Advances in Mathematics Education

Lyn D. English
Joanne T. Mulligan *Editors*

Reconceptualizing Early Mathematics Learning

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Advances in Mathematics Education

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Reconceptualizing Early Mathematics Learning

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Series Preface

The fifth volume of Advances in Mathematics Education focuses on an under addressed area of research in mathematics education, namely early mathematical thinking and learning. Despite the groundbreaking work of Piaget that led to the formulation of developmental theories, interest in further developing neo-Piagetian models of learning has waned since the 1980's. Three decades later, the community has come to realize that these developmental models do not take into consideration the sophisticated mathematical thinking that children are capable of, given the right mathematical activities to stimulate them into abstract reasoning.

The book, *Reconceptualizing Early Mathematics Learning*, edited by Lyn English and Joanne Mulligan presents studies that advance children's mathematical learning in ways we did not think was possible. The chapters focus on notions of early algebra, statistical thinking, beginning numeracy as well as the advocacy for the kinds of learning that are important for the 21st century. Several of the chapters also address the professional development of teachers necessary to promote early mathematical learning experiences. The theoretical foundations of this work are set in Newton and Alexander's chapter that surveys the state of the art. This is followed by empirical studies of Mulligan in Australia, Clements in the U.S. as well as alternative play-based classrooms of Wager. Data modeling is another theme explored by English with children in grades 1–3. Interdisciplinary approaches are also found in the work of Diefes-Dux that utilize model eliciting activities in art classrooms. The book provides a balance between theoretical foundations, empirical work with children that advance theories, as well as the importance of work with teachers to provide early mathematics learning and development.

An important feature to note in volume 5 is that the book series, *Advances in Mathematics Education*, has moved into topics not traditionally anchored in prior volumes of the connected journal, ZDM—*The International Journal on Mathematics Education*. This suggests that the series is open to research perspectives from the community that advance our field, without necessarily being anchored to ZDM.

We are deeply convinced that this book will make a strong contribution to the much needed diversity of theoretical advances in mathematics education.

Missoula, USA Bharath Sriraman
Hamburg, Germany Gabriele Kaiser
11 March 2013

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Perspectives on Reconceptualizing Early Mathematics Learning

Introduction

Lyn D. English and Joanne T. Mulligan

This edited volume emanated primarily from our concern that the mathematical capabilities of young children continue to receive inadequate attention in both the research and instructional arenas. Our research over many years has revealed that young children have sophisticated mathematical minds and a natural eagerness to engage in a range of mathematical activities. As the chapters in this book attest, current research is showing that young children are developing complex mathematical knowledge and abstract reasoning a good deal earlier than previously thought.

A range of studies in prior to school and early school settings indicate that young learners do possess cognitive capacities which, with appropriately designed and implemented learning experiences, can enable forms of reasoning not typically seen in the early years (e.g., Clements et al. 2011; English 2012; Papic et al. 2011; Perry and Dockett 2008). For example, young children can abstract and generalize mathematical ideas much earlier, and in more complex ways, than previously considered. Although there is a large and coherent body of research on individual content domains such as counting and arithmetic, there have been remarkably few studies that have attempted to describe characteristics of structural development in young students' mathematics.

The title of this volume, *Reconceptualizing Early Mathematics Learning*, captures the essence of each chapter. Collectively, the chapters highlight the importance of providing more exciting, relevant, and challenging 21st century mathematics learning for our young students. The chapters provide a broad scope in their topics and approaches to advancing young children's mathematical learning. They incorporate studies that highlight the importance of pattern and structure across the curriculum, studies that target particular content such as statistics, early algebra,

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and beginning number, and studies that consider how technology and other tools can facilitate early mathematical development. Reconceptualizing the professional learning of teachers in promoting young children's mathematics, including a consideration of the role of play, is also addressed. Although these themes are diffused throughout the chapters, we restrict our introduction to the core focus of each of the chapters.

To set the scene, the opening chapter by Newton and Alexander provides an in-depth historical analysis of the changing and, at times paradoxical, nature of early mathematics learning. By conceptualizing how perspectives on early mathematics learning have taken shape over the past century through the impact of both internal and external forces, Newton and Alexander highlight the changing character of early mathematics learning over the last century. They explore psychological, sociocultural, and neurophysiological developments that may have helped to shape these pedagogical trends in early mathematics education.

Emphasizing the importance of pattern and structure across the curriculum is the core feature of the chapters by Mulligan and her collaborators. Their classroom research with 4- to 8-year-old children reveals a focus on mathematical pattern and structure to be both critical and salient to young learners' mathematical development. They demonstrate how their construct, *Awareness of Mathematical Pattern and Structure*, generalizes across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding. The construct can bring more coherence to our understanding of mathematical development and the design of effective pedagogical approaches. For example, we report on an evaluation study that demonstrates the positive impact of a Pattern and Structure Awareness Program (PASMAP) in the first year of schooling.

Another approach to advancing early mathematics learning is offered by Clements and Sarama with their learning trajectories tool, which forms the core of their conceptual framework for developing curricula and teaching strategies. The learning trajectories describe how children learn major topics in mathematics and how teachers can support that learning, while their framework details criteria and procedures for creating scientifically based curricula using learning trajectories.

An alternative perspective on supporting early mathematics learning is proposed by Wager with her focus on play-based classrooms. She explores the application of 'focused' instruction that minimizes teacher-centered practices and promotes play—teachers plan and build on children's understanding, interests, and cultural practices, and recognize and respond to the mathematics that emerges in children's play.

A focus on play also features in the chapter by van Oers, who addresses ways in which we might help children explore their actual play situations from the perspective of number. Specifically, the chapter describes how translating number related problems into thinking tools that are accessible for mathematical refinement (i.e. mathematizing) can occur meaningfully within the context of young children's play. Such experiences are linked strongly to children's learning to communicate about number in a coherent way, rather than by instructing them on number operations.

Providing opportunities for children to direct their own learning is also highlighted in English's chapter, where she explores reconceptualizing young children's statistical experiences from the beginning years of formal schooling. Specifically, she addresses data modeling as a means of developing young children's abilities to impose structure on complex data, detect relationships between seemingly diverse concepts and representations, and organize, structure, visualize, and represent data. Ways in which young learners engaged in these processes during a longitudinal study of data modeling across grades one to three are described.

Other ways of enriching learning opportunities are offered in the chapters that address advances in technology. Ginsburg et al. show how cognitive psychology can inform the design and evaluation of software for early mathematics learning, and how the resulting software can provide new approaches to evaluating learning and enhance basic cognitive research. With examples from their *MathemAntics* software program, the authors illustrate the affordances of computer technology in fostering transformative improvements in early mathematics education. The development of such software has the potential to elicit advanced mathematical thinking and reveal unexpected deviations from known developmental trajectories.

Likewise, Goodwin and Highfield demonstrate the rich learning experiences that interactive technologies can provide for the mathematics learning of 3–8-year-olds. Their exemplars demonstrate how the pedagogic design of technologies can have a substantial impact on young children's development of basic mathematical concepts. In addition, Goodwin and Highfield provide evidence that different forms of multimedia offer unique opportunities for learners, whose responses challenge the widespread belief that young children are incapable of dealing with complex mathematical concepts.

Other didactic tools that have the potential to enhance early mathematical development include picture books, as seen in van den Heuvel-Panhuizen's and Elia's chapter. They present a framework of picture book characteristics that support kindergartners' learning of mathematics, and examine three reading book techniques investigated in their research. A major conclusion of their research is reading picture books can support substantially children's mathematical understanding and should thus have a significant place in the early curriculum. The use of picture books appears effective for a wide range of children including those of different ages, socio-economic backgrounds, and language and mathematical abilities.

An innovative, interdisciplinary approach to furthering early mathematical development is described in Diefes-Dux' chapter. She considers how art education, which is typically viewed solely as an opportunity to explore creative thinking, can be a powerful partner in advancing young children's problem solving, with a focus on mathematical modeling. The chapter describes a Draw-a-Monster activity that was created to adhere to design principles for a Model-Eliciting Activity and implemented in two art classrooms. Ways in which an activity of this nature can be linked to children's learning in other domains including mathematics, language arts, and engineering are explored. Preparing young children for more complex modeling situations, appearing increasingly in their world, are discussed.

Enhancing teachers' professional development is a core concern of the chapters by Warren and Miller, Papic, and Perry and Dockett. The first two chapters report on teacher programs designed to promote the early mathematics learning of disadvantaged, indigenous children. Warren and Miller's program for teachers resulted in improvements in their affective domain, with teachers becoming more confident in their ability to teach mathematics. These gains in teacher confidence led to improved pedagogical practices, and enriched mathematical content knowledge and instruction. In turn, these outcomes impacted positively on the children's confidence and learning.

Papic reports on a series of studies aimed at improving young indigenous children's learning opportunities, particularly in early algebra, patterning, and mathematical reasoning. The development of teachers' pedagogical and mathematical content knowledge was also a core aim achieved through ongoing, supportive professional development. The program was geared towards the broader goal of closing the gap in numeracy achievement for Australian indigenous children in rural and regional early childhood settings. The studies outlined in their chapter provide empirical evidence that, through scaffolding teacher's abilities to promote early mathematics learning, children's development of sophisticated concepts and skills emerges. In particular, prior to formal schooling young children are capable of abstracting, generalizing, and explaining patterns and pattern structures.

Perry and Dockett's chapter also addresses both teacher and student development, with a focus on findings from the *Early Years Numeracy Project* in South Australia. The development of a major artifact from the project, namely, the *Reflective Continua*, forms the focus of the chapter. Ways in which educators have used the Reflective Continua to stimulate the powerful mathematics learning of young children are reported. The Continua's rich contribution to teacher development lies in its frameworks that guide educator reflections on children's mathematical work and assist in the planning of future learning experiences.

In concluding, we thank the authors for their insightful and future-oriented perspectives on early mathematics learning and development. The book would not have been possible without their commitment to advancing the field; we hope their diverse collection of studies will provide a strong foundation for much needed future research.

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Early Mathematics Learning in Perspective: Eras and Forces of Change

Kristie J. Newton and Patricia A. Alexander

One of the endlessly alluring aspects of mathematics is that its thorniest paradoxes have a way of blooming into beautiful theories. Phillip J. Davis (1964)

As the opening chapter in this important volume that looks deeply at the changing and somewhat paradoxical nature of early mathematics learning, our goal is to position those shifting perspectives within a historical framework. By conceptualizing how views of early mathematics learning have taken shape over the past century through the pushes and pulls of both endogenous (internal) and exogenous (external) forces, one can better grasp the re-conceptualization of mathematics learning conveyed within the ensuing chapters. There are perhaps few who would argue with the underlying premise of this book; that the character of early mathematics education has changed dramatically over the last century not only in terms of the pedagogical approaches to teaching young children, but also in relation to the content and goals of that instruction. However, the progression of that change may be less evident and, consequently, worthy of scrutiny.

Changes in complex domains such as early childhood mathematics rarely happen abruptly or without inducement. Rather, such transformations seemingly unfold over the course of many years in response to internal and external conditions. Here we endeavor to unearth those inducements, some of which arise more directly from within the community of researchers and practitioners invested in early mathematics teaching and learning. Other of those inducements can be situated within the broader educational and psychological communities, reflecting varied if not conflicting theoretical orientations toward human learning and development (see Fig. 1). Thus, in this chapter, we attempt to identify six particular periods or eras associated with

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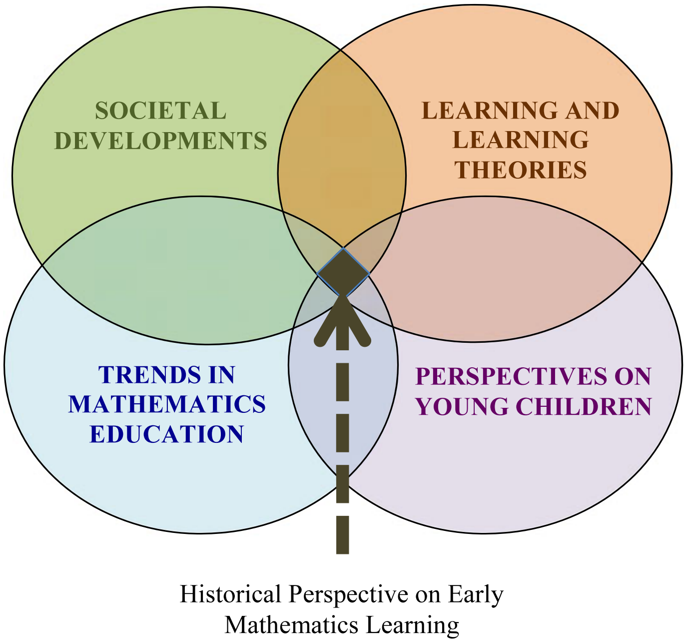

Fig. 1 The sources of evidence in constructing eras of change in early mathematics learning

mathematics learning in young children and seek to explore psychological, sociocultural, and neurophysiological developments that may have helped to shape those eras.

Before we begin this historical overview of early mathematics learning, we want to state frankly that there is not exactness to the characterization we present. Historical analysis of this sort is characteristically inferential. Consequently, others who would engage in a comparative historical examination of the early mathematics literature or judge the endogenous and exogenous forces that were at work within each time period may reach different conclusions. Moreover, the boundaries between historical eras are neither rigid nor fixed. Here we looked at 20-year periods as meaningful, generational units for analysis, but the trends underway within the era do not simply begin or end at the preset time points. Further, our own interests and empirical foci, such as our investment in the study of learning theories and children's understanding of fractions and their analogical reasoning, will undoubtedly color the perspectives we forge herein from the existing evidence.

Nonetheless, with these caveats in mind, the eras we identify derive from the theoretical and empirical literature of that period, and consider personages whose writings and thinking were particularly influential. We also signify some important events within mathematics, education, or the broader society that are anchored to these time periods. Also, we consider the views of children and the teaching of mathematics to children that were prevailing within each era, as well as the presence of alternative or competing perspectives that may have signaled subsequent changes on the horizon.

For example, early childhood educators have traditionally advocated for learning through play, but views of the nature and purpose of play have varied markedly among educational researchers and practitioners. In part, these variations exist as a function of the changing beliefs about children's cognitive capabilities and about the role of early educational experiences in enhancing the capabilities of young minds. They also mirror shifting psychological orientations toward learning and philosophical perspectives toward knowledge and knowing. An examination of the writings of such influential theorists as Dewey, Thorndike, Piaget, Vygotsky, Flavell, and Rogoff will serve to illuminate these shifting orientations toward young minds, mathematics, and the learning of mathematics. Drawing on these writings, we chart the course of early mathematics education in relation to these theoretical underpinnings, consider emerging trends, and address the implications for early mathematics research and practice.

Era of Experiential Learning (1900–1920)

Every historical analysis begins at some predetermined point in time. For our purposes, this analysis begins at the turn of the 20th century. So much of the world was undergoing change as the century dawned; civil conflicts in the United States and France had ended and the industrialization of much of Europe and the US was well underway. In terms of psychology, there was a growing interest in the nature of childhood itself—a concept that was not commonly considered prior to the industrialization of world powers—and an investment in developing mental faculties beginning at a young age (Elkind 1998). In the late 19th century, mathematics was considered important as a way to exercise the mental faculties in the developing mind (Sztajn 1995), and we see this notion carried forward into the early 20th century.

Those who were associated with early mathematics learning at the close of the 19th century, however, had not fully embraced the character of young children and their predisposition toward learning through free play that would become a feature of the next era. Rather, the available methods to teaching young children from that period were somewhat formal and structured in nature as perhaps best exemplified by the work of Friedrich Froebel. In particular, the techniques used by Froebel, who has been credited with introducing the concept of "kindergarten" into Western culture, involved concrete materials, such as geometric figures, that were deeply mathematical and which could be used to engage young children in the learning of mathematical concepts (Balfanz 1999). *Froebel Gifts*, as his educational materials were called, were very carefully devised and intended to be systematically used in the early childhood setting to foster particular ways of thinking and behaving. Froebel, as with Montessori who borrowed from his work, understood the role of "free work" or activities to teach the young. However, he saw these mathematical activities not as an end in and of themselves but as important ways of exercising the developing mind of the child. Thus, Froebel's approach could be described as rather formal and less spontaneous than we would see in the early 20th century in what we have labeled the *Era of Experiential Learning*.

By the early 1900s, the influence of Froebel was fading in favor of more holistic and less orchestrated conceptions of early childhood education (Balfanz 1999). Whereas Froebel's kindergarten was rather structured in its treatment of mathematics, the more child-centered orientations of this era focused on the child as a social being. "Free work" was still central to early mathematics learning in this era, but the child was given increased freedom to choose and to freely explore the mathematically associated objects and activities that populated the learning environment. Mathematics was not directly the focus of learning in this setting, but was rather understood as manifestations of young children's true interests that needed to be appropriately fed and actively nurtured through relevant and engaging experiences (Dewey 1903). Mathematical experiences existed but were informal in nature and embedded within children's exploratory activities. In other words, the learning of mathematics was somewhat more incidental than intentional and the consequence of learning *in* experience rather than learning *from* experience (Saracho and Spodek 2009).

Influential Personages

No name is more associated with this Era of Experiential Learning than that of John Dewey, the pragmatist and the father of progressive education. In his formulation of progressive education, Dewey was influenced by Montessori's idea of learning through activities and appreciated the efforts of Froebel to create learning environments for the young. Among the tenets of the progressive movement was Dewey's often-expressed idea that education is the process of living and not simply a preparation of later life (Dewey 1900/1990). Toward that end, Dewey argued that the content of learning should derive from the children's existing interests and draw meaningfully from the children's life in the broader social community. Mathematics was not to be dealt with as isolated content nor used as mental calisthenics, but was to be experienced fully and naturally by children through hands-on, project-based activities that built on children's existing interests and that pertained to activities (e.g., cooking, building) that were valued outside of school.

The focus on the teaching of mathematics to young children through engrossing experiences of value personally and socially coincided with the emergence of a new field, developmental psychology. Unlike the developmental psychology of today, this earliest manifestation of this field had more to do with systematic observation than experimental study. This focus can be clearly seen in the work of G. Stanley Hall (1907), considered the founder of developmental psychology, the first president of the American Educational Research Association, and the father of the child study movement. Perhaps best known for his fascination with *peculiar and exceptional* *children*, Hall devised methods for the detailed documentation of children's physical attributes and psychological behaviors.

The significance of developmental psychology, in general, and the child movement in particular was the now commonly accepted premise that young children are much more than miniature adults. Rather, they live and learn differently and those differences undergo systematic change over time. In terms of mathematics, this translated into critical questions about what the mind of the child was able to grasp mathematically and how best to harness the burgeoning knowledge about the physical and psychological development of children to teach them mathematical concepts and procedures appropriately (Alexander et al. 1989). Questions of developmental appropriateness and about how best to bring children and mathematics together reappear throughout the ensuing eras. For those of the Experiential Learning Era, that question was best answered by allowing young children to shape the educational agenda through the enactment of their interests and choices and by positioning the particulars of mathematical concepts and processes as background to the wants and desires of those children. Learning by doing remained the rule of the day, even as the uniqueness of young children was embraced.

Views of Children and the Teaching of Mathematics

In order to address growing concerns in the early part of the century about appropriate instruction in kindergarten, and in particular the appropriateness of Froebelian kindergarten, the International Kindergarten Union formed the Committee of Nineteen. A lack of consensus within the Committee meant that three reports on the content and goals of kindergarten were eventually issued, ranging from support of the Froebel method to endorsement of a more child-centered progressive orientation. Patty Smith Hill wrote the report supporting a child-centered approach, which ultimately won favor in academia (Beatty 1995). Following this report, play became a legitimate part of kindergarten programs and it was recommended that learning be guided by activities of interest to the child (Saracho and Spodek 2009).

Around this time, Margaret MacMillan established the first nursery school in England. Play was also an important part of this approach, but there was little concern for academic subjects. When academic subjects were introduced to older children, no particular approach to teaching was prescribed (Saracho and Spodek 2008). What was important instead to McMillan was the child's health and hygiene, with a particular concern for poor and working class children. As a result, outdoor play and good air ventilation were incorporated into the program, to the extent that her early buildings were only partially enclosed and the children sometimes even napped outside (Beatty 1995). As nursery schools gained popularity over the next few decades, their focus would broaden to include the child's general well-being and readiness for more formal learning in school.

Inspired by Dewey, Kilpatrick (1926) forwarded the Project Method, which engaged children in learning activities that were purposeful and practical. According to Kilpatrick, children would be naturally stimulated to learn if they were provided with interesting experiences that involved them in the community. This followed the principles of progressive education wherein it was expected that children are naturally curious and interested in the world around them and that those curiosities and true interests included mathematical concepts and processes. There was also the presumption that the mind of the child was highly capable of dealing with mathematical concepts and procedures if those concepts and procedures were embedded in activities that the children valued and that they could reasonable pursue. Thus, when engaged in the Project Method, children may count or measure objects as they worked toward a larger goal, such as raising chickens (Saracho and Spodek 2008, 2009).

Mathematics was not emphasized in its own right, but it was expected that children would learn some mathematical ideas as they participated in the projects. We see similar orientations to mathematics learning in contemporary sociocultural theories of learning and development, such as Rogoff's (1990) concept of legitimate peripheral participation. As with the aforementioned discussion, this overview of the Experiential Learning Era of early mathematics education introduces several themes about children and mathematics learning that will periodically reappear in our analysis. The first has to do with the perceived capacities of the young mind and whether the mind of the child is conceived as fertile ground for grasping basic mathematical concepts and procedures or not. The second has to do with the need to foreground the mathematical concepts or procedures or whether the mathematics should be embedded in socioculturally valued experiences or activities. For those in the Experiential Learning Era, children were perceived as highly self-directed and inquisitive learners who were able to acquire mathematical understanding if they were allowed to explore those ideas within the context of self-chosen, self-directed and socially valued activities.

Competing Views

Perhaps the most evident contrast to arise during this Experiential Learning Era was the all too familiar theme of traditional or basic skills education that has remained the counterpoint to progressive movements throughout the century. Specifically, Deweyan approaches to early childhood education were not the only ones that conflicted with the ideas forwarded by Froebel and his notion of kindergarten (Balfanz 1999). Another critic was Thorndike (1913), a behaviorist in terms of this theoretical orientation toward learning and development, whose work can be seen as the backbone for basic skills training and development. Thorndike purported that formal instruction in arithmetic was fruitless before second grade, and that even when mathematics was introduced, understanding was not a pre-requisite for acquiring mathematical skill (Baroody 2000). Rather than believing that mathematics could be learned incidentally through purposeful activities, Thorndike, who equated learning with behavioral change and manifestations, believed that mathematics must be systematically structured and practiced and, thus, had no place in early childhood education. Thorndike, along with many other critics, thought that the early years should be focused instead on social development and health (Balfanz 1999).

Era of Childhood Readiness (1920–1940)

Two trends that appeared within the Era of Experiential Learning—learning through play and the focus on early childhood as a particular period of development—carried forward into the Era of Childhood Readiness. What distinguished this new era from the previous, as we will discuss, was the acceptance of mathematics as not solely as means to an end, but as a curricular end in and of itself. These trends combined together positioned the early educational years as a time to prepare the child for the more formal study of the domain of mathematics—to ensure that they were "ready" to think and perform mathematically in subsequent years.

As more students began to attend schools during the early twentieth century, an increased focus was placed on mathematics that was considered to be practical for the average person. This was especially true during the Great Depression, since limited availability of jobs kept many students in school for longer. The resulting increase in high school enrollment meant that more students were focused on training for jobs rather than college. Mathematics was de-emphasized, and in some cases, the mathematics requirements for graduation were removed altogether (Walmsley 2007). Instead, courses such as home economics, art, and physical education increased in popularity at the secondary level. Partly in response to the decreased focus on mathematics in schools, the National Council of Teachers of Mathematics (NCTM) was founded in 1920 (Austin 1921). Meanwhile, a parallel trend was occurring in early childhood education. In particular, mathematics in the early years was extremely limited. The focus shifted to play, imagination, physical movement, and social skills in part because this is what many educators felt young minds were able to cognitively and physically address (Saracho and Spodek 2009).

Several reasons contributed to this reality, including dominant theoretical perspectives during this era coupled with the changes occurring in the later grades. Mathematics was included as part of the first and second grades, but the amount of time spent on mathematics instruction was a fraction of the time spent on reading, language, and even recess (Balfanz 1999). Likely influenced by the continuing arguments by Thorndike and his adherents, it was determined that to focus any more specifically or directly on mathematics in these young years would not prove fruitful. Rather, drawing on the work in child study of the prior decades, it was held that educators needed to ascertain whether young children showed signs in their play and interactions with others that they were cognitively predisposed for formal instruction, including formal instruction in mathematics in the years to come.

Personages

Two contemporaries and colleagues warrant particular recognition for their role in shaping this Era of Readiness, Arnold Gesell and Frances Ilg. As has often been the case in the history, especially in these early years, Arnold Gesell came to education from a different field. He had trained to be a physician, but become enamored with questions of nature versus nurture and the role that each played in the development of children with disabilities. From decades of systematic research with Frances Ilg, he argued that nurture had a significant role to play in children's developmental trajectory. Gesell and Ilg did not discount the power of nature but felt that there was much that could be done within the early years of life to stimulate cognitive capacity—to build on what nature had provided.

Gesell's writing on *The Mental Growth of the Preschool Child* (1925), and *The Preschool Child from the Standpoint of Public Hygiene and Education* (1923), as well as his work with Ilg on the development of early childhood assessments served to justify the time as one of nurturing the young child—of readying them for the more formalized instruction in mathematics and other contents that would follow. His influence also extended beyond the educational community to parents concerned with child development and child rearing. This influence was largely due to his highly cited volume that documented early childhood milestones, *An Atlas of Infant Behavior* (1934) and to the two guides for child rearing that he coauthored with Ilg, *Infant and Child in the Culture of Today* (1943), and *The Child from Five to Ten* (1946).

By the close of this era, many regarded Gesell as the foremost authority on child rearing and child development. Not only did he argue strongly for the influence of early nurturance at home, but he also became an advocate for a nationwide nursery school system that could provide the early stimulation and support that he promoted in his writings to educators and to parents. And it was these strongly held beliefs in the importance of readiness to later development that mark this era, particularly when coupled with the Thorndikian perspective that training in mathematics should be reserved for later elementary and not attempted within the early grades.

Views of Children and the Teaching of Mathematics

Views of young children and the teaching of mathematics during this era were shaped by an emerging interest into the inner workings of the human mind (cognition) in relation to the behavioral indicators of capability (behaviorism). In line with Thorndike's work, some theorized that formal instruction in mathematics was unnecessary—and perhaps even harmful—in the early years and should be delayed until the child was in a formal school setting (Balfanz 1999). As a result, the mathematics curriculum was limited in the early years. Nursery schools, which were popularized during this time, held little regard for academic subjects in general and even less for mathematics in particular. They instead encouraged dramatic play, physical movement, and even caring for animals (Saracho and Spodek 2009).

With more and more children attending kindergarten, it was gradually becoming linked to the public school system. As this happened, mathematics became even more de-emphasized in early childhood. One contributing factor was that textbooks were written with the assumption that kindergarten students had no prior knowledge of mathematics; arithmetic was limited or absent until first grade (Balfanz 1999). Given the emphasis in later grades on mathematics that would be useful to the average person, mathematics for young learners was perhaps meaningless. Instead, the purpose of kindergarten was readiness for more formal learning; for example, following directions and complying with rules were emphasized (Saracho and Spodek 2009).

Competing Views

The nature versus nurture discussion that Gesell and Ilg brought to the public attention in this period can be contrasted with that of another developmentalist, Jean Piaget (1926/1930, 1952, 1955). Like Gesell, Piaget came to his interests in education and human learning and development from an alternative profession. In Piaget's case, this profession was science and biology. Piaget, like G.S. Hall and Arnold Gesell, was an acute observer of nature. In fact, even as a child, it was apparent that Piaget had remarkable capacity to build upon direct, detailed observations; having published a book about birds that were found around his home in Switzerland before he was 10. What Piaget's observations of animal life and later human life led him to was a more stage-like perspective on human cognition and a stronger appreciation for the "nature" side of human development than his predecessors.

During the 1920s and 1930s, Piaget began publishing books based on observations and clinical interviews of young children which presented his ideas about the child's emerging ability to think logically (Piaget 1926/1930)—what became known as genetic epistemology. While many praised his use of naturalistic settings, some also criticized his research for not being "scientific" enough (Beatty 2009). Likewise, his ideas about the child's egocentric nature were initially met with mixed results among early childhood educators. His work enjoyed a brief reception in the United States during this time but did not fully take hold until a few decades later. Meanwhile in Geneva, he continued to conduct research with children and to develop a more fully-articulated theory of child development that would eventually gain widespread support and make a profound contribution to the mathematics education of young children.

The most influential aspect of Piaget's prolific work was the stages of development he conceptualized and the more focused attention on cognition rather than on behavioral manifestations. Although Piaget presented these as general stages and not fixed timeframes of development, their consequences, which carried over into the next era, was to think of the capacities of young mind's including mathematical capabilities as rather rigid or set.

Era of Cognitive Development (1940–1960)

The prioritizing of social skills over academic rigor that began early in the twentieth century led to growing concerns by mid-century that students were ill-prepared for technical jobs. This idea that was especially highlighted by War World II, when it became clear that soldiers often did not have the mathematical skills needed for their jobs in the military. Moreover, many critics felt that "practical" mathematics was no longer enough even for the average student and that, in order to flourish in an increasingly technological world, higher levels of mathematics needed to be required at the secondary level. This view was coupled with a shifting focus from memorization to understanding, most notably reflected in the work of Brownell (Kilpatrick 1992; Lambdin and Walcott 2007). By the end of this era, the New Math movement, which emphasized mathematical ideas and structures and included more rigorous mathematics than prior decades, had taken shape (Herrera and Owens 2001; Jones and Coxford 1970; Walmsley 2007). Launched in 1957, Sputnik further created a sense of urgency to update mathematics education at all levels and secured the momentum of the New Math movement in the decade to come.

With the external forces in play, fascination with the "black box" of the human mind moved into prominence within the educational community and strict behavioral theories of learning faded into the background (Newell et al. 1957, 1958). How the mind works and how the understanding of mental processing could be harnessed into better educational outcomes became the approach *de rigueur*. Increased attention was also being paid to the cognitive and mathematical skills of young children during this period, as evidenced by the significant amount of research that emerged in this area over the next several decades. This research supported the idea that young children can learn mathematical ideas if they are grounded in the child's experiences, and that children begin school with significant informal experiences on which to build. Moreover, building on these experiences can help students make sense of the new information they are encountering in school settings (Baroody and Ginsburg 1990).

Personages

As noted, no individual was more contributory to shaping the Era of Cognitive Development than Jean Piaget. While other children were playing with toys and enjoying their childhoods, Piaget was already immersed in scientific study of animals and fossils. More than his developmental predecessors, Piaget (1952, 1955) showed the world of living organisms—humans included—as growing and changing in systematic ways. But the changes that Piaget documented through his clinical methods were stage based, meaning that they were conceived as transformative, distinct, and dramatic rather than gradual and continuous. Those cursorily familiar with Piaget's theoretical and empirical writings are likely aware of his four developmental stages: sensorimotor, preoperational, concrete operational, and formal operational.

Even those who have moved away from the initial conceptualization of these stages or who are more liberal in their age-related characterizations (i.e., neo-Piagetians) still bow to Piaget's notion of stage-like development in young children (Case 1985, 1992; Flavell 1985; Flavell et al. 1993). A detailed discussion of these stages is beyond the scope and intention of this chapter. Yet, because of their relevance to this examination of the history of early mathematics education, we will briefly address the characterizations of the sensorimotor and preoperational stages and consider the implications for early mathematics education.

From the moment of their birth, children are thrust into a strange, new world that they must come to know through their senses—their primary tools for growth and development. Given the primary role of the senses in the first months after birth, it is understandable that Piaget would refer to this initial period of cognitive development that was conceived to run to about age two as the sensorimotor stage. Even in their rudimentary formation of mathematical conceptions and procedures, children at this young age remain dependent on direct, physical examination and exploration to survive and to understand their world and their place in that world (Alexander 2006). This sensory-physical exploration is aided by the fact that these young children are maturing neurologically and motorically, and they are acquiring the ability to express their wants and needs to others around them.

Toward the end of this initial developmental stage, according to Piaget, young children begin to realize that the things they see and do can be represented symbolically with words or numbers. They begin to think symbolically by language, in problem solving, and through imaginative play (Piaget 1955). Just as increased mobility and linguistic facility are keys to change, this acquisition of symbolic understanding becomes a catalyst for moving young children into a new realm of development—preoperational thinking. Thus, while there may be a genetic predisposition in the preoperational child to sense quantity or spatial orientation, critical to later mathematical development, this grasping of symbolic representation makes more conceptual growth possible. Along with the ability to use symbols and signs critical to mathematical thinking and learning, there are certain defining attributes of the preoperational mind—a period that runs from around ages 2 to 7. For one, young children become more skilled at engaging in conventional rather than idiosyncratic communication (Piaget 1955). Another cognitive achievement of the preoperational stage has to do with young children's conception of time and space (Piaget 1952).

Still, as noted, there were clear limitations within Piagetian theory to the nature of the thinking and reasoning of which preoperational children were presumed capable. Those limitations were associated with such processes of egocentrism (self-centeredness), conservation (maintenance of quantity or mass), and reversibility (reverse thinking). For example, according to Piagetian theory, young children must abandon their propensity toward idiosyncratic speech and become more facile in conventional language as they develop (Piaget 1955). To Piaget, idiosyncratic speech was evidence of egocentrism or children's tendency to view the world through their own knowledge and experiences without regard to other perspectives. Although egocentrism is often used pejoratively when applied to more mature individuals, this was not Piaget's intention. Rather, Piaget (1926/1930, 1952) sought to demonstrate that preoperational thinkers had an understandably limited sphere of existence that resulted in their interpretation events solely through their personal lens of experience.

Interestingly, for all the remarkable cognitive and mathematical accomplishments that young children realize in these initial two stages of development, there were seemingly negative consequences that arose from the adaptation of a Piagetian perspective—consequences not intended by Piaget. For one, many within the educational realm did not recognize that Piaget was describing typical behavior of children—what most children would be expected to do under most conditions. Instead, the idea of individual differences or variability got lost in the curricular programs that were devised. Activities were excluded or not introduced under the notion that children of a given age would not be able to benefit from them (White and Alexander 1986). Further, the adaptation of a stage model of development led to an all-or-nothing conception of learning. Children were either preoperational or concrete operational or not. There was less room for cognitive variability based on task or context.

Views of Children and the Teaching of Mathematics

Although Piaget believed that very young children were not capable of developing true number concepts, he insisted that they were curious and naturally interested in patterns. He also believed that developing understanding, rather than rote memorization, was most important for learning mathematics (Baroody 2000). Whereas his earlier work stressed the relation between language and logical thought, his later work supported the idea that logical thought is developed through children's activity (Beatty 2009).

Hence, activities that involved concrete materials were thought to help support the eventual development of number concepts. Learning activities or exercises were derived based on classical Piagetian tasks or the concepts and processes that those tasks exemplified. Through the manipulation of concrete materials, mathematical ideas could be discovered and later formalized through instruction (Lambdin and Walcott 2007). Unlike Froebel's materials, which were inherently mathematical, these concrete materials could include activities with everyday objects (e.g., sorting or counting cookies). Such an approach is based in the idea that rote memorization of information is not adequate for developing either deep understandings of or positive attitudes toward mathematics (Baroody and Ginsburg 1990).

Competing Views

At this point in the history of mathematical learning and development, there was no argument that there was a systematic nature to growth in young children. The controversies that existed had more to do with whether that change was continuous or discontinuous (stage-like), and, relatedly, whether the mind of the young child could benefit from early exposure to mathematical concepts and procedures or not. The rumblings for more continuous development had always been present over the decades but those rumblings became to grow louder toward to end of this era. Regrettably, some of the strongest and more compelling voices for continuous change (i.e., those of Vygotsky and Luria) were not heard until years later as a result of political circumstances.

Some approaches to early mathematics education during this time represented a merging of several perspectives. For example, the Reggio Emilia approach involved children in the sort of pre-number activities such as classifying and sorting that were consistent with Piaget's notions of developmental appropriateness. At the same time, mathematics was embedded in project work, or in-depth explorations of familiar contexts, an aspect inspired by Dewey. The approach was also inspired by the work of Bruner (1960, 1961), Bruner et al. (1956), who argued strongly for the power of scaffolding by others, and Vygotsky, who held that children were co-creators of knowledge and they use a variety of cultural tools to aid in this process (Dodd-Nufrio 2011; Linder et al. 2011). This latter perspective would take a strong foothold in the decades to come.

Era of Socially-Scaffolded Development (1960–1980)

During the next era in the history of mathematical teaching and learning, the debates over continuous or discontinuous models of development became coupled with an augmented awareness of international competitiveness and growing concerns over students' preparedness. For instance, the New Math movement continued through the 1960s, but dissatisfaction with students' computational skills spurred a public push for a shift "back to basics" during the 1970s. Coupled with this trend was an increased focus on standardized testing and teacher accountability (Walmsley 2007). However, some argued that early childhood educators could promote fluency with basic skills while still supporting children's thinking in a way that is consistent with theories of cognitive development. In other words, academic learning of skills was not necessarily viewed as incompatible with play (Havis and Yawkey 1977).

During this time, there was also heightened attention on poverty and its impact on academic learning; raising awareness of the power of nurture or the social context to impact young children's subsequent growth and development. As a result, Head Start was founded in 1965 to target poor children and their parents. The program was meant to be comprehensive, including services that ranged from health and nutrition to parent education and involvement. Although it was not without its critics in the beginning, Head Start is considered a beneficial initiative in part because of longitudinal studies that illustrated a variety of positive, long-term outcomes (Beatty 1995). As with much of the readiness work in the prior eras, the focus within Head Start in these early years was on preparing the child, socially and cognitively, for the demands of formal instruction in mathematics and other domains that would soon follow.

Personages

Two individuals merit recognition for the shaping of this Era of Socially-Scaffolded Development—one a contemporary of Piaget whose writings did not become widely distributed until decades after his death (i.e., Vygotsky) and another whose writings still influence educational research and practice (i.e., Bruner). While we would still regard the theory and research of these two giants within the field of education as cognitive—that is both were invested in understanding the workings of the individual mind so as to promote learning and development—they manifest more emphasis on the role of the social context and those who populate that social environment in fostering young children's growth than did Piaget and his adherents. For this reason, while Piaget has been called a cognitive constructivist, Vygotsky and Bruner would be more suitably regard as social constructivists (Murphy et al. 2012).

Volumes have been dedicated to the research and influence of Lev Vygotsky and we will not be able to do justice to that legacy here. However, as it pertains to early mathematics learning and teaching several conclusions are especially noteworthy. First, while Vygotsky admired Piaget and found much within his writings with which he agreed, he differed strongly with Piaget on several critical accounts (Vygotsky 1934/1986). Specifically, Vygotsky (1978) did not hold to a stage or discontinuous view of development and put much more weight on the influence of more knowledgeable others to guide and support development. In this way, Vygotsky (1978, 1934/1987) was more invested in understanding optimal rather than typical development. It was not a question of what children generally could do mathematically without guidance, but rather what a given child could potentially demonstrate when functioning within a rich and supportive environment.

Similarly, Jerome Bruner (1960, 1966, 1974) argued for guided discovery for young children and was credited with introducing the now often-used term "scaffolding" into the educational vernacular. Bruner felt that an interplay of various forms of representation, most notably symbolic, is what defined children's development; not the stage-like progression that Piaget contended. Further, through guided discovery, children have the opportunity to explore mathematical concepts and procedures but under the watchful eye of teachers who could help to orchestrate events in such a way as to maximize the child's learning. Consistent with the New Math movement, Bruner believed that mathematical concepts could be taught with integrity to young children, provided that tasks were carefully chosen to illustrate important ideas at an age-appropriate level (Herrera and Owens 2001; Lambdin and Walcott 2007).

Views of Children and the Teaching of Mathematics

Researchers during this time were beginning to question some of Piaget's ideas related to number development (Baroody 2000). According to Piaget, instruction for young children who cannot yet conserve number should be restricted to pre-number activities such as ordering and classifying (Clements 1984). In a training study comparing this approach to one that focused explicitly on number concepts, Clements (1984) found that the group trained in number concepts outperformed the other group on number skills yet performed as well on a test of the pre-number skills. In other words, it was thought that number skills cannot only be taught at this age, but doing so can also reinforce pre-number skills.

Other evidence for learning number skills can be found with Sesame Street, a children's show that was popularized during this time. This show was designed not just for entertainment but to actually teach social behaviors and early academic skills to young children, particularly to those with an economic disadvantage. Across a multitude of studies from several countries, positive effects have been found for both. For example, children who viewed Sesame Street generally entered kindergarten with a greater range of number skills than those who did not, with effects lasting for years (Fisch et al. 1999).

Competing Views

While the influence of the social context, especially in the form of more knowledgeable others was gaining prominence, there were two groups of theorists and researchers who offered contrasting perspectives on young children and their learning of mathematics. On the one hand, there were the neo-Piagetian's like Robbie Case (1985) and John Flavell (1985) who retained a more cognitive orientation toward young children's development. While differentially or more liberally interpreting Piaget's work, they still held to a discontinuous view of learning and development and effectively documented both the capabilities but limitations of the young mind in terms of dealing with symbolic representations that mathematics learning demanded. Further, Flavell (with Miller and Miller 1993), who studied with Piaget, argued convincingly that attempts to document what typically occurs in the course of development for most children under most circumstances did not preclude efforts to appreciate individual variability. In this way, the homogeneity and heterogeneity of young children's development could rightfully co-exist.

At the other end of the spectrum, there was an escalation in the number of scholars trained in social anthropology and cultural anthropology who began to focus their expertise and interests onto questions of education and learning (e.g., Lave and Wenger 1991; Rogoff 1990). With this escalation, a harbinger for the era that would follow, less concern existed for the operations or development of the individual mind. Rather, the attention was on society or communities and on the activities of the collective as they engaged in socially-valued and socially-supported practices that served as evidence of mathematical learning. The mathematical learning of young children was not constrained to schooled versions of mathematics but was opened to everyday cognitions that occurred within the course of living and functioning within sociocultural communities.

Era of Culturally-Nested Learning (1980–2000)

As the Era of Culturally-Nested Learning began to take shape, there appeared to be rather strong and contrasting perspectives on mathematics learning and teaching in juxtaposition. While contrasting or competing views have always defined any period in the history of mathematics education, this difference was somewhat unique. For one, there were quite varied theoretical and empirical orientations among educational researchers generally, including those holding to more cognitive constructivist and social constructivist perspectives. Moreover, there were clearly distinct and seemingly conflicting orientations toward learning within the community of mathematics education, as we will discuss. Further, the conceptualization and operationalization of mathematics teaching and learning espoused among educational researchers stood in sharp contrast to the conceptualizations and operationalizations of mathematics teaching and learning operating within the educational system.

For instance, within the mathematics education community, there were those who continued to give primacy to the individual mind (e.g., cognitive constructivists or radical constructivists), whereas others held more steadfastly to a sociocultural or sociocontextual frame (e.g., socioculturalists or situated cognitivists). These contrasting views were artfully captured in Sfard's (1998) provocative article on AM (acquisition metaphor) and PM (participation metaphor) perspectives on learning.

Yet, both of these metaphorical stances toward learning in mathematics and other complex domains stood in sharp contrast to what was ongoing with regard to mathematics education within this historical timeframe. Specifically, this era saw a rise in basic skill assessments within public schools and calls for teacher accountability. On the other hand, what should count as "basics" for school mathematics curricula was being questioned by some, and there was a concomitant rise in the mathematics requirements for high-school graduation during this period. In addition, mathematics educators saw a need to go beyond computational skills to include estimation, problem solving, and the use of technology (Lambdin and Walcott 2007; Walmsley 2007).

To this end, NCTM published the 1989 *Curriculum and Evaluation Standards for School Mathematics*, with other documents to follow. This document emphasized the importance of teacher-facilitated investigations for children, designed to help foster problem solving, deep understanding, and ownership of mathematical ideas. As an example, the *investigative approach* attempted to blend skills, concepts, and mathematical inquiry by presenting children with worthwhile, challenging tasks or projects that encourage exploration. Within this approach, students are encouraged to share their ideas, and the teacher prompts and guides students when they are struggling (Baroody 2004). This move within the mathematics education community was supported by emerging research that illustrated young children are capable of more sophisticated mathematical thought than was proposed by Piaget (Baroody 2000). However, heated debates about the balance between computational skill and problem solving, known as the "math wars," characterized the second half of this era (Herrera and Owens 2001).

The expanding popularity and capabilities of hypermedia technology, which would become even more apparent in the years to follow, were also having an effect on what was taught and how it was taught within schools and classrooms throughout the industrialized world. From graphing calculators to personal computers and from online communities to the proliferation of media sites dedicated to young children and mathematics, it was becoming unnecessary and, perhaps, impossible to contain children's interactions to the classroom. The universe of actual and virtual "others" who could afford scaffolding to children engaged in mathematics-related activities was expanding by leaps and bounds—altering the face of mathematics teaching and learning for all time (Shaffer and Kaput 1999).

Another circumstance adding to the "messiness" of this era was the globalization of society and commerce and the ensuing international comparisons of student academic performance. Specifically, international mathematics and science studies that began in the 1990s highlighted the need for reform at all levels. Mathematics educators and policymakers in Western countries were particularly alarmed at how their students performed relative to many of the Asian countries. These results inspired several countries, including New Zealand, Australia, and Canada, to create initiatives that targeted the early years (Young-Loveridge 2008).

Personages

Because of the influence of social and cultural anthropologists during this particular era, we want to describe the particular contributions of two such individuals, Jean Lave and Barbara Rogoff. In decades past, anthropologists like Margaret Mead brought their theoretical interests and research methodologies to the study of particular social and cultural groups (e.g., Samoans); often quite distinct from their own. Over the course of this era, however, those trained as anthropologists found compelling evidence of mathematical thinking and capabilities within certain communities of practice such as apprentice tailors, milk deliverers, or dieters (Carraher et al. 1985). In her groundbreaking volume on *Cognition in Practice* (1988), Lave brought this fascinating work to the attention of the wider educational community and argued compellingly that such everyday cognition should be valued since it demonstrated evidence of mathematical thinking and learning within "authentic" contexts.

