Early Awareness of Mathematical Pattern and Structure

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Introduction

One of the most fundamental challenges for mathematics education today is to inspire young students to develop "mathematical minds" and pursue mathematics learning in earnest. Current research shows that young students are developing complex mathematical knowledge and abstract reasoning much earlier than previously considered. A range of studies in prior to school and early school settings indicate that young students do possess cognitive capacities which, with appropriately designed and implemented learning experiences, can enable forms of reasoning not typically seen in the early years (e.g., Clarke et al. 2006; Clements et al. 2011; English 2012; Papic et al. 2011; Perry and Dockett 2008; Thomas et al. 2002; van den Heuvel-Panhuizen and van den Boogaard 2008; van Nes and de Lange 2007).

Our research aims to provide new insights into how young students can abstract and generalize mathematical ideas much earlier, and in more complex ways, than previously considered. Although there is a large and coherent body of research on individual content domains such as counting and arithmetic, there have been remarkably few studies that have attempted to describe general characteristics of structural development in young students' mathematics. The Australian *Pattern and Structure Project*, initiated in 2001, aims to develop a different approach to understanding mathematics learning, beginning with very young students, that reaches beyond basic numeracy to one that cultivates mathematical patterns and relationships. Over the past decade, a suite of studies with 4- to 8-year old students has found that an awareness of mathematical pattern and structure is both critical and salient to mathematical development among young students (Mulligan and Mitchelmore 2009). Our

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research aims to find reliable and consistent methods for describing the growth of students' awareness of mathematical structures and relationships over time. Utilizing this knowledge to develop quantitative reasoning at an optimum age, when they are eager to learn, is central to this project.

One purpose of this chapter is to describe the construct, *Awareness of Mathematical Pattern and Structure* (AMPS), which our research has shown generalizes across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding. It is our belief that a focus on AMPS could bring more coherence to our understanding of mathematical development and the development of effective pedagogical approaches. We refer to a range of studies with diverse samples in order to describe as explicitly as possible the bases for our identification of early developmental features of AMPS.

This chapter begins by focusing on the role of pattern and structure in early mathematical development. We then trace the development of our early studies on multiplicative reasoning, representations and spatial structuring that led to the Pattern and Structure Project, and we describe the seminal study that established the AMPS construct. The chapter concludes with several examples of developing structural awareness in young students.

Pattern and Structure in Early Mathematical Development

Of particular significance in young students' mathematical development are the reasoning processes they use in learning about their world, such as spatial and quantitative reasoning, deduction and induction, analogical reasoning, and statistical reasoning. In essence, effective mathematical reasoning involves the ability to note patterns and structure in both real-world situations and symbolic objects; such reasoning enables the formation of generalizations in which the abstraction of ideas and relationships can take place (National Council of Teachers of Mathematics 2010).

Virtually all mathematics is based on pattern and structure. As defined in our studies, mathematical *pattern* involves any predictable regularity involving number, space, or measure. Examples include friezes, number sequences, measurement, and geometrical figures. By *structure*, we mean the way in which the various elements are organized and related. Thus, a frieze might be constructed by iterating a single "unit of repeat"; the structure of a number sequence may be expressed in an algebraic formula; and the structure of a geometrical figure is shown by its various properties (Mulligan and Mitchelmore 2009). What we call structural thinking is more than simply recognizing elements or properties of a relationship; it involves having a deeper awareness of how those properties are used, explicated, or connected (Mason et al. 2009).

Spatial Structuring

The study of spatial structure has long been recognized as an important feature of constructing measurement units and geometric properties. Battista (1999) defined spatial structuring as:

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object (p. 418).

Battista and Clements (1996) and Battista et al. (1998) found that students' spatial structuring abilities provided the necessary input and structural organization for the numerical processes that the students used to calculate the number of squares in an array. This finding explains how attempts at enumeration sometimes engender spatial structuring, which in turn provides the input and organization for enumeration. Hence, spatial structuring is "an essential mental process underlying students' quantitative dealings with spatial situations" (Battista et al. 1998, p. 503).

