Chapter 3
Arguments, Proofs, and Dialogues

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To what extent do proofs fall within the scope of a theory of argumentation? In this chapter I shall try to provide an answer. To this end, several types of proof need to be distinguished. Proofs of most types will be seen to be arguments, and therefore amenable to analysis from the point of view of argumentation studies. The last section presents a dialectical view of proof as an argument in dialogue that meets certain supplementary conditions. These conditions can, however, be formulated in dialectical terms.

3.1 Proof and Argument

What is a proof? A set of solid reasons and lucid inferences that clinch the argument? Or do we need more, before we are prepared to accept the credentials of a supposed proof? Are proofs within the range of legitimate subjects for a theory of argumentation? Or must the territory be left to formal logicians?

The Latin word *argumentum* has 'proof' as one of its meanings; *argumentatio* means 'argumentation' or '(the furnishing of) proof;' the correlative verbs are *argumentor* and *arguo*: I argue, I prove. Does this mean that, fundamentally, arguing and proving are one and the same? If so, one may be puzzled by the existence of a branch of logic called 'proof theory' (or metamathematics). Obviously, proof theory and theory of argumentation are quite different disciplines.[1]

[1]To see what theory of argumentation is about, one may consult Barth and Martens (1982), and Van Eemeren et al. (1987, 1996); for proof theory see Prawitz (1981). The word 'argument,' in this

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Webster's Ninth New Collegiate Dictionary renders the relevant meaning of 'to prove' as follows:

> 3 a: to establish the existence, truth, or validity of (as by evidence or logic)

This covers proofs in mathematics as well as proofs in science and proofs in court. There is, indeed, no reason to postulate three radically different meanings for these cases. On the relevant meanings of 'arguing' the dictionary instructs us as follows:

> *vi* [...] **1**: to give reasons for or against something: REASON [...] *vt* [...] **2**: to consider the pros and cons of: DISCUSS **3**: to prove or try to prove by giving reasons: MAINTAIN **4**: to persuade by giving reasons: INDUCE

Thus the domains of application of the terms 'to prove' and 'to argue' overlap: whenever someone tries, by giving reasons, to establish the existence, truth, or validity of something and *succeeds* in doing so, both terms apply: what she has been doing was not just arguing a case, but proving her conclusion as well. But if she only *tried* to establish the existence, truth, or validity, but without success, she has been arguing all the same, but she has not been proving anything. Hence it is quite possible to argue without proving. Can one also prove without arguing?

A proof is to establish the correctness of a proposition, to justify a point of view or claim. But we are not told that argument is the only way to achieve this end. Suppose I claim to be capable of singing a song. You want a proof. I could start an argument about my reputation as a singer, but the easiest way to justify my claim would be just to burst into song. Similarly, a claim to the effect that it is snowing could be established by drawing back the curtains. If you utter 'I can pronounce an English sentence' you have laid down your claim and handed us a proof of it at one and the same time. In all these cases one justifies a claim without offering an argument. One is proving something, but not arguing for it. Such proofs are intuitively clear, and they are so immediately. We shall use the term *Immediate and Intuitive Proof* for this type of case.

The term 'proof' displays a process-product ambiguity (as does the term 'argument'). Sometimes the *process* of establishing the correctness of a proposition is meant, but at other times it is the *product* (often a text) that is meant. The same holds for 'immediate and intuitive proof,' though, in some cases, it may sound a bit peculiar when the term 'proof' is applied to the product, which could be a song, or a figure, or a gesture, or whatever.

3.2 Mathematical Proof

Among mathematical proofs the *Immediate and Intuitive* ones are at one extreme, whereas the *Formal Proofs* are at the other. In between one finds informal proofs

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paper, is not used in the technical, logical, sense of a premises-conclusion constellation, but refers to verbal and social means (especially the presentation of reasoning) to convince an addressee that a certain claim is justified. Cf. Walton (1990, esp. 411).