As with her contemporary Lave, Rogoff (1990) was invested in the sociocultural collective and was especially concerned with studying how children appropriate or master patterns of participation in group activities, including those activities that involved mathematical thinking and performing. For instance, in one of her classical studies, Rogoff and colleagues (Rogoff et al. 1995) investigated Girl Scouts engaged in the planning and tracking of orders and the delivery of cookie orders, offering evidence of important conceptual and procedural learning within this community valued activity.

There are several significant aspects to this sociocultural perspective for early mathematics teaching and learning. For one, there was a rejection of the Vygotskian notion of internalization because it signified separate psychological planes for the individual and the community (Sawyer 2004), when the "child and the social world are mutually involved" and, thus, cannot be "independently defineable" (Rogoff 1990, p. 28). As a consequence, determinations of young children's mathematical capabilities had to rely on an analysis of the collective actions and social interplay—not the assessments of any individual student. For another, classrooms and schools were not held as the conduits of formal, abstracted mathematical concepts or procedures but as sociocultural venues in which particular values, customs, and participatory structures are developed.

Views of Children and the Teaching of Mathematics

The shifting perspective toward more socially-nested forms of learning mathematics confronted differing orientations and agendas articulated by the educational and political establishments of this time. Specifically, the performance of young children in mathematics across the global community did not necessarily or consistently favor more participatory or social models of instruction as those advocated by socioculturalists. Rather the international profile was quite mixed; from the structured and more formal approach to early mathematics within certain countries (e.g., Singapore) to more informal and exploratory methods of others (e.g., New Zealand). However, international studies revealing these differences in teaching styles across various countries highlighted the idea that even *teaching is a cultural activity* (Stigler and Hiebert 1999). "Teaching, like other cultural activities, is learned through informal participation over long periods of time. It is something one learns to do by growing up in a culture rather than by formal study" (p. 2).

Despite the variable approaches to teaching mathematics, this era saw a rising interest in non-school factors that influence the learning of mathematics. For example, research with young children has illustrated that the degree to which students struggle with mathematics can be highly context dependent. In other words, knowledge learned in one situation does not necessarily transfer to other situations. For example, work by Carraher et al. (1985) showed that Brazilian children who demonstrated sophisticated thinking about arithmetic in contextualized settings often struggled with the same problems presented in numerical form (Sophian 1999). These findings support the earlier notions that children can and do gain much informal knowledge about mathematics outside of school, and that this knowledge should be a source for learning. Connecting this knowledge to teaching has been the focus of large-scale projects such as *Cognitively Guided Instruction* (Carpenter and Fennema 1991).

The home environment also was seen to play a role in the early learning of mathematics. For instance, in a study of four- to six-year olds, Blevins-Knabe and Musun-Miller (1996) found that increased number activities in the home generally predicted scores on standardized tests of mathematics achievement. However, this pattern did not hold across all ethnic groups or across groups of varying levels of education. More recently, Levine et al. (2010) further confirmed the importance of the home environment for later learning of mathematics. These researchers found that the amount of number talk from parents to their 14- to 30-month-olds predicted knowledge of cardinal number meanings at 46 months, even after controlling for socioeconomic status (SES).

At the same time, cross-cultural studies indicated that SES does play a role in early mathematical development. And while preschools do not inevitably close the SES gap, they have the potential to do so when they include high quality mathematics curricula (Starkey and Klein 2008). A contributing factor to growth in knowledge of mathematics during preschool is the degree to which teachers engage in math talk (Klibanoff et al. 2006).

Competing Views

Although the research on sociocultural influences on mathematical learning has been convincing, an opposing view has support as well. This view posits that humans are pre-disposed to acquiring skill with numbers. In a pioneering study, Starkey and Cooper (1980) were able to demonstrate that 16- to 30-week old infants are sensitive to changes in sets of up to four objects. Since that time, a significant amount of research has been conducted to substantiate the claim that humans have an innate sense of numerosity (Butterworth 1999, 2005; Dehaene 1997).

Further, those who retained a more cognitive view of early mathematics learning remained active during this period. Whether these examinations of young children's mathematical thinking and reasoning were conducted within research laboratories, classrooms, and in home environments, they focused on the mental processing and performance indicators that individual children demonstrated (e.g., Kerkman and Siegler 1997; Rittle-Johnson and Siegler 1998). Although such cognitive emphases were not commonplace among mathematics educators during this period, findings from such studies, combined with the rising interest in neuroscience and neurobiological served as omens to the now emergent era.

Emerging Era of Embodied Learning (2000–present)

There is always a tremendous risk involved in attempting to describe the current era. Some degree of distance is critical in making the appropriate determination. That being said, we will briefly consider what we see as signs or omens that distinguish this present phase in the history of early mathematics education. Historically, mathematics or the teaching of mathematics has been considered either not important or inappropriate for young children, but a significant body of research now suggests otherwise. Mounting evidence illustrates that young children have the interest and capacity to learn meaningful mathematics at early ages. That burgeoning research also suggests that adult guidance and support is needed to fully realize this potential and that the level of necessary support may vary across individuals (National

Research Council 2009). Moreover, there is significant evidence regarding the importance of early mathematics skills and their predictive power for later learning (Duncan et al. 2007).

In many ways, current beliefs about the learning of mathematics in early childhood represent an amalgamation of several perspectives highlighted during the past century. Psychological, social, and cultural perspectives each contribute to our current understandings of early mathematics education, and most recently, neurological perspectives have shed light on the nature of mathematical development (Butterworth 1999; Dehaene 1997). Specifically, the sociocultural orientations that arose in the prior era have remained evident within the educational research community, whereas the persistence of investment in basic skills development and the assessment of young children can still be identified within the K-12 experience.

We have chosen to label this final period as the Era of Embodied Learning because of the re-emergence of consideration of biological/neurological indicators of mathematics learning and development. Greater funding is being directed toward fMRI and ERP (event related potentials) studies of the young mathematical mind. Further, entire conferences and volumes are now dedicated toward the genetic and neurological foundations for mathematical learning and performance within very young populations (Butterworth 2005). For example, Blair et al. (2008) asserted that some children make arithmetic errors that reflect not only faulty procedural and conceptual knowledge but also a failure of executive functioning processes such as working memory and inhibitory control. Also considered to be an aspect of self-regulation, inhibitory control was found to be especially important in both mathematics and reading in the early years (Blair and Razza 2007). As a result, Blair and colleagues suggested that executive processes need explicit attention in the classroom.

Thus, the mind/body duality that had been characteristic of early decades has begun to give way to a realization that not only is mind nested in the sociocultural collective but also that the mind and the body of the child work as one. The more that is understood about the whole child—neurologically, biologically, and cognitive—individually *and* within the broader sociocultural context, then the better supportive and facilitative environments can be devised to support that child's mathematical learning and development. That is the bottom line of this emergent era.

Conclusions

Over the past century, we have seen the views of young learners and their capacity to understand and do mathematics shift as various theoretical perspectives have come to the forefront, while others fade into the background. Currently, the research community seems to have embraced multiple perspectives, simultaneously acknowledging individual, social, and cultural influences on mathematical thought. With these acknowledgments has come an increased awareness of the importance of mathematics for young children and a belief in their interest and capacity for learning it. *Mathematics for all* has been embraced by early childhood educators and researchers alike, and the result has been a strong movement toward the reconceptualization of mathematics learning for young children. As the ensuing chapters illustrate, great strides have already been made in this direction.

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Early Awareness of Mathematical Pattern and Structure

Joanne T. Mulligan and Michael C. Mitchelmore

Introduction

One of the most fundamental challenges for mathematics education today is to inspire young students to develop "mathematical minds" and pursue mathematics learning in earnest. Current research shows that young students are developing complex mathematical knowledge and abstract reasoning much earlier than previously considered. A range of studies in prior to school and early school settings indicate that young students do possess cognitive capacities which, with appropriately designed and implemented learning experiences, can enable forms of reasoning not typically seen in the early years (e.g., Clarke et al. 2006; Clements et al. 2011; English 2012; Papic et al. 2011; Perry and Dockett 2008; Thomas et al. 2002; van den Heuvel-Panhuizen and van den Boogaard 2008; van Nes and de Lange 2007).

Our research aims to provide new insights into how young students can abstract and generalize mathematical ideas much earlier, and in more complex ways, than previously considered. Although there is a large and coherent body of research on individual content domains such as counting and arithmetic, there have been remarkably few studies that have attempted to describe general characteristics of structural development in young students' mathematics. The Australian *Pattern and Structure Project*, initiated in 2001, aims to develop a different approach to understanding mathematics learning, beginning with very young students, that reaches beyond basic numeracy to one that cultivates mathematical patterns and relationships. Over the past decade, a suite of studies with 4- to 8-year old students has found that an awareness of mathematical pattern and structure is both critical and salient to mathematical development among young students (Mulligan and Mitchelmore 2009). Our

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research aims to find reliable and consistent methods for describing the growth of students' awareness of mathematical structures and relationships over time. Utilizing this knowledge to develop quantitative reasoning at an optimum age, when they are eager to learn, is central to this project.

One purpose of this chapter is to describe the construct, *Awareness of Mathematical Pattern and Structure* (AMPS), which our research has shown generalizes across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding. It is our belief that a focus on AMPS could bring more coherence to our understanding of mathematical development and the development of effective pedagogical approaches. We refer to a range of studies with diverse samples in order to describe as explicitly as possible the bases for our identification of early developmental features of AMPS.

This chapter begins by focusing on the role of pattern and structure in early mathematical development. We then trace the development of our early studies on multiplicative reasoning, representations and spatial structuring that led to the Pattern and Structure Project, and we describe the seminal study that established the AMPS construct. The chapter concludes with several examples of developing structural awareness in young students.

Pattern and Structure in Early Mathematical Development

Of particular significance in young students' mathematical development are the reasoning processes they use in learning about their world, such as spatial and quantitative reasoning, deduction and induction, analogical reasoning, and statistical reasoning. In essence, effective mathematical reasoning involves the ability to note patterns and structure in both real-world situations and symbolic objects; such reasoning enables the formation of generalizations in which the abstraction of ideas and relationships can take place (National Council of Teachers of Mathematics 2010).

Virtually all mathematics is based on pattern and structure. As defined in our studies, mathematical *pattern* involves any predictable regularity involving number, space, or measure. Examples include friezes, number sequences, measurement, and geometrical figures. By *structure*, we mean the way in which the various elements are organized and related. Thus, a frieze might be constructed by iterating a single "unit of repeat"; the structure of a number sequence may be expressed in an algebraic formula; and the structure of a geometrical figure is shown by its various properties (Mulligan and Mitchelmore 2009). What we call structural thinking is more than simply recognizing elements or properties of a relationship; it involves having a deeper awareness of how those properties are used, explicated, or connected (Mason et al. 2009).

Spatial Structuring

The study of spatial structure has long been recognized as an important feature of constructing measurement units and geometric properties. Battista (1999) defined spatial structuring as:

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object's nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the new object (p. 418).

Battista and Clements (1996) and Battista et al. (1998) found that students' spatial structuring abilities provided the necessary input and structural organization for the numerical processes that the students used to calculate the number of squares in an array. This finding explains how attempts at enumeration sometimes engender spatial structuring, which in turn provides the input and organization for enumeration. Hence, spatial structuring is "an essential mental process underlying students' quantitative dealings with spatial situations" (Battista et al. 1998, p. 503).

Battista et al. (1998), Outhred and Mitchelmore (2000), and Reynolds and Wheatley (1996) have all studied the development of students' structuring of rectangular figures and arrays. They found that most students learn to construct the row-by-column structure of rectangular arrays by about Grade 4 and have by that time also acquired the equal-groups structure required for counting rows and columns in multiples.

Further research has highlighted how structuring two- and three-dimensional space contributes to students' understanding of important mathematical procedures and concepts such as multiplication, patterning, algebra, and the recognition of geometric shapes and figures (see also Carraher et al. 2006; Clements and Sarama 2009; Mulligan and Mitchelmore 2009; Papic et al. 2011; van Nes and de Lange 2007).

Numerical Structuring

Structure has also been a growing theme in the past two decades of research on students' development of numerical concepts. Many studies have examined counting, subitizing, grouping, unitizing, partitioning, estimating, and notating as essential elements of numerical structure (e.g., Clark and Kamii 1996; Hiebert and Wearne 1992; Lamon 1996; Steffe 1994; Wright 1994). In their studies of the base ten system, Cobb et al. (1997) described first graders' coordination of units in terms of the structure of collections. Thomas et al. (2002) later identified structural elements of the base ten system (such as grouping, partitioning, and patterning) in students' images and recordings of the numbers 1 to 100. In a study of partitioning, Hunting (2003) found that students' ability to change focus from counting individual items to identifying the structure of a group was fundamental to the development of their number knowledge. Van Nes and de Lange (2007) also found a strong link between developing number sense and spatial structuring in Kindergartners' finger patterns and subitizing structures. Studies of partitioning and part-whole reasoning (Lamon 1996; Young-Loveridge 2002) indicate the importance of unitizing and spatial structuring in the development of fraction knowledge.

Extensive research has highlighted young students' strategies in recognizing the structure of word problems (Mulligan and Vergnaud 2006) as well as structural relationships such as equivalence, associativity, and inversion, and functional thinking (Warren and Cooper 2006, 2008). Moreover, studies of multiplication and division have indicated that composite structure is central to multiplicative reasoning (Steffe 1994).

Seminal work by our Australian colleague, Lyn English, has explored structural mapping in students' solutions to combinatorial problems, another multiplicative field (English 1993, 1999). She found that ten-year-old students often had difficulties explaining the structure of the problems and rarely identified the cross-multiplication feature.

Patterning and Data Representation

Much recent research has focused on students' patterning and analogical reasoning skills (Blanton and Kaput 2005; Carraher et al. 2006; English 2004; Papic et al. 2011). For example, the Dutch *Curious Minds* project highlights patterning and spatial skills beyond early numeracy (van Nes and de Lange 2007). There is also increasing evidence that early algebraic thinking develops from the ability to see and represent patterns and relationships such as equivalence and functional thinking in early childhood (Warren and Cooper 2008).

A recent focus of research has shown that data modeling, a developmental process that begins with inquiries and investigations of meaningful phenomena (Lehrer and Schauble 2005) also requires students to seek structure and recognize patterns. Findings of a longitudinal study of data modeling in Grade 1 (English 2012) indicate that students as young as six years old can successfully collect, represent, interpret, communicate, and argue about the structure of data provided they address familiar themes (see chapter *Cognitive Guidelines for the Design and Evaluation of Early Mathematics Software: The Example of MathemAntics*, p. 83, this volume).

The Pattern and Structure Project

Studies on Multiplicative Structure

Our studies on the role of structure in students' mathematical development can be traced to the mid-1990s. In a longitudinal study of students' intuitive models for multiplication and division (Mulligan and Mitchelmore 1997), we studied the strategies Grades 2–3 students use to solve a wide variety of multiplication and division word problems involving grouping, partitioning, counting and patterning. We found that the intuitive model employed to solve a particular multiplicative word problem did not necessarily reflect any specific problem feature but, rather, the mathematical structure that the student was able to impose on it (Mulligan and Watson 1998). Students acquired increasingly sophisticated strategies based on a developing awareness of the equal-groups structure. Many students focused on additive rather than multiplicative structure, which impeded the development of their solution strategies. Other students demonstrated remarkable understanding of structural relationships such as array structure, commutativity, and the inverse relationship between multiplication and division, even in combinatorial problems.

A 3-year longitudinal study investigated the development of students' representations of a range of numerical processes, including counting, grouping, base ten structure, multiplicative and proportional reasoning, between Grades 2 and 5 (Mulligan et al. 1997). We found that "low achievers" (as defined by their teachers) were more likely to produce poorly organized representations and were only able to replicate models of groups, arrays or patterns that had been produced by others. They tended to use unitary counting exclusively, and appeared unable to visualize part-whole relations. Moreover, they made little progress between Grade 2 and Grade 5. "High achievers", however, used abstract notational representations with well-developed grouping, partitioning, and unitizing strategies from the outset; often looked for similarities and differences between their representations; and made significant gains in their multiplicative thinking.

Structural Development of the Base Ten System

In other studies, we focused on students' representations and conceptual understanding of the structure of the base ten system. Thomas and Mulligan (1995) found that the representations of counting and base ten made by mathematically gifted students depicted robust numerical and spatial structures. We postulated two types of internal representation: dynamic (changing and/or encoding motion) and static. Students with high levels of understanding of numeration showed evidence both of dynamic imagery and of structural development in their representations of number.

A larger study of students aged 5–12 years followed, in which we explored the relationship between students' counting, grouping, partitioning, and place-value knowledge and their development of the base ten numeration system (Thomas et al. 2002). We were able to describe several mathematical structural features in their representations: counting and symbols, number patterns and sequences, groupings by tens, use of ten as an iterable unit, recursive grouping, and multiplicative structure supporting place value knowledge. We found a wider use of structure than we had anticipated. For example, Fig. 1 shows how two Grade 2 students represented the number 11. Both drawings show some attempt to utilize array structure, but one is far more sophisticated than the other.

One of the key findings of this study was that students who used a variety of images to represent counting and numeration were more flexible in their thinking and

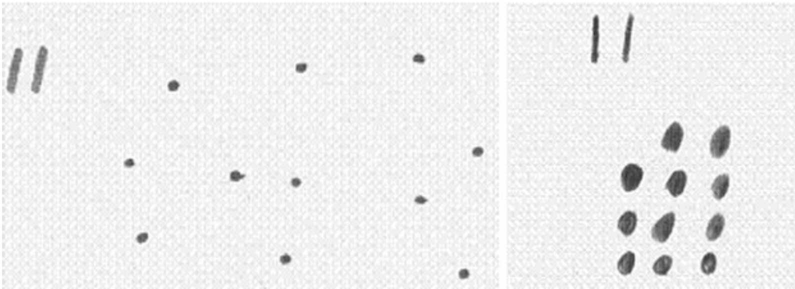


Fig. 1 Two Grade 2 students' representations of the number 11

tended to notice similarities and differences between representations. For example, such a student might recognize a 5 by 5 square array in a 10 by 10 hundred square.

All our studies were consistent with the literature on the differential effects of imagery use in the development of elementary arithmetic (Gray et al. 2000) and the finding that students who recognize the structure of mathematical processes and representations tend to acquire deep conceptual understanding (Pitta-Pantazi et al. 2004). We formed the hypothesis that *the more a student's internal representational system has developed structurally, the more coherent, well organized, and stable in its structural aspects will be their external representations and the more mathematically competent the student will be*. What was unclear at this stage of the research was whether students were aware of structural features and able to apply them to a variety of situations. If so, what role could the recognition of such structural features play in forming mathematical concepts? These research questions guided the next stage of our research.

Awareness of Mathematical Pattern and Structure (AMPS)

In the light of the above hypothesis, we conjectured that young students might possess a general characteristic called *Awareness of Mathematical Pattern and Structure* (AMPS). Students with high AMPS would recognize and operate well with a variety of early mathematical patterns and structures, whereas students with low AMPS would have difficulty recognizing such patterns. High AMPS students would tend to look for similarities and differences in new patterns and broaden their structural understanding accordingly, whereas low AMPS students were likely to focus on idiosyncratic, superficial features and not notice underlying structural features. AMPS was thus considered "to have two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyze patterns). Both are likely to be general features of how students perceive and react to their environment" (Mulligan and Mitchelmore 2009, p. 39).

We tested our conjecture with a large study of Grade 1 students (Mulligan and Mitchelmore 2009). In particular, we posed the following three research questions:

1. Can the structure of young students' responses to a wide variety of mathematical tasks be reliably classified into categories that are consistent across the range of tasks?
2. Do individuals demonstrate consistency in the structural categories shown in their responses?
3. If so, is the individual student's general level of structural development related to their mathematical achievement?

We devised a set of 39 tasks that we judged, on the basis of previous research, to be likely to show structural development in students' responses and combined them into the *Pattern and Structure Assessment* (PASA) interview. The tasks involved key processes such as subitizing, unitizing, partitioning, repetition, spatial structuring, multiplicative and proportional relationships, and transformation; they all required students to identify, visualize, represent, or replicate elements of pattern and structure. Some of the tasks extended well beyond state curriculum expectations (e.g., constructing a pictograph) but these were included in order to yield a wider range of responses than might otherwise be expected. Examples of these tasks, and illustrations of student responses, are given later in this chapter.

Analysis gave positive answers to all three research questions. Firstly, we were able to classify almost all student responses to all tasks reliably into four levels of structural awareness:

- *Prestructural*. Students pick on particular features that appeal to them but are often irrelevant to the underlying mathematical concept.
- *Emergent*. Students recognize some relevant features, but are unable to organize them appropriately.
- *Partial structural*. Students recognize most relevant features of the structure, but their representations are inaccurate or incomplete.
- *Structural*. Students correctly represent the given structure.

Secondly, students were remarkably consistent in the structural level they showed across the various tasks. For every student, there was a clear modal response level, with the modal class frequency having at least twice the frequency of each other class. Thirdly, there was an extremely high correlation between students' structural level and the total number of correct PASA responses, which we took to be a measure of their mathematical achievement level. Moreover, teachers identified all the prestructural students as low achievers and all the structural students as high achievers.

A follow-up study investigated structural development among the eight lowest-achieving students and the eight highest-achieving students over the subsequent 18 months (Mulligan et al. 2005). It was found that PASA responses could still be reliably classified using the same four structural levels, but it was possible to add an additional level (called *advanced structural*) to accommodate responses that also generalized the underlying feature to other contexts. Consistent with earlier results, substantial differences were found between the two groups of students. The high achievers made significant progress over the 18 months and many of their responses fell into the advanced structural level. By contrast, low achievers' responses varied

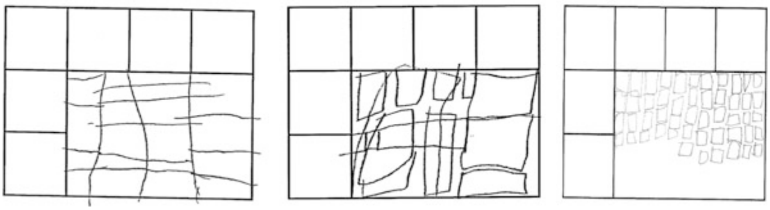


Fig. 2 A Grade 1 students' three attempts to complete a rectangular grid

from one interview to the next but did not show any progress towards greater use of mathematical structure. For example, Fig. 2 shows a low achiever's responses to the "complete the grid" task at the start, the middle, and the end of the study period. The student had some idea of shape, congruence, and collinearity, but was struggling to coordinate them. This student was judged to be at the emergent structural level on all three occasions.

We regarded the results of the two studies as strong support for the existence of AMPS as a psychological construct. In particular, they showed how low AMPS could interfere with the acquisition of fundamental mathematical understandings from an early age.

Examples of Structural Development

The remainder of this chapter is devoted to describing the development of structural awareness, drawing on examples from various studies in the Pattern and Structure Project. Students' drawn responses and explanations to six PASA tasks are illustrated. In each case, students were asked to visualize, then draw and explain their mental images. We use these examples to more clearly explicate the four levels of AMPS that we outlined above, together with the advanced level subsequently added.

We use examples from drawn responses because they most vividly illustrate students' development. Some caution should be exercised in interpreting young students' drawings because of possible problems with fine motor coordination at that age. However, it should be noted that students were routinely asked to explain the images they were attempting to represent and were offered the opportunity to repeat their drawings if they were not satisfied with them. Furthermore, a very similar pattern of development was observed in a wide variety of non-drawing tasks.

Structuring a Clock Face

In one PASA task, students are given a circle that is intended to represent an analogue clock and asked to complete the drawing.

Fig. 3 Prestructural representations of a clock face (drawn from memory by Grade 1 students)

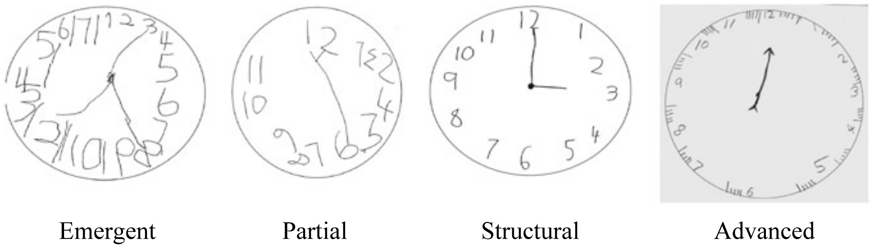


Fig. 4 Structural development in depictions of a clock face

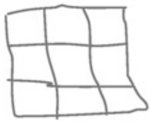
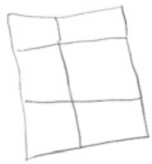
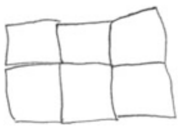
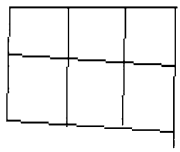
Figure 3 shows four students' attempts to represent a clock face. The first example only depicts one hand, and the next example suggests that the students knew there is also something around the circumference. The third and fourth examples are a little more advanced, suggesting the students were aware that a clock face has numbers on it and ticks around the edge. In none of these drawings is there any indication that the students perceived the structure of the clock face in terms of units of time.

Figure 4 shows some typical drawings at the subsequent levels, from which we can make inferences about students' developing awareness of structure. Students at the emergent level are aware that a clock face has some numbers in unitary counting order around the edge, starting at 1 near the top. At the partial structural level they realize that the numbers are restricted from 1 to 12, with 12 at the top. At the structural level, students have formed a clearer idea of how the numbers depicting hours are distributed in equal spaced intervals around the clock. At the advanced level, students find the positions of 12, 3, 6, and 9 before filling in the other numbers and may indicate one minute intervals between the numerals.

Structuring Rectangular Grids

Another task requires students to briefly view a rectangular grid (2×3 or 3×4), draw it from memory, find the total number of squares, and explain their strategy. Table 1 outlines the criteria for classification of the drawings, with examples of some typical responses to the 2×3 task. Development consists of increasing awareness and coordination of the shape of the squares, their number, and how they fit together.

Table 1 Typical student drawings of a 2 × 3 rectangular grid, by structural level

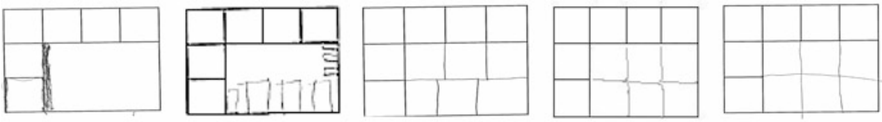
Structural level	Description	Examples
1. Prestructural	Scattered squares or a row of squares.	
2. Emergent	An attempt to draw a grid or border, but neither their numerical nor their spatial structure is correctly represented.	
3. Partial structural	A grid that is incomplete or inaccurate in terms of the number of squares drawn. For example: (a) a grid with an incorrect number of squares (b) a correct grid but drawn in the wrong orientation (c) an array with the correct number of squares, but with the squares not touching.	  
4. Structural	A grid of adjacent squares with the correct number of rows and columns, but with each square drawn separately.	
5. Advanced	A grid of the correct number of squares, drawn using continuous horizontal and vertical lines.	

Notice that drawings at both the structural and advanced levels are "correct", but the advanced level representation suggests a further advance in understanding that allows generalization to different tasks of a similar nature (see the next item).

Structuring Area

Students are shown a 3 × 4 rectangle with squares drawn along two adjacent sides and instructed to "finish drawing the squares, exactly like these, to cover all of this shape". We have already shown how one Grade 1 student responded to this task at the emergent level (Fig. 2). Figure 5 shows some typical responses at all five levels.

Structural development on this task is very similar to that shown by the previous task (see Table 1). Students at the prestructural level know only that there has to



Prestructural Emergent Partial Structural Advanced

Fig. 5 Structural development in grid completion task

be something in the empty space and do not always draw squares. Emergent students appear to have picked up on one or two features of grids (e.g., congruence, alignment, contiguity, or borders) and reproduce this feature in some form without filling the space systematically. Partial structural students attempt to fill the space, but do not draw the correct number of squares or do not draw them touching correctly. Students at this level may indicate the beginning of multiplicative thinking when they remark on the equal number of squares in each row or column. Structural students have coordinated all the crucial features of a grid and know exactly where the squares go; but they still draw them individually. By contrast, advanced structural students know that the edges of the contiguous squares form straight lines and use this fact to construct the grid very quickly. Students at this level draw grids of all sizes in the same way and are more likely to use multiplication to find the total number of squares.

Structuring a Triangular Array

In another item, students are briefly shown a card with a triangular pattern of six dots on it and asked to draw it from memory. The drawings in Fig. 6 show the progress of a single student over an 18 month period. The first drawing was made at the beginning of the Kindergarten year; it is classified as emerging because it is made up of dots and has some indication of a triangular outline. The second drawing was made by the same student at the end of the Kindergarten year and already shows the triangular pattern adequately. In the third drawing, made in the middle of Grade 1, the student has not only copied the given pattern accurately but has also extended it

Emergent Structural Advanced

Fig. 6 Three drawings of the triangular array made by the same student

Prestructural (Interview 1) Prestructural (Interview 2) Prestructural (Interview 3)

Fig. 7 Three drawings of the triangular array made by a different student

more or less accurately to a fourth row. It appears that the student could extend the pattern indefinitely; the drawing is therefore classified as advanced structural.

In contrast, Fig. 7 shows the very limited progression made by another student over the same 18-month period. There is no evidence of any progress. The student does not appear to recognize collinearity or triangularity, and would probably have great difficulty in any mathematical topic where such ideas occur.

Structuring Length

In a further item, students are shown a drawing of a long thin rectangle, asked to imagine that it is a ruler and asked to "draw things on it so that it becomes a ruler you can measure with". They are then asked to actually use it to measure the length of a pencil. Figure 8 shows typical drawings at each of the first four levels of structural development; the fifth level is achieved when a student explains how the number on the ruler gives the length in terms of a unit of measurement.

The progression on this task parallels that of the clock task. A vague awareness that there are some markings on a ruler gives way to knowledge that these are ticks and numbers, which are then coordinated until equal spacing is achieved.

Structuring Data

In another PASA task, students are given a table listing a collection of animals (7 dogs, 5 cats, and 3 birds) and asked to draw a graph to represent these data. Grade 1 students are expected to draw a horizontal pictograph, the only type they may have met in class. Figure 9 shows some typical drawings at the first four levels; the fifth level is achieved when a student explains that the largest number of animals is shown by the longest row (so that you do not need to count each row).

Prestructural drawings show one or more animals, not arranged in any order. Emergent responses pick up on one or two aspects of pictograms; the example in Fig. 9 shows an awareness that there are icons arranged in rows, but the numbers

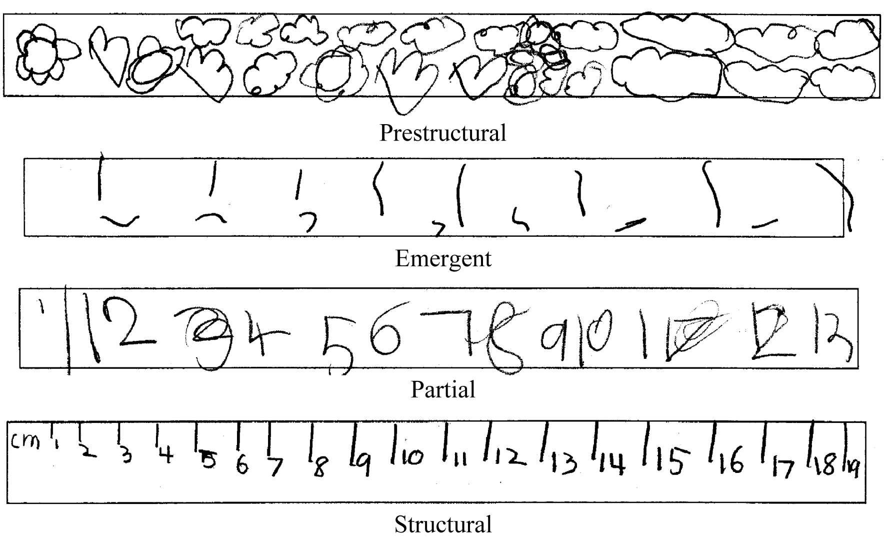


Fig. 8 Typical drawings of a ruler at the first four structural levels

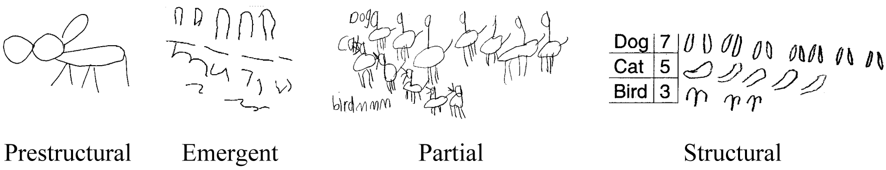


Fig. 9 Typical pictograms drawn at the first four structural levels

bear no relation to the data. In partial structural representations, the icons are arranged in rows with the correct numbers of icons, but the icons are not aligned. Finally, at the structural level the icons are aligned vertically so that the numbers of each type of animal can be compared using the lengths of the rows. (Notice the similarity to array structure.) From here, it is a small step to the basic principle of graphical representation that equal numbers are represented by equal lengths.

Discussion

A growing body of research has established that a large and significant portion of students' mathematical thinking in early childhood can be described in terms of a growing awareness of pattern and structure. Our research has enabled us to reliably describe this development in terms of a number of structural levels.

As can be seen from the examples above, young students initially show no evidence that they are aware of the mathematical structure implicit in a range of tasks. They tend to focus on superficial features, which they select and organize in their own way. Gradually, however, they become more and more aware of mathematically relevant characteristics—various elements (numbers, lines, shapes) and their arrangement (e.g., along a line or circumference, equally spaced, contiguous, aligned). In each task we have examined, it has been possible to reliably classify the growth of this awareness into four levels, starting with a single aspect of structure and culminating in a coordinated representation of all relevant aspects. Some young students then make one further step. They realize that the structure is more general than the specific task, enabling them to succeed on similar tasks or simple extensions. However, some young students do not progress through these levels at all. They may notice some aspects of mathematical pattern, but they do not become sufficiently aware of how these aspects are organized to be able to advance beyond the prestructural level. In fact, their representations often become more crowded and chaotic.

Our research shows that, not only can students' levels of structural development through these five levels be reliably categorized, but students who show a particular level of development on one task typical of the early mathematics curriculum also operate at a similar level on other such tasks. This finding indicates an underlying construct that we call Awareness of Mathematical Pattern and Structure (AMPS).

Our studies have also highlighted some big ideas underlying the development of structural understanding in early childhood. The first is that of generality. Number patterns are a prime example: There is always some general rule telling you how the pattern is to be continued, whether it be a simple repetition of what has come before or a continuation of some form of growth. Other generalizations that can arise from the study of numerical patterns include commutativity of addition and multiplication. The study of spatial patterns leads to concepts such as collinearity, congruence and symmetry and the formulation of general properties of basic two-dimensional figures. For all the PASA tasks, the highest structural level always corresponds to awareness of some such generalization. Finding and expressing such numerical and spatial generalizations is the beginning of algebra and geometry, respectively.

A second important big idea is that of equal grouping. The perception of a repeating pattern as a number of identical "chunks" leads to the idea of skip counting and then multiplication. Rectangular arrays and grids can be deconstructed into numbers of equal rows or columns, so the same counting strategies can be applied. Number lines and measurement scales represent replications of identical units and so are also multiplicative, as is the decimal numeration system. Even graphical representation requires equal grouping; for example, in a frequency chart each object must be represented by a fixed unit. Equal grouping is also required when a set of objects or a quantity is partitioned into equal parts, the foundation of the concepts of division and fractions.

Students with high AMPS are likely to have a better understanding of both these Big Ideas than those with low AMPS. They are likely to look for, remember and apply spatial and numerical generalizations and in particular are likely to grasp the multiplicative relationships that underlie the majority of the concepts in the elementary mathematics curriculum. Not only are they better placed to understand proportionality, but they have been primed to use mathematical reasoning in other areas.

So it should not be surprising that students with high AMPS do well in mathematics and that, conversely, students with low AMPS struggle (Mulligan 2011).

Conclusion

We regard the AMPS construct as a significant contribution to research into early mathematics education. It provides a single lens with which to examine student thinking in a wide variety of mathematical topics encountered in early childhood and thus yields a more uniform approach than can be obtained by examining each topic separately.

Our research has focused on students up to the second year of formal schooling, but our occasional explorations with older students suggest that the AMPS construct could also have a much wider application. In particular, there is the possibility that low AMPS in early childhood could predict poor performance in mathematics throughout a student's school career and even beyond. Extending the AMPS construct to the later years of schooling is a promising field for further research.

Our research has also thrown up two vital questions: Is AMPS a fixed trait or can it be taught? If AMPS can be improved by teaching, will there be a concomitant improvement in mathematics achievement? A positive answer to the second question would have particularly far-reaching implications for mathematics curriculum, pedagogy, and assessment. Both questions were addressed within the Pattern and Structure Project by developing the *Pattern and Structure Mathematics Awareness Program* (PASMAP) to teach AMPS. These studies—which eventually led to the large-scale evaluation study, *Reconceptualizing Early Mathematics Learning*, that inspired this volume—are described in chapter *Reconceptualizing Early Mathematics Learning: The Fundamental Role of Pattern and Structure*, p. 47.

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Reconceptualizing Early Mathematics Learning: The Fundamental Role of Pattern and Structure

Joanne T. Mulligan, Michael C. Mitchelmore, Lyn D. English, and Nathan Crevensten

In chapter *Early Awareness of Mathematical Pattern and Structure*, we introduced the Pattern and Structure Project, which focused broadly on the development of patterning and structural development among 4 to 8 year olds in this project. Our research has aimed to find reliable and consistent methods for measuring and describing the growth of students' structural development in mathematics. We provided a rationale for the construct, *Awareness of Mathematical Pattern and Structure* (AMPS), which our studies have shown generalizes across early mathematical concepts, can be reliably measured, and is correlated with mathematical understanding (Mulligan and Mitchelmore 2009). Our belief is that the development of AMPS can bring more coherence to mathematical development but this needs the support of an innovative pedagogical approach and framework.

The challenge was to identify core features of AMPS and to design pedagogy that explicitly improves students' awareness of pattern and structure. To that end, the *Pattern and Structure Mathematics Awareness Program* (PASMAP) was developed concurrently with the studies of AMPS and the development of the Pattern and Structure Assessment (PASA) interview. The culmination was a large-scale two-year longitudinal study, *Reconceptualizing Early Mathematics Learning* (REML), which was the inspiration for this volume.

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In this chapter, we summarize some early classroom-based teaching studies and describe the PASMAP that resulted. We then outline the REML project and discuss the consequences for our view of early mathematics learning.

Classroom-Based PASMAP Studies

PASMAP had its origin in a year-long numeracy initiative where our pattern and structure approach was trialed in a New South Wales metropolitan state school experiencing disadvantage and low achievement in numeracy. The project aimed to develop in students an Awareness of Mathematical Pattern and Structure (AMPS) based on structural aspects of mathematical development that had been identified in previous studies. A research team worked for a year with 27 primary teachers from Kindergarten to Grade 6 (683 students in total) to scaffold learning with small groups of students within regular classroom time (Mulligan et al. 2006). Priority was given to the professional learning and support of six lead teachers from Kindergarten to Grade 3.

Every teacher administered a PASA interview to all of their students and the results were then used to allocate students to small groups for instruction. The PASA data comprised PASA total scores, and students' strategies and drawn representations of solution processes. Data were summarized by the researchers for common response patterns for individual students, teachers and grade levels. A framework for developing and implementing a structural approach to learning mathematics was then developed by the research team in collaboration with participating teachers. The number system, counting patterns, multiplication and division, partitioning, and fractions comprised the main focus.

Several professional development meetings supported the planning and implementation of the PASA and PASMAP, assisted by input from the school's learning support and mentoring teams. PASMAP was implemented across the school for two consecutive terms, each of ten weeks duration. Teachers integrated PASMAP learning experiences into their regular mathematics program to varying extents, depending on the needs of the students and the support available. To assess progress, the PASA interviews were repeated at the end of the intervention.

The results showed a marked improvement in correct responses and an increased proportion of responses classified at the more advanced partial and structural levels of development.[1] The improvement was most marked in the Kindergarten and Grades 1 and 2 where the most intensive support had been focused. Figures 1 and 2 summarize the PASA data for Kindergarten and Grade 1 students.

Substantial improvements were also found in school-based and system-wide measures of numeracy achievement (NSW Department of Education and Training 2002), although they were less pronounced in the upper primary years. For example, on the Schedule of Early Number Assessment (SENA 1) 89 % of students were

[1]See chapter *Early Awareness of Mathematical Pattern and Structure*, page 12 for an explanation of the various structural levels.

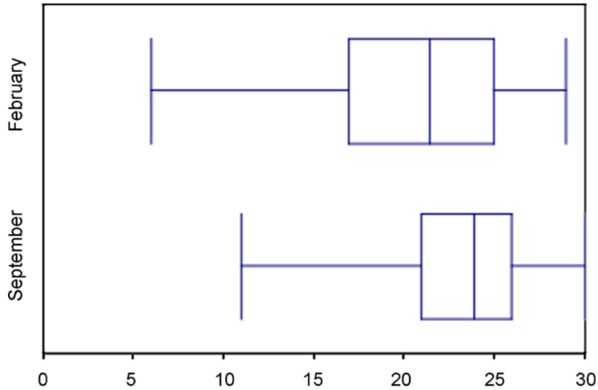


Fig. 1 Box and whisker plot of Kindergarten and Grade 1 students' (7 × Early Stage 1 classes) pre-(February) and post-assessment (September) PASA scores ($n = 134$)

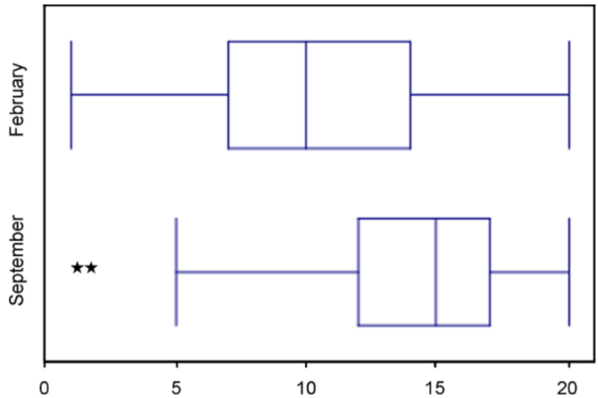


Fig. 2 Box and whisker plot of Grades 1 and 2 (6 × Stage 1 classes) students' pre-(February) and post-assessment (September) PASA scores ($n = 120$)

categorized at the first three levels of counting and arithmetic knowledge (emergent, perceptual and figurative counting) at pre-test; the post-test proportion of students at these lower levels was only 56 %. Similarly, on the NSW Basic Skills Testing Program Numeracy trend data, 23 % of Grade 3 students increased numeracy scores from Levels 1 and 2 to Levels 3 or 4 but a smaller proportion at Grade 5 (16 %) showed such an increase. The marked improvements shown in the SENA data were achieved mainly in the Kindergarten and Grades 1 through 3, possibly because the lead teachers were most consistent and given considerable support in comparison with the upper grades.

This classroom-based work allowed PASMAP to be trialled with a large number of students who struggled to achieve basic numeracy. Adopting the structural approach encouraged teachers and students to recognize similarities and differences in mathematical representations and to form simple generalizations. A focus on multiplicative concepts (including understanding the base ten system, grouping, and partitioning) was found integral to building structural relationships in early mathematics and spatial structuring was necessary to visualize and organize these structures. Teachers were therefore encouraged to focus students' attention more explicitly on spatial structuring in the development of number concepts including for example, the use of number patterns and the construction of base-ten knowledge. Teachers found this approach a novel way of teaching compared to the traditional focus on number concepts and skills in isolation.

Despite the promising results, a pilot project of this scale had many limitations and it was not possible to generalize the findings to other settings. Many teachers struggled to understand the goal of developing mathematical relationships and simple generalizations. However, the evaluation data obtained from the teachers were invaluable in informing the subsequent development and expansion of PASMAP.

Preschoolers' Patterning

A concurrent study by Marina Papic (Papic et al. 2011) was conducted in the prior-to-school context in the belief that the early development of patterning could provide a foundation for successful mathematical development. Papic found that preschoolers' awareness of pattern could be reliably assessed and that development could be scaffolded through a framework of patterning experiences. Since Papic's study is described in more detail in chapter *The Role of Picture Books in Young Children's Mathematics Learning*, we summarize some key findings.

In one pre-school, Papic worked with the teachers to develop a 6-month intervention focusing on mathematical patterns. A matched preschool acted as a comparison group. Individual task-based interviews were conducted before and after the intervention, and the children from both groups were followed up into their first year of formal schooling. Children in the intervention program showed more advanced patterning skills than the non-intervention sample at the end of the pre-school year. Compared to the non-intervention group, the intervention children created far more complicated patterns, they were able to solve growing pattern tasks (which had not been the included in the intervention), and they showed higher scores on a standard numeracy assessment, the SENA (Department of Education and Training 2002). It was also found that teachers in the intervention pre-school had spontaneously amended their whole cross-curricula activities to take advantage of many more patterning opportunities than had been included in their original curriculum.

In the patterning program, children were frequently exposed to the concept of a unit of repeat as configurations were broken down into identical "chunks". They also engaged in skip counting (e.g., "2, 4, 6") that promoted the language of multiplication (e.g., "3 times"). The patterning experiences may have promoted conceptual understanding of the idea of composite unit that is fundamental to multiplicative reasoning. This development may have made it easier for the children to use base-ten structure and other multiplicative concepts more effectively in Kindergarten. We suggest that the patterning program had in effect strengthened the preschoolers' AMPS—not only in terms of their understanding of fundamental concepts but, perhaps more importantly, in encouraging them to look for and analyze patterns. One result was a level of understanding that readily transferred to more complex patterning and counting tasks one year later.

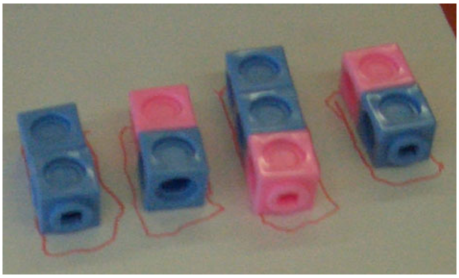


Fig. 3 Kindergarten child's incorrect attempt to make an AAB pattern in chunks

An Intervention Study with Kindergarten Students

Inspired by the promising results in the study with preschoolers, PASMAP was further developed as a connected set of instructional sequences that integrated patterning (repetitions and growing patterns) and functional thinking, units of space and measurement, spatial structuring and number sense, skip counting and multiplicative processes. Using a design study approach, Mulligan and colleagues explored the impact of PASMAP on mathematics learning with a group of ten students aged 4 to 6 years in the first year of formal schooling (Kindergarten in the state of NSW), who had been identified by teachers as needing additional support in numeracy (Mulligan et al. 2008). A specially trained, experienced classroom teacher engaged the students in PASMAP tasks over 15 weekly teaching episodes. Tasks were designed and modified continuously, and differentiated for individuals. Students were assessed pre- and post-intervention using a revised PASA interview and two sub-tests of the Woodcock-Johnson mathematics test (Woodcock et al. 2001).

