Reconceptualizing Early Mathematics Learning: The Fundamental Role of Pattern and Structure

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In chapter *Early Awareness of Mathematical Pattern and Structure*, we introduced the Pattern and Structure Project, which focused broadly on the development of patterning and structural development among 4 to 8 year olds in this project. Our research has aimed to find reliable and consistent methods for measuring and describing the growth of students' structural development in mathematics. We provided a rationale for the construct, *Awareness of Mathematical Pattern and Structure* (AMPS), which our studies have shown generalizes across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding (Mulligan and Mitchelmore 2009). Our belief is that the development of AMPS can bring more coherence to mathematical development but this needs the support of an innovative pedagogical approach and framework.

The challenge was to identify core features of AMPS and to design pedagogy that explicitly improves students' awareness of pattern and structure. To that end, the *Pattern and Structure Mathematics Awareness Program* (PASMAP) was developed concurrently with the studies of AMPS and the development of the Pattern and Structure Assessment (PASA) interview. The culmination was a large-scale two-year longitudinal study, *Reconceptualizing Early Mathematics Learning* (REML), which was the inspiration for this volume.

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In this chapter, we summarize some early classroom-based teaching studies and describe the PASMAP that resulted. We then outline the REML project and discuss the consequences for our view of early mathematics learning.

Classroom-Based PASMAP Studies

PASMAP had its origin in a year-long numeracy initiative where our pattern and structure approach was trialed in a New South Wales metropolitan state school experiencing disadvantage and low achievement in numeracy. The project aimed to develop in students an Awareness of Mathematical Pattern and Structure (AMPS) based on structural aspects of mathematical development that had been identified in previous studies. A research team worked for a year with 27 primary teachers from Kindergarten to Grade 6 (683 students in total) to scaffold learning with small groups of students within regular classroom time (Mulligan et al. 2006). Priority was given to the professional learning and support of six lead teachers from Kindergarten to Grade 3.

Every teacher administered a PASA interview to all of their students and the results were then used to allocate students to small groups for instruction. The PASA data comprised PASA total scores, and students' strategies and drawn representations of solution processes. Data were summarized by the researchers for common response patterns for individual students, teachers and grade levels. A framework for developing and implementing a structural approach to learning mathematics was then developed by the research team in collaboration with participating teachers. The number system, counting patterns, multiplication and division, partitioning, and fractions comprised the main focus.

Several professional development meetings supported the planning and implementation of the PASA and PASMAP, assisted by input from the school's learning support and mentoring teams. PASMAP was implemented across the school for two consecutive terms, each of ten weeks duration. Teachers integrated PASMAP learning experiences into their regular mathematics program to varying extents, depending on the needs of the students and the support available. To assess progress, the PASA interviews were repeated at the end of the intervention.

The results showed a marked improvement in correct responses and an increased proportion of responses classified at the more advanced partial and structural levels of development.[1] The improvement was most marked in the Kindergarten and Grades 1 and 2 where the most intensive support had been focused. Figures 1 and 2 summarize the PASA data for Kindergarten and Grade 1 students.

Substantial improvements were also found in school-based and system-wide measures of numeracy achievement (NSW Department of Education and Training 2002), although they were less pronounced in the upper primary years. For example, on the Schedule of Early Number Assessment (SENA 1) 89 % of students were

[1] See chapter *Early Awareness of Mathematical Pattern and Structure*, page 12 for an explanation of the various structural levels.

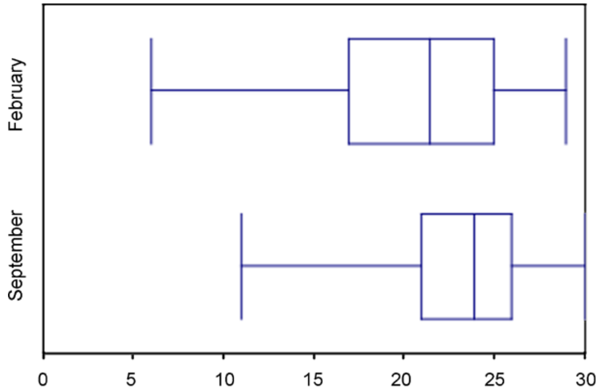

Fig. 1 Box and whisker plot of Kindergarten and Grade 1 students' (7 × Early Stage 1 classes) pre-(February) and post-assessment (September) PASA scores ($n = 134$)

