefficients a_0, a_2, a_4, b_2, and b_4 will not be zero, because they depend, respectively, on a_6 and b_6.
2. Another possibility is that the infinite determinant of the lower portion is not zero, requiring that the associated a_{even} and b_{even} coefficients vanish. With these as and bs zero, the upper portion of the system becomes independent of the lower. For a nontrivial solution for a_0, a_2, a_4, the upper portion of determinant 4.276 must vanish, whereas for a nontrivial solution for b_2 and b_4, the upper

portion of determinant 4.277 must vanish. For Eq. 4.276, this involves a 3×3 determinant and yields a cubic on c, while for Eq. 4.277, this involves a 2×2 determinant and gives a quadratic on c. Together, these yield five expressions for c in terms of the other parameters of the problem, which, if real, correspond to five transition curves. One of these passes through the c-axis at $c = 0$, and the other four produce tongues of instability emanating from $c = 4$ and $c = 16$, respectively.

A similar story holds for Eqs. 4.274 and 4.275. If $P(m) = 0$ for some integer m (positive, negative, or zero), then only a finite number of tongues will occur from among the infinite set of tongues that emanate from the points $c = (2n-1)^2, n = 1, 2, 3, \ldots$ on the c-axis.

As an example, let us return to Eq. 4.254, for which $a = b = \varepsilon/2$ and $d = 0$. The polynomials $Q(m)$ and $P(m)$ become, from Eqs. 4.270 and 4.271,

$$Q(m) = \frac{d}{2} + bm - 2am^2 = \frac{\varepsilon}{2}(m - 2m^2) = 0 \Rightarrow Q(0) = 0, Q\left(\frac{1}{2}\right) = 0, \quad (4.278)$$

$$\begin{aligned} P(m) &= \frac{d + b(2m-1) - a(2m-1)^2}{2} \\ &= \frac{\varepsilon}{4}\left[(2m-1) - (2m-1)^2\right] \Rightarrow P(1) = 0, P\left(\frac{1}{2}\right) = 0. \end{aligned} \quad (4.279)$$

The important results here are that $Q(0) = 0$ and $P(1) = 0$. When $Q(0) = 0$ is substituted into Eqs. 4.272 and 4.273, we see that the element c in the upper left corner of Eq. 4.282 becomes disconnected from the rest of the infinite determinant, which is itself identical to the infinite determinant in Eq. 4.273. From this, we may conclude that all the "even" tongues disappear.

And when $P(1) = 0$ is substituted into Eqs. 4.274 and 4.275, we see that the element in the upper left corner of both Eqs. 4.274 and 4.275 becomes disconnected from the rest of the infinite determinant, which itself is the same for both Eqs. 4.274 and 4.275. From this, we may conclude that only one "odd" tongue survives. It is bounded by the transition curves $c = 1 \pm P(0) = 1 \pm \frac{\varepsilon}{2}$.

4.4.4 Designing a System with a Finite Number of Tongues

By choosing the coefficients a, b, and d in Eq. 4.253 such that both $Q(m)$ and $P(m)$ have integer zeros, we may design a system that possesses a finite number of tongues of instability. For example, let us take $Q(-2) = 0$ and $P(3) = 0$. Since $P(m) = Q(m - 1/2)$ from Eq. 4.271, $P(3) = Q(5/2) = 0$, and we require a function $Q(m)$ which has zeros $m = -2, 5/2$,—that is, $Q(m) = (m+2)(m-5/2)$ $= m^2 - m/2 - 5$. Now since $Q(m) = d/2 + bm - 2am^2$ from Eq. 4.297, we may choose $a = -\varepsilon/2, b = -\varepsilon/2$, and $d = -10\varepsilon$, producing the ODE (Rand et al. 2005):

$$\left(1 - \frac{\varepsilon}{2}\cos 2t\right)\frac{d^2x}{dt^2} - \frac{\varepsilon}{2}\sin 2t\frac{dx}{dt} + (c - 10\varepsilon\cos 2t)x = 0. \tag{4.280}$$

From the reasoning presented above, we see from Eqs. 4.282 and 4.273 that $Q(-2) = 0$ produces a single tongue emanating from the point $c = 4, \varepsilon = 0$. Similarly, we see from Eqs. 4.274 and 4.275 that $P(3) = 0$ produces three tongues emanating from the points $c = 1, 9, 25, \ \varepsilon = 0$. Thus, Eq. 4.280 has four tongues of instability.

This result may be checked by generating series expansions for the transition curves and verifying that the tongue widths are zero for all tongues except for the four stated tongues.

4.4.5 Application

4.4.5.1 Application 1

In this section on Mathieu's equation, we saw that the stability of the x-mode of the particle in the plane was governed by the equation (Rand et al. 2005)

$$\frac{d^2v}{dt^2} + \left(\frac{1 - L - A^2\cos^2 t}{1 - A^2\cos^2 t}\right)v = 0. \tag{4.281}$$

Multiplying Eq. 4.281 by $1 - A^2\cos^2 t$ and using a trig identity, we obtain

$$\left(1 - \frac{A^2}{2} - \frac{A^2}{2}\cos 2t\right)\frac{d^2v}{dt^2} + \left(1 - L - \frac{A^2}{2} - \frac{A^2}{2}\cos 2t\right)v = 0. \tag{4.282}$$

Equation 4.282 may be put in the form of Ince's equation (4.253) by dividing by $1 - \frac{A^2}{2}$, in which case we obtain the following expressions for the parameters a, b, c, d:

$$a = d = \frac{\frac{-A^2}{2}}{1 - \frac{A^2}{2}}, \quad b = 0, \quad c = \frac{1 - L - \frac{A^2}{2}}{1 - \frac{A^2}{2}}. \tag{4.283}$$

Next, we use Eqs. 4.297 and 4.298 to compute $Q(m)$ and $P(m)$:

$$Q(m) = \frac{d}{2} + bm - 2am^2 = a\left(-2m^2 + \frac{1}{2}\right) \Rightarrow Q\left(\frac{1}{2}\right) = 0, Q\left(-\frac{1}{2}\right) = 0, \tag{4.284}$$

$$P(m) = \frac{d + b(2m-1) - a(2m-1)^2}{2} \\ = \frac{a}{2}(-(2m-1)^2 + 1) \Rightarrow P(0) = 0, P(1) = 0. \tag{4.285}$$

The important result here is that $P(0) = 0$ and $P(1) = 0$. Inspection of Eqs. 4.274 and 4.275 shows that the resulting linear algebraic equations on the coefficients $a_{\rm odd}$ are identical to those on $b_{\rm odd}$, so that coexistence occurs for all solutions of period 2π. Thus, all the "odd" tongues are absent. On the other hand, since the zeros of $Q(m)$ are not integers, we see that Eq. 4.281 exhibits an infinite number of "even" tongues that are bounded by transition curves on which there exist solutions of period π.

4.4.5.2 Application 2

A two-degree-of-freedom system consists of a particle of mass m and a disk having moment of inertia J, which are respectively restrained by two linear springs and a linear torsion spring. As is shown in the figure, the equations of motion can be written in the form (Rand et al. 2005)

$$(1 + \varepsilon y^2)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\varepsilon y \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} + p^2 x = 0, \tag{4.286}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \varepsilon y\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + y = 0. \tag{4.287}$$

This system has an exact solution called the y-mode:

$$y = A\sin t, \quad x = 0 \tag{4.288}$$

The stability of the y-mode is governed by the linear variational equation

$$\left(1 + \frac{\varepsilon A^2}{2} - \frac{\varepsilon A^2}{2}\cos 2t\right)\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \varepsilon A^2 \sin 2t \frac{\mathrm{d}u}{\mathrm{d}t} + p^2 u = 0. \tag{4.289}$$

Equation 4.289 can be put in the form of Ince's equation (4.253) by dividing by $1 + \frac{\varepsilon A^2}{2}$. The parameters a, b, c, d are found to be

$$b = -2a = \frac{\varepsilon A^2}{1 + \frac{\varepsilon A^2}{2}}, \quad c = \frac{p^2}{1 + \frac{\varepsilon A^2}{2}}, \quad d = 0. \tag{4.290}$$

Next, we use Eqs. 4.270 and 4.271 to compute $Q(m)$ and $P(m)$:

$$Q(m) = \frac{d}{2} + bm - 2am^2 = -2a(m^2 + m) \Rightarrow Q(0) = 0, Q(-1) = 0. \tag{4.291}$$

$$P(m) = \frac{d + b(2m-1) - a(2m-1)^2}{2} = \frac{-2a(2m-1) - a(2m-1)^2}{2} \Rightarrow P\left(\pm\frac{1}{2}\right) = 0. \tag{4.292}$$

The important result here is that $Q(0) = 0$ and $Q(-1) = 0$. Inspection of Eqs. 4.272 and 4.273 shows that $c = 0$ is a transition curve and that the linear equations on the other a_{even} coefficients are identical to those on the b_{even} coefficients, so that coexistence occurs for all solutions of period π. Thus, all the "even" tongues are absent. On the other hand, since the zeros of $P(m)$ are not integers, we see that Eq. 4.289 exhibits an infinite number of "odd" tongues that are bounded by transition curves on which there exist solutions of period 2π.

4.4.5.3 Application 3

This example involves an elastic pendulum—that is, a plane pendulum consisting of a mass m suspended under gravity g by a weightless elastic rod of unstretched length L and having spring constant k. Let the position of the mass be given by the polar coordinates r and φ. Then the kinetic energy T and the potential energy V are given by

$$T = \frac{m}{2}\left[\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right], \tag{4.293}$$

$$V = \frac{k}{2}(r-L)^2 - mgr\cos\phi. \tag{4.294}$$

Lagrange's equations for this system are

$$m\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - mr\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + k(r-L) - mgr\cos\phi = 0, \tag{4.295}$$

$$mr^2\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + 2mr\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + mgr\sin\phi = 0. \tag{4.296}$$

Equations 4.295 and 4.296 have an exact solution, the $r-$mode:

$$r = A\cos\omega t + L + \frac{mg}{k}, \quad \phi = 0, \quad \text{where} \quad \omega = \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}. \tag{4.297}$$

The stability of the r-mode is governed by the linear variational equation

$$(Ak\cos\omega t + mg + kL)\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - 2Ak\omega\sin\omega t\frac{\mathrm{d}u}{\mathrm{d}t} + gku = 0. \tag{4.298}$$

In order to put Eq. 4.297 in the form of Ince's equation (4.253), we set

$$\omega t = 2\tau, \tag{4.299}$$

which gives

$$(Ak\cos\omega t + mg + kL)\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} - 4Ak\omega\sin 2\tau\frac{\mathrm{d}u}{\mathrm{d}\tau} + \frac{4gk}{\omega^2}u = 0. \tag{4.300}$$

Equation 4.300 can be put in the form of Ince's equation (4.253) by dividing by $mg + kL$. The parameters a, b, c, d are found to be

$$a = -\frac{b}{4} = \frac{Ak}{mg + kL}, \quad c = \frac{4ag}{A\omega^2}, \quad d = 0. \tag{4.301}$$

Next, we use Eqs. 4.297 and 4.298 to compute $Q(m)$ and $P(m)$:

$$Q(m) = \frac{d}{2} + bm - 2am^2 = -2a(m^2 + 2m) \Rightarrow Q(0) = 0, Q(-2) = 0, \tag{4.302}$$

$$\begin{aligned} P(m) &= \frac{d + b(2m-1) - a(2m-1)^2}{2} \\ &= -2a(2m-1) - \frac{a}{2}(2m-1)^2 \Rightarrow P\left(\frac{1}{2}\right) = 0, P\left(-\frac{3}{2}\right) = 0. \end{aligned} \tag{4.303}$$

The important result here is that $Q(0) = 0$ and $Q(-2) = 0$. Inspection of Eqs. 4.272 and 4.273 shows that $c = 0$ is a transition curve and that the linear equations on the other a_{even} coefficients are identical to those on the b_{even} coefficients, so that coexistence occurs for all solutions of period π. Thus, all the "even" tongues are absent. Note that $c = 4$ is an exact transition curve, but because of coexistence, there is no associated tongue. On the other hand, since the zeros of $P(m)$ are not integers, we see that Eq. 4.300 exhibits an infinite number of "odd" tongues, which are bounded by transition curves on which there exist solutions of period 2π.

Problems

Solve the differential equations of motion for systems in this chapter using presented methods in previous chapters.

4.1 This problem concerns a nonlinear oscillator with quadratic nonlinearity

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x + \varepsilon x^2 = 0$$

and the initial condition $x(0) = 1$, $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$.

4.2 For the oscillator,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x + \varepsilon a x^2 + \varepsilon^2 b x^3 = 0$$

4.3 We studied the forced Duffing oscillator in the form

$$\frac{d^2x}{dt^2} + x + \varepsilon c\frac{dx}{dt} + \varepsilon\alpha x^3 = \varepsilon F\cos\omega t, \qquad \varepsilon \ll 1$$

4.4 Consider the free response of the undamped, SDOF system where it is shown in Fig. 4.11 that the restoring forces in the spring are given by

$$F_{sp} = -\left(k\,x(t) + \alpha\,x(t)^3\right)$$

with $\alpha > 0$. With this restoring force, the equation of motion of the system is

$$\ddot{x} + k\,x + \alpha\,x^3 = 0$$

where the equation of motion for this system with a cubic nonlinear stiffness is commonly known as Duffing's equation.

4.5 The equations of motion for the two identical coupled double-well Duffing oscillators that we are interested in are the following:

$$\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt} + x - x^3 + k(y-x) + f\cos(\Omega t)$$
$$\frac{d^2y}{dt^2} = -\alpha\frac{dy}{dt} + y - y^3 - k(y-x)$$

where α is the damping parameter, k the coupling parameter, f and the amplitude and the frequency of the driving force, respectively.

4.6 The Duffing equation near resonance at $\Omega = 3$, with weak excitation, is

$$\ddot{x} + 9x = \varepsilon(\gamma\cos t - \beta x + x^3).$$

4.7 The Duffing oscillator with equation

$$\ddot{x} + \varepsilon\kappa\dot{x} - x + x^3 = \varepsilon f(t),$$

is driven by an even $T =$ periodic function $f(t)$ with mean value zero. where $f(t)$ is

$$f(t) = \begin{cases} \gamma, & -\frac{1}{2} < t < \frac{1}{2}, \\ -\gamma, & \frac{1}{2} < t < \frac{3}{2}. \end{cases}$$

4.8 Consider the following generalization of Van der Pol's equation:

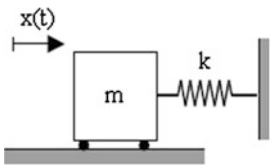


Fig. 4.11 Single-degree-of-freedom system

$$\frac{d^2x}{dt^2} + x - \varepsilon\left(1 - ax^2 - b\left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} = 0$$

4.9 This problem concerns the equation

$$\frac{d^2x}{dt^2} + x + \varepsilon\frac{dx}{dt}\left(1 - \left(\frac{dx}{dt}\right)^2 + \beta\left(\frac{dx}{dt}\right)^4\right) = 0$$

4.10 This problem concerns the equation

$$\frac{d^2x}{dt^2} + x + 0.035\frac{dx}{dt} + x^3 - 0.6x^2\frac{dx}{dt} + 0.1\left(\frac{dx}{dt}\right)^3 = 0,$$
$$\frac{d^2x}{dt^2} + x - \varepsilon\left(1 - ax^2 - b\left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} = 0.$$

4.11 This problem concerns the equation

$$\frac{d^2x}{dt^2} + x - \varepsilon\frac{dx}{dt}\left(1 + x - x^2\right) = 0, \qquad \varepsilon \gg 1$$

4.12 This problem concerns the equation

$$\frac{d^2z}{dt^2} + z = -A\frac{dz}{dt} + z^3 - z^2\frac{dz}{dt}$$

4.13 Consider the modified Van der Pol oscillator:

$$\frac{d^2x}{dt^2} - \varepsilon\left(1 - x^4\right)\frac{dx}{dt} + x = 0.$$

4.14 The Van der Pol equation is

$$\frac{d^2u}{dt^2} + \varepsilon\frac{du}{dt}\left(u^2 - 1\right) + u = 0.$$

4.15 Consider two Van der Pol oscillators with delay coupling:

$$\ddot{x}_1 + x_1 - \varepsilon(1 - x_1^2)\dot{x}_1 = \varepsilon a\dot{x}_2(t - \tau),$$
$$\ddot{x}_2 + x_2 - \varepsilon(1 - x_2^2)\dot{x}_2 = \varepsilon a\dot{x}_1(t - \tau),$$

where a is a coupling parameter, τ is the delay time, and $\varepsilon \ll 1$.

4.16 Consider a Van der Pol oscillator with a cubic nonlinear spring under harmonic forcing:

$$\ddot{x} - \varepsilon\dot{x}\left(1 - x^2\right) + x + \varepsilon x^3 = F\cos\frac{\Omega}{\omega_n}t.$$

4.17 Consider a model of two coupled Van der Pol oscillators that can be represented by the following differential equations:

$$\ddot{X}_1 + \omega_1^2 X_1 = \kappa(1 - \beta X_1^2)\dot{X}_1 + G(\dot{X}_2 - \dot{X}_1) + R_1 \sin \omega_e t$$
$$\ddot{X}_2 + \omega_1^2 X_2 = \kappa(1 - \beta X_2^2)\dot{X}_2 + G(\dot{X}_1 - \dot{X}_2) + R_2 \sin \omega_e t.$$

4.18 An autonomous modified van der Pol oscillator is described by the equation

$$M\ddot{x} + \Gamma(x^2 - 1)\dot{x} + \frac{2\pi b}{\lambda} \sin(2\pi x/\lambda) + Kx = 0$$

where M is the mass, Γ is damping coefficient, b is the strength of the periodic potential, λ is its period, and K is the stiffness constant.

4.19 Consider a general class of nonlinear Van der Pol oscillators:

$$\ddot{x} + \operatorname{sgn}(x)|x|^{\alpha} = \varepsilon\dot{x}\left(1 - x^2\right), \quad \alpha > 0,$$

where

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{for} \quad x > 0, \\ -1 & \text{for} \quad x > 0. \end{cases}$$

4.20 The one-dimensional, nonlinear elastic force Van der Pol oscillator equation is

$$\ddot{x} + x|x| = \varepsilon\dot{x}\left(1 - x^2\right),$$

where ε is a positive parameter.

4.21 Consider the system governed by

$$\ddot{x} + \mu \sin \dot{x} + x = 0$$

4.22 The equation for the nonrelativistic externally forced Van der Pol oscillator is

$$\ddot{x} + \alpha\dot{x}\left(x^2 - 1\right) + kx = g \cos \omega t,$$

where the right-hand side corresponds to an external driving force.

4.23 The system

$$\dot{x} = -\frac{1}{2}\alpha\left(1 - \frac{1}{4}r^2\right)x - \frac{\omega^2 - 1}{2\omega}y \quad \left(r^2 = x^2 + y\right),$$
$$\dot{y} = \frac{\omega^2 - 1}{2\omega}x + \frac{1}{2}\alpha\left(1 - \frac{1}{4}r^2\right)y + \frac{\Gamma}{2\omega}$$

occurs in the theory of forced oscillations of the Van der Pol equation:

$$(i)\ \alpha = 1, \Gamma = 0.75, \omega = 1.2;$$
$$(ii)\ \alpha = 1, \Gamma = 2.0, \omega = 1.6$$

4.24 The Van der Pol equation with parametric excitation is

$$\ddot{x} + \varepsilon\left(x^2 - 1\right)\dot{x} + (1 + \beta\cos t)x = 0.$$

4.25 The equations of a displaced Van der Pol oscillator are given by

$$\dot{x} = y - a,\ \dot{y} = -x + \delta\left(1 - x^2\right)y,$$

where $a > 0$ and $\delta > 0$. If the parameter a $= 0$, then the usual equations for the Van der Pol oscillator appear.

4.26 This problem concerns the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{1}{4} + \varepsilon k_1\right)x + \varepsilon x^3\cos t = 0, \qquad \varepsilon << 1$$

4.27 This question concerns the nonlinear Mathieu equation $\alpha > 0$:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + (\delta + \varepsilon\cos t)x + \varepsilon\alpha x^3 = 0, \qquad \alpha > 0$$

4.28 Damped Mathieu equation for $\delta = 1/4$:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{1}{4} + \varepsilon\cos t\right)x = 0$$

4.29 We consider the differential equation

$$\left(1 - \frac{\varepsilon}{2}\cos 2t\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{\varepsilon}{2}\sin 2t\frac{\mathrm{d}x}{\mathrm{d}t} + (c - 10\varepsilon\cos 2t)x = 0$$

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Chapter 5
Applied Problems in Dynamical Systems

In this chapter, we consider several important applied problems in the field of dynamics, vibrations, and oscillations that were analyzed by methods mentioned in previous chapters and solved by nonlinear dynamical teams from Babol Noshirvani University of Technology.

5.1 Problem 5.1. Displacement of the Human Eardrum

5.1.1 Introduction

As in the first problem, we consider the displacement equation of a human eardrum:

$$u'' + \omega^2 u + \varepsilon u^2 = 0, u(0) = A, u'(0) = 0. \tag{5.1}$$

5.1.2 Variational Iteration Method

According to the variational iteration method (VIM), the first three approximations of this equation can be written as follows:

$$u_0(t) = A \cos \alpha\omega t, \tag{5.2}$$

$$u_1(t) = a \cos \omega t - b + c \cos 2\alpha\omega t, \tag{5.3}$$

where

$$a = A + \frac{\varepsilon A^2}{2\omega^2} - \frac{\varepsilon A^2}{2\omega^2(4\alpha^2 - 1)}, b = \frac{\varepsilon A^2}{2\omega^2}, c = \frac{\varepsilon A^2}{2\omega^2(4\alpha^2 - 1)}, \tag{5.4}$$

S. H. H. Kachapi and D. D. Ganji, *Dynamics and Vibrations*,
Solid Mechanics and Its Applications 202, DOI: 10.1007/978-94-007-6775-1_5,

$$\alpha = \frac{1}{2}\sqrt{\varepsilon A/(\varepsilon A + 2\omega^2) + 1}. \tag{5.5}$$

$$u_2(t) = u_1(t) - \left(-b\omega^2 + \varepsilon b^2 + \frac{\varepsilon c^2}{2}\right) \times \frac{1}{\omega^2}(1 - \cos\omega t) + (-4\alpha^2\omega^2 c + \omega^2 c - 2bc\varepsilon) \times \frac{1}{\omega^2(4\alpha^2 - 1)}(\cos 2\alpha\omega - \cos\omega t) + \frac{c^2\varepsilon}{2\omega^2(16\alpha^2 - 1)}(\cos 4\alpha\omega t - \cos\omega t) \tag{5.6}$$

Now this equation can be solved using the other three methods.

5.1.3 Perturbation Method

Since this method does not have a high level of accuracy, the fourth order is used to develop accuracy, as follows.

Using a perturbation technique up to the fourth order of ε, we have

$$u = u_0 + \varepsilon^1 u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4. \tag{5.7}$$

Inserting Eq. 5.7 into Eq. 5.1 and equating the coefficients of powers of ε on both sides, we obtain the following equations:

$$\varepsilon^0 : \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_0(t)\right) + \omega^2 u_0(t) = 0, \tag{5.8}$$

$$\varepsilon^1 : \omega^2 u_1(t) + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_1(t)\right) + u_0(t)^2 = 0, \tag{5.9}$$

$$\varepsilon^2 : \omega^2 u_2(t) + 2u_0(t)u_1(t) + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_2(t)\right) = 0, \tag{5.10}$$

$$\varepsilon^3 : u_1(t)^2 + \omega^2 u_3(t) + 2u_0(t)u_2(t) + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_3(t)\right) = 0, \tag{5.11}$$

$$\varepsilon^4 : \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u_4(t)\right) + \omega^2 u_4(t) + 2u_1(t)u_2(t) + 2u_0(t)u_3(t) = 0. \tag{5.12}$$

Equations 5.8–5.12 are solved recursively for $u_i(t)(i = 0, 1, 2, 3, 4)$, with respect to the initial conditions (Eq. 5.1).

Consider the following initial conditions:

$$u_0(0) = A, \;\; u_0'(0) = 0. \tag{5.13}$$

Inserting them into Eq. 5.8, the solution is simply obtained as follows:

$$u_0(t) = A\cos\omega t. \tag{5.14}$$

Substituting Eq. 5.14 into Eq. 5.9, we have

$$\omega^2 u_1(t) + \left(\frac{d^2}{dt^2}u_1(t)\right) + A^2\cos(\omega t)^2 = 0. \tag{5.15}$$

Considering Eq. 5.15 and using the initial conditions

$$u_1(0) = 0,\ \ u_1'(0) = 0, \tag{5.16}$$

the solution of Eq. 5.15 is attained as follows:

$$u_1(t) = \frac{1}{3}\frac{\cos(\omega t)A^2}{\omega^2} + \frac{1}{6}\frac{A^2(-3+\cos(2\omega t))}{\omega^2}. \tag{5.17}$$

In the same way, we substitute Eqs. 5.14 and 5.17 into Eq. 5.10, and we obtain

$$\omega^2 u_2(t) + 2A\cos(\omega t)\left(\frac{1}{3}\frac{\cos(\omega t)A^2}{\omega^2} + \frac{1}{6}\frac{A^2(-3+\cos(2\omega t))}{\omega^2}\right) + \left(\frac{d^2}{dt^2}u_2(t)\right) = 0 \tag{5.18}$$

with the following initial conditions:

$$u_2(0) = 0, u_2'(0) = 0. \tag{5.19}$$

Now we solve Eq. 5.18 with the above initial conditions as follows:

$$\begin{aligned} u_2(t) = {} & \frac{1}{144}\frac{A^3\cos(\omega t)}{\omega^4} \\ & + \frac{1}{144}\frac{A^3(-48+16\cos(2\omega t)+30\cos(\omega t)+60\sin(\omega t)\omega t)}{\omega^4}. \end{aligned} \tag{5.20}$$

In the same way, we obtain $u_3(t)$ and $u_4(t)$ as

$$\begin{aligned} u_3(t) = {} & \frac{29}{432}\frac{A^4\cos(\omega t)}{\omega^6} + \frac{1}{432\omega^6}A^4(-225+90\cos(\omega t)+180\sin(\omega t)\omega t \\ & + 9\cos(3\omega t) + \cos(4\omega t) + 96\cos(2\omega t) + 60\omega t\sin(2\omega t)), \end{aligned} \tag{5.21}$$

$$\begin{aligned} u_4(t) = {} & \frac{37}{6912}\frac{\cos(\omega t)A^5}{\omega^8} + \frac{1}{20736\omega^8}A^5(-14400+6960\cos(\omega t)+13920 \\ & \sin(\omega t)\omega t + 1116\cos(3\omega t) + 64\cos(4\omega t) + 540\omega t\sin(3\omega t) + 6144 \\ & \cos(2\omega t) + 3840\omega t\sin(2\omega t) + 5\cos(5\omega t) - 1800\cos(\omega t)\omega^2 t^2) \end{aligned} \tag{5.22}$$

Substituting $u_1(t), u_2(t), u_3(t)$, and $u_4(t)$ into Eq. 5.7, $u(t)$ will be

$$\begin{aligned} u(t) = \frac{1}{20736\omega^8}(&-14400A^5\varepsilon^4 + 20736A\cos(\omega t)\omega^8 + 7071A^5\varepsilon^4\cos(\omega t) \\ &+ 1116A^5\varepsilon^4\cos(3\omega t) + 64A^5\varepsilon^4\cos(4\omega t) + 6144A^5\varepsilon^4\cos(2\omega t) + 5A^5\varepsilon^4\cos(5\omega t) \\ &- 10800A^4\varepsilon^3\omega^2 - 6912A^3\varepsilon^2\omega^4 - 10368A^2\varepsilon\omega^6 + 8640A^3\varepsilon^2\omega^5\sin(\omega t)t \\ &+ 8640A^4\varepsilon^3\omega^3\sin(\omega t)t + 540A^5\varepsilon^4\omega t\sin(3\omega t) + 2880A^4\varepsilon^3\omega^3 t\sin(2\omega t)) \\ &+ 6912A^2\varepsilon\omega^6\cos(\omega t) + 48A^4\varepsilon^3\omega^2\cos(4\omega t) + 4608A^4\varepsilon^3\omega^2\cos(2\omega t) \\ &+ 3456A^2\varepsilon\omega^6\cos(2\omega t) + 4176A^3\varepsilon^2\omega^4\cos(\omega t) + 432A^3\varepsilon^2\omega^4\cos(3\omega t) \\ &+ 5712A^4\varepsilon^3\omega^2\cos(\omega t) + 432A^4\varepsilon^3\omega^2\cos(3\omega t) + 2304A^3\varepsilon^2\omega^4\cos(2\omega t) \\ &+ 13920A^5\varepsilon^4\sin(\omega t)\omega t + 3840A^5\varepsilon^4\omega t\sin(2\omega t) - 1800A^5\varepsilon^4\cos(\omega t)\omega^2 t^2 \end{aligned} \tag{5.23}$$

The perturbation method (PM) is not exact enough for the fewer number of repetitions, so that we have to apply four stages, as seen above.

5.1.4 Homotopy Perturbation Method

According to the homotopy technique,

$$L = u'' + \omega^2 u, N = \varepsilon u^2 \tag{5.24}$$

We construct a homotopy as follows:

$$\left(\frac{d^2}{dt^2}v(t)\right) + \omega^2 v(t) - \left(\frac{d^2}{dt^2}u_0(t)\right) - \omega^2 u_0(t) + p\left(\left(\frac{d^2}{dt^2}u_0(t)\right) + \omega^2 u_0(t)\right) + p\varepsilon v^2(t) = 0 \tag{5.25}$$

Substituting $v(t) = v_0(t) + pv_1(t)$ into Eq. 5.25, we have

$$\omega^2 v_1(t) + \left(\frac{d^2}{dt^2}v_1(t)\right) + \left(\frac{d^2}{dt^2}u_0(t)\right) + \omega^2 u_0(t) + \varepsilon v_0^2(t) = 0. \tag{5.26}$$

Substituting $u_0(t) = v_0(t) = A\cos(\alpha\omega t)$ into Eq. 5.26, we obtain

$$\omega^2 v_1(t) + \left(\frac{d^2}{dt^2}v_1(t)\right) - A\cos(\alpha\omega t)\alpha^2\omega^2 + \omega^2 A\cos(\alpha\omega t) + \frac{1}{2}\varepsilon A^2\cos(2\alpha\omega t) = 0. \tag{5.27}$$

Now we solve Eq. 5.27 with the conditions of $v_1(0) = 0, v_1'(0) = 0$:

$$\begin{aligned} v_1(t) = \frac{A}{2\omega^2(-1+4\alpha^2)}(&8\omega^2\cos(\omega t)\alpha^2 + 4\varepsilon A\cos(\omega t)\alpha^2 - 2\omega^2\cos(\omega t) \\ &- 2\varepsilon A\cos(\omega t) + \varepsilon A\cos(2\alpha\omega t) - 4\varepsilon A\alpha^2 + 2\omega^2\cos(\alpha\omega t) - 8\cos(\alpha\omega t)\alpha^2\omega^2 + \varepsilon A) \end{aligned} \tag{5.28}$$

Therefore,

$$u(t) = \lim_{p \to 1}(v_0(t) + pv_1(t)) = \frac{1}{2}\frac{A^2\varepsilon(-\cos(2\alpha\omega t) + 4\alpha^2 - 1)}{\omega^2(-1 + 4\alpha^2)}. \quad (5.29)$$

Now we can obtain different values of $u(t)$ for different values of ε, ω, A, and α.

5.1.5 Numerical Solution

The numerical solution obtained from the Runge–Kutta method includes the results from Table 5.1, Figs. 5.1 and 5.2.

