Chapter 27
A Cross-Sectional Study About Modelling Competency in Secondary School

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Abstract The meaning of mathematical modelling and modelling competencies has been discussed frequently and, in parts also controversially, recently. There is still a debate about how to measure modelling competency and a comprehensive agreement is not yet in sight. In this chapter we applied six stages of mathematical modelling competency to classify student solutions according to their modelling performance and investigated 234 student solution approaches from grades 6 to 11 for the modelling task, *Restringing a Tennis Racket*. We identified four main solution approaches. In the study we analysed the correlation of modelling competency stage and students' grade as well as gender. The study shows that mathematical modelling competency is independent of gender issues in this specific case, and gives insights into the modelling behaviour of students of different grades.

1 Introduction

In the last 10 years there has been a busy and controversial discussion about how mathematical modelling competencies can be evaluated and, especially, how they can be improved (e.g., Blomhøj and Jensen 2003; Kaiser et al. 2011). For this reason a variety of treatments have been established which show, among others, that the modelling competency, as well as incorporated sub-competencies (see Fig. 27.1), can definitely be improved by many different forms of treatments (e.g., DISUM[1]-project). Modelling competency is a quite difficult construct, involving many

[1]Didaktische Interventionsformen für einen selbstständigkeitsorientierten aufgabengesteuerten Unterricht am Beispiel der Mathematik, W. Blum, R. Messner and R. Pekrun.

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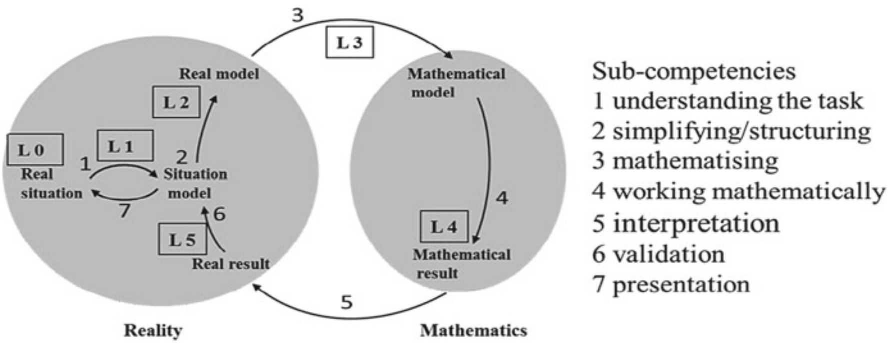


Fig. 27.1 Modelling cycle according to Blum and Leiß (2007) completed with modelling competency stages

factors, reaching from cognitive psychological aspects, like the influence of different thinking styles on the transition process from reality to mathematics (Borromeo Ferri 2010), through to problems concerning the analysis of empirical results, due to its limited measurability. In the context of measuring modelling competency there is still no consistent opinion. Isolated efforts have been made to operationalise modelling competency to have an assessable concept (e.g., Maaß 2007).

The study of Ludwig and Xu (2008) investigated solutions, documented by Chinese and German students of different grades, concerning the mathematical background of peeling a pineapple in a presented, characteristic way. Based on the possibility that there are differences among the two populations, students from grade 9 to 11 from Germany and Shanghai were asked to mathematically explain, that is, to set up a mathematical model of the spiral-shaped appearance of a peeled pineapple, as it is customary in China. Besides the result that the solution quality is correlated to the grade (9–11) of the students, the authors argued that the ability to solve the problem successfully increases with the grade. The research question for the present chapter results from this observation and addresses the assumption that the required mathematical knowledge needs to be cognitively consolidated before it can be used in a targeted manner. This phenomenon forms the basic question of the present chapter. How will students from a wider range of grades solve a modelling task? Can we expect multiple solution strategies? Is there any difference according to gender? To answer these questions we first have to analyse and evaluate the student solutions.

2 Theoretical Framework

This study is based on an evaluation scheme already applied in the study of Ludwig and Xu (2008) which assigns stages to the documented solutions of students, which represent their modelling progress in terms of the modelling cycle of Blum and Leiß (2007). The implementation of the study as a cross section study of students from different grades allows for an investigation of the relationship between modelling competency and grade. The same modelling task is given to all of the students, so that a consistent comparison of performance can be archieved. In addition, by extracting different solution approaches, we are better able to understand the modelling process of students.