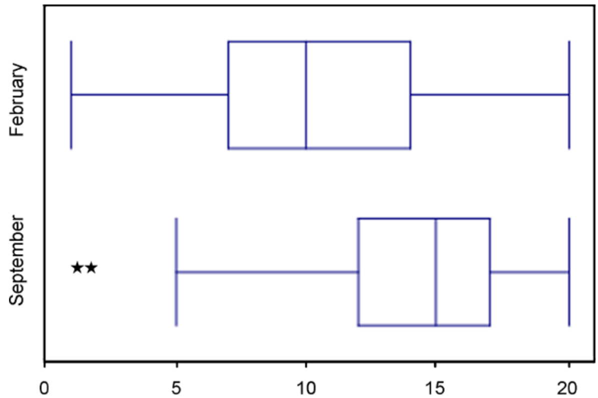

Fig. 2 Box and whisker plot of Grades 1 and 2 (6 × Stage 1 classes) students' pre-(February) and post-assessment (September) PASA scores ($n = 120$)

categorized at the first three levels of counting and arithmetic knowledge (emergent, perceptual and figurative counting) at pre-test; the post-test proportion of students at these lower levels was only 56 %. Similarly, on the NSW Basic Skills Testing Program Numeracy trend data, 23 % of Grade 3 students increased numeracy scores from Levels 1 and 2 to Levels 3 or 4 but a smaller proportion at Grade 5 (16 %) showed such an increase. The marked improvements shown in the SENA data were achieved mainly in the Kindergarten and Grades 1 through 3, possibly because the lead teachers were most consistent and given considerable support in comparison with the upper grades.

This classroom-based work allowed PASMAP to be trialled with a large number of students who struggled to achieve basic numeracy. Adopting the structural approach encouraged teachers and students to recognize similarities and differences in mathematical representations and to form simple generalizations. A focus on multiplicative concepts (including understanding the base ten system, grouping, and partitioning) was found integral to building structural relationships in early mathematics and spatial structuring was necessary to visualize and organize these structures. Teachers were therefore encouraged to focus students' attention more explicitly on spatial structuring in the development of number concepts including for example, the use of number patterns and the construction of base-ten knowledge. Teachers found this approach a novel way of teaching compared to the traditional focus on number concepts and skills in isolation.

Despite the promising results, a pilot project of this scale had many limitations and it was not possible to generalize the findings to other settings. Many teachers struggled to understand the goal of developing mathematical relationships and simple generalizations. However, the evaluation data obtained from the teachers were invaluable in informing the subsequent development and expansion of PASMAP.

Preschoolers' Patterning

A concurrent study by Marina Papic (Papic et al. 2011) was conducted in the prior-to-school context in the belief that the early development of patterning could provide a foundation for successful mathematical development. Papic found that preschoolers' awareness of pattern could be reliably assessed and that development could be scaffolded through a framework of patterning experiences. Since Papic's study is described in more detail in chapter *The Role of Picture Books in Young Children's Mathematics Learning*, we summarize some key findings.

In one pre-school, Papic worked with the teachers to develop a 6-month intervention focusing on mathematical patterns. A matched preschool acted as a comparison group. Individual task-based interviews were conducted before and after the intervention, and the children from both groups were followed up into their first year of formal schooling. Children in the intervention program showed more advanced patterning skills than the non-intervention sample at the end of the pre-school year. Compared to the non-intervention group, the intervention children created far more complicated patterns, they were able to solve growing pattern tasks (which had not been the included in the intervention), and they showed higher scores on a standard numeracy assessment, the SENA (Department of Education and Training 2002). It was also found that teachers in the intervention pre-school had spontaneously amended their whole cross-curricula activities to take advantage of many more patterning opportunities than had been included in their original curriculum.

In the patterning program, children were frequently exposed to the concept of a unit of repeat as configurations were broken down into identical "chunks". They also engaged in skip counting (e.g., "2, 4, 6") that promoted the language of multiplication (e.g., "3 times"). The patterning experiences may have promoted conceptual understanding of the idea of composite unit that is fundamental to multiplicative reasoning. This development may have made it easier for the children to use base-ten structure and other multiplicative concepts more effectively in Kindergarten. We suggest that the patterning program had in effect strengthened the preschoolers' AMPS—not only in terms of their understanding of fundamental concepts but, perhaps more importantly, in encouraging them to look for and analyze patterns. One result was a level of understanding that readily transferred to more complex patterning and counting tasks one year later.

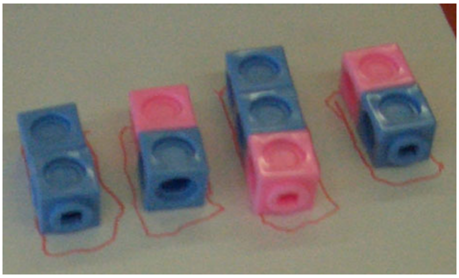

Fig. 3 Kindergarten child's incorrect attempt to make an AAB pattern in chunks

An Intervention Study with Kindergarten Students

Inspired by the promising results in the study with preschoolers, PASMAP was further developed as a connected set of instructional sequences that integrated patterning (repetitions and growing patterns) and functional thinking, units of space and measurement, spatial structuring and number sense, skip counting and multiplicative processes. Using a design study approach, Mulligan and colleagues explored the impact of PASMAP on mathematics learning with a group of ten students aged 4 to 6 years in the first year of formal schooling (Kindergarten in the state of NSW), who had been identified by teachers as needing additional support in numeracy (Mulligan et al. 2008). A specially trained, experienced classroom teacher engaged the students in PASMAP tasks over 15 weekly teaching episodes. Tasks were designed and modified continuously, and differentiated for individuals. Students were assessed pre- and post-intervention using a revised PASA interview and two sub-tests of the Woodcock-Johnson mathematics test (Woodcock et al. 2001).