Battista et al. (1998), Outhred and Mitchelmore (2000), and Reynolds and Wheatley (1996) have all studied the development of students' structuring of rectangular figures and arrays. They found that most students learn to construct the row-by-column structure of rectangular arrays by about Grade 4 and have by that time also acquired the equal-groups structure required for counting rows and columns in multiples.

Further research has highlighted how structuring two- and three-dimensional space contributes to students' understanding of important mathematical procedures and concepts such as multiplication, patterning, algebra, and the recognition of geometric shapes and figures (see also Carraher et al. 2006; Clements and Sarama 2009; Mulligan and Mitchelmore 2009; Papic et al. 2011; van Nes and de Lange 2007).

Numerical Structuring

Structure has also been a growing theme in the past two decades of research on students' development of numerical concepts. Many studies have examined counting, subitizing, grouping, unitizing, partitioning, estimating, and notating as essential elements of numerical structure (e.g., Clark and Kamii 1996; Hiebert and Wearne 1992; Lamon 1996; Steffe 1994; Wright 1994). In their studies of the base ten system, Cobb et al. (1997) described first graders' coordination of units in terms of the structure of collections. Thomas et al. (2002) later identified structural elements of the base ten system (such as grouping, partitioning, and patterning) in students' images and recordings of the numbers 1 to 100. In a study of partitioning, Hunting (2003) found that students' ability to change focus from counting individual items to identifying the structure of a group was fundamental to the development of their number knowledge. Van Nes and de Lange (2007) also found a strong link between developing number sense and spatial structuring in Kindergartners' finger patterns and subitizing structures. Studies of partitioning and part-whole reasoning (Lamon 1996; Young-Loveridge 2002) indicate the importance of unitizing and spatial structuring in the development of fraction knowledge.

Extensive research has highlighted young students' strategies in recognizing the structure of word problems (Mulligan and Vergnaud 2006) as well as structural relationships such as equivalence, associativity, and inversion, and functional thinking (Warren and Cooper 2006, 2008). Moreover, studies of multiplication and division have indicated that composite structure is central to multiplicative reasoning (Steffe 1994).

Seminal work by our Australian colleague, Lyn English, has explored structural mapping in students' solutions to combinatorial problems, another multiplicative field (English 1993, 1999). She found that ten-year-old students often had difficulties explaining the structure of the problems and rarely identified the cross-multiplication feature.

Patterning and Data Representation

Much recent research has focused on students' patterning and analogical reasoning skills (Blanton and Kaput 2005; Carraher et al. 2006; English 2004; Papic et al. 2011). For example, the Dutch *Curious Minds* project highlights patterning and spatial skills beyond early numeracy (van Nes and de Lange 2007). There is also increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships such as equivalence and functional thinking in early childhood (Warren and Cooper 2008).

A recent focus of research has shown that data modeling, a developmental process that begins with inquiries and investigations of meaningful phenomena (Lehrer and Schauble 2005) also requires students to seek structure and recognize patterns. Findings of a longitudinal study of data modeling in Grade 1 (English 2012) indicate that students as young as six years old can successfully collect, represent, interpret, communicate, and argue about the structure of data provided they address familiar themes (see chapter *Cognitive Guidelines for the Design and Evaluation of Early Mathematics Software: The Example of MathemAntics*, p. 83, this volume).