Consistent with this problem, when $t < 3$, all the methods lead to similar results, and as t increases, the results of the different methods increasingly diverge from each other.

Table 5.1 The numerical results of $u(t)$ for different values of time for Eq. 5.1 with the fixed values of $A = 1, \varepsilon = 0.1, \alpha = 0.51177$, and $\omega = 1$

t	$u(t)$	t	$u(t)$
0	1.00000	5.5	0.66070
0.5	0.86589	6	0.94711
1	0.50196	6.5	0.98055
1.5	0.00843	7	0.75138
2	−0.48824	7.5	0.32455
2.5	−0.87181	8	−0.18516
3	−1.06020	8.5	−0.65137
3.5	−1.01555	9	−0.96877
4	−0.74669	9.5	−1.07129
4.5	−0.30868	10	−0.93859
5	0.20169		

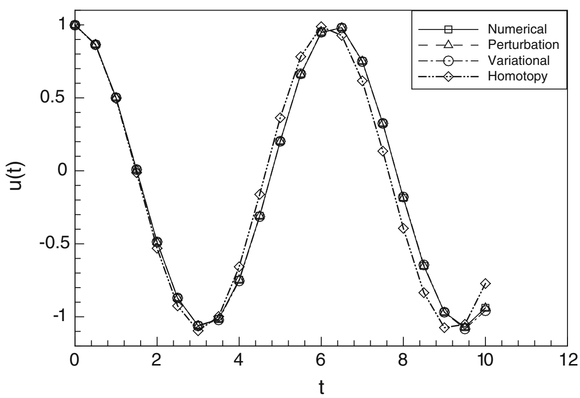


Fig. 5.1 The solution results of $u(t)$ by means of the four methods for Eq. 5.1 with the fixed values of $A = 1, \varepsilon = 0.1, \alpha = 0.51177$, and $\omega = 1$

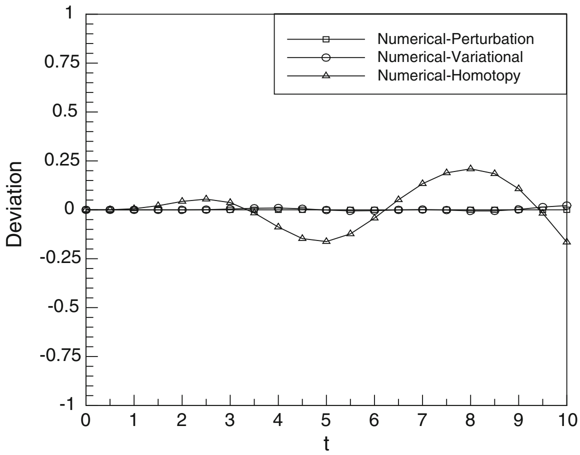


Fig. 5.2 Deviation of the three methods from the numerical results for Eq. 5.1 with the fixed values of $A = 1, \varepsilon = 0.1, \alpha = 0.51177$, and $\omega = 1$

5.2 Problem 5.2. Slides Motion Along a Bending Wire

5.2.1 Introduction

In this problem, we consider a bead of mass m that slides without friction along a wire that has the shape of a parabola $y = \lambda x^2$ with axis vertical in the earth's gravitational field g, as shown in Fig. 5.3. x is the generalized coordinate of the horizontal displacement. We can write down Lagrange's equation of motion as

$$(1 + 4\lambda^2 x(t)^2)\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\right) + 4\lambda^2 x(t)\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 + 2g\lambda x(t). \tag{5.30}$$

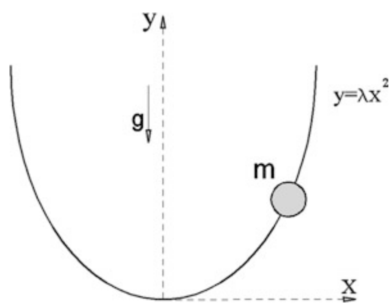


Fig. 5.3 Geometry of the problem

5.2.2 Energy Balance Method

Equation 5.30 can be rewritten as

$$\frac{d^2x(t)}{dt^2} + \omega_0^2\,x(t) + M\frac{d^2x(t)}{dt^2}\,x(t)^2 + Nx(t)\left(\frac{dx(t)}{dt}\right)^2 = 0, \tag{5.31}$$

where

$$N = M = 4\,\lambda^2 \text{ and } \omega_0^2 = 2\,g\,\lambda\,. \tag{5.32}$$

Initial conditions are

$$x(0) = A, \dot{x}\mathrm{Y}(0) = 0. \tag{5.33}$$

Its variational and Hamiltonian formulations for $N = M = 2$ can be easily obtained as

$$J(x) = \int_0^t \left\{-\frac{1}{2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}\,\omega_0^2\,x(t)^2 + x(t)^2\left(\frac{dx(t)}{dt}\right)^2\right\}dt. \tag{5.34}$$

Its Hamiltonian, therefore, can be written in the form

$$H = -\frac{1}{2}\left(\frac{dx(t)}{dt}\right)^2 + \frac{1}{2}\,\omega_0^2\,x(t)^2 + x(t)^2\left(\frac{dx(t)}{dt}\right)^2 - \frac{1}{2}\,\omega_0^2\,A^2. \tag{5.35}$$

Choosing the trial function $x(t) = A\,\cos(\omega t)$, the following residual equation will be obtained as

$$R = \frac{1}{2}\,A^2\,\sin(\omega t)^2\,\omega^2 + \frac{1}{2}\,A^2\,\omega_0^2\,\cos(\omega t)^2 + A^4\,\cos(\omega t)^2\,\sin(\omega t)^2\,\omega^2 - \frac{1}{2}\,\omega_0^2\,A^2. \tag{5.36}$$

If we collocate at $\omega t = \frac{\pi}{4}$, the following result will be achieved:

$$\omega = \frac{\omega_0\sqrt{1 + A^2}}{1 + A^2}. \tag{5.37}$$

The following approximate solution is obtained:

$$x(t) = A\,\cos\left(\frac{\omega_0\sqrt{1 + A^2}}{1 + A^2}\,t\right). \tag{5.38}$$

5.2.3 Variational Iteration Method

To solve Eq. 5.31 by means of the VIM, the arbitrary initial approximation is supposed as

$$x_0(t) = A\ \cos(\omega t). \tag{5.39}$$

Then we have

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -M\left(\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\right)x(t)^2 - Nx(t)\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 - \omega_0^2\, x(t) \tag{5.40}$$

or

$$\begin{aligned}\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} &= -M\left(\frac{\mathrm{d}^2(A\ \cos(\omega t))}{\mathrm{d}t^2}\right)(A\ \cos(\omega t))^2\\ &\quad - N(A\ \cos(\omega t))\left(\frac{\mathrm{d}(A\ \cos(\omega t))}{\mathrm{d}t}\right)^2 - \omega_0^2(A\ \cos(\omega t)).\end{aligned} \tag{5.41}$$

Integrating twice yields

$$\begin{aligned}x(t) = &-\frac{1}{9}\frac{1}{\omega^2}(A(-7MA^2\omega^2 + 2NA^2\omega^2 + 9\omega_0^2 + MA^2\omega^2\cos(\omega t)^3\\ &+ 6MA^2\omega^2\cos(\omega t) - NA^2\omega^2\cos(\omega t)\ \sin(\omega t)^2 - 2NA^2\omega^2\ \cos(\omega t)\\ &- 9\omega_0^2\ \cos(\omega t))).\end{aligned} \tag{5.42}$$

Equating the coefficients of $\cos(\omega t)$ in Eq. 5.42, we have

$$-\frac{1}{9}\frac{A\left(\frac{27}{4}\,MA^2\omega^2 - \frac{9}{4}\,NA^2\omega^2 - 9\omega_0^2\right)}{\omega^2} = A \tag{5.43}$$

or

$$\omega = \frac{2\,\omega_0\sqrt{(3MA^2 - NA^2\ + 4)}}{3MA^2 - NA^2 + 4}\ \text{rad/s}. \tag{5.44}$$

Therefore:

$$x_0 = A\cos\left(\frac{2\,\omega_0\sqrt{(3MA^2 - NA^2\ + 4)}}{3MA^2 - NA^2 + 4}t\right), \tag{5.45}$$

where $\delta\tilde{x}_n$ is considered as a restricted variation. Its stationary conditions can be obtained as

$$x_{n+1}(t) = x_n(t) + \int_0^t \gamma\left(\frac{d^2x(\zeta)}{d\zeta^2} + \omega_0^2 x(\zeta) - M\left(\frac{d^2x(\zeta)}{d\zeta^2}\right)x(\zeta)^2 - Nx(\zeta)\left(\frac{dx(\zeta)}{d\zeta}\right)^2\right)d\zeta, \tag{5.46}$$

where γ is a Lagrange multiplier. Its stationary conditions can be obtained as

$$\frac{\partial^2\gamma(t,\zeta)}{\partial\zeta^2} + \omega_0^2\gamma(t,\zeta) = 0, \tag{5.47}$$

$$1 - \frac{\partial\gamma(t,\zeta)}{\partial\zeta}|_{t=\zeta} = 0, \tag{5.48}$$

$$\gamma(t,\zeta)|_{t=\zeta} = 0. \tag{5.49}$$

The Lagrangian multiplier can there be identified as

$$\gamma = \frac{1}{\omega_0^2}\sin\omega_0^2(\zeta - t). \tag{5.50}$$

As a result, the following iteration formula is obtained:

$$x_{n+1}(t) = x_n(t) + \int_0^t \frac{1}{\omega_0^2}\sin\,\omega_0^2(\tau - t)\left(\frac{d^2x(t)}{dt^2} + M\left(\frac{d^2x(t)}{dt^2}\right)x(t)^2 + N\,x(t)\left(\frac{dx(t)}{dt}\right)^2 + \omega_0^2 x(t)\right)d\tau. \tag{5.51}$$

Using Eq. 5.51, other components can be obtained directly as follows:

$$x_1(t) = A\,\cos(\omega t) - \frac{1}{\omega_0^2(3MA^2 - NA^2 + 4)}(A^3(18M\cos(\omega t)^2 + 18N\cos(\omega t)^2 + 32M\cos(\omega t)^6 - 48M\cos(\omega t)^4 + 32N\cos(\omega t)^6 - 48N\cos(\omega t)^4 - M - N)(\cos(\omega_0^2 t) - 1)). \tag{5.52}$$

and so on, in the same way, the rest of the components of the iteration formula can be obtained.

5.2.4 Parameter Lindstedt–Poincaré Method

In order to use the parameter Lindstedt–Poincaré method (PL-PM), Eq. 5.31 can be re-written in the form

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) - \varepsilon\left(\left(\omega_0^2 - \omega^2\right)x(t) - M\frac{d^2x(t)}{dt^2}x(t)^2 - N x(t)\left(\frac{dx(t)}{dt}\right)^2\right) = 0. \tag{5.53}$$

The solution $x(t)$ and frequency ω are expanded as

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots, \tag{5.54}$$

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots. \tag{5.55}$$

Substituting Eqs. 5.54 and 5.55 into Eq. 5.53 and equating the terms with the identical powers of ε, gives

$$\varepsilon^0 : \ddot{x}_0(t) + \omega^2 x_0(t) = 0, \tag{5.56}$$

$$\varepsilon^1 : \frac{d^2x_1(t)}{dt^2} + \omega^2 x_1(t) + (\omega_0^2 - \omega^2)x_0(t) + M\frac{d^2x_0(t)}{dt^2}x_0(t)^2 + N x_0(t)\left(\frac{dx_0(t)}{dt}\right)^2 = 0. \tag{5.57}$$

Considering the initial conditions $x(0) = A, \dot{x}(0) = 0$, the solution of Eq. 5.56 is $x_0(t) = A\,\cos(\omega t)$.

Substituting $x_0(t)$ into Eq. 5.57 and simplifying it, we obtain

$$\frac{d^2x_1(t)}{dt^2} + \omega^2 x_1(t) + (\omega_0^2 - \omega^2)A\cos(\omega t) - MA^3\cos(\omega t)^3\omega^2 + NA^3\cos(\omega t)\sin(\omega t)^2\omega^2 = 0. \tag{5.58}$$

If, for this problem, the first-order approximation of solution and frequency are sufficient, then setting $\varepsilon = 1$ in Eqs. 5.54 and 5.55, we have

$$x(t) = x_0(t) + x_1(t), \tag{5.59}$$

$$\omega^2 = \omega_0^2 + \omega_1. \tag{5.60}$$

Based on of trigonometric functions properties, we have

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t). \tag{5.61}$$

Substituting Eq. 5.61 into Eq. 5.58 and eliminating the secular term, we have

$$(\omega_0^2 - \omega^2)A - \frac{3}{4}MA^3\omega^2 + \frac{1}{4}NA^3\omega^2 = 0 \tag{5.62}$$

or

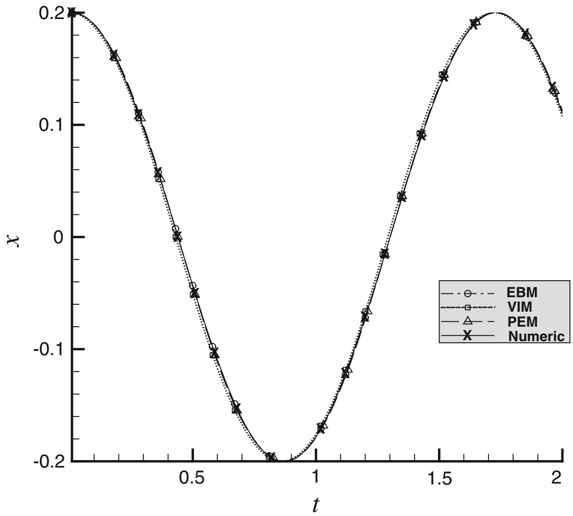


Fig. 5.4 The comparison of the obtained solutions with the numerical solution at $\lambda = \frac{\sqrt{2}}{2}, A = 0.2$

$$\omega = \frac{2\sqrt{(4 + 3MA^2 - NA^2)}}{4 + 3MA^2 - NA^2}\,\omega_0. \tag{5.63}$$

Solving the equation (5.58) yields

$$x_1(t) = \cos(\omega t)\left(\frac{1}{32}\,MA^3 + \frac{1}{32}\,NA^3\right) - \frac{1}{32}\,A^3(M + N)\,\cos(3\omega t). \tag{5.64}$$

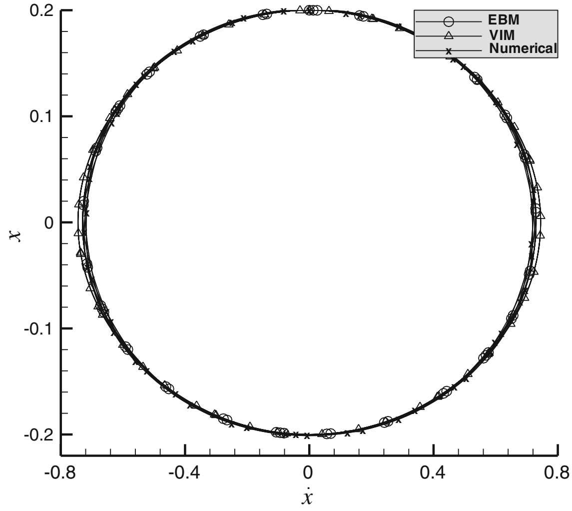


Fig. 5.5 Phase plane, at $\lambda = \frac{\sqrt{2}}{2}, A = 0.2$

Then we have (Figs. 5.4, 5.5)

$$x(t) = x_0(t) + x_1(t)$$
$$= A\cos(\omega t) + \cos(\omega t)\left(\frac{1}{32}MA^3 + \frac{1}{32}NA^3\right) - \frac{1}{32}A^3(M+N)\cos(3\omega t). \tag{5.65}$$

5.3 Problem 5.3. Movement of a Mass Along a Circle

5.3.1 Introduction

As is shown in Fig. 5.6, the motion of a mass m moving without friction along a circle of radius R that is rotating with a constant angular velocity Ω about its vertical diameter is considered. The forces acting on the particle are the gravitational force mg, the centrifugal force $m\Omega^2 R\sin(\theta(t))$, and the reaction force N. Taking moments about the center of the circle o and equating their summation to the rate of change of angular momentum of the particle about o, we obtain

$$mR^2\ddot{\theta}(t) - m\Omega^2R^2\sin(\theta(t))\cos(\theta(t)) + mgR\sin(\theta(t)). \tag{5.66}$$

In this problem, the nonlinearity is due to both inertia and large deformation. The initial conditions are

$$\theta(0) = A,\ \frac{d\theta}{dt}(0) = 0. \tag{5.67}$$

By using the approximations $\sin(\theta(t)) \approx \theta(t) - \frac{\theta(t)^3}{3!}, \cos(\theta(t)) \approx 1 - \frac{\theta(t)^2}{2!}$, the governing equation can be rewritten as

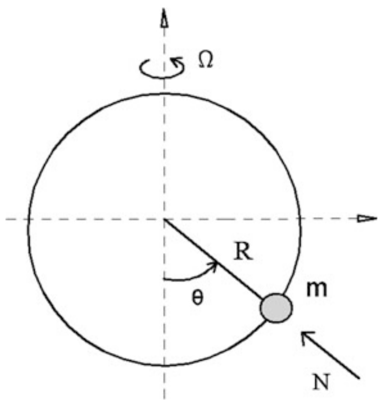


Fig. 5.6 Geometry of problem

$$\frac{d^2(\theta(t))}{dt^2} + \omega_0^2\theta(t) + G\,\theta(t)^3 - M\,\theta(t)^5 = 0,\ \ \theta(0) = A, \dot{\theta}(0) = 0, \tag{5.68}$$

where

$$It = m\,R^2, \omega_0^2 = \left(\frac{m\,g\,R}{It} - \frac{m\,\Omega^2\,R^2}{It}\right), G = \left(-\frac{m\,g\,R}{6\,It} + \frac{2\,m\,\Omega^2\,R^2}{3It}\right) \text{ and}$$
$$M = \frac{m\,\Omega^2\,R^2}{12\,It}. \tag{5.69}$$

5.3.2 Energy Balance Method

Variational and Hamiltonian formulations of Eq. 5.68 can be easily obtained as

$$J(x) = \int_0^t \left\{ -\frac{1}{2}\left(\frac{d\theta(t)}{dt}\right)^2 + \frac{1}{2}\omega_0^2\,\theta(t)^2 + \frac{G}{4}\,\theta(t)^4 - \frac{M}{6}\theta(t)^6 \right\} dt. \tag{5.70}$$

Its Hamiltonian, therefore, can be written in the form

$$H = -\frac{1}{2}\left(\frac{d\theta(t)}{dt}\right)^2 + \frac{1}{2}\omega_0^2\,\theta(t)^2 + \frac{G}{4}\,\theta(t)^4 - \frac{M}{6}\theta(t)^6 - \frac{1}{2}\omega_0^2 A^2 - \frac{G}{4}A^4 + \frac{M}{6}A^6. \tag{5.71}$$

Choosing the trial function $\theta(t) = A\,\cos(\omega t)$, the following residual equation can be obtained:

$$R = \frac{1}{2}A^2\,\sin(\omega t)^2\,\omega^2 + \frac{1}{2}A^2\,\omega_0^2\,\cos(\omega t)^2 + \frac{G}{4}A^4\,\cos(\omega t)^4 - \frac{M}{6}A^6\,\cos(\omega t)^6$$
$$-\frac{1}{2}\omega_0^2 A^2 - \frac{G}{4}A^4 + \frac{M}{6}A^6. \tag{5.72}$$

If we collocate at $\omega t = \frac{\pi}{4}$, the following result can be obtained:

$$\omega = \frac{1}{6}\sqrt{27\,GA^2 + 36\,\omega_0^2 - 21\,M\,A^4}. \tag{5.73}$$

The approximate solution can be obtained in the form

$$\theta(t) = A\,\cos\left(\frac{1}{6}\sqrt{27\,GA^2 + 36\,\omega_0^2 - 21\,M\,A^4}\,t\right). \tag{5.74}$$

5.3.3 Variational Iteration Method

According to VIM, the arbitrary initial approximation is supposed as

$$\theta_0(t) = A\ \cos(\omega t). \tag{5.75}$$

Then we have

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -G\,\theta(t)^3 + M\,\theta(t)^5 - \omega_0^2\,\theta(t) \tag{5.76}$$

or

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -G(A\ \cos(\omega t))^3 + M(A\ \cos(\omega t))^5 - \omega_0^2(A\ \cos(\omega t)). \tag{5.77}$$

Twice integrating yields

$$\begin{aligned}\theta(t) = &-\frac{1}{225}\frac{1}{\omega^2}(A(225\,\omega_0^2 + 175\,GA^2 - 149\,MA^4 - 225\,\omega_0^2\ \cos(\omega t)\\ &- 25\,GA^2\ \cos(\omega t)^3 - 150\,GA^2\ \cos(\omega t) + 9\,MA^4\ \cos(\omega t)^5\\ &+ 20\,MA^4\ \cos(\omega t)^3 + 120\,MA^4\ \cos(\omega t))).\end{aligned} \tag{5.78}$$

Equating the coefficients of $\cos(\omega t)$ in Eq. 5.78, we have

$$-\frac{1}{225}\frac{A\left(-\frac{675}{4}\,GA^2 + \frac{1125}{8}\,MA^4 - 225\,\omega_0^2\right)}{\omega^2} = A \tag{5.79}$$

or

$$\omega = \frac{1}{4}\sqrt{12\,GA^2 + 16\,\omega_0^2 - 10\,MA^4} \qquad \text{rad/s}. \tag{5.80}$$

Therefore:

$$\theta_0 = A\cos\left(\frac{1}{4}\sqrt{12\,GA^2 + 16\,\omega_0^2 - 10\,MA^4}\ t\right) \tag{5.81}$$

where $\delta\tilde{\theta}_n$ is considered as a restricted variation. Its stationary conditions can be obtained as

$$\theta_{n+1}(t) = \theta_n(t) + \int_0^t \lambda\left(\frac{\mathrm{d}^2\theta(\zeta)}{\mathrm{d}\zeta^2} + \omega_0^2\,\theta(\zeta) - M\,\theta(\zeta)^5 + G\,\theta(\zeta)^3\right)\mathrm{d}\zeta \tag{5.82}$$

where λ is a Lagrange multiplier. Its stationary conditions can be obtained as

$$\frac{\partial^2\lambda(t,\zeta)}{\partial\zeta^2} + \omega_0^2\,\lambda(t,\zeta) = 0, \tag{5.83}$$

$$1 - \frac{\partial \lambda(t,\zeta)}{\partial \zeta}|_{t=\zeta} = 0, \tag{5.84}$$

$$\lambda(t,\zeta)|_{t=\zeta} = 0. \tag{5.85}$$

The Lagrangian multiplier can there be identified as

$$\lambda = \frac{1}{\omega_0^2} \sin \omega_0^2 (\zeta - t). \tag{5.86}$$

As a result, the following iteration formula is obtained as

$$\theta_{n+1}(t) = \theta_n(t) + \int_0^t \frac{1}{\omega_0^2} \sin\, \omega_0^2(\tau - t) \left(\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} - M\,x(t)^5 + G\,\theta(t)^3\, + \omega_0^2\,\theta(t) \right) \mathrm{d}\tau. \tag{5.87}$$

Using Eq. 5.87, other components can be obtained directly as

$$\begin{aligned} \theta_1(t) = & A\, \cos(\omega t) - \frac{1}{8\,\omega_0^4}(A^3\, \cos(\omega t)(6\,G - 5\,M\,A^2 \\ & + 8\,M\,A^2\, \cos(\omega t)^4 - 5\,G\, \cos(\omega t)^2)(\cos(\omega_0^2 t) - 1)). \end{aligned} \tag{5.88}$$

In the same way, the rest of the components of the iteration formula can be obtained.

5.3.4 Parameter Lindstedt–Poincaré Method

In order to use the PL-PM, Eq. 5.68 can be re-written in the form

$$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \omega^2\,\theta(t) - \varepsilon((\omega_0^2 - \omega^2)\,\theta(t) - M\,\theta(t)^5 + G\,\theta(t)^3) = 0. \tag{5.89}$$

The solution $x(t)$ and frequency ω are expanded as

$$\theta(t) = \theta_0(t) + \varepsilon\,\theta_1(t) + \varepsilon^2\,\theta_2(t) + \cdots, \tag{5.90}$$

$$\omega^2 = \omega_0^2 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2^2 + \cdots. \tag{5.91}$$

Substituting Eqs. 5.90 and 5.91 into Eq. 5.89 and equating the terms with the identical powers of ε gives

$$\varepsilon^0 : \frac{\mathrm{d}^2\theta_0(t)}{\mathrm{d}t^2} + \omega^2\,\theta_0(t) = 0, \tag{5.92}$$

$$\varepsilon^1 : \frac{d^2\theta_1(t)}{dt^2} + \omega^2\,\theta_1(t) + (\omega_0^2 - \omega^2)\,\theta_0(t) - M\,\theta_0(t)^5 + G\,\theta_0(t)^3 = 0. \tag{5.93}$$

$$\vdots$$

Considering the initial conditions $\theta(0) = A, \dot{\theta}(0) = 0$, the solution of Eq. 5.92 is $\theta_0(t) = A\,\cos(\omega t)$. Substituting $\theta_0(t)$ into Eq. 5.93 and simplifying it, we obtain

$$\frac{d^2\theta_1(t)}{dt^2} + \omega^2\,\theta_1(t) + (\omega_0^2 - \omega^2)\,A\,\cos(\omega t) - M A^5 \cos(\omega t)^5 + G A^3\,\cos(\omega t)^3 = 0. \tag{5.94}$$

Similarly, setting $\varepsilon = 1$ in Eqs. 5.90 and 5.91, the first-order approximations of solution and frequency are

$$\theta(t) = \theta_0(t) + \theta_1(t), \tag{5.95}$$

$$\omega^2 = \omega_0^2 + \omega_1. \tag{5.96}$$

Based on of trigonometric functions properties, we have

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t). \tag{5.97}$$

Substituting Eq. 5.97 into Eq. 5.94 and eliminating the secular term leads to

$$(\omega_0^2 - \omega^2)\,A - \frac{5}{8}\,M A^5 + \frac{3}{4}\,G A^3 = 0 \tag{5.98}$$

or

$$\omega = \frac{1}{4}\sqrt{12\,G A^2 + 16\,\omega_0^2 - 10\,M A^4}. \tag{5.99}$$

Solving Eq. 5.94, we obtain

$$\begin{aligned}\theta_1(t) = {} & \frac{1}{96}\frac{\cos(\omega t)\,A^3(4\,M A^2 - 3G)}{\omega^2} - \frac{1}{384}\frac{1}{\omega^2}(A^3(15\,M A^2\cos(3\omega t) \\ & - 12\,G\,\cos(3\omega t) + M A^2\,\cos(5\omega t))).\end{aligned} \tag{5.100}$$

Then the first approximation solution can be written as (Figs. 5.7, 5.8)

$$\begin{aligned}\theta(t) &= \theta_0(t) + \theta_1(t) \\ &= A\,\cos(\omega t) + \frac{1}{96}\frac{\cos(\omega t)\,A^3(4\,M A^2 - 3G)}{\omega^2} - \frac{1}{384}\frac{1}{\omega^2}(A^3(15\,M A^2\cos(3\omega t) \\ &\quad - 12\,G\,\cos(3\omega t) + M A^2\,\cos(5\omega t))).\end{aligned} \tag{5.101}$$

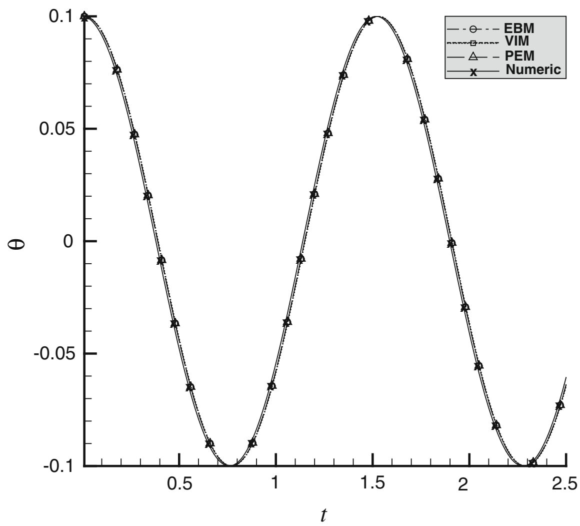


Fig. 5.7 Comparison of the obtained solutions with the numerical solution at $m = 3, R = 0.5, \Omega = \frac{\pi}{2}$ and $A = 0.1$.

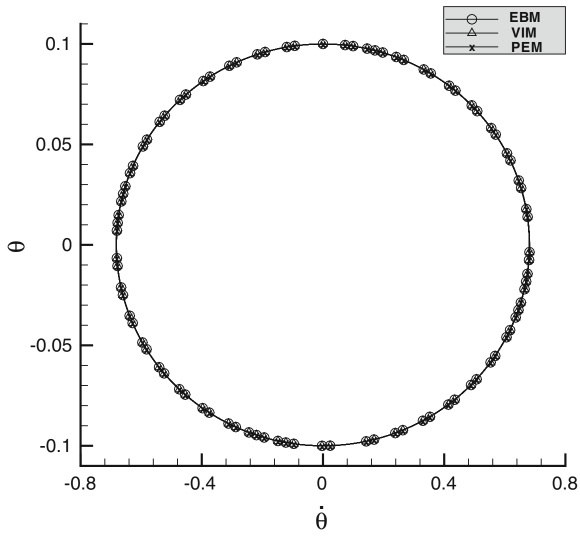


Fig. 5.8 Phase plane, at $m = 3, R = 0.5, \Omega = \frac{\pi}{2}$, and $A = 0.1$

5.4 Problem 5.4. Rolling a Cylinder on a Cylindrical Surface

5.4.1 Introduction

As is shown in Fig. 5.9, a cylinder with weight w and radius r rolls without slipping on a cylindrical surface of radius R. For no slipping, we have $r\,\phi = R\,\theta$. Its differential equation of motion about the lowest point is

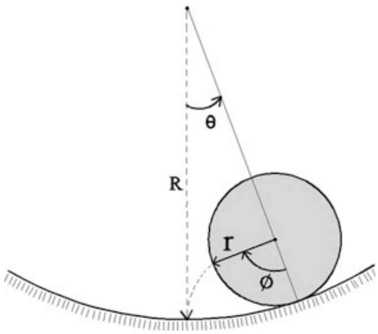


Fig. 5.9 Geometry of the problem

$$\frac{3}{2}\frac{w}{g}(R-r)^2\,\ddot{\theta}(t)+w(R-r)\,\sin(\theta(t))=0. \tag{5.102}$$

The initial conditions are

$$\theta(0)=A,\frac{\mathrm{d}\theta}{\mathrm{d}t}(0)=0. \tag{5.103}$$

At this step, we investigate Eq. 5.102 using an approximation for $\sin(\theta)$ as follows:

$$\sin(\theta(t))\approx\theta(t)-\frac{\theta(t)^3}{3!},-\frac{\pi}{2}\leq\theta(t)\leq\frac{\pi}{2}. \tag{5.104}$$

Substituting Eq. 5.104 into Eq. 5.102, and by some manipulation, Eq. 5.102 can be re-written in the form

$$\frac{\mathrm{d}^2(\theta(t))}{\mathrm{d}t^2}+\omega_0^2\theta(t)-\frac{g}{9(R-r)}\,\theta(t)^3=0,\ \theta(0)=A,\ \dot{\theta}(0)=0. \tag{5.105}$$

where

$$\omega_0^2=\frac{2\,g}{(3(R-r))}. \tag{5.106}$$

In the present problem, similar to the procedure mentioned in previous cases, the obtained results are as follows.