Every student showed improvement on PASA scores, with seven of the ten making marked improvements (Mulligan 2011). There were however no significant gains found on the Woodcock-Johnson test scores; possibly the limited 15-week period did not allow sufficient time to show such growth. An alternative explanation is that the test was not sensitive enough in scope or depth to detect conceptual problems related to mathematical pattern and structure. Advancement in structural development was clearly evident in students' solution strategies, their representations, and their explanations of their responses. There was evidence that students invented symbolizations and made emergent generalizations and marked growth in representing, symbolizing and translating simple and complex repetitions, structuring arrays and grids and unitizing area. However, these improvements were not necessarily consistent across tasks. It was concluded that PASMAP would have to be implemented over a longer period if it was to have a measurable effect on mathematical achievement.

Consistent with the work of Papic, students represented simple repetitions and growing patterns in a variety of forms. We explicitly focused on "chunking" (breaking the unit of repeat into sections) and placing in the pattern sequence (see Figs. 3, 4, and 5).

Figure 5 shows a student's drawing of an AAB repetition that they have made as a tower with two different colored blocks. The student symbolizes this pattern as a

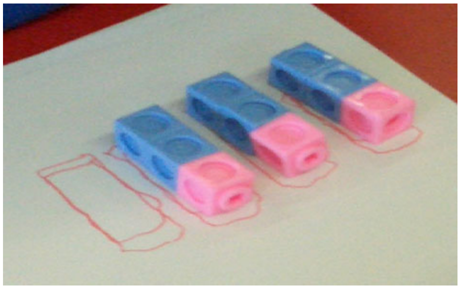

Fig. 4 Kindergarten child's correct attempt to break an AAB pattern of blocks into chunks

Fig. 5 Kindergarten student's representation of BBA repetition using two different symbolizations

'BBA' repetition and writes the correct sequence to the left hand side of the drawing. When asked if they could write this pattern in another way so that their friend could make the same pattern the student uses symbols 0 and X. They explain that these are the symbols you use when playing noughts and crosses but when making this pattern its '00X' repeated. The student has retained the initial pattern structure and developed a correct but different symbolization of the unit of repeat.

Similarly, improvements in recognition of subitizing patterns, counting in multiples in 2s, 3s and 4s and some partitive grouping strategies were also observed. This improvement could be explained by the varied and repeated PASMAP experiences in grouping and patterning using a unit of repeat. The development of spatial structuring through individuals' representations was encouraged, such as congruence of shapes, partitioning, and collinearity (see Figs. 6, 7 and 8).

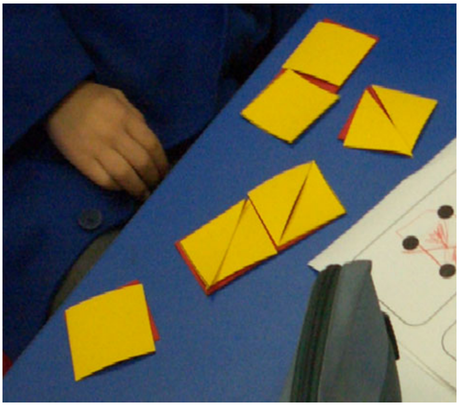

Fig. 6 Child correctly aligns squares and triangles congruently on square and rectangular cutout shapes placed underneath

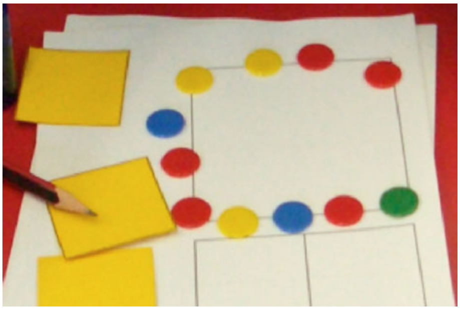

Fig. 7 Child places counters randomly around the border without noticing corners or sides of equal length [The smaller squares are later used to structure the square into quarters]

In Fig. 6 the child is required to place congruent triangles on single squares or rectangles so that the shapes are aligned. Recognizing corners and/or the symmetry of the triangles was observed.

In Fig. 7 the task required the child to place counters evenly spaced and aligned around the border of a square.

In Fig. 8 the task required the child to place counters on each of the corners; observation of the process by which the child noticed the corners and placed the counters, albeit unsystematically was observed.

Another important observation was that students were initially unable to represent simple arrays and grids beyond a pattern of four units but by the end of the program students could more readily represent the structure of rectangular grids and arrays Students' construction of a simple table of data showing functional thinking was also demonstrated in the final teaching episodes (Fig. 9). The students explained that "for each dog you have 4 legs, so its 4, 8, 12, 16, 20, . . . , for 5 dogs".