The Pattern and Structure Project

Studies on Multiplicative Structure

Our studies on the role of structure in students' mathematical development can be traced to the mid-1990s. In a longitudinal study of students' intuitive models for multiplication and division (Mulligan and Mitchelmore 1997), we studied the strategies Grades 2–3 students use to solve a wide variety of multiplication and division word problems involving grouping, partitioning, counting and patterning. We found that the intuitive model employed to solve a particular multiplicative word problem did not necessarily reflect any specific problem feature but, rather, the mathematical structure that the student was able to impose on it (Mulligan and Watson 1998). Students acquired increasingly sophisticated strategies based on a developing awareness of the equal-groups structure. Many students focused on additive rather than multiplicative structure, which impeded the development of their solution strategies. Other students demonstrated remarkable understanding of structural relationships such as array structure, commutativity, and the inverse relationship between multiplication and division, even in combinatorial problems.

A 3-year longitudinal study investigated the development of students' representations of a range of numerical processes, including counting, grouping, base ten structure, multiplicative and proportional reasoning, between Grades 2 and 5 (Mulligan et al. 1997). We found that "low achievers" (as defined by their teachers) were more likely to produce poorly organized representations and were only able to replicate models of groups, arrays or patterns that had been produced by others. They tended to use unitary counting exclusively, and appeared unable to visualize part-whole relations. Moreover, they made little progress between Grade 2 and Grade 5. "High achievers", however, used abstract notational representations with well-developed grouping, partitioning, and unitizing strategies from the outset; often looked for similarities and differences between their representations; and made significant gains in their multiplicative thinking.

Structural Development of the Base Ten System

In other studies, we focused on students' representations and conceptual understanding of the structure of the base ten system. Thomas and Mulligan (1995) found that the representations of counting and base ten made by mathematically gifted students depicted robust numerical and spatial structures. We postulated two types of internal representation: dynamic (changing and/or encoding motion) and static. Students with high levels of understanding of numeration showed evidence both of dynamic imagery and of structural development in their representations of number.

A larger study of students aged 5–12 years followed, in which we explored the relationship between students' counting, grouping, partitioning, and place-value knowledge and their development of the base ten numeration system (Thomas et al. 2002). We were able to describe several mathematical structural features in their representations: counting and symbols, number patterns and sequences, groupings by tens, use of ten as an iterable unit, recursive grouping, and multiplicative structure supporting place value knowledge. We found a wider use of structure than we had anticipated. For example, Fig. 1 shows how two Grade 2 students represented the number 11. Both drawings show some attempt to utilize array structure, but one is far more sophisticated than the other.

One of the key findings of this study was that students who used a variety of images to represent counting and numeration were more flexible in their thinking and

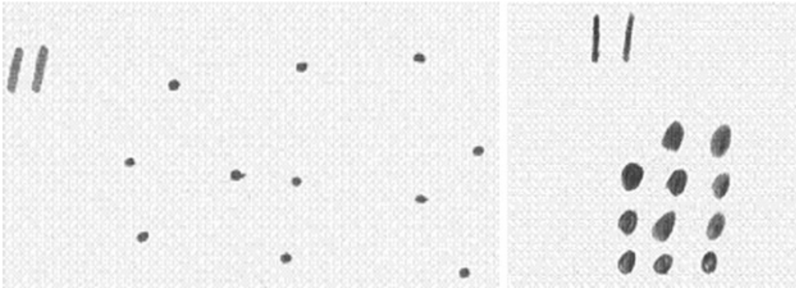


Fig. 1 Two Grade 2 students' representations of the number 11

tended to notice similarities and differences between representations. For example, such a student might recognize a 5 by 5 square array in a 10 by 10 hundred square.

All our studies were consistent with the literature on the differential effects of imagery use in the development of elementary arithmetic (Gray et al. 2000) and the finding that students who recognize the structure of mathematical processes and representations tend to acquire deep conceptual understanding (Pitta-Pantazi et al. 2004). We formed the hypothesis that *the more a student's internal representational system has developed structurally, the more coherent, well organized, and stable in its structural aspects will be their external representations and the more mathematically competent the student will be*. What was unclear at this stage of the research was whether students were aware of structural features and able to apply them to a variety of situations. If so, what role could the recognition of such structural features play in forming mathematical concepts? These research questions guided the next stage of our research.