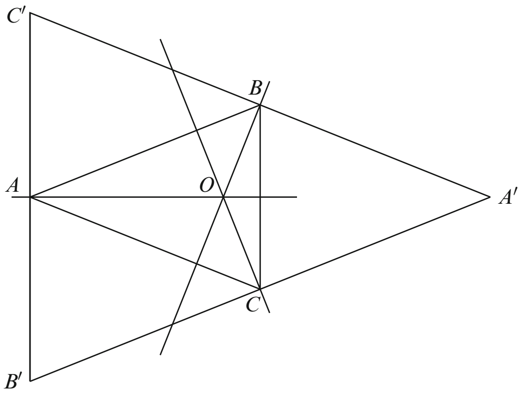

Fig. 3.1 The altitudes of a triangle *ABC* intersect in a single point *O*

in which arguments are brought into play, possibly within the context of an axiom system. The question is whether these informal proofs would be best regarded as special cases of arguing and argument, or as specimens of a completely different type of process (product). The following case study may clarify this issue:

Theorem 1. *The altitudes of a triangle intersect in a single point.*

As you may remember, each triangle has three altitudes. If A, B, and C are the vertices of a triangle, then the altitude from A is the line through A that is perpendicular to the side BC, etc. We take it to be self-evident that any *two* altitudes (e.g., the altitude from A and that from B) intersect in a single point (say O). The problem is to prove that the third altitude (that from C) passes through O as well.

A well-known proof of this theorem presents (or has the student draw) a figure (Fig. 3.1).

For some this figure, by itself, may constitute an immediate and intuitive proof. There is no fundamental difference, as far as the method of proof is concerned, between showing some figure (to prove a geometrical theorem) and drawing back the curtains in order to prove that it is snowing outside. But many will not be satisfied by Fig. 3.1 and demand an accompanying argument. Here is one (it is called a 'proof'):

Proof. Consider an arbitrary triangle ABC. Draw a line through A parallel to BC, one through B parallel to AC, and one through C parallel to AB. We have constructed a circumscribing triangle $A'B'C'$. It is not hard to see that the altitudes of the inner triangle ABC happen to be the perpendicular bisectors of the sides of the outer triangle $A'B'C'$ (this is left as an exercise). But, as we all know, the perpendicular bisectors of the sides of any triangle will intersect in a single point which is the center of the circumscribed circle of the triangle. Therefore, the altitudes of ABC will intersect in a single point, which is the center of the circumscribed circle of $A'B'C'$. Since ABC was chosen arbitrarily, the same result holds for any triangle: its altitudes will intersect in a single point (which may henceforth be called the *orthocenter* of the triangle), QED.

This illustrates what is meant by *Informal Proof*.[2] Among the characteristics of informal proof are the following: (i) it is an attempt to establish a conclusion by argument, (ii) which is addressed to an addressee who does not yet subscribe to the theorem, but who is willing to let himself be convinced of its truth, (iii) this addressee is assumed to be acquainted with a certain body of knowledge, reference to which is made in the proof (for instance, in the proof given above, the addressee was assumed to be familiar with the fact that the perpendicular bisectors of the sides of a triangle intersect in a single point), (iv) some parts of the proof are left as an exercise to the addressee, (v) some propositions are supposed to be immediately and intuitively obvious (for instance that there is a line through *A* parallel to *BC*), and (vi) some procedures of proof are presupposed as well (for instance, how to handle 'arbitrary objects' in order to obtain general conclusions). An informal proof, therefore, is just an argument of sorts, and hence an appropriate object of study for the theory of argumentation. If we take a dialectical view of argument in general, that is, if we look upon all argument as explicit or implicit critical discussion, then informal proofs are no exceptions: a critical discussion is (usually implicitly) contained in them.

An argument may or may not hold water. Even if it does not, it is an argument all the same. But 'proof' has another type of grammar. Ordinary usage suggests that shaky or fallacious arguments do not count as proofs. It is probably wise to follow ordinary usage in this respect as we are about to recommend some philosophical terminology. This means that we must differ from other, perhaps marginal, phenomena of usage, such as talk of proofs being 'wrong' or 'right.' On closer analysis, a 'wrong proof,' i.e., a 'proof' containing a fallacy or error, is no proof at all, just as a forged Vermeer is not a specific kind of Vermeer.