This project illustrated the rich and diverse learning experiences by ten young students in a program focused on structural awareness. However, the intervention was limited to a small group of students withdrawn for individualized instruction, and supported by specialist teachers and well-formulated resources, and we could not assume that the success of this program could be generalized to other settings. Nevertheless, our data did suggest that explicit assessment and teaching of struc-

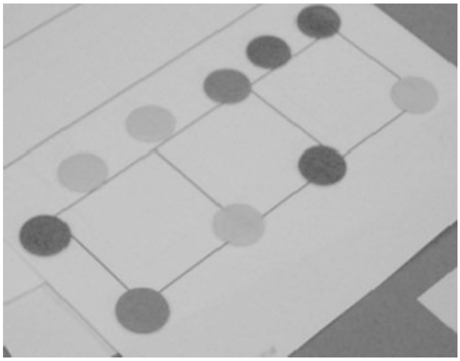

Fig. 8 Child places counters randomly on corner positions first but then attempts to fill the spaces in the border

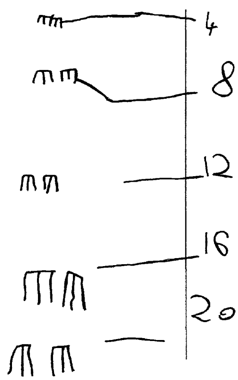

Fig. 9 Kindergarten student's attempt to represent the number of legs on a dog (4 legs) on increasing number of dogs

ture had the potential to effectively improve students' abstraction of mathematical processes.

Summary of Early Research Findings

The early studies suggested or confirmed several mechanisms whereby a focus on pattern and structure awareness promotes general mathematical development:

- Students become more aware of crucial structures such as rectangular arrays.
- Through the study of these structures, they more easily learn basic properties of number, space, and measurement.
- Students learn to break down unfamiliar large patterns into smaller patterns that they are familiar with—a process known as *unitizing* (Lamon 1996).
- The emphasis on reasoned comparisons, justifications, and generalizations focuses students' attention away from non-mathematical pattern features and develops a tendency to look for and explain patterns in new experiences.
- Identifying similarities and differences leads to abstraction and generalizations.

The early classroom-based studies discussed above provided strong support to the hypothesis that teaching young children about pattern and structure should lead to a general improvement in the quality of their mathematical understanding. However, none of the studies had a sufficiently large or representative sample, most lacked a comparison group and there was insufficient opportunity to track and describe in depth and the growth of structural development. A more comprehensive study was needed to evaluate these findings more systematically over a longer period of time within the regular school setting.

The *Reconceptualizing Early Mathematics Learning* Project

A new large study, the *Reconceptualizing Early Mathematics Learning (REML)*, was therefore designed to evaluate the effects of PASMAP on student mathematical development in the first year of formal schooling. The aims of the study were to:

- Evaluate the effectiveness of a school-entry PASMAP on student mathematical development and using classroom observations, interview-based student assessment and standardized assessment.
- Document in detail the impact of PASMAP on learning mathematics.
- Track and describe students' structural growth, particularly of high- and low-achievers, through fine-grained analysis of the growth of structural awareness.

PASMAP was also evaluated in terms of professional learning of teachers as they were supported in developing and evaluating the new approach.

The Sample

A purposive sample of four large primary schools, two in Sydney and two in Brisbane, Australia, comprising 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. At the follow-up assessment in September 2010, 303 students were retained. From pre-test data two focus groups of five students in each class were selected from the upper and lower quartiles, respectively. These 190 students were monitored closely by the teacher and research team throughout the study.

Procedure

Two different mathematics programs were implemented: In each school, two Kindergarten teachers implemented the PASMAP and two implemented their regular program. A researcher visited each teacher on a weekly basis and equivalent professional development was provided for all teachers. The PASMAP framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. Features of PASMAP were introduced by the research team incrementally, at approximately the same pace for each teacher, over three school terms (May–December 2009). However, implementation time varied considerably between classes and schools, ranging from one 40-minute lesson per week to more than five 1-hour lessons per week.

The PASMAP Components

Core components of the PASMAP and the pedagogical approach focused explicitly on the development of students' spatial structuring, multiplicative reasoning, and emergent generalizations rather than developing procedural skills or number concepts in isolation from other concepts. PASMAP provided an instructional approach where concepts were scaffolded and linked together to promote early algebraic thinking based on earlier approaches advocated by Blanton and Kaput (2005) and Carraher et al. (2006). Although the framework was in its developmental stage, these components could be described as potential trajectories of learning in a similar way to those described by Clements and Sarama (2009). Drawing on previous and current research on spatial structuring and early algebra, the PASMAP program comprised subitizing and spatial arrangements (Bobis 1996; Hunting 2003); simple and complex repetitions, growing patterns and functions (Warren and Cooper 2008); spatial structuring (Battista 1999; van Nes and de Lange 2007); the spatial properties of collinearity, congruence and similarity and transformation; the structure of measurement units and data representation, unitizing and multiplicative structure; the structure of counting sequences and base ten, and equivalence and inverse operations. Emphasis was also laid on the development of visual memory and justification for simple generalizations. Students were encouraged to seek out and represent pattern and structure across different concepts and transfer this awareness to other concepts.

This awareness was achieved through pattern-eliciting tasks that required students to use spatial structuring to copy or reproduce a model or other representations. (For examples see later sections in this chapter.) The teacher used probing questions to highlight important features of their models and drawings, to compare them with others, and to focus their attention on similarities and differences in crucial aspects of spatial and numerical structure. Tasks were modified and repeated regularly, reinforcing and extending generalizations and providing links to prior learning in a similar way to earlier studies.