5.4.2 Energy Balance Method Results

$$\omega=\frac{1}{2}\sqrt{\frac{g}{3(R-r)}\,A^2\;+4\,\omega_0^2}, \tag{5.107}$$

$$\theta(t) = A\,\cos\left(\frac{1}{2}\sqrt{\frac{g}{3(R-r)}\,A^2 + 4\,\omega_0^2}\;t\right). \tag{5.108}$$

5.4.3 Variational Iteration Method Results

$$\omega = \frac{1}{2}\sqrt{\frac{g}{3(R-r)}\,A^2\; + 4\,\omega_0^2}\,, \tag{5.109}$$

$$\theta(t) = A\,\cos(\omega t) - \frac{1}{4\,\omega_0^4}\left(\frac{g\,A^3}{9(R-r)}\cos(\omega t)(-3 + 4\,\cos(\omega t)^2)(\cos(\omega_0^2 t) - 1)\right). \tag{5.110}$$

5.4.4 Parameter Lindstedt–Poincaré Method Results

$$\omega = \frac{1}{2}\sqrt{\frac{g}{3(R-r)}\,A^2\; + 4\,\omega_0^2}\,, \tag{5.111}$$

$$\theta(t) = A\,\cos(\omega t) + \frac{g}{288(R-r)\,\omega^2}\,A^3\,\cos(\omega t)\; - \frac{g}{288(R-r)\,\omega^2}\,A^3\,\cos(3\omega t). \tag{5.112}$$

In this problem, the three resulting frequencies calculated using the three different methods are exactly the same (Figs. 5.10, 5.11).

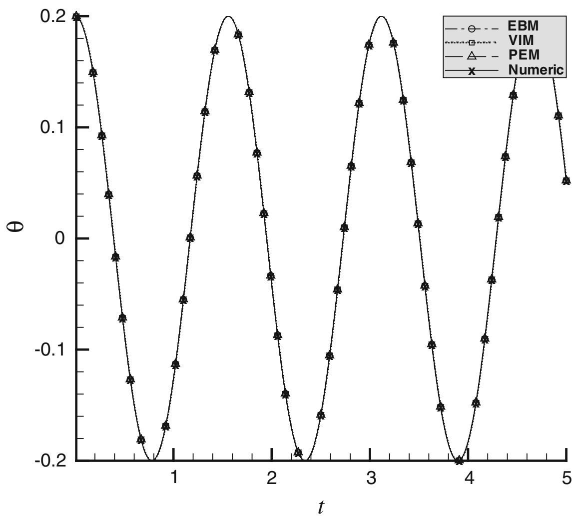


Fig. 5.10 Comparison of the obtained solutions with the numerical solution at $R = 0.5, r = 0.1$, and $A = 0.2$

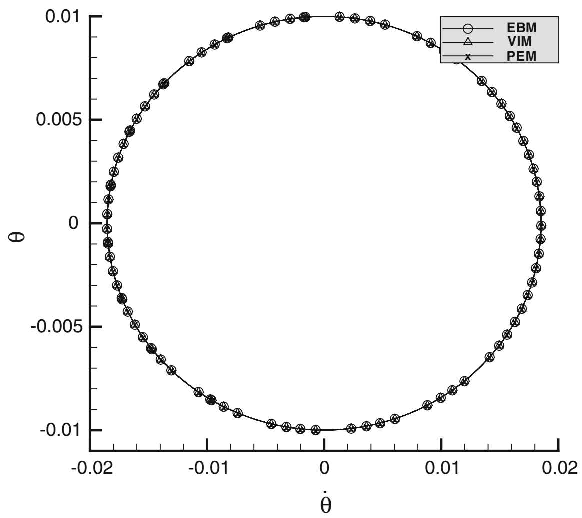


Fig. 5.11 Phase plane, at $R = 2, r = 0.1$, and $A = 0.01$

5.5 Problem 5.5. Movement of Rigid Rods on a Circular Surface

5.5.1 Introduction

The rigid rod rocks back and forth on the circular surface without slipping (Fig. 5.12).

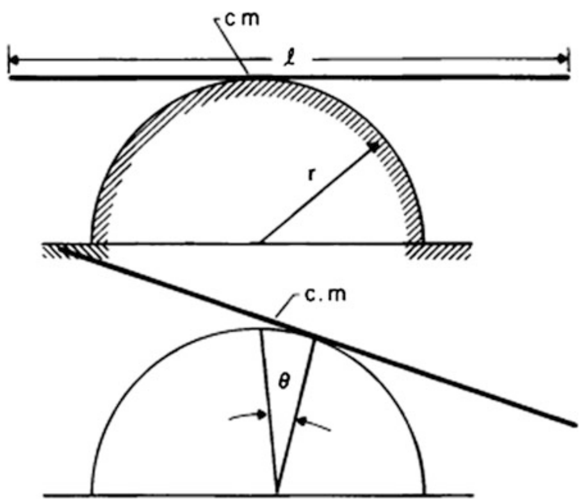


Fig. 5.12 Rigid rods on a circular surface

The equation governing θ is

$$\left(\frac{1}{12}l^2 + r^2\,\theta(t)^2\right)\ddot{\theta}(t) + r^2\theta(t)(\dot{\theta}(t))^2 + g\,r\,\theta(t)\,\cos(\theta(t)) = 0. \tag{5.113}$$

5.5.2 Energy Balance Method

We consider the above equation with the following initial conditions:

$$\theta(0) = A, \dot{\theta}(0) = 0. \tag{5.114}$$

By some manipulation, in Eq. 5.113, we have the following equation:

$$\ddot{\theta}(t) + M(\ddot{\theta}(t))\,\theta(t)^2 + N\,\theta(t)\,\dot{\theta}(t)^2 + a\,\theta(t) - W\,\theta(t)^3 = 0 \tag{5.115}$$

with the initial condition of Eq. 5.114, where

$$N = M = \frac{12\,r^2}{l^2}, W = \frac{6\,g\,r^2}{l^2}\text{ and }a = \frac{12\,g\,r}{l^2}, \tag{5.116}$$

its variational formulation for $N = M = 2$ or $\frac{r}{l} = \sqrt{6}$ can be easily established as

$$J(\theta) = \int_0^t \left\{-\frac{1}{2}\,\dot{\theta}^2 + \frac{1}{2}\,a\,\theta^2 \,+\, \theta^2\,\dot{\theta}^2 - \frac{1}{4}\,W\,\theta^4\right\}\mathrm{d}t. \tag{5.117}$$

Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{\theta}(t)^2 + \frac{1}{2}\,a\,\theta(t)^2 + \theta(t)^2\,\dot{\theta}(t)^2 - \frac{1}{2}a\,A^2 + \frac{1}{4}\,W\,A^4 \tag{5.118}$$

and

$$H_{t=0} = \frac{a}{2}\,A^2 - \frac{1}{4}\,W\,A^4, \tag{5.119}$$

$$H_t - H_{t=0} = \frac{1}{2}\dot{\theta}(t)^2 + \frac{\theta(t)^2}{2} + \frac{1}{2}\,\theta(t)^2\,\dot{\theta}(t)^2 - \frac{a}{2}\,A^2 + \frac{1}{4}\,W\,A^4 = 0. \tag{5.120}$$

We will use the trial function to determine the angular frequency ω, i.e.,

$$\theta(t) = A\,\cos(\omega t). \tag{5.121}$$

If we substitute Eq. 5.121 into Eq. 5.120, it results in the residual equation

$$\frac{1}{2}(-A\omega\,\sin(\omega t))^2+\frac{(A\,\cos(\omega t))^2}{2}+\frac{1}{2}(A\,\cos(\omega t))^2(-A\omega\,\sin(\omega t))^2-\frac{a}{2}A^2+\frac{1}{4}\,WA^4=0. \tag{5.122}$$

If we collocate at $\omega t=\frac{\pi}{4}$, we obtain

$$\frac{1}{4}A^2\omega^2+\frac{A^2\,\omega^2}{4}-\frac{\mathrm{a}}{4}A^2+\frac{3}{16}\,WA^4\;=0 \tag{5.123}$$

or

$$\omega\;=\frac{1}{2}\,\frac{\sqrt{-(1+A^2)(-4a+3WA^2)}}{1+A^2}. \tag{5.124}$$

Hence, the approximate period is

$$T\;=\frac{2\pi}{\left(\frac{1}{2}\,\frac{\sqrt{-(1+A^2)(-4a+3WA^2)}}{1+A^2}\right)} \tag{5.125}$$

and

$$\theta(t)=A\,\cos\left(\frac{1}{2}\,\frac{\sqrt{-(1+A^2)(-4a+3WA^2)}}{1+A^2}t\right). \tag{5.126}$$

5.5.3 Variational Iteration Method

To solve Eq. 5.115 by means of the VIM, we start with an arbitrary initial approximation

$$\theta_0(t)=A\,\cos(\omega t). \tag{5.127}$$

Then we have

$$\ddot{\theta}=-M\left(\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}\right)\theta(t)^2-N\,\theta(t)\left(\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\right)^2-a\,\theta(t)\;+W\,\theta(t)^3 \tag{5.128}$$

or

$$\ddot{\theta}=-M\left(\frac{\mathrm{d}^2(A\,\cos(\omega t))}{\mathrm{d}t^2}\right)(A\,\cos(\omega t))^2-N(A\,\cos(\omega t))\left(\frac{\mathrm{d}(A\,\cos(\omega t))}{\mathrm{d}t}\right)^2 \\ -a(A\,\cos(\omega t))\;+W(A\,\cos(\omega t))^3. \tag{5.129}$$

Integrating twice yields

$$\theta(t) = -\frac{1}{9}\frac{1}{\omega^2}(A(-7\,MA^2\omega^2 + 2\,NA^2\omega^2 + 9\,a - 7\,WA^2 + MA^2\omega^2\cos(\omega t)^3 + 6\,MA^2\omega^2\cos(\omega t) - NA^2\omega^2\cos(\omega t)\,\sin(\omega t)^2 - 2\,NA^2\,\omega^2\,\cos(\omega t) - 9\,a\,\cos(\omega t) + WA^2\,\cos(\omega t)^3 + 6\,WA^2\cos(\omega t))). \tag{5.130}$$

Equating the coefficients of $\cos(\omega t)$ in Eq. 5.131, we have

$$-\frac{1}{9}\frac{A\left(6\,MA^2\omega^2 - \frac{9}{4}\,NA^2\omega^2 - 9a + 6\,WA^2\right)}{\omega^2} = A \tag{5.131}$$

or

$$\omega = \frac{2\sqrt{-(8\,MA^2 - 3NA^2 + 12)(-3a + 2\,WA^2)}}{8\,MA^2 - 3NA^2 + 12}\ \text{rad/s}. \tag{5.132}$$

Therefore,

$$\theta_0 = A\cos\left(\frac{2\sqrt{-(8\,MA^2 - 3NA^2 + 12)(-3a + 2\,WA^2)}}{8\,MA^2 - 3NA^2 + 12}\,t\right). \tag{5.133}$$

We obtain the approximate period

$$T = \frac{2\pi}{\frac{2\sqrt{-(8\,MA^2 - 3NA^2 + 12)(-3a + 2\,WA^2)}}{8\,MA^2 - 3NA^2 + 12}}, \tag{5.134}$$

where $\delta\tilde{u}_n$ is considered as a restricted variation. Its stationary conditions can be obtained as

$$\theta_{n+1}(t) = \theta_n(t) + \int_0^t \lambda\left(\frac{d^2\theta(\zeta)}{d\zeta^2} + a\,\theta(\zeta) - M\left(\frac{d^2\theta(\zeta)}{d\zeta^2}\right)\theta(\zeta)^2 - N\,\theta(\zeta)\left(\frac{d\theta(\zeta)}{d\zeta}\right)^2 + W\,\theta(\zeta)^3\right)d\zeta. \tag{5.135}$$

Its stationary conditions can be obtained as

$$\frac{\partial^2\lambda(t,\zeta)}{\partial\zeta^2} + a\,\lambda(t,\zeta) = 0, \tag{5.136}$$

$$1 - \frac{\partial\lambda(t,\zeta)}{\partial\zeta}\Big|_{t=\zeta} = 0, \tag{5.137}$$

$$\lambda(t,\zeta)|_{t=\zeta} = 0. \tag{5.138}$$

The Lagrangian multiplier can be identified as

$$\lambda = \frac{1}{a}\sin a(\zeta - t). \tag{5.139}$$

As a result, we obtain the iteration formula

$$\theta_{n+1}(t) = \theta_n(t) + \int_0^t \frac{1}{a}\sin\, a(\tau - t)\left(\ddot{\theta} + M\left(\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2}\right)\theta(t)^2 + N\,\theta(t)\left(\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\right)^2 + a\,\theta(t)\; - W\,\theta(t)^3\right)\mathrm{d}\tau. \tag{5.140}$$

By use of Eq. 5.140, we can directly obtain other components as follows:

$$\begin{aligned}\theta_1(t) =& A\,\cos(\omega t) - \frac{1}{a^2}(A\,\cos(\omega t)(\omega^2 + M A^2\,\omega^2\,\cos(\omega t)^2 - N A^2\,\omega^2\,\sin(\omega t)^2 \\ &- a + W A^2\,\cos(\omega t)^2)(\cos(a\,t) - 1)).\end{aligned} \tag{5.141}$$

And so on, in the same way, the rest of the components of the iteration formula can be obtained.

5.5.4 Parametrized Perturbation Method

We rewrite the governing equation as

$$\begin{aligned}&\ddot{\theta}(t) + M(\ddot{\theta}(t))\,\theta(t)^2 + N\,\theta(t)\,\dot{\theta}(t)^2 + a\,\theta(t)\; - W\,\theta(t)^3 = 0,\\ &\theta(0) = A, \dot{\theta}(0) = 0.\end{aligned} \tag{5.142}$$

In order to apply the perturbation techniques, we introduce a small parameter ε by the transformation as

$$\theta(t) = \varepsilon v(t). \tag{5.143}$$

So the original Eq. 5.142 becomes a small parameter equation, which means that

$$\begin{aligned}&\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2} + \frac{12\,r^2\varepsilon^2}{l^2}v(t)^2\left(\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2}\right) + \frac{12\,r^2\varepsilon^2}{l^2}v(t)\left(\frac{\mathrm{d}v(t)}{\mathrm{d}t}\right)^2 - \frac{6\,g\,r\,\varepsilon^2}{l^2}v(t)^3 + a\,v(t) = 0,\\ &v(0) = \frac{A}{\varepsilon}, v'(0) = 0.\end{aligned} \tag{5.144}$$

We assume that ω_0^2 and the solution of Eq. 5.143 can be written in the forms

$$v(t) = v_0(t) + \varepsilon^2\,v_1(t) + \varepsilon^4 v_2(t) + \mathrm{K}, \tag{5.145}$$

$$a = \omega^2 + \varepsilon^2\,\omega_1 + \varepsilon^4\omega_2 + \mathrm{K}. \tag{5.146}$$

Substituting Eqs. 5.144 and 5.145 into Eq. 5.143 and equating coefficients of the same powers of ε results in the equations

$$\frac{d^2 v_0(t)}{dt^2} + \omega^2 v_0(t) = 0, v_0(0) = \frac{A}{\varepsilon}, \frac{dv_0}{dt}(0) = 0, \tag{5.147}$$

$$\frac{d^2 v_1(t)}{dt^2} + \frac{12\,r^2}{l^2} v_0(t)^2 \left(\frac{d^2 v_0(t)}{dt^2}\right) + \frac{12\,r^2}{l^2} v_0(t) \left(\frac{dv_0(t)}{dt}\right)^2 - \frac{6\,g\,r}{l^2} v_0(t)^3 + a\,v_1(t) + \omega_1 v_0(t) = 0,$$
$$v(0) = 0, \frac{dv(0)}{dt} = 0. \tag{5.148}$$

Solving Eq. 5.147 yields

$$v_0(t) = \frac{A}{\varepsilon}\cos(\omega t). \tag{5.149}$$

Substituting v_0 into Eq. 5.148 and eliminating the secular term gives us

$$\frac{\omega_1 A}{\varepsilon} - \frac{3}{4}\frac{M\,A^3\,\omega^2}{\varepsilon^3} + \frac{1}{4}\frac{N\,A^3\omega^2}{\varepsilon^3} - \frac{3}{4}\frac{W\,A^3}{\varepsilon^3} = 0 \tag{5.150}$$

or

$$\omega_1 = \frac{1}{4}\frac{A^2(3\omega^4\,M - N\,\omega^2 + 3W)}{\varepsilon^2}. \tag{5.151}$$

Where the angular frequency ω can be obtained from Eq. 5.146,

$$a = \omega^2 + \varepsilon^2\,\omega_1, \tag{5.152}$$

which leads to

$$\omega = \frac{\sqrt{-(4 + 3A^2M - A^2N)(-4a + 3A^2W)}}{4 + 3A^2M - A^2N}. \tag{5.153}$$

Hence, the approximate period is

$$T = \frac{2\pi}{\left(\frac{\sqrt{-(4+3A^2M-A^2N)(-4a+3A^2W)}}{4+3A^2M-A^2N}\right)}. \tag{5.154}$$

Solving Eq. 5.148, we obtain

$$\theta_1(t) = \frac{1}{32}\frac{\cos(\omega t)\,A^3(M\omega^2 + N\omega^2 + W)}{\varepsilon^3\omega^2} - \frac{1}{32}\frac{((M+N)\,\omega^2 + W)\,A^3\,\cos(3\omega t)}{\varepsilon^3\omega^2}. \tag{5.155}$$

If, for this problem, its first-order approximation is sufficient, then we have the first-order approximation solution of Eq. 5.142 (Figs. 5.13, 5.14, 5.15):

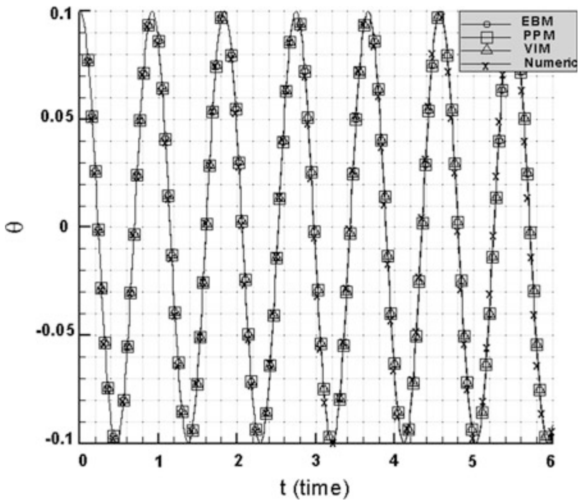


Fig. 5.13 Results of the energy balance method (EBM), parameterized perturbation method (PPM), and parameter expansion method (PEM) at $l = 1, r = \frac{\sqrt{6}}{6}\, l$ and $A = 0.1$

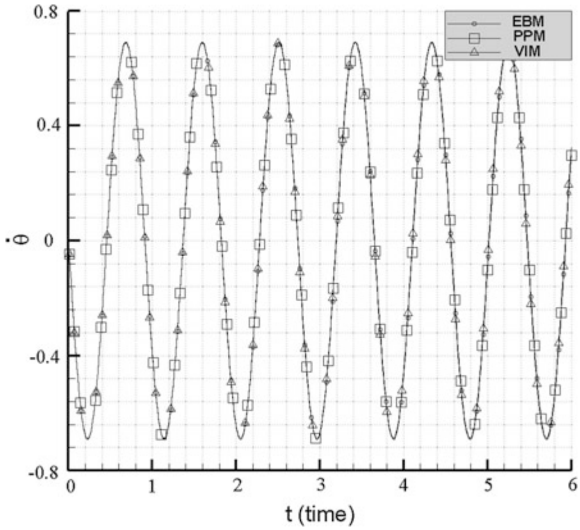


Fig. 5.14 Time history diagram of $\dot{\theta}$ at $l = 1, r = \frac{\sqrt{6}}{6} l$ and $A = 0.1$. *EBM* energy balance method, *PPM* parameterized perturbation method, *VIM* variational iteration method

$$
\begin{aligned}
\theta(t) &= \varepsilon(\theta_0(t) + \varepsilon^2\, \theta_1(t)) \\
&= A\, \cos(\omega t) + \frac{1}{32} \frac{\cos(\omega t)\, A^3 (M\omega^2 + N\omega^2 + W)}{\omega^2} - \frac{1}{32} \frac{((M+N)\,\omega^2 + W)\, A^3\, \cos(3\omega t)}{\omega^2}.
\end{aligned}
\tag{5.156}
$$

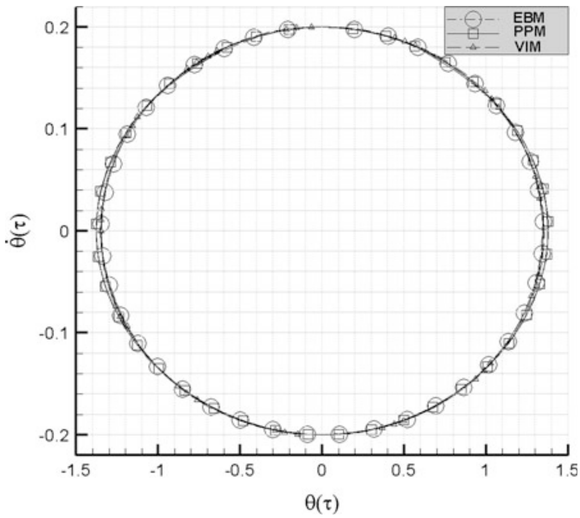


Fig. 5.15 Phase plane, in $l = 0.5, r = \frac{\sqrt{6}}{6}l, 0 \le t \le 40$ and $A = 0.2$.

5.6 Problem 5.6. Application of Two Degrees of Freedom Viscously Damped

5.6.1 Introduction

In this problem, a viscously damped spring–mass system having linear and nonlinear stiffness with two degrees of freedom has been considered, as shown in Fig. 5.16. The system consists of two blocks of mass m_1 and m_2. The block m_1 is connected to a nonlinear spring, the force displacement relation of which is $F_{\text{nonlinear spring}} = k_1 u(t) + k_3 u(t)^3$ and the same viscous damper with coefficient c that is connected to the latter. The block of mass m_2 is connected to a linear spring of stiffness k_2. The motion of the system is described by the coordinates $u(t)$ and $v(t)$, which define the positions of the masses m_1 and m_2 at any time t from the respective equilibrium positions. The application of Newton's second law of motion to each of the masses gives the equations of motion as

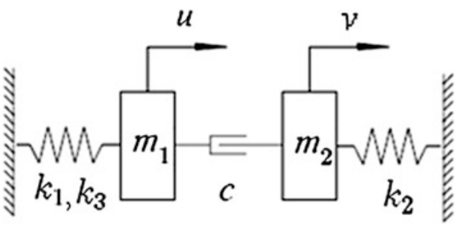


Fig. 5.16 Spring–mass damper system with two degrees of freedom

$$\begin{aligned}&\frac{\mathrm{d}^2u(t)}{\mathrm{d}t^2}+\frac{k_1}{m_1}u(t)+\frac{k_3}{m_1}u^3(t)+\frac{c}{m_1}\left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}-\frac{\mathrm{d}v(t)}{\mathrm{d}t}\right)=0,\\&\frac{\mathrm{d}^2v(t)}{\mathrm{d}t^2}+\frac{k_2}{m_2}v(t)+\frac{c}{m_2}\left(\frac{\mathrm{d}v(t)}{\mathrm{d}t}-\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right)=0,\end{aligned}\tag{5.157}$$

with the initial conditions of:

$$\begin{aligned}u_0(0)=A,\frac{\mathrm{d}u_0}{\mathrm{d}t}(0)=0,\\v_0(0)=B,\frac{\mathrm{d}v_0}{\mathrm{d}t}(0)=0.\end{aligned}\tag{5.158}$$

5.6.2 Application of the Homotopy Perturbation Method

Solving Eq. 5.157 considering the initial conditions (5.158) by means of the homotopy perturbation method (HPM), the following process, after separating the linear and nonlinear parts of each equation, is considered.A homotopy can be constructed as follows:

$$\begin{aligned}&H_1(u,p)=(1-p)\left(\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\frac{k_1}{m_1}u+\frac{c}{m_1}\frac{\mathrm{d}u}{\mathrm{d}t}\right)+p\left(\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\frac{k_1}{m_1}u+\frac{k_3}{m_1}u^3+\frac{c}{m_1}\left(\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{\mathrm{d}v}{\mathrm{d}t}\right)\right),\\&H_2(v,p)=(1-p)\left(\frac{\mathrm{d}^2v}{\mathrm{d}t^2}+\frac{k_2}{m_2}v+\frac{c}{m_2}\frac{\mathrm{d}v}{\mathrm{d}t}\right)+p\left(\frac{\mathrm{d}^2v}{\mathrm{d}t^2}+\frac{k_2}{m_2}v+\frac{c}{m_2}\left(\frac{\mathrm{d}v}{\mathrm{d}t}-\frac{\mathrm{d}u}{\mathrm{d}t}\right)\right).\end{aligned}\tag{5.159}$$

One can now try to obtain the solutions of system 5.159, in the form of

$$\begin{cases}u(t)=u_1(t)+u_2(t)+u_3(t)+\cdots,\\v(t)=v_1(t)+v_2(t)+v_3(t)+\cdots.\end{cases}\tag{5.160}$$

Substituting Eq. 5.160 into Eq. 5.159 and rearranging the resultant equations based on powers of p-terms, we obtain

$$p^0=\begin{cases}\dfrac{\mathrm{d}^2u_0}{\mathrm{d}t^2}+\dfrac{k_1}{m_1}u_0+\dfrac{c}{m_1}\dfrac{\mathrm{d}u_0}{\mathrm{d}t}=0,\quad u_0(0)=A,\dfrac{\mathrm{d}u_0}{\mathrm{d}t}(0)=0,\\\dfrac{\mathrm{d}^2v_0}{\mathrm{d}t^2}+\dfrac{k_2}{m_2}v_0+\dfrac{c}{m_2}\dfrac{\mathrm{d}v_0}{\mathrm{d}t}=0,\quad v_0(0)=B,\dfrac{\mathrm{d}v_0}{\mathrm{d}t}(0)=0,\end{cases}\tag{5.161}$$

$$p^1=\begin{cases}\dfrac{\mathrm{d}^2u_1}{\mathrm{d}t^2}+\dfrac{k_1}{m_1}u_1+\dfrac{k_3}{m_1}u_0^3+\dfrac{c}{m_1}\dfrac{\mathrm{d}u_1}{\mathrm{d}t}-\dfrac{c}{m_1}\dfrac{\mathrm{d}v_0}{\mathrm{d}t}=0,\quad u_0(0)=0,\dfrac{\mathrm{d}u_0}{\mathrm{d}t}(0)=0,\\\dfrac{\mathrm{d}^2v_1}{\mathrm{d}t^2}+\dfrac{k_2}{m_2}v_1+\dfrac{c}{m_2}\dfrac{\mathrm{d}v_1}{\mathrm{d}t}-\dfrac{c}{m_2}\dfrac{\mathrm{d}u_0}{\mathrm{d}t}=0,\quad v_1(0)=0,\dfrac{\mathrm{d}v_1}{\mathrm{d}t}(0)=0,\end{cases}\tag{5.162}$$

$$p^2 = \begin{cases} \frac{d^2u_2}{dt^2} + \frac{k_1}{m_1}u_2 + \frac{c}{m_1}\frac{du_2}{dt} - \frac{c}{m_1}\frac{dv_1}{dt} + \\ \qquad \frac{3k_3}{m_1}(u_0(t))^2 u_1(t) = 0, & u_0(0) = 0, \frac{du_0}{dt}(0) = 0, \\ \frac{d^2v_2}{dt^2} + \frac{k_2}{m_2}v_2 + \frac{c}{m_2}\frac{dv_2}{dt} - \frac{c}{m_2}\frac{du_1}{dt} = 0, & v_1(0) = 0, \frac{dv_1}{dt}(0) = 0, \end{cases} \tag{5.163}$$

where $u_1(t), v_1(t)$ can be obtained by solving these equations in terms of $u_0(t), v_0(t)$, respectively. To solve Eq. 5.161, the traditional approach is to assume each of the following solution forms:

$$u_0(t) = e^{st}, \tag{5.164}$$

$$v_0(t) = e^{wt}, \tag{5.165}$$

where w and s are complex parameters yet to be determined.

Here the solution procedure is described for obtaining $u_0(t)$ which can similarly be applied to obtain $v_0(t)$. By substituting Eq. 5.164 into the first equation of Eq. 5.161, we obtain

$$\left(s^2 + \frac{c}{m_1}s + \frac{k_1}{m_1}\right)e^{st} = 0, \tag{5.166}$$

which is satisfied for all values of t when the following equation, is known as the characteristic equation, holds:

$$s^2 + \frac{c}{m_1}s + \frac{k_1}{m_1} = 0. \tag{5.167}$$

Equation 5.9 yields the roots

$$s_{1,2} = -\frac{c}{2m_1} \pm \sqrt{\left(\frac{c}{2m_1}\right)^2 - \frac{k_1}{m_1}}. \tag{5.168}$$

Hence, the general solution of the first equation of Eq. 5.161 consists of the sum of two solutions of the form of Eq. 5.164 corresponding to the two roots of Eq. 5.168:

$$u_0(t) = Me^{s_1 t} + Ne^{s_2 t}, \tag{5.169}$$

or

$$u_0(t) = e^{-\left(\frac{c}{2m_1}\right)t}\left(M_1\, e^{\left(\sqrt{\left(\frac{c}{2m_1}\right)^2 - \frac{k_1}{m_1}}\right)t} + N_1\, e^{\left(-\sqrt{\left(\frac{c}{2m_1}\right)^2 - \frac{k_1}{m_1}}\right)t}\right), \tag{5.170}$$

where M_1 and N_1 are constants to be evaluated from initial conditions. The behavior of the terms in the parentheses depends on whether the numerical value within the radical is positive, zero, or negative. When the damping term $\left(\frac{c}{2m_1}\right)^2$ is larger than $\frac{k_1}{m_1}$, the exponents in the previous equation are real numbers, and no oscillation occurs. This case is said to be overdamped. When the damping term $\left(\frac{c}{2m_1}\right)^2$ is less than $\frac{k_1}{m_1}$, the exponent becomes an imaginary number, $\pm i\sqrt{\frac{k_1}{m_1}-\left(\frac{c}{2m_1}\right)^2}\,t$.