Every student showed improvement on PASA scores, with seven of the ten making marked improvements (Mulligan 2011). There were however no significant gains found on the Woodcock-Johnson test scores; possibly the limited 15-week period did not allow sufficient time to show such growth. An alternative explanation is that the test was not sensitive enough in scope or depth to detect conceptual problems related to mathematical pattern and structure. Advancement in structural development was clearly evident in students' solution strategies, their representations, and their explanations of their responses. There was evidence that students invented symbolizations and made emergent generalizations and marked growth in representing, symbolizing and translating simple and complex repetitions, structuring arrays and grids and unitizing area. However, these improvements were not necessarily consistent across tasks. It was concluded that PASMAP would have to be implemented over a longer period if it was to have a measurable effect on mathematical achievement.

Consistent with the work of Papic, students represented simple repetitions and growing patterns in a variety of forms. We explicitly focused on "chunking" (breaking the unit of repeat into sections) and placing in the pattern sequence (see Figs. 3, 4, and 5).

Figure 5 shows a student's drawing of an AAB repetition that they have made as a tower with two different colored blocks. The student symbolizes this pattern as a

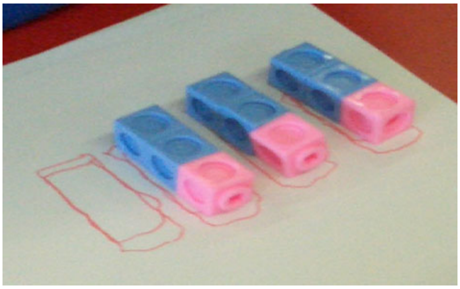


Fig. 4 Kindergarten child's correct attempt to break an AAB pattern of blocks into chunks

Fig. 5 Kindergarten student's representation of BBA repetition using two different symbolizations

'BBA' repetition and writes the correct sequence to the left hand side of the drawing. When asked if they could write this pattern in another way so that their friend could make the same pattern the student uses symbols 0 and X. They explain that these are the symbols you use when playing noughts and crosses but when making this pattern its '00X' repeated. The student has retained the initial pattern structure and developed a correct but different symbolization of the unit of repeat.

Similarly, improvements in recognition of subitizing patterns, counting in multiples in 2s, 3s and 4s and some partitive grouping strategies were also observed. This improvement could be explained by the varied and repeated PASMAP experiences in grouping and patterning using a unit of repeat. The development of spatial structuring through individuals' representations was encouraged, such as congruence of shapes, partitioning, and collinearity (see Figs. 6, 7 and 8).

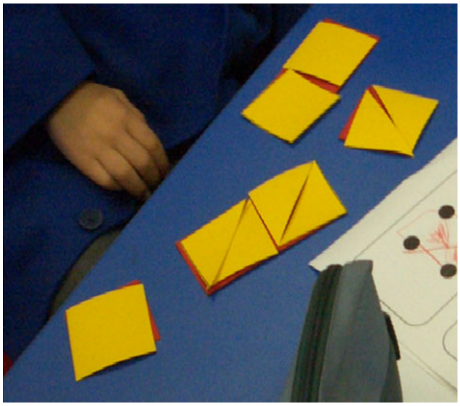


Fig. 6 Child correctly aligns squares and triangles congruently on square and rectangular cutout shapes placed underneath

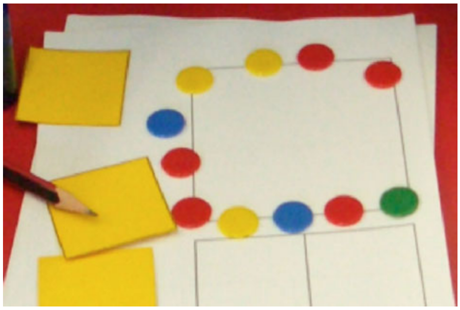


Fig. 7 Child places counters randomly around the border without noticing corners or sides of equal length [The smaller squares are later used to structure the square into quarters]

In Fig. 6 the child is required to place congruent triangles on single squares or rectangles so that the shapes are aligned. Recognizing corners and/or the symmetry of the triangles was observed.

In Fig. 7 the task required the child to place counters evenly spaced and aligned around the border of a square.

In Fig. 8 the task required the child to place counters on each of the corners; observation of the process by which the child noticed the corners and placed the counters, albeit unsystematically was observed.

Another important observation was that students were initially unable to represent simple arrays and grids beyond a pattern of four units but by the end of the program students could more readily represent the structure of rectangular grids and arrays Students' construction of a simple table of data showing functional thinking was also demonstrated in the final teaching episodes (Fig. 9). The students explained that "for each dog you have 4 legs, so its 4, 8, 12, 16, 20, . . . , for 5 dogs".

This project illustrated the rich and diverse learning experiences by ten young students in a program focused on structural awareness. However, the intervention was limited to a small group of students withdrawn for individualized instruction, and supported by specialist teachers and well-formulated resources, and we could not assume that the success of this program could be generalized to other settings. Nevertheless, our data did suggest that explicit assessment and teaching of struc-

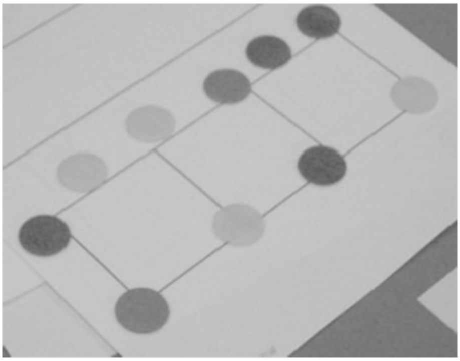


Fig. 8 Child places counters randomly on corner positions first but then attempts to fill the spaces in the border

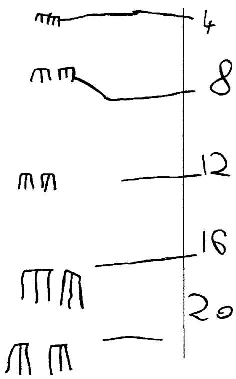


Fig. 9 Kindergarten student's attempt to represent the number of legs on a dog (4 legs) on increasing number of dogs

ture had the potential to effectively improve students' abstraction of mathematical processes.

Summary of Early Research Findings

The early studies suggested or confirmed several mechanisms whereby a focus on pattern and structure awareness promotes general mathematical development:

- Students become more aware of crucial structures such as rectangular arrays.
- Through the study of these structures, they more easily learn basic properties of number, space, and measurement.
- Students learn to break down unfamiliar large patterns into smaller patterns that they are familiar with—a process known as *unitizing* (Lamon 1996).
- The emphasis on reasoned comparisons, justifications, and generalizations focuses students' attention away from non-mathematical pattern features and develops a tendency to look for and explain patterns in new experiences.
- Identifying similarities and differences leads to abstraction and generalizations.

The early classroom-based studies discussed above provided strong support to the hypothesis that teaching young children about pattern and structure should lead to a general improvement in the quality of their mathematical understanding. However, none of the studies had a sufficiently large or representative sample, most lacked a comparison group and there was insufficient opportunity to track and describe in depth and the growth of structural development. A more comprehensive study was needed to evaluate these findings more systematically over a longer period of time within the regular school setting.

The *Reconceptualizing Early Mathematics Learning* Project

A new large study, the *Reconceptualizing Early Mathematics Learning (REML)*, was therefore designed to evaluate the effects of PASMAP on student mathematical development in the first year of formal schooling. The aims of the study were to:

- Evaluate the effectiveness of a school-entry PASMAP on student mathematical development and using classroom observations, interview-based student assessment and standardized assessment.
- Document in detail the impact of PASMAP on learning mathematics.
- Track and describe students' structural growth, particularly of high- and low-achievers, through fine-grained analysis of the growth of structural awareness.

PASMAP was also evaluated in terms of professional learning of teachers as they were supported in developing and evaluating the new approach.

The Sample

A purposive sample of four large primary schools, two in Sydney and two in Brisbane, Australia, comprising 316 students from diverse socio-economic and cultural contexts, participated in the evaluation throughout the 2009 school year. At the follow-up assessment in September 2010, 303 students were retained. From pre-test data two focus groups of five students in each class were selected from the upper and lower quartiles, respectively. These 190 students were monitored closely by the teacher and research team throughout the study.

Procedure

Two different mathematics programs were implemented: In each school, two Kindergarten teachers implemented the PASMAP and two implemented their regular program. A researcher visited each teacher on a weekly basis and equivalent professional development was provided for all teachers. The PASMAP framework was embedded within but almost entirely replaced the regular Kindergarten mathematics curriculum. Features of PASMAP were introduced by the research team incrementally, at approximately the same pace for each teacher, over three school terms (May–December 2009). However, implementation time varied considerably between classes and schools, ranging from one 40-minute lesson per week to more than five 1-hour lessons per week.

The PASMAP Components

Core components of the PASMAP and the pedagogical approach focused explicitly on the development of students' spatial structuring, multiplicative reasoning, and emergent generalizations rather than developing procedural skills or number concepts in isolation from other concepts. PASMAP provided an instructional approach where concepts were scaffolded and linked together to promote early algebraic thinking based on earlier approaches advocated by Blanton and Kaput (2005) and Carraher et al. (2006). Although the framework was in its developmental stage, these components could be described as potential trajectories of learning in a similar way to those described by Clements and Sarama (2009). Drawing on previous and current research on spatial structuring and early algebra, the PASMAP program comprised subitizing and spatial arrangements (Bobis 1996; Hunting 2003); simple and complex repetitions, growing patterns and functions (Warren and Cooper 2008); spatial structuring (Battista 1999; van Nes and de Lange 2007); the spatial properties of collinearity, congruence and similarity and transformation; the structure of measurement units and data representation, unitizing and multiplicative structure; the structure of counting sequences and base ten, and equivalence and inverse operations. Emphasis was also laid on the development of visual memory and justification for simple generalizations. Students were encouraged to seek out and represent pattern and structure across different concepts and transfer this awareness to other concepts.

This awareness was achieved through pattern-eliciting tasks that required students to use spatial structuring to copy or reproduce a model or other representations. (For examples see later sections in this chapter.) The teacher used probing questions to highlight important features of their models and drawings, to compare them with others, and to focus their attention on similarities and differences in crucial aspects of spatial and numerical structure. Tasks were modified and repeated regularly, reinforcing and extending generalizations and providing links to prior learning in a similar way to earlier studies.

Assessment Interviews and Classroom Data

All students were administered the *I Can Do Maths* (ICDM) standardized test of general mathematics achievement (Doig and de Lemos 2000) at the beginning and end of the 2009 school year and again in mid-2010. From the pre-test data, two focus groups were selected in each class consisting of five students from the upper and lower quartiles, respectively. These students were interviewed in more detail using the PASA in February 2009, December 2009, and September 2010, the number of students varying from 190 to 170. An additional "extension" version of PASA was also administered in September 2010. The PASA items were parallel on all three occasions, but increased somewhat in complexity to take account of students' development.

Other evaluation data included video for a sample of PASMAP lessons for evidence of AMPS and students' articulation of emergent generalizations. Analysis focused on the high ability and low ability focus students. Students' explanations and drawn representations, and photos of their responses to tasks were collected during the implementation of PASMAP and were coded immediately after each lesson for level of structural development. Evidence of student work, usually in the form of worksheets, was also collected for focus students in the regular classrooms. This evidence was digitally scanned and placed in individual profiles of learning. As well, teachers' views of the impact of the program on student learning and their own professional learning was collected and later analyzed.

Results

Quantitative Outcome Analysis

Analysis of the various PASA and ICDM scores showed the expected differences between ability levels and confirmed the equivalence of the two program groups. There was, however, a significant difference between the schools, with classes in the two Brisbane schools scoring lower than those in the two Sydney schools. No significant interactions were observed.

Total scores on the PASA and ICDM administered at the end of the intervention (December 2009) and at the retention point (September 2010) among the focus students were analyzed using analysis of covariance (ANCOVA). In each case, the covariates were the initial PASA and ICDM scores and the factors were school (one of four), ability (high vs. low) and program (PASMAP vs. non-PASMAP).

Analysis of the ICDM scores indicated no significant interactions or main effects apart from a school effect. In other words, the PASMAP and regular students made very similar gains on ICDM over the period of the study, but Sydney students gained more.

The analysis of the PASA scores also showed no significant interactions. However, there were two significant main effects at each point: a difference between schools, with the Sydney classes showing higher adjusted means than the Brisbane classes, and a difference between the program groups on each PASA assessment—modest at the end of the intervention ($p < 0.026$), highly significant at the retention point ($p < 0.002$), but only borderline ($p > 0.11$) for the extension section of the

Table 1 Analysis of covariance of PASA scores at retention point

Source	Type III sum of squares	df	Mean square	F	Sig.
Corrected model	1048.432[a]	17	61.672	10.380	0.000
Intercept	53.229	1	53.229	8.959	0.003
Covariate: PASA	158.346	1	158.346	26.650	0.000
Covariate: ICDM	14.071	1	14.071	2.368	0.126
School	117.125	3	39.042	6.571	0.000
Ability	15.259	1	15.259	2.568	0.111
Treatment	61.653	1	61.653	10.376	0.002
School * Ability	11.643	3	3.881	0.653	0.582
School * Treatment	43.663	3	14.554	2.450	0.066
Ability * Treatment	0.217	1	0.217	0.037	0.849
School * Ability * Treatment	13.589	3	4.530	0.762	0.517
Error	802.130	135	5.942		
Total	13412.000	153			
Corrected total	1850.562	152			

R squared = 0.567 (adjusted R squared = 0.512)

PASA. On each occasion, the PASMAP group scored higher than the regular group. Table 1 provides a summary the ANCOVA for the PASA at the retention point.

We inferred that the PASMAP treatment was effective in promoting the conceptual understanding of early mathematics, as measured by the PASA but not in improving mathematical achievement as measured by ICDM.

Rasch Scale Analysis

The PASA total scores and the ICDM scores were used to construct a single Rasch scale that incorporated all items along a continuum. The main advantage of using Rasch analysis for constructing the PASA scale was that it could be used to link different versions of the PASA used in this study (Andrich et al. 2001). The item map indicated that the PASA items and the students were reasonably well matched; in comparison, the ICDM items at the lower end of the scale did not sufficiently challenge the majority of students, although some more difficult ICDM items filled a gap between the PASA items (see Mulligan et al. 2011). The scale's order of item difficulty on PASA items provided a measure of the students' overall level of AMPS. Thus a conceptual analysis of the item and its position on the scale reflected the complexity of the task in terms of pattern and structure as well as the reasoning required to complete it successfully. What we aimed to achieve with the scale was

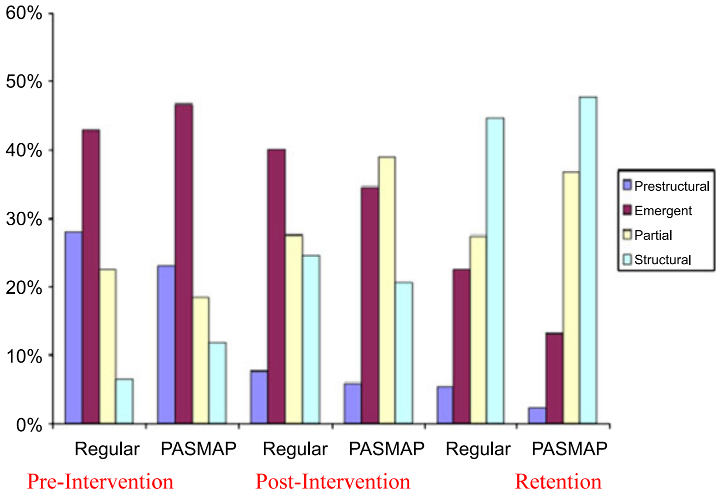


Fig. 10 Structural development across selected PASA items at three interview points Feb 2009 (Pre-intervention), Dec 2009 (Post-intervention), Sept 2010 (Retention) in two Sydney schools

a picture of how the PASA measure of AMPS fitted with a standardized measure of general numeracy ability over time.

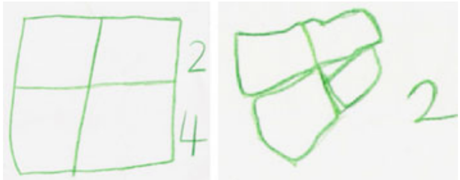
Structural Outcomes Analysis

To supplement the quantitative analysis, we provide some examples of the analysis of structural levels. Student responses on four PASA items requiring a drawn response at the three administrations were systematically coded for level of structural development (see chapter *Early Awareness of Mathematical Pattern and Structure*). Coding showed an inter-rater reliability of 0.91. Figure 10 summarizes the results for the Sydney students. It can be seen that the PASMAP students were initially slightly more advanced than the regular program students, with about 5 % more students in the partial structure and structural levels than the regular students. However, this difference grew in the subsequent administrations, reaching about 20 % at the retention point.

The following examples show how PASMAP learning experiences led to a deeper structural understanding of mathematical concepts and encouraged the development of emergent generalizations.

After a sequence of tasks focused on repetition and spatial patterns (Papic et al. 2011), there was a focus on constructing and analyzing simple grids. In the first of these, students were shown a 2×1 grid for a few seconds and then asked to draw it. The teacher then gave them a 2×1 grid and two matching squares and asked how many squares were needed to cover the grid. Different strategies for placing the squares were discussed, and students were also asked to fold the grid to explore

Fig. 11 Two contrasting representations of a 2 × 2 grid



Heela Lateh

the structure. The teacher then asked, “What’s the same?” and “What’s different?” and students encountered ideas such as counting, shape, sides and vertices, rotation (turning), congruence (same size and shape), and fractions (half). The grid and squares were then removed and students drew the grid from memory in both horizontal and vertical orientations. After sharing and discussing their drawings, the class summarized what they had learnt and looked for links to their earlier tasks (e.g., in the towers they had made from unifix cubes). This may have been a very elementary task, but it was fundamental to developing spatial structure and many students found it quite challenging.

The next lesson moved on to 2 × 2 grids (called “windows”), following a similar procedure. Previous ideas were reviewed and extended, and further ideas of rows and columns, clockwise and anticlockwise, vertical and horizontal, diagonals, and even quarters were encountered. The difference between the high- and low-ability students already became apparent, and student responses indicated to the teacher how perceptive some students were in terms of recognizing structural features while others paid little or no attention to mathematical features. Figure 11 shows two such contrasting drawings. Heela[2] had already recognized that she did not need to draw separate squares, whereas Lateh struggled to draw congruent squares in the standard orientation.

In subsequent lessons, the task was extended to larger rectangles. By repeatedly looking at what is the same and what is different between a given grid and their drawings, and by seeking generalizations from their observations, students gradually learned that a grid can be drawn using equally spaced, perpendicular lines. Each task reinforced the basic generalization that we call the ‘spatial structure’ of the grid. Discussion of similarities and differences between student’s drawings highlighted the crucial fact that a square grid contains the same number of equally sized rows and columns; further, the development of multiplication and commutativity emerged as well as area measurement. These ideas were further developed through a sequence of tasks focused on the pattern of squared numbers using square tiles and grid cards.

Students were initially given small plastic squares and asked to use them to make as many large squares as possible, in order of size, and to say how many small squares were in each larger square. To explore the structure of the pattern of squares students were given two sets of square grid cards (1 × 1, 2 × 2, 3 × 3, 4 × 4 and 5 × 5). After exploring systematic ways in which they could be fitted next to or

[2]Pseudonyms are used to preserve anonymity.

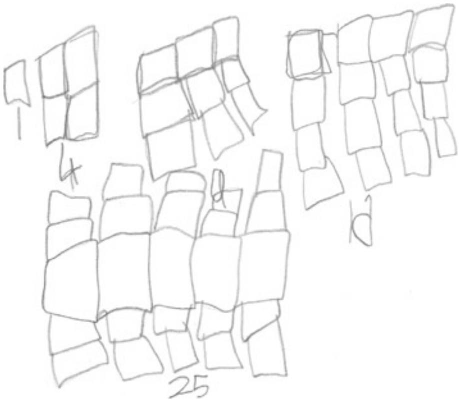


Fig. 12 'Low ability' Kindergarten student's drawing of emergent structure of the pattern of squares from memory

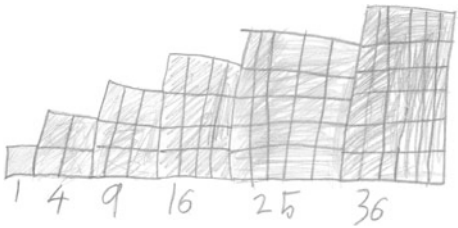


Fig. 13 'High ability' Kindergarten student's drawing of the pattern of squares from memory

on top of each other, or in various formations or sequence, the teacher posed the questions, "Can you see a pattern? How many small squares are there on each card? What is the best way to find out?" Students then cut up a second set of grid cards into rows or columns, place the cut-outs on top of the first set of cards, and discussed the numbers of rows or columns and the number of small squares in each. After examining the resulting number pattern (1, 4, 9, 16, 25), the teacher removed all the grid cards and cut-outs and challenged the students to reproduce the visual pattern from memory, first on grid paper and then on plain paper.

Figures 12 shows attempts by a 'low ability' student to draw the pattern from memory but the partial structure of the grid was counted and added as individual units. The student does, however, recognize the growing pattern of squares. In this case the student is assisted to use grid paper to form the squares in a sequence and to trace the rows and columns so as to develop collinearity. Figure 13 shows a 'high ability' student's structural development of the pattern of increasingly larger squares using the alignment of the growing squares. He visualizes and generalizes the pattern as "it goes up by one row and one column every time and it must be a square"; he also explains the numerical sequence as multiplicative "1, 2 by 2, 3 by 3, 4 by 4". His learning is extended by tasks such as "Can you work out what the tenth square will look like and can you continue the pattern? Can you make a growing pattern using triangles?"

In a follow-up task, students were given a 1×1 square and a 2×2 square and asked how many small squares fit on to the larger one. They were then given further 2×2 squares and asked to find the number of small squares in total, thus

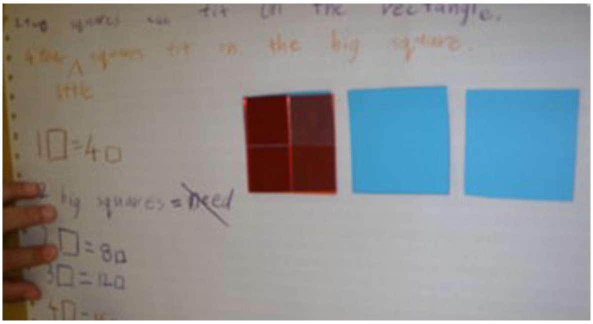


Fig. 14 Heela's use of the composite unit of 4 squares as a functional relationship

constructing the sequence 4, 8, 12, ... Finally, they were asked to generalize their findings. Heela invented a perfectly good means of symbolizing her results that closely resembles algebraic notation (see Fig. 14). In fact, she was treating the task as a functional relationship rather than a simple pattern continuation. Asked what she had learnt from the exercise, she said "I made a pattern so 1 big square is 4 little squares. So it's 4 for each square. Every time you use the square it's a four." Further tasks showed that she had generalized the relationship to all sizes of square and, indeed, any type of rectangle.

Other tasks extended the basic (multiplicative) generalization to rectangles. For example, students were asked to relate the number of unit squares needed to cover a rectangle to the size of the unit.

During the PASMAP intervention students demonstrated development of AMPS throughout the learning episodes in their representations and explanations. There were particular gains found in the PASA items requiring extension of a growing pattern and use of ten as a composite unit. More advanced responses were found in the related areas of simple and complex repetitions, growing patterns, multiplicative thinking (skip counting, partitioning and fractions), equivalence, and structuring area and drawing of grids with collinearity. Students in the regular program did not focus on growing patterns, multiplicative ideas or structuring measurement so their responses were limited to simple repetitions, unitary counting and additive thinking, and conservation of area. One of the most promising findings was that the PASMAP focus students categorized as low ability were able to develop structural responses over a relatively short period of time. The same gains were not evident for the regular group.

Discussion

The PASMAP students (and participating teachers) were systematically guided through related teaching/learning experiences so that deep connections between concepts were formed. This was in contrast to the regular program students where the pedagogy changed focus, sometimes on a daily basis, from one concept to another without opportunity for development of structural understanding and without focusing on the relationships between concepts.

Qualitative analyses of students' profiles and the classroom observation data showed stark differences in the way that the PASMAP students developed mathematical concepts and reasoning skills. PASMAP explicitly focused on the promotion of students' awareness of pattern and structure: the analysis of students' learning showed that all the PASMAP students developed AMPS to varying extents and greater gains were made than for the regular students.

Because PASMAP focused intently on developing structural relationships and spatial structuring from the outset, the PASMAP students made direct connections between numerical, measurement and spatial mathematical ideas, and formed emergent generalizations such as those described in the previous section. For example, students began to link simple skip counting to more complex multiples and arrays through their experience of the unit of repeat in patterning and measurement contexts. The most able PASMAP students used particular spatial features of pattern and structure to build more complex ideas. For example, they partitioned a 10×10 square into quarters and recognized that each of these squares formed a 5×5 array, and knew that this quarter contained 25 squares from their experience of the growing pattern of squares. Regular students could also solve tasks requiring multiplicative thinking but these were considered by the students as separate mathematical skills; for example, they learnt the skip counting pattern of 5s in isolation from all other activities. These students could not explain what was similar or different, what was the connection between ideas, or form simple generalizations.

A small proportion of students in the regular program did produce structural responses in the post-intervention PASA interview although they had apparently not been given opportunities to describe or explain their thinking in class. It would seem therefore that more advanced students may develop AMPS regardless of the instruction they receive. However, our results are indicative that such students are likely to make greater progress in a program that encourages them to look for patterns and explain their structure.

We must interpret our findings in light of one possible confounding factor: the amount of time that individual PASMAP teachers devoted to the program implementation. Some PASMAP teachers completed only half of the program components while others completed almost the entire program and revisited concepts regularly. Thus, further analysis of the impact of PASMAP must consider individual teacher effect, at least in terms of time on task, in order to evaluate the program's full impact on developing AMPS.

Conclusions and Implications for Further Research and Teaching

The study produced a valid and reliable interview-based measure and scale of mathematical pattern and structure that revealed new insights into students' mathematical capabilities at school entry. The PASA interview data indicated significant differences between groups in students' levels of structural development (AMPS) at the second and third assessments. Students participating in the PASMAP program showed higher levels of AMPS than for the regular group, made connections between mathematical ideas and processes, and formed emergent generalizations.

There were no significant differences found between groups on the standardized measure, ICDM. There are two possible reasons for this. The ICDM assessed numeracy in a limited way using traditional multiple-choice paper and pencil format and was quite different to the PASA interview. Secondly, the content of ICDM was limited in scope and depth: these multiple choice tasks focused on unitary counting sequences, recognizing simple two-dimensional and three-dimensional shapes and informal units of measure. There were no items that assessed pattern and structure.

Our studies show encouraging results, but further longitudinal research is needed with larger samples and more diverse samples, as well as utilizing digital learning tools. In particular, research is needed to determine whether an explicit focus on pattern and structure could later promote robust algebraic development—for example, in functional thinking—as well as in other related areas of learning.

A successive longitudinal project *Transforming Children's Mathematical and Scientific Development*[3] *2011–2013*, is in progress which extends the initial study and employs the same research team with some students tracked through from the 2009–2010 study reported in this chapter. This new project explores the role of pattern and structure in mathematics and science learning in Grades 1 to 3. In particular, the role of AMPS in structuring data is being investigated. Students are engaged in an innovative program, usually withdrawn in small groups and taught by the research team in collaboration with the teacher on a weekly basis for a 2-year period. This research integrates English's research on data modeling with the study of pattern and structure (English 2012). As a result, it will be possible to describe the structural development of young children's mathematical and scientific thinking extended to a wider range of concepts than previously studied.

Related studies at Macquarie University have also investigated structural development in studies of preschoolers' use of virtual manipulatives and dynamic interactive software in constructing patterns (Highfield and Mulligan 2007). A recent design study describes the use of programmable robotic toys in terms of young children's representational structure of the dynamic pathways constructed in problem-solving tasks (Highfield and Mulligan 2009). Further, Goodwin studied the effect of digital media on young children's representations of fractions (Goodwin 2009). These studies suggest further possibilities for exploring early mathematics learning through digital technologies [see chapter *A Framework for Examining Technologies and Early Mathematics Learning*, this volume]. We question the impact of such technologies on children's developing AMPS.

Further research on the developmental precursors of AMPS is needed to determine why some children develop powerful mathematical structures and relationships in the prior to school years, while others may be impeded by idiosyncratic imagery throughout their early schooling. Further studies need to articulate the learning trajectories of very young children whose structural development is enhanced

[3] Australian Research Council Discovery Project DP110103586 (2011–2013).

by the PASMAP approach. There are many other factors that need investigating, for example, the impact of different early child rearing practices, approaches to learning in early childhood and early schooling, and possible cognitive-neuroscientific aspects—an emerging field of research in relation to mathematics learning (van Nes and de Lange 2007).

Teaching and learning mathematics through a pattern and structure approach may require fundamental changes to the way that mathematics learning, pedagogy, curriculum and assessment is conceptualized, structured, and implemented. The PASMAP approach promotes conceptual knowledge that is interrelated and pedagogical strategies that scaffold these interrelationships. Supporting teachers to implement a structural approach may require professional learning support to promote deeper understanding of key mathematical concepts and to develop increased teacher pedagogical content knowledge. The importance of pattern and structure in mathematics learning is reflected to some extent in the new *Australian Curriculum–Mathematics* under the Proficiencies (Understanding, Fluency, Problem Solving and Reasoning), which support mathematics learning as patterns, relationships and generalizations (ACARA 2012). However, the key interrelationships between concepts incorporated across the three stands of the *Australian Curriculum–Mathematics* are not foregrounded. A structural approach could support the development of deep conceptual understanding well beyond early algebra, and provide a framework for developing these Proficiencies more effectively.

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Reconceptualizing Statistical Learning in the Early Years

Lyn D. English

Introduction

This chapter argues for the need to restructure children's statistical experiences from the beginning years of formal schooling. The ability to understand and apply statistical reasoning is paramount across all walks of life, as seen in the variety of graphs, tables, diagrams, and other data representations requiring interpretation. Young children are immersed in our data-driven society, with early access to computer technology and daily exposure to the mass media. With the rate of data proliferation have come increased calls for advancing children's statistical reasoning abilities, commencing with the earliest years of schooling (e.g., Langrall et al. 2008; Lehrer and Schauble 2005; Shaughnessy 2010; Whitin and Whitin 2011).

Several articles (e.g., Franklin and Garfield 2006; Langrall et al. 2008) and policy documents (e.g., National Council of Teachers of Mathematics 2006) have highlighted the need for a renewed focus on this component of early mathematics learning, with children working mathematically and scientifically in dealing with real-world data. One approach to this component in the beginning school years is through data modelling (English 2010; Lehrer and Romberg 1996; Lehrer and Schauble 2000, 2007).

Data modelling is a developmental process, beginning with young children's inquiries and investigations of meaningful phenomena, progressing to identifying various attributes of the phenomena, and then moving towards organising, structuring, visualising, and representing data (Lehrer and Lesh 2003). As one of the major thematic "big ideas" in mathematics and science (Lehrer and Schauble 2000, 2005), data modelling should be a fundamental component of early childhood curricula. Limited research exists, however, on such modelling and how it can be fostered in the early school years. The bulk of the research has focused on secondary and tertiary levels, with the assumption that primary school children are unable to develop

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their own models and sense-making systems for dealing with complex situations (Greer et al. 2007).

Recent research has indicated that young children do possess many conceptual resources that, with appropriately designed and implemented learning experiences, can be bootstrapped toward sophisticated forms of reasoning not typically seen in the early grades (e.g., Clarke et al. 2006; Clements et al. 2011; English and Watters 2005; Lesh and English 2005; Papic et al. 2011; Perry and Dockett 2008). Most research on early mathematics learning has been restricted to an analysis of children's actual developmental level, which has failed to illuminate children's potential for learning under stimulating conditions that challenge their thinking. "Research on children's current knowledge is not sufficient" (Ginsburg et al. 2006, p. 224). This sentiment was further expressed by Perry and Docket (2008):

> . . . young children have access to powerful mathematical ideas and can use these to solve many of the real world and mathematical problems they meet. These children are capable of much more than they are often given credit for by their families and teachers. . . The biggest challenge. . . is to find ways to utilise the powerful mathematical ideas developed in early childhood as a springboard to even greater mathematical power for these children as they grow older. . . (p. 99).

This chapter is structured as follows. Consideration is given to data modelling with a specific focus on structuring and representing data including the use of conceptual and metarepresentational competence, informal inference (making predictions), and the role of context. A longitudinal study of data modelling in grades one to three is then addressed followed by a selection of findings.

Data Modelling

The starting point for developing statistical reasoning through data modelling is with the world and the problems it presents, rather than with any preconceived formal models. Data modelling is a developmental process (Lehrer and Schauble 2005) that begins with young children's inquiries and investigations of meaningful phenomena, progressing to deciding what aspects are worthy of attention and how these might be measured, and then moving towards structuring, organising, analysing, visualising, and representing data (as indicated in Fig. 1). Conceptual competence and metarepresentational competence play a significant role in the overall modelling process, as indicated later in this chapter.

The model created, which provides a solution to the children's original question/s, is repeatedly tested and revised, and ultimately allows children to draw informal inferences and make recommendations from the original problem and later, similar problems. Children's generation, testing, and revision of their models, which lie at the core of what it means to reason statistically, is an important developmental process.

A data modelling approach to statistical reasoning differs in several ways from what is typically done in early classroom experiences with data. In particular, data

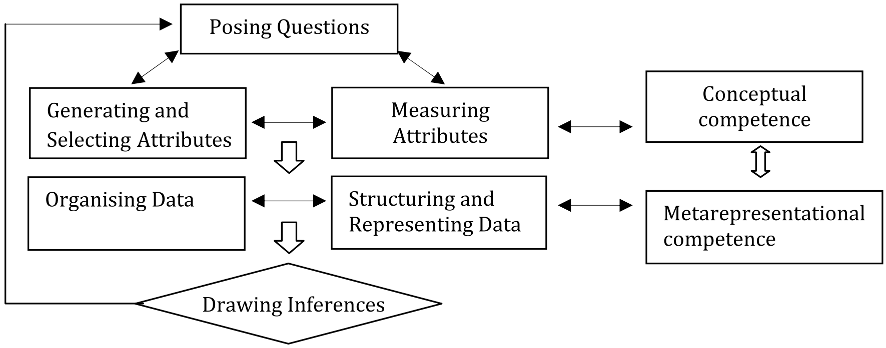

Fig. 1 Components of data modelling (adapted from Lehrer and Schauble 2004)

modelling engages children in problems that evolve from their own questions and reasoning; there is a move away from isolated tasks with restricted data to comprehensive thematic experiences involving multiple data considerations; and the components of data modelling involve foundational statistical concepts and processes that evolve over time and are tightly interactive (as indicated in Fig. 1), rather than rigidly sequential.

In the remainder of this section, I consider briefly some of the core components of data modelling, namely, structuring and representing data, informal inference (specifically, making predictions), and conceptual and metarepresentational competence. I also consider the role of task context in data modelling.

Structuring and Representing Data

Models are typically conveyed as systems of representation, where structuring and displaying data are fundamental—"Structure is constructed, not inherent" (Lehrer and Schauble 2007, p. 157). However, as Lehrer and Schauble indicated, children often have difficulties in imposing structure consistently and often overlook important information that needs to be included in their representations or alternatively, they include redundant information. Providing opportunities for young children to structure and display data in ways that they choose, and to analyse and assess their representations is important in addressing these early difficulties.

The need for classroom experiences that provide such opportunities has been emphasised over the years (e.g., Cengiz and Grant 2009; Curcio and Folkson 1996; Friel et al. 2001; Hutchison et al. 2000; Lehrer and Schauble 2007; Russell 1991), yet young children's typical exposure to data structure and displays has been through conventional instruction on standard forms of data representation. The words of Russell (1991) are still timely today:

> We have two choices in undertaking data analysis work with students: we can lead them to organising and representing their data in a way that makes sense to us, or we can support

> them as they organise and represent their data in a way which makes sense to them. In the first case, they learn some rules—and they learn to second-guess what they are supposed to do. In the second case, they learn to think about their data. Students need to construct their own representations and their own ways of understanding, even when their decisions do not seem correct to adults (p. 160).

Metarepresentational and Conceptual Competence

Russell's advice is reflected in the research of diSessa and his colleagues (e.g., diSessa 2004; diSessa et al. 1991), who proposed the term, *metarepresentational competence*, to indicate the range of students' capabilities in constructing and using external representations. The prefix, *meta*, was used to caution against the typically limited views of representational competence that have been assumed (and continue to be assumed) of young learners. Unlike the standard representational techniques students might have learned from specific instruction, metarepresentational competence encompasses students' "native capacities" (diSessa 2004, p. 294) to create and re-create their own forms of representation. Indeed, such competence appears to exist before instruction and to develop independently of it.

As indicated in this chapter, young children's metarepresentational competence involves generating their own forms of inscription. By the start of school, children already have developed a wide repertoire of inscriptions, including common drawings, letters, numerical symbols, and other referents. Children's developing inscriptional capacities provide a basis for their mathematical activity. Indeed, inscriptions are mediators of mathematical learning and reasoning; they not only communicate children's mathematical thinking but they also shape it (Lehrer and Lesh 2003; Olson 1994). As Lehrer and Schauble (2006) stressed, developing a repertoire of inscriptions, appreciating their qualities and use, revising and manipulating invented inscriptions and representations, and using these to explain or persuade others, are essential for data modelling.

Another issue that has received limited attention with respect to children's metarepresentational competence is the joint development of metarepresentational and conceptual competence (diSessa 2004). As diSessa noted, research is scant here and the role of student-created representations in conceptual development is rather complex. Assuming that representations can "strongly mediate learning," it follows that metarepresentational competence can indirectly influence the development of conceptual competence as students generate their own representations (diSessa 2004, p. 304). However, as diSessa noted, it may be that metarepresentational competence can *directly* influence conceptual learning and vice versa. In essence, the questions that warrant attention include how certain strengths or limits of metarepresentational competence might advance or hinder conceptual competence, and whether metarepresentational competence and conceptual competence develop jointly.

Informal Inference: Making Predictions

There has been limited research on young children's abilities to make predictions based on data, an important component of beginning, informal inference. Although young children obviously do not have the mathematical background to undertake formal statistical tests, they nevertheless are able to draw informal inferences based on various types of data (Watson 2007). Predictions can be based on aspects of the problem scenario and context, and children's understanding of the data presented. As pointed out by Watson (2006), one of the aims of statistics education is to help students make predictions that have a high probability of being correct. Yet in the real world, decisions are required where there is uncertainty and where several alternatives might be reasonable. Hence, young children's exposure to informal inference involving uncertainty is an important learning foundation if a meaningful introduction to formal statistical tests is to take place in the secondary school.

The Role of Context

The nature of task design, including the task context, is a key feature of data modelling activities. Children need to appreciate that data are numbers in context (Langrall et al. 2011; Moore 1990), while at the same time abstract the data from the context (Konold and Higgins 2003). As Moore noted, a data problem should engage students' knowledge of context so that they can understand and interpret the data rather than just perform arithmetical procedures.

The need to consider carefully task design is further highlighted in research showing that the data presentation and context of a task itself have a bearing on the ways students approach problem solution; presentation and context can create both obstacles and supports in developing students' statistical reasoning (Cooper and Dunne 2000; Pfannkuch 2011).

In the remainder of this chapter, I address a longitudinal study of data modelling across grades one to three and consider a selection of findings focusing on children's predictions for missing data, their structuring and representing of data, and their metarepresentational and conceptual competence.

A Longitudinal Study of Data Modelling

A three-year longitudinal study of data modelling was conducted from 2009 through to 2011 in an inner-city Australian school, situated in a middle socio-economic area, with an enrolment of approximately 500 students from Prep-7. The three first-grade classes (mean age of 6 years 8 months) continued into the second year (mean age of 7 years 10 months), while only two of the classes were able to participate in the third year (mean age of 8 years 8 months). A seventh-grade class (age range of 12–13 years) also participated in one of the activities during the second year of the study.

Table 1 Items Taken to Baxter Brown's Picnic

	Liver Straps	Beef Discz	Dentastix	My Dog Gourmet Beef	Bones	Oinkers
Baxter B.	3	5	2	1	3	
Monty	2	7	1	2	1	
Fleur	4	0	3	4	5	
Daisy	3	1	4	3	2	
Lilly	5	3	0	2	4	
Pierre	7	5	2	6	10	

Activities and Procedures

Literature was used as a basis for the problem context in each of the activities implemented across the three years. It is well documented that storytelling provides an effective context for mathematical learning, with children being more motivated to engage in mathematical activities and displaying gains in achievement (van den Heuvel-Panhuizen and van den Boogaard, 2008). Storybooks, both purposefully created and commercially available ones, were read to the children at the beginning of each activity and referred to during the course of the activities. A central character in many of the activities was Baxter Brown (a West Highlander X toy poodle), given that the children requested further stories about him during the second and third years.

For *Baxter Brown's Picnic* (second year of the study), the class was presented with a table of six different items that he and each of his five canine friends chose to take on their picnic. The final column of the table was left blank, as indicated in Table 1. After discussing what they noticed about the values and variation in values across the table, the children were invited to predict the number of Oinkers that Baxter Brown and each of his friends might take on the picnic.

For *Planning a Picnic* (grades 2 and 7; second year of study), an initial class discussion focused on questions the children might ask about planning a class picnic. In their groups, the children then listed five items they would like to take on the picnic, which were recorded by the teacher in a table on an interactive white board. The children were subsequently asked what might be done with the data and what questions they might ask about the data. Each group's question was recorded on the board, with brief discussion on how some of the questions might be refined. In their groups, the children proceeded to answer their question and were to display their findings using whatever representation they liked. They were provided with a range of recording material including blank chart paper, grid paper, and chart paper displaying a circle shape. The children could use whatever of these materials they liked; no encouragement was given to use any specific recording material. On completion of the activity, the groups reported back to their class peers on how they answered their question.

The grade 2 children were subsequently asked how their responses might compare with those of the other grade 2 classes, and were then invited to consider how the grade 7 classes in their school might respond to the activity. On the suggestion of one of the second-grade teachers, we administered the *Planning a Picnic* activity in one seventh-grade class. We then brought together the teachers and students from the second-grade class and the seventh-grade class for a sharing of how they worked the activity.

For the *Investigating and Planning Playgrounds* activity (third year of the study), which centred around the school's new playground for the younger grades, the children posed questions that might help them find out more about their classmates' thoughts on their new play area. In their groups, the children created four survey questions and were to provide four answer options for each question (e.g., one group posed the question, *How long do you spend on each piece of equipment*? with the response options of 30 mins, 15 mins, 5 mins, and 20 mins). On answering their own questions, each group chose one focus question to which the other groups were to respond. The children were to predict initially how their focus question might be answered by the remaining groups. Each group subsequently analysed all their collected data for their focus question and were to display their findings using their choice of representation. As before, the children were encouraged to represent their findings in more than one way, with no specific direction given.

Data Collection and Analysis

All class discussions, group work, and the grade 2/7 sharing of models were videotaped and audiotaped and subsequently transcribed. All artifacts were collected and analysed along with the transcripts. Where appropriate, iterative refinement cycles for analysis of children's learning (Lesh and Lehrer 2000) were used, together with constant comparative strategies (Strauss and Corbin 1990) in which data were coded and examined for patterns and trends.

In the next section, consideration is given to selected findings from two activities in the second year of the study, namely, *Baxter Brown's Picnic* and *Planning a Picnic*, and an activity in the third year, namely, *Investigating and Planning Playgrounds*. For the second-year activities, attention is given to the nature of the three grade 2 classes' predictions for the missing values in the table of data (Table 1) for *Baxter Brown's Picnic*, and the questions posed and the representations created by one second-grade class and one seventh-grade class in *Planning a Picnic*. The metarepresentational and conceptual competence displayed in the sharing of products between the second-grade class and the seventh-grade class is then addressed. Finally, consideration is given to the second-grade children's application of these competencies in undertaking the third-year activity, *Investigating and Planning Playgrounds*.

Selection of Findings

Grade Two Children's Predictions for Baxter Brown's Picnic

In contrast to the children's use of informal inference in the first year of the study (English 2012), where they used the variation and range of values in a table of data to predict unknown values, the context of the present activity appeared to inhibit the children's ability to abstract the data from the context (Konold and Higgins 2003). Each class initially identified the blank column as the first feature they noticed, with one child explaining, "Nobody wants Oinkers."

In predicting how many Oinkers each of the dogs might take to the picnic, the children predicted small values less than 10, with their reasoning mainly based on the total number of other items each dog was bringing and the fact that if a larger number of Oinkers were brought to the picnic, the dogs "might get sick," "get a tummy ache," or "get fat." One child suggested zero, "because there has to be something that he doesn't like."

There were some responses however, indicating an awareness of the need to consider the nature of the existing values, such as, "Because he (Monty) doesn't eat that much of anything else so he mustn't eat that much." In response to a child who predicted that Baxter Brown would take zero Oinkers, because he already has many other items, the teacher accepted the response as a reasonable prediction. Another student, however, disagreed, stating, "I don't think it's reasonable because he's pretty of a greedy guts so I think he would have more" (basing her decision on the existing item values for Baxter Brown).

On asking each class to consider the scenario of Baxter Brown taking 26 Oinkers, Monty 33, Fleur 50 etc., the majority of children used the task context to decide that these values were inappropriate. Comments such as, "They're um too big, the dogs would probably get a tummy ache and get sick" and "It's too heavy for them to carry to the picnic," were common. On the other hand, other responses suggested that some children were aware of the need to focus on the nature of the data, for example, "They would be bigger than all the numbers," "Ten is the highest number you can go up to," "There's only one two-digit number," and "Because there would be too much."

Children's Questions and Representations for Planning a Picnic

The findings reported here focus on the responses of the selected second-grade class and the seventh-grade class. The table created by the grade 2 class appears below; a comparable table was developed by the grade 7 students.