This does not at all make it easier to apply the term '(informal) proof' correctly. Suppose we are given an alleged proof. It is an argument for sure, but we cannot tell whether the argument is a proof without knowing whether the argument holds water. However, with arguments, this cannot be seen from the text: we have to study the context. Informal mathematical proof is no exception. What counts as a proof for one person may not count as proof for someone else. The argument given above for the existence of an orthocenter with each triangle can only take effect as a proof for you if you are familiar with certain presupposed facts and methods: you have to be acquainted with perpendicular bisectors and with the theorem that they intersect in a single point. Moreover the homework exercises must be within your reach. And you must be able to understand the use of such phrases as 'consider an arbitrary triangle *ABC*,' etc.[3]

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[2] The term *Informal Proof* is here used in a narrow sense, excluding those informal proofs (in a broad sense) that can be classified either as immediate and intuitive or as informal axiomatic proofs (see below). A more explicit, but cumbersome, name for this type of proof would be: *Informal Argumentative Nonaxiomatic Proof*.

[3] Cf. Corcoran: 'A linear chain of reasoning that is cogent for one person need not, and normally will not, be cogent for all other persons.' (1989, 34).

Informal proofs are just arguments. Even so, their authors do not usually announce them as 'arguments,' but as 'proofs.' Why? One reason could be that the author is sincerely convinced that his argument is impeccable, that it should satisfy any member of his intended audience, does not admit of rational objections or reservations, and perhaps fulfills a number of supplementary conditions (see Sect. 3.4). By calling his argument a 'proof,' the author underlines these matters and reminds his audience of the supplementary conditions that an informal mathematical proof should satisfy. In itself there is nothing objectionable to this use of the word 'proof.' But, of course, calling an argument 'sound,' 'correct,' or 'conclusive' does not provide the argument with these meritorious qualities. The self-praise implied in the announcement of an argument as a 'proof' is liable to induce one to overstep the boundaries of sober argument into the realm of propaganda and intimidation. Another message is tagged to the announcement: 'Don't try to find any objections to this, for my argument isn't just an argument, but a proof, meaning a clincher, and if you don't agree, that only goes to show that you failed to grasp the whole thing. . . ' The use of the term 'proof' for these ends is fallacious, it is an *argumentum ad verecundiam*.

Van Eemeren and Grootendorst (1987) reserve the term 'fallacy' for speech acts which hinder in any way the process of conflict resolution in a critical discussion (284). A fallacy violates a rule that has to be observed in a critical discussion. Van Eemeren and Grootendorst (1987) present ten such rules and it is not hard to see which rule is violated by an intimidating use of the term 'proof' to describe one's own argument. The rule formulates what may be called *The Principle of Burden of Proof*:

Rule II: Whoever advances a standpoint is obliged to defend it if asked to do so (285).

This principle applies, not only to the theorem to be proved, but also to the data adduced in the proof and to the methods of proof. For all of these, there is a burden of proof as soon as there is a challenge. Primarily, the principle refers to situations that are explicitly dialogical, for instance, situations in which certain specific parts of an alleged proof are explicitly criticized. But the challenge can also remain implicit. In that case there may, nevertheless, be an identifiable burden of proof. Thus, in a situation where it is reasonable to suppose that a number of addressees will be unable to follow a published argument as it stands, there is, one may say, an implicit challenge: the mathematician is asked (implicitly) to back up her argument. Calling her argument a 'proof,' however, she evades the burden. The move owes its efficacy to the diffidence (*verecundia*) felt when confronted with 'proof' (*argumentum ad probandi verecundiam*).

A strong form of the fallacy would even result in a threat, and thus violate the following dialectical rule formulated by Van Eemeren and Grootendorst (to be called *The Principle of Parrhesia*):

Rule I: Parties must not prevent each other from advancing or casting doubt on standpoints (284).