2.1 The Stages

When elaborating on a mathematical modelling task, one passes through different phases of the modelling cycle, according to Blum and Leiß (2007) (see Fig. 27.1). The transition from one modelling phase to another, that is a progression in terms of the modelling cycle, requires a successful overcoming of cognitive obstacles (Blum 2007). Although we cannot assume a linear cycling through the modelling cycle by the students, as it was verified by Borromeo Ferri (2006) that students follow individual modelling routes when being confronted with modelling tasks, most solutions indicate a more or less detailed solution process. Since it was not part of the study to reveal the actual modelling progress of the students, we concentrated on how far the students have gone through the modelling cycle reflected by their solutions. We assigned so called *modelling competency stages* to each student solution, based on their solution progress within the modelling cycle. We applied a scale of six consecutive stages to categorise the student solutions (Ludwig and Xu 2008), which can be easily integrated in the modelling cycle of Blum and Leiß (2007) (see Fig. 27.1). However we cannot preclude that the students would not have been able to reach a higher stage when having had more time to work on the task. This fact is represented by the terminology "stage" since "stage" reflects a part of a progress which implies a potential achievement of a higher stage.

The definition of *modelling competency stage*, as we use it in this chapter in relation to modelling competency, is based on the competency concept of Weinert (2001), where he points out that competency is an ability which is subject to assessment and used by a person explicitly. In a broader sense this understanding of modelling competency coheres with the definition of Blomhøj and Jensen where they pointed out that modelling competency is "headed for action" (Blomhøj and Jensen 2003).

In the following we will explain the stages as they are applied in this study. For a comparative explanation see Ludwig and Xu (2008). We have to note, that the stages are not distributed equidistantly in general. The reason for this lies in the fact, that the underlying scale is not metric.

- Stage 0 indicates that the student does not understand the task or is not willing to solve it. There are no sketches or notes on the worksheet.
- Stage 1 shows that the student understands the given real situation, but the student is neither able to structure nor to simplify it. The student is also not able to find a connection of real situation and mathematical ideas. There are some reasonable sketches on the worksheet but no simplifications or mathematical formations are identifiable.

- Stage 2 encompasses the development of a real model by simplifying and structuring the real situation. However, the student is not able to transfer this model into a mathematical model. There are sketches of the situation and assumptions to simplify it.
- Stage 3 contains the transfer of the real model into a mathematical model. The student is able to work to a limited extent with this model and produces a concrete result. However, the student is not able to generalise the solution process. The real model is set up, which implies a completion of any sketch with mathematical notations. In addition, reasonable formula approaches are obvious and a solution in terms of a numeric value is identifiable.
- Stage 4 includes a generation of a mathematical question from the real situation. The student is able to work within the mathematical context and to establish a general formula; but this formula is not analysed or validated yet.
- Stage 5 indicates that the student analysed or validated the solution to better adjust the formula to the given situation. Thus, the student gives suggestions for improvement.

3 The Study

3.1 The Real Situation

The string of a tennis racket can only be replaced as a whole, in contrast to the strings of a guitar, which can be easily exchanged individually. In addition, the string needed for the racket has to be cut off from a big string coil, such that the total length must be known in advance. Of course the string should not be cut off too short, since then it is unusable. On the other hand, string should not be wasted.

Based on these considerations the question arises, how the minimal string length of a tennis racket can be determined using the dimension data of the racket. To be able to repair different sorts of rackets, as for example also badminton rackets, a general formula has to be established which is independent of the actual dimension data given on the worksheet. This problem has also been considered by Ludwig (2008); however we focus here on the student solutions, their performance and correlation with gender aspects. An extended version has been published in Ludwig and Reit (2013b).

3.2 Sample and Study Implementation

The study was performed in grammar schools in Bavaria and Baden-Wuerttemberg in grades 6, 8, 9, 10 and 11 with students from the age of 12–17. We involved a total of 234 students, all not having had special training in mathematical modelling before.

