

# Broadcasting and Embedding Algorithms for a Half Hypercube Interconnection Network

Mi-Hye Kim, Jong-Seok Kim and Hyeong-Ok Lee

**Abstract** The half hypercube interconnection network, has been proposed as a new variation of the hypercube, reducing its degree by approximately half with the same number of nodes as an  $n$ -dimensional hypercube,  $Q_n$ . This paper proposes an algorithm for one-to-many broadcasting in an  $n$ -dimensional half hypercube,  $HH_n$ , and examines the embedding between hypercube and half hypercube graphs. The results show that the one-to-many broadcasting time of the  $HH_n$  can be accomplished in  $n + 1$  when  $n$  is an even number and in  $2 \times \lceil n/2 \rceil$  when  $n$  is an odd number. The embedding of  $HH_n$  into  $Q_n$  can be simulated in constant time  $O(n)$  and the embedding of  $Q_n$  into  $HH_n$  in constant time  $O(1)$ .

**Keywords** Half hypercube · One-to-many broadcasting · Embedding

## 1 Introduction

There is increasing interest in parallel processing as a technique for achieving high-performance owing to the need for many computations with real-time data processing in modern applications [1, 2]. A parallel processing system can connect hundreds of thousands of processors with their own memory via an interconnection

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network. The overall performance of the system is dependant on the performance of each processor and the architecture of the interconnection network used [1–3]. Many interconnection network topologies have been described in the literature, such as star, mesh, bubble-sort, and pancake graphs.

The hypercube,  $Q_n$ , is a typical topology and is an  $n$ -regular and node- and edge-symmetric graph with  $2^n$  nodes and diameter  $n$  ( $n \geq 2$ ). The hypercube  $Q_n$  has simple routing algorithms and recursive structures with maximum fault-tolerance. In addition, it has the advantage that its network structure can easily be embedded in various types of commonly used interconnection networks. With such advantages, it is widely used in various application areas [1, 4, 5]. However, its network cost is increased considerably in relation to the increased degree when the number of nodes increases. To resolve this drawback, several hypercube variations have been introduced. We have proposed the half-hypercube interconnection network, reducing its degree by approximately half, even though it has the same number of nodes as a hypercube  $Q_n$ . In this paper, we propose an algorithm for one-to-many broadcasting in an  $n$ -dimensional half hypercube,  $HH_n$ , and analyze the embedding method of a half hypercube graph into a hypercube graph and vice versa.

The most common properties for measuring the performance of interconnection networks include degree, diameter, connectivity, fault tolerance, broadcasting, and embedding [6, 7]. In [1], we analyzed the degree, diameter, connectivity, and fault-tolerance parameters of the half hypercube. Here, we examine the broadcasting and embedding properties of a  $HH_n$  to strengthen its effectiveness. Broadcasting is one of the major primitives for communication of parallel processing involving message disseminating from an origin node to all the other nodes (one-to-many broadcast) or among the nodes (many-to-many broadcast) in an interconnection network. Embedding is to evaluate the relative performance of two arbitrary interconnection networks. This is of interest, because the properties and algorithms developed in a certain topology can easily be adapted to another network at less cost [8].

The organization of this paper is as follows. Section 2 presents the definition of the half hypercube  $HH_n$ . Section 3 proposes and analyzes a broadcasting algorithm for an  $n$ -dimensional half hypercube,  $HH_n$ . Section 4 examines the embedding algorithms between hypercube and half hypercube graphs. Section 5 concludes the paper.

## 2 Definition of the Half Hypercube

The half hypercube  $HH_n$  ( $n \geq 3$ ) is defined as an  $n$ -dimensional binary cube where the nodes of  $HH_n$  are all binary  $n$ -tuples in the same way as the hypercube  $Q_n$  ( $n \geq 2$ ). That is, an  $n$ -dimensional half hypercube is denoted as  $HH_n$  and each node is represented with  $n$  binary bits. The degree and node connectivity of  $HH_n$  are  $\lceil n/2 \rceil + 1$  and  $\lceil n/2 \rceil + 1$  ( $n \geq 3$ ), respectively. An  $HH_n$  graph is expanded with recursive structures and has  $2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor$ -dimensional hypercube structures with maximum fault-tolerance [1].