Assessment Interviews and Classroom Data

All students were administered the *I Can Do Maths* (ICDM) standardized test of general mathematics achievement (Doig and de Lemos 2000) at the beginning and end of the 2009 school year and again in mid-2010. From the pre-test data, two focus groups were selected in each class consisting of five students from the upper and lower quartiles, respectively. These students were interviewed in more detail using the PASA in February 2009, December 2009, and September 2010, the number of students varying from 190 to 170. An additional "extension" version of PASA was also administered in September 2010. The PASA items were parallel on all three occasions, but increased somewhat in complexity to take account of students' development.

Other evaluation data included video for a sample of PASMAP lessons for evidence of AMPS and students' articulation of emergent generalizations. Analysis focused on the high ability and low ability focus students. Students' explanations and drawn representations, and photos of their responses to tasks were collected during the implementation of PASMAP and were coded immediately after each lesson for level of structural development. Evidence of student work, usually in the form of worksheets, was also collected for focus students in the regular classrooms. This evidence was digitally scanned and placed in individual profiles of learning. As well, teachers' views of the impact of the program on student learning and their own professional learning was collected and later analyzed.

Results

Quantitative Outcome Analysis

Analysis of the various PASA and ICDM scores showed the expected differences between ability levels and confirmed the equivalence of the two program groups. There was, however, a significant difference between the schools, with classes in the two Brisbane schools scoring lower than those in the two Sydney schools. No significant interactions were observed.

Total scores on the PASA and ICDM administered at the end of the intervention (December 2009) and at the retention point (September 2010) among the focus students were analyzed using analysis of covariance (ANCOVA). In each case, the covariates were the initial PASA and ICDM scores and the factors were school (one of four), ability (high vs. low) and program (PASMAP vs. non-PASMAP).

Analysis of the ICDM scores indicated no significant interactions or main effects apart from a school effect. In other words, the PASMAP and regular students made very similar gains on ICDM over the period of the study, but Sydney students gained more.

The analysis of the PASA scores also showed no significant interactions. However, there were two significant main effects at each point: a difference between schools, with the Sydney classes showing higher adjusted means than the Brisbane classes, and a difference between the program groups on each PASA assessment—modest at the end of the intervention ($p < 0.026$), highly significant at the retention point ($p < 0.002$), but only borderline ($p > 0.11$) for the extension section of the

Table 1 Analysis of covariance of PASA scores at retention point

Source	Type III sum of squares	df	Mean square	F	Sig.
Corrected model	1048.432[a]	17	61.672	10.380	0.000
Intercept	53.229	1	53.229	8.959	0.003
Covariate: PASA	158.346	1	158.346	26.650	0.000
Covariate: ICDM	14.071	1	14.071	2.368	0.126
School	117.125	3	39.042	6.571	0.000
Ability	15.259	1	15.259	2.568	0.111
Treatment	61.653	1	61.653	10.376	0.002
School * Ability	11.643	3	3.881	0.653	0.582
School * Treatment	43.663	3	14.554	2.450	0.066
Ability * Treatment	0.217	1	0.217	0.037	0.849
School * Ability * Treatment	13.589	3	4.530	0.762	0.517
Error	802.130	135	5.942		
Total	13412.000	153			
Corrected total	1850.562	152			

R squared = 0.567 (adjusted R squared = 0.512)

PASA. On each occasion, the PASMAP group scored higher than the regular group. Table 1 provides a summary the ANCOVA for the PASA at the retention point.

We inferred that the PASMAP treatment was effective in promoting the conceptual understanding of early mathematics, as measured by the PASA but not in improving mathematical achievement as measured by ICDM.

Rasch Scale Analysis

The PASA total scores and the ICDM scores were used to construct a single Rasch scale that incorporated all items along a continuum. The main advantage of using Rasch analysis for constructing the PASA scale was that it could be used to link different versions of the PASA used in this study (Andrich et al. 2001). The item map indicated that the PASA items and the students were reasonably well matched; in comparison, the ICDM items at the lower end of the scale did not sufficiently challenge the majority of students, although some more difficult ICDM items filled a gap between the PASA items (see Mulligan et al. 2011). The scale's order of item difficulty on PASA items provided a measure of the students' overall level of AMPS. Thus a conceptual analysis of the item and its position on the scale reflected the complexity of the task in terms of pattern and structure as well as the reasoning required to complete it successfully. What we aimed to achieve with the scale was