Using Euler's Formula, we have

$$e^{\pm i\sqrt{\frac{k_1}{m_1}-\left(\frac{c}{2m_1}\right)^2}\,t}=\cos\sqrt{\frac{k_1}{m_1}-\left(\frac{c}{2m_1}\right)^2}\,t\pm i\sin\sqrt{\frac{k_1}{m_1}-\left(\frac{c}{2m_1}\right)^2}\,t. \tag{5.171}$$

Hence, the terms of Eq. 5.169 within the parentheses are oscillatory; we refer to this case as underdamped. A typical response history of an underdamped system is shown in Fig. 5.17. In the limiting case between the oscillatory and nonoscillatory motion, $\left(\frac{c}{2m_1}\right)^2=\frac{k_1}{m_1}$, and the radical is zero. The damping corresponding to this case is called *critical damping*, denoted by c_c^u, where

$$c_c^u=2m_1,\ \omega_n=2\sqrt{k_1m_1}, \tag{5.172}$$

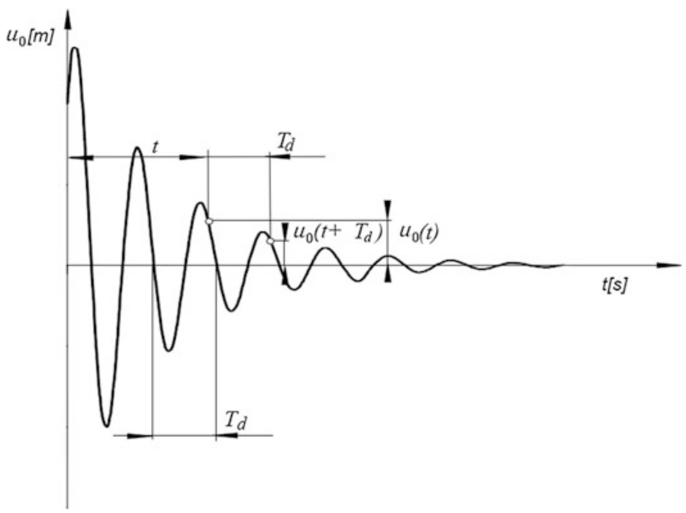


Fig. 5.17 Characteristic response history of the underdamped system

and

$$\omega_n = \sqrt{\frac{k_1}{m_1}}.$$

ω_n, is called the *natural frequency* of the undamped system. Any damping can then be expressed in terms of the critical damping by a nondimensional number ζ, called the *damping ratio* or *damping factor*:

$$\zeta = \frac{c}{c_c^u}. \tag{5.173}$$

Similar to those of $u_0(t)$, it holds for $v_0(t)$, that is

$$w_{1,2} = -\frac{c}{2\,m_2} \mp \sqrt{\left(\frac{c}{2\,m_2}\right)^2 - \frac{k_2}{m_2}}, \tag{5.174}$$

$$v_0 = e^{-\left(\frac{c}{2\,m_2}\right)t}\left(M_2\,e^{\left(\sqrt{\left(\frac{c}{2\,m_2}\right)^2 - \frac{k_2}{m_2}}\right)t} + N_2\,e^{\left(-\sqrt{\left(\frac{c}{2\,m_2}\right)^2 - \frac{k_2}{m_2}}\right)t}\right), \tag{5.175}$$

$$\xi = \frac{c}{c_c^v}, \tag{5.176}$$

$$c_c^v = 2\,m_2\,\varpi_n = 2\,\sqrt{k_2\,m_2}\,, \tag{5.177}$$

where

$$\varpi_n = \sqrt{\frac{k_2}{m_2}}.$$

We shall consider the third case, since it is the only case that leads to an oscillatory motion.

Transformation of the rearranged resultant equations on the basis of powers of p-terms into the standard form yields

$$p^0 = \begin{cases} \dfrac{\mathrm{d}^2u_0(t)}{\mathrm{d}t^2} + \omega_n^2 u_0(t) + 2\,\varsigma\,\omega_n\,\dfrac{\mathrm{d}u_0(t)}{dt} = 0, \quad u_0(0) = A, \dfrac{\mathrm{d}u_0}{\mathrm{d}t}(0) = 0, \\ \dfrac{\mathrm{d}^2v_0(t)}{\mathrm{d}t^2} + \varpi_n^2 v_0(t) + 2\,\xi\,\varpi_n\,\dfrac{\mathrm{d}v_0(t)}{\mathrm{d}t} = 0, \quad v_0(0) = B, \dfrac{\mathrm{d}v_0}{\mathrm{d}t}(0) = 0, \end{cases} \tag{5.178}$$

$$p^1 = \begin{cases} \dfrac{d^2u_1(t)}{dt^2} + \omega_n^2 u_1(t) + 2\,\varsigma\,\omega_n\,\dfrac{du_1(t)}{dt} - 2\,\varsigma\,\omega_n\,\dfrac{dv_0(t)}{dt} \\ \qquad + \alpha\,\omega_n^2\,u_0^3(t) = 0, & u_1(0) = 0\,, \frac{du_1}{dt}(0) = 0, \\ \dfrac{d^2v_1(t)}{dt^2} + \varpi_n^2 v_1(t) + 2\,\xi\,\varpi_n\,\dfrac{dv_1(t)}{dt} - 2\,\xi\,\varpi_n\,\dfrac{du_0(t)}{dt} = 0, & v_1(0) = 0\,, \frac{dv_1}{dt}(0) = 0\,, \end{cases} \tag{5.179}$$

$$p^2 = \begin{cases} \dfrac{d^2u_2(t)}{dt^2} + \omega_n^2 u_2(t) + 2\,\varsigma\,\omega_n\,\dfrac{du_2(t)}{dt} - 2\,\varsigma\,\omega_n\,\dfrac{dv_1(t)}{dt} \\ \qquad + 3\alpha\,\omega_n^2\,u_0^2(t)\,u_1(t) = 0, & u_2(0) = 0\,, \frac{du_2}{dt}(0) = 0, \\ \dfrac{d^2v_2(t)}{dt^2} + \varpi_n^2 v_2(t) + 2\,\xi\,\varpi_n\,\dfrac{dv_2(t)}{dt} - 2\,\xi\,\varpi_n\,\dfrac{du_1(t)}{dt} = 0, & v_2(0) = 0\,, \frac{dv_2}{dt}(0) = 0, \end{cases} \tag{5.180}$$

where

$$2\,\varsigma\,\omega_n = \frac{c}{m_1}, 2\,\xi\,\varpi_n = \frac{c}{m_2} \text{ and } \alpha = \frac{k_3}{k_1}.$$

A spring for which α is positive is called a *hardening* spring, and a spring for which α is negative is called a *softening* spring. The problem under consideration is the former case. Each of Eq. 5.178 is known as the mathematical model of the linear vibration of the system with one degree of freedom and is classified as a linear homogeneous ordinary differential equation of the second order.

In the underdamped case the characteristic roots are given by Eq. 5.167 and may be written as

$$s_{1,2} = -\omega_n\,\varsigma \pm i\,\omega_d, \tag{5.181}$$

and similarly for $v_0(t)$, we have

$$w_{1,2} = -\varpi_n\,\xi \pm i\,\varpi_d, \tag{5.182}$$

where

$$\omega_d = \omega_n\,\sqrt{1-\varsigma^2} \text{ and } \varpi_d = \varpi_n\,\sqrt{1-\xi^2}.$$

The quantity ω_d is called the *frequency* of the damped vibration.

Substitution of each of the characteristic roots defined by Eqs. 5.181 and 5.182 into corresponding Eq. 5.178 gives two solutions in the following form:

$$u_0(t) = C\,e^{-\varsigma\,\omega_n t} e^{\pm i\,\omega_d t}, \tag{5.183}$$

$$v_0(t) = D\,e^{-\xi\,\varpi_n t} e^{\pm i\,\varpi_d t}. \tag{5.184}$$

Hence, we have:

$$u_0(t) = e^{-\varsigma\,\omega_n t}\left(C_1\, e^{i\,\omega_d t} + C_2\, e^{-i\,\omega_d t}\right), \tag{5.185}$$

$$v_0(t) = e^{-\xi\,\varpi_n t}\left(D_1\, e^{i\,\varpi_d t} + D_2\, e^{-i\,\varpi_d t}\right), \tag{5.186}$$

where C_1, C_2, D_1 and D_2 are determined by the initial conditions of $u_0(0) = X_0^u, \frac{\mathrm{d}u_0}{\mathrm{d}t}(0) = V_0^u$ and $v_0(0) = X_0^v, \frac{\mathrm{d}v_0}{\mathrm{d}t}(0) = V_0^v$.

Equations 5.185 and 5.186 may also be expressed in the equivalent form as

$$u_0(t) = Q_1\, e^{-\varsigma\,\omega_n t}\,\cos(\omega_d t - \phi_1), \tag{5.187}$$

$$v_0(t) = Q_2\, e^{-\xi\,\varpi_n t}\,\cos(\varpi_d t - \phi_2), \tag{5.188}$$

where

$$Q_1 = X_0^u\sqrt{1 + \frac{\left((V_0^u/\omega_n X_0^u) + \varsigma\right)^2}{1-\varsigma^2}},\ \phi_1 = \tan^{-1}\left(\frac{(V_0^u/\omega_n X_0^u) + \varsigma}{\sqrt{1-\varsigma^2}}\right),$$

$$Q_2 = X_0^v\sqrt{1 + \frac{\left((V_0^v/\varpi_n X_0^v) + \xi\right)^2}{1-\xi^2}},\ \phi_2 = \tan^{-1}\left(\frac{(V_0^v/\varpi_n X_0^v) + \xi}{\sqrt{1-\xi^2}}\right)$$

It may be seen that the solutions of Eqs. 5.187 and 5.188 correspond to harmonic oscillation whose amplitudes Q_1 and Q_2 decline with time at the rate $\varsigma\,\omega_n$ and $\xi\,\varpi_n$, respectively. In this case, the motion is not periodic, but the time T_d (see Fig. 5.17) between every second zero-point is constant and is called the *period* of the damped vibration. It is easy to see from Eq. 5.170 that, $T_d^u = \frac{2\pi}{\omega_d}$ or $T_d^v = \frac{2\pi}{\varpi_d}$. The frequency of a damped oscillation is lower, and hence, the corresponding period is longer than that of the undamped system ($\varsigma = 0, \xi = 0$). Thus, it is seen that damping tends to slow the system down, as might be anticipated. The two amplitude coefficients Q_1, Q_2 and the phase angles ϕ_1 and ϕ_2 of Eqs. 5.187 and 5.188 can be determined by the initial conditions of $u_0(0) = A, \frac{\mathrm{d}u_0}{\mathrm{d}t}(0) = 0$, and $v_0(0) = B, \frac{\mathrm{d}v_0}{\mathrm{d}t}(0) = 0$ as follows:

$$Q_1 = A\sqrt{1 + \frac{(\varsigma)^2}{1-\varsigma^2}},\ \phi_1 = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right), \tag{5.189}$$

$$Q_2 = B\sqrt{1 + \frac{(\xi)^2}{1-\xi^2}},\ \phi_2 = \tan^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\right). \tag{5.190}$$

The solution terms of $u_1(t)$ and $v_1(t)$ are too long to be shown in this problem. But, for current purposes, by letting $A = 0.01$, $B = 0.02, k_1 = 5\times 10^5$, $k_2 = 1.5\times 10^5$, $k_3 = 8\times 10^5$, $m_1 = 40, m_2 = 35$ and $c = 20$ these values will be calculated as

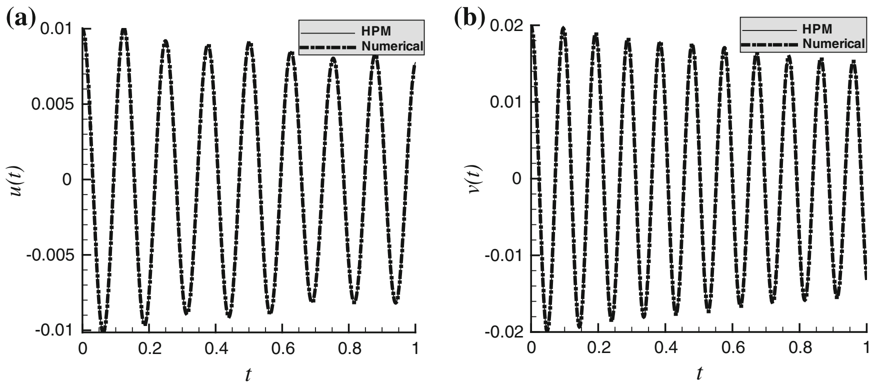


Fig. 5.18 The comparison of the solutions obtained by the homotopy perturbation method (HPM) and the numerical method for $A = 0.01, B = 0.02, k_1 = 10^5, k_2 = 1.5 \times 10^5, k_3 = 8 \times 10^5, m_1 = 40, m_2 = 35$ and $c = 20$

$$\begin{cases} u_0(t) = 0.01000002500\, e^{-0.1118033988\sqrt{5}t} \cos(49.99987500\,\sqrt{5}t - 0.002236069840), \\ v_0(t) = 0.02000001904\, e^{-0.1971615884\sqrt{210}t} \cos(14.28570068\sqrt{210}t - 0.001380131557), \end{cases} \tag{5.191}$$

In the same manner, the rest of the components are obtained. A significant achievement of this work is that, if higher numbers of iterations are applied, the solution tends toward a closed form, completely similar to the exact solution. According to the HPM, we can conclude that

$$\begin{cases} u(t) = \lim\limits_{p \to 1}(u_0(t) + p\, u_1(t) + \cdots), \\ v(t) = \lim\limits_{p \to 1}(v_0(t) + p\, v_1(t) + \cdots). \end{cases} \tag{5.192}$$

In Fig. 5.18 the comparison of the solutions between the HPM and numerical results is shown.

5.7 Problem 5.7. Application of Viscous Damping with a Nonlinear Spring

5.7.1 Introduction

Here, a system consisting of a block of mass m that hangs from a viscous damper with coefficient c and a nonlinear spring of stiffness k_1 and k_3 is considered. The force displacement for the spring is $F_{\text{spring}} = k_1\, x(t) + k_3\, x(t)^3$, as shown in

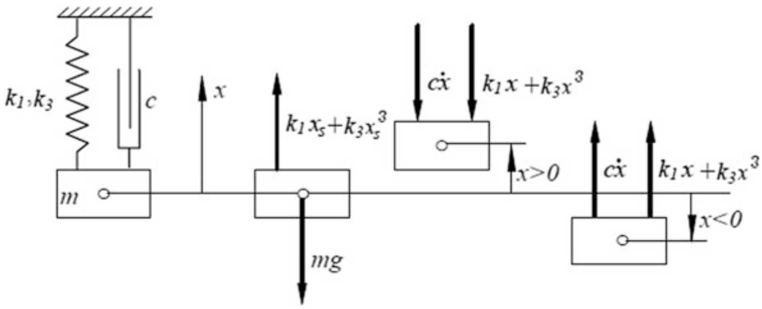


Fig. 5.19 Spring–mass damper system with one degree of freedom

Fig. 5.19. The system is considered to be in equilibrium. To develop the mathematical model, we take advantage of Newton's generalized equations. This requires introduction of the absolute system of coordinates. We are assuming that the origin of the absolute system of coordinates coincides with the center of gravity of the body while the body stays at its equilibrium position, as shown in Fig. 5.19. Assuming that the system is out of the equilibrium position (see Fig. 5.19) by a distance $x(t)$, the equation of motion is given by the nonlinear differential equation

$$\frac{d^2x(t)}{dt^2} + \frac{k_1}{m}x(t) + \frac{k_3}{m}x^3(t) + \frac{c}{m}\frac{dx(t)}{dt} = 0, \tag{5.193}$$

with the initial conditions

$$x_0(0) = A, \frac{dx_0}{dt}(0) = 0. \tag{5.194}$$

5.7.2 Application of Homotopy Perturbation Method

To solve Eq. 5.193 by means of the HPM, the following process, after separating the linear and nonlinear parts of the equation, is considered.

An HPM can be constructed as

$$H(x,p) = (1-p)\left(\frac{d^2x}{dt^2} + \frac{k_1}{m}x + \frac{c}{m}\frac{dx}{dt}\right) + p\left(\frac{d^2x}{dt^2} + \frac{k_1}{m}x + \frac{k_3}{m}x^3 + \frac{c}{m}\frac{dx}{dt}\right). \tag{5.195}$$

Substituting $x(t) = x_0 + p\,x_1 + p^2 x_2 + \cdots$ into Eq. 5.195 and rearranging the resultant equation on the basis of powers of p-terms, we obtain

$$p^0 : \frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + \frac{k_1}{m} x_0(t) + \frac{c}{m}\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = 0, \quad x_0(0) = A, \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0, \tag{5.196}$$

$$p^1 : \frac{\mathrm{d}^2 x_1(t)}{\mathrm{d}t^2} + \frac{k_1}{m} x_1(t) + \frac{c}{m}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} + \frac{k_3}{m} x_0^3(t) = 0, \quad x_1(0) = 0\,, \frac{\mathrm{d}x_1}{\mathrm{d}t}(0) = 0, \tag{5.197}$$

Transformation of the above equations into the standard form yields

$$p^0 : \frac{\mathrm{d}^2 x_0(t)}{\mathrm{d}t^2} + \omega_n^2 x_0(t) + 2\,\varsigma\,\omega_n\,\frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = 0, \quad x_0(0) = A, \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0, \tag{5.198}$$

$$p^1 : \frac{\mathrm{d}^2 x_1(t)}{\mathrm{d}t^2} + \omega_n^2 x_1(t) + 2\,\varsigma\,\omega_n\,\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} + \alpha\,\omega_n^2\,x_0^3(t) = 0, \quad x_1(0) = 0\,, \frac{\mathrm{d}x_1}{\mathrm{d}t}(0) = 0, \tag{5.199}$$

where

$$\omega_n = \sqrt{\frac{k_1}{m}},\ 2\,\varsigma\,\omega_n = \frac{c}{m} \text{ and } \alpha = \frac{k_3}{k_1}.$$

ω_n, is called the *natural frequency* of the undamped system, and ς is called the *damping factor* or *damping ratio*. A spring for which α is positive is called a *hardening* spring, and a spring for which α is negative is called a *softening* spring. The problem under consideration is the former case.

Equation 5.198 is known as the mathematical model of the linear vibration of the system with one degree of freedom and is classified as a linear homogeneous ordinary differential equation of the second order.

To solve Eq. 5.198, we assume the solution in the form

$$x_0(t) = Ce^{st}, \tag{5.200}$$

where C and s are (complex) parameters that are yet to be determined. Substitution of Eq. 5.200 into Eq. 5.198 results in the characteristic equation

$$s^2 + 2\,\omega_n\,\varsigma\,s + \omega_n^2 = 0, \tag{5.201}$$

which yields the roots

$$s = -\omega_n\,\varsigma \pm \omega_n\,\sqrt{\varsigma^2 - 1}. \tag{5.202}$$

The solution of Eq. 5.198 is thus made up of the sum of two solutions of the form of Eq. 5.200 corresponding to the two roots of Eq. 5.202. It is evident from Eqs. 5.200 and 5.202 that the solution of Eq. 5.198 is characterized by whether the damping ratio is less than, greater than, or equal to unity. Since the resulting solution of the main problem is affected from the solution of Eq. 5.198 directly, it is necessary to consider the solutions of Eq. 5.198 by choosing different values of parameter ς. Then, we shall consider each case separately.

5.7.3 Underdamped System $\left(\varsigma^2 < 1 \text{ or } \frac{c}{2m} < \sqrt{\frac{k}{m}}\right)$

In this case, the characteristic roots given by Eq. 5.202 may be written as

$$s = -\omega_n \varsigma \pm i \omega_d, \tag{5.203}$$

where

$$\omega_d = \omega_n \sqrt{1 - \varsigma^2}.$$

The quantity ω_d is called the *frequency* of the damped vibration. Substituting each of the characteristic roots defined by Eq. 5.203 into Eq. 5.200 gives two solutions of the form

$$x_0(t) = C e^{-\varsigma \omega_n t} e^{\pm i \omega_d t}. \tag{5.204}$$

The general solution for Eq. 5.198 consists of a linear combination of these two solutions. Hence, we have

$$x_0(t) = e^{-\varsigma \omega_n t}(C_1 e^{i \omega_d t} + C_2 e^{-i \omega_d t}), \tag{5.205}$$

where C_1 and C_2 are determined by the initial conditions of $x_0(0) = x_0$, $\frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = v_0$.

Equation 5.205 may also be expressed in the equivalent form as

$$x_0(t) = Q e^{-\varsigma \omega_n t} \cos(\omega_d t - \phi), \tag{5.206}$$

where

$$Q = x_0 \sqrt{1 + \frac{((v_0/\omega_n x_0) + \varsigma)^2}{1 - \varsigma^2}}, \phi = \tan^{-1}\left(\frac{(v_0/\omega_n x_0) + \varsigma}{\sqrt{1 - \varsigma^2}}\right).$$

It may be noticed that the solution of Eq. 5.198 corresponds to the harmonic oscillation the amplitudes of which, Q, decay with time at the rate $\varsigma \omega_n$. In this case, the motion is not periodic, but for the time T_d (see Fig. 5.20), every two-zero point is constant, and it is called the *period* of the damped vibration. It is easy to see from Eq. 5.206 that $T_d = \frac{2\pi}{\omega_d}$. The frequency of the damped oscillations is lower, and hence, the corresponding period is longer than that of the undamped system ($\varsigma = 0$). It is thus obvious that damping tends to slow the system down, as might be anticipated. For $\omega_n = 1 \left[\frac{1}{s}\right], x_0(0) = 1\,[m], \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 1 \left[\frac{m}{s}\right]$, and $\varsigma = 0.1$.. The free motion is shown in Fig. 5.20.

The amplitude coefficient Q and phase angle ϕ of Eq. 5.198 can be determined by applying the initial conditions of $x_0(0) = A, \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0$ to Eq. 5.206 as

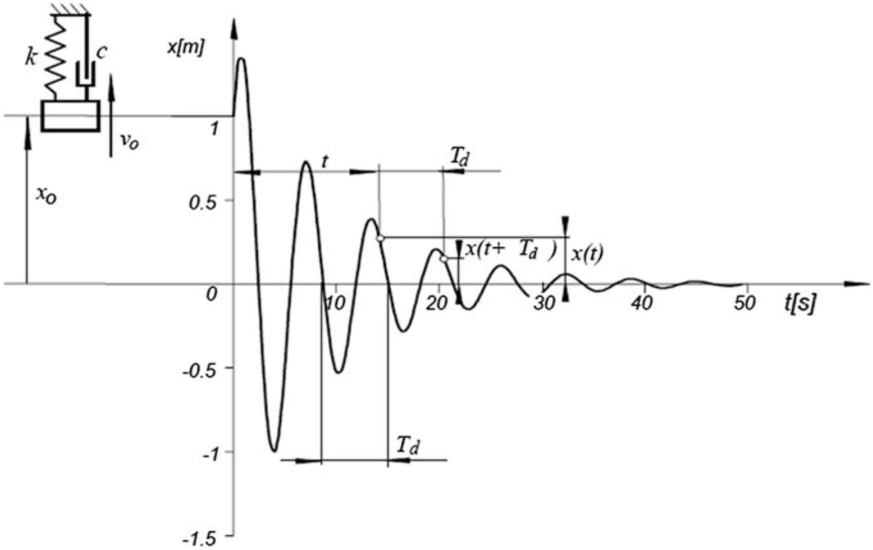


Fig. 5.20 Free motion of the underdamped system

$$Q = A\sqrt{1 + \frac{(\varsigma)^2}{1 - \varsigma^2}}, \phi = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1 - \varsigma^2}}\right). \tag{5.207}$$

In the same manner, substituting $x_0(t)$ into the right-hand side of Eq. 5.199, $x_1(t)$ can be obtained. The resulting solution is too long to be shown here. However, by letting $A = 0.01, k_1 = 200\,, k_3 = 400, c = 5$, and $m = 1$ the values of $x_1(t)$ and $x_0(t)$ can be obtained as

$$x_0(t) = 0.0101600101\,e^{-2.5t}\cos(13.91941090\,t - 0.1777106008) \tag{5.208}$$

$$\begin{aligned} x_1(t) = &- 0.000002490047909e^{-2.5\,t}\sin(13.91941091\,t) \\ &+ 7.280000035\,10^{-7}e^{-2.5\,t}\cos(13.91941091\,t) \\ &+ 2.097550492\,10^{-57}(1.043955815\,10^{51}\sin(13.91941090\,t \\ &- 0.1777106009) + 8.351646536\,10^{48}\sin(41.75823270\,t \\ &- 0.5331318027) - 1.875000011\,10^{50}\cos(13.91941090\,t \\ &- 0.1777106009) + 3.049999995\,10^{49}\cos(41.75823270\,t \\ &- 0.5331318027))\,e^{-7.5\,t}. \end{aligned} \tag{5.209}$$

According to the HPM, we can conclude that

$$\begin{aligned} &x(t) = \lim v(x,t) = x_0(t) + x_1(t) + \cdots \\ &p \to 1. \end{aligned} \tag{5.210}$$

Therefore, if the first-order approximation of the solution is sufficient, substituting the values of $x_0(t)$ and $x_1(t)$ from Eqs. 5.208 and 5.209 into Eq. 5.210 yields

$$\begin{aligned} x(t) = {} & 0.0101600101\, e^{-2.5\,t} \cos(13.91941090\,t - 0.1777106008) \\ & - 0.000002490047909\, e^{-2.5\,t} \sin(13.91941091\,t) \\ & + 7.280000035\, 10^{-7} e^{-2.5\,t} \cos(13.91941091\,t) \\ & + 2.097550492\, 10^{-57} (1.043955815\, 10^{51} \sin(13.91941090\,t \\ & - 0.1777106009) + 8.351646536\, 10^{48} \sin(41.75823270\,t \\ & - 0.5331318027) - 1.875000011\, 10^{50} \cos(13.91941090\,t \\ & - 0.1777106009) + 3.049999995\, 10^{49} \cos(41.75823270\,t \\ & - 0.5331318027))\, e^{-7.5\,t}. \end{aligned} \tag{5.211}$$

Comparisons of the results obtained by HPM and the exact solutions are given in Table 5.2 and Fig. 5.21.

For different values of k, the accuracy of the resulting solutions has been shown in the following Figs. 5.22 and 5.23.

In Figs. 5.22 and 5.23, the comparisons are made with the fixed values of $k_1 = 100, k_3 = 50, c = 5, m = 1$ and various values of A in the four cases: (a) $A = 0.01$, (b) $A = 0.05$, (c) $A = 0.1$, (d) $A = 0.5$.

In Fig. 5.24, the comparisons are made with the fixed values of $k_1 = 100, k_3 = 50, m = 1, A = 0.01$ and various values of c in the four cases: (a) $c = 10$, (b) $c = 12$, (c) $c = 14$, (d) $c = 18$.

These comparisons are an indication of the accuracy of the HPM as applied to this particular problem and show that it provides an excellent approximation to the solution of Eq. 5.193.

Table 5.2 The comparison of the results of the underdamped system for $k_1 = 200, k_3 = 400, c = 5, m = 1, A = 0.01$

t(s)	HPM solutions	Numeric solutions	Error presentation
0	0.010000000	0.010000000	0.0
0.25	−0.005368253	−0.005368262	0.00016
0.5	0.002555940	0.002555945	0.00020
0.75	−0.001043101	−0.001043097	0.00038
1	0.000321098	0.000321088	0.00306
1.25	−0.000025411	−0.000025394	0.06446
1.5	−0.000066335	−0.000066353	0.02745
1.75	0.000074269	0.000074287	0.02489
2	−0.000055996	−0.000056013	0.03020
2.25	0.000035270	0.000035284	0.04050
2.5	−0.000019574	−0.000019585	0.05560

Note HPM homotopy perturbation method

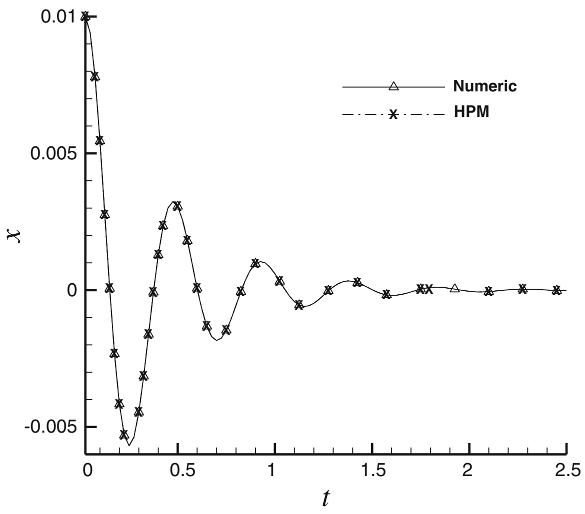


Fig. 5.21 Free motion of the underdamped system. *HPM* Homotopy PerturbationMethod

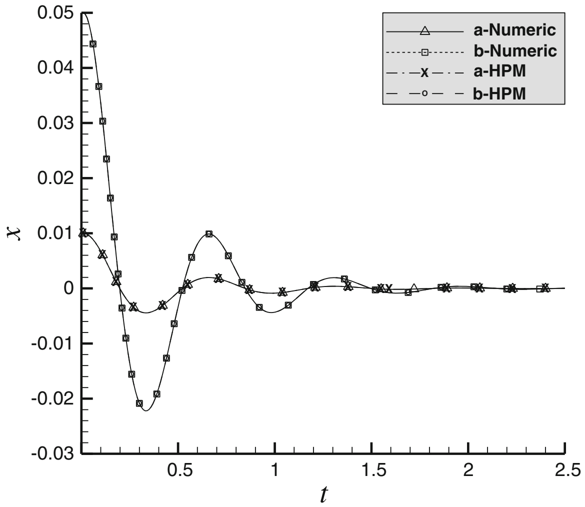


Fig. 5.22 The comparison of the results of the underdamped system (cases: *a* and *b*). *HPM* Homotopy PerturbationMethod

5.7.4 Overdamped System $\left(\varsigma^2 > 1 \text{ or } \frac{c}{2m} > \sqrt{\frac{k}{m}}\right)$

For such systems, characteristic roots given by Eq. 5.202 are all real. Substitution of these roots into Eq. 5.200 gives the solution for the overdamped case as

$$x_0(t) = C_1\, e^{-(\varsigma-\gamma)\,\omega_n t} + C_2\, e^{-(\varsigma+\gamma)\,\omega_n t}, \tag{5.212}$$

where $\gamma = \sqrt{\varsigma^2 - 1}$ and C_1 and C_2 are determined by the initial conditions of

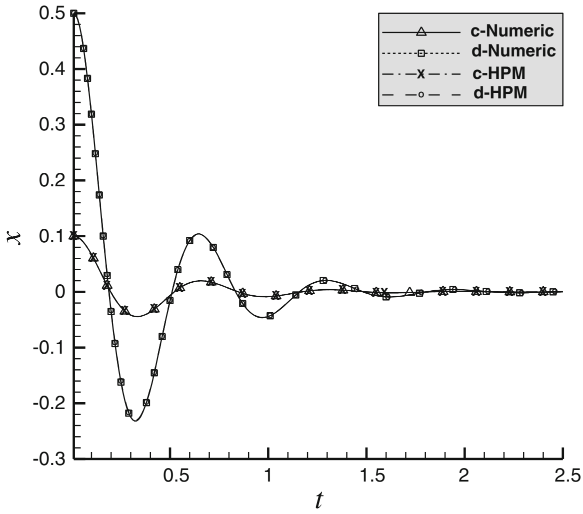


Fig. 5.23 The comparison of the results of the underdamped system (cases: *c* and *d*). *HPM* Homotopy PerturbationMethod

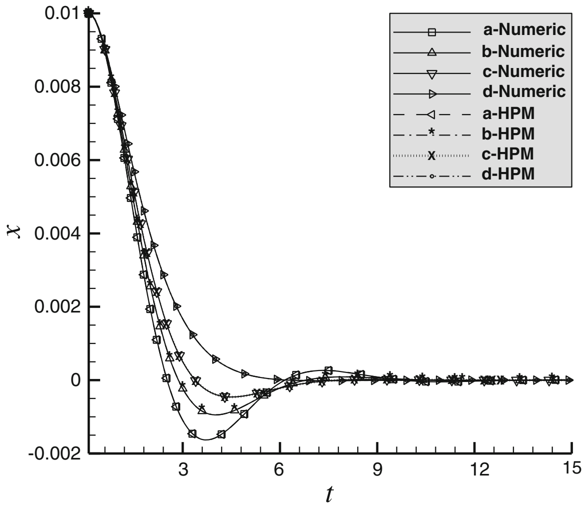


Fig. 5.24 The comparison of the results of the underdamped system (cases: *a*, *b*, *c*, and *d*). *HPM* Homotopy PerturbationMethod

$x_0(0) = x_0\,, \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = v_0.$

An equivalent form of the solution is easily obtained with the aid of $\mathrm{e}^{\pm\alpha} = \cosh\alpha \pm \sinh\alpha$ as

$$x_0(t) = e^{-\varsigma\,\omega_n t}\left(x_0\,\cosh(\gamma\,\omega_n t) + \frac{(v_0 + \varsigma\,\omega_n t)}{\omega_n\,\gamma}\sinh(\gamma\,\omega_n t)\right). \tag{5.213}$$

Consideration of the exponential form of the solution, Eq. 5.212, shows that both terms of the solution decay exponentially.