Awareness of Mathematical Pattern and Structure (AMPS)

In the light of the above hypothesis, we conjectured that young students might possess a general characteristic called *Awareness of Mathematical Pattern and Structure* (AMPS). Students with high AMPS would recognize and operate well with a variety of early mathematical patterns and structures, whereas students with low AMPS would have difficulty recognizing such patterns. High AMPS students would tend to look for similarities and differences in new patterns and broaden their structural understanding accordingly, whereas low AMPS students were likely to focus on idiosyncratic, superficial features and not notice underlying structural features. AMPS was thus considered "to have two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyze patterns). Both are likely to be general features of how students perceive and react to their environment" (Mulligan and Mitchelmore 2009, p. 39).

We tested our conjecture with a large study of Grade 1 students (Mulligan and Mitchelmore 2009). In particular, we posed the following three research questions:

1. Can the structure of young students' responses to a wide variety of mathematical tasks be reliably classified into categories that are consistent across the range of tasks?
2. Do individuals demonstrate consistency in the structural categories shown in their responses?
3. If so, is the individual student's general level of structural development related to their mathematical achievement?

We devised a set of 39 tasks that we judged, on the basis of previous research, to be likely to show structural development in students' responses and combined them into the *Pattern and Structure Assessment* (PASA) interview. The tasks involved key processes such as subitizing, unitizing, partitioning, repetition, spatial structuring, multiplicative and proportional relationships, and transformation; they all required students to identify, visualize, represent, or replicate elements of pattern and structure. Some of the tasks extended well beyond state curriculum expectations (e.g., constructing a pictograph) but these were included in order to yield a wider range of responses than might otherwise be expected. Examples of these tasks, and illustrations of student responses, are given later in this chapter.

Analysis gave positive answers to all three research questions. Firstly, we were able to classify almost all student responses to all tasks reliably into four levels of structural awareness:

- *Prestructural*. Students pick on particular features that appeal to them but are often irrelevant to the underlying mathematical concept.
- *Emergent*. Students recognize some relevant features, but are unable to organize them appropriately.
- *Partial structural*. Students recognize most relevant features of the structure, but their representations are inaccurate or incomplete.
- *Structural*. Students correctly represent the given structure.

Secondly, students were remarkably consistent in the structural level they showed across the various tasks. For every student, there was a clear modal response level, with the modal class frequency having at least twice the frequency of each other class. Thirdly, there was an extremely high correlation between students' structural level and the total number of correct PASA responses, which we took to be a measure of their mathematical achievement level. Moreover, teachers identified all the prestructural students as low achievers and all the structural students as high achievers.

A follow-up study investigated structural development among the eight lowest-achieving students and the eight highest-achieving students over the subsequent 18 months (Mulligan et al. 2005). It was found that PASA responses could still be reliably classified using the same four structural levels, but it was possible to add an additional level (called *advanced structural*) to accommodate responses that also generalized the underlying feature to other contexts. Consistent with earlier results, substantial differences were found between the two groups of students. The high achievers made significant progress over the 18 months and many of their responses fell into the advanced structural level. By contrast, low achievers' responses varied

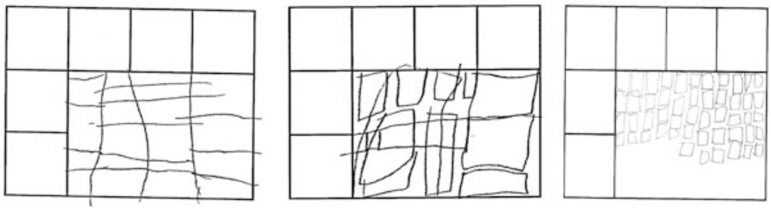


Fig. 2 A Grade 1 students' three attempts to complete a rectangular grid

from one interview to the next but did not show any progress towards greater use of mathematical structure. For example, Fig. 2 shows a low achiever's responses to the "complete the grid" task at the start, the middle, and the end of the study period. The student had some idea of shape, congruence, and collinearity, but was struggling to coordinate them. This student was judged to be at the emergent structural level on all three occasions.