Not surprisingly, the grade 2 students' questions were less sophisticated than their older counterparts, resulting in a few difficulties in answering their questions and representing their findings. Nevertheless, the children's questions reflected a basic conceptual competence in their consideration of the frequency and mode of

Table 2 Picnic Items Chosen by the Grade 2 Class

Group 1	Group 2	Group 3	Group 4	Group 5
choc chip cookies	sandwiches	blanket	food	cup cakes
fruit	fizzy drinks	fruit	picnic basket	cake
sausage rolls	cookies	cake	sunscreen	juice
cordial	fruit	esky	drinks	fruit soft drink
sandwiches	fruit pudding	soft drinks	chairs	carrots

the data they had generated, with their questions including: Is there a most popular food? Is there a most popular item? What are the different types of items? Did everybody choose the same items? The grade 7 students' questions, on the other hand, indicated an understanding of percent and percentage together with a more sophisticated awareness of descriptive statistics. Their questions included: How many different picnics brought 2 or more healthy foods? What percentage of foods are unhealthy? What is the most popular item on the list, soft drink or sandwich? How many items are processed foods in each picnic? What percentage of groups brought fruit on their picnic? What percentage of groups chose sandwiches compared to groups who chose fruit to bring on their picnic? What food group does the majority of food from all of the picnics come from?

Although the grade 2 children's questions were less sophisticated in terms of the conceptual competence suggested, they nevertheless displayed metarepresentational competence that appeared to rival that of the older students. The grade 2 children generated a wider range of representations, with each group using inscriptions in their analysis of the data of Table 2. For example, one group who addressed their question, "Did everybody choose healthy items?" placed an X on what they considered to be unhealthy items, a * on healthy items, a 0 on "things that aren't food," and a created symbol of mixed shapes for "fruit/sugar." This group also drew a food pyramid, with a focus on healthy and unhealthy items, and followed this with a third representation, a circle divided into halves displaying drawings and labels of "junk food" and "healthy foods." Four of the 17 grade 2 groups made a list of selected items, before constructing a bar graph (3 groups) or a circle graph (cut into thirds; 1 group). It is interesting to note the apparent interaction between children's metarepresentational competence and conceptual competence in the following group's realisation of how their representation (bar graph) changed their initial response to their question, "Is there a most popular food?"

> *There is, there is, the answer was, there is not any popular food because there were, there's 3 ... we recorded how many different stuff there was and on one square (of their bar graph) it means that um, it means that there was one thing, on two squares it means that's there's two things and it keeps on going up to 6. And then we found out that there was no most popular food. There were 3 tying, drinks, cakes and picnic stuff... We wrote first that there was a popular thing but then when we ended up doing the graph, it ended up that there was, um, three populars.*

In contrast to the grade 2 children, all but one of the 11 grade 7 groups chose only one representation, with vertical bar graphs and circle graphs being equally popular (each chosen by 5 groups, with the display of percentages prominent). One grade 7 group who created a circle graph also made a tally chart first. From this group's explanation of how they generated their final model, it appeared that their conceptual competence in determining the number of foods in each group and calculating percentages facilitated their subsequent application of metarepresentational competence in displaying their data: "1. Study the collected data. 2. Discuss which food groups the food falls into. 3. Make a list of our data. 4. Tally the number of foods in each group and find percentages. 5. Record data on a pi-graph (sic) and show which food group has the biggest percentage."

The remaining grade 7 group created a line graph, displaying metarepresentational competence in explaining why they selected a line graph in preference to a bar graph: "Well we thought because there are so many foods, drawing bars to make them seeable would be quite squishy; we just thought it would be easier to read if it was a line graph."

Sharing Models for Planning a Picnic

Prior to the grade 2 class sharing their models with their grade 7 counterparts, the grade 2 teacher asked her class to recall how they predicted the grade 7 students might work the activity. The children's responses suggested an awareness of differences in the competencies between the two grades, with comments such as, "They won't have the same ideas" and "We said that they might be better because they'd had more years."

As the grade 7 class presented their models to their younger counterparts, there were several displays of metarepresentational and conceptual competence at both grade levels. For example, one grade 7 group reported that they solved their question using a bar graph that showed percentages of the particular items targeted in their question. When asked why they chose this representation, the group explained, "We tried a pie graph but we couldn't like split it into the right amount of groups," suggesting awareness that their lack of conceptual competence prevented their construction of such a representation. Other examples of metarepresentational competence occurred when the grade 7 students were invited to define a circle graph for the grade 2 children. One group member explained, "A pie graph is a circle that you put lines into and then colour sections which is what, yeah, is what you chose." Another grade 7 group displayed metarepresentational competence in justifying their selection of a bar graph in preference to a circle graph in answering their question, "How many different picnics brought two or more healthy foods?" The group explained, "Cause if you did like a pie graph... you wouldn't really show each group and how many items each individual group brought."

Although indicating an awareness of their conceptual limitations in interpreting the more sophisticated models of the grade 7 class, the grade 2 children nevertheless demonstrated metarepresentational competence in their interpretations. For example, when asked to compare their bar graph representations with those of the older students, they responded that theirs was easier to read as, "They (grade 7) used percentages and we don't know about percentages yet." In responding to how they knew which were the most and least popular items in one of the grade 7 models, one child explained, "Cause it's got the names at the bottom (labels under X axis). I was looking at the fruit one and I knew that it was the most... cause it's got the highest thing (bar) that goes up."

A follow-up grade 2 class discussion on how their working of the activity compared with the grade 7 students again suggested further application of conceptual and metarepresentational competence, where the younger children explained that they did not know how to construct circle graphs but they nevertheless knew that the circle represented 100 %. The children could also interpret the grade 7 representations, explaining: "We took more healthy food than they did;" "They were really bad choices;" "They did pie graphs and we didn't know like how to;" and (they did) "The line graph." In a follow-up question, the grade 2 children commented that 100 % means "all of it" (circle) and "to understand the pie, we can look at it and see if it adds up to 100 %."

Children's Conceptual and Metarepresentational Competence in Investigating and Planning Playgrounds

A particularly interesting finding during the project was the grade 2 children's transfer of conceptual and metarepresentational competence on progressing into the third year. Twelve months later, building on their developments in the second year, the children showed increased sophistication in the use of multiple representations for the playground activity. The classroom teachers had not focused specifically on any one representation and, in particular, circle graphs were not a part of the third-grade curriculum. The children not only displayed different approaches to creating a circle graph, but they also insisted on using percentages as well, albeit, mostly inaccurately. Of the nine grade 2 student focus groups who progressed to the third year, seven tried to apply their awareness of this concept. The children had not formally studied percentages but were transferring their learning from their experience with the grade 7 class (e.g., "Do we have to do percentages like the Year 7s did last year?") and sharing their ideas with those group members who were not involved in the grade 2/7 experience. Children's application of their conceptual competence here is revisited subsequently in this section.

In attempting to represent their data in a circle graph, the focus groups primarily used a ruler and/or estimation. For example, one group argued over how to estimate a sector for each response option, with one child insisting that "You have to find the middle first. That's the first thing you actually do." Using his conceptual competence to guide him, he then placed the ruler through the centre of the circle and drew a small sector to represent the two "for exercise" responses to the focus question, "Why do you like the equipment you chose?"

The group's explanation of their actions suggested an interactive application of conceptual and metarepresentational competence: "Two will only be like this (drawing a small sector and recording "2")... cause it's a very small amount." When asked how many "pieces of the pie" they needed, the group quickly replied "four, cause there's four of them (response options)." The group also commented that the four sectors would not be the same size "because if there's two people, this would have to be a smaller piece to fit two people and a bigger piece to fit nine people in." In estimating the size of the sector to represent the nine responses of "It's challenging," one child claimed that "nine would be half of it" (there was a total of 20 responses to their focus question). After much discussion, the group decided "no, no, no, that nine can't be that big (half of the circle)" taking into account the frequencies of the other response options (six, three, and two). One child subsequently tried to measure the sectors with his fingers to make the nine sector smaller than the total of the other response options (11), explaining, "Yeah, it actually does have to be a bit smaller."

Returning to the children's conceptual competence in dealing with percent, it was interesting how the children negotiated the representation of 0 % (for the response option that received zero votes, namely, *fits more people than the oval*) and 1 % (for the response option, *good views*). Although most of the groups did not have the conceptual competence to use percentages to construct their circle graphs, they nevertheless demonstrated some metarepresentational competence in trying to display 0 % and 1 %. For example, after instructing Kim to "Write 12 percent" in one sector of the circle graph (representing 12 responses), Belinda said, "Maybe you could rule a little bit off it... that could be zero percent." She further noted that the response option of *good views* "only has one percent..." and "has to be really small, like that small." This group further struggled with their display of 0 %, claiming that there was insufficient space to label the option of *fits more people than the oval*. When their teacher asked, "How can you show 0 %," Belinda responded that "You should just rub that out... cause that got nothing." But then Kim was puzzled by "How would you do zero?" to which Belinda replied "Rub it out, rub it out."

Another group, however, demonstrated an emerging conceptual competence in applying their understanding of percentage to their circle graph construction. When two group members recommended recording the number of focus question responses (to the question, "Why do you like the Spider Web") in the circle graph segments they had drawn, Hugh disagreed, saying "Do percent". When the research assistant queried the group on how they intended determining this, they explained that they knew that the circle graph represents 100 % and that of the 20 votes for their focus question, there were two responses that each received five votes, one that received four, and one that scored six. Hugh explained that instead of recording the actual number of focus question responses, percentages should be shown: "That is 100 %, so we needed to do 25 %; that's 25 (%), so that should be 24 (%), and that should be 25 (%) and that one should be 26 (%)." Further explanation when queried by the class teacher demonstrated how they used both their conceptual and metarepresentational competence in constructing their circle graph:

> Hugh: Well here we had six here, five here and five here and four there so we wrote down 5, 5, 6, and 4.
> Belinda: And she put 20.
> Hugh: And we like, we measured up with the ruler to know which line goes where, and we just put a 20 in front of it.

Hamish: Cause it was 26 and 24.
Allya: And we knew that this 5 and 5 had to be half so it had to be 25 and 25 and then we had to work out this has to be 26 and 24 which is, that plus that equals 50 and then that plus that equals 50.

Discussion and Concluding Points

As the chapters of this book attest, when we engage young children in challenging experiences that extend beyond the curriculum, their learning potential often exceeds our expectations. This chapter has provided one example of such potential. In reconceptualizing early statistical learning, the present study has shown how young children can deal effectively with experiences in data modelling. Specifically, the children demonstrated skills in posing and researching their own questions; in gathering, structuring, and representing data in ways they chose; and in drawing informal inferences (making predictions) when faced with missing data. The children's application of conceptual and metarepresentational competence facilitated their achievements here.

Four main issues arising from the data modelling activities are worth highlighting—the role of task context, the posing of investigative questions, the application of conceptual and metarepresentational competence, and the role of model sharing in learning and the transfer of learning.

With respect to the first issue, as previously noted, children need to appreciate that data are numbers in context, while at the same time abstract the data from the task context. Although context provides meaning in statistics (Garfield and Ben-Zvi 2008), it can create both obstacles and supports in student's statistical reasoning (Pfannkuch 2011). The purposefully created context of Baxter Brown and his canine friends organising a picnic appeared to hinder the children's analysis of the table of data (Table 1). Only a few children justified their predictions by considering the nature (range and/or variation) of the values displayed, with the majority making contextual inferences such as the need to consider the dogs' health. The role and impact of task context require careful consideration in designing statistical activities; clearly a good deal more research is needed here to guide the development of data modelling in the early years.

Posing questions about the class selection of picnic items was a comparatively new experience for the second-grade children, presenting a challenge as to how some questions might be refined. Such difficulties can be expected—transforming initial questions into more specific statistical questions is not an easy step, especially for young children (Konold and Higgins 2003). Not surprisingly, the grade 7 students generated more sophisticated questions, applying the conceptual competence that they had developed during their additional years of schooling. Nevertheless, the younger children displayed substantial metarepresentational competence in generating a wider range of representations including the use of inscriptions. Such competence also came to the fore, in conjunction with their conceptual competence, when the children realised that their initial response to a question was inappropriate once they had displayed their data on a bar graph.

The children's competencies in constructing circle graphs again revealed the extent of their potential for engaging in data modelling. The children had not been taught to construct such representations yet chose different approaches in doing so, in both the second and third years of the study. The interactions between conceptual and metarepresentational competence were especially evident here, with children's graph construction facilitated by their knowledge that the sectors should be proportional to the values being represented.

Further development of these competencies evolved from the children's sharing of their models with their older peers. Experiencing the models produced by the grade 7 students generated an interest in the notion of percent and calculating percentages—children added the percent inscription to their representations with some also trying to calculate percentages to determine sector size. Although most of the younger children could not calculate correct percentages, they nevertheless displayed conceptual competence in their understanding of percent and its role in constructing representations.

The sharing of models between the younger and older students was a rich learning experience for both, providing opportunities for appreciating different approaches to dealing with data and for questioning, explaining, and interpreting the data models of others. Consideration should be given to creating such sharing opportunities across grade levels.

The present study leaves a number of issues worthy of further investigation in young children's statistical learning. These include the nature and role of context in the design of data modelling activities, the factors in a task context that either support or hinder children's statistical reasoning, and ways in which we might capitalise on literature and on children's interests in data collection, display, and analysis (cf. Whitin and Whitin 2011). Further consideration also needs to be given to how we might advance young children's conceptual and metarepresentational competence in data modelling, the ways in which we might promote children's skills in posing effective research questions, and the types of statistical experiences that can be shared effectively by children across grade levels together with the nature of the learning that takes place in such shared experiences.

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Cognitive Guidelines for the Design and Evaluation of Early Mathematics Software: The Example of *MathemAntics*

Herbert P. Ginsburg, Azadeh Jamalian, and Samantha Creighan

Introduction

This chapter shows how cognitive psychology can inform the design and evaluation of software for early mathematics education, and how the resulting software can provide new approaches to evaluation of learning and to basic cognitive research.

The need for improved early mathematics education is abundantly clear. Too many children (at least in the U.S.) perform poorly in mathematics from the earliest days of school (Mullis et al. 1997). The problem is especially acute in the case of underprivileged, low-income children, who start behind (Starkey and Klein 2008) and fall further behind as they grow older (Cunha et al. 2006; Duncan et al. 2007).

Although there are many contributors to low academic achievement, ranging from poverty to teacher pay, the primary factor is *not* young children's inability to learn mathematics. A very large body of research shows that children naturally develop a surprisingly proficient and complex "everyday mathematics" (Sarama and Clements 2009b), which provides a useful foundation on which mathematics education can build (Baroody 2004). The research also shows that quality early mathematics education can have long term positive impacts on achievement and can provide substantial benefits for those who need the most help, namely underprivileged, low-income children (Cross et al. 2009).

We propose that the affordances of computer technology, although hardly a panacea, offer the possibility of transformative improvements in early mathematics education. It is possible and desirable, we argue, to design software to help children learn mathematics, to help teachers teach it, and to eliminate the need for textbooks as we know them.

Our optimism (some concerns will follow) is fueled by two developments. One is that education authorities now accept that computer technology has an important place in education from elementary school through postgraduate studies. The

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second, and more important, is that touch screen devices, notably the iPad, have become ubiquitous in a short period of time. Software developers have been releasing very large numbers of mathematics apps for young children from age 3 upwards. In July 2012 there were more than 20,000 education apps designed for the iPad, and more than 1.5 million iPads in U.S. schools (Brian 2012). It is not uncommon to see very young children using touch screen devices with reasonable proficiency. Even toddlers navigate the screen, access their favorite app icons, and play with the software for long periods of time.

Although only in their infancy (like some of their users), the new touch screen devices seem ideal for the population on which this paper focuses: young children from roughly age three to six whose reading ability is limited or non-existent and who may be unable to move a mouse with facility but can nevertheless touch and manipulate attractive virtual objects on a brilliantly vivid screen. Touch screen devices that can talk to young children (when they tap on a word or numeral or perform certain actions) and that allow them to touch and manipulate virtual objects can set the stage for—but definitely do not guarantee—dramatic advances in the quality of software.

Yet there is reason for concern. The availability of large numbers of apps does not in itself solve the educational problem. Individual apps by the hundreds or thousands seem to emerge with no evident plan, rhyme or reason. Some of the most popular apps have a limited focus. For example, Space *Math* (Mclean 2012) drills students on number facts but is little more than an efficient worksheet that does not promote conceptual knowledge. Although drill can be useful, there seem to be few examples of mathematics software for young children that promote conceptual understanding and non-trivial problem solving.

Further, although rigorous evaluations are rare, our informal observations suggest that the quality of mathematics apps and other software is generally not impressive. Many designers and publishers claim that their software is of high quality because it is "research-based." Yet these assertions need to be taken with a very large grain of salt. Educational researchers have argued that research is seldom used in meaningful and effective ways in software development (Sarama and Clements 2002). There are exceptions, an example of which is *The Number Race*, designed for 7- to 9-year-olds with mathematical difficulties, which focuses on improving number sense through number comparison tasks (Wilson et al. 2006a, 2006b). Another exception is *Dots2Track* (Butterworth and Laurillard 2010), which also targets children with mathematics learning difficulties. Both of these software programs derive from a research-based analysis of children's learning difficulties, and show promise of correcting cognitive deficiencies and promoting mastery of early mathematics.

We propose that cognitive science (which for purposes of this paper we define as including contributions from the overlapping and sometimes vaguely defined disciplines of cognitive developmental psychology, educational psychology, learning science, cognitive psychology, and mathematics education) can and should play an essential role in exploiting the powerful affordances of computers in the service of education. First, cognitive principles can provide a framework for the design of educationally rich software, ensuring that its pedagogy effectively promotes mathematical proficiency in the broad sense, including both the motivated learning of rich content and the development of genuine mathematical thinking. Such software can help teachers to change their roles and improve their teaching, and eventually will make traditional textbooks obsolete. Second, the cognitive principles can contribute to the usability testing and meaningful evaluation of software designs. The cognitive principles can also guide the formative and summative assessment of children's learning, as well as the evaluation of achievement and the effectiveness of software. And finally software based on cognitive principles can return the favor by setting the stage for new kinds of basic research into children's mathematics learning in rich environments.

Cognitive Principles for the Design of Software

We begin with what the computer can do. We know that software affordances can support learning (Sawyer 2006) in several ways. Computers can represent abstract knowledge in interactive visual models; touch screens allow children to manipulate virtual objects; computer tools provide opportunities for learners to explore and develop solutions to interesting problems; and computers allow building a community of learners and collaborative learning. Software can also engage children in "microworlds"—artificially designed "mathemagenic" environments that entail and stimulate the exploration of mathematical ideas, foster the development of thought and skill, and offer children powerful tools to do significant mathematics (Hoyles and Noss 2009; Hoyles et al. 2002; Papert 1980). Microworlds can include goal-driven activities, virtual objects, tools, representations, scaffolds, feedback, pedagogical agents, interaction, fantasy, challenges, communication and collaboration, record keeping and reporting.

We propose 6 cognitive design principles that can exploit these affordances.

- Engage children in cognitively and mathematically appropriate activities.
- Develop effective models for representing abstract ideas.
- Encourage accurate and efficient strategies.
- Identify and eliminate bugs and other misconceptions.
- Design appropriate physical interactions.
- Integrate narratives and stories with mathematical concepts.

For each design principle, we outline important issues that software developers need to consider, and then we discuss examples of how cognitive principles can inform design.

Engage Children in Cognitively and Mathematically Appropriate Activities

The first step in designing learning activities is to clearly define the content to be learned and taught. The new Common Core State Standards Initiative (2010) provides broad guidance, based extensively on cognitive developmental research. For example, the Kindergarten Standards propose that children should learn the number names and count sequence to 100; count to tell the number of objects; and compare numbers.

To those unfamiliar with cognitive psychology, teaching this content would appear to be easy. After all, what can young children learn beyond a little counting, names of shapes, and memorized facts? But neither the mathematical content nor its learning and teaching are simple. Software developers have produced many apps and programs designed to help children learn these important topics. Unfortunately too many apps focus on the rote teaching of number words and simple enumeration without regard to its meaning (for example, *Toddler Counting*, iTot Apps, LLC 2011). Although knowing the number words and being able to recite the list in the correct order is important, children need to learn a great deal more about number than current software apps teach. Rich content should include thinking as well as facts and procedures.

To develop effective software that can help implement the Core Standards, designers need to understand the specific cognitive processes and obstacles involved in young children's learning the counting words, enumeration (determining the number of a set of objects), and number comparisons of various types. And it goes without saying that designers need to have a deep knowledge of the mathematics itself, which in the case of young children is far from trivial: it deals with fundamental concepts of number theory.

Fortunately, a substantial body of research can provide guidance for software development in the area of early number. Gelman and colleagues (Cordes and Gelman 2005; Gelman and Gallistel 1978; Gelman 1993; Gelman and Gallistel 1986) show that learning to enumerate involves several components. The child must acquire three "how-to-count" principles: (1) stable order principle, (2) the one-to-one correspondence principle, and (3) the cardinal principle. The Stable Order Principle requires that symbols have a consistent order across counting occurrences. The One-to-one Correspondence Principle explains that for every object in the counting set only one and one counting symbol is applied. The Cardinal Principle refers to the fact that the last symbol of a count represents number of objects in the set that has been counted. A child who counts 6 candies as "1, 2, 3, 4, 5, 6" and says, "I have 6 candies," knows the Cardinal Principle, at least for the number 6. Gelman and Gallistel (1986) also define two "what-to-count" principles: (1) the Abstraction Principle states that any combination of discrete objects could be counted (e.g. a set of heterogeneous farm animals or abstract entities such as months in a year), (2) the Order Irrelevance Principle states that a set can be counted in any order (left to right, right to left, or any other order) and yet the cardinality of the set does not change.

Children also need to understand the meaning underlying the distinctive symbolic language of mathematics and more generally to appreciate that mathematics is meaningful. Vygotsky (1978) proposed that early mathematics education should help children learn to synthesize their "spontaneous concepts" (everyday mathematics) with "scientific concepts" (the organized, formal mathematics that constitutes the accumulated cultural wisdom and that schools try to teach). "The strength of scientific concepts lies in their conscious and deliberate character. Spontaneous concepts, on the contrary, are strong in what concerns the situational, empirical, and practical" (Vygotsky 1986, p. 194). This approach, similar to Dewey's (1976), aims to integrate the best of the vital and spontaneous with the best of the rigorous and scientific so as to produce meaningful knowledge. Another way of saying this is we should help children to mathematize their meaningful everyday mathematics.

To create useful software, designers need to draw upon cognitive research to develop clear answers to the following three questions relating to content:

1. What should be the specific mathematical and cognitive content of the activity?
2. What is the developmental trajectory for mastering the concepts and skills in question?
3. What prior knowledge is required to master understanding of the new concept?
4. How does the software promote understanding of symbols?

To illustrate how this can be done, we consider one set of *MathemAntics* activities designed to help pre-school children learn about numerical relations and comparison of set sizes. To design the content for the activities, we first analyzed the developmental trajectory for mastering the number relation concepts.

2- and 3-year-old children begin to learn relations such as *more* and *same* (Clements and Sarama 2007, 2008; Fuson 1992a, 1992b; Ginsburg 1989). Children can compare two sets using subitizing (*seeing* the number immediately without counting), length, or density strategies (Mix 1999). We based the content of the simplest *Equivalence* activities in *MathemAntics* on what we know about children's knowledge, and synthesized activity content with their mathematical understanding. The simplest activity encourages the child to visually inspect two sets of 3 or fewer objects arranged in a row or in a column, with the length of the rows congruent with the cardinal values. In higher levels, larger sets with numbers of objects beyond the subitizing range are presented to the child for comparison, and the child may still use length or density strategies to compare these sets. In the *Hens Laying Eggs* activity, for example, children are asked whether Fluffy and Fancy Pants have the same number of eggs (see Fig. 1). The eggs for each hen are arranged in a row, and the length of each row is congruent with the cardinality of that row (i.e. the set with fewer eggs is arranged in a shorter row).

Another simple activity called *Bedtime* (see Fig. 2) encourages children to compare two sets by matching objects from one set to the other set. Children put each animal in a bed to help it get ready to sleep, and ring a bell when they are done. The one-to-one activity not only emphasizes the matching strategy but also highlights the one-to-one counting principle.

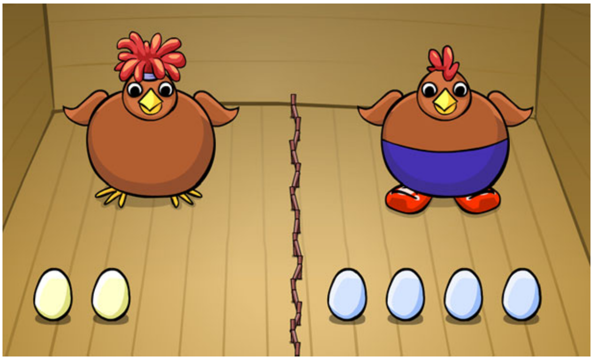


Fig. 1 Level 1 of Hens Laying Eggs activity

Subitizing, length, or density strategies are not efficient for comparing two sets that are larger than 3, have close to a 1:1 ratio, or have incongruent physical and numerical attributes (e.g., two equal sets with different densities). By the age of 4, children should be able to use counting and matching strategies to compare the size of sets with 5 or fewer objects (Cross et al. 2009). To use counting successfully, children need to be able to count each set accurately, remember the cardinality of the first set while counting the second set, and then compare the cardinalities of the two sets. Fluency in counting is important. Without it children may forget their first count result by the time they have counted the second set. Children also need to know order relations of cardinal numbers—the further numbers are along in their counting list, the larger quantities they represent (e.g., Fuson et al. 1982). Therefore, fluency in counting (i.e. mastering all 5 counting principles) is the required prior knowledge for using counting to compare two sets. For example, to compare the two schools of fish in the *Pop the Bubble* activity, children need to count each set separately and pop the bubble that has more fish in it (see Fig. 3).

In addition to counting fluency and knowledge of order relations of cardinal numbers, executive function skills are critical in the successful use of counting strategies. Inhibitory skills, working memory, and cognitive flexibility (Diamond 2008) are of particular importance. To compare two sets, children need to be able to count one set, stop, and start from number 1 to count the other set. In fact, in our own usability research we observed that some 4 year olds fail to stop after counting the first set and continue counting the second set together with the first set. Children who are able to count two sets separately may also fail to compare the two sets based on their counting results due to working memory limitation: by the time they count the second set they may forget the counting result for the first set. To assist their memory and to teach them the meaning of the symbol, we provide the option to have the numeral for each set written at the top of the bubble. So, when the child indicates how many fish are in a bubble, the numeral will appear on top of the bubble and stay on the screen (see Fig. 4). This helps the child to learn to read the written numerals and understand what they mean.

By the age of 5, children should be able to use counting strategy to find out which is more and which is less for two numbers ≤ 10 (Cross et al. 2009). In the Pop the Bubble activity for example, we ask children to compare two sets of fish and pop

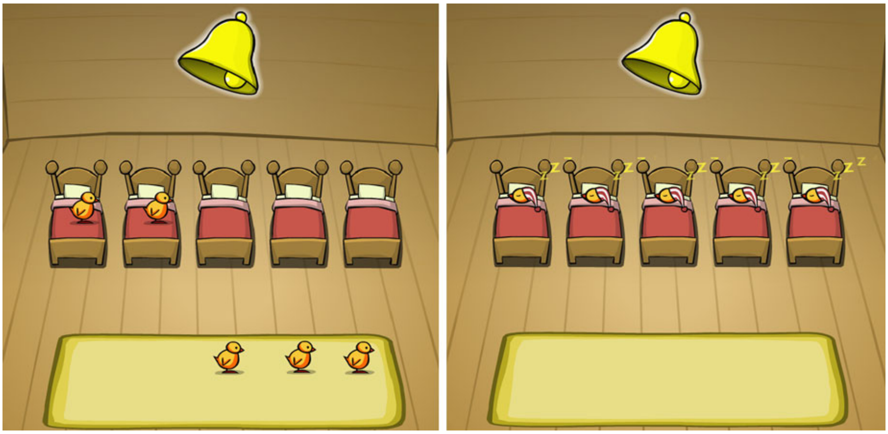
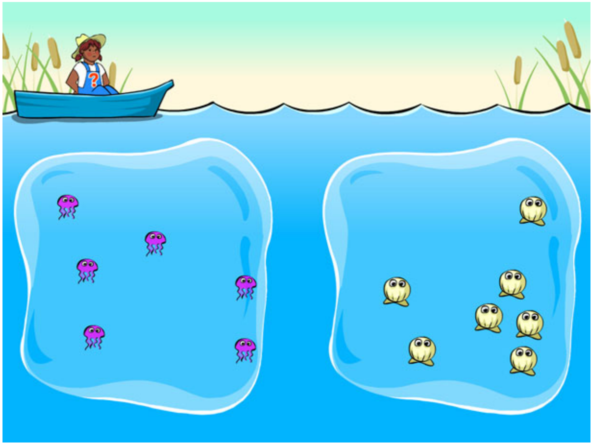


Fig. 2 Bedtime activity

Fig. 3 Pop the Bubble activity



the bubble with fewer or more fish in it. At this age, children also need to learn the meaning of the equal and non-equal symbols, and we introduce these symbols as part of the activities so that children will be able to achieve a synthesis between their everyday knowledge (this group has more than that) and the appropriate symbol. For example, in the Quick Compare activity, we ask the child to compare number of animals in two barns by clicking on the equal or non-equal signs (see Fig. 5).

By first grade, not only do children need to know which set has more, but they should know "how many more", or "fewer" objects one set has compared to another (Cross et al. 2009). For example, in more difficult levels of the Hens Laying Eggs activity we ask children to make Fluffy have 2 more eggs than Fancy Pants. The child may remove eggs by clicking on them, or touch a chicken's belly to have her lay an egg. At this age, children are again introduced to "more than" or "fewer than" symbols and number sentences in the context of their own everyday mathematics.

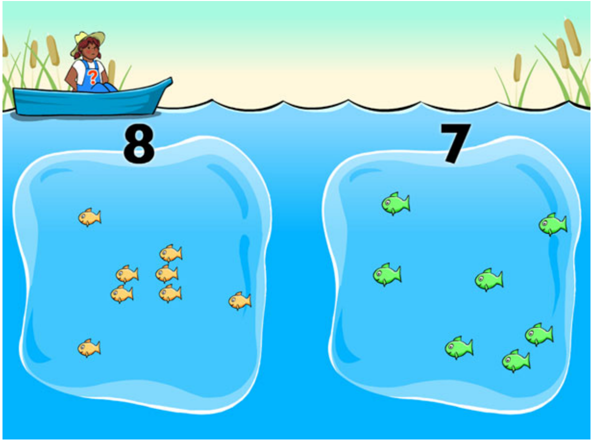


Fig. 4 Pop the Bubble activity with number symbols

Conclusion We have seen that cognitive psychology provides detailed and specific information concerning the cognitive content of the mathematics to be learned—cardinality, number comparison, and the rest—and the struggles children have in learning it. Key learning and teaching principles should inform the design of software. For example, to produce a meaningful synthesis between informal knowledge and the formal symbolism of mathematics, we linked the eggs in the Hens Laying Eggs activity with the number sentence (8 > 6 in Fig. 6).

Fig. 5 Quick Compare activity

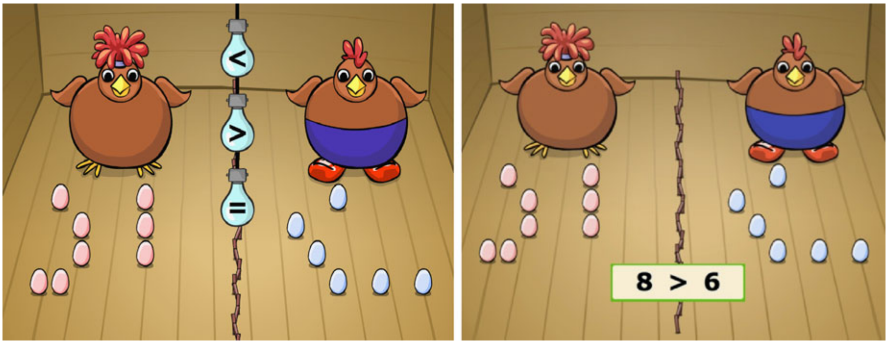


Fig. 6 Hens Laying Eggs activity with number sentence

Develop Models Representing Abstract Ideas

Perhaps the most important educational goal is to promote deep understanding of mathematical principles. Mathematics educators use a variety of instructional aids to help children learn abstract concepts. These aids can take the form of visual representations, models, tools, or manipulatives. Whatever the name, the goal is the same—to represent a mathematical concept or relationship in a meaningful way. Many mathematical ideas can be modeled through such representations, such as representing the number five as five objects, a block with five discrete units, or a continuous line shading five units on a number line. The way in which a concept is modeled can directly influence the way the child conceptualizes and understands the mathematics. For example, modeling the multiplication problem 3×4 as three equal groups of four is different than representing the problem as the area of a rectangle that is 3 units long and 4 units wide. Though an adult understands that both of these representations will yield the same total, 12, a child new to the concept may struggle to synthesize multiple representations. And indeed the two representations are in fact different. The rectangle representation can be used to deal with continuous area while the group of objects representation must involve discrete quantities.

A powerful way to teach children about abstract mathematical ideas is through manipulatives embodying various kinds of representations (Mix 2010). The basic idea is that manipulatives—things that hands can touch, feel and move—can help students to create mental representations of mathematical ideas and procedures. Manipulatives are not just things to play with, but artifacts designed to help the learner construct sound ideas. Manipulatives are successful when they can be abandoned because they are no longer needed to support the ideas.

Used properly, manipulatives can advance our general educational principles. They are responsive to the child's current cognitive state in that they may involve, for example, manipulating sets of objects instead of symbolic statements. Manipulatives elaborate on everyday knowledge by helping the child to perfect judgments of numerical magnitude (for example, by providing practice in comparing the numbers of sets of objects). Activities involving manipulatives can help the child mathematize by showing the numerals corresponding to various numbers of objects (for example, the numeral 10 next to a collection of 10 blocks). Manipulatives can also promote a synthesis of everyday mathematics (this is a much taller tower of blocks than that tower) with the symbolic (this tower has 13 blocks but that tower has only 5).

Concrete manipulatives like Cuisenaire Rods or Unifix Cubes have been used for a long time to teach fundamental concepts of number. Virtual manipulatives (as defined by Moyer et al. 2002) and models can also help visually represent mathematical ideas and relationships (Mix 2010). Indeed computer technology can be used to create virtual manipulatives that in some ways may be more powerful than their concrete counterparts. Although it is virtually impossible to have the child work with 5,000 blocks, doing so can be child's play on the computer screen. The representation can be visual or pictorial (as when a child sees a randomly arrayed collection of animals on the screen), which can be manipulated by a mouse or by the fingers on a touch screen (for example, into groups of 3 animals in a line), and can be connected in real time to symbolic representations (for example, if the line is 3 long the numeral is 3, but if an object is added to the line the numeral changes to 4) (Moyer-Packenham et al. 2008). To fully understand and embody the mathematics, a child must be able to flexibly connect and utilize multiple representations, understanding that a virtual group of five chicks can be represented by a five block (comprised of 5 connected unit blocks) and by the numeral 5. If the child sees the numeral 5, she should be able to represent it with the 5 block or the 5 virtual chicks. This is the kind of flexible synthesis entailed in deep understanding.

Although work with manipulatives and models is widespread and often acclaimed as beneficial, research regarding their effectiveness shows that their use does not necessarily ensure deep understanding (Moyer-Packenham et al. 2008; Mix 2010). The issue is not simply whether or not manipulatives are used. It should come as no surprise that the fundamental question seems to lie in the circumstances in which particular models and manipulatives can be used most effectively for particular purposes. Moreover, children need enough time to interact with a specific representation to fully understand the underlying mathematical content, must be carefully introduced to new representations, and must be provided with scaffolds to connect multiple representations in meaningful ways (Sarama and Clements 2009a; Mix 2010).

Unfortunately, current software seldom uses virtual representations in meaningful ways. Some utilize multiple representations for the same concept but lack the scaffolding to promote synthesis. Others provide representations that are conceptually confusing (for example, representing one as eight fish forming the numeral one!) or are inappropriate for the intended age (having instructions appear as written text only for preschoolers!). Other mathematics software programs stress rote learning of number facts and make almost no use of virtual manipulatives that offer useful representations.

Given the potential importance of virtual manipulatives, designers need answers to the following three questions:

1. What models can be used to represent a concept?

2. What are their possible benefits?
3. How can they be connected with other models?

What Models Can be Used to Represent a Concept? When designing educational software, designers must first consider the different ways in which the concept could be modeled. Inspiration may be drawn from existing school curricula or from the cognitive psychology literature. The most effective models clearly and appropriately represent and help the child form a useful mental representation of the abstract mathematical concept. For example, the multiplication problem 3×4 can be represented as an array of 3 rows of 4 dots each; a rectangular array of 3 by 4 squares; 3 jumps of length 4 on a number line; and 3 plants with 4 flowers on each. These are all different ways of thinking about 3 times 4, and each has its specific strengths and weaknesses. As mentioned earlier, a discrete array cannot represent area, and neither can jumps on a number line, which in turn cannot represent the idea of "number of elements per unit" as well as the flowers can. Given these possible representations that are useful for different purposes, designers may be able to transform the appropriate model into a powerful virtual manipulative.

Benefits Although we have highlighted the importance of using multiple models, we will provide an in-depth look at one, the number line. The number line is versatile in that it can be thought of as a visual representation of number, but it can also model mathematical relationships (such as modeling how 80 is larger than 30), and serve as a mathematical tool to solve problems (such as solving $3 + 5$ by making jumps on a number line). Psychological research has shown that the number line is very powerful in representing mathematical concepts such as numeral identification, counting, and most importantly numerical magnitudes. Working with the number line can build number sense and contribute to the development of a mental number line (Jordan et al. 2006).

As children begin to study the symbolic number system, they may learn to map magnitudes onto a mental number line. They may know that 3, 4, and 5 are pretty close to one another, but that 100 is far away. However, their mental number line is imperfect. It follows a logarithmic representation for larger and unfamiliar quantities. In other words, children have a pretty good idea of where the smaller numbers fall on the line, but after a certain limit, the larger numbers tend to be jumbled together as very big. To examine the mental number line, researchers have used a task that involves giving a child to place various target numbers on a blank number line with only the endpoints marked (Booth and Siegler 2006; Laski and Siegler 2007; Moeller et al. 2008). Instruction and exposure to larger numbers, as well as feedback on performance on the number line estimation task, help children to shift from a logarithmic representation to a more accurate linear mental number line. Several interventions that focus on the number line—for example, Early Learning in Mathematics (Chard et al. 2008), The Number Race (Wilson et al. 2006a), and The Great Race (Ramani and Siegler 2008)—have reported successful outcomes. Siegler and Booth (2004) also found that instruction of this type may have long-term benefits in other areas of mathematics: the likelihood of using a linear representation of number is highly predictive of later mathematical achievement.

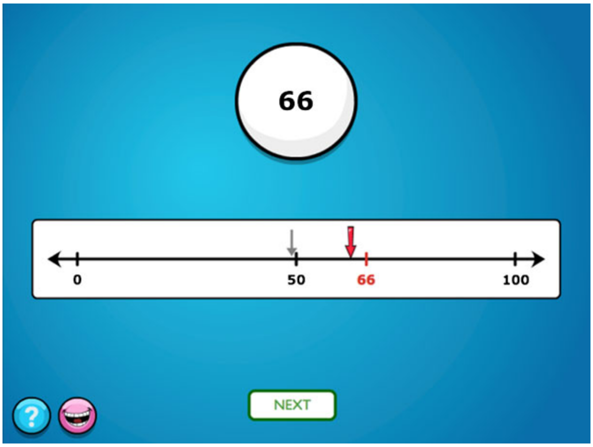


Fig. 7 Basic Number Line activity

When designing *MathemAntics* we aimed to use the number line as a powerful virtual manipulative. For example, in *MathemAntics*, as in the traditional number line estimation task, the child places various target numbers on a number line. But then the software gives the child multiple attempts and specific verbal and visual feedback, which has been shown to assist children in forming an accurate linear mental representation of number (Laski and Siegler 2007). And of course, the computer allows the educator to set various parameters such as the length of the line, the presence or absence of hash marks in any places.

Figure 7 shows an example of the basic number line activity in *MathemAntics*. The gray arrow denotes the child's incorrect first attempt; the red arrow denotes the child's current attempt; and the correct position of the target, 66, is shown in red as visual feedback.

We have also designed an activity that allows children to build their own number line by positioning numerals on the line. This activity scaffolds children to pay careful attention to the order and spacing of the numbers. When finished, the child can compare his number line to a correct number line above, and fix the former accordingly. This activity is intended to encourage children to explore and learn number relations.

Adjusting the parameters of the number line allows modeling of other important mathematical concepts such as number operations (modeling addition as jumps on a number line), place value (labeling the decades only on a 0–100 number line) (Moeller et al. 2008), measurement (Petitto 1990), and rational number (Schneider and Siegler 2010). The number line is a highly versatile model capable of conveying many mathematical concepts. However, designers should be aware that simply including a number line does not guarantee the child will master all of these concepts. This reiterates the importance of the first cognitive design principle, knowing the specific developmentally appropriate conceptual content. We will revisit this issue in the discussion of later principles and provide examples of how designers can use the number line in useful ways to encourage deep understanding.

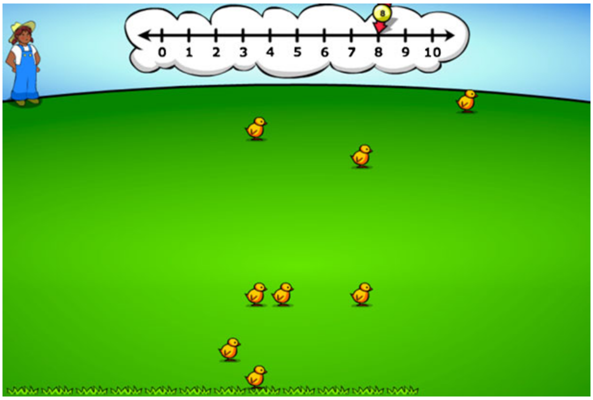


Fig. 8 Enumeration activity using the number line as response mechanism

Ways to Connect Multiple Representations As we have seen, the number line can be a useful tool for teaching many concepts, from order to rational number. It can also be used to integrate multiple representations, thus promoting the synthesis of different areas of mathematical knowledge.

To achieve this goal, the number line appears in other *MathemAntics* activities as a response mechanism or tool. In early enumeration activities, a child may explore the learning environment by adding or taking away chicks in a field. The total number of chicks is simultaneously represented with a yellow magnitude bar on a number line that changes in real time based on the child's actions. In a later enumeration activity, the child indicates his answer by clicking the corresponding numeral on the number line (see Fig. 8).

Feedback or scaffolding can highlight and count the animals out loud while a yellow magnitude bar represents the quantity on the number line, thus aiming to carefully connect the representations for the child. Once the child masters enumeration, the concrete representation of chicks in a field may be taken away leaving only the number line. In later activities, the child may choose to use various tools, like the number line, to solve problems involving mathematical operations. In this advanced case, the child has already been introduced to multiple representations, provided with appropriate scaffolds to connect them, and may now choose how to represent the problem.

Conclusion Many kinds of manipulatives can be used to represent mathematical ideas. As a virtual manipulative, the number line can be used to promote number sense and a more accurate mental number line; to teach ideas of order and rational number (as well as any of the operations on the whole numbers); and to promote a synthesis between the written numbers and important mathematical ideas.

Encourage Accurate and Efficient Strategies

Learning mathematics is not only about providing correct answers to problems, but also using efficient strategies to solve those problems. Strategies may be efficient or inefficient, accurate or inaccurate. One major goal of mathematics education is to encourage children to use increasingly effective strategies for solving problems.

When designing learning activities to teach new strategies, two points are important to consider. First, adoption of new strategies is often quite slow: children may continue using prior strategies even if newly adopted ones have clear advantages that the children themselves can explain (Siegler and Jenkins 1989; Alibali 1999; Chen and Klahr 1999; Goldin-Meadow and Alibali 2002; Granott 2002; Kuhn et al. 1995). Second, rate of adoption of new strategies varies in relation to two factors: accuracy and efficiency (Siegler and Svetina 2006). If the newly adopted strategy significantly improves accuracy relative to a prior strategy (generally from consistently incorrect to consistently correct), it will quickly replace the old one. If both the new and old strategies result in correct performance, children adopt new strategies that boost speed or dramatically reduce processing more quickly than do strategies with less substantial advantages. Thus, differences in both accuracy and efficiency of the new and old strategies contribute to the rate of adaptation, with accuracy being the more powerful factor.

Very few mathematics apps for children consider strategy use. Many apps employ a multiple-choice format and simply ask children to provide an answer. Although these apps may help children to practice and improve their mastery of number facts and calculation skills, effective mathematics education requires more apps that promote the adoption of effective strategies for problem solving.

Prior to designing an app to promote strategy learning, three questions should be answered:

1. What are the possible strategies to solve the problem?
2. Which strategies lead to more accurate answers and under what conditions?
3. What strategies are more efficient than the others for a specific problem?

To answer these questions, we draw upon findings of cognitive psychology, as well as our clinical interviews and observations during usability and play testing. Usability testing involves determining whether the user can do whatever he/she is supposed to do. For example, can a 3 year old drag objects on a touch screen? Play testing involves determining whether the child uses software as it was designed to be used. For example, if the software offers a potentially useful tool, does the child use it for the intended purpose? We then illustrate answers to these questions by considering the design of *MathemAntics* activities.

Different strategies may be employed to compare magnitudes of two sets depending on the features of the problem. Perceptual subitizing, length, or density strategies are the most basic. If the set sizes are less than 3 or 4, the child may simply subitize—quickly "see"—the number of objects in each set and evaluate if they are the same or one is more. For young children, fast subitizing is only efficient and accurate if the number of objects in each set is less than 4 (Sarama and Clements

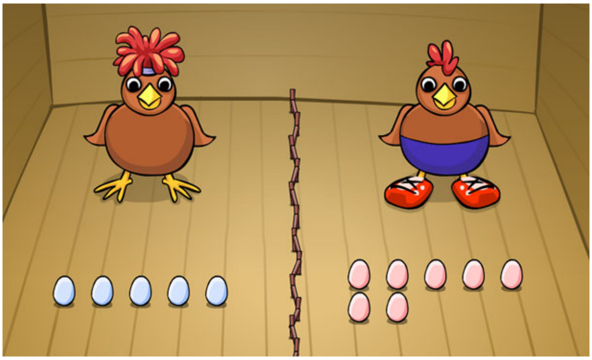


Fig. 9 Encouraging visual comparison in Hens Laying Eggs activity

2009b). When comparing large sets of objects (more than 4), children may employ visual comparison strategies employing length or density. Children judge a group of objects to be greater in number when it is either longer or denser than another group. Visual comparison strategies are efficient and relatively accurate if the sizes of objects in the two sets are similar, and the area occupied by each set is congruent to its numerical value (Mix 1999). Further, the visual comparison strategies are accurate within a ratio limit (Feigenson et al. 2004). Thus, it is easier to see the difference between 4 and 5 than between 94 and 95. However, comparing cardinality of two sets based on visual attributes such as length and density is not accurate if the two sets have a close to 1:1 ratio, or if the visual attributes are incongruent with the numerical property of the sets. Conceivably counting was developed to deal with relatively difficult problems like these.