The study was to be integrated into the normal school routine, so the task had to be able to be processed in a 45-min class, as is usual in Germany. We developed a modelling task which is, according to the German curricula, equally well suited for

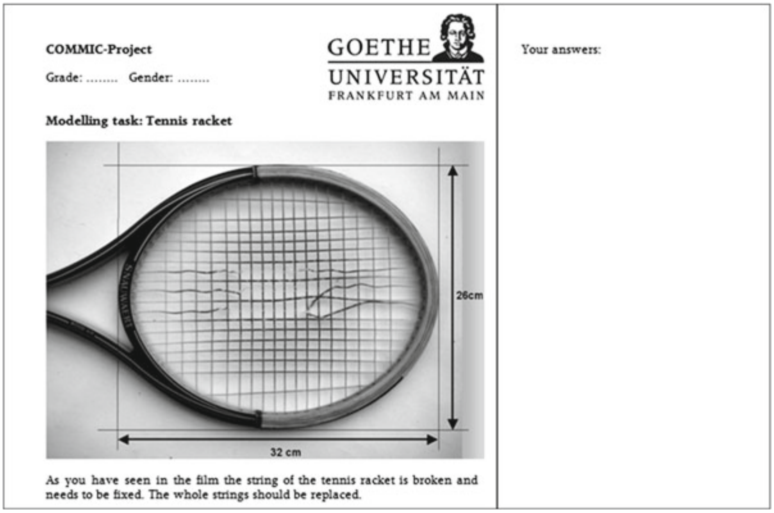


Fig. 27.2 Worksheet with the modelling task

all participants. The underlying mathematical knowledge, for example performing an approximation of the tennis racket by a rectangle, should already be available to 5th graders, according to German curricula.

At the beginning of the lesson a 90 s clip was shown, illustrating the situation of the modelling task. The second phase of the lesson contained working on the actual modelling task in a seatwork setting. Therefore, the students received a worksheet in A4 landscape format (see Fig. 27.2); on the left side there is a picture of the tennis racket with the broken string. This picture is equipped with original specifications, indicating the horizontal and vertical length of the racket area. Additionally, the following two mathematical questions are below the picture (not shown in Fig. 27.2).

(a) It is now up to you to estimate the total length of the string you need for this racket in a mathematical way. Perhaps the dimensions in the picture will help you.
(b) Can you specify a simple formula that an employee in a sports shop can use to calculate the total length of the string of different rackets? The formula can use racket data, which are easy to determine.

4 Solution Approaches

4.1 Approach by Direct Measurement

With a direct measurement approach the student directly measures the string length illustrated on the worksheet with a ruler and translates these values subsequently into real values by calculating the scale using the dimensions given in the picture.

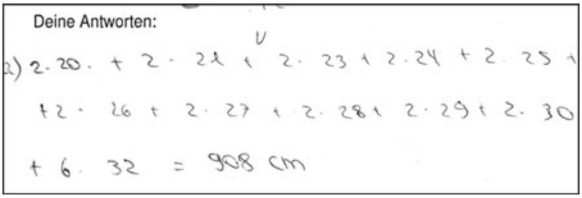


Fig. 27.3 A student solution using direct measurement (rated stage 3)

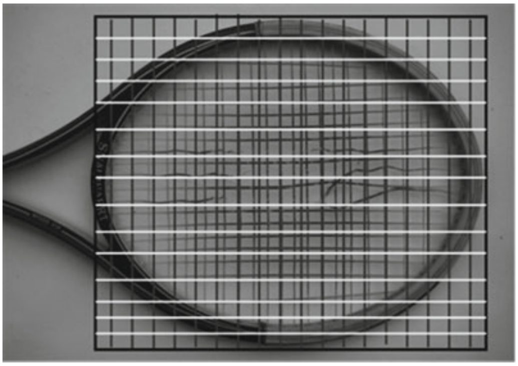


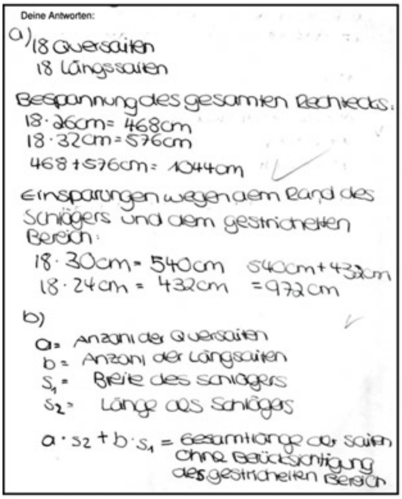
Fig. 27.4 The *rectangle* net overlaying the elliptical racket

This process can also be observed vice versa, that is, calculating the scale first and then summing up the real string lengths to obtain the total length (see Fig. 27.3). The main problem of the direct measurement approach is that it is not possible to set up a general formula based on this approach so it must be classified as stage 3. Most students using this approach stopped after subtask (a), having stated a numeric value for the total string length. Subtask (b) requiring aspects of generalisation, had often not been elaborated to some extent.