We regarded the results of the two studies as strong support for the existence of AMPS as a psychological construct. In particular, they showed how low AMPS could interfere with the acquisition of fundamental mathematical understandings from an early age.

Examples of Structural Development

The remainder of this chapter is devoted to describing the development of structural awareness, drawing on examples from various studies in the Pattern and Structure Project. Students' drawn responses and explanations to six PASA tasks are illustrated. In each case, students were asked to visualize, then draw and explain their mental images. We use these examples to more clearly explicate the four levels of AMPS that we outlined above, together with the advanced level subsequently added.

We use examples from drawn responses because they most vividly illustrate students' development. Some caution should be exercised in interpreting young students' drawings because of possible problems with fine motor coordination at that age. However, it should be noted that students were routinely asked to explain the images they were attempting to represent and were offered the opportunity to repeat their drawings if they were not satisfied with them. Furthermore, a very similar pattern of development was observed in a wide variety of non-drawing tasks.

Structuring a Clock Face

In one PASA task, students are given a circle that is intended to represent an analogue clock and asked to complete the drawing.

Fig. 3 Prestructural representations of a clock face (drawn from memory by Grade 1 students)

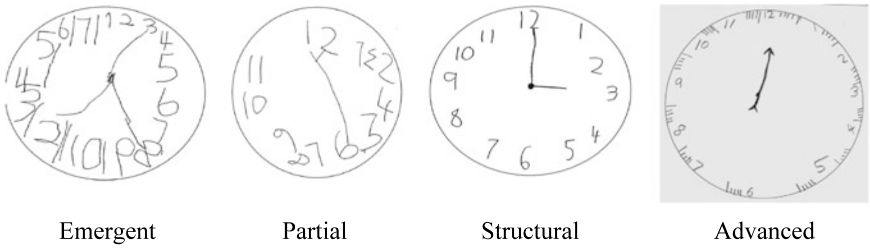


Fig. 4 Structural development in depictions of a clock face

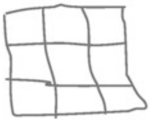
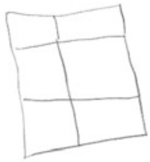
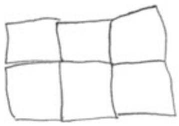
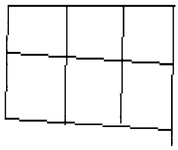
Figure 3 shows four students' attempts to represent a clock face. The first example only depicts one hand, and the next example suggests that the students knew there is also something around the circumference. The third and fourth examples are a little more advanced, suggesting the students were aware that a clock face has numbers on it and ticks around the edge. In none of these drawings is there any indication that the students perceived the structure of the clock face in terms of units of time.

Figure 4 shows some typical drawings at the subsequent levels, from which we can make inferences about students' developing awareness of structure. Students at the emergent level are aware that a clock face has some numbers in unitary counting order around the edge, starting at 1 near the top. At the partial structural level they realize that the numbers are restricted from 1 to 12, with 12 at the top. At the structural level, students have formed a clearer idea of how the numbers depicting hours are distributed in equal spaced intervals around the clock. At the advanced level, students find the positions of 12, 3, 6, and 9 before filling in the other numbers and may indicate one minute intervals between the numerals.

Structuring Rectangular Grids

Another task requires students to briefly view a rectangular grid (2×3 or 3×4), draw it from memory, find the total number of squares, and explain their strategy. Table 1 outlines the criteria for classification of the drawings, with examples of some typical responses to the 2×3 task. Development consists of increasing awareness and coordination of the shape of the squares, their number, and how they fit together.