For instance, our mathematician could claim that she has a full proof to fall back on, safely stored in her desk, implying that any doubt would be futile, and that those of her critics that persist in casting doubt on parts of the proof risk a severe loss of prestige in the near future (*argumentum ad baculum*).[4]

The efficacy of the high-sounding word 'proof' in an *argumentum ad verecundiam* is enhanced by the existence of some rather special, but prestigious, meanings of this term. One of these is the concept of *axiomatic proof* in mathematics. Primarily, an axiomatic proof is a proof within an axiom system, such as (a specified axiomatization of) Euclid's geometry. Working within the confines of a specific axiom system, one has no need to prove the axioms. (Nor is one to prove the system's definitions.) Dependent upon one's epistemology, the axioms are viewed as self-evident, as a matter of choice or convention, or as principles that can be justified from outside the system. Starting from the axioms (and perhaps some basic definitions) the mathematician proves, within the system, one theorem after another. All along, new terms are introduced by definition. In this process, the order of proofs and definitions is crucial. Each proof may fall back only upon axioms, on definitions that were introduced before, and on theorems that were proved before. Definitions may depend upon theorems, but these theorems must precede them.

Proofs within an axiom system often strike one as highly technical, for obviously they utilize much symbolism and quite subtle methods of deduction. Nevertheless these proofs remain *informal*, as long as they are expressed in a language that was never formalized. We shall call such proofs *Informal Axiomatic Proofs*. An informal axiomatic proof is an argument directed at an audience that accepts the axioms and has 'gone through' all the proofs of earlier theorems and the definitions used in them. Such proofs are arguments and are therefore suitable objects of study for a theory of argumentation. The special context in which these arguments are proffered, moreover, makes them especially interesting from an argumentative point of view. There are fallacies and brands of criticism that are peculiar to this context, such as certain forms of the *circulus vitiosus in probando/in definiendo*, the criticism of (perhaps benign) loops in proofs or definitions, and the criticism of inelegance.

3.3 Formal Proof

Formal Proofs are quite different from any kind of informal proof, however technical, in that they presuppose a formalized language. Stipulations that define a formalized language must precede formal proofs formulated within that language. A formalization of the theorem on altitudes and its proof requires a previous specification of a formalized language for geometrical thought. If you wonder about

[4]Another relevant rule would be *The Principle of Pertinence*: '*Rule IV*: A standpoint may be defended only by advancing argumentation relating to that standpoint' (Van Eemeren and Grootendorst, 1987, 286). But this rule applies to the argumentation stage and therefore assumes that our mathematician has already acknowledged the existence of critical doubt and agreed to accept a burden of proof. She could then use another type of *ad verecundiam* to try to discharge this burden, e.g. by reference to her expertise in proof construction. This involves more than merely a claim to have a proof.

how to conceive of such a language, imagine something similar to a language for predicate logic with identity in which there are some fixed predicate letters assigned to the 'primitive notions of geometry' (Px: x is a point; Lx: x is a line; Ixy: x is on y, etc.). Alphabet and syntax of the language must be precisely specified. The sentences of such a language are often called *formulas*. *Axioms*, too, are formulas. *Formal proofs* are sequences or tree diagrams of formulas constructed according to syntactically specified rules of derivation such as the well-known *Rule of Modus Ponens* (Rule of Detachment or Arrow-Elimination Rule). Axioms and rules of derivation taken together define a *formal system*. A formula is a *theorem* of a formal system if and only if there is a formal proof for it within the system. To check whether an alleged proof really is a proof, one does not need to know anything about geometry. Nor does one have to be schooled in the theory of argumentation. A purely syntactical check suffices.

Why have formal proofs? For what purpose? Gottlob Frege (1879) certainly had a use for them. He needed them in his attempt to establish that all of mathematics would be derivable from logical principles (logicism). For in order to show this, he had to derive the (completely evident) principles of arithmetic from (equally evident) logical principles, without falling back, inadvertently, upon the use of arithmetic itself. Therefore it was not sufficient to go by intuitively evident steps, as one would go about it in ordinary informal proofs. Everything, including the rules of derivation, needed to undergo a complete formalization:

> Damit sich hierbei nicht unbemerkt etwas Anschauliches eindrängen könnte, musste Alles auf die Lückenlosigkeit der Schlusskette ankommen [To see to it that in this process no intuitive content might without being noticed insert itself, everything had to depend upon having a chain of deductions without any gaps (transl. EK)] (*x*).