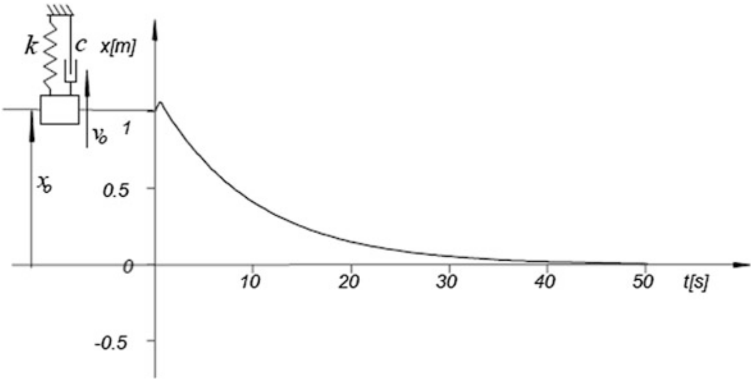


Fig. 5.25 Free motion of the underdamped system

For $\omega_n = 1\left[\frac{1}{s}\right], x_0(0) = 1\,[m], \frac{dx_0}{dt}(0) = 1\left[\frac{m}{s}\right]$, and $\varsigma = 5$, the free motion is shown in Figs. 5.21 and 5.25.

In this case for the initial conditions $x_0(0) = x_0, \frac{dx_0}{dt}(0) = v_0$, $x_0(t)$ can be obtained as

$$x_0(t) = e^{-\varsigma\,\omega_n t}\left(A\,\cosh(\gamma\,\omega_n t) + \frac{(\varsigma\,\omega_n t)}{\omega_n\,\gamma}\sinh(\gamma\,\omega_n t)\right). \tag{5.214}$$

By substituting $x_0(t)$ into the right-hand side of Eq. 5.199, $x_1(t)$ is obtained. For $A = 0.01, k_1 = 100, k_3 = 400, c = 100$, and $m = 1$, the values of $x_1(t)$ and $x_0(t)$ can be obtained as

$$x_0(t) = e^{-50\,t}(0.01\,\cosh(5\,\sqrt{2}\sqrt{46}t) + 0.001086956522\sqrt{46}\sqrt{2}\,\sinh(5\,\sqrt{2}\sqrt{46}t)), \tag{5.215}$$

$$\begin{aligned}
x_1(t) = & -0.000001085153262\,e^{-2.041684770\,t} - 1.163392757\,10^{-7}\,e^{-97.95831523t} \\
& -2.956253516\,10^{-10}\,e^{-97.95831523t}(-163.6313736\,\cosh(91.8332609t) \\
& +0.000466207519\,e^{95.916633046\,t}\cosh(291.8332609\,t) \\
& -163.6313736\sinh(91.8332609\,t) \\
& -0.0004662075196\,e^{95.916633046\,t}\sinh(291.8332609\,t) \\
& +(-3680.000672\,e^{95.916633046\,t} - 230.0999566)\cosh(-4.083369540\,t) \\
& +(-0.0999566\,e^{95.916633046\,t} - 0.000694)\cosh(195.9166305\,t) \\
& +(-3680.000672\,e^{95.916633046\,t} - 230.0999566)\sinh(-4.083369540\,t) \\
& +(0.0999566\,e^{95.916633046\,t} + 0.000694)\sinh(195.9166305\,t) \\
& +(9.395831522\,e^{95.916633046\,t} + 0.195831524)(\cosh(100\,t) - \sinh(100\,t))).
\end{aligned} \tag{5.216}$$

Therefore, substituting the values of $x_0(t)$ and $x_1(t)$ from Eqs. 5.215 and 5.216 into Eq. 5.210 yields

$$\begin{aligned} x(t) = {} & e^{-50t}(0.01\cosh(5\sqrt{2}\sqrt{46}t) + 0.001086956522\sqrt{46}\sqrt{2}\,\sinh(5\,\sqrt{2}\sqrt{46}t)) \\ & - 0.000001085153262\,e^{-2.041684770\,t} - 1.163392757\,10^{-7}\,e^{-97.95831523t} \\ & - 2.956253516\,10^{-10}e^{-97.95831523t}(-163.6313736\cosh(91.8332609\,t) \\ & + 0.000466207519\,e^{95.916633046\,t}\cosh(291.8332609\,t) \\ & - 163.6313736\sinh(91.8332609\,t) \\ & - 0.0004662075196e^{95.916633046\,t}\sinh(291.8332609\,t) \\ & + (-3680.000672e^{95.916633046\,t} - 230.0999566)\cosh(-4.083369540\,t) \\ & + (-0.0999566e^{95.916633046\,t} - 0.000694)\cosh(195.9166305\,t) \\ & + (-3680.000672e^{95.916633046\,t} - 230.0999566)\sinh(-4.083369540\,t) \\ & + (0.0999566e^{95.916633046\,t} + 0.000694)\sinh(195.9166305\,t) \\ & (9.395831522e^{95.916633046\,t} + 0.195831524)(\cosh(100\,t) - \sinh(100\,t))). \end{aligned} \tag{5.217}$$

The comparisons between the results obtained by HPM and the exact solutions are given in Table 5.3 and Fig. 5.26.

In Fig. 5.27, comparisons are made with the fixed values of $k_1 = 50, k_3 = 100, c = 100, m = 1$ and various values of A in the three cases: (a) $A = 0.01$, (b) $A = 0.05$, (c) $A = 0.1$.

In Fig. 5.28, comparisons are made with the fixed values of $k_1 = 50, k_3 = 100, m = 1, A = 0.01$ and various values of c in the three cases: (a) $c = 150$, (b) $c = 200$, and (c) $c = 300$.

Table 5.3 Comparison of the results of the overdamped system for $k_1 = 100, k_3 = 400, c = 100, m = 1, A = 0.01$

t(s)	HPM solutions	Numeric solutions	Error presentation
0	0.010000000	0.010000000	0.0
0.25	0.007846639	0.007847636	0.01272
0.5	0.006095397	0.006095865	0.00767
0.75	0.004735004	0.004735224	0.00463
1	0.003678228	0.003678332	0.00281
1.25	0.002857309	0.002857357	0.00170
1.5	0.002219604	0.002219626	0.00099
1.75	0.001724225	0.001724236	0.00062
2	0.001339406	0.001339411	0.00038
2.25	0.001040472	0.001040475	0.00022
2.5	0.000808256	0.000808258	0.00019

Note HPM homotopy perturbation method

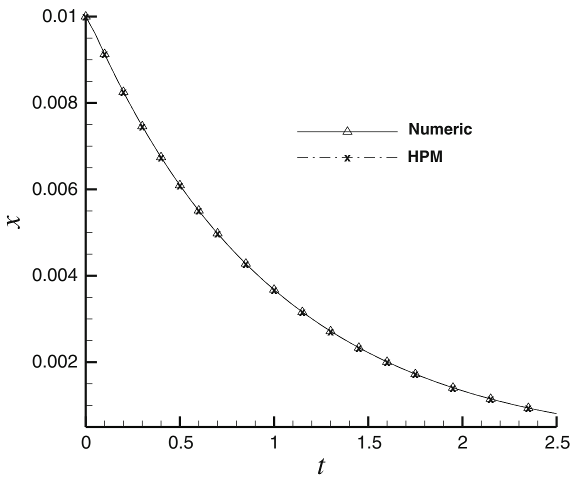


Fig. 5.26 Free motion of the overdamped system. *HPM* Homotopy Perturbation Method

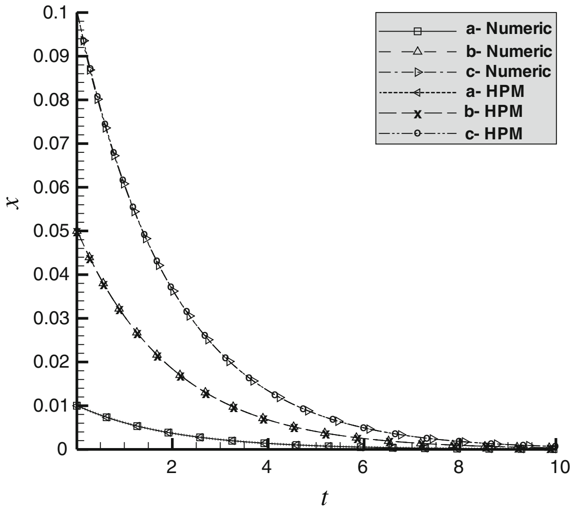


Fig. 5.27 Comparison of the results of the overdamped system (cases: *a*, *b*, and *c*). *HPM* Homotopy Perturbation Method

5.7.5 Critically Damped System $\left(\varsigma^2 = 1 \text{ or } \frac{c}{2m} = \sqrt{\frac{k}{m}}\right)$

For the critically damped system, the characteristic roots given by Eq. 5.202 will reduce to

$$s = -\omega_n, -\omega_n. \tag{5.218}$$

Substitution of these roots into Eq. 5.200 yields the solution for the critically damped case as

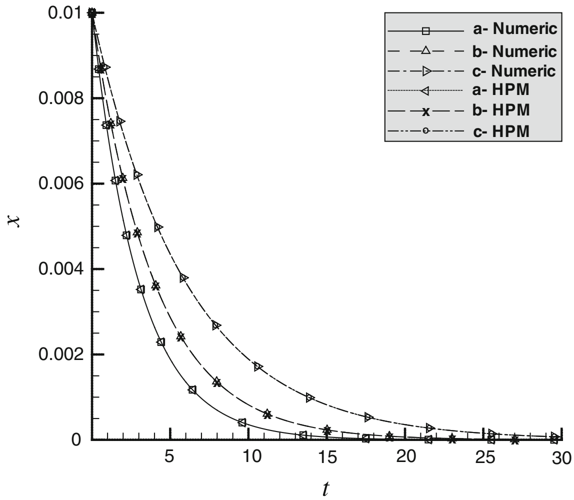


Fig. 5.28 Comparison of the results of the overdamped system (cases: *a*, *b*, and *c*). *HPM* Homotopy Perturbation Method

$$x_0(t) = (C_1 + C_2 t)e^{-\omega_n t}. \tag{5.219}$$

Imposition of the initial conditions, $x_0(0) = x_0, \frac{dx_0}{d_t}(0) = v_0$, renders the response given by Eq. 5.219 as

$$x_0(t) = (x_0 + (v_0 + \omega x_0)\, t)e^{-\omega_n t}. \tag{5.220}$$

For $\omega_n = 1\left[\frac{1}{s}\right]$, $x_0(0) = 1\,[m]$, $\frac{dx_0(0)}{d_t} = 1\left[\frac{m}{s}\right]$, and $\varsigma = 5$, the free motion is shown in Fig. 5.29.

Critical damping offers a possibly faster return to the system's equilibrium position.

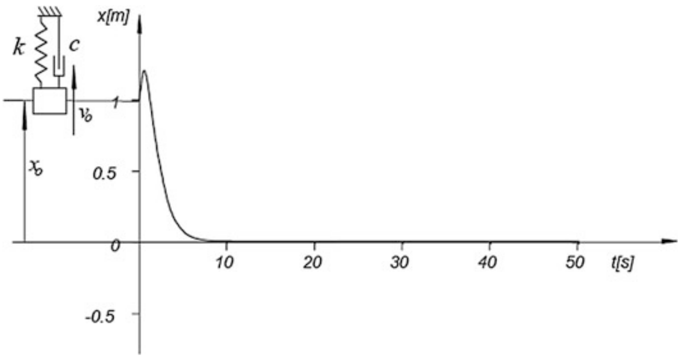


Fig. 5.29 Free motion of the critically damped system

In this case, for the initial conditions, $x_0(0) = A$, $\frac{dx_0}{dt}(0) = 0$, $x_0(t)$ can be obtained as

$$x_0(t) = A(1 + \omega_n t)e^{-\omega_n t}. \tag{5.221}$$

By substitution $x_0(t)$ into the right-hand side of Eq. 5.199, $x_1(t)$ is obtained. For $A = .01, k_1 = 200, k_3 = 400, c = 20\sqrt{20}$, and $m = 10$ the values of $x_1(t)$ and $x_0(t)$ can be obtained as

$$x_0(t) = e^{-2\sqrt{5}t}(0.01 + 0.02\sqrt{5}t), \tag{5.222}$$

$$\begin{aligned} x_1(t) = {} & 0.00000575\, e^{-4.472135954\,t} - 0.0000212426457\, e^{-4.472135954\,t}t \\ & + ((-51.42956347 - 7244.860245\,t^2 - 61715.47617\,t^4 - 35777.08763\,t^6 \\ & - 960\,t - 28400\,t^3 - 72000\,t^5)e^{-13.41640786\,t})/(8.944271908\,10^6 \\ & + 5.366563145\,10^8\,t^2 + 1.2\,10^8\,t + 8\,10^8\,t^3). \end{aligned} \tag{5.223}$$

Similarly, the resultant solution can be written as

$$\begin{aligned} x(t) = {} & e^{-2\sqrt{5}t}(0.01 + 0.02\sqrt{5}t) \\ & + 0.00000575\, e^{-4.472135954\,t} - 0.0000212426457\, e^{-4.472135954\,t}t \\ & + ((-51.42956347 - 7244.860245\,t^2 - 61715.47617\,t^4 - 35777.08763t^6 \\ & - 960\,t - 28400\,t^3 - 72000\,t^5)e^{-13.41640786\,t})/(8.944271908\,10^6 \\ & + 5.366563145\,10^8\,t^2 + 1.2\,10^8\,t + 8\,10^8\,t^3). \end{aligned} \tag{5.224}$$

Comparisons between the results obtained by HPM and the exact solutions are given in Table 5.4 and Fig. 5.30.

In Fig. 5.31, comparisons have been made with the fixed values of $k_1 = 50, k_3 = 100$, $A = 0.01$ and various values of c and m in the three cases: (a) $c = 10\sqrt{10}$, $m = 5$, (b) $c = 20\sqrt{5}$, $m = 10$, and (c) $c = 20\sqrt{10}$, $m = 20$.

In Fig. 5.32, comparisons are made with the fixed values of $k_3 = 600$, $A = 0.01$, $m = 10$ and various values of c and k_1 in three cases: (a) $c = 20\sqrt{10}$, $k_1 = 100$, (b) $c = 40\sqrt{5}$, $k_1 = 200$, and (c) $c = 40\sqrt{10}$, $k_1 = 400$.

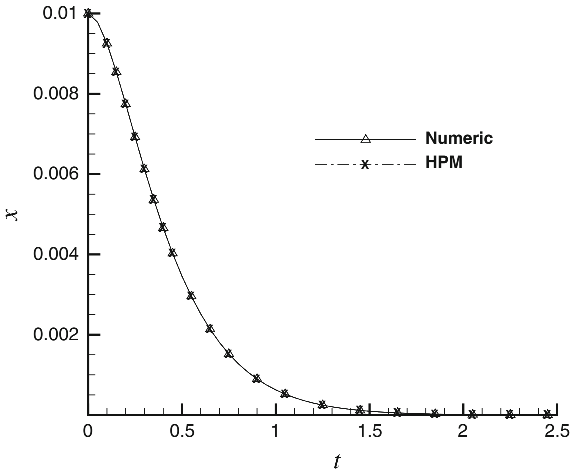
5.7.6 Discussion and Conclusion

The underdamped case is very important in the study of mechanical vibrations, since it is the only case that leads to an oscillatory motion. Thus, we investigate the details of this case further. The accuracy of the results is shown in Table 5.5 and demonstrates the acceptability of greater values for the parameter k.

Table 5.4 Comparison of the results of the critically damped system for $k_1 = 200, k_3 = 400, c = 40\sqrt{5}, m = 10, A = 0.01$

t(s)	HPM solutions	Numeric solutions	Error presentation
0	0.010000000	0.010000000	0.0
0.25	0.006923840	0.006923844	0.00006
0.5	0.003458071	0.003458076	0.00013
0.75	0.001520996	0.001520999	0.00022
1	0.000624899	0.000624900	0.00022
1.25	0.000246025	0.000246020	0.00189
1.5	0.000094074	0.000094060	0.01455
1.75	0.000035215	0.000035204	0.03154
2	0.000012971	0.000012961	0.07157
2.25	0.000004717	0.000004711	0.12444
2.5	0.000001698	0.000001694	0.24157

Fig. 5.30 Free motion of the critically damped system. *HPM* homotopy perturbation method



The obtained results are sensitive to the values of the parameter A, and for $A > 0.5$, the accuracy of the results decreases. Letting $A = 1, k_1 = 200, k_3 = 400, c = 5$, and $m = 1$, the result has been compared in Fig. 5.33, in which the decrement of accuracy is obvious.

Another point is that, in the case of the underdamped system with linear vibration, the logarithmic decrement is introduced, which is defined as the natural logarithm of ratio of two successive displacements, $x(t)$ and $x(t + T_d)$, that are one period apart, as shown in Fig. 5.20. The expression for the logarithmic decrement can be obtained as

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}. \tag{5.225}$$

Fig. 5.31 Comparison of the results of the critically damped system (cases: *a*, *b*, and *c*). *HPM* homotopy perturbation method

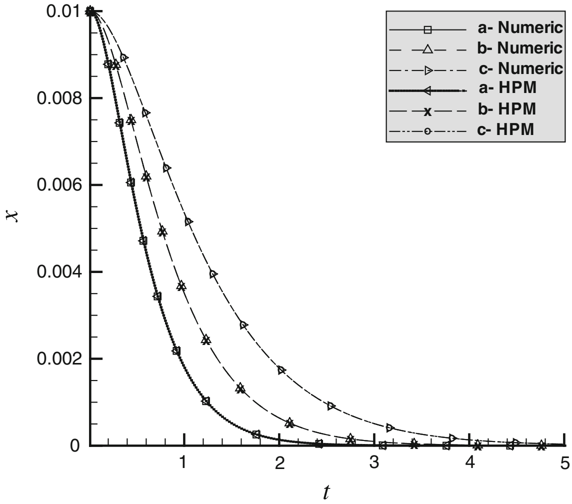
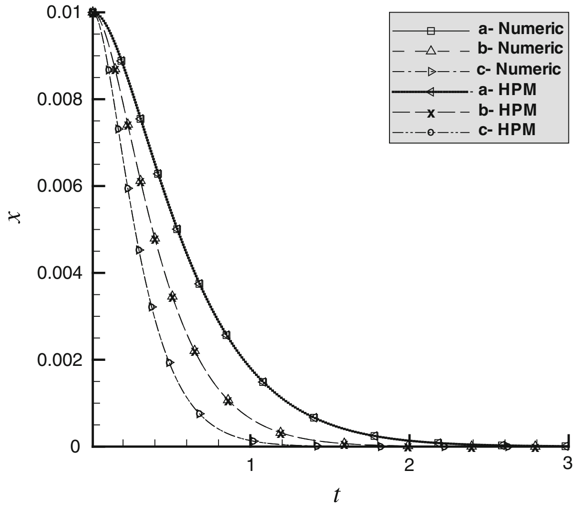


Fig. 5.32 Comparison of the results of the critically damped system (cases: *a*, *b*, and *c*). *HPM* homotopy perturbation method



A convenient way to determine the amount of damping present in a system is to measure the rate of decay of free oscillations. The larger the damping, the greater will be the rate of decay. If, as in the present problem, the damper coefficient c is constant, it is expected that the logarithmic decrement will remain constant. Values of a logarithmic decrement for the present nonlinear problem have been calculated and compared with the logarithmic decrement of the equivalent linear system $(k_3 = 0)$ in Table 5.6.

Table 5.5 Comparison of the results of the underdamped system for $k_1 = 1000000, k_3 = 100000, c = 50, m = 5, A = 0.02$

t(s)	HPM solutions	Numeric solutions	Error presentation
0	0.02000000000	0.02000000000	0.0
0.1	0.00907624026	0.00907642982	0.00209
0.2	0.00079943390	0.00079949857	0.00809
0.3	−0.00262014085	−0.00262026831	0.00486
0.4	−0.00264907980	−0.00264928302	0.00767
0.5	−0.00141720697	−0.00141736011	0.01080
0.6	−0.00029931667	−0.00029936832	0.01725
0.7	0.00025231886	0.00025234637	0.01090
0.8	0.00033691011	0.00033696699	0.01688
0.9	0.00021001020	0.00021005720	0.02237
1	0.00006482633	0.00006484737	0.03245
1.1	−0.00001898810	−0.00001899149	0.01317
1.2	−0.00004091661	−0.00004093236	0.03847
1.3	−0.00002979243	−0.00002980983	0.05838
1.4	−0.00001172668	−0.00001173796	0.09615
1.5	0.00000004194	0.00000004174	0.49024

Note HPM homotopy perturbation method

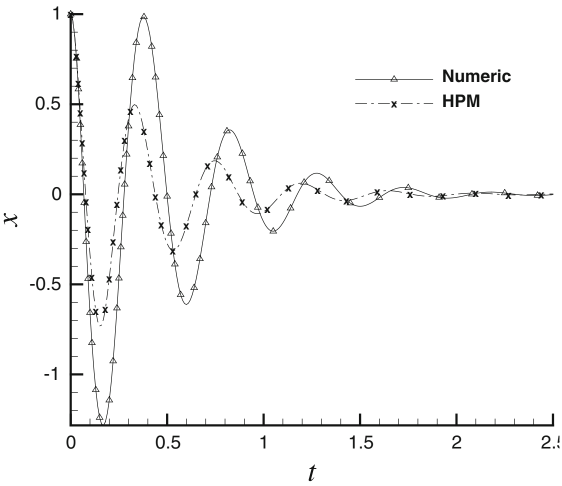


Fig. 5.33 Comparison of the approximate solution obtained by the homotopy perturbation method (HPM) with the exact one in the case of the underdamped system for $A = 1, k_1 = 200, k_3 = 400, c = 5$, and $m = 1$

5.8 Problem 5.8. Application of Cubic Nonlinearity

5.8.1 Introduction

A system having cubic nonlinearities with the following governing equation is considered:

Table 5.6 Comparison of the logarithmic decrement of the underdamped system

A	K_1	K_3	C	m	T_d	$(T_d)_{\text{linear}}$	δ	$(\delta)_{\text{linear}}$
0.01	800	1000	5	1	0.223014 ~ 7	0.223017	0.55751 ~ 4	0.55754
0.01	100	150	14	1	0.879821924 ~ 5	0.879821925	6.1586 ~ 7	6.1587
0.02	100000	1000000	50	5	0.0140503 ~ 4	0.014050507	0.070250 ~ 2	0.07025253

Note The symbol "~" denotes that the last digit varies between the two digits connected by this symbol

$$\frac{d^2u(t)}{dt^2} + \omega_1^2\, u(t) = -2\,\mu_1\, \frac{du(t)}{dt} + \alpha_1\, u(t)^3 + \alpha_2\, u(t)^2\, v(t) + \alpha_3\, u(t)\, v(t)^2 + \alpha_4\, v(t)^3, \tag{5.226}$$

$$\frac{d^2v(t)}{dt^2} + \omega_2^2\, u(t) = -2\,\mu_2\, \frac{dv(t)}{dt} + \alpha_5\, u(t)^3 + \alpha_6\, u(t)^2\, v(t) + \alpha_7\, u(t)\, v(t)^2 + \alpha_8\, v(t)^3, \tag{5.227}$$

with the initial conditions

$$u_0(0) = A,\ \frac{du_0}{dt}(0) = 0, \tag{5.228}$$

$$v_0(0) = B,\ \frac{dv_0}{dt}(0) = 0. \tag{5.229}$$

We seek the approximate solution of Eqs. 5.226 and 5.227 for small but finite amplitudes when $\mu_1 = 0$ and $\mu_2 = 0$.

By means of the MHPM, the periodic solutions and frequency–amplitude relations have been obtained for each component of a two-degrees-of-freedom system separately.

All the solutions to Eqs. 5.228 and 5.229 are periodic, and the angular frequencies of these oscillations are denoted by Ω_u and Ω_v. Note that one of our major tasks is to determine the functional behavior of frequencies as a function of the initial amplitudes A and B.

For Eqs. 5.226 and 5.227, the following homotopy can be established:

$$(1-p)\left(\frac{d^2u(t)}{dt^2} + \omega_1^2\, u(t)\right) + p\left(\frac{d^2u(t)}{dt^2} + \omega_1^2\, u(t) - (\alpha_1\, u(t)^3 + \alpha_2\, u(t)^2\, v(t) + \alpha_3\, u(t)\, v(t)^2 + \alpha_4\, v(t)^3)\right) = 0, \tag{5.230}$$

$$(1-p)\left(\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2}+\omega_2^2\, v(t)\right)+p\left(\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2}+\omega_2^2\, v(t)-((\alpha_5\, u(t)^3+\alpha_6\, u(t)^2\, v(t)\right.$$
$$\left.+\,\alpha_7\, u(t)\, v(t)^2+\alpha_8\, v(t)^3)\right)=0, \tag{5.231}$$

where p is the homotopy parameter. When $p=0$, Eqs. 5.230 and 5.231 become linear differential equations for which an exact solution can be calculated. For $p=1$, these equations then become the original problem. Now, the homotopy parameter p is used to expand the solutions $u(t)$, $v(t)$ and the square of the unknown angular frequencies Ω_u and Ω_v as

$$u(t)=p^0\, u_0(t)+p^1\, u_1(t), \tag{5.232}$$

$$v(t)=p^0\, v_0(t)+p^1\, v_1(t). \tag{5.233}$$

As was discussed above, the angular frequencies of each component of the system will be obtained separately. These frequencies are denoted by Ω_u and Ω_v where Ω_u, Ω_v are frequencies for $u(t)$ and $v(t)$, respectively. In the frequency expansion, two general cases have been considered. First, it can be assumed that $\Omega_u=\Omega_v=\Omega$, which means that all components of the system are oscillating with the same frequency. Next, it is assumed that components of the system are oscillating with different frequencies.

5.8.2 *First Assumption*

In this case, frequencies can be expanded as

$$\omega_1^2=\Omega^2+p^1\,\omega_u, \tag{5.234}$$

$$\omega_2^2=\Omega^2+p^1\,\omega_v, \tag{5.235}$$

where ω_u and ω_v should be determined.

Substituting Eqs. 5.232–5.235 into Eqs. 5.230 and 5.231 and collecting the terms of the same power of p, a series of linear equations is obtained:

$$\begin{aligned}&\frac{\mathrm{d}^2 u_0(t)}{\mathrm{d}t^2}+\Omega^2\, u_0(t)=0,\\ &u_0(0)=A,\ \frac{\mathrm{d}u_0}{\mathrm{d}t}(0)=0,\end{aligned} \tag{5.236}$$

P^0:

$$\frac{d^2 v_0(t)}{dt^2} + \Omega^2 v_0(t) = 0, \qquad v_0(0) = B, \frac{dv_0}{dt}(0) = 0, \tag{5.237}$$

$$\frac{d^2 u_1(t)}{dt^2} + \Omega^2 u_1(t) + \omega_u u_0(t) - \alpha_4 v_0(t)^3 - \alpha_1 u_0(t)^3 - \alpha_3 u_0(t) v_0(t)^2 - \alpha_2 u_0(t)^2 v_0(t) = 0, \qquad u_1(0) = 0, \frac{du_1}{dt}(0) = 0, \tag{5.238}$$

P^1 :

$$\frac{d^2 v_1(t)}{dt^2} + \Omega^2 v_1(t) + \omega_v v_0(t) - \alpha_5 u_0(t)^3 - \alpha_8 v_0(t)^3 - \alpha_7 u_0(t) v_0(t)^2 - \alpha_6 u_0(t)^2 v_0(t) = 0, \qquad v_1(0) = 0, \frac{dv_1}{dt}(0) = 0. \tag{5.239}$$

Solutions of Eqs. 5.236 and 5.237 are

$$u_0(t) = A \cos(\Omega t), \tag{5.240}$$

$$v_0(t) = B \cos(\Omega t). \tag{5.241}$$

On the basis of trigonometric functions properties, we have

$$\cos^3(\Omega t) = 1/4 \cos(3\Omega t) + 3/4 \cos(\Omega t). \tag{5.242}$$

Substituting Eqs. 5.240 and 5.241 into Eqs. 5.238 and 5.239 yields

$$\frac{d^2 u_1(t)}{dt^2} + \Omega^2 u_1(t) + \omega_u A \cos(\Omega t) - \alpha_4 B^3 \cos(\Omega t)^3 - \alpha_1 A^3 \cos(\Omega t)^3 - \alpha_3 AB^2 \cos(\Omega t)^3 - \alpha_2 BA^2 \cos(\Omega t)^3 = 0 \qquad u_1(0) = 0, \frac{du_1}{dt}(0) = 0 \tag{5.243}$$

$$\frac{d^2 v_1(t)}{dt^2} + \Omega^2 v_1(t) + \omega_v B \cos(\Omega t) - \alpha_5 A^3 \cos(\Omega t)^3 - \alpha_8 B^3 \cos(\Omega t)^3 - \alpha_7 AB^2 (\cos(\Omega t))^3 - \alpha_6 A^2 B \cos(\Omega t)^3 = 0 \qquad v_1(0) = 0, \frac{dv_1}{dt}(0) = 0 \tag{5.244}$$

On the basis of the trigonometric function properties of Eq. 5.242, Eqs. 5.243 and 5.244 can be rewritten in the following forms:

$$\frac{d^2u_1(t)}{dt^2}+\Omega^2\,u_1(t)+\left(\omega_uA-\frac{3}{4}\alpha_1A^3-\frac{3}{4}\alpha_4B^3-\frac{3}{4}\alpha_2A^2B-\frac{3}{4}\alpha_3AB^2\right)\cos(\Omega t)$$
$$-\frac{1}{4}\alpha_4B^3\cos(3\Omega t)-\frac{1}{4}\alpha_1A^3\cos(3\Omega t)-\frac{1}{4}\alpha_2A^2B\cos(3\Omega t)-\frac{1}{4}\alpha_3AB^2\cos(3\Omega t)=0, \tag{5.245}$$

$$\frac{d^2v_1(t)}{dt^2}+\Omega^2\,v_1(t)+\left(\omega_vB-\frac{3}{4}\alpha_5A^3-\frac{3}{4}\alpha_8B^3-\frac{3}{4}\alpha_6A^2B-\frac{3}{4}\alpha_7AB^2\right)\cos(\Omega t)$$
$$-\frac{1}{4}\alpha_8B^3\cos(3\Omega t)-\frac{1}{4}\alpha_5A^3\cos(3\Omega t)-\frac{1}{4}\alpha_6A^2B\cos(3\Omega t)-\frac{1}{4}\alpha_7AB^2\cos(3\Omega t)=0. \tag{5.246}$$

To avoid secular terms in $u_1(t)$ and $v_1(t)$, we must eliminate contributions proportional to $\cos(\Omega t)$ on the left-hand side of Eqs. 5.245 and 5.246:

$$\omega_uA-\frac{3}{4}\alpha_1A^3-\frac{3}{4}\alpha_4B^3-\frac{3}{4}\alpha_2A^2B-\frac{3}{4}\alpha_3AB^2=0, \tag{5.247}$$

$$\omega_vB-\frac{3}{4}\alpha_5A^3-\frac{3}{4}\alpha_8B^3-\frac{3}{4}\alpha_6A^2B-\frac{3}{4}\alpha_7AB^2=0. \tag{5.248}$$

From these equations, the solutions ω_u and ω_v can be easily obtained as

$$\omega_u=+\frac{3}{4}\alpha_1A^2+\frac{3}{4}\frac{\alpha_4B^3}{A}+\frac{3}{4}\alpha_2AB+\frac{3}{4}\alpha_3\,B^2, \tag{5.249}$$

$$\omega_v=+\frac{3}{4}\frac{\alpha_5A^3}{B}+\frac{3}{4}\alpha_8B^2+\frac{3}{4}\alpha_6A^2+\frac{3}{4}\alpha_7AB. \tag{5.250}$$

Letting $p=1$ in Eqs. 5.234 and 5.235, the approximate frequency can be easily obtained:

$$\Omega=\frac{\sqrt{-A(4\,\omega_u^2\,A+3\alpha_3AB^2+3\alpha_1\,A^3+3\alpha_4\,B^3+3\alpha_2\,A^2B)}}{2\,A}, \tag{5.251}$$

$$\Omega=\frac{\sqrt{-B(4\,\omega_v^2\,B+3\alpha_7AB^2+3\alpha_5\,A^3+3\alpha_8\,B^3+3\alpha_6\,A^2B)}}{2\,B}. \tag{5.252}$$

Equations 5.251 and 5.252 are frequency–amplitude relations for $u(t)$ and $v(t)$, respectively. In this approach, the obtained results for frequency–amplitude relations are similar to what is obtained in the nonlinear oscillatory systems with only one degree of freedom, where frequency is a function of initial amplitude; here, also, frequency–amplitude relations are functions of both initial amplitudes of A and B.