Children may also use a matching strategy when comparing two relatively small sets. For example, a child may examine whether there are the same number of beds as there are animals by corresponding each animal with one bed. However, matching is not efficient when comparing 20 beds and 20 animals or when sets are not in close proximity. Counting is a more efficient and sophisticated strategy for comparing the quantities in situations where visual comparison strategies are not efficient.

Software has unique affordances to encourage the use of strategies by highlighting the advantages of new strategies or limiting the resources needed to use another strategy. Following are five design scenarios.

Visual Representations The visual properties of a problem can be set to encourage certain strategies over others. For example, to encourage visual comparison strategies to compare the quantity of two sets, objects in each set could be positioned in rows and columns inviting the child to compare the length of the rows: see Fig. 9.

As another example, to encourage use of the "count on" strategy to solve an addition problem, the bigger addend could be represented with a number, and the smaller addend with discrete objects, as in this hypothetical example (see Fig. 10).

Tools Tools can encourage the use of specific strategies to solve problems. For example, to encourage comparing quantity by matching, in *MathemAntics* children

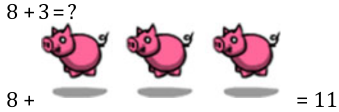


Fig. 10 Encouraging "Count On" strategy for addition

may have access to a "line-up" tool that arranges objects in a row and thus provides an organized visual comparison (see Fig. 11).

Note that the same tool that is designed to highlight a strategy may hinder appropriate use of another strategy. For example, if the activity aims to encourage comparing quantity by counting, the line-up tool should not be accessible. In higher levels of the activity, the "line-up" tool is no longer available, but the child may drag objects and arrange them in a certain way. To encourage counting, the dragging option could be made inaccessible. Thus, tools and virtual manipulatives can focus student attention on specific mathematical relations and processes in order to encourage one strategy over another, a method called "focused constraint" (Moyer-Packenham and Westenskow 2013).

Setting Parameters of the Problem Parameters—special features of a problem, like the number of objects or their spatial arrangement—can be adjusted to encourage certain strategies or discourage others. Petitto (1990) used number line estimation and specially designed rulers to explore the development of numerical proportion abilities and understanding of units of measurement. She found that in the number line estimation task children use many different (often inefficient) strategies that suggest limited understanding of proportional reasoning. For example, younger children often utilize counting strategies (either counting up from left to right or counting down from right to left), oftentimes with inaccurate units. By contrast, older children begin to correctly utilize the midpoint strategy when appropriate (e.g. counting on from 50 to get to 53 on a 0–100 line).

The specific parameters that can be manipulated in the MathemAntics version of the number line estimation task, such as providing tick marks of varying inter-

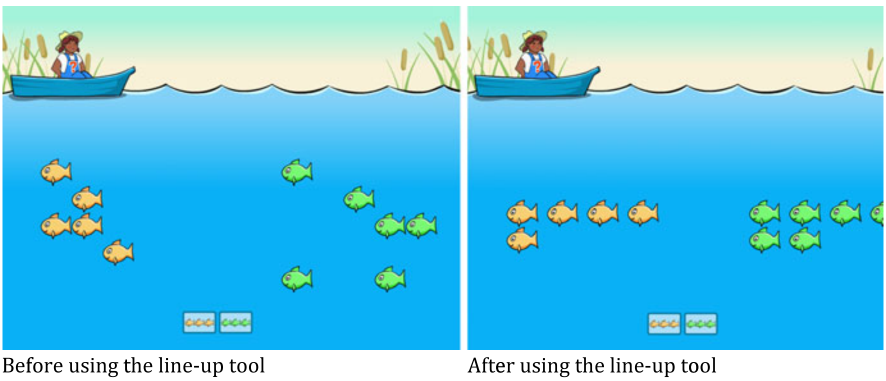


Fig. 11 Using the Line-Up tool to encourage visual comparison strategy

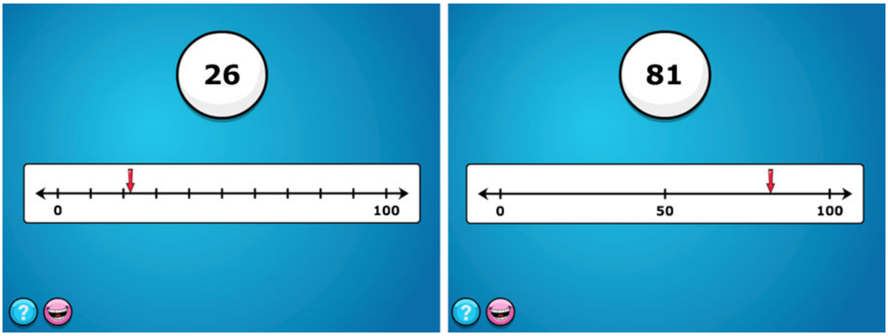


Fig. 12 Adjusting parameters of the number line to encourage different strategies

vals that can be labeled or not, may encourage the use of different effective strategies. However, the presence of tick marks may also encourage precise but tedious counting rather than estimation based on number sense. Further, showing the tick marks may prevent a child from shifting to using a more advanced midpoint strategy. Hence, the parameters of the number line may also be set so that the midpoint (rather than the tick marks) is always visible. In the left image of Fig. 12, we see a number line set to display tick marks encouraging precision and counting-based strategies whereas in the right image we see a number line with only the midpoint visible, encouraging the midpoint and more approximation-based strategies.

Feedback Simply indicating correctness or incorrectness of an answer may not be the best way to respond to a child's efforts. Instead, feedback may be designed to encourage a certain strategy. For example, if the goal of an activity is to urge children to compare sets by counting, the software feedback to an incorrect response could be "Oops, you were wrong this time. Let me show you how I know that. Fluffy has 1, 2, 3, 4—4 eggs altogether, but Fancy pants has 1, 2, 3, 4, 5—5 eggs altogether. They don't have the same number of eggs." On the other hand, if the activity aims to encourage comparing quantity by matching, the feedback could be a simple animation, matching eggs of the two hens and highlighting the leftovers.

Scaffolding Carefully designed scaffolds could also help a child in adopting a new strategy. Scaffolding may highlight the features of the problem that are relevant to a strategy, ease the difficulties involved in adaptation of a new strategy, and model the strategy for the child. These scaffolds may be implemented in the software as feedback to the child's wrong answer. The *MathemAntics* version of the number line activity provides scaffolds designed to help children to utilize certain strategies. In the standard version of the activity, the computer provides the child with a target number that the child must place on a number line. The child must click on the number line where he thinks the target number belongs. Once the child clicks, an arrow appears. The child can move the arrow to adjust his answer using the arrow keys or clicking in a different spot on the line. Once the child believes he has found the target number, he presses the space bar to submit his answer and receive feedback.

To scaffold a specific strategy, the arrow can already be presented to the child in a specified location. The arrow serves as a visual cue to help the child focus his attention on the relevant portion of the number line. Then, the child may move the arrow left or right to find the target number and submit his answer using the spacebar. For example, if a child must place 5 on a 0 to 100 line, the arrow will appear close to the left endpoint whereas if a child must place 80, the arrow will appear close to the right endpoint. Similarly, if a child must place 53, the arrow may appear close to 50, making it easier to use the midpoint strategy and encouraging the use of the most appropriate strategy based on the target number.

Conclusion Learning strategies is at least as important as memorizing facts or learning procedures. Cognitive psychology has identified the major strategies children use as well as the conditions under which the strategies can be effective. Software designers should aim to promote those strategies using the special affordances of computers, including visual representation, tools, setting problem parameters, feedback, and scaffolding.

Identify and Eliminate Bugs and Other Misconceptions

Children (and adults) make mistakes, some of which are unsystematic. Children may guess an answer if they don't understand the question, or if they don't know how to solve a problem. Unsystematic errors could also result from boredom or carelessness. On the other hand, children's mistakes may result from systematic use of an inefficient strategy or rule—a "bug"—that may have sensible origins (Ginsburg 1989). Computer programs are not the only operating systems that may suffer from bugs! For example, Sarah may say that 12 + 35 is 38, 23 + 13 is 54, and 1 + 25 is 17. At first glance, the child's answers may seem to be absurd and nonsensical, and one may speculate that the child is simply guessing the answers. However, a closer look clarifies that Sarah is systematically using an incorrect rule to find the answers. She first adds the digits of the double-digit numbers, and then puts results together to form a new number: $12 + 35 = (1 + 2) + (3 + 5) = 38$.

When Bob was asked to identify the hen with more eggs, (see Fig. 9), he always responded with the hen on the right side. Was he guessing or did he simply like the hen on the right side (Fancy Pants) more than the one on the left side (Fluffy)? Or, is it possible that his mistake is based on an incorrect strategy use? We asked Bob to show us how he knows Fancy Pants (who had 4 eggs) has more eggs than Fluffy (who had 5). Starting on the left side, he counted all the eggs for both hens together, and said "Fluffy has 5, and Fancy Pants has 9, so Fancy Pants has more." Because systematic errors are fundamentally different from random mistakes, designers and educators must examine not only accuracy but also attempt to discover *why* the child was incorrect.

To create useful software, designers need to draw upon cognitive research to develop clear answers to the following three questions related to errors and misconceptions:

1. What are the possible bugs and other misconceptions underlying mistakes?
2. How do we assess the bugs, misconceptions and other sources of errors?
3. What can be done to help children eliminate the errors?

Possible Bugs and Their Sources The cognitive psychology literature (Brown and VanLehn 1982) provides detailed information concerning children's "buggy" solutions in arithmetic. For example, given the problem

$$\begin{array}{r} 21 \\ \underline{-19} \end{array}$$

children often get the answer 18, because they always "take away" the smaller number from the larger. Or given the problem

$$\begin{array}{r} 21 \\ \underline{+19} \end{array}$$

they get the answer 310 because they fail to regroup the ten (or "carry").

In our own work with *MathemAntics*, we found many systematic errors and misconceptions when children compare quantities of two sets. Some of these systematic errors were already described in the literature as far back as Piaget (1952a). Some children had no difficulty identifying the set with more objects when the spatial attributes and numerical magnitude were congruent (as when both the length and number of objects in one row is greater than the length and number of objects in the other row), but failed to answer correctly when the spatial and numerical properties of the sets were incongruent (as when the shorter row has more objects than the longer). These children were inappropriately using a visual comparison strategy.

What are the sources of errors? In many cases it is hard to choose among several possibilities. Some of the errors and misconceptions are directly related to the child's mathematical understanding. Saxe et al. (2010) found that although the number line affords representing powerful mathematical ideas, and many children have used the number line in the classroom, they have misconceptions about its key principles. When asked to place various targets on the line (e.g., 9, 12, 13) children placed the targets in the correct numerical order from left to right, but failed to space them correctly according to linear units. Our informal pilot work revealed similar findings. Second-grade children placed all targets (1–9) on an open number line marked 0–10 equidistant in relation to one endpoint while ignoring the other endpoint, resulting in a line clustered close to the left endpoint. These children reported that the number lines they constructed were correct, indicating they were attending to the order of numbers on the number line while ignoring the proportionality. In this case, the misconception stems from lack of understanding of the key features of the number line. Through a carefully designed tutorial integrating the number line with additive numerical units, Saxe et al. (2010) were able to mitigate these misconceptions.

However, other errors may be due to limitations in procedural and utilization skills, or components of executive functioning such as working memory, inhibition, task-switching, and updating (Bull and Scerif 2001) rather than a lack of understanding the mathematical concepts. It is important to consider domain general cognitive skills of a child and how these skills contribute to the child's understanding (or misunderstanding) of a topic. For example, a child whose comparison judgment does not match his counting may have forgotten the counting results by the time he judges which set has more (hence, limitation in working memory). Another explanation is that the child does not know the meaning of all the counting words (hence has not mastered the counting principles), or the child simply does not understand the question that has been asked.

To highlight another example, when completing the *MathemAntics* version of the number line estimation task, one first-grader worked with a number line marked from 0–10 on his first few sessions. Once he mastered the 0–10 line, he moved on to work on a number line marked from 0–100. When presented with 10 as a target number, he immediately placed it at the right endpoint. While he correctly identified the right endpoint as being 100 he continued to make a perseveration error: he was unable to inhibit the response of placing 10 at the right end point. This seems to indicate difficulty with inhibition and task switching as much as difficulty with the mathematics itself.

Identifying and Assessing Errors

In developing *MathemAntics*, we used careful observation to identify the bugs, misconceptions, and other sources of children's errors. We also conducted informal "clinical interviews" with the children as they used the software. We asked questions like, "How did you get that answer?" or "How did you know? Show me." In a later section, we expand on the rationale for use of the clinical interview.

We recorded our observations and interviews with *Silverback*© software (Clearleft Limited 2010), which keeps a video record of everything the child does on the computer screen and also makes a video of the child's face and anything said during the session. At the end, *Silverback*© exports a video showing both the child's work on the screen as well as her behavior, including her responses to clinical interview questions. This is an extremely efficient tool for recording, identifying, and later coding children's strategies and concepts as they use the software.

Having used observations and the clinical interview to obtain useful information about children's thinking, we are able to design "stealth assessment" of the child's learning. For example, if we learn that children typically use certain bugs, such as always subtract the smaller number from the larger, we can have the computer identify the wrong answers as those likely to have been produced by that bug: thus, $12 - 5 = 13$ and $12 - 3 = 11$. This method originates in the seminal work of Brown and Burton (1978).

To investigate whether a child uses area rather than number to compare the quantity of two sets, the software could analyze the child's answers to the problems in which area and number magnitudes are congruent and in those that are not congruent. If the child is always correct for the congruent problems, and not correct for the incongruent problems, the child is using a visual comparison strategy inappropriately. Subsequently, appropriate feedback and scaffolding may be provided to eliminative such systematic inappropriate strategy use. If we fail to conduct this kind of analysis, we might incorrectly conclude that the child's performance is simply a result of random response.

Helping Children Overcome the Errors Having identified children's systematically incorrect thinking (like buggy strategies), well-designed software needs to help children to understand the error of their ways (and the ways of their errors). Direct scaffolding is one approach. The software can first identify the child who fails to choose the hen with more eggs when area and numerical magnitudes are incongruent. Next the software can tell the child that he needs to count each set separately and then use this information to identify the set with more objects. Scaffolding with explanation and clarification is an even better approach. For example the software could arrange the eggs in rows and columns, match the eggs from one hen to the other, and point to the hen that has more eggs. Then, it could explain that greater area does not necessarily mean the higher number, and to know which hen has more, counting is a better strategy.

In some other scenarios, there is even a better approach, one that involves active participation of the child. The boy who failed to inhibit the response to place 10 at the right endpoint of a hundreds number line inspired the design of a "count by 10s" scaffold. The *MathemAntics* software can identify a child who is making this type of perseveration error (e.g. placing 2 in the place of 20), show him a number line with the decades marked, and then instruct him to count by 10s, as each decade number highlights along with an audio that speaks the decades name.

The simple theory underlying our approach is that you need to understand children's errors before you can help children to overcome them. And if the software can identify the thinking underlying the errors, then we can program the software to help the child construct a better understanding that will in turn eliminate the errors.

Conclusion Cognitive psychology can provide detailed information concerning children's systematic errors (or "bugs") in arithmetic. Guided by the cognitive principle of starting with the child's current state, designers need to produce software that can identify the errors and then correct them. Furthermore, designers should use clinical interviews in usability testing to identify systemic errors connected with the software itself. The basic principle is to identify, understand, and respond to the systemic ways in which users misunderstand the mathematics or the software.

Design Appropriate Physical Interactions

Until now, we have introduced each principle with important lessons from cognitive psychology and have stressed how to apply them to designing educational software. In the case of interactions, we must first begin with the possibilities that the technology affords. With computer software, a child may click, drag, or use the keyboard to interact with the program. As new technologies emerge, especially multi-touch devices, the ways in which the child can interact with the program evolve as well. Designers can now deploy a plethora of interface interactions such as touch, drag, swipe, squish, expand, rotate, blow, tilt, or shake, fling, and flick all of which can add to the interest, functionality, effectiveness, and creativity of the software.

Decisions about which interface interactions to include in the design are not trivial. Children are physically and cognitively different from adults and these differences have important implications for software design. Some design decisions are related to physical usability. Can the young child touch and drag an object on the screen? Can the child use her fingers to pinch objects so as to reduce their size? Other design decisions must be based on cognitive psychology. We focus here on how cognitive psychology can inform the following question regarding interface interactions: What are the cognitive benefits and limitations of different kinds of interface interactions, particularly on touch screen devices?

Benefits and Limitations Clearly, new and more complex interface interactions may help make the software more engaging and motivating, but can children use and learn from them? Shuler (2009) warns that the limiting physical attributes of mobile devices (such as restricted text entry and small screen size) as well as the usability difficulties of poorly designed interface interactions may distract children from their learning goals. Poorly designed applications fail to consider usability issues and motor development of children at various ages.

Young children may have physical difficulty with some standard (not touch-screen) computer interfaces. Using the mouse to click on an icon, drag items on a screen, rotate, tilt, and draw requires a degree of fine motor development typically lacking in young children. Hourcade et al. (2004) found that preschool children were slower and less accurate than older children in using the mouse for point-and-click tasks, especially when the target size was small. This difficulty can be attributed to under-development of fine-motor skills and is likely to distract from learning. Fisher (2012) recommends providing large icons with "large hotspots" around the target, and avoiding drag-and-drop if possible. Further, in the case of touch-screen design, she recommends leaving a "safe-zone" around the edges to avoid any accidental touch when holding the device, and having the software respond to the child's touch only when an object is meant to be touched. The object then may make a noise, change color, wiggle, or make something happen in the activity.

Physical activity is important for designers not only because it can become a limiting factor for usability but also because it can promote the development of thinking. In Piaget's (1952b) view, children's cognitive development begins with physical activities like sorting objects, causing things to happen (like making an object move), comparing objects' size and weight, and pointing to things during counting. Gradually these physical activities become internalized as cognition. Now the child can think about sorting, about causality, and comparisons. Now the child can do mental counting and addition.

Complex cognitive processes appear to be grounded in the body's interactions with the learning environment (Wilson 2002). Multi-touch devices afford gestural interactions that build upon children's natural inclinations. Fingers are a child's first tools in learning mathematics. Children use their fingers to represent quantities, they point to objects to count them, and they use their fingers to add and subtract numbers (Butterworth 1999). A substantial body of literature calls attention to the significance of gesturing in learning and development (Dick et al. 2012; Goldin-Meadow and Alibali 2013; Ping and Goldin-Meadow 2008), and especially in learning and explaining mathematics (Cook et al. 2012; Goldin-Meadow et al. 2009). Gesturing might encourage children to extract meaning implicit in their hand movements (Goldin-Meadow et al. 2009); gesturing might help children to structure abstract ideas and map them metaphorically onto the visual space, the screen with which they are working (Kessel and Tversky 2006); gesturing might help children to focus their attention and embody the relevant information in the task.

Note that gesturing and hand movements matter, but also the form of gesturing is important in changing thought. In one study, Jamalian and Tversky (2012), showed that circular gesturing while explaining a cyclic sequence of events (such as the growth from seed to flower) helps adults to overcome their linear bias and instead to adopt more complex cyclical thinking. In another study, Segal (2011) investigated children's performance in discrete (single-digit addition) and continuous (number line estimation) mathematics problems while controlling for the type of gesture. In the continuous mathematics problem condition, 1st and 2nd grade children were asked to estimate a quantity on a number line shown on a computer interface. In this condition, children became more accurate when they dragged their finger or mouse on the number line as opposed to when they simply pointed and clicked on the number line. The gesture and the nature of the problem are both continuous. On the other hand, when solving addition problems, children were more accurate and adopted more advanced strategies when they pointed or clicked on each object to be counted/added comparing to clicking on a single button representing the total number of objects to be added. Segal argues that performance improves when the form of gesture is congruent with the problem-solving thinking process, and worsens when gesture is incongruent.

The educator's job is to help the child progress from physical to mental activity. Work with physical and virtual manipulatives should be designed to promote thinking, not merely playing with physical and virtual objects, no matter how attractive they may be. Engagement with physical and virtual objects is educationally useful only insofar as it eliminates the need for itself because what was on the "plane of action" is now transformed in the "plane of thought."

Given all this, a fundamental cognitive design principle is that the type and nature of gestures and other physical movements on both touch and mouse dominated screens can have an important influence on children's strategies and accuracy. This principle guided our design of the *MathemAntics* version of the number line activity. To allow flexible strategy use, we included many ways the child can place targets on the line. The child may use the mouse to click anywhere on the line, or use the left and right arrow keys to make small adjustments. Usability and play testing revealed that some children used the arrow keys to count discrete units from zero to reach the target number. Such counting strategies are useful, but inefficient and time-consuming, especially for larger target numbers (such as 98 on a 1–100 number line). To discourage the use of this strategy, we revised the activity so that the arrow keys still make small adjustments, but these adjustments no longer correspond to correct discrete units. Through feedback, the child will quickly discover that using the arrow keys to count to the target number will not lead to the correct answer and will encourage strategies based on approximate reasoning.

When implementing interactions, designers must also consider how the child will indicate he is ready to submit his answer. Far too many applications allow the child to succeed "by accident." For example, one simple enumeration app asks the child to put out 4 tokens, and the child may begin dragging tokens. But a close look at the child's strategy raises the possibility that he may be unsystematically dragging as many tokens as he can without counting or may be counting incorrectly. Yet once the child puts out 4 tokens, the trial ends and the child is given positive feedback about his performance, despite the fact that the child may have been trying to put out more than 4. This child does not have the option to carry out his plans and thus is missing an important learning opportunity that might result from his failure. We considered this issue in the design of *MathemAntics* and included a way for the child to indicate he is done working and ready to submit his answer. In our number line activities, the child has the flexibility of moving the cursor along the number line until he chooses to press the spacebar, which submits the answer and then results in the appropriate feedback.

Conclusion Computers allow children to interact with the computer interface in many different ways, from clicking and dragging to pinching with the fingers. Designers need to pay close attention to usability issues resulting from young children's limited fine motor skills. But more importantly, cognitive psychology suggests that children's gestures and other movements have important influences on what they learn and how they think, and that computer software needs to engage children in meaningful gestural interactions that promote deep learning.

Integrate Narratives and Stories with the Mathematical Concepts

Cognitive development in general and development of mathematical concepts in particular should take place in the context of supporting environments (Gelman et al. 1991). These environments can be social, cultural, natural (Gelman et al. 1991), or virtual. Gelman and Gallistel (1986) highlight the importance of meaningful context for measuring and promoting young children's number competence. Carpenter and Moser (1984) show that children's solutions to addition and subtraction problems reflect the semantic structure of the problem. Carpenter and Moser (1984) report that different contexts for subtraction word problems such as separating, joining, part-part-whole, comparison, influence the child's strategy use and responses. Other researchers (Barlow and Harmon 2012) point out benefits of problem contexts such as balancing and jig-saw problems for thinking about equality and the meaning of the equal sign.

Well-designed software can make learning meaningful through use of narratives. Narratives can help children to elaborate their everyday mathematics, for example by considering the number of monkeys falling off a bed one at a time. Narratives can help children synthesize concepts from the ordinary world with those in the abstract world of formal mathematics, for example by mathematizing the unfortunate monkeys as a series of subtraction statements. Narratives can give meaning to otherwise hard-to-comprehend symbols and operations, helping children to appreciate mathematics as a problem-solving discipline as opposed to a collection of meaningless facts and procedures to be memorized. In order to be successful, the goals of the narrative should be congruent with the goals of the mathematics. An effective narrative will assist the child in focusing attention on the relevant mathematical information, rather than distracting from it (Lepper and Malone 1987).

Stories, narratives, and fantasy also have the potential to enhance the learning experience by increasing motivation and engaging the learner (Malone and Lepper 1987). A meaningful narrative may help a child identify with the story and connect with the characters on an emotional level. Thus, the child's interest in the activity grows which in turn may build intrinsic motivation—interest in the activity for its own sake.

As promising as narratives are for learning, some designers fail to use them effectively. All too often the narrative is violent frosting on the number fact cake, as when the child is asked to shoot down correct number facts floating in the mathematical sky. Sometimes developers use a narrative that does not support or interferes with a meaningful learning process, as in the case of Rocket Math (Russell-Pinson 2012). In this game, students build a rocket and launch their rocket into space for math-based missions like solving addition or multiplication equations. Students earn "money" for their successful completed missions and they can use that money to upgrade and improve their rockets. The game is fun to play for many children, but in our own testing, we realized that students often spend more time buying boosters, fins, and decorations for their rocket than solving the math problems. Some children have even difficulties in building a rocket in the first place to launch into space so their rockets cannot even fly to solve the missions; for these children, the game-play gets focused on "how to build a good rocket" rather than learning the math facts.

We propose that it is important to consider the following three questions regarding to stories and contexts when designing learning activities.

1. What are the different contexts in which to situate an abstract concept?
2. What does the context add to the learning experience?
3. How could we integrate stories and the mathematical concept?

We illustrate the role of narratives and context with examples of *MathemAntics* activities focused on numerical relations and the equivalence of two sets. By definition, two sets are numerically equivalent if for every object in one set there is one and only one object in the other set.

Recognizing numerical equivalence between two sets, despite any visual differences, is a fundamental numerical competence (Mix 1999) that many adults may take for granted, but young children find cognitively challenging. Narratives can

Fig. 13 Bedtime activity

situate this abstract concept in a meaningful context, helping children to better comprehend it. Containers, for example, are a natural match for objects. Children could start thinking about one-to-one correspondence by placing objects in containers and checking whether all objects have a container to go in to. In *MathemAntics*, we have chicks and beds, birds and nests, apples and baskets (see Fig. 13). Children are asked to put each chick in a bed, each bird in a nest, each apple in a basket, and check whether there are the same in number.

Feeding animals is another natural context in which the concept of numerical equivalence could be situated. Children are asked to pick as many apples as there are goats to feed, and ring the bell when they are done. After ringing the bell, apples get distributed among the goats and the child can check if he picked more, fewer, or just the same number of apples as there are goats (see Fig. 14).

In another activity we use the social schema of "ownership" and "having" to situate the concept of equivalence. Remember Fluffy and Fancy Pants? In earlier levels of the activity we ask children to judge whether the two hens *have* the same number of eggs. In more advanced levels of the activity, children are asked to make Fluffy and Fancy Pants have the same number of eggs, or make one of them have more or fewer eggs than the other. Children can touch the chicken's belly to lay an egg or touch an egg to hatch it.

"Friendship" in the sense of pairing is another everyday social schema that can bring up the idea of equivalence. In the *Matcher* activity children are in charge of finding friends for the chicks who are new to the farm (see Fig. 15). The child is asked to open the corral that has the same number of animals inside it as there are chicks so that each chick will have a friend. If the child opens the horses' corral, they come out, pair up with the chicks and dance in pairs. If the child opens the goats' corral, they come out, pair up with chicks, and the two extra goats weep complaining that they do not have any friends.

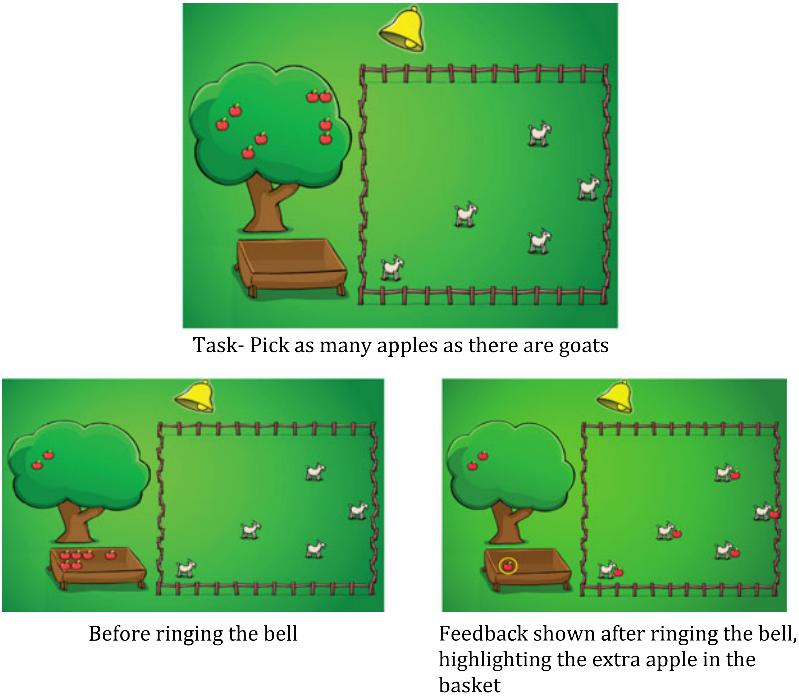


Fig. 14 Feeding Animals activity

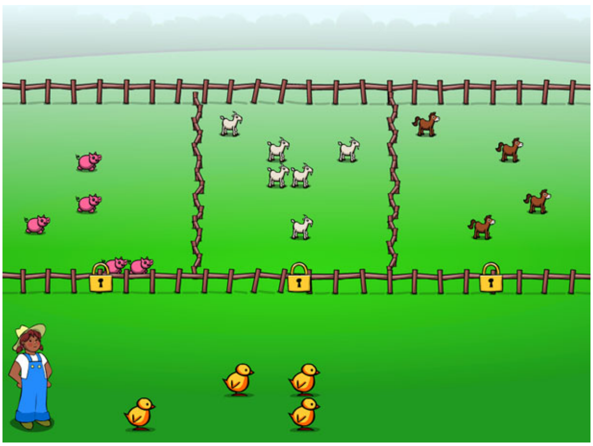


Fig. 15 Match Friends activity

In more advanced activities we ask the child to directly compare the number of objects in two sets. Children are asked whether there is the same number of yellow fish as green fish, to pop the bubble that has more fish in it, or pop the bubble with fewer fish. (See Fig. 16. Note that the green swordfish pops the bubble.)

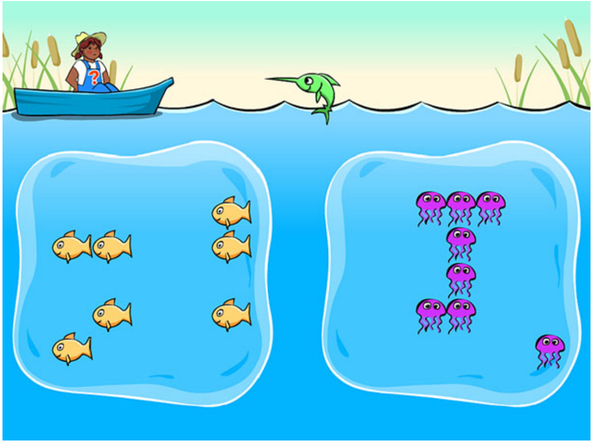


Fig. 16 Fish in the pond narrative in the Pop the Bubble activity

In all these activities, we introduce formal mathematical language such as "same", "more", "fewer", "most", "fewest", "greater", "less", verbal and written number words, mathematical symbols such as "$> = \neq <$", and number sentences. The formal mathematics is thus integrated into the overall narrative as part of instruction, feedback, scaffolding, or response mechanism.

Elements of narratives other than the content may also enhance the learning experience. For example, pedagogical agents could be characters in the story and play a central role in the learning process. In *MathemAntics*, we have a farmer who asks children to feed animals in the farm, count number of chicks in the barn, play with the fish in the pond, make the hens lay equal numbers of eggs, or estimate the amount of dough required for making cookies. The farmer provides feedback and scaffolding and rewards children with tools such as the "line-up" tool, or the "pair-up" tool, which children can use to solve more challenging problems.

Conclusion For learning to be effective, it may benefit from being situated in a meaningful context that gives it a purpose. Stories can promote and motivate mathematics learning. Unfortunately, some software employs narratives that are essentially irrelevant to the mathematics and also entail unnecessary violence, like shooting down rockets. But narratives and characters can promote meaningful mathematics learning if they are congruent with the mathematical ideas to be learned.

Evaluations and Assessments

Cognitive principles can guide evaluation and usability testing of software designs, and can also provide useful approaches to formative and summative assessment and to the evaluation of achievement and software effectiveness.

Cognitive Evaluation and Usability Testing of Software

At the beginning, designers should determine whether the software under development is congruent with the cognitive design principles described above. For example, does the software deal with the appropriate cognitive content? Does it promote thinking as well as procedure and memorized facts? Does it employ powerful visual models? Does it offer powerful tools? Does it identify bugs and misconceptions underlying errors? We propose that designers not conversant with the cognitive psychology of children's thinking and learning cannot create educationally effective software. The lack of this knowledge is one reason why many apps and software programs are of poor quality. What to do? One solution is to collaborate with psychologists. Every design company should hire at least one person knowledgeable in cognitive psychology!

Even if the software seems to pass the cognitive psychology design test, designers should conduct usability testing to see whether the software "works" in several senses. Designers want to determine whether the software contains any bugs, whether the children understand the function of a particular feature (like an icon that must be pressed to submit an answer), whether children can navigate through the material effectively (first use this icon and then use that), and whether they misinterpret the purpose of the activity. All good designers are well aware of the need for early and fast usability testing. The common practice is to closely observe the user interact with the software; users may ask questions and are encouraged to express their difficulties, feelings, or comments; however, designers are encouraged not to interfere with the experience and not to answer any of those questions. The goal is to investigate whether the user interacts with the software as predicted, if the user enjoys the interaction, and whether there are any usability or navigation problems.

Although observational usability testing can yield important information, we propose that observation is not sufficient and must be supplemented by intensive questioning—a "clinical interview usability method" that in turn leads to methods for helping children to understand the task and to perform more effectively.

The clinical interview method, originally developed by Piaget (1976), involves a flexible questioning of the subject, child or adult (Ginsburg 1997). The interviewer attempts to identify the thinking underlying the subject's overt behavior by asking relevant questions, modifying them, and following up with probing questions such as: "Why did you do that?," "How did you know?," and "How did you figure it out?" These kinds of questions need to be guided by as much knowledge of children's thinking as possible. For example, the interviewer who has extensive knowledge of the bugs that typically infest children's calculation methods is more likely to ask penetrating questions than the interviewer ignorant of them. In general, the more you already know about children's thinking (or almost anything else) the more can you learn about it. But you have to start somewhere, and the question "How did you do it?" is a good first step.

The challenge for designers becomes how to embed clinical interview methods into the usability testing of the software. Again, the main solution is to collaborate with cognitive psychologists. Every design company should hire at least one person knowledgeable in the clinical interview (as well as in cognitive psychology as described above—ideally the same person).

Formative and Summative Assessments

As we showed earlier, the computer software can be used to conduct "stealth assessment" of the child's learning. For example, if we learn from cognitive psychology or pilot work that children consistently believe that physically larger sets are more numerous than smaller ones (for example, 3 elephants are "more" than 5 ants), we can program the computer to identify these "size bugs" by tracking wrong answers in response to similar problems contrasting size and number. The software can also identify such important behaviors as accuracy; latency in response to different types of problems (some of which benefit from speed, like retrieval of number facts) and some of which, by contrast, require careful contemplation (as in the case of word problems); and the frequency of certain kinds of choices (as when the child is given the opportunity to employ different tools and strategies).

Stealth assessment can provide teachers with valuable "formative assessment" reports on individual students or the whole class, as well as suggestions of activities that might benefit individual students or even the whole class. For example, if a student can accurately retrieve "number facts" (like $2 + 3$) but has no strategies to figure out facts that have not been memorized, the computer can offer activities designed to promote use of strategies. By contrast, the student who possesses good strategies for figuring out number facts but cannot recall them quickly and accurately may benefit from drill. The suggestions need not be limited to computer activities, but could also involve textbook lessons, work with physical manipulatives, or even a mandate to avoid spending too much time working with computers and instead to go out and play in the non-virtual world of sun and fresh air.

Stealth assessment can also be extremely valuable for "summative" data by tracking progress during the school year and providing pre- and post-test indicators of student achievement. Unlike most conventional achievement tests, stealth assessments also have the ability to describe the trajectory of the individual child's (and the class of children's) learning. The assessments can depict the evolution of accuracy, concepts, and strategies, and even engagement and interest (Rodrigo and Baker 2011). Moreover, these portraits are based on many data points—literally thousands for children using the computer on a daily basis. The volume of data can overcome the inevitable noise produced by children's fluctuating attention and other sources of error variance. Moreover, the assessments draw upon problems of obvious educational relevance and thus have good "face validity." Conventional achievement tests may correlate highly with other conventional achievement tests, but this may be merely an artificial dance in which the blind lead the blind. What could be more "valid" than reports using very large amounts of data to depict the development of children's performance, strategies and concepts?

Finally, summative assessments of this type can serve as the foundation for evaluating particular software programs or apps. The summative data can be used to compare the overall effectiveness of different software programs, but more importantly can be used to compare the effectiveness of particular activities within a software program. For example, one activity within a comprehensive computer program may be successful at teaching concepts but not so effective at promoting recall, whereas an activity within another program may promote recall but not concepts. Assessments of this type can help a school district evaluate educational software in a more nuanced manner than can a conventional evaluation providing pre- and post-tests of dubious validity. And they can help designers improve specific activities within a software package.

Conclusion Cognitive principles can provide a preliminary basis for evaluating software. They can reveal whether the software as constituted seems to focus on important aspects of learning and use sound pedagogy. The cognitive principles and the method of clinical interview can also guide usability research. Sound software based on these principles can provide teachers with formative assessments of student performance, learning and thinking, and suggest approaches to instruction. The software's summative assessments can serve as valid indicators of student achievement, and might even eliminate the need for traditional evaluation measures or at least assign them a secondary role.

From Software to Research

We have shown that cognitive psychology can and should guide software design, evaluation and assessment. Here we argue that high-quality software offers new ways to conduct cognitive research and a powerful lens for examining the child's mind.

Most research studies on the development of mathematical thinking use a cross-sectional approach to investigate children's current cognitive status at several age levels. The outcomes of this kind of research have been enormously useful for both psychology and education, and have indeed revolutionized our views of children's mathematical development. We know a good deal, for example, about young children's everyday addition and how it influences the understanding of the symbolic addition taught in school. The research is not only of theoretical interest but can be used to guide teaching and assessment.

At the same time, this important research is limited in a key respect. Vygotsky (1978) makes a distinction between the "actual developmental level" (p. 85), which is the child's current state, vs. the level of "potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). The essence of the distinction is the idea that the child's current performance may not indicate underlying competence and potential for learning.

Vygotsky stresses the importance of examining the "...dynamic mental state, allowing not only for what has been achieved developmentally but also for what is in the course of maturing" (p. 87).

We think it is fair to say that most research studies on cognition generally, and on the trajectories of mathematical development in particular, do not follow Vygotsky's advice. Instead the research often uses cross-sectional studies to investigate children's current cognitive status at several age levels. Because this enormously valuable research largely does not deal with children's "dynamic mental state," we need to take the idea of "developmental trajectories" with a dash of Vygotskian salt. It is possible that current views of developmental trajectories underestimate what children can do and learn.

One way to investigate this issue is by closely examining children's mathematical thinking in the context of stimulating software such as *MathemAntics*. As noted earlier, Papert (1980) made the argument that children's mathematical thinking may look very different in rich "microworlds"—artificially designed "mathemagenic" environments—than it would in the context of the not-very-dynamic school or home. This argument is especially important in the case of disadvantaged children and those with disabilities. Our own preliminary observations suggest that low-SES children may perform much better than expected when engaged in *MathemAntics* software. And some early research on LOGO suggests that mathemagenic computer environments enable severely disabled individuals to perform at unexpected levels (Weir et al. 1982).

Not only does the mathemagenic environment provide the context for dynamic learning, but the software also supplies the means for studying it. *MathemAntics* includes a control panel, hidden from the child, that allows the adult to vary key problem features, like the number, size and shape of virtual objects, the availability of tools, and the nature and timing of feedback. Other software (unless it is extremely simple and rigid) must have controls like these as well, even if they are hidden from the user and anyone else. The control panel is an experimenter's delight. It allows for easy manipulation of all or any of the available variables. In effect, the panel provides the opportunity for factorial designs. (We plan to make our *MathemAntics* software and panel available to researchers willing to share data with us.) And because children do the activities repeatedly, over time, the software provides the opportunity for microgenetic research—the investigation of learning over a not too brief time period of say a few weeks or months. Of course, if children are engaged in the various activities over a period of years, the opportunities for longitudinal research are evident.

Further, as we have seen, the software can help to measure children's performance, accuracy, latencies, and even the kinds of strategies they employ. This stealth assessment can be useful for the researcher, as well as the teacher. But in either event, the development of stealth assessment requires considerable pilot research. Some may require close observation of children working with the software, as we described earlier. Of course, stealth assessment may not produce a complete picture, and therefore may need to be supplemented by other kinds of measures (like coding video segments of children as they work with computers). But the software does make possible computer generated descriptive reports, like frequencies of strategy use, or the conditional probability of strategy use given certain conditions. For example, does the child use the counting on strategy for addition on the problems for which it is most appropriate? The reports can be imported into a statistical program so as to permit instant analysis of very large amounts of data. In principle, the software could easily generate reports and analyses of thousands (why not millions?) of children working with the software.

Conclusion The development of exciting, meaningful and motivating software provides researchers with the opportunity to study children's learning in a context that may elicit from them more advanced performance and thinking than they display in standard experimental tasks. The software may reveal surprising deviations from known developmental trajectories, especially in disadvantaged and handicapped children. In any event, the software makes it feasible to conduct microgenetic and longitudinal studies, to gather data on very large numbers of children, and conduct stealth assessments that reveal much more than accuracy. These possibilities may in turn require the development of new statistical approaches to apply to massive microgenetic research studies.

Final Remarks

We have argued that cognitive principles can and should shape the development of software, which in turn can improve learning, teaching and testing. We used examples from *MathemAntics* to show that such software is not a figment of our imagination. But examples—even though providing a kind of existence theorem—are insufficient in the absence of solid data. Skepticism is appropriate because many educational innovations fail, and computers are no exception to the rule.

But suspend disbelief and suppose that we can create an effective and comprehensive system of early mathematics software guided by cognitive principles. The possibilities for transforming education are enormous.

The software will enable children, especially the disadvantaged, to reach higher levels of mathematics achievement than they do now. They will learn both concepts and skills; they will think as well as remember; and they will use sensible strategies as well as apply standard algorithms. And they will enjoy learning meaningful mathematics.

The software will enable children to explore and learn on their own and at their own pace. This can be a lifeline for students with weak teachers—who are unfortunately more numerous than we would like. The software will also help children work with each other on common problems, engage in productive argument, and share solutions.

Children will not make a strong distinction between literature and mathematics. They will read and engage with mathematical stories and enjoy both the mathematics and the stories.

Comprehensive mathematics education software will supplement, eliminate, or transform traditional textbooks. Static two-dimensional representations on paper are clearly ineffective for portraying many kinds of transformations (like subtraction) and other mathematical processes (like splitting a segment on a number line into a thousand pieces) and for providing opportunities for interactive learning. At the very least, comprehensive mathematics software can supplement traditional textbooks. But two other possibilities are more attractive. One is replacing textbooks entirely, especially for young children who cannot read well. Older children might be better served by a hybrid textbook/software system that preserves the written word but also offers interactive software. In other words, textbooks can take the form of e-books that embed the mathematics software in a coherent manner.

As children in the class work on computers, the teacher can attend to the needs of individual children more frequently and effectively than is possible in a large classroom without computers.

The software-based formative assessments will help teachers learn to understand children as individuals and will guide teaching. The assessments can be seen as a kind of professional development for teachers who know little about underlying principles of student learning and thinking and who do not know how to conduct effective assessments of individual students.

The assessments can change the practice of evaluation. Because it provides so much rich data, stealth assessment should largely replace other assessment and testing procedures. What could be more ecologically valid and useful than detailed portraits of children's learning over the course of a year or longer?

Software can set the stage for a different kind of basic research that examines children's learning and thinking in rich and adaptive educational environments over long periods of time. The number of children and number of "trials" will be enormous—for example, hundreds or even thousands of children working with computers all year long and providing many thousands of data points. In this kind of adaptive learning environment, no two children may have exactly the same learning experiences or the same number of them. To deal with these complexities, use of novel statistical methods will be essential. Analyses created for basic factorial designs will not be helpful. Use of effect size to determine the significance of mean differences in achievement will become obsolete.

And finally we observe that apps can introduce a large amount of whimsical fun (think Fluffy and Fancy Pants) into children's mathematics learning. This is not trivial: the antic (as in *MathemAntics*) is intended not only to amuse but to show that thinking needs to be liberated from dull convention.

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Rethinking Early Mathematics: What *Is* Research-Based Curriculum for Young Children?

Douglas H. Clements and Julie Sarama

How many times have you heard, "Our mathematics curriculum is based on research"? Have all these curricula (including early childhood education programs, educational software, teaching strategies, etc.) been similar? Have they all been effective? Many who publish a curriculum, or write or speak about a teaching approach, claim their approach is based on research, even though they vary widely (Battista and Clements 2000; Clements 2007; Senk and Thompson 2003). Such claims are often vacuous, including general statements about what "the research says." Unfortunately, such overuse of the phrase "research-based" undermines attempts to create a shared research foundation for the development of, and informed choices about, classroom curricula (National Research Council 2002, 2004).

We believe that researchers and practitioners can work together to ameliorate this situation and develop, evaluate, and use valid research-based approaches. To support such collaborative activity, we have developed two major conceptual tools. The first is a set of *learning trajectories* that describe how children learn major topics in mathematics and how teachers can support that learning. Based upon studies in fields ranging from cognitive and developmental psychology to early childhood and mathematics education, these guide the creation of standards, curricula, and teaching strategies. They also are at the core of the second conceptual tool, a framework for developing curricula and teaching strategies. This framework describes criteria and procedures for creating scientifically-based curricula.