Table 1 Typical student drawings of a 2 × 3 rectangular grid, by structural level

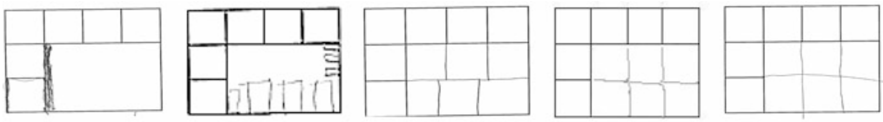
Structural level	Description	Examples
1. Prestructural	Scattered squares or a row of squares.	
2. Emergent	An attempt to draw a grid or border, but neither their numerical nor their spatial structure is correctly represented.	
3. Partial structural	A grid that is incomplete or inaccurate in terms of the number of squares drawn. For example: (a) a grid with an incorrect number of squares (b) a correct grid but drawn in the wrong orientation (c) an array with the correct number of squares, but with the squares not touching.	  
4. Structural	A grid of adjacent squares with the correct number of rows and columns, but with each square drawn separately.	
5. Advanced	A grid of the correct number of squares, drawn using continuous horizontal and vertical lines.	

Notice that drawings at both the structural and advanced levels are "correct", but the advanced level representation suggests a further advance in understanding that allows generalization to different tasks of a similar nature (see the next item).

Structuring Area

Students are shown a 3 × 4 rectangle with squares drawn along two adjacent sides and instructed to "finish drawing the squares, exactly like these, to cover all of this shape". We have already shown how one Grade 1 student responded to this task at the emergent level (Fig. 2). Figure 5 shows some typical responses at all five levels.

Structural development on this task is very similar to that shown by the previous task (see Table 1). Students at the prestructural level know only that there has to



Prestructural Emergent Partial Structural Advanced

Fig. 5 Structural development in grid completion task

be something in the empty space and do not always draw squares. Emergent students appear to have picked up on one or two features of grids (e.g., congruence, alignment, contiguity, or borders) and reproduce this feature in some form without filling the space systematically. Partial structural students attempt to fill the space, but do not draw the correct number of squares or do not draw them touching correctly. Students at this level may indicate the beginning of multiplicative thinking when they remark on the equal number of squares in each row or column. Structural students have coordinated all the crucial features of a grid and know exactly where the squares go; but they still draw them individually. By contrast, advanced structural students know that the edges of the contiguous squares form straight lines and use this fact to construct the grid very quickly. Students at this level draw grids of all sizes in the same way and are more likely to use multiplication to find the total number of squares.

Structuring a Triangular Array

In another item, students are briefly shown a card with a triangular pattern of six dots on it and asked to draw it from memory. The drawings in Fig. 6 show the progress of a single student over an 18 month period. The first drawing was made at the beginning of the Kindergarten year; it is classified as emerging because it is made up of dots and has some indication of a triangular outline. The second drawing was made by the same student at the end of the Kindergarten year and already shows the triangular pattern adequately. In the third drawing, made in the middle of Grade 1, the student has not only copied the given pattern accurately but has also extended it

Emergent Structural Advanced

Fig. 6 Three drawings of the triangular array made by the same student

Prestructural (Interview 1) Prestructural (Interview 2) Prestructural (Interview 3)

Fig. 7 Three drawings of the triangular array made by a different student

more or less accurately to a fourth row. It appears that the student could extend the pattern indefinitely; the drawing is therefore classified as advanced structural.

In contrast, Fig. 7 shows the very limited progression made by another student over the same 18-month period. There is no evidence of any progress. The student does not appear to recognize collinearity or triangularity, and would probably have great difficulty in any mathematical topic where such ideas occur.

Structuring Length

In a further item, students are shown a drawing of a long thin rectangle, asked to imagine that it is a ruler and asked to "draw things on it so that it becomes a ruler you can measure with". They are then asked to actually use it to measure the length of a pencil. Figure 8 shows typical drawings at each of the first four levels of structural development; the fifth level is achieved when a student explains how the number on the ruler gives the length in terms of a unit of measurement.