The German mathematician David Hilbert wanted to provide finitary foundations for mathematics. He never explained what, precisely, we are to understand by 'finitary,' but clearly principles and types of reasoning belonging to a very elementary part of arithmetic are meant. Traditionally, mathematics surpasses finitary bounds. For instance, any proof that refers to the set of natural numbers as a completed, actually infinite, totality (and not merely to the sequence of natural numbers as potentially infinite, every natural number n being followed by $n+1$) would count as nonfinitary. The nonfinitary part of mathematics, however, is not isolated from the finitary part, for a nonfinitary proof may very well have a finitary conclusion. The problem is whether such nonfinitary proofs yield reliable results from a finitary point of view. Hilbert wanted to show (by finitary means) that they do, in other words, he wanted to show that the traditional nonfinitary proofs never yield a conclusion that would be incorrect from a finitary point of view.[5] For this end he needed to enter into certain (finitary) mathematical investigations of formal proofs. This started a discipline called *proof theory* or *metamathematics*. Prawitz (1971, 1981) pointed out

[5]Given certain assumptions, this formulation of Hilbert's program is equivalent to the better-known version: to prove arithmetic consistent by finitary means. Cf. Van Dalen (1978, 58 f.) and Prawitz (1981, 235 f.).

that proof theory (taken in a broad sense) does not start with Hilbert (or with Frege) but has been part of logic since Aristotle. His term is *general proof theory*, whereas he uses the term *reductive proof theory* to refer to those studies that are connected with programs like Hilbert's. I hope it will be clear from these remarks that though proof theory is concerned, primarily, with formal proofs, it may, by formalization, yield insights into proof and possibilties of proof that are highly relevant to the study of informal (axiomatic or nonaxiomatic) proof. Formal systems of proof can serve as 'models' for certain techniques of reasoning and arguing. In a sense they give us argumentation-theoretical models (models of argument).

Thus formal proofs can be useful in several respects, but they can never replace informal proofs. For instance, metamathematical proofs themselves (i.e., proofs about formal proofs) are usually informal, and if they are formal they impose a need for a metametamathematics, and so on. In the end, one will find, in each actual case, a level of informality.

Now that we have surveyed a number of types of proof (*Immediate and Intuitive Proofs*, *Informal Proofs*, *Informal Axiomatic Proofs*, *Formal Proofs*) and their uses, it may have become clear that we should drop the idea of an 'absolute' notion of proof. What happens to be a proof for one audience or within the confines of one system, need not be one for some other audience or within some other system. What happens to constitute a proof here and now for a particular audience may not maintain this status forever.[6]

With respect to the theory of argumentation, there are two obvious conclusions:

1. With the exception of *Immediate and Intuitive Proofs* and *Formal Proofs*, every mathematical proof is an argument, and therefore a suitable object of study for a theory of argumentation.[7]
2. *Formal Proof* occurs in systems that can be interpreted as models of reasoning or arguing and of which the theory of argumentation may avail itself.[8]

3.4 The Surplus Value of a Proof

Not every argument is a proof. Setting aside intentions to impress or intimidate, whenever the term 'proof' is applied to an argument, it must be understood to refer to one or more supplementary conditions which the argument is supposed to fulfill.

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[6]Dummett, when discussing the philosophical and semantical aspects of intuitionistic views on implication and proof, points out the possibility that 'mathematics becomes a subject where results are fallible and liable to revision' (1977, 402).

[7]Presumably, similar observations hold for proofs in science and for proofs in court.

[8]If formal systems of proof (e.g., systems for natural deduction) provide models of certain aspects of argument, the same holds *a fortiori* for formal systems of dialogue rules, such as the dialogue games introduced by Lorenzen and Lorenz (1978), in which dialectical interaction is explicitly taken into account. Cf. also Barth and Krabbe (1982), Haas (1984), Krabbe (1985), Stegmüller and Varga von Kibéd (1984).

These conditions may be different in different contexts of use. Consequently, one should take heed if someone utters the word.