4.2 Rectangle Model

The rectangle model includes the approximation of the tennis racket area by a rectangle (see Fig. 27.4). Doing this, the respective students lengthen the strings beyond the frame of the racket. They count the vertical and horizontal strings (in our case 18 horizontal and vertical strings) and multiply these values by the given side length (here 32 cm) and width (here 26 cm) of the rectangle, or tennis racket respectively (see Fig. 27.5).

The calculated result of 1,044 cm total string length in Fig. 27.5 qualifies the solution approach for stage 3. Furthermore, the student argues, that the rectangle model is sufficiently exact, because the resulting overestimation intuitively includes the extra string needed to fasten the string on the frame of the racket. This improves the student solution to stage 4. The establishment of variables for the number of horizontal and vertical strings and length and width of the racket leads to a general formula, which is valid for different rackets. This gives rise to rating the student solution at stage 5.



Translation:

Your answers:

a) *18 vertical strings*
18 horizontal strings

Covering of the total rectangle:

18·26cm = 468cm, 18·32cm = 576cm

468 + 576cm = 1044cm

Conservations resulting from the edge of the racket and the dashed area:

18·30cm = 540cm

18·24cm = 972cm 540cm + 432cm = 972cm

b)

a = number of crosses

b = number of mains

s_1 *= width of the racket*

s_2 *= length of the racket*

$a \cdot s_1 + b \cdot s_2$ *= total length of the string regardless of the dashed area*

Fig. 27.5 A rectangle solution of a student in grade 10 (rated stage 5)

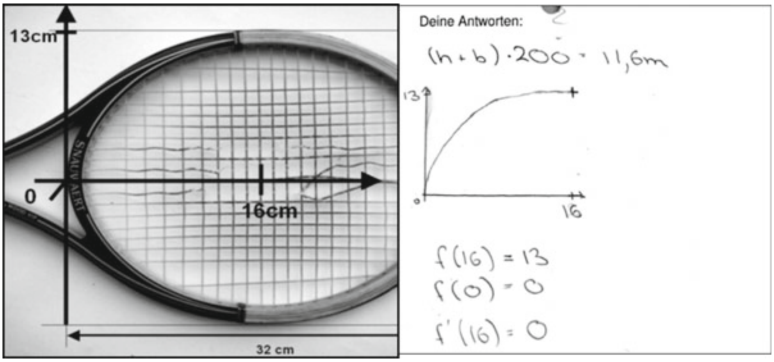


Fig. 27.6 A solution using the function approach (rated stage 2)

4.3 Functional Model

By applying the functional model some students tried to approximate the shape of the elliptical tennis racket with a graph of a function (see Fig. 27.6). To do so, the student implicitly inserted a coordinate system into the picture on the worksheet such that three constraints ($f(16)=13$, $f(0)=0$, $f'(16)=0$) can be set up. However, as long as the function type is unknown, the three equations are more or less unusable, even if they are indeed comprehensible. An explicit calculation could not be performed by the student which leads to a rating with stage 2, but the subsequent strategy was outlined perfectly. The student argued that the inverse function of the sketched function is necessary and that the outputs belonging to integer *x*-values provide the single string lengths.

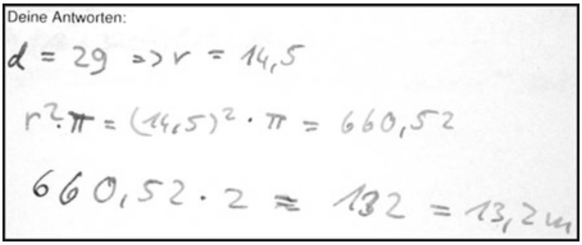


Fig. 27.7 A student solution using a circular model and ignoring units (rated stage 2)