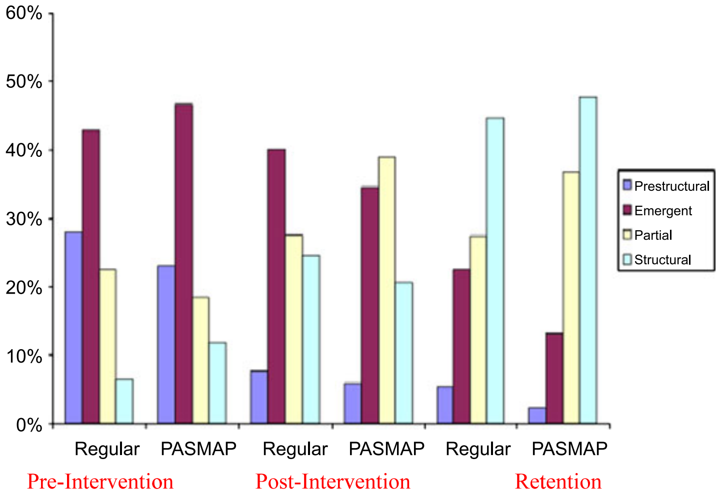

Fig. 10 Structural development across selected PASA items at three interview points Feb 2009 (Pre-intervention), Dec 2009 (Post-intervention), Sept 2010 (Retention) in two Sydney schools

a picture of how the PASA measure of AMPS fitted with a standardized measure of general numeracy ability over time.

Structural Outcomes Analysis

To supplement the quantitative analysis, we provide some examples of the analysis of structural levels. Student responses on four PASA items requiring a drawn response at the three administrations were systematically coded for level of structural development (see chapter *Early Awareness of Mathematical Pattern and Structure*). Coding showed an inter-rater reliability of 0.91. Figure 10 summarizes the results for the Sydney students. It can be seen that the PASMAP students were initially slightly more advanced than the regular program students, with about 5 % more students in the partial structure and structural levels than the regular students. However, this difference grew in the subsequent administrations, reaching about 20 % at the retention point.

The following examples show how PASMAP learning experiences led to a deeper structural understanding of mathematical concepts and encouraged the development of emergent generalizations.

After a sequence of tasks focused on repetition and spatial patterns (Papic et al. 2011), there was a focus on constructing and analyzing simple grids. In the first of these, students were shown a 2×1 grid for a few seconds and then asked to draw it. The teacher then gave them a 2×1 grid and two matching squares and asked how many squares were needed to cover the grid. Different strategies for placing the squares were discussed, and students were also asked to fold the grid to explore

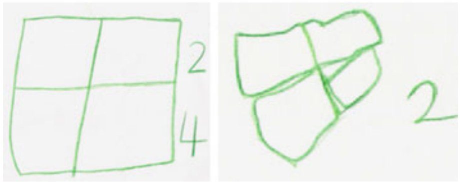

Fig. 11 Two contrasting representations of a 2 × 2 grid

the structure. The teacher then asked, "What's the same?" and "What's different?" and students encountered ideas such as counting, shape, sides and vertices, rotation (turning), congruence (same size and shape), and fractions (half). The grid and squares were then removed and students drew the grid from memory in both horizontal and vertical orientations. After sharing and discussing their drawings, the class summarized what they had learnt and looked for links to their earlier tasks (e.g., in the towers they had made from unifix cubes). This may have been a very elementary task, but it was fundamental to developing spatial structure and many students found it quite challenging.

The next lesson moved on to 2 × 2 grids (called "windows"), following a similar procedure. Previous ideas were reviewed and extended, and further ideas of rows and columns, clockwise and anticlockwise, vertical and horizontal, diagonals, and even quarters were encountered. The difference between the high- and low-ability students already became apparent, and student responses indicated to the teacher how perceptive some students were in terms of recognizing structural features while others paid little or no attention to mathematical features. Figure 11 shows two such contrasting drawings. Heela[2] had already recognized that she did not need to draw separate squares, whereas Lateh struggled to draw congruent squares in the standard orientation.

In subsequent lessons, the task was extended to larger rectangles. By repeatedly looking at what is the same and what is different between a given grid and their drawings, and by seeking generalizations from their observations, students gradually learned that a grid can be drawn using equally spaced, perpendicular lines. Each task reinforced the basic generalization that we call the 'spatial structure' of the grid. Discussion of similarities and differences between student's drawings highlighted the crucial fact that a square grid contains the same number of equally sized rows and columns; further, the development of multiplication and commutativity emerged as well as area measurement. These ideas were further developed through a sequence of tasks focused on the pattern of squared numbers using square tiles and grid cards.

Students were initially given small plastic squares and asked to use them to make as many large squares as possible, in order of size, and to say how many small squares were in each larger square. To explore the structure of the pattern of squares students were given two sets of square grid cards (1 × 1, 2 × 2, 3 × 3, 4 × 4 and 5 × 5). After exploring systematic ways in which they could be fitted next to or

[2]Pseudonyms are used to preserve anonymity.