Aristotle, for example, requires the premises of a proof (*apodeixis*) to be (i) true, (ii) themselves indemonstrable, (iii) better knowable than the conclusion, and (iv) giving the cause of the conclusion (*Anal. post.*, I.2, 71b17–33; Aristotle, 1976, 31). The conclusion has to be obtained from the premises by deductive argument (syllogism). Consequently, according to Aristotle, arguments that do not comply with any or some of these conditions are not proofs, but those that do comply have several *surplus values*, such as giving the cause of the conclusion, and can thus yield knowledge.

A recent proposal can be found in a paper by John Corcoran (1989). According to Corcoran, the term 'proof' makes 'tacit reference to a participant or to a community of participants' for whom the alleged proof would constitute a proof (22). In other words, what is a proof for one person is not necessarily a proof for some other person.[9] Not every argument is a proof for everyone, or even for anyone. In order to be a proof for someone, an argument should have the right surplus value for that person:

> Critical evaluation of an argumentation to determine whether it is a proof for a given person reduces to two basic issues: are the premises known to be true by the given person? And does the chain of reasoning deduce the conclusion from the premise-set for the given person? (25).

For special purposes, very special conditions may come into play. We already mentioned Frege's need for proofs starting from purely logical principles that lead, without any gaps, to arithmetical conclusions. Dummett (1977) distinguishes 'mere demonstration' (mathematical proof in a broad sense) from 'canonical proof' (mathematical proof in a narrow sense).[10] Informal proofs are mere demonstrations, but in order to define the concept of an informal proof one needs to refer to the concept of a canonical proof. For canonical proofs, there are supplementary conditions. For instance, no canonical proof is to contain a statement that is more complex than the proof's conclusion (Dummett, 1977, 395). The task of an informal proof (demonstration) is to show that a canonical proof exists for its conclusion (392). Hence an informal proof has some surplus value over mere arguments, and a canonical proof has a surplus value again over a mere demonstration.

According to the philosophy of mathematics in Lakatos (1976), proofs are placed near the beginning, rather than at the end, of the 'method of proofs and refutations,' a heuristic pattern of mathematical discovery. As a stage in this pattern, proof is preceded only by the formulation of a problem and a conjecture (Lakatos, 1976, 127). Once a proof has been obtained, the process of discovery proceeds by criticism and analysis of the proof, by correction of the conjecture, and so on. Hence, according to Lakatos, proof is not primarily a matter of argument: a proof serves to open possibilities for criticism and thus to advance the inquiry, but not to convince

[9] Cf. Note 3.

[10] The distinction is important for an intuitionistic explication of the meaning of logical constants.

someone else of the correctness of a theorem. From this point of view, proof may have a surplus value over argument, but so has argument over proof.

These examples may suffice to show that the surplus value of a proof over mere argument can be specified in radically different ways and that each account of this surplus value is closely linked to further philosophical positions taken by its author. Any theorist of argumentation who discusses proof should, therefore, take care to make clear which concept of proof is intended, i.e., what supplementary conditions there are for an argument to count as proof.

3.5 Proof and Implicit Dialogue

In a dialectical theory of argumentation, a monological argument is viewed as an implicit discussion aiming at the resolution of a conflict (explicit or implicit) concerning the acceptability of a point of view.[11] The author defending a point of view in a letter to the editor, for instance, knows or at least assumes that his point of view is not automatically shared by all readers. So he assumes that there is or may be a conflict of opinion about this point of view. To resolve this conflict, the author needs to have a critical discussion with his opponents. But as long as the critics do not actually participate in writing up the letter in a dialogical format, this critical discussion has to remain implicit, whereas the explicit format will be that of monologue. Underlying this monologue, however, there is an implicit critical discussion to which the argumentation analyst refers.

If a proof is to be an argument with a certain surplus value, then obviously one should turn to the underlying discussion to find a good candidate for this surplus value. If the underlying discussion fulfills certain supplementary conditions (over and above compliance with rules that hold for any critical discussion), then the (monological) argument is to be called a (monological) proof, but otherwise it is not. What supplementary conditions would be appropriate?