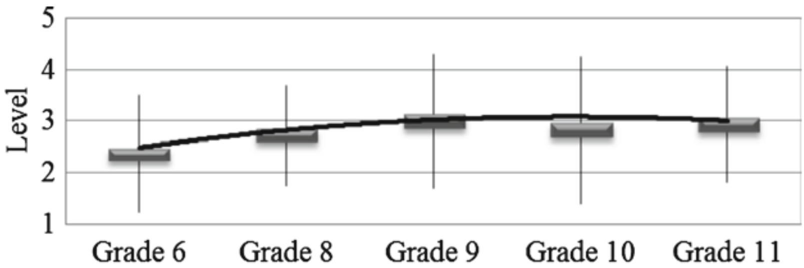


Fig. 27.8 Distribution of stages per grade

4.4 Area Model

The area model, as shown in Fig. 27.7, was used rather infrequently. It became clear, that the student had difficulties in transforming the area of an object with the string length (see Fig. 27.7). One explanation may be the difficult accessibility of a circle or other geometric object by a functional term.

5 Results

The 234 solutions have been analysed by two independent raters classifying each solution approach according to our stages. They found the four main solution approaches and classified them using the six stages. With Cohen's reliability coefficient, $\kappa = 0.789$ and Pearson's correlation coefficient, $r = 0.84$, there was a good inter-rater reliability.

In Fig. 27.8, it is obvious that the students improve their modelling skills between grade 6 and grade 9, at least related to the given task. From grade 9 this development passes over to saturation. A possible assumption is that so long as the required mathematical knowledge is not cognitively consolidated to this point, we have a continuous process of improvement concerning modelling skills. This increase in modelling competency then reaches its maximum and stays at this stage.

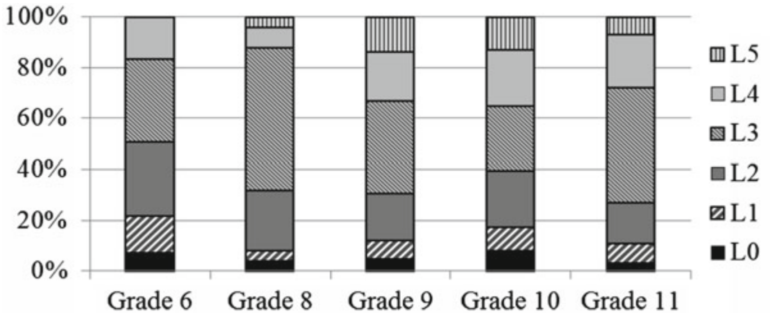
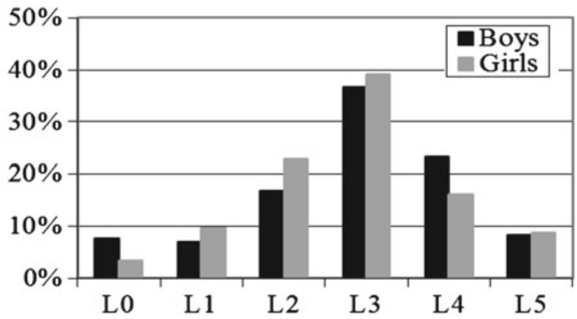


Fig. 27.9 Percentage of students by grade reaching various stages

Fig. 27.10 Percentage of boys and girls stopping at various stages



5.1 Stages Within Grades

When looking at the distribution of stages within the grades, it is apparent that no student of grade 6 reached stage 5 and more than 80 % of the students of grade 6 and 8 did not reach stage 4 (see Fig. 27.9). Grade 6 and 8 students appear to have had more problems with subtask (b), that is generalising their solution of subtask (a), than the others, since the elaboration of subtask (b) was prerequisite to reach stage 4. The distribution of stages from grade 9 to 11 is not conspicuously different, consistent with the findings of Fig. 27.8.

5.2 Performance by Gender

A Mann-Whitney-*U*-Test to detect differences in the modelling competency among boys and girls was not statistically significant ($z = -0.89$, $p = 0.374$) (see also Fig. 27.10).

Both, girls and boys have a maximum at stage 3, suggesting that stage 3 is a kind of barrier for the students. Figure 27.9 also clearly shows that most students reached stage 3. The main step from stage 3 to 4 is the establishment of a general formula. It is known that generalisations are often very hard to understand and to accomplish by students (c.f. Stillman et al. 2007). However, this shows that the transfer to the actual modelling task, namely subtask (b), is still difficult for students.