Learning Trajectories

Research-based learning trajectories are tools educators can use to improve mathematics learning and teaching (Simon 1995). Our learning trajectories are based on

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evidence that children generally follow natural paths—sequences of increasingly sophisticated levels of thinking—as they learn mathematics topics (Clements and Sarama 2009; Sarama and Clements 2009b). These sequences can be described as *developmental progressions*. When teachers understand these developmental progressions, and use them in selecting and sequencing instructional activities, they can build more effective mathematics learning environments.

A complete learning trajectory has three parts: a goal, a developmental progression, and instructional activities. To attain a certain mathematical competence within a given domain (the goal), children typically learn each successive level of thinking (the developmental progression), aided by activities (instructional tasks) designed to build the mental actions-on-objects that enable thinking at each higher level (Clements and Sarama 2004b).

Early Addition and Subtraction: An Example

The main *goal* of the counting-based addition and subtraction learning trajectory is that children learn to solve different types of arithmetic problems (Carpenter et al. 1988) and develop accuracy and eventually fluency with arithmetic combinations. The second component of the learning trajectory is the *developmental progression*, which describes a typical counting-based trajectory children follow in developing understanding and skill in arithmetic. The left column in Fig. 1 describes several levels of thinking in the learning trajectory and provides examples of children's behavior for each level. The right column provides examples of *instructional tasks*, matched to each of the levels of thinking in the developmental progression. These tasks are designed to help children learn the ideas and skills needed to achieve that level of thinking. However, instructional tasks are always examples—many tasks and approaches to teaching are possible. Therefore, curriculum developers and teachers should translate developmental progressions and instructional tasks for specific cultural, school, and individual contexts. That is, to re-think mathematics education, we must also re-consider the cultural and sociopolitical contexts children experience (Wager and Carpenter 2012, discuss these issues at length). Thus, there is no single or "ideal" developmental progression, and thus learning trajectory. The following presents just one example.

Summarizing, learning trajectories describe the goals of learning, the developmental progression through which children pass, and the learning activities in which children might profitably engage. They are based on research first because the *sources* of the developmental progressions are extensive research reviews and empirical work (Sarama and Clements in press). They are also research-based because whenever possible, the instructional tasks are guided by this same empirical work and by classroom-based research and the wisdom of expert teacher practice. Although it is beyond the scope of this chapter to present this body of research (see Sarama and Clements 2009b), along with the complex, cognitive actions-on-objects that underlie all the example behaviors in Fig. 1, we will provide one illustration of

***Goal:* Children solve different types of arithmetic problems and develop accurate and eventually fluent competencies with arithmetic combinations**

***Developmental Progression* (including Example Behaviors for each Level of Thinking)**	***Instructional Tasks***
Nonverbal +/− Adds and subtracts very small collections nonverbally. Shown 1 object then 1 object going under a cover, identifies or makes a set of 2 objects to "match."	"Blocks in the Box": Children play a game in which, for example, 2 blocks then 1 block go into a box, and try to "guess" how many are in the box. The cover is taken off and the blocks counted to check.
Small Number +/− Finds sums for joining problems up to $3 + 2$ by counting-all with objects. Asked, "You have 2 balls and get 1 more. How many in all?" counts out 2, then counts out 1 more, then counts the total.	"Word Problems (Join result unknown or separate, result unknown (take-away) problems, numbers < 5)": "You have 2 balls and get 1 more. How many in all?" "Finger Word Problems": Tell children to solve simple addition problems with their fingers. Use very small numbers. Children should place their hands in their laps between each problem. To solve the problems above, guide children in showing 3 fingers on one hand and 2 fingers on the other and reiterate: How many is that altogether? Ask children how they got their answer and repeat with other problems.

Fig. 1 Samples from the Learning Trajectory for Counting-based Arithmetic (addition and subtraction, adapted from Clements and Sarama 2009, 2012; Sarama and Clements 2009b)*

both the cognitive actions-on-objects that underlie the levels of thinking and how different trajectories grow not in isolation, but interactively.

Consider learning a critical competence for early arithmetic—counting on, used especially at the *Counting Strategies* level in Fig. 1. Children need to develop competencies from three learning trajectories to learn to count on meaningfully. Two provide support: (1) counting (Fuson 1988) and (2) subitizing, the quick recognition of the number in small sets without counting (e.g., Antell and Keating 1983; Kobayashi et al. 2004). (These two learning trajectories are described in Clements and Sarama 2009; Sarama and Clements 2009b.) The third, of course, is the arithmetic learning trajectory from Fig. 1.

Find Result +/−
Finds sums for joining (you had 3 apples and get 3 more, how many do you have in all?) and part-part-whole (there are 6 girls and 5 boys on the playground, how many children were there in all?) problems *by direct modeling, counting-all, with objects*.
Solves take-away problems by separating with objects.

Asked, "You have 2 red balls and 3 blue balls. How many in all?" counts out 2 red, then counts out 3 blue, then counts all the balls.
Asked, "You have 5 balls and give 2 to Tom. How many do you have left?" counts out 5 balls, then takes away 2, and then counts remaining 3.

"Word Problems": Children solving all the above problems types using manipulatives or their fingers to represent objects.
For Separate, result unknown (take-away),

"You have 5 balls and give 2 to Tom. How many do you have left?" Children might counts out 5 balls, then takes away 2, and then counts remaining 3.

For Part-part-whole, whole unknown problems, they might solve

"You have 2 red balls and 3 blue balls. How many in all?"

"Places Scenes (Addition—Part-part-whole, whole unknown problems)": Children play with a toy on a background scene and combine groups. For example, they might place 4 tyrannosaurus rexes and 5 apatosauruses on the paper and then count all 9 to see how many dinosaurs they have in all.

Fig. 1 (Continued)

From the counting learning trajectory, children learn to count forward starting with any number. Then, they learn to understand explicitly and apply the idea that each number in the counting sequence includes *the number before, hierarchically*. That is, 5 includes 4, which includes 3, and so forth. For example, 3-year-old Abby could count up past 20, but always had to start at one. Asked to start at four, she hesitated, then said, "One, two three four, five. . . ." Her teacher played informal games with her such as placing a couple of blocks in a box, asking, "How many are in the box now?", adding one, and repeating the question. She also had Abby work on the computer activities in Fig. 2a. In *Build Stairs 2*, children slowly count, clicking on the next number. In the next level, they have to see that 4 comes after 3.

Counting Strategies +/−
Finds sums for joining (you had 8 apples and get 3 more…) and part-part-whole (6 girls and 5 boys…) problems with finger patterns and/or by counting on.

Counting-on. “How much is 4 and 3 more?” “Four… five, six, seven [uses rhythmic or finger pattern to keep track]. Seven!”
Counting-up-to May solve missing addend (3 + _ = 7) or compare problems by counting up; e.g., counts “4, 5, 6, 7” while putting up fingers; and then counts or recognizes the 4 fingers raised.
Asked, “You have 6 balls. How many more would you need to have 8?” says, “Six, seven [puts up first finger], eight [puts up second finger]. Two!”

“How Many Now?”: Have the children count objects as you place them in a box. Ask, “How many are in the box now?” Add one, repeating the question, then check the children’s responses by counting all the objects. Repeat, checking occasionally. When children are ready, sometimes add two, and eventually more, objects.

Variations: Place coins in a coffee can. Declare that a given number of objects are in the can. Then have the children close their eyes and count on by listening as additional objects are dropped in.

“Double Compare.” Students compare sums of cards to determine which sum is greater. Encourage children to use more sophisticated strategies, such as counting on.

“Easy as Pie”: Students add two numerals to find a total number (sums of one through ten), and then move forward a corresponding number of spaces on a game board. The game encourages children to count on from the larger number (e.g., to add 3 + 4, they would count “four … 5, 6, 7!”)

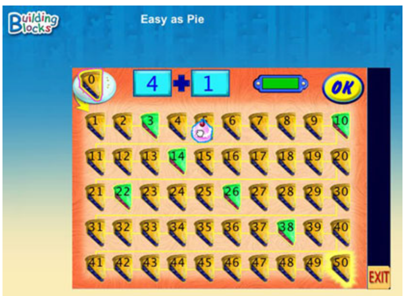


Fig. 1 (Continued)

Deriver +/− Uses flexible strategies and derived combinations (e.g., "7 + 7 is 14, so 7 + 8 is 15) to solve all types of problems. Includes Break-Apart-to-Make-Ten (BAMT). Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. Asked, "What's 7 plus 8?" thinks: 7 + 8 —> 7 + [7 + 1] —> [7 + 7] + 1 = 14 + 1 = 15. Or, using BAMT, thinks, 8 + 2 = 10, so separate 7 into 2 and 5, add 2 and 8 to make 10, then add 5 more, 15.	(The BAMT strategy is taught here.) "Tic-Tac-Total": Draw a tic-tac-toe board and write the numbers 1 to 10. Players take turns crossing out one of the numbers and writing it in the board. Whoever makes 15 first wins. "21": Play cards, where Ace is worth either 1 or 11 and 2 to 10 are worth their values. Dealer gives everyone 2 cards, including herself. On each round, each player, if sum is less than 21, can request another card, or "hold." If any new card makes the sum more than 21, the player is out. Continue until everyone "holds." The player whose sum is closest to 21 wins.
Problem Solver +/− Solves all types of problems, with flexible strategies and known combinations.	"Word Problems (all types of problem structures for single-digit problems)"

* Note that these counting-based strategies are only one of the paths to arithmetic (see Chap. 6 in each of two companion books, Clements and Sarama 2009; Sarama and Clements 2009b)

Fig. 1 (Continued)

Both activities benefit from computer technology. Young children find computer interactions motivating, especially in narrative contexts in which they help animals (Sarama and Clements 2002). The computer manipulatives are just as easy for them to use, and often provide better supports for learning (Sarama and Clements 2009a). Finally, children receive immediate, patient feedback on their actions (Clements and Sarama 2002). After these experiences, Abby could start at any number up to 10 and count forward or backward.

From the subitizing learning trajectory, children learn to quickly recognize the number in visual sets, such as a triangle pattern of blocks or a spatial pattern of three fingers. Importantly, they also learn *rhythmic patterns*. For example, they learn the rhythm of three ("Doo—Day—Doo" or hearing three taps, etc.). In the same time period as she learned the counting skills, Abby engaged in a series of activities that developed her ability to subitize small numbers. Figure 2b illustrates the type of activity that helped Abby become fluent in this ability. The activity "Snapshots"

a. Counting from any number

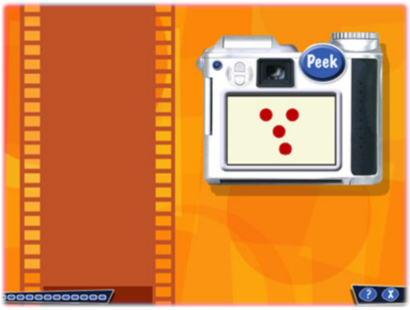
"Build Stairs 2": Students identify the appropriate stacks of unit cubes to fill in a series of staircase steps.

"Build Stairs 3": Students identify the numeral that represents a missing number in a sequence.

b. Subitizing

In "Snapshots": (b-1) Children are shown an arrangement of dots for 2 seconds. (b-2) They are then asked to click on the corresponding numeral. They can "peek" for 2 more seconds if necessary. (b-3) They are given feedback verbally and by seeing the dots again.

b-1



b-2

b-3

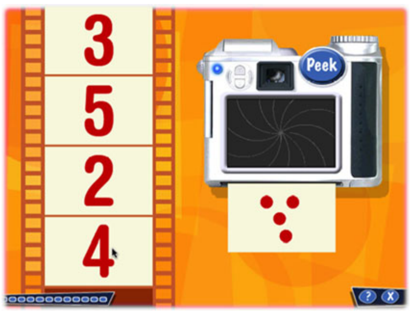


Fig. 2 Teaching the levels of thinking from different learning trajectories that help children learn to count on

c. The main addition and subtraction learning trajectory

(c-1) “Pizza Pazzazz 4”: Students add and subtract numbers up to totals of 3, (with objects shown, but then hidden) matching target amounts.

(c-2) “Dinosaur Shop 3”: Customers at the shop asks students to combine their two orders and add the contents of two boxes of toy dinosaurs (number frames) and click a target numeral that represents the sum.

(c-3) “Bright Idea”: Students are given a numeral and a frame with dots. They count on from this numeral to identify the total amount, and then move forward a corresponding number of spaces on a game board.

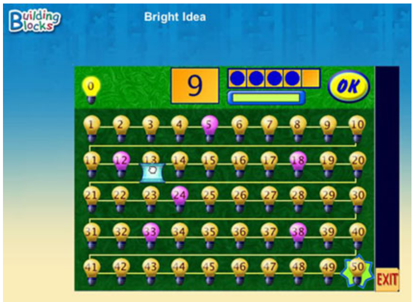


Fig. 2 (Continued)

from *Building Blocks* (Clements and Sarama 2007a) moves along a learning trajectory from the smallest numbers (1–2) to slightly larger sets (3–5) and also from matching exact dot arrangements to different dot arrangements to matching dots to numerals, as shown in Fig. 2b. Children can ask for a “Peek” to see the set again before giving their response (but only once more).

From the addition and subtraction learning trajectory, children learn to interpret additive situations mathematically, such as interpreting a real-world problem as a "part-part-whole" situation. Examining a small part of the developmental progression, at the earliest level of thinking (see *Nonverbal* +/− in Fig. 1), children use initial bootstrapping abilities (inchoate premathematical and general cognitive competencies and predispositions at birth or soon thereafter) and intuitive competencies based on mental images of very small sets. Children later learn to use counting to determine the number in each part and in the whole, originally needing to directly model the situation, using one object for each element in the problem and counting each part and the whole starting from "one" each time (see the subsequent two levels, *Small Number* +/− and *Find Result* +/−). Abby had quickly worked through the *Nonverbal* +/− level activity, "Pizza Pazzazz 4" (Fig. 2-c-1) up to the *Find Result* +/− activity "Dinosaur Shop 3." Generally, she counted all objects. That is, she counted out 5 green toy dinosaurs, then 3 red dinosaurs, then counted them all, starting at 1, clicking on the "8," thereby showing she understood the task.

Next, through the constructive synthesis of the levels of thinking from these three learning trajectories, counting, subitizing, and addition and subtraction, children learn to solve problems such as, "You have three blue blocks and seven red blocks. How many blocks do you have in all?" by modeling the problem situation and counting on Carpenter and Moser (1984). They understand that these numbers are two parts and that they need to find the whole. They also understand that the order of numbers does not matter in addition. They know, in practice, that the sum is the number that results by starting at the first number and counting on a number of iterations equal to the second number. They can use counting to solve this, starting by saying "seven…" because they understand that word can stand for the counting acts from 1 to 7 (because 7 includes 6, and 6 includes 5…). The elongated pronunciation may be substituting for counting the initial set one-by-one. It is *as if* they counted a set of 7 items. Finally, they know *how many more* to count because they use the subitized rhythm of three, so they then say, "*eight*, *nine*, *ten*!"

To develop this level of thinking, Abby engaged in many activities. Illustrated in Fig. 2-c-3, "Bright Idea" is a game in which, for the first time in a series of similar board games, not all quantities were represented by sets of dots. Instead, one of the addends is represented by a numeral (and a large single-digit at that), which research (Siegler and Jenkins 1989) shows encouraged Abby to count on: "9, 10, 11, 12, 13!"

Abby learned more quickly than most, but the *Building Blocks Software*'s (Clements and Sarama 2007a) automatic movement along the learning trajectory supported her learning and illustrates the great potential children have to learn mathematics. By four years of age, Abby was given five train engines. She walked in one day with three of them. Her father said, "Where's the other ones?" "I lost them," she admitted. "How many are missing?" he asked. "I have 1, 2, 3. So [pointing in the air] four, five…two are missing, four and five. [pause] No! I want these to be [pointing at the three engines] one, three, and five. So, two and four are missing. Still two missing, but they're numbers two and four." Abby thought about counting and numbers—at least small numbers—abstractly. She could assign 1, 2, and 3 to the three engines, or 1, 3, and 5! Moreover, she could count the numbers. That is, she applied counting… to counting numbers.

Reflection: How Learning Trajectories Require Rethinking Early Mathematics

Developing and implementing learning trajectories such as these has several implications for reconceptualizing early childhood mathematics education, some of which may be more apparent than others. We mention just a few.[1]

- *Rethinking goals.* Counting and arithmetic are standard curriculum content. However, *subitizing* has too often been ignored. Although it is the first-developing numerical competence (e.g., Antell and Keating 1983), the lack of attention to it may result in children *regressing* in their subitizing skills (Wright 1994). Not only is it a valuable competence itself, but also the brief discussion of arithmetic showed how it supports later learning of other topics. As a second example, goals for children have often been thought of mostly as procedural skills. The levels of thinking presented within learning trajectories combine conceptual knowledge, skills, and problem-solving competencies.
- *Rethinking curricular sequences.* A typical traditional sequence of instruction is teaching counting in kindergarten and then introducing addition and subtraction the next year (first grade or Year 1). In contrast, research underlying the learning trajectories indicates that counting and arithmetic begin in the first years of life (e.g., Kobayashi et al. 2004; Wynn 1992), and develop in parallel, gradually becoming increasingly intertwined and connected (e.g., Baroody 2004; Fuson 1988, 2004).
- *Rethinking goals for specific age children.* Children may benefit from working on topics previously thought too difficult for their age (National Research Council 2009). Examples include the incorporation of much larger numbers, activities involving challenging reasoning, and rich geometry (e.g., symmetry, composition, motions, notions to which we return) (Carpenter et al. 1988; Clements and Sarama 2009; Sarama and Clements 2009b; Zvonkin 2010). Although Abby was an exceptional learner, her thinking makes it clear that mathematical goals need to be reconsidered, as they often underestimate what young children can learn.
- *Rethinking curriculum and teaching strategies.* Recognition of the sequence of levels of *thinking* (as opposed to simple accumulation of facts and skills) implies a different view of curriculum and teaching (Fuson et al. 2000). Further, available research can sometimes give quite specific guidance on teaching strategies.

Let's examine two examples of such specific guidance for early arithmetic, addressing two critical points in the learning trajectory, learning to count on and learning arithmetic combinations. Regarding the former, most children can invent counting on in environments in which children's inventions and discussions of strategies

[1] These are not limited to recent work on learning trajectories, of course. They have been raised by other projects, such as cognitively-guided instruction (Carpenter and Franke 2004) on which our notion of learning trajectories are based (for a discussion of these roots, see Sarama and Clements in press).

are encouraged. However, for a variety of reasons, some individuals may have difficulty learning counting on. The teaching sequence in Fig. 3 has proven effective (El'konin and Davydov 1975; Fuson 1992). These understandings and skills are reinforced with additional problems and focused questions.

Besides carefully addressing necessary ideas and subskills, this instructional activity is successful because it promotes *psychological curtailment* (Clements and Burns 2000; Krutetskii 1976). Curtailment is an encapsulation process in which one mental activity gradually "stands in for" another mental activity. Children must learn that it is not necessary to enumerate each element of the first set. The teacher explains this, then demonstrates by naming the number of that set with an elongated number word and a sweeping gesture of the hand before passing on to the second addend. El'konin and Davydov (1975) claim that such abbreviated actions are not eliminated but are transferred to the position of actions which are considered as if they were carried out and are thus "implicit." The sweeping movement gives rise to a "mental plan" by which addition is performed, because only in this movement does the child begin to view the group as a unit. The child becomes aware of addition as distinct from counting. This construction of counting on must be based on physically present objects. Then, through introspection (considering the basis of one's own ways of acting), the object set is transformed into a symbol (El'konin and Davydov 1975).

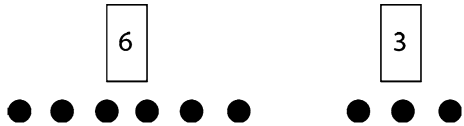
Our second example of instructional activities supported by specific research evidence is found in the next level in Fig. 1, *Deriver* $+/-$. The goal is to build fluency with basic combinations while maintaining understanding. Teaching the BAMT Strategy actually consists of a series of instructional activities involving several interrelated learning trajectories (from Murata 2004). BAMT stands for Break-Apart-to-Make-Ten. Before lessons on BAMT, children work on several related learning trajectories. They develop solid knowledge of numerals and counting (i.e., move along the counting learning trajectory). This includes the number structure for teen numbers as $10 +$ another number, which is more straightforward in Asian languages than English ("thirteen" is "ten and three"—note that U.S. teachers must be particularly attentive to this competence). They learn to solve addition and subtraction of numbers with totals less than 10, often chunking numbers into 5 (e.g., 7 as 5-plus-2) and using visual models.

With these levels of thinking established, children develop several levels of thinking within the composition/decomposition developmental progression. For example, they work on "break-apart partners" of numbers less than or equal to 10. They solve addition and subtraction problems involving teen numbers using the 10s structure (e.g., $10 + 2 = 12$; $18 - 8 = 10$), and addition and subtraction with three addends using 10s (e.g., $4 + 6 + 3 = 10 + 3 = 13$; $15 - 5 - 9 = 10 - 9 = 1$).

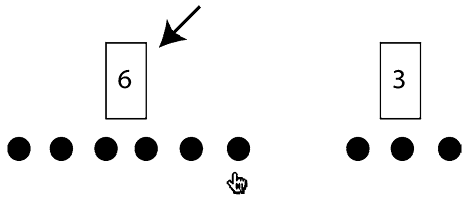
Teachers then introduce problems such as $9 + 6$. They first elicit, value, and discuss child-invented strategies (such as counting on) and encourage children to use these strategies to solve a variety of problems. Only then do they proceed to the use of BAMT. They provide supports to connect visual and symbolic representations of quantities. In the example $9 + 4$, they show 9 counters (or fingers) and 4 counters, then move one counter from the group of four to make a group of ten. Next, they

1. Make sure the child can count starting at numbers other than one (see the "Counter from N ($N+1, N-1$)" level in the counting learning trajectory, Clements and Sarama 2009; Sarama and Clements 2009b).
2. Try larger numbers: several such as $22+1$, then problems such as $1+18$. Although smaller numbers are beneficial to children in many situations, here, the child's desire to find a simpler way can often motivate the invention and use of counting on.
3. If these fail, teach individuals or small groups each component of counting on as follows.

Lay out the problem ($6+3$) with numeral cards. Count out objects into a line below each card.

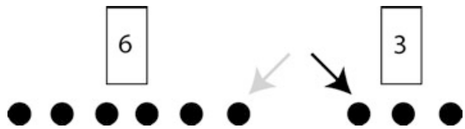


Ask child to count out a set of 6. When they reach the sixth object, point to numeral card and say, "See this is six also. It tells how many dots there are here" (gesture around all 6 dots).



Solve another problem. If the child counts the first set starting with one again, interrupt them sooner and ask what number they will say when they get to the last object in the first set. Emphasize it will be the same as the numeral card.

Point to the last dot and say (using $6+3$ again for this example) "See, there are six here, so this one (exaggerated jump from last object in the first set to first object in the second set) gets the number *seven*."



Repeat with new problems. As necessary, interrupt child's counting of the first set with questions: "How many are here (first set)? So this (last of first) gets what number? ("Six!") And what number for this one"? ("Seven!")

Fig. 3 Teaching counting on skills to children who need assistance to use counting on, or do not spontaneously invent this strategy

The line slants between the numbers, indicating that we need to find a partner for 9 to make 10.	Four is separated into two partners, 1 and 3.	The ring shows how the numbers combine to make 10.	Ten and three are shown to add to 13.

Fig. 4 *Teaching BAMT* (modified from Murata 2004)

highlight the three left in the group. Then children are reminded that the 9 and 1 made 10. Last, children see 10 counters and 3 counters and think ten-three, or count on "ten-one, ten-two, ten-three." Later, representational drawings serve this role, in a sequence such as shown in Fig. 4.

Teachers spend many lessons ensuring children's understanding and skill using the BAMT strategy. Children are asked why the strategy works and what its advantages are. Extensive use of BAMT to solve problems helps children develop fluency.

Not all instructional tasks are as specific as these just outlined. In many cases, the instructional tasks presented with the learning trajectories are simply illustrations of the kind of effective activities that would be appropriate to reach a certain level of thinking. For example, the problems suggested for each level should be changed for different children, but the *type* of problem is important.

A final observation regarding our Fig. 1 discussions is that learning trajectories promote learning skills and concepts together, as mentioned previously (see "Rethinking goals"). Learning skills before developing understanding can lead to learning difficulties (Baroody 2004; Fuson 2004; Kilpatrick et al. 2001; Sophian 2004; Steffe 2004). Further, effective curricula and teaching often build on children's thinking, provide opportunities for both invention and practice, and ask children to explain their various strategies (Hiebert 1999). Such programs facilitate conceptual growth and higher-order thinking without sacrificing skill learning. Effective teachers also consistently integrate real-world situations, problem solving, and mathematical content (Fuson 2004). Making connections to real-life situations also enhances children's knowledge and positive beliefs about mathematics (Perlmutter et al. 1997). Thus, a critical task for teachers is to adapt activities such as those in Fig. 1 so that they are relevant and appropriate to their own students.

Rethinking Curriculum Development: What Is a Research-Based Curriculum?

Children's progress through learning trajectories is profoundly influenced by their first educational experiences. Indeed, "the early grades may be precisely the time that schools have their strongest effects" (Alexander and Entwisle 1988). Research also suggests that early childhood classrooms underestimate children's ability to learn mathematics and are too often ill suited to help them learn due to lack of knowledge of the variety of learning trajectories. Thus, children actually *regress* on some math skills during kindergarten (Wright 1994). For example, perhaps because they do not understand the idea and importance of subitizing, kindergarten teachers often told children who had already subitized a small collection correctly, to "*Count* them!" thus undermining children's use of a valuable practice. We need more structured, sophisticated, better-developed and well-sequenced mathematics in early childhood education. How do we do that well? How do we avoid the problem mentioned in the introduction: that it is difficult to know how to evaluate if a curriculum truly is "based on research"?

A Framework for Research-Based Curricula

Based on a review of research and expert practice (Clements 2008), we constructed and tested a framework for the construct of research-based curricula. Our "Curriculum Research Framework" (CRF, Clements 2007) rejects the sole use of commercially-oriented "market research" and "research-to-practice" strategies. Although included in the CRF, such strategies alone are inadequate. For example, research-to-practice strategies are flawed in their presumptions because they employ one-way translations of research results, are insensitive to changing goals in the content area, and are unable to contribute to a revision of the theory and knowledge. Such knowledge building is—alongside the development of a scientifically-based, effective curriculum—a critical objective of a scientific curriculum research program. Indeed, a valid scientific curriculum development program should address two basic questions—about effects and conditions—in three domains-practice, policy, and theory. For example, such a program should address the practical question of whether the curriculum is effective in helping children achieve specific learning goals, but also under what conditions it is effective. Theoretically, the research program should also address why it is effective and why certain sets of conditions decrease or increase the curriculum's effectiveness.

To address all these issues, the CRF includes three broad categories of research and development work, within which there are ten phases through which a curriculum should be subjected to warrant the claim that it is based on research. The three categories are: (1) reviewing existing research (A Priori Foundations), (2) building models of children's thinking and learning in a domain (Learning Trajectories), and (3) appraising the effectiveness and general worth of the result (Evaluation, both formative, leading to revisions, and summative, to determine the effects of the completed curriculum). The categories and phases within them are outlined in Table 1. The categories are described in the leftmost column. The questions addressed are provided in the middle column, and the specific methodologies to address these questions within each phase are described in the rightmost column.

Table 1 *Categories and Phases of the* Curriculum Research Framework (*adapted from Clements 2007*)

Categories	Questions Asked	Phases
A Priori Foundations: In variants of the research-to-practice model, extant research is reviewed and implications for the nascent curriculum development effort drawn.	What is already known that can be applied to the anticipated curriculum?	Established review procedures and content analyses are employed to gather knowledge concerning the specific subject matter content, including the role it would play in students' development (*phase 1*); general issues concerning psychology, education, and systemic change (*phase 2*); and pedagogy, including the effectiveness of certain types of activities (*phase 3*).
Learning Trajectories: Activities are structured in accordance with empirically-based models of children's thinking and learning in the targeted subject-matter domain.	How might the curriculum be constructed to be consistent with models of students' thinking and learning?	In *phase 4,* the nature and content of activities is based on models of children's mathematical thinking and learning. Specific learning trajectories are built for each major topic.
Evaluation: In these phases, empirical evidence is collected to evaluate the curriculum, realized in some form. The goal is to evaluate the appeal, usability, and effectiveness of an instantiation of the curriculum.	How can market share for the curriculum be maximized?	*Phase 5* focuses on marketability, using strategies such as gathering information about mandated educational objectives and surveys of consumers.
	Is the curriculum usable by, and effective with, various student groups and teachers?	Formative phase*s 6 to 8* seek to understand the meaning that students and teachers give to the curriculum objects and activities in progressively expanding social contexts; for example, the usability and effectiveness of specific components and characteristics of the curriculum as implemented by a teacher who is familiar with the materials with individuals or small groups (*phase 6*) and whole classes (*phase 7*) and, later, by a diverse group of teachers (*phase 8*). The curriculum is altered based on empirical results, with the focus expanding to include aspects of support for teachers.

The first curriculum to be developed using the Curriculum Research Framework (CRF) was Building Blocks (Clements and Sarama 2003, 2007b, 2013), a NSF-funded PreK to grade 2 mathematics research and curriculum development project that was one of the first to develop materials that comprehensively address recent standards for early mathematics education for all children (e.g., Clements and Conference Working Group 2004; National Council of Teachers of Mathematics 2000,

Table 1 (Continued)

Categories	Questions Asked	Phases
	What is the effectiveness (e.g., in affecting teaching practices and ultimately student learning) of the curriculum, now in its complete form, as it is implemented in realistic contexts?	Summative phases 9 and 10 both use randomized field trials and differ from each other most markedly on the characteristic of scale. They both examine the fidelity or enactment, and sustainability, of the curriculum when implemented on a small (*phase 9)* or large (*phase 10)* scale, with *phase 10* also investigating the critical contextual and implementation variables that influence its effectiveness. Experimental or carefully planned quasi-experimental designs, incorporating observational measures and surveys, are useful for generating political and public support, as well as for their research advantages. In addition, qualitative approaches continue to be useful for dealing with the complexity and indeterminateness of educational activity.

2006). We will illustrate the CRF by giving concrete descriptions of how the phases were enacted in the development of the Building Blocks preschool curriculum.

A Priori Foundations

The first category includes three variants of the research-to-practice strategy, in which existing research is reviewed and implications for the nascent curriculum development effort are drawn.

(1) In *General A Priori Foundation*, developers review broad philosophies, theories, and empirical results on learning and teaching. Based on theory and research on early childhood learning and teaching (e.g., National Research Council 2001), we determined that *Building Blocks*' basic approach would be finding the mathematics in, and developing mathematics from, children's activity. That is, we wanted to "mathematize" everyday activities, such as puzzles, songs, moving, and building. For example, teachers might help children mathematize moving their bodies in many ways. Children might count their steps as they walk. They might also move in patterns: step, step, hop; step, step, hop. . . . They might do both, counting as they walk, "one, *two*, three, *four*, five, *six*,. . .". These examples show that mathematizing means representing and elaborating everyday activities mathematically. Children create models of everyday situations with mathematical objects, such as numbers and shapes; mathematical actions, such as counting or transforming shapes; and their structural relationships—and use those understandings to solve problems. They learn to increasingly see the world through mathematical lenses.

(2) In *Subject Matter A Priori Foundation*, developers review research and consult with experts to identify topics that make a substantive contribution to children's mathematical development, are generative in children's development of future mathematical understanding, and are interesting to children. We determined the topics that fit those criteria by considering what mathematics is culturally valued and empirical research on what constituted the core ideas and skill areas of mathematics for young children (Baroody 2004; Clements and Battista 1992; Clements and Conference Working Group 2004; Fuson 1997). We then organized for the development of learning trajectories in the domains of number (counting, subitizing, sequencing, arithmetic), geometry (matching, naming, building and combining shapes), patterning, and measurement.

(3) In *Pedagogical A Priori Foundation,* developers review empirical findings on making activities educationally effective—motivating and efficacious—to create general guidelines for the generation of activities. As an example, research using computer software with young children (Clements et al. 1993; Clements and Swaminathan 1995; Steffe and Wiegel 1994) showed that preschoolers can use computers effectively and that software can be made more effective by employing animation, children's voices, and clear feedback. Although such software is only a small component of the Building Blocks curriculum, it makes a significant contribution, because research was used in its development, giving the developers information on how to make the software targeted and effective.

Learning Trajectories

In the second category, developers structure activities in accordance with theoretically- and empirically-based models of children's thinking in the targeted subject-matter domain. This phase involves the creation of research-based learning trajectories. Figure 1 illustrated a part of our arithmetic learning trajectory. We turn to one in geometry so we do not give the misimpression that learning trajectories only apply to numerical domains.

When we were working with kindergartners, one of us (Sarama) observed that several children followed a similar progression in choosing and combining shapes (e.g., rhombi or equilateral triangles) to make another shape (e.g., to cover a hexagon as in Fig. 5) (Sarama et al. 1996). Initially, they merely appreciated how one pattern block could be made using other pattern blocks, but their efforts to cover a hexagon with other pattern blocks was by trial-and-error. Later, they explicitly recognized the hexagon could be made with 2 trapezoids, followed by other combinations. Sarama reviewed the behaviors of all the kindergarten children. She found several similar sequences and noted that, throughout the study, children's development appeared to move from placing shapes separately to considering shapes in combination; from manipulation- and perception-bound strategies to the formation of mental images (e.g., decomposing shapes imagistically); from trial and error to intentional and deliberate action and eventually to the prediction of succeeding placements of shapes;

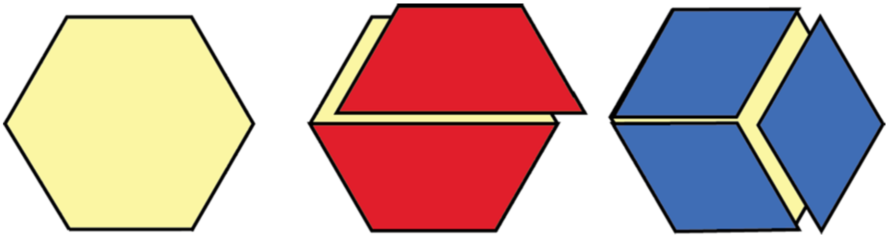


Fig. 5 How children might cover a pattern block hexagon with other pattern block shapes

and from consideration of visual "wholes" to a consideration of side length, and, eventually, angles.

Based on these observations, we wrote a tentative set of levels of thinking, the first draft of the developmental progression, then revised it as we studied the results of other researchers (Mansfield and Scott 1990; Sales 1994; Vurpillot 1976). At this point, we involved several additional teachers, because any learning trajectory should "speak" to practitioners as well as researchers. Eight volunteers who were helping us develop the *Building Blocks* curriculum worked with us for six months testing and refining activities (for the complete story, see Clements et al. 2004). Their case studies indicated that the work of about four-fifths of the children studied was consistent with the developmental progression. Finally, we conducted a study with David Wilson, utilizing 72 randomly selected children from pre-K to grade 2. Analyses again indicated child progress consistent with the developmental progression (Clements et al. 2004).

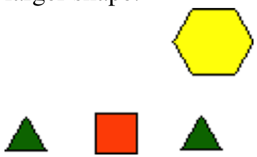
In this way, through cycles of curriculum revision and observations, we created the complete learning trajectory, including the developmental progression and a set of instructional tasks, which include on- and off-computer puzzles, that appeared to facilitate growth for children at different points along the trajectory (as embodied now in the *Building Blocks* concrete puzzles). Figure 6 illustrates several early levels of the learning trajectory.

Again, computer technology makes a substantive contribution. First, the *Building Blocks Software* (Clements and Sarama 2007a) moves children forward (or backward) along the learning trajectory automatically based on children's performance. Second, the tasks are designed to fit the trajectory precisely. For example, children's work is initially scaffolded by the inclusion of internal line segments in most cases, but these are faded in subsequent puzzles. Third, analyses of children's responses are often superior to situations using physical manipulatives. For example, children will often place physical shapes so that they cover a puzzle but also "hang over" outside of the puzzle—and they and their teachers rarely notice this. Computers detect every error. Fourth, when such errors are detected, the feedback is immediate and in some ways superior. For example, the shapes placed by the child can be made translucent, clearly showing the mismatch between the child's solution and the actual puzzle. In these ways, work with computers provides a unique and substantial contribution to children's learning.

Goal: **Children compose geometric shapes intentionally to create a superordinate shape, building understanding of part-whole relationships as well as the properties of the original and composite shapes.**

Developmental Progression

Pre-Composer Manipulates shapes as individuals, but is unable to combine them to compose a larger shape.



Instructional Tasks

This level is not an instructional goal. However, several preparatory activities may orient 2- to 4-year-old children to the task, and move them toward the next levels that do represent (some) competence. In "Shape Pictures," children play with pattern blocks and Shape Sets, often making simple pictures.

In "Mystery Toys" children match shapes, but the *result* of their work is a pictured made up of other shapes—a demonstration of composition.

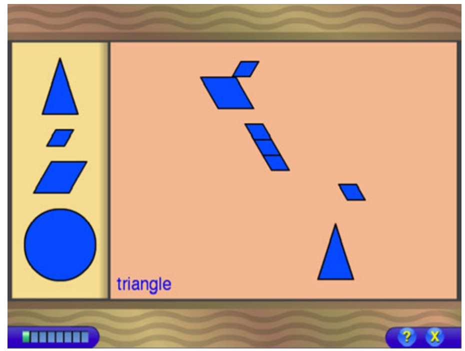


Fig. 6 Samples from the Learning Trajectory for the Composition of 2D Shapes (adapted from Clements and Sarama 2009; Sarama and Clements 2009b)

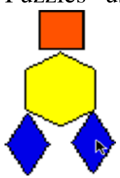
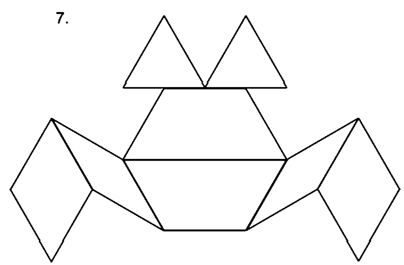
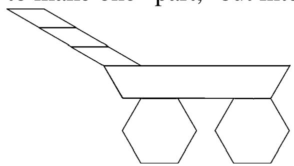
Piece Assembler Makes pictures in which each shape represents a unique role (e.g., one shape for each body part) and shapes touch. Fills simple "Pattern Block Puzzles" using trial and error. 	In the first "Pattern Block Puzzles" tasks, each shape is not only outlined, but touches other shapes only at a point, making the matching as easy as possible. Children merely match pattern blocks to the outlines. Then, the puzzles moved to those that combine shapes by matching their sides, but still mainly serve separate roles.	
Picture Maker Puts several shapes together to make one part of a picture (e.g., two shapes for one arm). Uses trial and error and does not anticipate creation of new geometric shape. Chooses shapes using "general shape" or side length.	The "Pattern Block Puzzles" at this level start with those where several shapes are combined to make one "part," but internal lines are still available. 	

Fig. 6 (Continued)

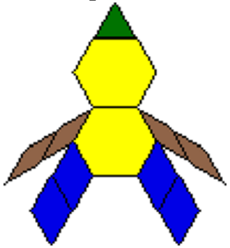
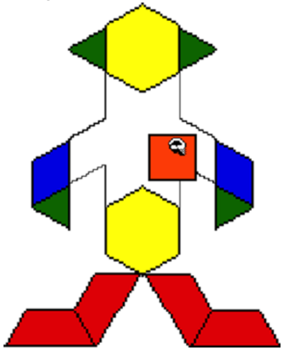
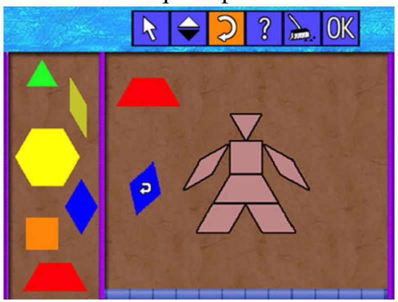
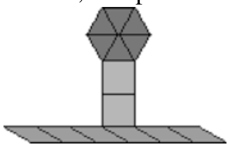
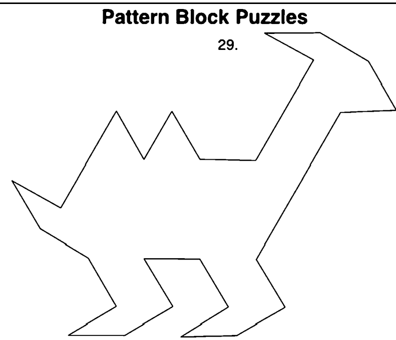
Fills "easy" "Pattern Block Puzzles" that suggest the placement of each shape (but note below that they child is trying to put a square in the puzzle where its right angles will not fit). Make a picture 	Later puzzles in the sequence require combining shapes to fill one or more regions, without the guidance of internal line segments.  "Piece Puzzler 3" is a similar computer activity. In the first tasks, children must concatenate shapes, but are helped with internal line segments in most cases; these internal segments are faded in subsequent puzzles. 

Fig. 6 (Continued)

Shape Composer. Composes shapes with anticipation ("I know what will fit!"). Chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. In the "Pattern Block Puzzles" below, all angles are correct, and patterning is evident.



The "Pattern Block Puzzles" and "Piece Puzzler" activities have no internal guidelines and larger areas; therefore, children must compose shapes accurately.



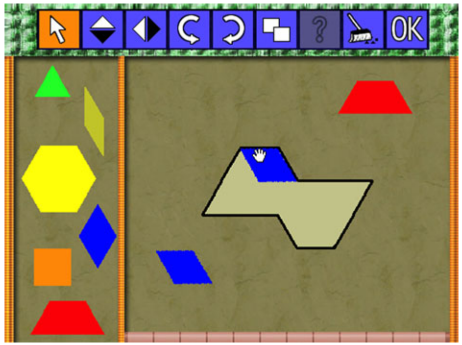


Fig. 6 (Continued)

Evaluation

In the third category of the CRF, developers collect empirical evidence to evaluate the appeal, usability, and effectiveness of a version of the curriculum. Past phase (5) *Market Research* is (6) *Formative Research: Small Group*, in which developers conduct pilot tests with individuals or small groups on components (e.g., a particular activity, game, or software environment) or sections of the curriculum. Although teachers are involved in all phases of research and development, the process of curricular enactment is emphasized in the next two phases. Studies with a teacher who participated in the development of the materials in phase (7) *Formative Research: Single Classroom*, and then teachers newly introduced to the materials in phase (8) *Formative Research: Multiple Classrooms*, provide information about the usability of the curriculum and requirements for professional development and support materials. We conducted multiple case studies at each of these three phases (e.g., Clements and Sarama 2004a; Sarama 2004), revising the curriculum multiple times, including two distinct published versions (Clements and Sarama 2003, 2007c).

In the last two phases, (9) *Summative Research: Small Scale* and (10) *Summative Research: Large Scale*, developers evaluate what can actually be achieved with typical teachers under realistic circumstances. To avoid the misconstrual that the CRF privileges scientific research of a limited nature, note that *the first 8 phases involve only qualitative research and the last two combine quantitative and qualitative research.* The CRF uses a wide range of methods, omitting no genre of educational research.

An initial phase-9 summary research project (Clements and Sarama 2007d) yielded effect sizes between 1 and 2 (standard deviation units). However, this study only involved 4 classrooms. Thus, we moved to phase 10, which also uses randomized trials, which provide the most efficient and least biased designs to assess causal relationships (Cook 2002), where the curriculum is implemented in a greater number of classrooms, with more diversity, and less ideal conditions. In a larger study (Clements and Sarama 2008), we randomly assigned 36 classrooms to one of three conditions. The experimental group used Building Blocks (Clements and Sarama 2007b). The comparison group used a different preschool mathematics curriculum—the same as we previously used in the Preschool Curriculum Evaluation Research (Preschool Curriculum Evaluation Research Consortium 2008) research (mainly Klein et al. 2002). The control used their schools' existing curriculum ("business as usual"). Two observational measures indicated that the curricula were implemented with fidelity and that the experimental condition had significant positive effects on classrooms' mathematics environment and teaching. From the beginning to end of the school year, the experimental group score increased significantly more than the comparison group score (effect size, .47) and the control group score (effect size, 1.07). Focused early mathematical interventions, especially those based on a comprehensive model of developing and evaluating research-based curricula, can increase the quality of the mathematics environment and teaching, and can help preschoolers develop a foundation of informal mathematics knowledge (Clements and Sarama 2008). We believe that these positive effects, even when compared to another curriculum supported equivalently, were due to *Building Blocks'* development within the CRF and especially the core use of learning trajectories.

Conclusion

Through our collaboration with teachers, administrators, and other researchers, we believe we have developed and evaluated truly research-based approaches. The two major conceptual tools, sets of learning trajectories and the Curriculum Research Framework, have shown their effectiveness in a number of studies. We hope others test whether these and other similar tools (see Maloney et al. in press) contribute to a scientific base for early mathematics education.

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Reflecting on Young Children's Mathematics Learning

Bob Perry and Sue Dockett

Over the last 6 years, South Australian preschool and first years of school educators[1] have come together to consider how they can facilitate young children's[2] learning of powerful mathematical ideas without jeopardizing the well-established benefits of young children learning through play. The chapter begins with a brief discussion around the recognition of young children as powerful mathematicians and how this recognition is facilitated through the documentation of children's mathematical learning using learning stories. It then introduces the *Early Years Numeracy Project* in South Australia and reviews the development of a major artifact from the project—the *Reflective Continua*. Ways in which educators have used the *Reflective Continua* to stimulate the powerful mathematics learning of young children complete the chapter.

Young Children as Powerful Mathematicians

It is well known that young children can be powerful mathematicians and that they are able to demonstrate this power through their actions in both play and structured learning (Hunting et al. 2012; Kilpatrick et al. 2001; Lee and Ginsburg 2007; Perry and Dockett 2008; Sarama and Clements 2009; Thomson et al. 2005). The Australian Association of Mathematics Teachers (AAMT) and Early Childhood Australia (ECA) (2006, p. 1) state that

[1]The term 'educators' is used throughout this chapter to designate anyone working with children in prior-to-school or school settings who may impact on the children's learning. For example, 'educators' may be teachers, assistants, preschool directors, and school principals.

[2]The term 'children' is used throughout this chapter rather than 'students'. This allows the one word to be used for young people in preschool and school settings.

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> all children in their early childhood years are capable of accessing powerful mathematical ideas that are both relevant to their current lives and form a critical foundation to their future mathematical and other learning. Children should be given the opportunity to access these ideas through high quality child-centred activities in their homes, communities, prior-to-school settings and schools.