Along these lines, one may consider the following definition:

Definition 1. A monological argument is a *monological proof* for X if (i) the underlying discussion complies with the rules of a dialectical system that has been accepted by X whereas (ii) X is commited in the strongest sense to the initial concessions in that discussion, and (iii) all possible chains of criticism (i.e., chains of arguments that the Opponent may select) are followed through, and finally (iv) the discussion is won by the Proponent.[12]

[11] Cf. Van Eemeren et al. (1983, 9).

[12] Cf. Barth and Krabbe (1982, Ch. III, esp. III.6, III.8, and III.13) for the idea of a discussion as consisting of several *chains of arguments* and for the concepts of *winning or losing* a chain of arguments or the discussion as a whole. The concepts of a *chain of arguments* and of a *chain of criticism* (Dutch: kritieklijn) were introduced by E. M. Barth.

In this definition, '*X*' may stand either for a person or for a company or community. The first condition merely states that an argument, in order to count as a proof, should be dialectically correct, i.e., not fallacious. It is a necessary, but not a sufficient condition, which implies that *X* is committed to the premises of the argument (the initial concessions of the underlying discussion), at least to the extent that there is no reason for *X* to object to them. The second condition adds that this commitment is to be stronger than a commitment to mere concessions would require: the premises of the argument are to count as *assertions* of *X*'s, implying that *X* carries a burden of proof for them, if they happen to be challenged.[13] The third condition stipulates that the argument, in order to count as a proof, must deal thoroughly with all relevant objections, possible cases, potential counterexamples, etc. The last condition, together with the third, implies that the underlying discussion shows how a Proponent of the conclussion of the argument can always win *vis-à-vis* an Opponent that grants the premises, no matter how this Opponent selects her moves in the discussion. In other words: there is a *winning strategy* for the Proponent, and this strategy is reconstructible from the explicit (monological) argument.[14]

Notice that this definition is thoroughly nonpsychological and nonepistemical. There is no implication that the rules of the dialectical system in question are rationally or epistemically justified, or that *X* knows the premises to be true, or that *X* recognizes a proof for *X* as such. It is possible for another person, *Y*, to know that something actually is a proof for *X*, even though *X* in all sincerity, denies the fact.

3.6 Proof and Explicit Dialogue

The term 'argument' also refers to the argumentation presented by the Proponent (Protagonist) in the argumentation stage of an explicit dialogue.[15] In this case, as well, we may hunt around for some supplementary conditions that would justify us in saying that a certain critical discussion provides a proof for the Opponent (Antagonist). This would give us a concept of proof as a kind of dialogue: a concept of *dialectical proof*. The simplest choice of conditions would be to copy them from Definition 1, with the Opponent as *X*:

Definition 2. The Proponent in a critical discussion provides a *dialectical proof* of his point of view for the Opponent, if (i) the critical discussion complies with the rules of a dialectical system that has been accepted by the Opponent whereas (ii) the Opponent is committed in the strongest sense to the initial concessions in

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[13] For an exposition of different types of commitment in dialogue, see Walton and Krabbe (1995, esp. Sect. 5.4).

[14] Winning strategies are studied in Barth and Krabbe (1982, Ch. V).

[15] For the stages of a discussion, see Van Eemeren et al. (1983, Hstk. 2), Van Eemeren and Grootendorst (1984, 85ff), and Van Eemeren and Grootendorst (1992, 35).

that discussion, and (iii) all possible chains of criticism (i.e., chains of arguments that the Opponent may select) are followed through, and finally (iv) the discussion is won by the Proponent.

The two kinds of proof, monological and dialectical, are straightforwardly linked: a monological argument is a monological proof for X, if and only if the discussion underlying the argument provides a dialectical proof for X, with X in the role of Opponent.

In practice, it will often be very hard to establish that a certain monological argument constitutes proof for a given person X. For one thing, one is to provide a reconstruction of the underlying dialogue. Then, one is to be sure about X's commitments, both with respect to the initial concessions and to the rules of dialogue on which the underlying discussion is based. If too much uncertainty remains on either of these accounts, it will remain unsettled whether the purported proof is a proof for X or not.