6 Discussion

The investigation of solutions of a single modelling task and the comparison to performance has not been done before explicitly in this manner and gives interesting insights into modelling behaviour of students. A first quite pleasing result of this study was that nearly all students were able to document a more or less useful solution to this modelling task in contrast to PISA 2000 where German students were not yet able to work on mathematical problem solving tasks in a satisfactory manner (Artelt et al. 2001).

As expected, especially students in younger grades seem to have had more difficulties in solving this modelling task. This strengthens the assumption that mathematical modelling competencies are enhanced by experience and practice, even though the more advanced mathematical knowledge might also be an issue. Since the present study cannot uniquely determine the actual reasons for the differences in performance of grades, more data must be collected to clarify this. However, for all grades there is an obstacle proceeding from the explicit numeric result to its generalisation and this could be commensurate with similar difficulties in purely mathematical contexts.

In contrast to the widespread stereotype of mathematically science oriented boys and linguistically gifted girls we did not find statistical evidence of differences in the performance of boys and girls in elaborating on a modelling task. This coincides well with the findings of Ludwig and Reit (2013a) where tests support a general comparability of boys' and girls' modelling competency with reference to the tennis racket task. To explain this in more detail, some more task oriented research is needed. The correlation between solution model and performance has not been investigated in this study. Moreover, with the evaluation of solution approaches it is possible to draw conclusions about the coherence of subject matter and approach used.

References

Artelt, C., Baumert, J., Klieme, E., Neubrand, M., Prenzel, M., Schiefele, U., et al. (2001). *PISA 2000 Zusammenfassung zentraler Befunde*. Berlin: Max-Planck-Institut für Bildungsforschung.

Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: conceptual clarification and educational planning. *Teaching Mathematics and Its Applications, 22*(3), 123–138.

Blum, W. (2007). Modellierungsaufgaben im Mathematikunterricht – Herausforderungen für Schüler und Lehrer. In H. Humenberger et al. (Hrsg.), *Festschrift für Hans-Wolfgang Henn* (pp. 8–22). Hildesheim: Franzbecker.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with mathematical modelling problems? The example "Sugarloaf". In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling ICTMA 12: Education, engineering and economics* (pp. 222–231). Chichester, UK: Horwood.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM – The International Journal on Mathematics Education, 38*(2), 86–95.

Borromeo Ferri, R. (2010). On the influence of mathematical thinking styles on learners' modelling behaviour. *Journal für Mathematik-Didaktik* (Journal for Didactics of Mathematics), *31*(1), 99–118.

Kaiser, G., Blum, W., Borromeo Ferri, R., & Stillman, G. (2011). *Trends in teaching and learning of mathematical modelling: ICTMA 14*. New York: Springer.

Ludwig, M. (2008). *Mathematik + Sport: Olympische Disziplinen im mathematischen Blick*. Wiesbaden: Vieweg + Teubner Verlag.

Ludwig, M., & Reit, X.-R. (2013a). Comparative study about gender differences in mathematical modelling. In G. Nagarjuna, A. Jamakhadia, & E.M. Sam (Eds.), *Proceedings of EPISTEME 5* (pp. 48–54). Mumbai: Cinnamon Teal.

Ludwig, M., & Reit, X.-R. (2013b). Eine empirische studie zum mathematischen modellieren im sport. In R. Borromeo Ferri, G. Greefrath, & G. Kaiser (Eds.), *Mathematisches modellieren für schule und Hochschule – Theoretische und didaktische hintergründe*. Berlin: Springer Vieweg.

Ludwig, M., & Xu, B. (2008). A comparative study of modelling competencies among Chinese and German students. In M. Blomhøj & S. Carreira (Eds.), *Mathematical applications and modelling in the teaching and learning of mathematics: Proceedings from TSG 21 at ICME 11* (pp. 197–206). Roskilde: IMFUFA, Roskilde University.

Maaß, K. (2007). *Mathematisches modellieren – Aufgaben für die Sekundarstufe*. Berlin: Cornelsen Verlag Scriptor GmbH & Co. KG.

Stillman, G., Galbraith, P., Brown, J., & Edwards, I. (2007). A framework for success in implementing mathematical modelling in the secondary classroom. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 688–697). Adelaide: MERGA.

Weinert, F. E. (2001). Vergleichende Leistungsmessung in Schulen – eine umstrittene Selbstverständlichkeit. In F. E. Weinert (Ed.), *Leistungsmessungen in Schulen*. Weinheim: Beltz.