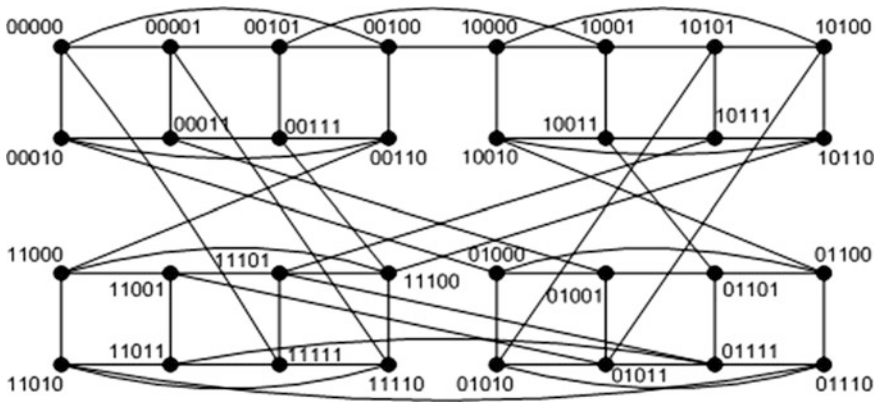


Fig. 1 Example of a 5-dimensional half hypercube ( $HH_5$ ) [1]

In  $Q_n$ , an edge exists between two arbitrary nodes  $S$  and  $S'$  if and only if their corresponding  $n$ -tuples differ in exactly one  $k$  position of the bit strings of  $S$  and  $S'$  ( $1 \leq k \leq n$ ) [9]. On the other hand, in  $HH_n$ , two types of edge exist: the  $h$ -edge, which connects node  $S$  to a node that has the complement in one bit at exactly one  $h$  position ( $1 \leq h \leq \lfloor n/2 \rfloor$ ), and the *swap*-edge (shortly, *sw*-edge), which connects node  $S$  to a node where the  $\lfloor n/2 \rfloor$  leftmost bits of the bit string of  $S$  and the  $\lfloor n/2 \rfloor$  bits on the right-side of  $S$  starting from the  $\lfloor n/2 \rfloor$  position are exchanged ( $h = \lfloor n/2 \rfloor$ ). However, if two parts of  $\lfloor n/2 \rfloor$  bits to swap in the bit string of node  $S$  are the same, the node  $S$  connects to node  $\bar{S}$ , which is the one's complement of the binary number of node  $S$  [1]. Note that  $\bar{S}$  indicates the one's complement of the binary number of node  $S$  in this paper.

At node  $S(=s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1} s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2 s_1)$  of  $HH_n$ , the address of node  $S'$  adjacent to node  $S$  via an *sw*-edge is considered using two cases depending on whether  $n$  is even or odd. For instance, when  $n$  is even, node  $S(=s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1} s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2 s_1)$  is adjacent to node  $S'(=s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2 s_1 s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1})$  where  $(s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1})$  and  $(s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2 s_1)$  of  $S$  are swapped. When  $n$  is odd, node  $S$  is adjacent to node  $S'(=s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2 s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1} s_1)$  where  $(s_n s_{n-1} \dots s_{\lfloor n/2 \rfloor + 1})$  and  $(s_{\lfloor n/2 \rfloor} s_{\lfloor n/2 \rfloor - 1} \dots s_2)$  of  $S$  are swapped. Figure 1 presents a 5-dimensional half hypercube ( $HH_5$ ) [1].

### 3 Broadcasting Algorithm for $HH_n$

Broadcasting is a basic data communication technique for interconnection networks involving message transmission between nodes and is used by parallel algorithms [7, 10]. There are two types of broadcasting communication: one-to-many transmission, which transmits messages from a node to all the other nodes,

and many-to-many transmission, which transmits messages among nodes. Here, we will demonstrate that the one-to-many broadcasting time of  $HH_n$  is  $n + 1$  when  $n$  is even and the broadcasting time is  $2 \times \lceil n/2 \rceil$  when  $n$  is odd.