In the argumentation stage of some kinds of critical discussion, the Proponent is allowed to present, not only such argumentation as is directly relevant to the Opponent's preceding challenge, but also longer stretches of reasoning that in fact constitute monological arguments, even though they are put forward in a dialogical context. Such monological arguments, presented within an explicit discussion, can themselves be analysed by reference to (another) implicit discussion. Suppose that this implicit discussion happens to provide a dialectical proof for its Opponent. Then the monological argument will be a monological proof for the Opponent of the explicit discussion. This may sound complicated, but it really only goes to show that monological proofs may function also *within* critical discussions. This is yet another example of the ways in which proof ties up with dialogue.

3.7 The Genesis of Proof

If there were no proofs, one would have to invent them. Suppose a certain company of discussants has agreed upon a specific dialectical system S_0 for the regulation of critical discussions among them. Many fruitful and provoking discussions have taken place among the members of this company, all of them regulated by the rules of S_0. Suppose that, after acquiring a certain body of experience with S_0, it has become clear and obvious for the members of this company that whosoever concedes both *if A then B* and *if B then C* can be forced, in a discussion conforming to S_0, to refrain from opposition to the proposition *if A then C*. The company has discovered an important law of dialogical logic that was already implicit in the rules of their agreed dialectical system S_0: whenever the Opponent concedes *if A then B* and *if B then C*, there is a winning strategy for the Proponent of *if A then C*! This company would then be well-advised to skip this type of dialogical fragment and in its stead to adopt a rule of inference:

$$\textit{if } A \textit{ then } B,\ \textit{if } B \textit{ then } C \;\Rightarrow\; \textit{if } A \textit{ then } C.$$

(The rule is applied by substituting sentences for the variables A, B, and C. To the left of the arrow one then finds the premises of the application, and to the right of the arrow its conclusion.) Adoption of this new rule amounts to a change of dialectical system. S_0 is traded for S_1. Within S_1 it is no longer possible to criticize the steps that conform to the rule of inference just stated. (That is to say, one is not allowed to criticize the relative conclusiveness of the premises for the conclusion in this step, whereas it is of course still possible to criticize the tenability of the premises themselves.) The company has discovered and adopted the rule of the Hypothetical Syllogism!

In the same way this company may go on, discovering ever more laws of dialogical logic relating to the existence of winning strategies in certain situations, and extending its dialogical system, by adding ever more rules of inference so as to obtain the systems $S_2, S_3, \ldots$ Finally, the company winds up with a dialectical system in which a whole system of rules of inference has been incorporated.

Given such a system one may introduce a somewhat special notion of proof:

Definition 3. Let D be a discussion, according to the rules of S between two parties X and Y. Let A be an argument, advanced by Y in the course of this discussion D, in order to convince X of a conclusion C. At moment t, argument A counts as a proof of C in D if and only if the following two conditions hold:

1. Each ultimate premise of A was asserted by X, before t, and not withdrawn by X before t.[16]
2. Each separate argumentative step of A is explicitly sanctioned by one of the inference rules incorporated in S.

Let us suppose that S allows of more ways of arguing than those sanctioned by the incorporated body of rules of inference. And let us further assume that, although every assertion in dialogue counts as a concession, not every commitment to a concession carries a burden of proof which would make it equal to an assertion.[17] Then Definition 3 shows how a meaningful distinction can be introduced between, on the one hand, a merely successful and nonfallacious argument in dialogue, and, on the other hand, a proof in dialogue. Proof is seen to have a surplus value, both with respect to its ultimate premises and with respect to the separate steps of which it consists. Further, it is seen how proof ties up with rules of inference. But the roots of these rules are again dialectical. This concept of proof is, moreover, fully externalized and independent of the concepts of 'knowledge' and 'truth'.[18]

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[16] As was pointed out in Section 3.5, this implies that X has a burden of proof for these premises (if challenged, and if X has not discharged this burden before).

[17] See Note 13.

[18] On 'externalization', see Barth and Krabbe (1982, 32–33, 60).

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