The first curriculum framework for Australian early childhood education—*Early Years Learning Framework for Australia* (Department of Education, Employment and Workplace Relations (DEEWR), 2009, p. 38) provides a list of these powerful mathematical ideas:

> Spatial sense, structure and pattern, number, measurement, data, argumentation, connections and exploring the world mathematically are the powerful mathematical ideas children need to become numerate.

Australia has also recently introduced its first national curriculum in mathematics (Australian Curriculum, Assessment and Reporting Authority (ACARA) 2011) which contains a similar list of powerful mathematical ideas: number and algebra; measurement and geometry; statistics and probability; understanding; fluency; problem solving; and reasoning. These powerful mathematical ideas form an important foundation for the *Reflective Continua* to be investigated in this chapter. Another foundation is the notion of *learning stories.*

Learning stories (Carr 2001; Carr and Lee 2012)

> are qualitative snapshots, recorded as structured written narratives, often with accompanying photographs that document and communicate the context and complexity of children's learning (Carr 2001). They include relationships, dispositions, and an interpretation by someone who knows the child well. They are "structured observations in everyday or 'authentic' settings, designed to provide a cumulative series of snapshots" (Carr and Claxton 2002, p. 22). Learning stories acknowledge the multiple intelligences and holistic nature of young children's learning, educators' pedagogy, and the context in which the learning takes place. Educators use their evaluation of the learning story to plan for future, ongoing learning (Perry et al. 2007a).

Such learning stories have allowed the components of the *Reflective Continua* to be demonstrated through the use of 'work samples' created by children and educators in authentic contexts. Links between a young child's observed and documented activity, and powerful mathematical ideas are illustrated through the following example of a learning story, written by a preschool educator, which considers 4-year-old Rachel's interest in shapes (Fig. 1).

> Rachel likes to play with shapes and was laying out the shapes on the table and putting them together so there were no gaps. She used the colours and the shapes to create a pattern. She stood some of the blocks upright to make a border around her pattern. Rachel could tell me [the educator] what colours she had used and how she had made her pattern. When I asked her why she thought it was a pattern she said "It goes green triangle, then red square, then red square and back to green triangle". I asked her why she had put the shapes standing up. She replied "This is the border and they are all the same shape".

Rachel has a clear view of what a pattern is and how patterns can be used in her play and learning. She is able to explain and justify her decisions and to use the results of her investigations to extend her thinking. Powerful ideas in mathematics

Fig. 1 Rachel—powerful mathematician

and Rachel's play have coalesced to provide an opportunity for learning in a relevant and meaningful context. How educators recognise these powerful mathematical ideas and undertake their own pedagogical inquiry about planning future mathematical learning experiences for their children provide the focus of this chapter.

The Early Years Numeracy Project

The *Early Years Numeracy Project* (EYNP) ran in various guises from 2004 to 2011 in the state of South Australia using a collaborative approach to professional development for both preschool and first years of school educators (Perry 2011; Perry et al. 2007b). While there have been many positive outcomes from this project, the two most tangible artefacts have been the *Numeracy Matrix* and the *Reflective Continua*. The development and use of the *Numeracy Matrix* have been reported elsewhere (Harley et al. 2007; Perry et al. 2012). In this chapter, we consider the development and use of the *Reflective Continua*.

The Reflective Continua

The final phase of EYNP, from 2009 to 2011, had the following aims:

- to develop the mathematics content knowledge and pedagogical content knowledge of the site-based early childhood educators;
- to build on previous work that had led to the development of the numeracy matrix linking powerful mathematical ideas to developmental learning outcomes; and
- to develop and trial reflective continua based on the numeracy matrix and the use of learning stories (Carr 2001) to guide mathematics teaching and learning for children aged 3–8 years in preschools and the first years of school.

This chapter reports on the fulfilment of the third of these aims.

For the development of the *Reflective Continua*, the researchers worked with four Numeracy Leaders—three from the first years of school, and one from preschool.

The Numeracy Leaders' role was to work with a total of 45 site-based early childhood educators in preschools and schools to introduce and implement the *Reflective Continua*.

From July, 2009 until December, 2010, regular professional learning meetings were held for the four Numeracy Leaders. These meetings were led by the researchers and centred on powerful mathematical ideas and links to both school and preschool curricula. The professional learning meetings also focused on the role of Numeracy Leaders in the development of collaborative partnerships with colleagues. The role of the Numeracy Leaders was challenging, as each was responsible for engaging and guiding both preschool and school educators in their clusters. Ongoing support from each other and the project leaders was a critical element of the Numeracy Leaders' programs.

Educator reflection on their own pedagogical practice had been a key component of the EYNP from its inception (Perry et al. 2007a, 2006). Hence, the development of a *Reflective Continuum* for each of the powerful mathematical ideas that encapsulated the pedagogies of their settings was a natural consequence of the earlier work.

Through an iterative approach involving the Numeracy Leaders and the educators in the clusters, the *Reflective Continua* were created.

What Are the Reflective Continua?

In their final form, the *Reflective Continua* consist of a set of seven tables—one for each of the strands and competencies (powerful mathematical ideas) in the *Australian Curriculum—Mathematics* (ACARA 2011): number and algebra; measurement and geometry; statistics and probability; understanding; fluency; problem solving; and reasoning—which provide frameworks to guide educator reflections on children's mathematical work. Such reflective practice is designed to assist educators plan future learning experiences for their children. Each *Reflective Continuum* highlights a progression of development and engagement with the relevant powerful mathematical idea. These progressions have been developed by the site educators under the tutelage of the Numeracy Leaders and the researchers, using the current curriculum documents for guidance.

Four levels of development and engagement are used to demonstrate children's progression—*Emerging, Investigation, Application and Generalisation*. The meanings given to each of these levels are:

- *Emerging*: the learner is beginning to understand the basic concepts involved in the powerful mathematical idea but is not yet able to use this early understanding;
- *Investigation*: the learner is confident enough in her/his understanding of the powerful mathematical idea to explore problems and real-life situations;
- *Application*: the learner's knowledge and understanding of the powerful mathematical idea can be applied to find solutions to problems;

- *Generalisation*: the learner is able to transfer her/his knowledge and understandings between powerful mathematical ideas and/or powerful ideas from other key learning areas.

An example of one Reflective Continuum is given in Fig. 2.

For each level in each *Reflective Continuum*, a number of 'indicators' are provided as guidance to educators about what might be expected to be observed from children working at the level. As well, there are children's names in each column which are hyperlinked to work samples and/or learning stories. For example, clicking on Example 2: Sol in the 'Emerging' column leads to the following learning story:

> Sol went over to the toy tray and pulled out the stacking cups. Very carefully, with lots of balance, she began to stack the cups in sizes. She tried different sizes at first realising a smaller cup would disappear inside a larger one. With trial and error she built a big tower. Sol is showing reflective thinking and problem solving skills which are basic numeracy skills. We could extend her numeracy skills by introducing basic shapes.

This story provides one example of what the Emerging level could look like in Measurement and Geometry. For every level in every continuum, there is at least one example from preschool and one from the first years of school, thus illustrating that all levels are possible in both settings.

The *Reflective Continua* are designed to allow educators to ascertain quickly at which level each of their children is demonstrating her/his knowledge of the powerful mathematical ideas. Children's work samples are provided to illustrate how each level might present in preschools or the first years of school. For example, a preschool child at the Application level of the powerful idea Measurement and Geometry might 'describe position in relation to surroundings' by applying proximity terminology such as 'next to' or 'close'. On the other hand a child in the first years of school might use units of measure or directions to describe her/his position.

The *Reflective Continua* help educators make sense—in terms of powerful mathematical ideas—of their observations of learners that they make as part of the normal routine of each day. By reflecting on observations of learner actions and coupling these to the educators' own understandings of hypothetical learning trajectories (Sarama and Clements 2009), progress towards developing and understanding the seven powerful mathematical ideas can be observed.

There is no suggestion that any child would perform at the same level for each of the powerful mathematical ideas and, therefore, be labelled with one of these level names. As well, there was no attempt to link the *Reflective Continua* levels with either curriculum or Year/age stages. Instead the levels provide a hypothetical learning trajectory (Sarama and Clements 2009) for each of the powerful mathematical ideas. It would be eminently possible for a child in preschool to be demonstrating behaviours that would suggest positioning her/him at, say, the Application level for one of the powerful mathematical ideas while, by contrast, a child in Year 2 might be engaging with this idea at the Emerging level.

There were a number of other issues raised by the Numeracy Leaders and their cluster colleagues during the development of the *Reflective Continua.* Four are of interest here.

Measurement and geometry

Emerging	Investigation	Application	Generalisation
Uses mathematical language without making comparisons of measurable attributes	Investigates the use of units in the measurement process	Applies knowledge of the measurement processes to estimate and measure using standard units	Transfers the understanding of measurement processes using standard units to measure various attributes of an object
Incidentally compares measurement attributes	Uses relevant non-standard units to measure through comparing, counting and describing the results in appropriate language	Applies knowledge of shape and space to help them communicate aspects of their own lives and environment	Uses directions and/or landmarks to plan and describe a journey
Shows awareness of space as it relates to one's movement	Sorts objects according to shape	Describes position in relation to surroundings	Employs (mental) visualisation of shapes to solve problems
Describes their own position in their environment using words such as 'over', 'under', 'beside', etc.	Investigates properties of shapes and the ways they can be manipulated	Creates and uses visual representations of various environments	
	Explores understandings of directional language and mapping		
Example 1: Leon Example 2: Sol Example 3: Kamal	Example 1: Eliza Example 2: Bolek Example 3: Dani	Example 1: Fatima Example 2: Elke Example 3: Katrina	Example 1: Corbie Example 2: Lexie Example 3: Thein

Fig. 2 Reflective continuum for measurement and geometry

Profiling of Children

The possibility of using the *Reflective Continua* to produce individual profiles of children across the seven powerful mathematical ideas was considered but was determined not to be of sufficient value to the educators and children to justify the time that would be necessary to develop profiles.

Local Development of Indicators of Development and Engagement

The possibility that levels of development and engagement with each of the powerful mathematical ideas could be rewritten or substituted by individual educators in order to reflect the particular contexts of the learners was considered. This consideration was seen by the Numeracy Leaders as a positive engagement by the cluster educators with the *Reflective Continua*. However, it was felt that educators needed to become familiar with the continua and use them in their own contexts before they would be able to change the indicators with reliability. Hence, it was communicated to educators in the clusters that they should remain with the published indicators initially, with the aim of changing them to suit only after more extensive use.

Consistency of Judgement

Consistency of educator judgement has been a critical issue in classroom assessment for many years in Australia. It is well known that "a variety of influences and knowledges impact on teacher judgement" (Connolly et al. 2011). The importance of moderation through consultation, negotiation and use of standards in achieving consistency of judgement has also been established (Klenowski and Adie 2009; Wyatt-Smith et al. 2010). Such consistency was an issue for some of the Numeracy Leaders and many of the educators in the clusters. Many of the cluster educators were concerned that they might make different judgements from other educators and that they 'would not get it right'. However, it was determined that, given the primary purpose of the continua was to encourage and facilitate reflection on the part of educators, rather than assessment of the children's mathematics, it did not matter whether one educator made precisely the same judgement based on a particular work sample as others. In the documentation introducing the *Reflective Continua*, it is stated explicitly that the decision about placement of a child on a continuum should be made by an educator who knows the child and has observed the learning experience being judged. As the purpose of the decision is to reflect on the child's work within the learning experience and to answer the question 'Where to next?', there is no need for consistency of judgement. What is important is that the educator making the judgement is able to use that judgement to facilitate further learning by the child.

Expectations of Levels of Development and Engagement for Preschool and First Years of School Children

Not surprisingly, many educators assume that younger children will not be able to develop and engage with powerful mathematical ideas to the same levels as older children (Hunting et al. 2012). This was the case for many of the cluster educators. These educators had not taken into consideration that children's development and engagement with powerful mathematical ideas is determined not only by what the children could do but also by what the educators did. Children are unlikely to demonstrate their full potential unless they are provided with the challenging and supportive contexts that allow them to do so (Bobis et al. 2012; Hunting et al. 2012). The Numeracy Leaders were adamant that the final version of the reflective continua had to demonstrate that children in preschool and the first years of school could perform at all four of the levels. The way of demonstrating this was through the provision of work samples and learning stories from children and educators in both sectors.

Learning Stories

Learning stories and other work samples have been used in the *Reflective Continua* to provide guidance for educators about how children might indicate the level of their development and engagement with each of the powerful mathematical ideas; and, then, what both the educators and the children might do next. At each level of each *Reflective Continuum*, work samples and learning stories are provided from both preschool and school children, illustrating what performance at this level might look like in these settings. We conclude this chapter with several examples of the links between work samples/learning stories and levels in the *Reflective Continua*, using the measurement continuum introduced earlier as an exemplar. All of the judgements concerning the levels illustrated by the samples have been made by cluster preschool and school educators under the guidance of the Numeracy Leaders.

Emerging An example from this level (Sol and the stacking cups) has already been provided earlier in the chapter. However, another is added here for completion.

This learning story reports on a group activity in a first year of school class when the children had been set the task of making gnocchi.

> Leon and some other children were making gnocchi using a recipe that had to be read before cooking the potatoes and mixing them with the flour. While rolling the dough, the children compared size and shape to see if they were on the right track. Leon was particularly keen to make the gnocchi into uniform shapes (same shape and size). We can look for other opportunities to make uniform shapes.

Investigation The following learning story (Fig. 3) developed by a preschool teacher provides evidence of the indicator, 'Uses relevant non-standard units to

Fig. 3 Using non-standard units of measurement—investigation

measure through comparing, counting and describing the results in appropriate language'. The educator also notes in the learning story that the children have demonstrated their learning with other powerful mathematics ideas such as Number and Algebra.

> Today we gave the children the opportunity to measure shapes with some non-standard units of measurement—pebbles, seed pods, shells and dried beans. The children enjoyed chatting to each other about how many of the resources were needed to go around the perimeter of each shape. Once they had measured with one of the resources they would choose another and compare their results. The children were encouraged to record their findings and they did this by writing down the number of shells, pebbles etc they had used and then they drew the shape they had measured. It was interesting to note that they then started to form patterns around the perimeters of the shapes using the pebbles, pods, shells and beans. Bolek went out into the garden and found sticks and rosemary twigs and used these to measure around the shapes, showing that he had an understanding of the fact that many things can be used to measure. The children also noticed that more small objects and less large objects were required to measure the perimeter of a given shape. I need to provide more opportunities for them to explore this further.

Application The use of standard units—represented here by the materials available to the child—and the applications required to undertake estimation in measurement are illustrated by Katrina's understanding of the measurement of area in this learning story (Fig. 4).

> Today the children were given the opportunity to measure the area of different sized leaves using Unifix blocks. Katrina set to work and covered a leaf with blocks and discovered that she had used 18 blocks to cover the area of the leaf. She then chose a smaller leaf and I asked her how many blocks she thought she would need to cover this leaf. I explained to

Fig. 4 Learning about area—application

her that this was called estimating and she predicted that the leaf would be covered with 8 blocks. Katrina's estimation was very good as the area of the leaf was in fact 7 blocks. She continued to choose leaves and cover them with blocks, estimating each time the number of blocks she thought she would need. For a middle sized leaf she estimated 9 and the area was in fact 10, for a bigger leaf she estimated 28 and the area was in fact 25.
Through this activity Katrina demonstrated that she can use relevant units to measure attributes of objects, through comparison and counting and describe the results in appropriate language. She is also aware that number has meaning in everyday worlds, can count in rote and she also recognises patterns of numbers to quantify small collections without counting.

Generalisation This example from a Year 4 class provides an insight into how school educators might utilise learning stories and the power it can give the educator to reflect on children's learning. Even though the *Reflective Continua* were developed for use in early childhood settings, it would seem that they might have application with older children.

Corbie was working in a group of five children and their task was to measure and give directions from the classroom to the flying fox, but they had to travel around the hitting wall. They were asked to choose appropriate measuring tools and estimate as they went. The group decided to bring a trundle wheel and a measuring tape.
The first suggestion for the first step was, "Go out the door and walk down the ramp."
Corbie didn't think this should be the first step as there was no measurement involved.
Corbie: We could say go through the door and turn to the left and then we could measure all the way to the end of the art room because we don't have to change direction until then.
Educator: How far to the left would you need to turn?
Another group member: A quarter turn?
C: Can I write 90° instead?
E: Is it the same as a quarter of a turn?
C: Yeah.
E: Then of course you can use it in your recording.
The group then estimated how far they thought it was to the end of the art room and measured it with the trundle wheel. Corbie took great care to ensure that the trundle wheel was stopped precisely on the measurement he recorded and started from that exact same spot.
As they worked their way towards the hitting wall, there were a couple of measurements that were much shorter. Corbie suggested we use the tape measure for these measurements as it was far more accurate as it had mm on it. They then reached a point that they would have to make continual quarter turns and walk short distances in between each turn.
E: Is there a simpler way to get there rather than making all these quarter turns?
C: Yeah, we could make half a quarter turn and that would take us straight to the flying fox.
E: What would half a quarter turn be?
C: An eighth of a turn.
E: How do you know that half of a quarter is an eighth?
C: I learnt ages ago that if you halve a fraction, you double the denominator.
When looking back at Corbie's work, I noticed that instead of recording this as an eighth of a turn, he had changed it to a 45° turn to fit in with the way he had recorded his other turns.
I think Corbie is working at Generalisation level because he makes strong links between Spatial Sense and Geometric Reasoning and Measurement. He uses the generalisation about fractions that he has picked up from previous learning and applies it to help solve the problem.

The examples illustrate the potential of the *Reflective Continua* to support educators in the interpretation of their observations and reflections. While not all observations need to be transformed into learning stories, this format does provide a rich perspective from which educators can develop future plans for children's mathematics learning. By providing a stimulus and structure for reflection, the *Reflective Continua* provide a tool for educators to use in their quest for excellence in the development of young children's powerful mathematical ideas.

Conclusion

During the Early Years Numeracy Project and afterwards, both the *Numeracy Matrix* and the *Reflective Continua* have been utilised by educators in South Australia to assist them in facilitating the learning of mathematics in both preschools and the first years of school. At the conclusion of the Early Years Numeracy Project, one of the Numeracy Leaders reported that she was using the *Reflective Continua* to assist her identify different levels of mathematical thinking and that "this tool has really motivated me to explore how children think and ways of extending their thinking" (personal communication). Another cluster educator working in the first years of school wrote:

> I know that my time with the numeracy project supported me to develop a wider view of the learner, to learn more about the child's context, needs and abilities. It is a model that I can use in my current site with my own class, of inquiring into what I am doing to make a difference for children learning mathematics.

Another of the cluster educators summed up her experiences with the Early Years Numeracy Project in the following words.

> For many years I have craved the opportunity to be challenged in my thinking and professional practice. I have been constantly trying to do this myself but it is difficult without a structure and time for reflection and professional dialogue. This project, and in particular the reflective continua have provided the most wonderful opportunity for me to receive this challenge to my professional practice. … It has been an awesome scaffold for that professional dialogue and also that self/professional reflection. I believe that I am a better practitioner as a result and will continue to strive to better myself.

The need for educators to reflect on their own pedagogy is well recognised (Grossman and McDonald 2008; Moyles et al. 2006) but many early childhood educators find this difficult to achieve in mathematics because they lack sufficient knowledge of, and confidence in, their own mathematics (Anthony and Walshaw 2007; Perry and Dockett 2008). While they might be able to see their children as powerful mathematicians, they do not necessarily see themselves in this way. The *Reflective Continua* have provided participants in the Early Years Numeracy Project with a scaffold through which to structure their reflections and their pedagogical actions. The impact within the project and beyond suggests that such an approach to pedagogical inquiry can be very beneficial to educators, and children.

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Practices that Support Mathematics Learning in a Play-Based Classroom

Anita A. Wager

The importance of providing young children with opportunities to learn mathematics is well documented in this volume and elsewhere. Studies have found that success in early mathematics is a predictor of later learning in both mathematics (Jordan et al. 2009) and overall achievement (Duncan et al. 2007; Romano et al. 2010). This evidence has led to increased attention to early mathematics learning, yet growing international concerns that young children are not prepared academically to enter school has resulted in calls for more explicit mathematics instruction, particularly for children from historically marginalized communities (Fleer 2011; Zigler and Bishop-Josef 2006). As standards-based accountability works its way to early childhood mathematics, the field is confronted with the challenge of maintaining play-based pedagogy that is developmentally appropriate rather than adopting academically oriented programs (Fleer 2011).

In a report summarizing a conference on early childhood mathematics [Berkeley Pathways Report], Schoenfeld and Stipeck (2012) recommended that preK teachers devote 30 minutes a day to focused mathematics instruction. This recommendation and other calls for an increase in focused mathematics instruction (National Research Council [NRC] 2009) may be interpreted as a shift away (or at least some time away) from the play-based curriculum that scholars in early childhood education have identified as most appropriate. From an early childhood perspective these demands for mastery are incongruent with what is developmentally appropriate and may be "insensitive to individual, cultural, and linguistic variation in young children" (Bredekamp 2004, p. 78). Yet, there is evidence that children engage in "powerful mathematical ideas" in play based preK settings and that by attending to these ideas, teachers can support children's mathematical learning and preparation for more academic schooling (Perry and Dockett 2008a, 2008b; Perry et al. 2012). In this chapter I report on a case study of one teacher's play-based preK classroom

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to uncover ways to "foster the prerequisites for the academic skills [ ] through play" (Bodrova 2008, p. 358). Here I provide an interpretation of 'focused' instruction that minimizes teacher-centered practices and privileges play.

Learning Mathematics in Play

Mathematics learned in and through play has a long history of study. Comenius, Pestalozzi, Froebel, Reggio Emilio, and Montessori each extolled the benefits of and opportunities to learn mathematic through play (Shapiro 1983). In the past century new perspectives on play, including developmental (Piaget 1962), naturalistic (Dewey 1944) and social-constructivist (Vygotsky 1978), further highlighted the need to understand how children learn mathematics through play and how teachers should support that learning. More recently research in early childhood recommended that instruction be rooted in play in order to provide the most developmentally appropriate approach and support young children's growth in multiple domains (Bodrova 2008; Copple and Bredekamp 2009).

According to social constructivist theory the benefits of play go beyond socio-emotional development to mediate young children's learning (Jones and Reynolds 2011; Vygotsky 1978). In describing a Vygotskian approach to teaching, Bodrova (2008) suggested that make-believe play is both a source of development and a prerequisite to learning. Fleer (2011) argued that children's flexible movement between the real world and imaginary situations reflect their learning and that imagination is "the bridge between play and learning" (p. 224). From a cultural-historical perspective the "bridge" should be mediated by the teacher. Van Oers (2010) builds on the principle that young children can learn mathematics when adults (teachers) mathematize unintentional mathematical engagement in play. I suggest these scholars are considering play from the viewpoint of what it 'means for' children rather than what it 'does to' them (Wood 2010). Using Wood's perspective, learning mathematics in a play-based classroom suggests that children have regular opportunities to engage in mathematics throughout the day and throughout the classroom. Teaching young children with an eye toward what it 'means for' them is not easy; teachers must do so in an integrated, culturally responsive way. In the literature review that follows, I refine Wood's (2010) view of an "integrated pedagogical approach" to describe teaching mathematics that (a) plans and prepares for mathematics learning; (b) builds on children's understanding, interests and cultural practices; and (c) recognizes and responds to mathematics that emerges in play.

Planning and Preparing for Mathematics Learning Free play alone is not sufficient to support young children's mathematics learning (Ginsburg and Ertle 2008; van Oers 2010). An integrated pedagogical approach to teaching mathematics in a play-based classroom includes both child- and teacher-initiated learning experiences (Siraj-Blatchford 2009). Teacher-initiated practices include activities to introduce new content and vocabulary (Clements 2004). The ideas that are introduced during these more structured activities are often taken up during play (Klibanoff et al. 2006). Teacher-initiated practices also include planning for the incorporation of mathematics practice during daily routines and transitions (Ginsburg 2006; NRC 2009; Perry and Dockett 2011).

Whereas teacher-initiated activities offer space for introduction to new ideas, play has been shown to provide space to explore and develop these skills (Parks and Chang 2012). Preparing the classroom space to encourage exploration and development of skills requires careful planning in the selection and introduction of materials (Clements 2004; NRC 2009; Saracho and Spodek 2008). This purposeful attention to materials and the arrangement of the classroom encourages initiative and engagement (Copple and Bredekamp 2009). Although not necessarily a component of imaginary play (and for some, falls outside the definition of play), materials such as puzzles and games provide a natural connection to mathematics. Research suggests that children's spatial development is highly dependent on their spatial experiences with materials such as puzzles (NRC 2009). Linear board games have been shown to support number development and counting skills (Siegler 2009; Siegler and Ramani 2008). In addition to the games, puzzles, and blocks that are explicitly designed to support mathematics learning, a wide range of materials that may provoke engagement with mathematics should be available in all interest areas throughout the classroom (see Bennett and Weidner 2012). Thoughtful inclusion of materials that encourage mathematical play in various interest areas is important. When children manipulate materials they "construct many different kinds of mathematical relationships" (Moomaw 2011, p. 9).

Building on Children's Understanding, Interests and Cultural Practices Planning and preparing for mathematics instruction should be based on an understanding of children's mathematical understanding (Carpenter et al. 1989; National Council of Teachers of Mathematics [NCTM] 2000; NRC 2009). Children have intuitive ideas about mathematics and connecting new knowledge to those ideas and existing knowledge is a corner stone of teaching for understanding (Baroody 2004; Carpenter and Lehrer 1999; Hiebert and Carpenter 1992). To further support sense making, mathematics should be connected to children's interests and everyday activities (Clements 2004; Hedges 2011; NCTM 2000). Finally, teachers need to attend to the practices and understandings that children bring from their out-of-school experiences that may vary widely from teachers' own experiences (González et al. 2001; Tudge and Doucet 2004; Wager 2012). This is easier said than done, as understanding the diverse learning opportunities children experience in their homes and communities requires a significant time commitment. However, without knowing about and connecting to out-of-school practices, children will start to develop the notion that there are two systems for mathematics—one for school and one outside (Masingila 2002).

Responding in the Moment The mathematical experiences that children encounter and engage in through play should be supported not only by their teachers' design of the environment but also, the ways in which teachers extend children's encounters with mathematics (Clements 2004; Lee and Ginsburg 2009). Thus, the mathematics that emerges naturally through play should be recognized and reinforced (mathematized) by the teacher. Children have "explicit *interest* in mathematical ideas" (Ginsburg and Ertle 2008, p. 53). In a mathematically rich classroom, children demonstrate and develop their own natural inclination to engage in a wide range of mathematical activities such as counting, patterning, and developing spatial relationships (Geist 2001; Saxe et al. 1987; Seo and Ginsburg 2004). When these activities are observed, teachers should be prepared to reinforce the learning (Clements 2004; Ginsburg and Ertle 2008; Perry and Dockett 2008a, 2008b). "Teachers most important role [ ] should be finding frequent opportunities to help children reflect on and extend the mathematics that arises in their everyday activities, conversation, and play" (Clements 2004, p. 59).

In this chapter I examine the case of one teacher to provide an example of the ways children learn mathematics in a purposefully designed play-based classroom. The case study supports literature that suggests children have multiple opportunities to learn powerful mathematics when the teacher purposefully plans and responds to learning (Perry and Dockett 2011; van Oers 2010). It is significant in that it offers an approach to mathematics learning that is both developmentally and academically responsive and offers a possible place of convergence between views that separate child-initiated play and teacher-initiated learning (Pramling Samuelsson and Asplund Carlsson 2008). In the approach described here, the teacher included *brief* opportunities for intentional mathematics teaching, purposefully seeded the environment with mathematical tools, and observed and responded to the mathematics that emerged naturally through play. Play, for the purposes of this study, included those activities children engaged in when given choice. At times this was imaginative play that often took place in the dramatic play and block areas but also throughout the room. Other times this included playing with games or puzzles. These were considered play in this case because the children chose to play with them and often created their own rules for how to play.

Although the 'combination' of teacher- and child-initiated activities evidenced in this case can be found in many preK classrooms, it is the proportion of time devoted to each that identifies this classroom as play-based. I suggest that this case provides an interpretation of recent recommendations for focused mathematics time that maintains the integrity of play-based classrooms and provides opportunities for children to engage in rich mathematics.

Play-Based Public PreK

A local district in the Midwestern United States recently began a public four-year-old kindergarten program (4K). The half-day play-based program is available to all four-year-olds in the district. This chapter describes the opportunities for mathematics learning in one of these 4K classrooms. Marie's class was in an elementary school predominately serving a white, middle and working class community.

Approximately 36 % of the students in the school were provided with subsidized meals; Marie's classroom reflected this population. There were 15 children in her afternoon class; the maximum number permitted for 4K in the district. Marie was participating in her second year of a professional development program, sponsored by the local University, the district, and the National Science Foundation, to support 4K teachers in adopting culturally and developmentally responsive practices for counting and number.

Marie's teaching experience included over 25 years in preK-5 classrooms (20 in kindergarten) and two years as an elementary school librarian. Marie sought out the opportunity to teach in 4K because she disagreed with the changes in the way kindergarten was taught in the district as a result of increased emphasis on standards. She believed in developmentally appropriate practices and viewed mathematics in 4K as woven through play. For example, in describing how she provided children the opportunity to use manipulatives she said, "put it on the table and let them explore, ask questions and then make sure they know where they can find it [the manipulatives] later" (interview, Sept. 2010). Marie also drew on her experience as a librarian and owned many books explicitly related to counting.

This is an instrumental case study of Marie's classroom to provide insight into play-based mathematics classrooms rather than to understand Marie's teaching in particular (Stake 2005). Marie was selected as a case study because of her many years teaching kindergarten. Most of the 4K teachers in the district had experience in K-2 and I was interested in understanding how teachers with experience teaching in more academic environments adapted their practice to a play-based curriculum. I observed in Marie's afternoon classroom two times each month for a total of approximately 50 hours of observation. The intent of the observations was to capture, as much as possible, the interactions between teacher and children but with a particular focus on children's engagement with mathematics; thus, I often followed children as they engaged in mathematics in play when Marie may have been in another part of the room. To maximize observations of the mathematics that occurred in play, each observation lasted the duration of the class period or three hours. Following each observation, I wrote full field notes using photographs and video I had taken to supplement the details. As part of my role as observer, I also developed "a membership identity" in the classroom (Angrosino 2008, p. 167). I regularly interacted with the children during play yet balanced this with times in which, "I was writing" and they knew not to disturb me.

During the process of writing field notes and observing in the classroom, I identified the three spaces in which children learned and engaged with mathematics: (a) instructional time; (b) engagement with mathematical games, manipulatives, or other mathematical objects; and (c) free play.[1] The teaching practices that aligned with these spaces were: (a) teacher-initiated explicit instruction of mathematics;

[1] I distinguish engagement with mathematical games, manipulatives, or other mathematical objects because these have been intentional placed in the classroom by the teacher. I do, however, consider this 'play' as children may choose to engage with them or not.

Broad code	Sub code	Description:
Intended	Daily Routine	Mathematics practices incorporated into a teacher's daily routine, typically held on the rug as a start to the day.
	Transition	Mathematics initiated by the teacher during transition times, often a tool for classroom management and usually involving number.
	Explicit Activity	Intentionally planned whole group or small group instruction on a mathematics topic.
	'Aha' Moments	Unexpected time when mathematics emerges from an activity that was planned but not intended to be mathematical.
Seeded	Games	Children playing with commercial or teacher created games that incorporate mathematics.
	Manipulatives	Child interactions with counters, geoboards, and other mathematical tools that teachers introduce and strategically place in various interest areas in the classroom.
	Interest areas	Spaces in the classroom in which the teacher has designed opportunities to engage with mathematics.
Child-initiated	Spontaneous	Seemingly out of the blue engagement with mathematics.
	Linked to Activity	When children connect their play to a previously introduced mathematical activity.

Fig. 1 Mathematics in play-based preK

(b) seeding various interest areas with materials to encourage mathematical thinking; and (c) observing and responding to (mathematizing) children's mathematics that occurred in play. I analyzed the data systematically using these three spaces as my initial codes. After an initial round of coding, it became apparent that there were sub-codes within each code that further refined or defined the learning space. Figure 1 outlines the codes and definitions.

Learning Mathematics in Marie's Play-Based Classroom

Marie's child-centered classroom provided a balance of teacher- and child-initiated practices in which engagement in mathematics was often evident in play. By providing a combination of explicit practices in mathematics with a rich mathematical environment, Marie fostered children's learning of and interest in mathematics.

Intentional Mathematics Practices

Intentional practices are those that teachers purposefully plan. They include everyday routines, transitions, explicitly planned mathematics activities, and responses to situations that emerge in planned 'non-mathematical' activities. These activities are intentionally designed to introduce new math content and/or provide children with the opportunity to engage with mathematics. Though not particularly unique to this classroom, Marie's pedagogical approach provided evidence of how intentional practices supported an environment and established norms for mathematics to get taken up during 'unintentional' times.

Daily Routine Every day after lunch, the children gathered on the large circle rug as Marie guided them through the daily routine. The daily routine provided the children with frequent practice of those elements of the numbers core identified as appropriate in preK: cardinality, the number word list, one-to-one correspondence, written number symbols, and subitizing (NRC 2009). There were two ongoing activities in Marie's classroom routine: calendar and number of the day. Over the course of the year, Marie made several adaptations to this routine—adding to the level of difficulty as children progressed and responding to children's modifications.

A common practice in US kindergarten and pre-school classrooms is the Daily Calendar. Despite research that suggests a calendar is not an appropriate tool for teaching mathematics to preschoolers because they are not developmentally ready to understand time (Beneke et al. 2008), many teachers continue to use the calendar. They, and others argue that the use of calendar for the sole purpose of practicing counting and patterning is appropriate (Ethridge and King 2005). Marie is one of these teachers—for her, calendar time is not about dates but about rote counting and one-to-one correspondence. At the beginning of the routine, a 'helper of the day' is selected to lead the rest of the children in counting the days of the month as she/he points to a calendar with a wand or pointer. In September (the first month of class), Marie led the counting—particularly as the month progressed and numbers got higher. During this choral counting, not all children participated but all appeared engaged. Some called out the numbers with loud voices, others whispered, others mouthed the words, while some stayed silent but nodded their heads for each number as the class called it out. Starting in October, Marie did not count at all unless the 'helper' needed support. By the end of the year, many of the children could count up to 31 and all the children participated in the oral counting.

Number of the day was introduced in early October, when Marie showed the children the different numbers on each side of a 6 inch cube. In mid-November a second die was added and in January, many of the children started counting on from the larger number. At the beginning of May a third die was incorporated. After the helper rolled the die (dice) and led the children in counting the number of dots, she/he went to the board and moved the same number of magnets into a circle. In early December, Marie started leaving the magnets in the circle from the morning class and the children in the afternoon class had to decide how many to add or take away to get the number of the day. This activity provided children the opportunity to engage in change unknown problems (Carpenter et al. 1989). Next, the helper went to a large counting frame and moved the appropriate number of beads as the class counted. In early January, Marie noticed that the children were no longer just sliding beads on the top rung but instead making patterns. The alternating rung-by-rung patterns were similar to those observed by Seo and Ginsburg (2004) who found that play often involved patterns. Marie acknowledged the children's patterns and encouraged others to try by asking the class each day, "what patterns do you think [the helper of the day] will use?" This question provided other children, not just the helper to think about how they could present the number of the day in a pattern on the counting frame. Finally, the helper chose an action or rolled a die with actions on each face (jumping, blinking, snapping, etc.). All the children stood up and jumped, blinked, or snapped as they counted out the number on the die/dice.

Chen et al. (2008) suggest that 'helper of the day' may not be developmentally appropriate when the 'helper' does not have any choice. There were some elements of the routine in Marie's room that provided student choice but others did not. Another concern raised in research on 'helper of the day' is the 'helper' is the only one who learns. This may be true in some classrooms, but in Marie's room the helper was the leader who engaged the other children to count with him/her. It is also argued that routines take too much time and without checking my watch, I worried about this during observations. Yet, the routine just described generally took less than five minutes, something Marie tested one day when she discussed the sand timers that were in the classroom:

> Marie says that today we are going to do a science experiment [she had told me that the children were recently very interested in the large sand timers that the district had provided]. For the science experiment Marie shows the top of green timer, which is one minute; the yellow timer, which is three; the blue, which is five; and the orange, which is ten. Marie asks, "which do you think is faster, can you kids figure it out?...
>
> Randy picks helper of the day, Kara. Kara uses pointer to count to nine. The kids aren't looking at the timers but Marie points out that, "the green one finished first so calendar took us one minute to finish" [she writes 1 on the board]. Next, Kara rolls the dice to complete the number of the day routine [counting the dots on the die, moving the gumballs in or out of the machine, sliding the balls over on the 'abacus', and having everyone jump up and down]. As soon as they finish, Marie points out that the yellow timer has finished. "So how long does it take to do calendar and number of day?" Some children respond, "three minutes". [she writes 3 on the board] (*Field notes, Dec. 9, 2011*)

Transitions The children in Marie's afternoon class were frequently exposed to math during transitions. This happened most often when waiting in the hall outside the restrooms. At the beginning of the year Marie would ask children to "show me 4" and they would hold up 4 fingers. By the end of the year, she would pose questions such as "we have 9 students standing in the hall, how many are in the bathroom?" Other examples of Marie's use of transitions for some quick mathematics practice included times when children were making their way to the rug from lunch or free play. During these occasions, she would say ‘show me 5’ to support the idea of five fingers on one hand; or point to some triangles she had drawn on the board and ask children what shape they were; or have children put a tally mark on a graph.

Explicitly Planned Mathematics Activities In half of the class periods observed, Marie had an explicitly planned mathematics activity beyond the daily routine. These activities included brainstorming and planning for dramatic play; several literature connections; and introductions to a variety of mathematical ideas. Each of the activities introduced in whole group setting was later made available for children to choose during free play.

The dramatic play area took on many designs over the year. In October Marie shifted from the ‘Pizza Parlor’ to the ‘Spooky Café’ because the children started talking about Halloween. While in whole group, she invited children to brainstorm prices for menu items. Marie encouraged children to think about dollar values that ‘made sense’ for a menu thereby connecting the play opportunities in the dramatic play area to real life. Each time the dramatic play area changed, Marie introduced the children to the new theme (often based on their interests) and brainstormed ideas to support it.

After recess one day, Marie introduced the children to the concept of similarity and difference using hula-hoops and large attribute blocks. She started by putting a red circle in one hula-hoop and a green circle in the other. The children took turns sorting by color. Next Marie said,

> I am going to make it a little bit trickier and see if you guys can figure out. One hula-hoop had a red circle and one had a yellow square. All the kids put circles in one and squares in the other, Althea was only one that didn’t know what to put in. Marie then asks if anybody could think of another way we could do it? Randy says “triangles”. And they fill in one by one. Marie then puts small shapes in one side and large shapes in the other and immediately Jay says, “small and big”. Randy then points out a different way of thinking about it and did thin and thick shapes. Marie says, “these are different ways but both are right. If have a thin one in front of you put it in here (pointing to one of the hula hoops). The children fill in both hula-hoops. (*Field notes*)

After exploring similarity and difference with the whole class, Marie had smaller attribute blocks available with the math manipulatives but also left the large blocks and hula-hoops out on the rug so children could engage with them during free play. Many of the children sorted the small and large attribute blocks during play.

Marie also brought out math manipulatives such as geoboards and dice during free play (I consider this an explicit practice as Marie, rather than the children, got out the manipulative). After she introduced the geoboards and rubber bands to a small group of children, and worked with them to make and name different shapes, other children who saw the geoboards took them out on their own during later class periods. Similar to the ways that children learn from listening in on others solving problems, the children in Marie’s class learned from observing others and integrated those ideas in their play (Mills et al. 2012).

Unplanned Response to an Activity As in any classroom, Marie encountered unplanned teachable moments. There were occasions when she saw an opportunity to engage children with mathematics embedded in another activity she planned. In late March Marie read the children's book *Yikes!* (Florczak 2003). Her intention in selecting the book was to discuss the various emotions the main character expressed as he came across different animals in the jungle. She showed each page as the children described the emotion and Marie described the animal (the country and environment it lived in, how big it was, and what it ate). When she got to the Bengal tiger she told the children it could grow to be 10 feet long. Although it was not something she planned, she recognized that four-year-olds might not understand how long 10 feet are. She quickly found rulers and the following was observed:

> Marie: "one ruler is one foot, so how many of these do we need for ten feet?" There are many guesses starting at 2 and counting up and Randy says, "ten". Marie stops the story to show the kids how long 10 feet is, she's worried she doesn't have 10 rulers; she finds 6 right away and has the children start at one end of the rug and lay the rulers down.
>
> Marie can't find any more rulers so she gets the yardstick and explains that it is three feet or three rulers. She puts the yardstick down next to three rulers so they can see. Next she says that she thinks the rug is 10 feet so we can see how long the tiger is.
>
> The next page is the Crocodile and he can be twenty feet. Marie says, "We are going to figure out how long the Crocodile could be. If we know the rug is ten feet and we need to get to twenty, how many to twenty?"
> Keith: "two"
>
> Marie says we'll measure at the end of the story. She finishes the story and asks again how many rulers we need to get from 10 to 20. The children aren't coming up with an answer but Marie points out that 20 is our number of the day. She gathers up the rulers that are on the floor and hands them out to children. "We know the rug was ten", so she has kids lay down rulers one at a time doesn't have enough so replaces three rulers with yardstick [note: this is a bit confusing which she acknowledges wishing for more rulers]. Keeps counting them up to twenty. Marie has Jay stand at one end of the rug and Althea at the end of the rulers to see how long the croc is. The kids are awed. (*Field notes, March 20, 2012*)

This example showed how Marie's actions in the moment enabled the children to engage in both counting and measurement and made a mathematics connection to real life. She later told me that had she planned this ahead of time, she would have had enough rulers on hand. The yardstick was complicated for most children but stretched those such as Randy.

Other examples of mathematics that emerged from an activity often came from books in which mathematics was not the basis of or integral to the story; rather the mathematics emerged through natural connections (Shih and Giorgis 2004). These were situations in which Marie did not plan ahead for mathematics instruction but recognized the need in the moment and responded.

Seeding the Environment

Children in Marie's classroom had at least one hour of free play every day in addition to 30 minutes outside on the playground. In preparing the various interest areas in the classroom, Marie explicitly provided materials to support mathematics learning. All interest areas in the class were purposefully seeded with materials or activities to engage with mathematics. In addition to the block area (a space where several children regularly played) Marie had math games in the game shelves, a variety of manipulatives placed at the sensory table or in the game shelves, and often incorporated math learning opportunities in the other interest areas in the classroom.

Games The game shelves in Marie's room were filled with a variety of puzzles and board games to support learning in literacy and mathematics. The puzzles designed to promote literacy also provided support for spatial awareness—for example, a 2-piece picture puzzle with pictures of words that start with the same letter not only had children think about the initial letter sound but also the shape of pieces and how they fit together. Chutes and Ladders™ and other board games supported children's counting and problem solving skills. Marie also had a variety of teacher made games, particularly linear number board games.

Although designed to encourage counting, I observed how one student exhibited multiple problem-solving skills when playing a linear number board game.

> Kara and Kevin get out one of the number race games that Marie has made. Kevin has moved up the board until he is one away. I ask how many he needs to win and he says 1. He spins and stops the spinner with his finger at 1. They continue playing more games but Kevin keeps winning.
>
> Kara says: "you always beat me, let's play something else".
> Kevin tells her: "you have to go again".
> Kara: "let's do something else."
> Kevin: "It has to be a pattern (patter-*n*) with the winning pieces."
> He organizes the game pieces that have finished into a purple, orange, purple pattern. Kara seems satisfied with the change and they continue playing again to do pattern game. Kevin spins it lands on line between two and six and he shoves it to six and wins again. As they keep going, Kevin needs to get a two, so he again stops the spinner with his finger. Kara quits, and Kevin continues to play by himself.
>
> I ask Kevin if he'd like me to join him and he is beating me every time even without cheating very much. Then I realize his spinner goes up to six, mine (and Kara's) only goes to three.
>
> Researcher: "Do you notice any differences in the spinners?"
> Kevin, with a wry smile: "mine has 4, 5, 6 and yours has two 3, 2, 1"
> Researcher: "who do you think will win?"
> Kevin smiling more: "me!"

This vignette provided three examples of mathematical engagement that occurred during the game. First, Kevin recognized the number he needed to get in order to win and nudged or stopped the spinner to get it. During this exchange, I observed but did not react to Kevin's cheating. As Anderson and Gold (2006) point out, children's cheating in a game can be interpreted in multiple ways. Here, I was interested in how Kevin was cheating in order to get the desired number. In order to know he needed 2 on the spinner, he had to count ahead on the game board. Second, Kevin adapted the game in a way to keep Kara engaged. He shifted to goal of the game to making patterns. During the pattern making, Kevin supported Kara's understanding of what color counter would come next. Third, in purposefully selecting the spinner with 1–6 and making sure his opponent had the spinner with 2 sets of 1–3, he was exhibiting early understandings of probability. Marie's initial intent when she placed the two different spinners in the game shelf was to differentiate. Although Marie was in another area of the room and did not witness the incident, I shared with her my field notes from that day and she commented that she wished she could have been there to ask Kevin why the 1–6 spinner let him win more often.

Manipulatives There were a wide variety of manipulatives in the room including multiple containers of counters (bears, farm animals, wild animals, people), links, numeral magnets, geoboards, balance scales, pegboards, and other district provided items. Marie also brought in collections of things that she placed in different interest areas of the room. One day, Marie put out a box containing a variety of multi-sided dice. The children took out the dice and rolled them in the then-empty sensory table to see who got the highest number. The game evolved into a spinning contest using the 10-sided dice but the children still called out the number that landed 'up' after a long spin. Later in the year, Joel got out the same box of dice and selected one die with dots and one with numerals. We played a game trying to get the same number of dots as the numeral rolled. In February the sensory table was filled with water and the children used various measuring tools. This is a good example of the way Marie had described her perspective on using manipulatives—she provided the introduction, then allowed the children to choose.

Dramatic Play Whenever commerce occurred in the dramatic play area, Marie provided materials to encourage mathematical thinking. For example, the 'Italian Restaurant', 'Spooky Café', and 'Grocery Store' all involved the use of money (made from scraps of paper) and a cash register. A child operating the cash register would decide how much to charge for each purchase and make change; on those occasions when Marie was close by and noticed the activity, she would ask children to explain their thinking. Further children regularly counted as they prepared meals, for example, how many 'bats' were needed to prepare bat stew. These meals were served to adults in the room during which time Marie asked not only 'how many' questions (e.g. how many cookies are on the plate) but asked who had more or less, how many more or less a plate might have, and noticed and asked about any patterns created.