**Theorem 1** *When  $n$  is even, the one-to-many broadcasting time of  $HH_n$  is  $n + 1$  and when  $n$  is odd, the one-to-many broadcasting time of  $HH_n$  is  $2 \times \lceil n/2 \rceil$ .*

*Proof* Each cluster of  $HH_n$  represents a hypercube and the one-to-many broadcasting time of a hypercube  $Q_m$  is  $m$ . A cluster is connected to all the other clusters in  $HH_n$  by an external *sw*-edge. The broadcasting process is divided into three phases as follows:

- (1) Phase 1: Node  $S$  transmits messages to all the other nodes within its cluster
- (2) Phase 2: All nodes within the cluster to which node  $S$  belongs, including node  $S$  transmit messages to an arbitrary node in all the other clusters of  $HH_n$  using an external *sw*-edge.
- (3) Phase 3: Repeat the process of Phase 1 in each cluster of  $HH_n$ .

When  $n$  is even, the one-to-many broadcasting time is as follows. As the broadcasting time of an internal cluster of  $HH_n$  is the same as the one-to-many broadcasting time of a hypercube, the broadcasting time of Phase 1 is  $n/2$ . As broadcasting is performed only once in Phase 2, its broadcasting time is 1. As Phase 3 repeats the process of Phase 1, the broadcasting time is  $n/2$ . Therefore, the one-to-many broadcasting time of  $HH_n$  is  $n/2 + 1 + n/2 = n + 1$  when  $n$  is even.

When  $n$  is odd, the one-to-many broadcasting time is as follows. As the broadcasting time of an internal cluster of  $HH_n$  is the same as the one-to-many broadcasting time of a hypercube, the broadcasting time of Phase 1 is  $n/2$ . As broadcasting is performed only once in Phase 2, its broadcasting time is 1. If  $n$  is odd, the number of the *sw*-edges connecting clusters is 2 or 4. Thus, the number of nodes that initiate a message transmission is 2 or 4 in Phase 3. If the number of start nodes is 2, the one-to-many broadcasting time of a hypercube is reduced by 1. Therefore, the broadcasting time of Phase 3 is  $\lceil n/2 \rceil - 1$ . Consequently, the one-to-many broadcasting time of  $HH_n$  is  $\lceil n/2 \rceil + 1 + \lceil n/2 \rceil - 1 = 2 \times \lceil n/2 \rceil$  when  $n$  is odd.

## 4 Embedding Between Hypercube and Half Hypercube Graphs

Numerous parallel processing algorithms are being designed to solve many problems in a variety of interconnection network structures. Whether such algorithms designed for a specific interconnection network structure can be run on different interconnection network structures is an important issue in parallel processing. One of the most widely used measuring methods for this issue is embedding [10, 11], which involves mapping the processors and communication links of an interconnection network into those of another interconnection network.

We can represent an interconnection network as a graph  $G(V, E)$ , where  $V(G)$  and  $E(G)$  are the set of nodes and edges of graph  $G$ , respectively, and the set of paths of graph  $G$  is  $P(G)$ . The embedding of an interconnection network  $G(V, E)$  into another interconnection network  $G'(V', E')$  is defined as a function  $(\Phi, \rho)$ , where  $\Phi$  maps the set of vertices  $V(G)$  one-to-one into the set of vertices  $V'(G')$  and  $\rho$  maps the set of edges  $E(G)$  into the set of paths  $P'(G')$ ; that is,  $\Phi: V \rightarrow V'$  and  $\rho: E \rightarrow P'(G')$ . The representative measurement parameters for embedding costs are dilation and congestion. Dilation is the length of the shortest path from node  $S'$  to node  $T'$  in  $G'$  when the nodes  $S$  and  $T$  of an edge  $(S, T)$  in  $G$  are mapped to nodes  $S'$  and  $T'$  of  $G'$ ; i.e., the number of edges comprising the shortest path from node  $S'$  to node  $T'$  in  $G'$ . Congestion is the number of edges in  $G$  that pass an edge  $e$  in  $G'$  when  $G$  is mapped to  $G'$  [3, 4]. In this section, we analyze the embedding between a hypercube  $Q_n$  and a half hypercube  $HH_n$  using dilation.