The progression on this task parallels that of the clock task. A vague awareness that there are some markings on a ruler gives way to knowledge that these are ticks and numbers, which are then coordinated until equal spacing is achieved.

Structuring Data

In another PASA task, students are given a table listing a collection of animals (7 dogs, 5 cats, and 3 birds) and asked to draw a graph to represent these data. Grade 1 students are expected to draw a horizontal pictograph, the only type they may have met in class. Figure 9 shows some typical drawings at the first four levels; the fifth level is achieved when a student explains that the largest number of animals is shown by the longest row (so that you do not need to count each row).

Prestructural drawings show one or more animals, not arranged in any order. Emergent responses pick up on one or two aspects of pictograms; the example in Fig. 9 shows an awareness that there are icons arranged in rows, but the numbers

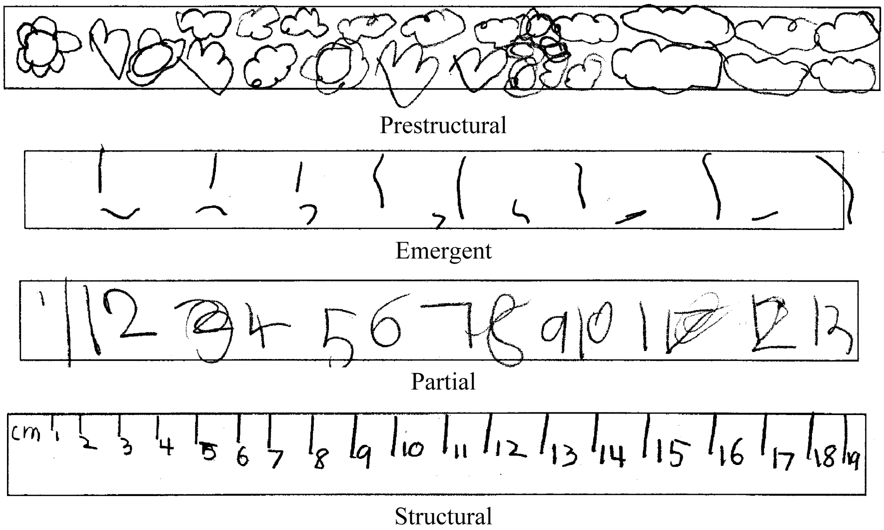


Fig. 8 Typical drawings of a ruler at the first four structural levels

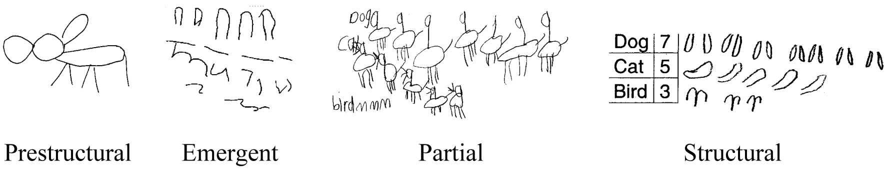


Fig. 9 Typical pictograms drawn at the first four structural levels

bear no relation to the data. In partial structural representations, the icons are arranged in rows with the correct numbers of icons, but the icons are not aligned. Finally, at the structural level the icons are aligned vertically so that the numbers of each type of animal can be compared using the lengths of the rows. (Notice the similarity to array structure.) From here, it is a small step to the basic principle of graphical representation that equal numbers are represented by equal lengths.

Discussion

A growing body of research has established that a large and significant portion of students' mathematical thinking in early childhood can be described in terms of a growing awareness of pattern and structure. Our research has enabled us to reliably describe this development in terms of a number of structural levels.