The correction term $u_1(t)$ for the periodic solution $u_0(t)$ can be obtained as

$$\begin{aligned} u_1(t) = & \frac{1}{96\,\Omega^2}(\cos(\Omega t)(3\alpha_3 AB^2 + 3\alpha_2 A^2 B + 3\alpha_1 A^3 + 3\alpha_4 B^3)) \\ & - \frac{1}{96\,\Omega^2}(\cos(3\Omega t)(3\alpha_3 AB^2 + 3\alpha_2 A^2 B + 3\alpha_1 A^3 + 3\alpha_4 B^3)), \end{aligned} \tag{5.253}$$

$$\begin{aligned} v_1(t) = & \frac{1}{96\,\Omega^2}(\cos(\Omega t)(3\alpha_7 AB^2 + 3\alpha_6 A^2 B + 3\alpha_5 A^3 + 3\alpha_8 B^3)) \\ & - \frac{1}{96\,\Omega^2}(\cos(3\Omega t)(3\alpha_7 AB^2 + 3\alpha_6 A^2 B + 3\alpha_5 A^3 + 3\alpha_8 B^3)). \end{aligned} \tag{5.254}$$

Therefore, from Eqs. 5.232 and 5.233, the approximation to the periodic solution is given by

$$\begin{aligned} u(t) = & A\cos(\Omega t) + \frac{1}{96\,\Omega^2}(\cos(\Omega t)(3\alpha_3 AB^2 + 3\alpha_2 A^2 B + 3\alpha_1 A^3 + 3\alpha_4 B^3)) \\ & - \frac{1}{96\,\Omega^2}(\cos(3\Omega t)(3\alpha_3 AB^2 + 3\alpha_2 A^2 B + 3\alpha_1 A^3 + 3\alpha_4 B^3)) \end{aligned} \tag{5.255}$$

$$\begin{aligned} v(t) = & B\cos(\Omega t) + \frac{1}{96\,\Omega^2}(\cos(\Omega t)(3\alpha_7 AB^2 + 3\alpha_6 A^2 B + 3\alpha_5 A^3 + 3\alpha_8 B^3)) \\ & - \frac{1}{96\,\Omega^2}(\cos(3\Omega t)(3\alpha_7 AB^2 + 3\alpha_6 A^2 B + 3\alpha_5 A^3 + 3\alpha_8 B^3)) \end{aligned} \tag{5.256}$$

According to obtained results, it can be seen that when $\omega_1 = \omega_2, m_1 = m_2$, $\alpha_1 = \alpha_5$, $\alpha_2 = \alpha_6$, $\alpha_3 = \alpha_7$, and $\alpha_4 = \alpha_8$, then if $A = B$ and $A = -B$, the components oscillate in equal frequencies. The comparisons between the results obtained by the MHPM and the numerical solutions are given in Tables 5.7 and 5.8 for $u(t)$ and $v(t)$ in the first 3 s of oscillation, respectively.

Table 5.7 Comparison between obtained solutions for $u(t)$ with a numerical one for $\omega_1^2 = \omega_2^2 = 5\pi, \alpha_1 = \alpha_5 = 0.1, \alpha_2 = \alpha_6 = 0.15, \alpha_3 = \alpha_7 = 0.2, \alpha_4 = \alpha_8 = 0.25, A = 0.05, B = -0.05$

t(s)	MHPM solution	Numerical solution	Error presentation
0	0.050000000	0.050000000	0.0
0.5	−0.019970739	−0.019970753	0.0000668
1	−0.034046692	−0.034046745	0.0001549
1.5	0.047168342	0.047168448	0.0002253
2	−0.003632802	−0.003632795	0.0002042
2.5	−0.044266333	−0.044266510	0.0003985
3	0.038994110	0.038994275	0.0004228

Table 5.8 Comparison between obtained solutions for $v(t)$ with a numerical one for $\omega_1^2 = \omega_2^2 = 5\pi, \alpha_1 = \alpha_5 = 0.1, \alpha_2 = \alpha_6 = 0.15, \alpha_3 = \alpha_7 = 0.2, \alpha_4 = \alpha_8 = 0.25, A = 0.05, B = -0.05$

t(s)	MHPM solution	Numerical solution	Error presentation
0	−0.050000000	−0.050000000	0.0
0.5	0.019972211	0.019969735	0.0000667
1	0.034048497	0.034048550	0.0001549
1.5	−0.047167186	−0.047167292	0.0002250
2	0.003628098	0.003628091	0.0002065
2.5	0.044269121	0.044269296	0.0003977
3	−0.038989728	−0.038989893	0.0004218

5.8.3 Second Assumption

When the components of the system oscillate with different frequencies, the following expansion can be used:

$$\omega_1^2 = \Omega_u^2 + p^1\omega_u, \tag{5.257}$$

$$\omega_2^2 = \Omega_v^2 + p^1\,\omega_v, \tag{5.258}$$

where Ω_u and Ω_v are the frequency–amplitude relations of $u(t)$ and $v(t)$, respectively. Similarly, substituting Eqs. 5.257 and 5.258 into Eqs. 5.230 and 5.231 yields

$$u_0(t) = A\,\cos(\Omega_u t), \tag{5.259}$$

$$v_0(t) = B\,\cos(\Omega_v t). \tag{5.260}$$

Substituting Eqs. 5.259 and 5.260 into Eqs. 5.238 and 5.239, yields

$$\begin{aligned}&\frac{d^2u_1(t)}{dt^2} + \Omega_u^2\,u_1(t) + \omega_u A\,\cos(\Omega_u t) - \alpha_4 B^3\cos(\Omega_v t)^3 - \alpha_1 A^3\cos(\Omega_u t)^3\\ &- \alpha_3 AB^2\cos(\Omega_u t)\cos(\Omega_v t)^2 - \alpha_2\,BA^2\cos(\Omega_u t)^2\cos(\Omega_v t) = 0,\end{aligned} \tag{5.261}$$

$$\begin{aligned}&\frac{d^2v_1(t)}{dt^2} + \Omega_v^2\,v_1(t) + \omega_v A\,\cos(\Omega_v t) - \alpha_8 B^3\cos(\Omega_v t)^3 - \alpha_5 A^3\cos(\Omega_u t)^3\\ &- \alpha_7 AB^2\,\cos(\Omega_u t)\cos(\Omega_v t)^2 - \alpha_6\,BA^2\cos(\Omega_u t)^2\,\cos(\Omega_v t) = 0.\end{aligned} \tag{5.262}$$

To analyze the particular solution of Eqs. 5.261 and 5.262, we need to distinguish between two cases: $\Omega_v \approx 3\Omega_u$, $\Omega_v \approx \frac{\Omega_u}{3}$ (case 1) and Ω_v away from $3\Omega_u$ and $\frac{\Omega_u}{3}$ (case 2). In this problem, the first case (case 1) has been considered. In the second case (case 2), the only terms that produce secular terms are the terms proportional to $\cos(\Omega_u t)$ in Eq. 5.261 and the terms proportional to $\cos(\Omega_v t)$ in

Eq. 5.262. On the basis of trigonometric function properties of Eq. 5.242, Eqs. 5.261 and 5.262 can be rewritten in the form

$$\frac{d^2u_1(t)}{dt^2}+\Omega_u^2\,u_1(t)+\left(\omega_u A-\frac{3\alpha_1A^3}{4}-\frac{\alpha_3AB^2}{2}\right)\cos(\Omega_u t)-\alpha_4B^3\cos(\Omega_v t)^3$$
$$-\frac{\alpha_1A^3}{4}\cos(3\Omega_u t)-\frac{\alpha_3AB^2}{2}\cos(\Omega_u t)\cos(2\,\Omega_v t)-\alpha_2\,BA^2\cos(\Omega_u t)^2\,\cos(\Omega_v t)=0, \tag{5.263}$$

$$\frac{d^2v_1(t)}{dt^2}+\Omega_v^2\,v_1(t)+\left(\omega_v\,B-\frac{\alpha_6\,BA^2}{2}-\frac{3\alpha_8B^3}{4}\right)\cos(\Omega_v t)-\frac{\alpha_8B^3}{4}\cos(3\Omega_v t)$$
$$-\alpha_5A^3\cos(\Omega_u t)^3-\alpha_7AB^2\,\cos(\Omega_u t)\cos(\Omega_v t)^2-\frac{\alpha_6\,BA^2}{2}\cos(2\,\Omega_u t)\cos(\Omega_v t)=0. \tag{5.264}$$

Eliminating the secular terms in Eqs. 5.263 and 5.264 yields

$$\omega_u A-\frac{3}{4}\alpha_1A^3-\frac{1}{2}\alpha_3AB^2=0, \tag{5.265}$$

$$\omega_v B-\frac{3}{4}\alpha_8B^3-\frac{1}{2}\alpha_6A^2B=0. \tag{5.266}$$

And similarly, the approximate frequency can be obtained as

$$\Omega_u=\frac{\sqrt{4\,\omega_u^2+2\,\alpha_3B^2-3\alpha_1\,A^2)}}{2}, \tag{5.267}$$

$$\Omega_v=\frac{\sqrt{4\,\omega_v^2\,+3\alpha_8B^2-2\,\alpha_6\,A^2)}}{2}. \tag{5.268}$$

By solving Eqs. 5.263 and 5.264 after eliminating the secular terms, $u_1(t)$ and $v_1(t)$ are obtained and are shown in the Appendix. Then, the approximation to the periodic solutions can be written from Eqs. 5.232 and 5.233. In the first case, in addition to the terms proportional to $\cos(\Omega_u t)$ and $\cos(\Omega_v t)$, secular terms are produced by the terms proportional to $\cos(\pm(\Omega_u-2\Omega_v)t), \cos(\pm3\Omega_u t)$, and $\cos(\pm3\Omega_v t)$ in each of Eqs. 5.261 and 5.262, but MHPM will not give accurate solutions in this case.

In the second case, the obtained approximate results are accurate for the small range of amplitudes and coefficients of nonlinear terms. For this problem, in Figs. 5.34 and 5.35, comparisons have been made with $\omega_1^2=\pi, \omega_2^2=4\,\pi, \alpha_1=10, \alpha_2=1.5, \alpha_3=2, \alpha_4=2.5, \alpha_5=3, \alpha_6=3.5, \alpha_7=4, \alpha_8=4.5, A=0,$ and $B=-0.05$.

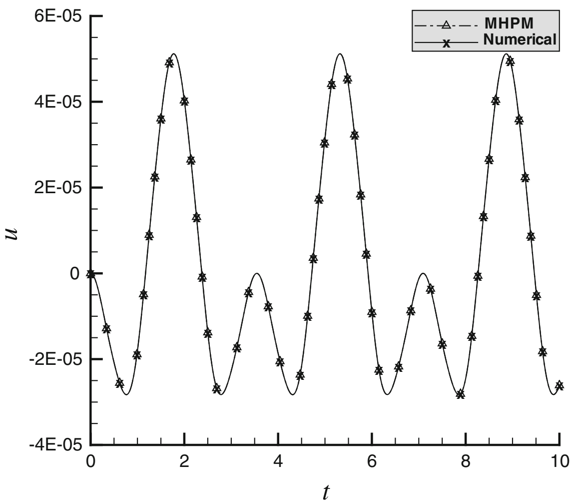


Fig. 5.34 Comparison of the result of the second assumption for $u(t)$

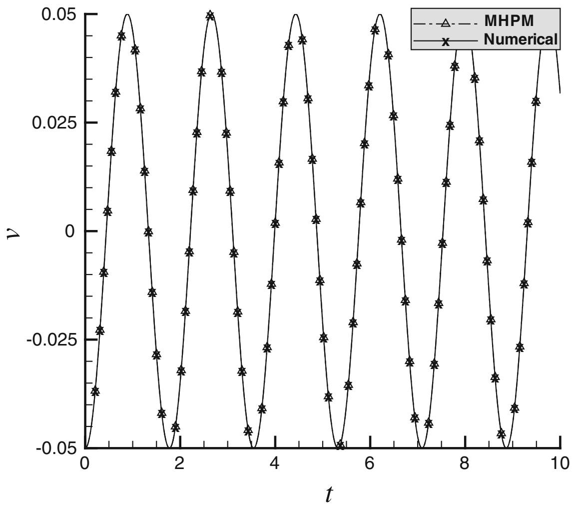


Fig. 5.35 Comparison of the result of the second assumption for $v(t)$

5.9 Problem 5.9. Van der Pol Oscillator

5.9.1 Introduction

In this problem, we consider the following Van der Pol oscillator:

$$u'' + \varepsilon(u^2 - 1)u' + u = 0,\ u(0) = 1 \text{ and } u'(0) = 0 \tag{5.269}$$

For every nonnegative value of the parameter ε, the solution of Van der Pol's equation for the initial conditions given above is periodic, and the corresponding phase plane trajectory is a limit cycle to which the other trajectories converge. The straightforward expansion solution of the above problem is

$$u = \cos t + \varepsilon\left(\frac{3}{8}t\cos t - \frac{1}{32}\sin 3t - \frac{9}{32}\sin t\right) + \cdots \quad (5.270)$$

It is worth noting that Eq. 5.270 is not asymptotically valid for t equal to or greater than ε^{-1}.

We will study the mathematical model of these methods (PM, HPM, VIM, Adomian's decomposition method [ADM]) and then their applications in the Van der Pol oscillator, and at the end, we will compare their solutions with the numerical solution (using Maple software to get a BPV form numerical solution method).

5.9.2 The Application of PM in the Van der Pol Oscillator

We assume that

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 \quad (5.271)$$

After substituting u from Eq. 5.271 into Eq. 5.269 and some simplification and substitution, we have:

$$\left(2u_0u_1\frac{du_0}{dt} + u_2 + u_0^2\frac{du_1}{dt} + \frac{d^2u_2}{dt} - \frac{du_1}{dt}\right)\varepsilon^2 + \left(-\frac{du_0}{dt} + u_1 + u_0^2\frac{du_0}{dt} + \frac{d^2u_1}{dt^2}\right)\varepsilon + u_0 + \frac{d^2u_0}{dt^2} = 0, \quad (5.272)$$

where

$$\varepsilon^3 = \varepsilon^4 = \cdots = 0. \quad (5.273)$$

After a rearrangement based on power of ε-terms, we have

$$\varepsilon^0 : \frac{d^2u_0}{dt^2} + u_0 = 0,\ u_0(0) = 1, \frac{du_0(0)}{dt} = 0 \quad (5.274)$$

$$\varepsilon^1 : \frac{d^2u_1}{dt^2} + u_1 + u_0^2\frac{du_0}{dt} - \frac{du_0}{dt} = 0,\ u_1(0) = 0, \frac{du_1(0)}{dt} = 0 \quad (5.275)$$

$$\varepsilon^2 : \frac{d^2u_2}{dt^2} + u_2 + u_0^2\frac{du_1}{dt} - \frac{du_1}{dt} + 2u_0u_1\frac{du_0}{dt} = 0,\ u_2(0) = 0, \frac{du_2(0)}{dt} = 0. \quad (5.276)$$

To determine u, the above equations should be solved. Solving Eq. 5.274 considering the appropriate initial conditions, we have

$$u_0 = \cos t. \quad (5.277)$$

Then we have

$$u_1 = -\frac{1}{4}\sin t - \frac{1}{8}\cos^2 t\sin t + \frac{3}{8}t\cos t \tag{5.278}$$

$$u_2 = \frac{103}{768}\cos t - \frac{9}{64}t\cos^2 t\sin t - \frac{11}{256}t\sin t + \frac{3}{128}t^2\cos t - \frac{5}{192}\cos^5 t - \frac{83}{768}\cos^3 t. \tag{5.279}$$

So u will be generally as follows (see Eq. 5.271):

$$u(t) = \cos t + \varepsilon\left(-\frac{1}{4}\sin t - \frac{1}{8}\cos^2 t\sin t + \frac{3}{8}t\cos t\right) + \varepsilon^2\left(\frac{103}{768}\cos t - \frac{9}{64}t\cos^2 t\sin t - \frac{11}{256}t\sin t + \frac{3}{128}t^2\cos t - \frac{5}{192}\cos^5 t - \frac{83}{768}\cos^3 t\right) \tag{5.280}$$

5.9.3 Homotopy Perturbation Method

Assuming $\frac{d^2u_0}{dt^2} = u_0 = 0$ and substituting v, some simplifications and rearranging based on powers of p-terms, we obtain

$$p^0 : \frac{d^2v_0}{dt^2} + v_0 = 0 \qquad v_0(0) = 1, \frac{dv_0(0)}{dt} = 0 \tag{5.281}$$

$$p^1 : \frac{d^2v_1}{dt^2} + v_1 + \varepsilon v_0\left(\frac{dv_0}{dt}\right) - \varepsilon\left(\frac{dv_0}{dt}\right) = 0 \qquad v_1(0) = 0, \frac{dv_1(0)}{dt} = 0 \tag{5.282}$$

$$p^2 : \frac{d^2v_2}{dt^2} + v_2 + \varepsilon v_0^2\left(\frac{dv_1}{dt}\right) - \varepsilon\frac{dv_1}{dt} + 2\varepsilon v_0 v_1\left(\frac{dv_0}{dt}\right) = 0 \quad v_2(0) = 0, \frac{dv_2(0)}{dt} = 0. \tag{5.283}$$

Solving Eqs. 5.281–5.283 considering appropriate initial conditions, we have

$$v_0 = \cos t \tag{5.284}$$

$$v_1 = \varepsilon\left(-\frac{1}{4}\sin t - \frac{1}{8}\cos^2 t\sin t + \frac{3}{8}t\cos t\right) \tag{5.285}$$

$$v_2 = \varepsilon^2\left(\frac{103}{768}\cos t - \frac{9}{64}t\cos^2 t\sin t - \frac{11}{256}t\sin t + \frac{3}{128}t^2\cos t - \frac{5}{192}\cos^5 t - \frac{83}{768}\cos^3 t\right) \tag{5.286}$$

So $u(t)$ will be generally expressed as

$$\begin{aligned}u(t) &= \lim_{p\to 1}\left(v_0 + pv_1 + p^2 v_2\right)\\ &= \cos t + \varepsilon\left(-\frac{1}{4}\sin t - \frac{1}{8}\cos^2 t\sin t + \frac{3}{8}t\cos t\right) + \varepsilon^2\left(\frac{103}{768}\cos t - \frac{9}{64}t\cos^2 t\sin t\right.\\ &\quad \left. - \frac{11}{256}t\sin t + \frac{3}{128}t^2\cos t - \frac{5}{192}\cos^5 t - \frac{83}{768}\cos^3 t\right)\end{aligned} \tag{5.287}$$

5.9.4 Application of VIM in the Van der Pol Oscillator

Its correction variational functional in t can be expressed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{\frac{d^2 u_n}{d\xi^2} + u_n + \varepsilon\left(\tilde{u}_n^2 - 1\right)\frac{d\tilde{u}_n}{d\xi}\right\}d\xi, \tag{5.288}$$

where λ is a general Lagrange multiplier and can be identified optimally by the variational theory.

After some calculations, we obtain the following stationary conditions:

$$\lambda''(\xi) + \lambda(\xi) = 0 \tag{5.289a}$$

$$1 - \lambda'(\xi)|_{\zeta=t} = 0 \tag{5.289b}$$

$$\lambda(\xi)|_{\xi=t} = 0. \tag{5.289c}$$

Equation 5.289a is called the Lagrange–Euler equation, and Eqs. 5.289b and 5.289c are natural boundary conditions.

Therefore, the Lagrange multiplier can be identified as $\lambda = \sin(\xi - t)$, and the variational iteration formula is obtained in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \sin(\xi - t)\left\{\frac{d^2 u_n}{d\xi^2} + u_n + \varepsilon(\tilde{u}_n^2 - 1)\frac{d\tilde{u}_n}{d\xi}\right\}d\xi. \tag{5.290}$$

We start with the initial approximation of $u_0(t)$ given by Eq. 5.269. Using the above iteration formula (5.290), we can directly obtain the other components:

$$u_0(t) = \cos(t) \tag{5.291}$$

$$u_1(t) = \cos t + \varepsilon\left(\frac{3}{8}t\cos t - \frac{1}{8}\sin t\cos^2 t - \frac{1}{4}\sin t\right) \tag{5.292}$$

$$u_2(t) = \cos t + \varepsilon\left(\frac{3}{8}t\cos t - \frac{1}{8}\sin t\cos^2 t - \frac{1}{4}\sin t\right) + \varepsilon^2\left(\frac{-11}{256}t\sin t + \frac{3}{128}t^2\cos t - \frac{5}{192}\cos^5 t - \frac{83}{768}\cos^3 t - \frac{9}{64}t\sin t\cos^2 t + \frac{103}{768}\cos t\right) + \cdots \tag{5.293}$$

5.9.5 Application of ADM in the Van der Pol Oscillator

By applying the procedure explained in the last section to the Van der Pol equation, we have

$$u(t) = f(t) - \varepsilon L_t^{-1}(u(t)^2 u_t(t)) + \varepsilon L_t^{-1}(u_t(t)) - u(t), \tag{5.294}$$

where, in Eq. 5.294,

$$L_t^{-1}(.) = \int_0^t \int_0^t (.)\mathrm{d}t\mathrm{d}t. \tag{5.295}$$

Substituting the Adomian polynomial, $u(t)$ is obtainable in the form

$$u(t) = \sum_{n=0}^{\infty} u_n(t) = f(t) + L_t^{-1}\left[\varepsilon\left(\sum_{n=0}^{\infty} A_n\right) - \left(\sum_{n=0}^{\infty} u_n(t)\right) + \varepsilon\left(\sum_{n=0}^{\infty} u_n(t)\right)_t\right] \tag{5.296}$$

Finally,

$$u_0(t) = f(t) \tag{5.297}$$

and

$$u_{n+1}(t) = L_t^{-1}[\varepsilon A_n - u_n(t) + \varepsilon u_{n\,t}(t)]. \tag{5.298}$$

By the latest replicable relation, we are able to compute all components of $u_n(t)$. Therefore, we consider the Van der Pol equation with initial conditions:

$$u(0) = 1, \quad u_t(0) = 0. \tag{5.299}$$

With these initial conditions, $f(t)$ can be calculated as

$$f(t) = 1. \tag{5.300}$$

Hence,

$$u_0(t) = f(t) = 1. \tag{5.301}$$

By means of the recursive relation obtained in the previous section, Adomian polynomials A_n and the component of decomposition series $u_n(t)$ for $n = 3$ are yielded and listed as below, respectively:

$$\begin{aligned} A_0 &= 0 \\ A_1 &= -t \\ A_2 &= -\varepsilon t^2 + \frac{7}{6}t^3 \\ A_3 &= -\varepsilon^2 \frac{2}{3}t^3 + \varepsilon \frac{25}{12}t^4 - \frac{61}{120}t^5 \end{aligned} \tag{5.302}$$

and

$$\begin{aligned} u_0(t) &= 1 \\ u_1(t) &= -\frac{1}{2}t^2 \\ u_2(t) &= -\varepsilon \frac{1}{3}t^3 + \frac{1}{24}t^4 \\ u_3(t) &= -\varepsilon^2 \frac{1}{6}t^4 + \varepsilon \frac{1}{12}t^5 - \frac{1}{720}t^6. \end{aligned} \tag{5.303}$$

Lastly, $u(t)$ given by creating a decomposition series for $n = 3$ is

$$\begin{aligned} u(t) &= \sum_{n=0}^{3} u_n(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) \\ &= 1 - \frac{1}{2}t^2 - \varepsilon \frac{1}{3}t^3 + \frac{1}{24}t^4 - \varepsilon^2 \frac{1}{6}t^4 + \varepsilon \frac{1}{12}t^5 - \frac{1}{720}t^6. \end{aligned} \tag{5.304}$$

As is shown in Figs. 5.36, 5.37, 5.38 and 5.39, the differences between the results of these four methods and the numerical solution (using Maple software to get a BPV form numerical solution method) are very small, and all of them closely mirror the numerical solution with great accuracy. Note that VIM is the easiest of them, PM and the HPM are similar together, and, importantly, the answer obtained through ADM, as shown in Fig. 5.39, is valid only at a certain area (0 up to 2), which is a limitation. As is shown in Fig. 5.40, these are differences between the results of three methods (PM, HPM, VIM) and the numerical solution. All of them have the same very small margin of curve error.

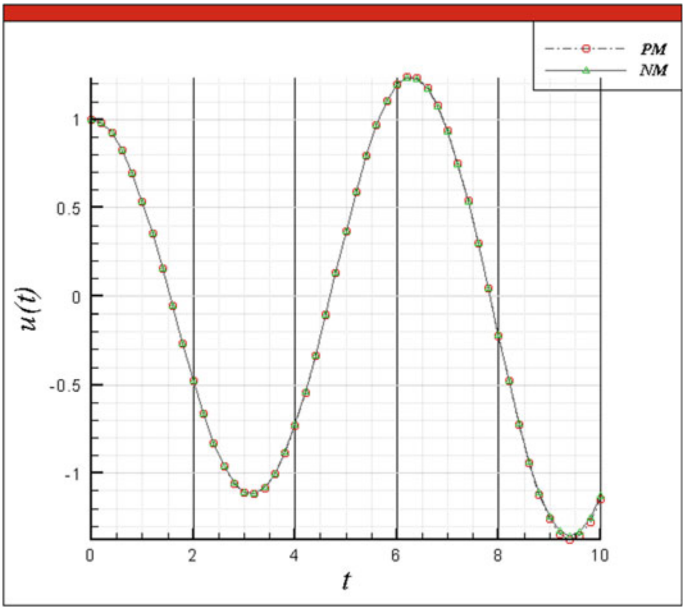


Fig. 5.36 Comparison of the answer resulted by perturbation method (PM) and the numerical solution (NM) at $\varepsilon = 0.1$

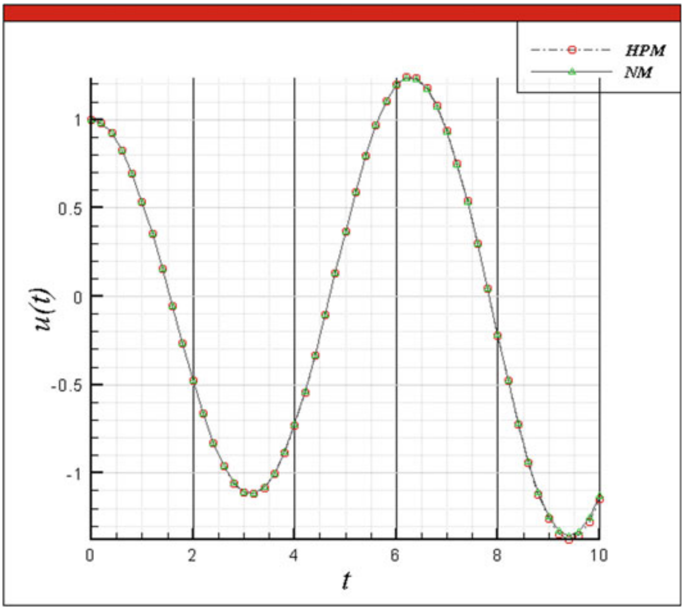


Fig. 5.37 Comparison of the answer resulting from the homotopy perturbation method (HPM) and numerical solutions (NM) at $\varepsilon = 0.1$

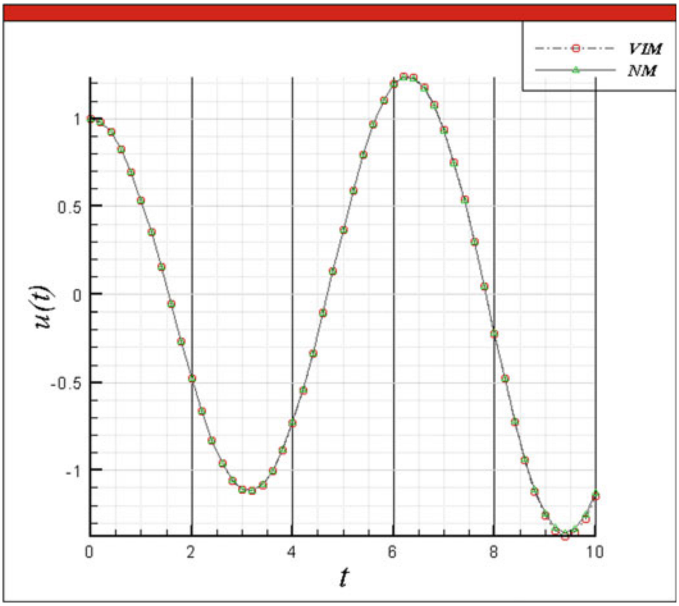


Fig. 5.38 Comparison of the answer obtained through the variational iteration method (VIM) and the numerical solution (NM) at $\varepsilon = 0.1$

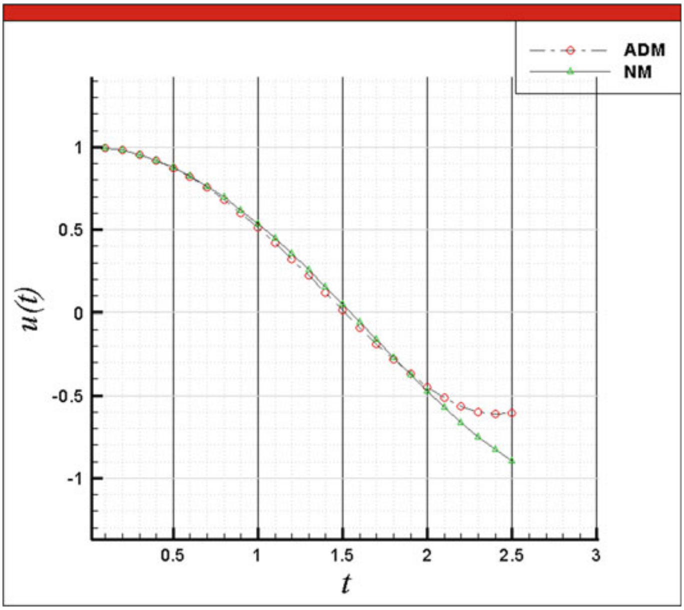


Fig. 5.39 The comparison of the answer obtained through ADM and the numerical solution at $\varepsilon = 0.1$

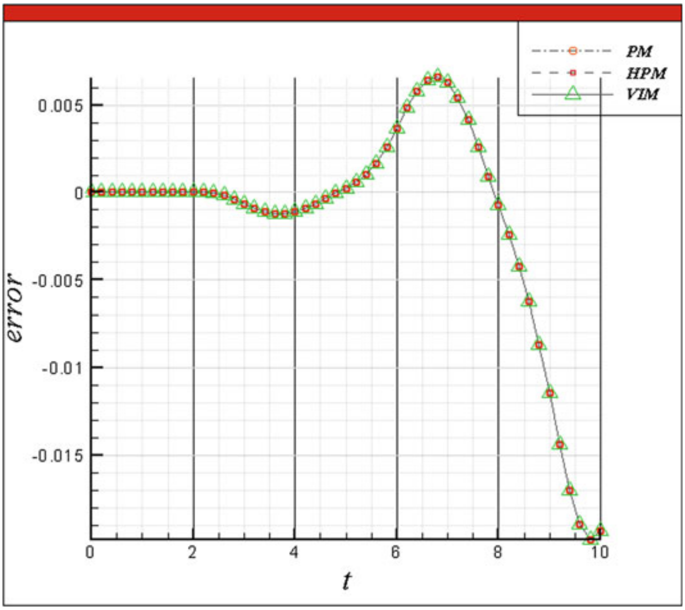


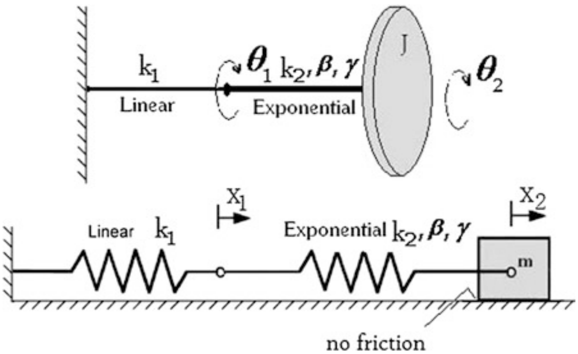
Fig. 5.40 Error in answers obtained through the perturbation method (PM), homotopy PM (HPM), and variational iteration method (VIM) in comparison with the numerical solution at $\varepsilon = 0.1$

5.10 Problem 5.10. Application of a Slender, Elastic Cantilever Beam

5.10.1 Introduction

Many engineering structures can be modeled as a slender, elastic cantilever beam carrying a concentrated mass at an intermediate point along its span. This system can be simulated by a mass with serial linear, nonlinear, or exponential stiffness, as shown in Fig. 5.41.