**Theorem 2** *An  $n$ -dimensional hypercube  $Q_n$  can be embedded into an  $n$ -dimensional half hypercube  $HH_n$  with dilation 3.*

*Proof* We can analyze the dilation of this mapping through the number of edges of  $HH_n$  required to map the  $k$ -dimensional edge ( $1 \leq k \leq n$ ), which represents the adjacent relationships of the nodes in  $Q_n$ , into edges in  $HH_n$ . Theorem 4 is proven by dividing the  $k$ -dimensional edge of  $Q_n$  into two cases depending on the dimension of  $k$ .

Case 1  $k$ -dimensional edge,  $1 \leq k \leq \lceil n/2 \rceil$

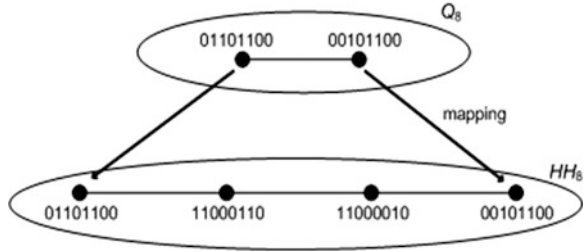
It can be easily observed that the  $k$ -dimensional edge of hypercube  $Q_n$  ( $1 \leq k \leq \lceil n/2 \rceil$ ) are the same as the  $h$ -dimensional edge of half hypercube  $HH_n$  ( $1 \leq h \leq \lceil n/2 \rceil$ ). Therefore, it is clear that the embedding of an  $n$ -dimensional hypercube  $Q_n$  into an  $n$ -dimensional half hypercube  $HH_n$  is possible with dilation 1 when the two adjacent nodes via a  $k$ -dimensional edge in  $Q_n$  are mapped to two adjacent nodes through an  $h$ -dimensional edge in  $HH_n$ .

Case 2  $k$ -dimensional edge,  $\lceil n/2 \rceil + 1 \leq k \leq n$

The address of node  $S'$  adjacent to an arbitrary node  $S(=s_n s_{n-1} s_{n-2} \dots s_k \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1)$  of  $Q_n$  via a  $k$ -dimensional edge has the complement at exactly one  $k$  position of the bit string of node  $S$  (i.e., bit  $s_k$ ). An edge of  $HH_n$  that has the same role as the  $k$ -dimensional edge of hypercube  $Q_n$  can be presented by sequentially applying the following edge sequence:  $\langle sw\text{-edge}, (k - \lceil n/2 \rceil)\text{-edge}, sw\text{-edge} \rangle$  ( $\lceil n/2 \rceil + 1 \leq k \leq n$ ). That is, it reaches a node with the address  $s_n s_{n-1} s_{n-2} \dots \bar{s}_k \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1$  when the edge sequence  $\langle sw\text{-edge}, (k - \lceil n/2 \rceil)\text{-edge}, sw\text{-edge} \rangle$  is applied sequentially to node  $S(=s_n s_{n-1} s_{n-2} \dots s_k \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1)$  of  $HH_n$ . Let a node  $S'$  adjacent to node  $S(=s_n s_{n-1} s_{n-2} \dots s_k \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1)$  of hypercube  $Q_n$  via a  $k$ -dimensional edge ( $\lceil n/2 \rceil + 1 \leq k \leq n$ ) be  $S'(=s_n s_{n-1} s_{n-2} \dots \bar{s}_k \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1)$ .