As can be seen from the examples above, young students initially show no evidence that they are aware of the mathematical structure implicit in a range of tasks. They tend to focus on superficial features, which they select and organize in their own way. Gradually, however, they become more and more aware of mathematically relevant characteristics—various elements (numbers, lines, shapes) and their arrangement (e.g., along a line or circumference, equally spaced, contiguous, aligned). In each task we have examined, it has been possible to reliably classify the growth of this awareness into four levels, starting with a single aspect of structure and culminating in a coordinated representation of all relevant aspects. Some young students then make one further step. They realize that the structure is more general than the specific task, enabling them to succeed on similar tasks or simple extensions. However, some young students do not progress through these levels at all. They may notice some aspects of mathematical pattern, but they do not become sufficiently aware of how these aspects are organized to be able to advance beyond the prestructural level. In fact, their representations often become more crowded and chaotic.

Our research shows that, not only can students' levels of structural development through these five levels be reliably categorized, but students who show a particular level of development on one task typical of the early mathematics curriculum also operate at a similar level on other such tasks. This finding indicates an underlying construct that we call Awareness of Mathematical Pattern and Structure (AMPS).

Our studies have also highlighted some big ideas underlying the development of structural understanding in early childhood. The first is that of generality. Number patterns are a prime example: There is always some general rule telling you how the pattern is to be continued, whether it be a simple repetition of what has come before or a continuation of some form of growth. Other generalizations that can arise from the study of numerical patterns include commutativity of addition and multiplication. The study of spatial patterns leads to concepts such as collinearity, congruence and symmetry and the formulation of general properties of basic two-dimensional figures. For all the PASA tasks, the highest structural level always corresponds to awareness of some such generalization. Finding and expressing such numerical and spatial generalizations is the beginning of algebra and geometry, respectively.

A second important big idea is that of equal grouping. The perception of a repeating pattern as a number of identical "chunks" leads to the idea of skip counting and then multiplication. Rectangular arrays and grids can be deconstructed into numbers of equal rows or columns, so the same counting strategies can be applied. Number lines and measurement scales represent replications of identical units and so are also multiplicative, as is the decimal numeration system. Even graphical representation requires equal grouping; for example, in a frequency chart each object must be represented by a fixed unit. Equal grouping is also required when a set of objects or a quantity is partitioned into equal parts, the foundation of the concepts of division and fractions.

Students with high AMPS are likely to have a better understanding of both these Big Ideas than those with low AMPS. They are likely to look for, remember and apply spatial and numerical generalizations and in particular are likely to grasp the multiplicative relationships that underlie the majority of the concepts in the elementary mathematics curriculum. Not only are they better placed to understand proportionality, but they have been primed to use mathematical reasoning in other areas.

So it should not be surprising that students with high AMPS do well in mathematics and that, conversely, students with low AMPS struggle (Mulligan 2011).

Conclusion

We regard the AMPS construct as a significant contribution to research into early mathematics education. It provides a single lens with which to examine student thinking in a wide variety of mathematical topics encountered in early childhood and thus yields a more uniform approach than can be obtained by examining each topic separately.

Our research has focused on students up to the second year of formal schooling, but our occasional explorations with older students suggest that the AMPS construct could also have a much wider application. In particular, there is the possibility that low AMPS in early childhood could predict poor performance in mathematics throughout a student's school career and even beyond. Extending the AMPS construct to the later years of schooling is a promising field for further research.

Our research has also thrown up two vital questions: Is AMPS a fixed trait or can it be taught? If AMPS can be improved by teaching, will there be a concomitant improvement in mathematics achievement? A positive answer to the second question would have particularly far-reaching implications for mathematics curriculum, pedagogy, and assessment. Both questions were addressed within the Pattern and Structure Project by developing the *Pattern and Structure Mathematics Awareness Program* (PASMAP) to teach AMPS. These studies—which eventually led to the large-scale evaluation study, *Reconceptualizing Early Mathematics Learning*, that inspired this volume—are described in chapter *Reconceptualizing Early Mathematics Learning: The Fundamental Role of Pattern and Structure*, p. 47.

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