Fig. 5.41 System of mass with serial linear and exponential stiffness



For this system, the governing differential equation of the motion is of the following form:

$$k_1 x_1 - k_2(x_2 - x_1) - \beta(x_2 - x_1)e^{\gamma(x_2 - x_1)} = 0, \tag{5.305a}$$

$$m\ddot{x}_2 + k_2(x_2 - x_1) + \beta(x_2 - x_1)e^{\gamma(x_2 - x_1)} = 0. \tag{5.305b}$$

Using the new variables, u and v can be defined as

$$u = x_1,$$
$$v = x_2 - x_1.$$

Then, Eq. 5.305 can be put into a different form:

$$k_1 u - k_2 v - \beta v e^{\gamma v} = 0, \tag{5.306a}$$

$$m(\ddot{u} + \ddot{v}) + k_2 v + \beta v e^{\gamma v} = 0. \tag{5.306b}$$

Solving Eq. 5.306a for u yields

$$u = \frac{k_2}{k_1} v + \frac{\beta}{k_1} v e^{\gamma v}, \tag{5.307}$$

If Eq. 5.307 is differentiated twice with respect to time and substituted into Eq. 5.306b, one finds that

$$\left(1 + \frac{k_2}{k_1}\right)\ddot{v} + \frac{\beta}{k_1}(1 + \gamma v)\ddot{v}e^{\gamma v} + \frac{\beta\gamma}{k_1}(2 + \gamma v)\dot{v}^2 e^{\gamma v} + \frac{k_2}{m} v + \frac{\beta}{m} v e^{\gamma v} = 0 \tag{5.308}$$

or

$$\ddot{v} + \mu v + [(1 + \gamma v)\xi\ddot{v} + (2 + \gamma v)\xi\gamma\dot{v}^2 + \eta v]e^{\gamma v} = 0, \tag{5.309}$$

where

$$\varepsilon = \frac{k_1}{k_1 + k_2},$$
$$\xi = \varepsilon\frac{\beta}{k_1},$$
$$\mu = \frac{k_2}{m}\varepsilon,$$
$$\eta = \frac{\beta}{m}\varepsilon.$$

5.10.2 Solution Using the First Case of the Homotopy Perturbation Method

We write Eq. 5.309 as

$$\begin{aligned}&\ddot{v}+\mu v+p[(1+\gamma v)\xi\ddot{v}+(2+\gamma v)\xi\gamma\dot{v}^2+\eta v]e^{\gamma v}=0.\\&v(0)=A,\ \dot{v}(0)=0.\end{aligned} \tag{5.310}$$

To illustrate the new modified HPM, we expand the solution *v(t)* and the square of the unknown angular frequency ω as

$$v=v_0+pv_1+p^2v_2+\cdots, \tag{5.311}$$

$$\mu=\omega^2-p\alpha_1-p^2\alpha_2-\cdots, \tag{5.312}$$

where α_i ($i=1, 2,\ldots$) are to be determined.

Substituting Eq. 5.311 into Eq. 5.310, and replacing Eq. 5.312 with the coefficient of $v(t)$ in Eq. 5.310, we have

$$\begin{aligned}&\ddot{v}_0+p\ddot{v}_1+p^2\ddot{v}_2+\cdots+(\omega^2-p\alpha_1-p^2\alpha_2-\cdots)(v_0+pv_1+p^2v_2+\cdots)\\&\quad+p[\xi(1+\gamma(v_0+pv_1+p^2v_2+\cdots))(\ddot{v}_0+p\ddot{v}_1+p^2\ddot{v}_2+\cdots)\\&\quad+\xi\gamma(2+\gamma(v_0+pv_1+p^2v_2+\cdots))(\dot{v}_0+p\dot{v}_1+p^2\dot{v}_2+\cdots)^2\\&\quad+\eta(v_0+pv_1+p^2v_2+\cdots)]e^{\gamma(v_0+pv_1+p^2v_2+\cdots)}=0.\end{aligned} \tag{5.313}$$

Equating the terms with identical powers of p, we obtain the following set of linear differential equations:

$$p^0:\ \ddot{v}_0+\omega^2v_0=0,\ v_0(0)=A,\dot{v}_0(0)=0. \tag{5.314}$$

$$\begin{aligned}&P^1:\ddot{v}_1+\omega^2v_1-\alpha_1v_0+[\xi(1+\gamma v_0)\ddot{v}_0+\xi\gamma(2+\gamma v_0)\dot{v}_0^2+\eta v_0]e^{\gamma v_0}=0,\\&v_1(0)=0,\dot{v}_1(0)=0.\end{aligned} \tag{5.315}$$

$$\begin{aligned}&p^2:\ddot{v}_2+\omega^2v_2-\alpha_1v_1-\alpha_2v_0+[\xi(\ddot{v}_1+\gamma(v_0\ddot{v}_1+\ddot{v}_0v_1))+\xi\gamma(4\dot{v}_0\dot{v}_1)\\&\qquad+\xi\gamma^2(2v_0\dot{v}_0\dot{v}_1+v_1\dot{v}_0^2)+\eta v_1]e^{\gamma(v_0+v_1)}=0,\\&v_2(0)=0,\ \dot{v}_2(0)=0.\end{aligned} \tag{5.316}$$

$$\begin{aligned}&p^3:\ddot{v}_3+\omega^2v_3-\alpha_3v_0-\alpha_2v_1-\alpha_1v_2+[\xi(\ddot{v}_2+\gamma(v_1\ddot{v}_1+v_2\ddot{v}_0+v_0\ddot{v}_2))\\&\qquad+\xi\gamma(2\dot{v}_1^2+4\dot{v}_2\dot{v}_0)+\xi\gamma^2(v_0\dot{v}_1^2+2v_0\dot{v}_2\dot{v}_0+2v_1\dot{v}_1\dot{v}_0+\dot{v}_0v_2)+\eta v_2)]e^{\gamma(v_0+v_1+v_2)}=0,\\&v_3(0)=0,\ \dot{v}_3(0)=0.\end{aligned} \tag{5.317}$$

The solution of Eq. 5.314 is

$$v_0(t) = A\cos(\omega t). \tag{5.318}$$

Substitution of this result into Eq. 5.315 gives

$$\begin{aligned} &\ddot{v}_1 + \omega^2 v_1 - \alpha_1 A\cos(\omega t) + [-\xi(1+\gamma A\cos(\omega t))A\omega^2\cos(\omega t) \\ &\quad + \xi\gamma(2+\gamma A\cos(\omega t))A^2\omega^2\sin(\omega t)^2 + \eta A\cos(\omega t)]e^{\gamma A\cos(\omega t)} = 0. \end{aligned} \tag{5.319}$$

It is possible to generate the Fourier series expansion

$$\begin{aligned} &\Big[-\xi(1+\gamma A\cos(\omega t))A\omega^2\cos(\omega t) + \xi\gamma(2+\gamma A\cos(\omega t))A^2\omega^2\sin(\omega t)^2 \\ &+\eta A\cos(\omega t)]e^{\gamma A\cos(\omega t)} = \sum_{n=0}^{\infty} a_{2n+1}\cos((2n+1)) = a_1\cos(\omega t) + a_3\cos(3\omega t) + \mathrm{K} \end{aligned} \tag{5.320}$$

where

$$\begin{aligned} a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} &[-\xi(1+\gamma A\cos(\omega t))A\omega^2\cos(\omega t) + \xi\gamma(2+\gamma A\cos(\omega t))A^2\omega^2\sin(\omega t)^2 \\ &+ \eta A\cos(\omega t)]e^{\gamma A\cos(\omega t)}(\cos(2n+1)\varphi)\,\mathrm{d}\varphi, \end{aligned}$$

where $\omega = \sqrt{\mu}$

(5.321)

Substituting Eq. 5.320 into Eq. 5.319, we have

$$\ddot{v}_1 + \omega^2 v_1 + \sum_{n=0}^{\infty} a_{2n+1}\cos((2n+1)\omega t) - \alpha_1 A\cos(\omega t) = 0, \tag{5.322}$$

or

$$\ddot{v}_1 + \omega^2 v_1 + \sum_{n=1}^{\infty} a_{2n+1}\cos((2n+1)\omega t) + (a_1 - \alpha_1 A)\cos(\omega t) = 0.$$

The absence of secular terms in $v_1(\mathrm{t})$ requires eliminating contributions proportional to $\cos(\omega t)$ in Eq. 5.322, and we obtain

$$\alpha_1 = \frac{a_1}{A}. \tag{5.323}$$

Considering Eqs. 5.323 and 5.322, we rewrite Eq. 5.322 in the form

$$\ddot{v}_1 + \omega^2 v_1 = -\sum_{n=1}^{\infty} a_{2n+1}\cos((2n+1)\omega t), \tag{5.324}$$

with initial conditions $v_1(0) = 0$ and $\dot{v}_1(0) = 0$. The periodic solution to Eq. 5.324 can be written as

$$v_1(t) = \sum_{n=0}^{\infty} b_{2n+1} \cos((2n+1)\omega t) = b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \cdots. \quad (5.325)$$

Substituting Eq. 5.325 into Eq. 5.324, we obtain

$$\omega^2 \sum_{n=0}^{\infty} b_{2n+1}(1 - (2n+1)^2) \cos((2n+1)\omega t) = -\sum_{n=1}^{\infty} a_{2n+1} \cos((2n+1)\omega t). \quad (5.326)$$

We can write the following expression for the coefficients b_{2n+1}:

$$b_{2n+1} = \frac{a_{2n+1}}{((2n+1)^2 - 1)\omega^2} = \frac{a_{2n+1}}{4n(n+1)\omega^2}, \text{ for } n \geq 1. \quad (5.327)$$

Taking into account that $v_1(0) = 0$, Eq. 5.327 gives

$$b_1 = -\sum_{n=1}^{\infty} b_{2n+1}. \quad (5.328)$$

The function $v_1(\mathrm{t})$ has an infinite number of harmonics, and it is difficult to solve the new differential equation; however, we can truncate the series expansion at Eq. 5.325 and write an approximate equation $v_1^{(\mathrm{N})}(\mathrm{t})$ in the form

$$v_1^{(N)}(t) = b_1^{(N)} \cos(\omega t) + \sum_{n=1}^{N} b_{2n+1} \cos((2n+1)\omega t), \quad (5.329)$$

where

$$b_1^{(N)} = -\sum_{n=1}^{N} b_{2n+1}. \quad (5.330)$$

Equation 5.329 has only a finite number of harmonics. It is possible to make this approximation because the absolute value of the coefficient b_{2n+1} decreases when n increases, as we can easily verify from Eqs. 5.320 and 5.327. Comparing Eqs. 5.325 and 5.329 and Eqs. 5.328 and 5.329, it follows that

$$v_1(t) = \lim_{n \to \infty} v_1^{(N)}(t), b_1 = \lim_{n \to \infty} b_1^{(N)}. \quad (5.331)$$

In the simplest case, we consider $N = 1$ ($n = 0, 1$) in Eqs. 5.329 and 5.330, and we obtain

$$v_1^{(1)}(t) = b_3(-\cos(\omega t) + \cos(3\omega t)). \quad (5.332)$$

From Eq. 5.327, the following expression for the coefficient b_3 is obtained:

$$b_3 = \frac{a_3}{8\omega^2}. \tag{5.333}$$

And from Eqs. 5.323 and 5.312, writing $p = 1$, we can find that the first-order approximate frequency is

$$\omega_1(A) = \sqrt{\mu + \alpha_1} = \sqrt{\mu + \frac{a_1}{A}}. \tag{5.334}$$

Substituting Eqs. 5.332 and 5.318 into Eq. 5.316 gives the following equation for $v_2(t)$:

$$\begin{aligned}
&\ddot{v}_2 + \omega^2 v_2 - \alpha_1 b_3(-\cos(\omega t) + \cos(3\omega t)) - \alpha_2 A\cos(\omega t) + [\xi(b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) \\
&\quad + \gamma(A\cos(\omega t) b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) - A\omega^2 b_3\cos(\omega t)(-\cos(\omega t) + \cos(3\omega t)))) \\
&\quad + \xi\gamma(4Ab_3\omega^2\sin(\omega t)(\sin(\omega t) - 3\sin(3\omega t))) + \xi\gamma^2(-2A^2 b_3\omega^2\cos(\omega t)\sin(\omega t)(\sin(\omega t) \\
&\quad - 3\sin(3\omega t)) + b_3 A^2\omega^2\sin(\omega t)^2(-\cos(\omega t) + \cos(3\omega t))) \\
&\quad + \eta b_3(-\cos(\omega t) + \cos(3\omega t))]\, e^{\gamma(A\cos(\omega t) + b_3(-\cos(\omega t) + \cos(3\omega t)))} = 0.
\end{aligned} \tag{5.335}$$

It is possible to generate an expansion of the Fourier series

$$\begin{aligned}
&-\alpha_1 b_3\cos(3\omega t) + [\xi(b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) + \gamma(A\cos(\omega t) b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) \\
&- A\omega^2 b_3\cos(\omega t)(-\cos(\omega t) + \cos(3\omega t)))) + \xi\gamma(4Ab_3\omega^2\sin(\omega t)(\sin(\omega t) - 3\sin(3\omega t))) \\
&+ \xi\gamma^2(-2A^2 b_3\omega^2\cos(\omega t)\sin(\omega t)(\sin(\omega t) - 3\sin(3\omega t)) + b_3 A^2\omega^2\sin(\omega t)^2(-\cos(\omega t) + \cos(3\omega t))) \\
&+ \eta b_3(-\cos(\omega t) + \cos(3\omega t))]e^{\gamma(A\cos(\omega t) + b_3(-\cos(\omega t) + \cos(3\omega t)))} \\
&= \sum_{n=0}^{\infty} c_{2n+1}\cos((2n+1)\omega t) = c_1\cos(\omega t) + c_3\cos(3\omega t) + \cdots
\end{aligned} \tag{5.336}$$

where

$$c_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \begin{aligned}
&\{-\alpha_1 b_3\cos(3\omega t) + [\xi(b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) + \gamma(A\cos(\omega t) b_3\omega^2(\cos(\omega t) - 9\cos(3\omega t)) \\
&- A\omega^2 b_3\cos(\omega t)(-\cos(\omega t) + \cos(3\omega t)))) + \xi\gamma(4Ab_3\omega^2\sin(\omega t)(\sin(\omega t) - 3\sin(3\omega t))) \\
&+ \xi\gamma^2(-2A^2 b_3\omega^2\cos(\omega t)\sin(\omega t)(\sin(\omega t) - 3\sin(3\omega t)) + b_3 A^2\omega^2\sin(\omega t)^2(-\cos(\omega t) + \cos(3\omega t))) \\
&+ \eta b_3(-\cos(\omega t) + \cos(3\omega t))]e^{\gamma(A\cos(\omega t) + b_3(-\cos(\omega t) + \cos(3\omega t)))}]\,\}\cos((2n+1)\varphi)\,\mathrm{d}\varphi,
\end{aligned} \tag{5.337}$$

Where $\omega = \sqrt{\mu + \alpha_1}$.

By substituting equations, we have

$$\ddot{v}_2 + \omega^2 v_2 + \sum_{n=0}^{\infty} c_{2n+1} \cos((2n+1)\omega t) - \alpha_2 A \cos(\omega t) + \alpha_1 b_3 \cos(\omega t) = 0,$$

or

$$\ddot{v}_2 + \omega^2 v_2 + \sum_{n=1}^{\infty} c_{2n+1} \cos((2n+1)\omega t) + (\alpha_1 b_3 - \alpha_2 A + c_1) \cos(\omega t) = 0. \tag{5.338}$$

The absence of secular terms in $v_2(t)$ requires eliminating contributions proportional to $\cos(\omega t)$ in Eq. 5.338, and we obtain

$$\alpha_2 = \frac{c_1 + \alpha_1 b_3}{A}. \tag{5.339}$$

Taking into account Eqs. 5.339 and 5.338, we rewrite Eq. 5.338 in the form

$$\ddot{v}_2 + \omega^2 v_2 = -\sum_{n=1}^{\infty} c_{2n+1} \cos((2n+1)\omega t), \tag{5.340}$$

with initial conditions $v_2(0) = 0$ and $\dot{v}_2(0) = 0$. The periodic solution to Eq. 5.340 can be written as:

$$v_2(t) = \sum_{n=0}^{\infty} d_{2n+1} \cos((2n+1)\omega t) = d_1 \cos(\omega t) + d_3 \cos(3\omega t) + \mathrm{K}. \tag{5.341}$$

Substituting Eq. 5.341 into Eq. 5.340, we obtain:

$$\omega^2 \sum_{n=0}^{\infty} d_{2n+1} (1 - (2n+1)^2) \cos((2n+1)\omega t) = -\sum_{n=1}^{\infty} c_{2n+1} \cos((2n+1)\omega t). \tag{5.342}$$

We can write the following expression for the coefficients b_{2n+1}:

$$d_{2n+1} = \frac{c_{2n+1}}{((2n+1)^2 - 1)\omega^2} = \frac{c_{2n+1}}{4n(n+1)\omega^2}, \text{ for } n \geq 1. \tag{5.343}$$

Taking into account that $v_2(0) = 0$, Eq. 5.343 gives

$$d_1 = -\sum_{n=1}^{\infty} d_{2n+1}. \tag{5.344}$$

With the same procedure as was used for approximate x_1, we obtain the following expression for x_2:

$$v_2^{(2)}(t) = -(d_3 + d_5) \cos(\omega t) + d_3 \cos(3\omega t) + d_5 \cos(5\omega t). \tag{5.345}$$

And from Eqs. 5.323and 5.339, writing $p = 2$, we can find that the second-order approximate frequency is

$$\omega_2(A) = \sqrt{\mu + \alpha_1 + \alpha_2} = \sqrt{\mu + \frac{a_1}{A} + \frac{c_1}{A} + \frac{a_1 b_3}{A^2}}. \tag{5.346}$$

5.10.3 Solution Using Second Case of the Homotopy Perturbation Method

In this method, firstly, Eq. 5.309 is written as

$$\begin{aligned} &\ddot{v} + v = v - \frac{\mu v + [(1+\gamma v)\xi\ddot{v} + (2+\gamma v)\xi\gamma\dot{v}^2 + \eta v]e^{\gamma v}}{\omega^2}, \\ &v_0(0) = A,\ \dot{v}_0(0) = 0,\ v_n(0) = 0,\ \dot{v}_n(0) = 0, \end{aligned} \tag{5.347}$$

where $\varphi = \omega t$.

Expanding $v(\varphi)$ and ω^2 corresponding to the HPM, and applying the artificial parameter method, one can easily obtain, at $O(p^0)$,

$$\ddot{v}_0 + v_0 = 0, v_0(0) = A,\ \dot{v}(0) = 0, \tag{5.348}$$

the solution of which is

$$v_0(\varphi) = A\cos(\varphi). \tag{5.349}$$

At $O(p^1)$, one obtains

$$\begin{aligned} \ddot{v}_1 + v_1 =& A\cos(\varphi) - (\mu A\cos(\varphi) + [-(1+\gamma A\cos(\varphi))\xi A\omega^2\cos(\varphi) \\ &+ (2+\gamma A\cos(\varphi))\xi\gamma A^2\omega^2\sin(\varphi)^2 + \eta A\cos(\varphi)]e^{\gamma A\cos(\varphi)})/\omega_0^2, \\ v_1(0) =& 0, \dot{v}_1(0) = 0. \end{aligned} \tag{5.350}$$

By the same manipulation as illustrated previously, we obtain

$$\begin{aligned} &\ddot{v}_1 + v_1 = A\cos(\varphi) + \frac{1}{\omega_0^2}(a_1\cos(\varphi) + a_3\cos(3\varphi)), \\ &v_1(0) = 0, \dot{v}_1(0) = 0, \end{aligned} \tag{5.351}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \begin{array}{l}[-(\mu A\cos(\varphi)+[-(1+\gamma A\cos(\varphi))\xi A\omega^2\cos(\varphi)+\\ (2+\gamma A\cos(\varphi))\xi\gamma A^2\omega^2\sin(\varphi)^2+\eta A\cos(\varphi)]e^{\gamma A\cos(\varphi)}](\cos(2n+1)\varphi)\mathrm{d}\varphi.\end{array} \tag{5.352}$$

The absence of secular terms in $v_1(\varphi)$ requires that

$$\omega_0^2 = -\frac{a_1}{A}, \tag{5.353}$$

and Eq. 5.351 will be

$$\ddot{v}_1 + v_1 = \frac{a_3}{\omega_0^2}\cos(3\varphi),\ v_1(0)=0,\ \dot{v}_1(0)=0. \tag{5.354}$$

The solution of Eq. 5.354 is

$$v_1(\varphi) = \frac{a_3}{8\omega_0^2}(\cos(\varphi)-\cos(3\varphi)). \tag{5.355}$$

At $O(p^2)$, one obtains

$$\begin{aligned}\ddot{v}_2 + v_2 = &\left(1-\frac{\mu+(2\gamma\xi\ddot{v}_0+\gamma^2\xi v_0\ddot{v}_0+3\gamma^2\xi\dot{v}_0^2+\gamma^3\xi v_0\dot{v}_0^2+\eta+\eta v_0\gamma)e^{\gamma v_0}}{\omega_0^2}\right)v_1\\ &+\frac{2\omega_1}{\omega_0^3}(\mu v_0+(\xi\ddot{v}_0+\gamma\xi v_0\ddot{v}_0+2\gamma\xi\dot{v}_0^2+\gamma^2\xi v_0\dot{v}_0^2+\eta v_0)e^{\gamma v_0})-\frac{2(2+\gamma v_0)\xi\gamma\xi\dot{v}_0e^{\gamma v_0}}{\omega_0^2}\dot{v}_1\\ v_2(0)=0,&\,\dot{v}_2(0)=0.\end{aligned} \tag{5.356}$$

By the same manipulation, we obtain

$$\begin{aligned}&\ddot{v}_2 + v_2 = b_1\cos(\varphi)+b_3\cos(3\varphi)+b_5\cos(5\varphi),\\ &v_2(0)=0,\ \dot{v}_2(0)=0\end{aligned} \tag{3.357}$$

where

$$\begin{aligned}b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2}&\left[\left(1-\frac{\mu+(2\gamma\xi\ddot{v}_0+\gamma^2\xi v_0\ddot{v}_0+3\gamma^2\xi\dot{v}_0^2+\gamma^3\xi v_0\dot{v}_0^2+\eta+\eta v_0\gamma)e^{\gamma v_0}}{\omega_0^2}\right)v_1\right.\\ &+\frac{2\omega_1}{\omega_0^3}(\mu v_0+(\xi\ddot{v}_0+\gamma\xi v_0\ddot{v}_0+2\gamma\xi\dot{v}_0^2+\gamma^2\xi v_0\dot{v}_0^2+\eta v_0)e^{\gamma v_0})\\ &\left.-\frac{2(2+\gamma v_0)\xi\gamma\xi\dot{v}_0e^{\gamma v_0}}{\omega_0^2}\dot{v}_1\right](\cos(2n+1)\varphi)\mathrm{d}\varphi.\end{aligned} \tag{5.358}$$

The absence of secular terms in $v_2(\varphi)$ requires that $b_1=0$, and we have

$$\omega_1 = \frac{\int_0^{\pi/2} [-(\omega_0^2 - \mu + (2\gamma\xi\ddot{v}_0 + \gamma^2\xi v_0\ddot{v}_0 + 3\gamma^2\xi\dot{v}_0^2 + \gamma^3\xi v_0\dot{v}_0^2 + \eta + \eta v_0\gamma)e^{\gamma v_0})\omega_0 v_1 - \omega_0 2(2+\gamma v_0)\xi\gamma\xi\dot{v}_0 e^{\gamma v_0}\dot{v}_1]\cos(\varphi)\mathrm{d}\varphi}{\int_0^{\pi/2} [2(\mu v_0 + (\xi\ddot{v}_0 + \gamma\xi v_0\ddot{v}_0 + 2\gamma\xi\dot{v}_0^2 + \gamma^2\xi v_0\dot{v}_0^2 + \eta v_0)e^{\gamma v_0})]\cos(\varphi)\mathrm{d}\varphi.} \quad (5.359)$$

Equation 5.357 will be

$$\begin{aligned} &\ddot{v}_2 + v_2 = b_3\cos(3\varphi) + b_5\cos(5\varphi), \\ &v_2(0) = 0, \dot{v}_2(0) = 0. \end{aligned} \quad (5.360)$$

The solution of Eq. 5.360 is

$$v_2(\varphi) = \left(\frac{b_3}{8} + \frac{b_5}{24}\right)\cos(\varphi) - \frac{\cos(\varphi)}{24}[16b_5\cos(\varphi)^4 + 12b_3\cos(\varphi)^2 - 20b_5\cos(\varphi)^2 - 9b_3 + 5b_5]. \quad (5.361)$$

Finally, the solution of Eq. 5.309 and its frequency for $p = 2$ are

$$v(t) = v_0(t) + v_1(t) + v_2(t), \quad (5.362)$$

$$\omega^2 = \omega_0^2 + \omega_1^2. \quad (5.363)$$

5.11 Problem 5.11. Dynamic Behavior of a Flexible Beam Attached to a Rotating Rigid Hub

5.11.1 Introduction

In this problem, the dynamic behavior of a flexible beam attached to a rotating rigid hub with torque is investigated. The beam model is widely used in engineering structures such as turbines, compressors, helicopter blades, satellite antennas, and robotic arms. These beam systems have complicated dynamic responses.

The system under consideration is shown in Fig. 5.42. The hub is assumed to be a rigid disc with radius R, and mass M and rotating at an angular velocity $\dot{\theta}$ about the inertial z-axis. Therefore, torque T effect on the hub causes it only to rotate. The X, Y, Z is a system of fixed rectangular Cartesian coordinate axes with origin at the center of the hub. The x, y, z and x', y', z' are two sets of rectangular Cartesian coordinate axes rotating with the hub with common origin at the root of the beam. The beam mid-plane $y'z'$ is inclined to the plane of rotation yz at an angle Ψ called the *setting* angle. The beam is assumed to be initially straight along the x'-axis clamped at its base to the hub surface, having a uniform cross-sectional area A_b, flexural rigidity EI, constant length l, mass m, and mass density ρ. The beam

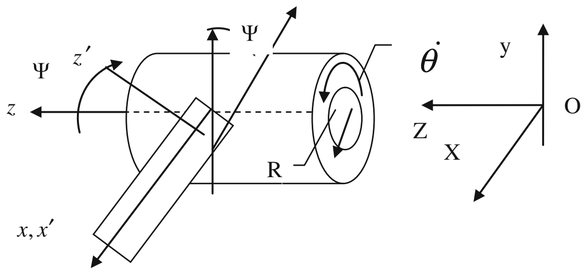


Fig. 5.42 Arm hub schematic diagram

thickness is assumed to be small, as compared with its length, so that the effects of shear deformation and rotary inertia can be ignored. The mathematical model used to describe the dynamics of the mentioned beam system is a special case of original work:

$$w'' + w + w^2 w'' + w w'^2 + w^3 = 0, \tag{5.364}$$

with initial conditions

$$w(0) = A \qquad w'(0) = 0. \tag{5.365}$$

5.11.2 Application of the Homotopy Perturbation Method

To solve Eq. 5.364 by means of the HPM, we consider the following process after separating the linear and nonlinear parts of the equation. *A* homotopy can be constructed as

$$H(w,p) = (1-p)[w'' + w] + p[w'' + w + w^2 w'' + w w'^2 + w^3], \tag{5.366}$$

where prime denotes differentiation with respect to t.

We consider w to be

$$w = w_0 + p w_1 + p^2 w_2 + p^3 w_3 + p^4 w_4 + \cdots. \tag{5.367}$$

Substituting Eq. 5.367 into Eq. 5.366 and rearranging the resultant equation on the basis of powers of p-terms, one has

$$p^0 : w_0'' + w_0 = 0, w_0(0) = A = 1\,, w_0'(0) = 0 \tag{5.368}$$

$$p^1 : w_0^2 w_0'' + w_1 + w_0 w_0'^2 + w_1'' + w_0^3 = 0, w_1(0) = 0\,, w_1'(0) = 0 \tag{5.369}$$

$$\begin{aligned} &p^2 : 2w_0 w_0' w_1' + w_2 + 3w_0^2 w_1 + 2w_0 w_1 w_0'' + w_2'' + w_1 w_0'^2 + w_0^2 w_1'' = 0, \\ &w_2(0) = 0, w_2'(0) = 0 \end{aligned} \tag{5.370}$$

$$p^3: 2w_0w_0''w_2 + 2w_0w_0'w_2' + w_2w_0'^2 + 3w_0w_1^2 + w_0w_1'^2 + 2w_1w_1'w_0' + w_1^2w_0'' + w_0^2w_2'' + 2w_0w_1w_1'' + w_3'' + w_3 + 3w_0^2w_2 = 0, w_3(0) = 0\,, w_3'(0) = 0 \tag{5.371}$$

$$p^4: w_1^2w_1'' + 2w_0w_1'w_2' + w_4 + w_1w_1'^2 + 2w_0w_2w_1'' + 2w_0w_1w_2'' + 2w_2w_0'w_1' + w_1^3 + 2w_1w_0'w_2' + w_4'' + 3w_0^2w_3 + 6w_0w_1w_2 + 2w_1w_2w_0'' + 2w_0w_0'w_3' + w_3w_0'^2 + w_0^2w_3'' + 2w_0w_3w_0'' = 0, w_4(0) = 0\,, w_4'(0) = 0 \tag{5.372}$$

where $w(t)$ may be written as follows by solving Eqs. 5.368–5.372 with the initial condition $w(0) = A = 1$:

$$w_0(t) = \cos(t), \tag{5.373}$$

$$w_1(t) = -\frac{1}{8}\sin(t)(t - \cos(t)\sin(t)), \tag{5.374}$$

$$w_2(t) = \frac{1}{256}\cos(t) + \frac{3}{64}\cos^2(t)\sin(t)t + \frac{9}{256}\sin(t)t - \frac{1}{128}\cos(t)t^2 + \frac{5}{64}\cos^5(t) - \frac{21}{256}\cos^3(t), \tag{5.375}$$

$$w_3(t) = \frac{411}{32768}\cos(t) - \frac{103}{1536}\cos^7(t) + \frac{461}{6144}\cos^5(t) - \frac{25}{512}\cos^4(t)\sin(t)t + \frac{1}{98304}\left(320 + 364t^2\right)\cos^3(t) + \frac{3}{256}\cos^2(t)\sin(t)t + \frac{1}{98304}\left(-48t^2 - 2337\right)\cos(t) + \frac{1}{98304}\left(32t^3 - 1872t\right)\sin(t) \tag{5.376}$$

And:

$$w_4(t) = -\frac{1407}{262144}\cos(t) + \frac{273}{4096}\cos^9(t) - \frac{1389}{16384}\cos^7(t) + \frac{721}{12288}\cos^6(t)\sin(t)t + \frac{1}{786432}\left(9900 - 12000t^2\right)\cos^5(t) - \frac{665}{24576}\cos^4(t)\sin(t)t + \frac{1}{786432}\left(8520t^2 - 3687\right)\cos^3(t) + \frac{1}{786432}\left(-864t^3 - 3444t\right)\sin(t)\cos^2(t) + \frac{1}{786432}\left(12264 - 2142t + 8t^4\right) + \frac{1}{786432}\left(9963t - 88t^3\right)\sin(t) \tag{5.377}$$

Then,

$$w(t) = w_0(t) + w_1(t) + w_2(t) + w_3(t) + w_4(t) + \cdots. \tag{5.378}$$

5.11.3 Application of the Energy Balance Method

In order to assess the advantages and the accuracy of the energy balance method (EBM), we will apply this method to the discussed system. Its variational formulation can be easily established:

$$J(w) = \int_0^t \left(-\frac{1}{2}w'^2(1+w^2) + \frac{1}{2}w^2 + \frac{1}{4}w^4\right)dt, \tag{5.379}$$

in which w and t are generalized dimensionless displacement and time variables, respectively.