The address of node  $sw(S)$  adjacent to node  $S(=s_n s_{n-1} s_{n-2} \dots s_k \dots s_{\lceil n/2 \rceil} \dots s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1)$  of  $HH_n$  via an  $sw$ -edge is  $s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1 s_n s_{n-1} s_{n-2} \dots s_k \dots s_{\lceil n/2 \rceil + 1}$ . To invert

**Fig. 2** Embedding example between  $Q_8$  and  $HH_8$  with dilation 3



the bit  $s_k$  in the bit string of node  $sw(S)$  to the complement, we take a node  $S''(=s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1 s_n s_{n-1} s_{n-2} \dots \bar{s}_k \dots s_{\lceil n/2 \rceil + 1})$  adjacent to node  $sw(S)$  via a  $(k - \lceil n/2 \rceil)$ -dimensional edge. The address adjacent to node  $S''$  through an  $sw$ -edge is  $s_n s_{n-1} s_{n-2} \dots \bar{s}_k \dots s_{\lceil n/2 \rceil + 1} s_{\lceil n/2 \rceil} \dots s_3 s_2 s_1$ . Therefore, the embedding of a  $k$ -dimensional edge of  $Q_n$  into a half hypercube  $HH_n$  is possible with dilation 3. Figure 2 presents an example of embedding between  $Q_8$  and  $HH_8$  with dilation 3.

**Theorem 3** A half hypercube  $HH_n$  can be embedded into a hypercube  $Q_n$  with dilation  $n$ .

*Proof* There exist two types of edges in half hypercube  $HH_n$ . Thus, we proved Theorem 5 by dividing it into two cases: h-edge and sw-edge.

Case 1 h-edge,  $1 \leq h \leq \lceil n/2 \rceil$

The address of node  $S'$  adjacent to node  $S(=s_n s_{n-1} s_{n-2} \dots s_{n/2} \dots s_3 s_2 s_1)$  of half hypercube  $HH_n$  via an h-edge is  $s_n s_{n-1} s_{n-2} \dots s_{n/2} \dots \bar{s}_h \dots s_3 s_2 s_1$  ( $1 \leq h \leq \lceil n/2 \rceil$ ). The address of node  $S'$  adjacent to node  $S(=s_n s_{n-1} s_{n-2} \dots s_k \dots s_{n/2} \dots s_3 s_2 s_1)$  of hypercube  $Q_n$  through a  $k$ -dimensional edge ( $1 \leq k \leq n$ ) is  $s_n s_{n-1} s_{n-2} \dots \bar{s}_k \dots s_{n/2} \dots s_3 s_2 s_1$ . Accordingly, the dilation of this embedding is 1 because the h-edge in half hypercube  $HH_n$  and the  $k$ -dimensional edge in  $Q_n$  are equivalent ( $1 \leq h, k \leq \lceil n/2 \rceil$ ).

Case 2 sw-edge

The sw-edge of half hypercube  $HH_n$  can be divided into two cases depending on the address of node  $S(=s_n s_{n-1} s_{n-2} \dots s_{n/2} \dots s_3 s_2 s_1)$ . Here, we prove the case with dilation  $n$ . If the  $n$ -address bits of node  $S$  are all binary 0, the  $n$ -address bits of node  $S'$  adjacent to node  $S$  through an sw-edge are all binary 1. The shortest path from node  $S(=s_n s_{n-1} s_{n-2} \dots s_{n/2} \dots s_3 s_2 s_1)$  to node  $S'(=s_n s_{n-1} s_{n-2} \dots s_{n/2} \dots s_3 s_2 s_1)$  in hypercube  $Q_n$  is the same as the path to which the  $k$ -dimensional edges are all applied. Therefore, a half hypercube  $HH_n$  can be embedded into a hypercube  $Q_n$  with dilation  $n$  ( $1 \leq k \leq n$ ).

## 5 Conclusion

This paper proposes a one-to-many broadcasting algorithm for the half hypercube interconnection network,  $HH_n$  that we proposed in [1], and proved that the one-to-many broadcasting time is  $n + 1$  when  $n$  is even and  $2 \times \lceil n/2 \rceil$  when  $n$  is odd in  $HH_n$ . We also showed that it is possible to embed an  $n$ -dimensional hypercube  $Q_n$  into an  $n$ -dimensional half hypercube  $HH_n$  with dilation 3, and that it is possible to embed  $HH_n$  into  $Q_n$  with dilation  $n$ . These results suggest that our half hypercube interconnection network  $HH_n$  has potential for implementation in large-scale systems for parallel processing.

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