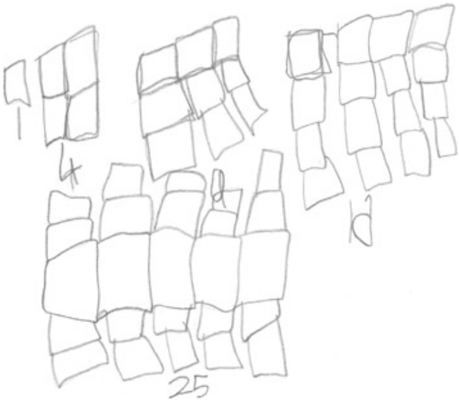

Fig. 12 'Low ability' Kindergarten student's drawing of emergent structure of the pattern of squares from memory

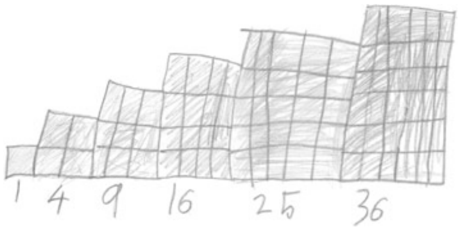

Fig. 13 'High ability' Kindergarten student's drawing of the pattern of squares from memory

on top of each other, or in various formations or sequence, the teacher posed the questions, "Can you see a pattern? How many small squares are there on each card? What is the best way to find out?" Students then cut up a second set of grid cards into rows or columns, place the cut-outs on top of the first set of cards, and discussed the numbers of rows or columns and the number of small squares in each. After examining the resulting number pattern (1, 4, 9, 16, 25), the teacher removed all the grid cards and cut-outs and challenged the students to reproduce the visual pattern from memory, first on grid paper and then on plain paper.

Figures 12 shows attempts by a 'low ability' student to draw the pattern from memory but the partial structure of the grid was counted and added as individual units. The student does, however, recognize the growing pattern of squares. In this case the student is assisted to use grid paper to form the squares in a sequence and to trace the rows and columns so as to develop collinearity. Figure 13 shows a 'high ability' student's structural development of the pattern of increasingly larger squares using the alignment of the growing squares. He visualizes and generalizes the pattern as "it goes up by one row and one column every time and it must be a square"; he also explains the numerical sequence as multiplicative "1, 2 by 2, 3 by 3, 4 by 4". His learning is extended by tasks such as "Can you work out what the tenth square will look like and can you continue the pattern? Can you make a growing pattern using triangles?"

In a follow-up task, students were given a 1×1 square and a 2×2 square and asked how many small squares fit on to the larger one. They were then given further 2×2 squares and asked to find the number of small squares in total, thus

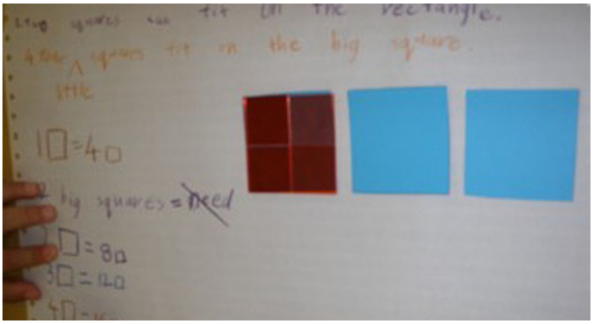

Fig. 14 Heela's use of the composite unit of 4 squares as a functional relationship

constructing the sequence 4, 8, 12, ... Finally, they were asked to generalize their findings. Heela invented a perfectly good means of symbolizing her results that closely resembles algebraic notation (see Fig. 14). In fact, she was treating the task as a functional relationship rather than a simple pattern continuation. Asked what she had learnt from the exercise, she said "I made a pattern so 1 big square is 4 little squares. So it's 4 for each square. Every time you use the square it's a four." Further tasks showed that she had generalized the relationship to all sizes of square and, indeed, any type of rectangle.

Other tasks extended the basic (multiplicative) generalization to rectangles. For example, students were asked to relate the number of unit squares needed to cover a rectangle to the size of the unit.

During the PASMAP intervention students demonstrated development of AMPS throughout the learning episodes in their representations and explanations. There were particular gains found in the PASA items requiring extension of a growing pattern and use of ten as a composite unit. More advanced responses were found in the related areas of simple and complex repetitions, growing patterns, multiplicative thinking (skip counting, partitioning and fractions), equivalence, and structuring area and drawing of grids with collinearity. Students in the regular program did not focus on growing patterns, multiplicative ideas or structuring measurement so their responses were limited to simple repetitions, unitary counting and additive thinking, and conservation of area. One of the most promising findings was that the PASMAP focus students categorized as low ability were able to develop structural responses over a relatively short period of time. The same gains were not evident for the regular group.

Discussion

The PASMAP students (and participating teachers) were systematically guided through related teaching/learning experiences so that deep connections between concepts were formed. This was in contrast to the regular program students where the pedagogy changed focus, sometimes on a daily basis, from one concept to another without opportunity for development of structural understanding and without focusing on the relationships between concepts.

Qualitative analyses of students' profiles and the classroom observation data showed stark differences in the way that the PASMAP students developed mathematical concepts and reasoning skills. PASMAP explicitly focused on the promotion of students' awareness of pattern and structure: the analysis of students' learning showed that all the PASMAP students developed AMPS to varying extents and greater gains were made than for the regular students.

Because PASMAP focused intently on developing structural relationships and spatial structuring from the outset, the PASMAP students made direct connections between numerical, measurement and spatial mathematical ideas, and formed emergent generalizations such as those described in the previous section. For example, students began to link simple skip counting to more complex multiples and arrays through their experience of the unit of repeat in patterning and measurement contexts. The most able PASMAP students used particular spatial features of pattern and structure to build more complex ideas. For example, they partitioned a 10×10 square into quarters and recognized that each of these squares formed a 5×5 array, and knew that this quarter contained 25 squares from their experience of the growing pattern of squares. Regular students could also solve tasks requiring multiplicative thinking but these were considered by the students as separate mathematical skills; for example, they learnt the skip counting pattern of 5s in isolation from all other activities. These students could not explain what was similar or different, what was the connection between ideas, or form simple generalizations.