Its Hamiltonian, therefore, can be written as

$$H = \frac{1}{2}w'^2\left(1+w^2\right) + \frac{1}{2}w^2 + \frac{1}{4}w^4 = \frac{1}{2}A^2 + \frac{1}{4}A^4 \tag{5.380}$$

or

$$R(t) = \frac{1}{2}w'^2\left(1+w^2\right) + \frac{1}{2}w^2 + \frac{1}{4}w^4 - \frac{1}{2}A^2 - \frac{1}{4}A^4 = 0 \tag{5.381}$$

Oscillatory systems contain two important physical parameters—that is, the frequency ω and the amplitude of oscillation A. So let us consider such initial conditions:

$$w(0) = A \quad w'(0) = 0 \tag{5.382}$$

Assume that its initial approximate guess can be expressed as

$$w = A\cos(\omega t). \tag{5.383}$$

Substituting Eq. 5.384 into Eq. 5.382 yields

$$\frac{1}{2}(-A\omega\sin\omega t)^2\left(1+(A\cos\omega t)^2\right) + \frac{1}{2}(A\cos\omega t)^2 + \frac{1}{4}(A\cos\omega t)^4 - \frac{1}{2}A^2 - \frac{1}{4}A^4 = 0 \tag{5.384}$$

If we collocate at $\omega t = \frac{\pi}{4}$, we obtain

$$\frac{A^2\omega^2}{4}\left(1+\frac{A^2}{2}\right) + \frac{A^2}{4} + \frac{A^4}{16} - \frac{A^2}{2} - \frac{A^4}{4} = 0 \tag{5.386}$$

or

$$\omega = \sqrt{1+\frac{A^2}{4+2A^2}} \tag{5.387}$$

Substituting Eq. 5.386 into Eq. 5.384 yields

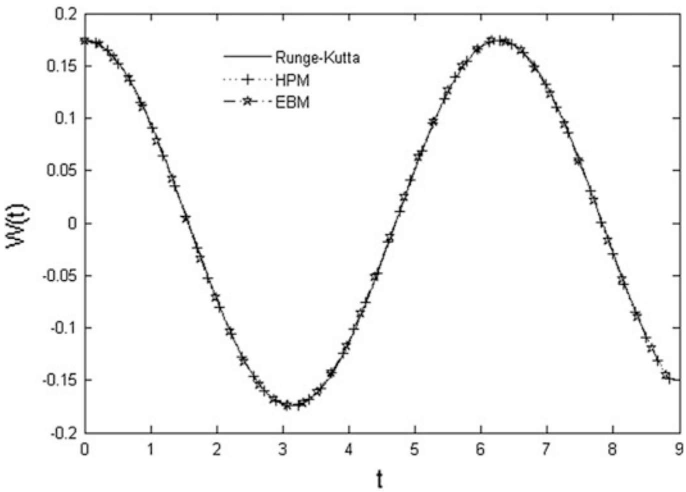


Fig. 5.43 Comparison of homotopy perturbation method (HPM) and energy balance method (EBM) with Runge–Kutta fourth order ($A = \pi/18$)

$$w = A\cos\left(\sqrt{1 + \frac{A^2}{4 + 2A^2}}\, t\right). \tag{5.388}$$

5.11.4 Results

Figure 5.43 shows the comparison solution of motion from the HPM and EBM with the Runge–Kutta 4th method. It can be observed that there is a significant agreement between the results obtained from the HPM and EBM and those of Runge–Kutta.

5.12 Problem 5.12. The Motion of a Ring Sliding Freely on a Rotating Wire

5.12.1 Introduction

A problem of a single-degree-of-freedom conservative system has been considered and is described in an equation as follows (see Problem 5.2). The motion of a ring of mass m sliding freely on the wire described by the parabola $z = rx^2$, which rotates with a constant angular velocity Ω about the z-axis, is shown in Fig. 5.44.

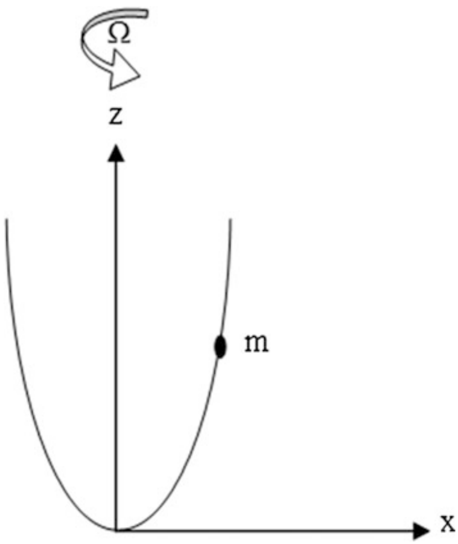


Fig. 5.44 Geometry of the problem

It is convenient to write the equation of motion of a ring by using a Lagrange formulation. For a conservative holonomic system, it can be expressed by the kinetic and potential energies T and V in terms of what is usually called *generalized coordinate* q, where q is a vector the elements of which are the independent coordinates needed to describe the system under consideration. For the present problem, the kinetic and potential energies are

$$T = \frac{1}{2} m(x'^2(t) + \Omega^2 x^2(t) + z'(t)^2), \tag{5.389}$$

$$V = mgz(t). \tag{5.390}$$

Using the concentrate $z = rx^2$, the above equations can be rewritten as

$$T = \frac{1}{2} m((1 + 4r^2 x^2(t))x'^2(t) + \Omega^2 x^2(t)), \tag{5.391}$$

$$V = mgrx^2(t). \tag{5.392}$$

Substituting T and V into the Lagrange equation yields

$$L = \frac{1}{2} m\,[(1 + 4r^2 x^2(t))x'^2(t) + \Omega^2 x^2(t)] - mgrx^2(t). \tag{5.393}$$

Finally, the equation of motion is

$$(1 + 4r^2 x^2(t))x'^2(t) + Ax(t) + 4r^2 rx'^2(t)x(t) = 0, \tag{5.394}$$

where A is

$$A = 2gr - \Omega^2. \tag{5.395}$$

5.12.2 Application of HPEM

According to the PEM, Eq. 5.394 can be rewritten as

$$1\,\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + Ax(t) - 4r^2[(x^2(t)x''(t) + x'^2(t)x(t))] = 0, \tag{5.396}$$

and the initial conditions are

$$x(0) = \lambda, \quad x'(0) = 0. \tag{5.397}$$

The form of the solution and the constant one in Eq. 5.394 can be expanded as

$$x(t) = x_0(t) + p\,x_1(t) + p^2x_2(t) + \mathrm{K}, \tag{5.398}$$

$$1 = 1 + pa_1 + pa_2 + \mathrm{K}, \tag{5.399}$$

$$A = \omega^2 + pb_1 + pb_2 + \mathrm{K}, \tag{5.400}$$

$$4r^2 = pc_1 + pc_2 + \mathrm{K}. \tag{5.401}$$

Substituting Eqs. 5.398–5.401 into Eq. 5.396, and processing as the standard PM, we have

$$x_0''(t) + \omega^2x_0(t) = 0, x_0(0) = \lambda, x_0'(0) = 0, \tag{5.402}$$

$$\begin{aligned}&\frac{\mathrm{d}^2x_1(t)}{\mathrm{d}t^2} + c_1x_0^2(t)\left(\frac{\mathrm{d}x_0^2(t)}{\mathrm{d}t^2}\right) + c_1x_0(t)\left(\frac{\mathrm{d}x_0(t)}{\mathrm{d}t}\right)^2 + \omega^2x_1(t) + b_1x_0(t) = 0,\\&x_1(0) = 0,\ x_1'(0) = 0.\end{aligned} \tag{5.403}$$

The solution of Eq. 5.397 is

$$x_0(t) = \lambda\cos(\omega t). \tag{5.404}$$

Substituting $x_0(t)$ from the above equation into Eq. 5.403 results in

$$\frac{d^2x_1(t)}{dt^2} - c_1\lambda^3\omega^2\cos^3(\omega t) + c_1\lambda^3\omega^2\cos(\omega t)\sin 2(\omega t) + \omega^2 x_1(t) + b_1\lambda\cos(\omega t) = 0. \quad (5.405)$$

But from Eqs. 5.400 and 5.401 and considering just two first terms, we have

$$b_1 = \frac{A - \omega^2}{p} \quad (5.406)$$

and

$$c_1 = \frac{4r^2}{p}. \quad (5.407)$$

After $p = 1$, eliminating the secular term requires that

$$b_1\lambda - \frac{5}{2}c_1\lambda^3\omega^2 = 0. \quad (5.408)$$

Two roots of this particular equation can be obtained as

$$\omega = \pm\sqrt{\frac{A\lambda}{\lambda - 10r^2\lambda^3}}. \quad (5.409)$$

Replacing ω from Eq. 5.409 into Eq. 5.404 yields (Fig. 5.45)

$$x(t) = x_0(t) = A\cos\left(\sqrt{\frac{A\lambda}{\lambda - 10r^2\lambda^3}}t\right). \quad (5.410)$$

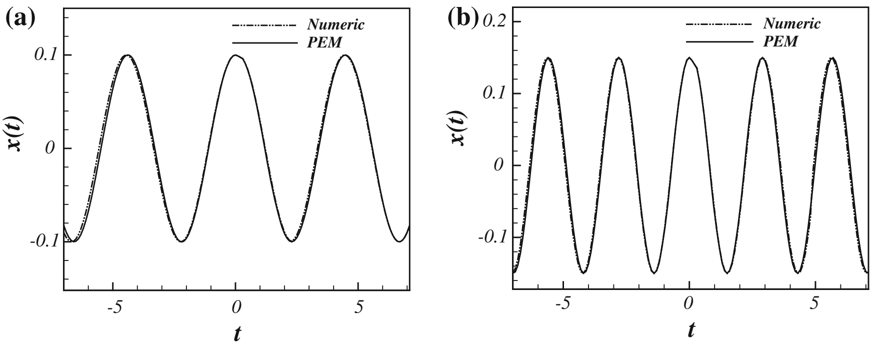


Fig. 5.45 The effects of constant parameters on position and velocity: **a** $A = 2$, $\lambda = 0.1$, $r = 0.5$; **b** $A = 5$, $\lambda = 0.15$, $r = 0.25$. *PEM* parameter expansion method

5.13 Problem 5.13. Application of a Rotating Rigid Frame Under Force

5.13.1 Introduction of Case 1

The rigid frame is forced to rotate at the fixed rate Ω. While the frame rotates, the simple pendulum oscillates (see Fig. 5.46).

By using the Lagrange method, the governing equation can be easily obtained as

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + (1 - A\cos(x(t)))\sin(x(t)) = 0. \tag{5.411}$$

Here, by using the Taylor's series expansion for $\cos(x(t))$ and $\sin(x(t))$, the above equation reduces to

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + (1 - A)x(t) - \left(\frac{1}{6}x^3(t) + \frac{2}{3}Ax^3(t) - \frac{1}{2}Ax^5(t)\right) = 0, \tag{5.412}$$

with the boundary conditions

$$x(0) = \lambda, \quad x'(0) = 0, \tag{5.413}$$

where A is $\frac{\Omega^2 r}{g}$.

5.13.2 Application of HPEM

According to the HPEM, Eq. 5.412 can be rewritten as

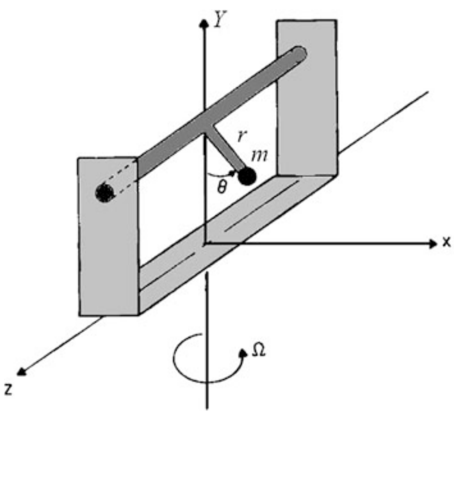


Fig. 5.46 Geometry of a rotating rigid frame under force

$$1\frac{d^2x(t)}{dt^2}+(1-A)x(t)-1\left(\frac{1}{6}x^3(t)+\frac{2}{3}Ax^3(t)-\frac{1}{2}Ax^5(t)\right)=0, \tag{5.414}$$

and the initial conditions are

$$x_0(0)=\lambda, \quad x_0'(0)=0. \tag{5.415}$$

The form of the solution and the constant one in Eq. 5.414 can be expanded as

$$x(t)=x_0(t)+p\,x_1(t)+p^2x_2(t)+\mathrm{K}, \tag{5.416}$$

$$1=1+pa_1+pa_2+\mathrm{K}, \tag{5.417}$$

$$1-A=\omega^2+pb_1+pb_2+\mathrm{K}, \tag{5.418}$$

$$1=pc_1+pc_2+\mathrm{K}. \tag{5.419}$$

Substituting Eqs. 5.416–5.419 into Eq. 5.420 and processing as the standard PM, it can be written as

$$x_0''(t)+\omega^2x_0(t)=0,\ x_0(0)=\lambda,\ x_0'(0)=0, \tag{5.420}$$

$$\frac{d^2x_1(t)}{dt^2}+b_1x_0^2(t)\left(\frac{dx_0^2(t)}{dt^2}\right)-\frac{1}{2}Ax_0^5(t)+\frac{2}{3}Ax_0^3(t)+\omega^2x_1(t)+\frac{1}{6}x_0^3(t)=0,$$
$$x_1(0)=0,\ x_1'(0)=0. \tag{5.421}$$

The solution of Eq. 5.420 is

$$x_0(t)=\lambda\cos(\omega t). \tag{5.422}$$

Substituting $x_0(t)$ from the above equation into Eq. 5.421 results in

$$\frac{d^2x_1(t)}{dt^2}+\frac{1}{6}\lambda^3\cos^3(\omega t)b_1\lambda\cos(\omega t)+\frac{2}{3}A\lambda^3\cos^3(\omega t)+\omega^2x_1(t)-\frac{1}{2}A\lambda^5\cos^5(\omega t)$$
$$=0. \tag{5.423}$$

But from Eqs. 5.418 and 5.419,

$$b_1=\frac{1-A-\omega^2}{p} \tag{5.424}$$

and

$$c_1=\frac{1}{p}. \tag{5.425}$$

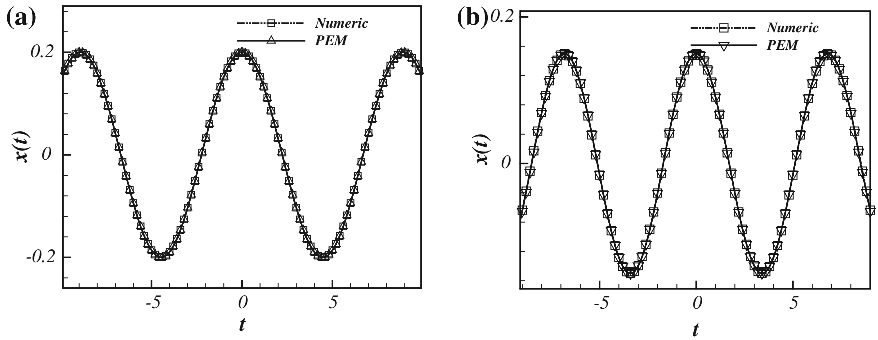


Fig. 5.47 The effects of constant parameters on position and velocity: **a** $A = 0.5$, $\lambda = 0.2$; **b** $A = 0.15$, $\lambda = 0.15$. *PEM* parameter expansion method

On the basis of trigonometric functions properties, we have

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t), \tag{5.426}$$

$$\cos^5(\omega t) = 1/6[\cos(5\omega t) - 5\cos(3\omega t) + 20\cos(\omega t)]. \tag{5.427}$$

After $p = 1$ and eliminating the secular term, ω has been obtained as

$$\omega = \pm\sqrt{\frac{1}{8}\lambda^2 + \frac{5}{18}A\lambda^4 - \frac{1}{2}A\lambda^2 + 1 - A}. \tag{5.428}$$

Replacing ω from Eq. 5.428 into Eq. 5.429 yields (Fig. 5.47)

$$x(t) = x_0(t) = A\cos\left(\sqrt{\frac{1}{8}\lambda^2 + \frac{5}{18}A\lambda^4 - \frac{1}{2}A\lambda^2 + 1 - At}\right). \tag{5.429}$$

5.13.3 Introduction of Case 2

The governing equation (5.411) can be derived with respect to θ as

$$\frac{d^2\theta}{dt^2} + (1 - \Lambda\cos\theta)\sin\theta = 0, \theta(0) = A, \frac{d\theta}{dt}(0) = 0. \tag{5.430}$$

where $\Lambda = \frac{\Omega^2 r}{g}$, we can use centrifugal acceleration from the rotating reference frames. It is a readily observable physical fact. The magnitude and direction of the centrifugal acceleration varies from place to place in the rotating frame. Therefore, we call it the *centrifugal field*. It is zero at the pivot. When we go farther from the pivot, it gets stronger. It is everywhere directed radially outward from the pivot. Centrifugal force is related to the Coriolis force in rotating frameworks.

Pendulums require great mechanical stability: a length change of only 0.02 %, 1/5 mm in a clock pendulum, will cause an error of a minute per week.

5.13.4 Solution of Case 2 Using Frequency Formulation

Having introduced the above mentioned approach, it is applied here to the nonlinear differential equation. The equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t) + (1 - \Lambda \cos u(t)) \sin u(t). \tag{5.431}$$

The trial functions are chosen as

$$u_1(t) = A \cos t, \tag{5.432}$$

$$u_2(t) = A \cos 2t. \tag{5.433}$$

Substituting u_1, u_2 into Eq. 5.431 separately results in R_1, R_2, where

$$\begin{aligned} R_1 &= \frac{\partial^2}{\partial t^2}(A \cos t) + (1 - \Lambda \cos(A \cos t)) \sin(A \cos t) \\ &= -A \cos t + (1 - \Lambda \cos(A \cos t) \sin(A \cos t) \end{aligned} \tag{5.434}$$

and

$$\begin{aligned} R_2 &= \frac{\partial^2}{\partial t^2}(A \cos 2t) + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) \\ &= -4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t). \end{aligned} \tag{5.435}$$

So

$$\omega^2 = \frac{-4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) + 4A \cos t - 4(1 - \Lambda \cos(A \cos t) \sin(A \cos t)}{-4A \cos 2t + (1 - \Lambda \cos(A \cos 2t)) \sin(A \cos 2t) + A \cos t - (1 - \Lambda \cos(A \cos t) \sin(A \cos t)}. \tag{5.436}$$

Now we form this equation and put $\cos t = \cos 2t = k$, so

$$R_1 = -Ak + (1 - \Lambda \cos(Ak)) \sin(Ak), \tag{5.437}$$

$$R_2 = -4Ak + (1 - \Lambda \cos(Ak)) \sin(Ak). \tag{5.438}$$

The period may be obtained as

$$\omega = \sqrt{-\frac{(-1 + \Lambda \cos(Ak)) \sin(Ak)}{Ak}}. \tag{5.439}$$

Write the following integral as Eq. 5.428:

$$\int_0^{\frac{T}{4}} \left[\left(\left(-\frac{(-1+\Lambda\cos(Ak))\sin(Ak)\times\cos\left(\sqrt{-\frac{(-1+\Lambda\cos(Ak))\sin(Ak)}{Ak}}\times\right)t}{k} \right) - \left(1-\Lambda\cos\left(A\cos\left(\sqrt{-\frac{(-1+\Lambda\cos(Ak))\sin(Ak)}{Ak}}\times t\right)\right)\right) \times \sin\left(A\cos\left(\sqrt{-\frac{(-1+\Lambda\cos(Ak))\sin(Ak)}{Ak}}\times t\right)\right)\right)\times\cos\omega t \right] \mathrm{d}t. \tag{5.440}$$

If we equate Eq. 5.430 to zero, we can find K as $K =$ -0.8657421586. We may then substitute K in Eq. 5.439) to find

$$\omega = 0.7599599730\sqrt{\frac{2.\sin(0.8657421586.A) - \Lambda.\sin(1.731484317.A)}{A}}.$$

Using EBM, we can find the parameter above as

$$\omega_{\mathrm{EBM}} = \frac{2}{A}\sqrt{\cos\left(\frac{\sqrt{2}}{2}A\right) - \cos A + \frac{\Lambda}{2}\left(\cos^2 A - \cos^2\left(\frac{\sqrt{2}}{2}A\right)\right)}, \tag{5.441}$$

where we assumed that

$$g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}, r = 1\,\mathrm{m}, \Omega = 3\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

The comparison of results obtained through the amplitude frequency formulation (AFF) and EBM is illustrated in Fig. 5.48.

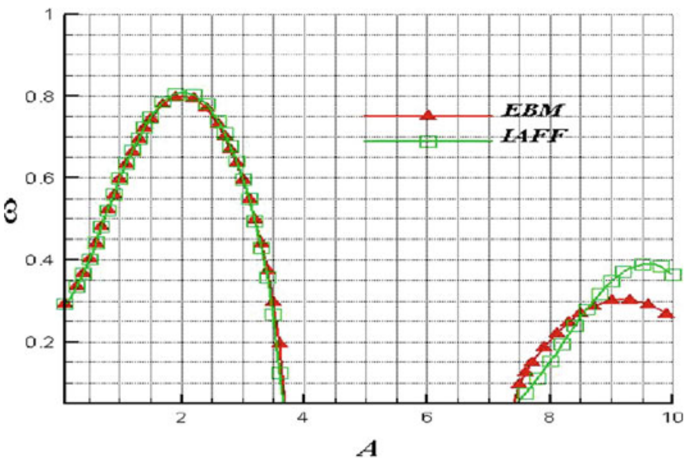


Fig. 5.48 Comparison of periods between two methods (case 2)

As was discovered earlier, we cannot consider every quantity for A, because for some amounts of A, the value of ω^2 becomes negative. By continuing this procedure, the amplitudes of ω decrease and tend to zero.

5.14 Problem 5.14. Application of a Nonlinear Oscillator in Automobile Design

5.14.1 Introduction

Consider the nonlinear oscillator shown in Fig. 5.49.

This oscillator is very applicable in automobile design, where a horizontal motion is converted into a vertical one or vice versa.

The equation of motion and appropriate initial conditions for this case can be formulated as

$$\begin{aligned}&(1+Ru(t)^2)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t)\right)+Ru(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right)^2+\omega_0^2u(t)+\frac{1}{2}\frac{Rgu(t)^3}{l}=0\\&u(0)=A,\frac{\mathrm{d}u}{\mathrm{d}t}(0)=0,\end{aligned}\tag{5.442}$$

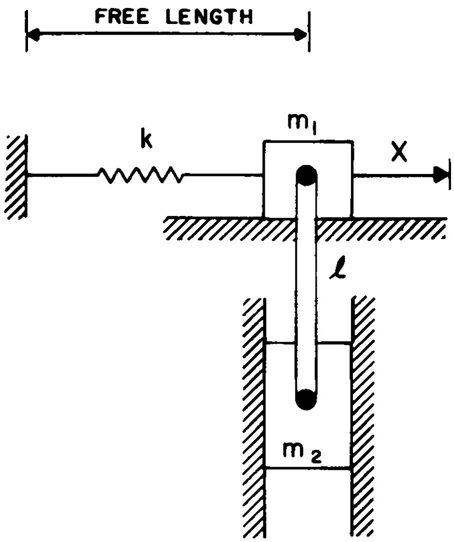


Fig. 5.49 Geometry of the problem

where

$$\omega_0^2 = \frac{k}{m_1} + \frac{Rg}{l}, \ R = \frac{m_2}{m_1} \tag{5.443}$$

The specific dynamics of a spring–mass system are described mathematically by the simple harmonic oscillator, and the regular periodic motion is known as *simple harmonic motion*. In the spring–mass system, oscillations occur because, at the static equilibrium displacement, the mass has kinetic energy that is converted into potential energy stored in the spring at the extremes of its path. The spring–mass system illustrates some common features of oscillation, namely the existence of equilibrium and the presence of a restoring force getting stronger when the system deviates from equilibrium.

5.14.2 Solution Using the Amplitude Frequency Formulation

According to He's AFF, we choose two trial functions: $u_1(t) = A\cos(t), u_2(t) = A\cos(2t)$ Substituting u_1, u_2 into Eq. 5.442, we will obtain the following residuals, respectively:

$$\begin{aligned} R_1 =& (1 + RA^2\cos(t)^2)\left(\frac{\partial^2}{\partial t^2}(A\cos(t))\right) + RA\cos(t)\left(\frac{\partial}{\partial t}(A\cos(t))\right)^2 \\ &+ \omega_0^2 A\cos(t) + \frac{1}{2}\frac{RgA^3\cos(t)^3}{l} - (1 + RA^2\cos(t)^2)A\cos(t) + RA^3\cos(t) \\ &\sin(t)^2 + \omega_0^2 A\cos(t) + \frac{1}{2}\frac{RgA^3\cos(t)^3}{l} \end{aligned} \tag{5.444}$$

and

$$\begin{aligned} R_2 =& (1 + RA^2\cos(2t)^2)\left(\frac{\partial^2}{\partial t^2}(A\cos(2t))\right) + RA\cos(2t)\left(\frac{\partial}{\partial t}(A\cos(2t))\right)^2 \\ &+ \omega_0^2 A\cos(2t) + \frac{1}{2}\frac{RgA^3\cos(2t)^3}{l} - 4(1 + RA^2\cos(2t)^2)A\cos(2t) + 4RA^3\cos(2t) \\ &\sin(2t)^2 + \omega_0^2 A\cos(2t) + \frac{1}{2}\frac{RgA^3\cos(2t)^3}{l} \end{aligned} \tag{5.445}$$

The angular rate is

$$\omega = \Big(-4(1+RA^2\cos(2t)^2)A\cos(2t) + 4RA^3\cos(2t)\,\sin(2t)^2 + \omega_0^2A\cos(2t) + \frac{1}{2}\frac{RgA^3\cos(2t)^3}{l} + 4(1+RA^2\cos(t)^2)A\cos(t) - 4RA^3\cos(t)\sin(t)^2 - 4\omega_0^2A\cos(t) - \frac{2RgA^3\cos(t)^3}{l}\Big)\Big/\Big(-4(1+RA^2\cos(2t)^2)A\cos(2t) + 4RA^3\cos(2t)\sin(2t)^2 + \omega_0^2A\cos(2t) + \frac{1}{2}\frac{RgA^3\cos(2t)^3}{l} + (1+RA^2\cos(t)^2)A\cos(t) - RA^3\cos(t)\sin(t)^2 - \omega_0^2A\cos(t) - \frac{1}{2}\frac{RgA^3\cos(t)^3}{l}\Big) \tag{5.446}$$

Now we form this equation and put $\cos t = \cos 2t = k$, so

$$\omega = \frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}}{2} \tag{5.447}$$

We choose $u(t) = A\cos(\omega t)$ and put it into the AFF equation to determine the integral

$$\int_0^{T/4}\left[\left(\left(1+RA^2\cos\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right)^2\right)\left(\frac{\partial^2}{\partial t^2}\left(A\cos\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right)\right)\right) + RA\cos\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right)\left(\frac{\partial}{\partial t}\left(A\cos\left(\frac{1}{2}\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right)\right)\right)\right)\right) + \omega_0^2A\cos\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right) + \frac{1}{2}\frac{RgA^3\cos\left(\frac{\sqrt{2}\sqrt{\frac{2\omega_0^2l+RgA^2k^2}{l(1+2RA^2k^2-RA^2)}}\times t}{2}\right)^3}{l}\right)\times\cos(\omega t)\Bigg]\mathrm{d}t \tag{5.448}$$

We can find K if we put Eq. 5.448 equal to zero. So we have

$$k^2 = \frac{3}{4} \tag{5.449}$$

Now, the exact expression for ω can be found by substituting K into Eq. 5.447. Subsequently, we have

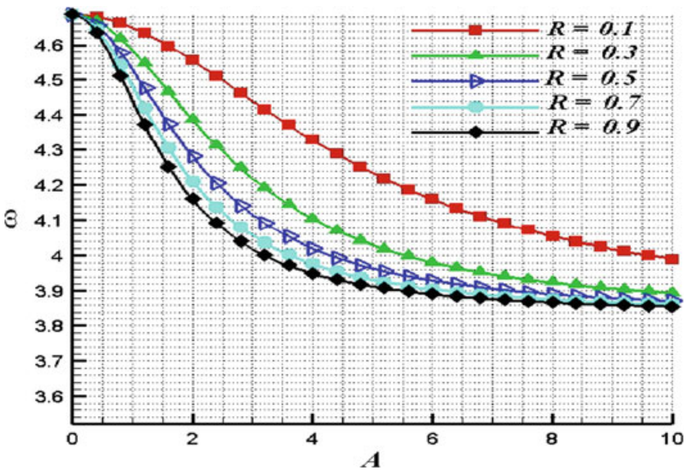


Fig. 5.50 Comparison of periods by method

$$\omega_{\mathrm{IAFF}} = \frac{1}{2} \times \sqrt{\frac{8\omega_0^2 l + 3A^2 Rg}{l(2 + RA^2)}} \tag{5.450}$$

by considering the constants $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}, k = 200\,\frac{\mathrm{N}}{\mathrm{m}}, m_1 = 10\,\mathrm{kg},\ m_2 = 1\,\mathrm{kg},$ $l = 0.5\,\mathrm{m}$.

To evaluate the utility of the method and show how the value of mass ratio affects frequency, variations of the frequency versus R are demonstrated in Fig. 5.50.

The results obtained here show the high-efficiency and accuracy of the AFF, although this method is very simple.

Note

Applied problems used in this chapter were written by a number of nonlinear dynamics teams in the Mechanical Engineering Department of Babol Noshirvani University of Technology. For more details on this chapter, refer to http://ddganji.com/publish/publish.htm.