A small proportion of students in the regular program did produce structural responses in the post-intervention PASA interview although they had apparently not been given opportunities to describe or explain their thinking in class. It would seem therefore that more advanced students may develop AMPS regardless of the instruction they receive. However, our results are indicative that such students are likely to make greater progress in a program that encourages them to look for patterns and explain their structure.

We must interpret our findings in light of one possible confounding factor: the amount of time that individual PASMAP teachers devoted to the program implementation. Some PASMAP teachers completed only half of the program components while others completed almost the entire program and revisited concepts regularly. Thus, further analysis of the impact of PASMAP must consider individual teacher effect, at least in terms of time on task, in order to evaluate the program's full impact on developing AMPS.

Conclusions and Implications for Further Research and Teaching

The study produced a valid and reliable interview-based measure and scale of mathematical pattern and structure that revealed new insights into students' mathematical capabilities at school entry. The PASA interview data indicated significant differences between groups in students' levels of structural development (AMPS) at the second and third assessments. Students participating in the PASMAP program showed higher levels of AMPS than for the regular group, made connections between mathematical ideas and processes, and formed emergent generalizations.

There were no significant differences found between groups on the standardized measure, ICDM. There are two possible reasons for this. The ICDM assessed numeracy in a limited way using traditional multiple-choice paper and pencil format and was quite different to the PASA interview. Secondly, the content of ICDM was limited in scope and depth: these multiple choice tasks focused on unitary counting sequences, recognizing simple two-dimensional and three-dimensional shapes and informal units of measure. There were no items that assessed pattern and structure.

Our studies show encouraging results, but further longitudinal research is needed with larger samples and more diverse samples, as well as utilizing digital learning tools. In particular, research is needed to determine whether an explicit focus on pattern and structure could later promote robust algebraic development—for example, in functional thinking—as well as in other related areas of learning.

A successive longitudinal project *Transforming Children's Mathematical and Scientific Development*[3] *2011–2013*, is in progress which extends the initial study and employs the same research team with some students tracked through from the 2009–2010 study reported in this chapter. This new project explores the role of pattern and structure in mathematics and science learning in Grades 1 to 3. In particular, the role of AMPS in structuring data is being investigated. Students are engaged in an innovative program, usually withdrawn in small groups and taught by the research team in collaboration with the teacher on a weekly basis for a 2-year period. This research integrates English's research on data modeling with the study of pattern and structure (English 2012). As a result, it will be possible to describe the structural development of young children's mathematical and scientific thinking extended to a wider range of concepts than previously studied.

Related studies at Macquarie University have also investigated structural development in studies of preschoolers' use of virtual manipulatives and dynamic interactive software in constructing patterns (Highfield and Mulligan 2007). A recent design study describes the use of programmable robotic toys in terms of young children's representational structure of the dynamic pathways constructed in problem-solving tasks (Highfield and Mulligan 2009). Further, Goodwin studied the effect of digital media on young children's representations of fractions (Goodwin 2009). These studies suggest further possibilities for exploring early mathematics learning through digital technologies [see chapter *A Framework for Examining Technologies and Early Mathematics Learning*, this volume]. We question the impact of such technologies on children's developing AMPS.

Further research on the developmental precursors of AMPS is needed to determine why some children develop powerful mathematical structures and relationships in the prior to school years, while others may be impeded by idiosyncratic imagery throughout their early schooling. Further studies need to articulate the learning trajectories of very young children whose structural development is enhanced

[3] Australian Research Council Discovery Project DP110103586 (2011–2013).

by the PASMAP approach. There are many other factors that need investigating, for example, the impact of different early child rearing practices, approaches to learning in early childhood and early schooling, and possible cognitive-neuroscientific aspects—an emerging field of research in relation to mathematics learning (van Nes and de Lange 2007).

Teaching and learning mathematics through a pattern and structure approach may require fundamental changes to the way that mathematics learning, pedagogy, curriculum and assessment is conceptualized, structured, and implemented. The PASMAP approach promotes conceptual knowledge that is interrelated and pedagogical strategies that scaffold these interrelationships. Supporting teachers to implement a structural approach may require professional learning support to promote deeper understanding of key mathematical concepts and to develop increased teacher pedagogical content knowledge. The importance of pattern and structure in mathematics learning is reflected to some extent in the new *Australian Curriculum–Mathematics* under the Proficiencies (Understanding, Fluency, Problem Solving and Reasoning), which support mathematics learning as patterns, relationships and generalizations (ACARA 2012). However, the key interrelationships between concepts incorporated across the three stands of the *Australian Curriculum–Mathematics* are not foregrounded. A structural approach could support the development of deep conceptual understanding well beyond early algebra, and provide a framework for developing these Proficiencies more effectively.

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