Observing Free Play for Child-Initiated Mathematics

Sometimes counting seemed to come out of nowhere, at other times; child-initiated mathematics was linked to the daily routine or to an activity that had been introduced earlier.

Spontaneous Noah was sitting at the lunch table one day and started counting out loud. It appeared that he was pointing at something so Marie asked what it was he was counting. There was an alphabet border over the chalkboard and he told her he was counting the letters. Marie said, "Let's see, are you counting the big and the little letters, you can count a lot that way" (*fieldnotes, October 11, 2011*). The idea of counting the alphabet letters went viral and other children at the table started pointing at the letters and counting. Interestingly, this was early in the year and few of the children could count up to 10 and only one could count to 20. None-the-less, they were eager to try. Other incidents of spontaneous counting occurred regularly in Marie's classroom. Spontaneous counting generally occurred in one of two ways—as children (or adults) were distributing items or when there were collections of items. Children counted puzzle pieces as they pulled them from the box or stirring sticks as they were handed out for an art project.

Linked to Activity Introduced Another way in which child-centered mathematics occurred was when the children built on an idea introduced earlier in the day. For example, in January, the children worked on art projects in which they made pigs out of different shapes (circles, triangles, and rectangles). After the projects were completed and it was time for free play, Noah went to the art table and started drawing pigs using the same shapes. After the incident mentioned earlier during which Marie pulled out yardsticks to measure 20 feet (the length of a Nile crocodile), Will sought out the yardsticks during free play to measure various items around the room.

There were occasions when the children extended ideas they learned during the daily routine, particularly when using the counting frame.

> There was a new idea with the counting frame as the children had started moving the counting beads over in different patterns. Jay shifted 4 beads on the top row and 4 on the bottom. Marie wondered, "How did he move the beads to get 8?"
> The children cried out, "4 and 4."
> Marie, "yes, but *how* did he move them? He made a pattern, first top row then bottom." (*Field notes, January 17, 2012*)

In this example, Jay extended the regular routine providing an opportunity to decompose eight as well as think about patterns. Marie responded to his spontaneous patterning, and as discussed previously, continued to encourage other children to make patterns on the counting frame. In another instance, Joel went over to the counting frame prior to daily routine starting and moved five beads on each rung so that each rung has five beads on each side. He then announced, "They are the same". I asked what he meant and he said there was the same number on each side. Joel had taken up an idea introduced by Jay and 'played' with it.

Opportunities to Learn with an Integrated Pedagogical Approach

In examining Marie's classroom through my refined view of Wood's (2010) "integrated pedagogical approach", I found that Marie (a) planned and prepared for mathematics; (b) built on children's understanding, interests, and, to a lesser extent, cultural practices; and (c) recognized and responded to mathematics that emerged in play.

Planning and Preparing for Mathematics Learning Marie explicitly planned mathematics instruction to introduce children to new topics and concepts such as shapes and similarity/difference or materials such as geoboards and attribute blocks. This practice occurred two to three days a week and generally lasted between five and ten minutes. The only day I evidenced a longer 'lesson' was during the discussion of attribute blocks and hula-hoops that lasted 20 minutes. Marie had not planned to extend the lesson that long but some of the children kept things going. This activity provided children with the opportunity to examine shapes to identify the similarities and differences.

The mathematical ideas that Marie introduced were often taken up by children during free play, what I described in the findings as child-initiated mathematics linked to an activity. This finding is consistent with what others have found (Klibanoff et al. 2006) and affirms the importance of making new ideas accessible in free play. Marie's careful seeding of the environment provided children with the opportunity to bump into mathematics during imaginary play and engage with mathematics in game play and puzzles. Measuring tools in the sand table and water table, cash registers in dramatic play, items for sorting in the discovery area were all taken up by children as they interacted with mathematics (Bennett and Weidner 2012). The teacher-made linear board games, such as the one Kevin and Kara played, have been shown to support number development (Siegler 2009; Siegler and Ramani 2008) and, in this case, problem solving and patterning.

Building on Children's Understanding, Interests and Cultural Practices As in any classroom, Marie's children had a wide range of skills and understandings. She built on their understanding by providing activities with multiple entry points and materials that could be used in a variety of ways (Carpenter et al. 1989; NRC 2009). Marie moderated her involvement in the daily routine as children developed their counting and problem solving skills. She also added to the level of difficulty as she saw children master certain skills. For example, over the course of the year, she went from one die to three and supported children by permitting them to count all or count on depending on the children's understanding. Thus Marie was building on what her children knew—a corner stone of teaching mathematics with understanding (Carpenter and Lehrer 1999) and a key recommendation of the NRC (2009), the NCTM/National Association for the Education of Young Children joint position statement (2002), and the Berkeley Pathways report (2012). There were multiple occasions when Marie built on children's interests and connected activities to real world experiences (Clements 2004; NCTM 2000). After noticing children's frequent use of the sand timers, she incorporated the timers into the daily routine. Children's interests drove ideas for dramatic play, and Marie included them in decisions on topics such as menus choices and prices. In terms of drawing on children's cultural practices and out-of-school experiences, the only place I observed this connected to mathematics was when Marie incorporated children's home language in counting.

Responding in the Moment I observed several occasions when Marie responded to child-initiated mathematics thereby extending children's understanding or making explicit connections to their ideas and mathematics (Clements 2004; Ginsburg and Ertle 2008; Lee and Ginsburg 2009; Perry and Dockett 2008a, 2008b). Marie responded to the mathematics children took up during daily routine and free play. When the children started making patterns with the counting beads during daily routine, Marie mathematized this for both the child making the pattern and other children when she asked what patterns they noticed. During free play, Marie often observed and extended children's thinking through questioning. There were multiple occasions when I observed her asking children about the number of items in the 'meals' they served, how much something cost, what patterns they saw, and who had more or less. The day Noah burst out counting, a frequent occurrence with young children (Saxe et al. 1987), Marie talked about it with him and other children as they joined in.

In addition to mathematizing child-initiated mathematics, Marie was responsive to opportunities to engage in mathematics that emerged unexpectedly. This was particularly true when she read to children or during transitions. The book *Yikes!* set the stage for an exploration of measurement and engagement with measurement tools. Marie was conscious of opportunities during transitions to count and compare.

What Did the Children Learn?

With 15 children in a classroom it is easy for some of them to avoid counting during daily routine (this did in fact occur upon occasion). For some children the numbers are too easy or too difficult, some are not confident in their knowledge or language skills, others are just shy. But, over the course of the year, all the children came to participate verbally in the daily routine. Further, all of the children demonstrated an increase in numeracy and problem solving skills. Marie's daily routine and counting during transitions established a norm that there are many different things to count—numerals, dots, balls, jumps, blinks, magnets, letters, etc.—and that one can count any time. This practice established a community that fostered positive dispositions towards mathematics, an important aspect of young children's mathematical development (Anthony and Walshaw 2009). The mathematics in Marie's classroom extended well beyond early numeracy to include geometry, spatial thinking, patterning, and measurement. Some children regularly gravitated to the block area where they constructed elaborate buildings. Others frequently pulled out the geoboards to create different shapes, and others used the variety of manipulatives and tools. Children's engagement with materials to recognize and identify the properties of shapes supported their intuitive knowledge and development of spatial awareness (Sarama and Clements 2008). Marie encouraged children to share their description of patterns they created during routines and free play, a skill which has been shown to contribute to the development of pre-algebraic reasoning (Papic and Mulligan 2005; Papic et al. 2011). Although I hesitate to rely on a standardized measure because I do not believe it accurately portrays children's full potential; all of the children in Marie's classroom passed the kindergarten screener used in the district (the one exception was a child with a severe cognitive disability).

Concluding Thoughts

Counter to the research suggesting that preK teachers rarely mathematize children's activities, Marie recognized children's mathematical understanding and either extended, modified or supported them in daily routine. She was aware of the developmental needs of four-year-olds who might compliantly sit on the floor for 20 minutes for whole group time but she knew that was developmentally inappropriate for that age. Marie considered the linguistic resources that her multi-lingual children brought by having them share their expertise counting in Spanish or Chinese, but did not push them to do so if the children were not comfortable.

Anthony and Walshaw (2009) called for "policy makers and mathematics educators need to be increasingly informed by research that bridges the early years divide" in order to identify pedagogical practices that offer rich mathematical learning opportunities in play-based classrooms (p. 117). Although not necessarily generalizable, I suggest that Marie's case offers an example of this bridge by employing three elements: (1) *brief* intentional mathematics instruction that, over the course of the year, covers mathematics topics appropriate for early childhood and responds to children's understanding and interests; (2) carefully seeded interest areas that encourage multiple opportunities to engage with mathematics tools, manipulatives, vocabulary, and ideas; and (3) observing, recognizing, and responding to mathematics that emerges through play.

As a participant in professional development explicitly focused on developmentally and culturally responsive mathematics, Marie was particularly attentive to the opportunities for mathematics learning in her 4K classroom. This case offers the early childhood mathematics education community an example of the possibilities for learning in play given particular pedagogical practices. As such, mathematics educators are tasked with the responsibility of providing both practicing teachers and teacher candidates with the professional learning opportunities that support practices such as this. To do so, teachers need: a solid grasp of mathematical knowledge for teaching preK (Baroody 2004; Ginsburg and Ertle 2008; NRC 2009); to understand how to build on children's mathematical understandings (Carpenter et al. 1989); to know where each child is developmentally (Copple and Bredekamp 2009); and understand the multiple resources children from home.

Current policy documents suggest teachers spend an increasing amount of time on focused mathematics instruction (Schoenfeld and Stipeck 2012; NRC 2009). If 'focused' time is interpreted as predominately teacher-initiated mathematics instruction, time for play and the mathematics (and other) learning that occurs there will be reduced. However, if we use Marie's purposeful approach as an example of focused mathematics, children will have multiple opportunities to engage with and learn meaningful mathematics in a play-based environment.

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Communicating About Number: Fostering Young Children's Mathematical Orientation in the World

Bert van Oers

It is the meaning that is important, not the sign. We may change the sign, but retain the meaning. Vygotskij (1983), *Sobranie sočinenij, T. 5*, p. 74.

Towards Autonomous Citizenship and Mathematical Proficiency

One of the prominent features of western cultural history is the increasing 'mechanisation' in the ways we tend to conceive of our reality. Both the physical and the social world is increasingly seen as a rule-governed system that is driven by mechanisms, procedures, rules and tool-based actions and operations, which bring unity and predictability in the workings of the system. In his magnificent description of the history of science the Dutch mathematician Dijksterhuis (1961) has demonstrated that this cultural evolution is strongly related to the development and increasing use of mathematics in all parts of science (and life in general for that matter).

Mathematics nowadays indeed has become a core element in people's cultural functioning and everybody is supposed to be able to accomplish and understand the basic procedures of mathematical thinking (addition, subtraction, multiplication, division). OECD (2012) referred to this capacity as 'mathematical literacy', which was defined as 'the capacity to identify, understand and engage in mathematics, and to make well-founded judgements about the role that mathematics plays in an individual's current and future private life, occupational life, social life with peers and relatives, and life as a constructive, concerned and reflective citizen' (OECD 2012, glossary, p. 29). Mathematical literacy is more than just being able to perform mathematical operations without errors, but also contains reflective discourse about mathematics as a cultural activity. All manifestations of mathematical literacy somehow are based on the use of symbolic means and semiotic devices, and on

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learning to employ them like a mathematician. As Pimm (1987) pointed out, basic mathematical proficiency implies 'learning to speak like a mathematician'. Mathematical language includes all types of semiotic devices to be used for dealing with mathematical meanings (oral and written language, symbols, gestures and graphic constructions; see also Cobb et al. 2000; and the notion of 'representational repertoire' in Lehrer and Schauble 2008, p. 10).

With 'mathematical literacy' increasingly becoming a cultural exigency for autonomous citizens, most governments around the world nowadays demand that their state's educational provisions bring their citizens to a basic mathematical understanding and proficiency. Such mathematical literacy is generally supposed to be an indispensible competence for autonomous citizenship, for getting jobs and continued studies, or more general: for contributing to the maintenance and innovation of society.

Promoting Mathematical Literacy from an Early Age

Over the past decades, educational systems over the world have seriously picked up this challenge of promoting mathematical literacy in all children. In the approaches for the achievement of this goal, we can identify three trends: a focus on operational proficiency, an advancement of the start of formal mathematical thinking to the younger ages, and an attempt to make mathematics more meaningful for (young) children by embedding their problem solving in number tasks in everyday contexts that make sense to them. Recent discussions, however, have expressed reservations on the expectations regarding these trends. Research has demonstrated that operational proficiency does not automatically foster *understanding* in the older pupils in primary school (see for example Bruin-Muurling 2010 with regard to understanding of fractions), nor does it stimulate the development of problem solving abilities (Kolovou 2011). The latter researchers have demonstrated that meaningful, context-based realistic mathematics education (Gravemeijer 1994) hasn't always been able to solve these problems to a satisfying degree in everyday classrooms.

As to the wide-spread introduction of number-focused tasks and assignments in early education (counting, dealing with written symbols, learning number facts etc.) research has also shown that there is reason to be careful. Teaching young children to manipulate formal symbols correctly was found to make little sense to pupils (see among others Bryant 1997; Munn 1998) and not conducive to a feeling of ownership of the number concept in many young pupils (Ekeblad 1996). Children evidently demonstrate behaviour (like counting) that looks mathematical from the outside (as it is fairly in conformity with adult mathematical operations). These children, however, are often unable to apply this 'knowledge' in new situations, or answer questions about it, e.g. concerning the cardinal or ordinal aspects of number (see for example Bryant 1997; Hughes et al. 2000). Such a formal introduction of young children into the domain of mathematical thinking might even cause serious problems in some children's mathematical thinking development (Hughes 1991).

How, then, to achieve this cultural objective of helping children to appropriate mathematical literacy that can support them in their present and future autonomous participation in a wide range of cultural practices? The aim of the chapter is to build an argument that supports the claim of the importance of young children's ability to communicate about number as a core aspect of the development of mathematical literacy. The chapter first concentrates on the role of language in general for the development of mathematical thinking, and more specifically on the role of communication for the orientation in the world from a mathematical point of view. Young children's need for such communication arises from their involvement in cultural practices. As shown by the examples discussed later in this chapter, teachers can regulate the development of such a need by confronting children with demands from the situation that require translation of experiences into mathematical language and objects. However, rather than focusing directly on the accomplishment of mathematical actions or the application of mathematical rules in those situations, it is claimed here that more priority should be given to (collaborative) orientation *in a situation* from a mathematical point of view, and develop children's mathematical understanding by assisting them in learning how to communicate with others in situations that have to do with number, quantity, space and relations. The chapter finally argues that young children's development of communication on mathematical aspects of reality can be distinguished in two different processes: the improvement of the communicative tools appropriate for communication in the mathematical domain, and reflection on the properties of number.

For an understanding of the relevance of communication in the mathematical domain, it is necessary to explain first in more general terms the relationship between language and mathematical thinking. This topic will be addressed in the next section.

The Role of Language in Mathematical Thinking Development

Many researchers have already emphasised the importance of language for the development of mathematical thinking (e.g. Pimm 1987, 1995), and have been able to demonstrate empirically that relationships do exist between mathematical thinking and narrative competence from an early age (see Burton 2003; Krumheuer 1997). As Mix et al. (2005) have persuasively pointed out, most current models of number development fail to capture the complex interaction between verbal and nonverbal processes. They argue for a conception of early number development as a fluid and multi-faceted process in which verbal and non-verbal processes are tightly interwoven. In their view, number development in young children should be based on interactions in which children are "exposed to number language and develop some mastery of it without completely understanding it". The authors emphasise that such partial understandings are to be considered as "useful, indeed crucial, contributions to children's learning" (Mix et al. 2005, p. 324). From their review of empirical evidence on the relevant dimensions in number development they conclude that the following five conditions seem to be essential for progress in the process of number development: (1) it is contextualised; (2) it is piecemeal; (3) it is socially scaffolded; (4) it differs across individuals; and (5) it uses domain general processes (i.e. is partly based on non-mathematical cognitive processes) (Mix et al. 2005, p. 326). In many cases numerical development is rooted in non-numerical experiences with area or contour length (Mix et al. 2002). The outcomes of these studies support assumptions about the fundamental role of language and communication in number development. It is, however, important to emphasise here that this does not imply the primacy of language over numeracy! The relevance of the argument of Mix and her colleagues is that numerical development co-develops with the progress in language development.

From a Vygotskian point of view we may say that the child from the first day of his life is involved in and benefits from a cognitively distributed cultural world that is imbued with mathematics and mathematical language. As long as the child himself is not able to deal effectively with quantity and number, people from his environment can take care of this and scaffold the child's development step by step and sensitively into the mathematical world view. By 'imitative participation' (see van Oers 2012c) the child will gradually learn the gist of mathematical thinking and appropriate the relevant concepts and operations through communications with more knowledgeable others.

In our own research, we studied the relationship between mathematical skill (arithmetic) and narrative competence (see van Houten 2011) in 89 primary school children between 6 and 9 years old, controlling for vocabulary ability.[1] It turned out in a regression analysis of the data that both vocabulary ability and narrative competence had significant predictive value for mathematical skill. We found a correlation between narrative competence and vocabulary (0.46), and hence we should take into account that there is a risk of collinearity of the variables that may make the regression analysis unreliable. However, a check for this condition demonstrated that the reliability of our regression analysis was not seriously endangered by collinearity (Tolerance: 0.79; VIF: 1.27). Further analysis of our data showed that narrative competence was a far better predictor than vocabulary. When we calculated partial correlation between narrative competence and mathematical ability (controlling for vocabulary), the correlation still remained significant, and (according to the regression analysis) vocabulary didn't add a significant proportion to the explanation of the dependent variable (mathematical skill). The other way around, the partial correlation between vocabulary and mathematical skill, controlling for narrative competence, became non-significant and narrative competence contributed significantly to the explanation of the dependent variable (mathematical skill) in the regression analysis (study reported by van Houten 2011).

[1]All measurements were made with reliable and valid tests. Mathematical skill and Vocabulary were measured with instruments published by the Dutch national test institute (CITO). Narrative competence was measured by an instrument that was produced and validated in several studies in our own department.

We concluded from this study, that an important factor in mathematical thinking development is the ability to use language in a coherent way (as in narrative competence), relating a starting point (e.g. question) via a coherent reasoning process (narrating) to a conclusion (plot). I take this as a basis for the assumption that mathematical thinking and its development is related to the ability of making 'mathematical texts' ('mathematical stories', Forman and Ansell 2005), and to the ability to communicate coherently about number, space, relations to oneself (as in thinking), or to others (as in mathematical discourse). The found correlations between narrative competence and mathematical thinking, as well as the theoretical assumptions about communication and mathematical thinking (see for example Sfard 2008) call for further study of young children's ways of communicating about quantity and number.

The studies discussed above suggest that the textual aspect of language, that is, the facility to build organised systems of utterances (propositions), is related to the development of mathematical thinking. We must keep in mind here that mathematical texts are not confined to verbal language in the ordinary sense, but include all kinds of semiotic tools (symbols, gestures, formula, schematic models, diagrams, graphs etc.). However, confirming the relationship between organised language use ('texts') and mathematics is not enough for the understanding of this relationship. In the following I will point out that the function of language as a semiotic tool for orientation in the mathematical aspects of reality is essential for the development of mathematical thinking.

Orientation, Mathematising, and Communication

Orientation is a psychological process that tries to find appropriate fit between (human) actions and the requirements of a situation. The decision to act for instance on the basis of mathematical knowledge is probably most successful when a previous orientation process has suggested that this intended action is relevant, considering the demands of the situation.

Every human action is part of a situated activity. Actions always require an orientation process of examining the situation and the psychological conditions (e.g. one's personal goals) in order to be appropriate to the situation at hand. By such explorations an actor can find out how to act, or decide how to approach the situation. This can be a preliminary reflection or a process of continuous monitoring, but one way or the other actions always need exploratory processes to control the fit to the situation at hand. This process of probing situations in order to find out the nature of the situation, its affordances and rules, and in order to know what to do or what to say is called '*orientation*' in an Activity Theory perspective (see for example van Oers 1996a). This process is often instantaneous (when we immediately see what to do and how), but in many situations it also needs a more expanded reflective activity. According to Gal'perin (1976; see also van Oers 2006) 'orientation' is the essence of human consciousness.

When, for instance, we find ourselves in a bakery, we are supposed to order something like bread (rather than a beer), wait for our turn to pay rather than immediately taking a loaf of bread, slicing it and eating it on the spot (like one could do at home). People wouldn't be able to conduct proper behaviour, if they weren't able to identify the nature of the situation, its affordances and demands, and decide what is proper to do and to say there. Likewise, when confronted with a problem, actors always have to find out how to approach the problem, mathematically or otherwise. More experienced people immediately see how a situation is to be dealt with (this is characteristic for 'disciplined perception'—see Stevens and Hall 1998). In social interactions young children can benefit from this 'disciplined perception' of others. By being involved in shared orientation activities and being guided by appropriate communications on mathematical issues, they can learn to recognise which situations should be addressed in a mathematical way and how.

Orientation can have different forms depending on the situation and the level of the actor's development or intentions. When we have bought two loaves of bread of € 1.10 each, most adults know immediately what they have to pay at the checkout. No need to figure out extensively that a mathematical approach is needed, and to decide which operations should be executed. Orientation is immediate and often so quickly processed that we don't realise that it ever took place. A different situation can be experienced when we have to decide whether to buy a packet of butter (250 gram) for € 0.75 or another one (300 gram) for € 0.85. In such situations more orientation with the help of mathematical knowledge and abilities is needed in order to find out what to buy. Seeing that a mathematical approach is needed is mostly not the main problem here, but finding out how to decide which one is the best buy. Even when people decide that the mathematics is too complicated (or too much work) for mental calculation, and prefer to use other arguments to make the decision (see Lave 1988) some mathematical orientation is still unavoidable.

Orientation in a situation to find out what to do from a mathematical point of view often implies a process of translating some characteristics of the given situation into mathematical terms. Freudenthal (1973) referred to this process as '*mathematising*' and describes this as the human activity of organising a field (be it conceptual or material) into a structure *that is accessible for mathematical refinement* (Freudenthal 1973, p. 133). Organising a field of mathematical objects like numbers into the category of even or odd numbers, or natural and rational numbers are examples of mathematising at a conceptual level, but the recognition of the growth of a plant in early childhood classrooms as a measurement problem is an act of mathematising too. Mathematising is the activity of producing structured objects that allow further elaborations in mathematical terms through problem solving and (collective) reasoning/argumentation. Mathematising is the basis of mathematical thinking that underlies both operational-procedural thinking and mathematical problem-solving. Helping (young) children to appropriate strategies of mathematising is a core element of the stimulation of their mathematical literacy.

Mathematising is closely related to language as a tool for orientation and communication. People need a proper language for communicating about quantities, spatial positions or relations from a mathematical point of view. Especially they need an appropriate level of narrative competence when they ask help from others, or when they are involved in a collective process of (mathematical) orientation in a problem situation. The language for communication can take different forms. With regard to mathematical reasoning and the construction of a mathematical space for focused communication, a number of researchers have pointed at the relevance of gestures (gesticulations) as a means for communication in a mathematical discourse or teaching process (see among others Bjuland et al. 2008; Yoon et al. 2011). Similar suggestions have recently been forwarded with regard to picture books as a communicative means for the stimulation of young children's mathematical thinking (see for example Elia et al. 2010). There is also a body of research that indicates the value of graphic representations for communication and orientation in the mathematical aspects of reality (see for example Lehrer and Schauble 2008; Lehrer and Pritchard 2002; Nemirovsky and Monk 2000; van Oers 1994, 2012a; van Oers and Poland 2007).

Inventing or looking for useful symbolic means to communicate about number, number operations or space, are common practice in many play activities of young children. Traditional decisions to impose formal symbolic language (numeral symbols) on children for their communication about number and numerosity is not only detrimental for many children's mathematical thinking development (see Hughes 1991), it also fails to appreciate children's abilities to use different languages (pictures, oral language, metaphors, gestures etc.) for effective communication of their experiences with reality. Basically it denies one of the most powerful qualities of mathematics itself, which exploits the values of rewriting expressions and mapping from one symbolic system into another (as Descartes elegantly has shown when he mapped geometry into algebraic functions, leading to the nowadays still powerful branch of mathematical thinking called Cartesian (or analytic) geometry). Vygotskij (1983) aptly expressed this versatility of languages when he wrote: 'It is the meaning that is important, not the sign; we may change the sign and retain the meaning" (see the motto of this chapter). Coding and recoding meanings in different expressions ('representational redescription' as Karmiloff-Smith has once called it) is a basic dynamic aspect in mathematical thinking and its development (Karmiloff-Smith 1995). Therefore, in the guidance of young children into proficient mathematical thinking, the experience with this flexible recoding is to be considered one of the basic educational objectives in this area. Children's ability to be involved in this process is strongly dependent on their communicative ability with regard to the mathematical aspects of reality. The development of this ability is the topic of the next section.

Learning to Communicate from a Mathematical Point of View

As explained elsewhere (van Oers 2012c), learning to adopt a specific point of view, and acting or speaking consistently from that point of view is characteristic for the development of abstract thinking. Therefore, the widespread opinion that mathematics is abstract, is correct, as it represents a historically developed ability to look consistently at (physical or cultural) reality from *one* point of view (embodied as the legacy of the mathematical community), neglecting all other points of view that might be taken.

For the introduction of children into this abstract way of looking at (and talking about) the world, we have to get children involved as participants in activities that are being organised from this point of view. Evidently, there can be a variety of reasons for adopting a mathematical point of view: in traditional schools the mathematical point of view is adopted since this practice requires it (O'Connor 1998). Doing arithmetic in the classroom is imposed in such cases by the traditional demands of the situation (e.g. the textbook). A mathematical point of view can also be triggered in children through interactional processes, when someone else (e.g. the teacher) asks questions like 'how much?", "Which one is more?". Finally, through their interactions with more knowledgeable peers or adults, (young) children also learn to recognise situational cues as referents to mathematical language or actions, or as demands for orientation through mathematising. Either of these starting points of mathematical activity may lead to mathematising and the accomplishment of the mathematical operations that finalise the process by offering an acceptable solution to a problem.

One of the first processes of mathematising in young children has to do with challenges of how to refer to quantity or spatial relations or positions. The child's proficiency to take part in collective processes of mathematising (or engaging in an activity of individually accomplishing this process) is dependent on his ability to use available language for communicating his ideas about numerosity aspects of the world. For some time the child can use these words correctly (e.g. when he explains how old he is, how old his sister is and explain who of the two is the youngest). Likewise he can also state that we need four glasses of lemonade when his (four) friends are coming to visit him. And check it! In these cases children can use these everyday words correctly to communicate with others efficiently (though not always perfectly) from this numerosity point of view. He is communicating *with* numbers in his everyday conversations, similar to his skill to use of the word 'table' to refer to a specific piece of furniture. This is an essential first step towards the development of meaningful mathematical thinking. It is important to keep in mind, however, that young children's communication with number words is by itself not necessarily a sign of mathematical thinking.

Through interactions with more knowledgeable others, this language development in young children goes hand in hand with a process of "mathematicalisation" of everyday language, which is closely related to interactions with more knowledgeable others (see van Oers 2002). Step by step the counting words that children have picked up in everyday conversations, get a partially mathematical meaning, when the child starts to realise that a name of a numeral (on his house, on the bus, on the birthday hat of his little sister etc.) may refer to different things: sometimes it is only a name (like on the bus), but sometimes it refers to numerosity, but at other moments it refers to order (like in the supermarket: who was first, who is next?). Although still quite incomplete in the beginning (as Piaget has already shown), the differentiation in the use of numerals shows emergent reflections on the meaning of numerals and indicates the commencement of a new mathematical content in the child's everyday words.

Genuine mathematical thinking is, however, more than communicating *with numbers*. Mathematical thinking essentially requires *reflection* on (the relationships among) mathematical objects, i.e. relate different mathematical objects (like numbers), explain operations with number, evaluate the use of mathematical notions, in short: it calls for the ability to communicate with oneself or other people *about* mathematical objects and their interrelationships. With regard to number, we can say that learning to communicate from a mathematical point of view requires a transition from communication *with number* (for achieving an everyday goal of referencing) to communication *about* number for justifying a communicative claim about the quantitative or spatial dimensions of reality.

Although the details of this developmental trajectory are far from clear as yet, systematic observations of young children have produced at least three important moments and processes that seem conducive to the gradual evolution of mathematical thinking in young children. I will illustrate and discuss some pieces of evidence for these moments in the following section.

Fostering Children's Development Towards Communicating About Number

Ample observations since the work of Piaget (1952) have demonstrated that young children themselves often spontaneously invent ways to communicate their ideas and intentions regarding to numerosity (see Carruthers and Worthington 2006; Gelman and Gallistel 1978). Such spontaneous processes in children are premature processes of mathematising. In our own research over the past decades we could establish that triggering mathematising in young children, encouraging them to invent tools for communication, and questioning them about numbers are accessible and engaging activities for young children when they make sense in the context of their play. Fostering such processes constitutes a basis for later mathematising and communication *about* number.

Stimulating Early Mathematising

From a very young age children spontaneously invent ways to deal with the quantitative or spatial aspects of their environments, and communicate about these aspects with the symbolic expressions they have invented. The magnificent work of Carruthers and Worthington is pioneering and seminal here (see for example Carruthers and Worthington 2006). Through a huge wealth of observations they have revealed young children's potential of mark making and have produced evidence for the concurring evolution of children's meanings in matters of quantity, relation and number. Their situated interpretations of the children's marks and related narratives show that even at an early age children's mark making expresses the child's logic in dealing with quantity and number. The children organise their private experiences in symbolic ways that open ways for refinement. In other words, these children are really involved in mathematising (as defined above), even though their means of communication are not always conventional or drawn from the ‘official’ mathematical register. These children are definitely communicating their take on reality from a mathematical point of view.

Although Carruthers and Worthington have offered several rich examples of children using their own (self-invented) marks, or children using more conventional number symbols, or both, we need to explore these processes in greater detail to find out how children merge their own understandings with the disciplinary rule-based mathematising. For expanding our understanding of the process, we have to scrutinise the communication processes and their evolution in detail. We need more understanding of the ways how young children use the conventional mathematical symbols as codes for the expression of their personal notions of number and spatial relations, and how these evolve into mathematical concepts accepted by the wider mathematical community.

There are several pedagogical ways to get children involved in communication about numbers. One of them is asking questions that encourage children to reflect on their use, utterances or manipulation with numbers, or ask for arguments that explain their ideas. That is why such questions for children like “Are you sure?” can be very productive for the stimulation of mathematical thinking, even with young children (see for example van Oers 1996b). Figuring out answers to these questions in the context of children’s play contributes to the sense of these questions as it enhances children’s ability to participate more successfully in the play with other children.

In the next section some of the evidence will be presented for two phenomena in the development of communication *about* number, which seem crucial in the process of mathematical thinking development. The observational data were gathered and recorded in early years classrooms in a play-based curriculum in the Netherlands.

Improving Tools for Precise Reference in Communication

In their engagement in play, children often wish to find solutions for communicative problems they encounter, for example, when playing supermarket. When the teacher encourages children to solve play-related problems, children can often suggest possible solutions, try them out, share and discuss them. In other words, they start to *communicate* in order to generate and examine possible solutions.

An illustrative example comes from our own classroom observations, showing how an object (in this case a drawing) can be transformed into an object that opens and refines possibilities for mathematical actions. As a matter of fact, children are often challenged in their play to reflect on the adequacy of the symbols they use for the communication of their ideas about quantities.

One morning an early years teacher (grade 2; 5-year-old children) had told a story about a king who wanted a new castle. After the story the teacher talked with the children about castles. In this conversation the teacher showed the children pictures and tried to find out what the children knew about castles. She introduced a

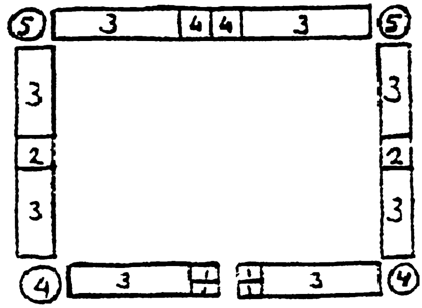

Fig. 1 Construction plan of a castle

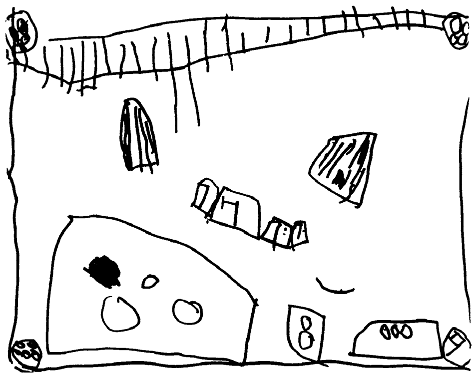

Fig. 2 Drawing of a castle by one of the 5 year old children

drawing of a ground-plan of the king's castle (see Fig. 1) and verified in a preparatory whole-class conversation whether all children correctly understood and could read the drawing. Many children had visited castles during their vacations with their families. Finally, the teacher invited a number of children (in dyads) to build a castle with different construction materials on the basis of the teacher's construction plan.

Following this construction plan, two boys (5 years old) had made a big castle together using blocks from the construction corner. Other children in their classroom showed interest and came looking at it. During their building activity, however, the boys noticed that they had not enough round blocks for the exact replication of the towers from the teacher's design. So they couldn't exactly copy the teacher's design and had to change their building to cope with the numbers of blocks available.

When the boys were finished, the teacher asked questions about their castle and invited the boys to make a drawing of it. That would be helpful for the communication and handy in case other children ever would want to build a castle like that. Note that the teacher actually invites the boys with this request to *communicate* as clearly as possible with future others about their castle. The boys found it a good idea and started drawing. One of the boys produced a drawing of their castle as represented in Fig. 2 (see also van Oers 2008 for further semiotic analysis).

Initially, it was a drawing with four walls with towers in the corners, very much similar to the construction plan the teacher had provided (with the walls, and the towers as small circles at the corners), but with new aspects added (e.g. some little

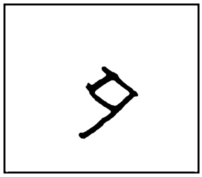

Fig. 3 A 5 year old's attempt at writing 'four'

houses in the inner court yard). However, in the course of the drawing the child's representation gradually evolved into a drawing that also included aspects of a frontal view (the battlements of the castle represented as stripes on the top line, and a few windows, the striped objects). Obviously, the drawing is a mixture between a frontal view and a view from above. It is still remotely related to the teacher's construction plan, but not a copy of it. It was the boy's own solution for the representation of the castle he had built with his peer.

Different analyses can be made of the boy's attempt to make a representation of the castle. For the present argument, however, I will only use this case as an illustration of the boy's struggle for perfection in the communication of the mathematical aspect of his drawing with the help of symbols. In the conversation between the boys, their attention was drawn to the construction of the walls and towers, and the boy who made this drawing (Fig. 2), wanted to indicate how many blocks should be put on top of each other, like in the teacher's model. He starts with the tower at the right (bottom) and copies the symbol from the teachers model (see Fig. 3).

He is, however, obviously not satisfied with it and shifts to an analogous inscription (based on the number of little circles equal to the number of bricks). By itself it is remarkable that the boy does not continue copying the symbols of the teacher to the best of his abilities, or doesn't draw the required number of bricks (as quadrangles). Instead, he invents new symbolic means (circles representing the quantities) for the communication of the required numbers of bricks. The boy continues with this drawing of quantities in the other towers (four circles for four blocks etc.) and in the polygons representing the walls. He draws four circles in the front left polygon (the big one), but immediately noticed that he drew one circle too much and he crosses out one of them (see black area in left corner below). By so doing, he shows that he is really performing a reflective mathematical act of making one-to-one correspondences: he thinks *about* the two quantities and their relationship, but especially about the adequacy of his symbols for correctly representing part of the situation.

The boy in this example obviously is looking for ways to communicate his knowledge of the quantitative aspects of reality (his castle) to others. As the conventional symbols were obviously too difficult for him to write, he invents new analogous means which open a way for him to refine his communication about reality. Without these means it would have been impossible for him to enter the world of meaningful mathematising, that is to say, accomplish a way of mathematising that makes sense for him and others. In this activity the boy was highly engaged in the communicative activity (he was eager to show the viewers the numbers of his drawing), he evidently followed a number of rules (e.g. a rule of correspondence between the number of blocks and the structure of his representation), and made use of his freedom by changing to new communicative means. This example of mathematical communication grew meaningfully out of the children's play. Inventing appropriate symbols to refer as precisely as possible to this number aspect of reality, is a necessary step towards communicating about numbers and an important step in the development of mathematical thinking, as it helps children to understand the function of the conventional mathematical symbols in due time too.

Reflection on the Properties of Numbers

When children master the conventional symbols for number as a way of reflectively referring to quantities (like in the previous example), it is important to reflect with them on their meanings and interrelationships, especially when we want them to discover new properties of numbers. Making representations (like the number line or other types of diagrams) can be usefully introduced to support communications about number (see van Oers 2012a). The following is an illustration of how a teacher introduces the notion of 'odd' and 'even' as a property of numbers. As a start she got children involved in a play of the post office and the mail man, making use of the fact that in the Netherlands the numbers of the houses are even at one side of the street and odd at the other. For preparing the mail man's round, children categorise the letters for the left side and for the right side of the street separately.

This is the situation:

Teacher with three 5/6 year old children (two girls; one boy), sitting at a table.

A big number line is displayed on the wall of the classroom, lining up the numbers on differently coloured cards: the numbers 10 and 20 on a yellow card, the other numbers either green (odd) or red (even). They had been talking about a post office play and about the work of the mailman.

There is a mailbox in the classroom that the children had made in the weeks before. There is also a chest with two columns of drawers, where the children can keep their own stuff like their drawings and other work. In the previous days the children have numbered all drawers of this chest with number symbols so that they can function as personal letter boxes as well.

The boy in this small group is asked to empty the mailbox of the classroom. He brings all the letters from the mailbox to the table where the teacher and the two girls are seated. The following conversation between the teacher and the three children ensued:

Conversation:[2]

Teacher: "How can we sort these envelopes? Do you know how to do that?" [to one of the girls].

[2]Thanks to Niko Fijma who communicated this event to me on a video clip.

Girl 1: "you have to put things together"
Teacher: "How can we do this with the envelopes?"
Girl 1: "you look at all the addresses and suddenly you see two of them that are the same, and you can put them together".
Teacher: "OK, so if the mailman sees 'Hoorn' [the name of the hometown of the children], then he puts everything with 'Hoorn' together in one group?"
Girl 1: "No, he looks at the addresses and puts the same addresses together"
Teacher: "All right". [The teacher suggests that the pupils sort the envelopes and emphasises that the children work together. The children discover, however, that all of their envelopes are addressed to children in group 3a. So there is only one group.] "What shall we do now?"
Girl 2: "We can put the numbers on the envelopes in different groups, as with the drawers"
Teacher: "Yeah! Yesterday we have numbered all the drawers of the chest. We had a red row of numbers, and a green one. Do you remember what the difference was?"
Girl 1: "they go in steps of two... I was number 1 and then you go to the right and I thought that Alex would be the next, but he wasn't. He was in the other row, below me"
Teacher: "Yes, Alex was in the row below yours, wasn't he? Each row made steps of two with the numbers. What was the difference between the two rows?"
... ...
Girl 1: "we had a row with the numbers 1–3–5–7–, and one row with 2–4–6–8–"
Teacher: "OK, but what was the difference between the two rows? Who can remember how we called the different numbers?"
Girl 1: "Odd numbers..."
Teacher: "Odd and..."
Girl 1: "and even"
Teacher: "do also you remember which of the number lines was even and which one odd?"

[Pupils look at the number line on the wall.]

Girl 1: "Red is odd"
Teacher: "Red is odd? Take a good look at the number line again". "We said yesterday that 2–4–6 were even... and which colour is that?"

[Children take a look at the number line on the wall.]

Children: "Red, yes red is even and green is odd, then"

[The teacher shows a red card with the word "even" and a green card with "odd".]

Teacher: "That may be handy when we are going to sort the envelopes. And to which group belong the yellow numbers? Do you know that, Collin?" [the boy].
Boy: "eh... Those are the numbers... eh..." [long pause]
Teacher: "Do they belong to the red group or to the green group? What do you think? Take a look again at the number line"

Boy: “To... the... green one”
Teacher: “To the green one? Why do you think so?”

[inaudible]

Teacher: “What do you think?” [to the girls]
Girls 2: “they belong to the red group”
Teacher: “You think these are even numbers? Why do you think so?”
Girls 2: [inaudible] “... it is between two greens”
Teacher: “Yes he is placed between to greens, isn’t it?”
Boy: “Do we have to put the even ones here on the table” [point with his right hand to the table], “and the red ones here” [points with his left hand to another point on the table]
Teacher: “Yes good idea” [The teacher puts the red and green card on the table, as suggested by the boy]. “So here we can put all even numbers and here all odds. OK, let’s do it. How are you going to do it?”

[Children distribute the envelopes; and with the help of the number line on the wall they can allocate all envelops in the even or odd group.]

Teacher: “Now we have to check the two piles; and put the envelopes in a correct order.”

[The envelope numbered “2” is already on the table.]

Teacher: “What comes after two?”
Boy: [looks at the pile of envelopes and says] “6” [which is on top of the pile]
Teacher: “We should make steps of two, remember?”
Boy: “4”
Teacher: “If each of you takes some of the envelopes to look for the numbers. . . .”

Children put the even and the odd group in the correct order. They know the even numbers by heart. To identify the odd numbers in line they sometimes mention the first one aloud, say the next in the ordinary number line softly for themselves and name the next one (spoken aloud) as the following odd number. The yellow numbers (10 and 20) are correctly allocated to the even group.

The children check the piles of envelopes by predicting the number of the next envelope that the teacher shows one by one. Finally the boy distributes all envelopes to the personal letter boxes of all children in their group.

In this project the teacher with the children are not using the numbers just as a communicative means to refer to quantities, but have constructed numbers as objects that can be talked about, reflected and refined with new properties (odd and even). The teacher’s goal was obviously not to try achieve a formal definition of even or odd, but to support children’s collaborative reflection on the properties of numbers. By talking about the numbers with the help of the number line, they discovered that the (order of the) numbers can be differentiated into two classes. Interestingly, in this stage the children have learned new words to refer to subtypes of numbers and relate these to the number line. They also noticed that these two types of number alternate in the number line. It would take a further activity of communicating *about* odd and even numbers to discover their distinctive properties. As we can see, a beginning understanding of ‘odd’ and ‘even’ is emerging when the children notice that odd and even are always ‘neighbours’. With this property they find out that the differently coloured numbers (10 and 20) should be even “as they are in between two odds”.

Conclusion

For the moment this chapter did not want to go beyond valid casuistic evidence for the claim that mathematical communication (*with* number representations, and *about* number properties), can be accomplished with young children, under the guidance of a teacher. Both processes in mathematical communication open ways for communicative refinement of mathematical symbols and ideas, and as such they represent early steps in the development of mathematising and children’s mathematical orientation in the world.

This chapter explored some of the processes in young children’s development of mathematising that may foster children’s mathematical orientation in the world. From the perspective of Cultural-Historical Activity Theory, this development is theoretically interpreted as a process of guided appropriation of mathematical tools for orientation in the world, especially for coping appropriately with its numerical, quantitative and spatial aspects. The conception of mathematical thinking development as a basically narrative and communicative process, draws attention to questions about the early stages in children’s communication regarding the mathematical aspects of reality. More specifically this leads to questions about how children build the tools for referring to these mathematical aspects, and how communication *with* these symbolic tools can be used for helping children to communicate *about* numbers and discover their properties.

According to the Cultural-Historical Activity Theory, play activities are the appropriate contexts for young children’s communication and their learning (see for example van Oers 2012b). Given the rule-governed nature of activities, participation in (playful) activities provides a basis for paying attention to mathematical (number-related) rules as well (van Oers 2012a). The degrees of freedom inherent in play activities, allows young children to invent their own (unconventional) ways of construction rules and means for reference to the numeral aspects of their play activity.

The research into children’s mathematical communications described in this chapter, was conducted in a play-based curriculum that has shown to be productive in triggering young children’s communication about number, especially when more knowledgeable others are involved for the guidance of the children in the perspective of mathematical thinking development (Fijma 2012). In this chapter, two classroom settings for young children in a play-based curriculum have been presented to demonstrate how Activity Theory works out in practice, and how these settings can be used as data sources for studying how young children improve their communication regarding numerical aspects of their reality. From a research point of view, the whole enterprise can be seen as a theory-driven construction of data for getting a deeper understanding of children’s communication about number.

The qualitative data-analysis (summarised above) supported the claim that communicating about mathematical aspects of reality can engage young children in mathematising and mathematical narratives from an early age. The studies showed that early mathematising may entail both reflectively constructing codes for referring as precisely as possible to number and quantity, and constructing numbers as objects to talk about for the discovery of their properties and for the refinement of the children's ideas and representations of number and quantity. Hence, in addition to communicating with number, our observations have demonstrated that 5-year old children can get meaningfully involved in activities of communicating *about* number.

However, for the enhancement of mathematical literacy more is needed than initiation in mathematics as a cultural domain and strengthening children's way of communicatively orienting in the world from a numerical point of view. Other developments must be promoted in the future (see also van Oers 2001). A topic for further concern is the problem of attitude. Learning to communicate about number will probably contribute only modestly to the enhancement of mathematical literacy, if no attitude for mathematising and discussing number problems is developed. Further study of the relationships between communicating about number, attitude and the play-based curriculum is required.

Within its focus on mathematical communication in young children, the conclusions drawn above might be promising, but should not obscure the fact that this is still only a limited outcome, when we aim at the development of mathematical literacy in the broad sense. At least four issues can be mentioned here that articulate the limitations of the research described above and that at the same time hint at topics for future research and approaches:

(1) The first pertains to the development of higher levels of communication for the improvement of mathematical thinking, i.e. communication about how to communicate validly and productively in the mathematical domain. Reflections about rules for communication about mathematical objects (such as 'be consistent', or 'define your concepts unambiguously') is an essential dimension of mathematical thinking. In the mathematical community such reflections are often also carried out in communication with mathematical peers (see for example Cobb et al. 1993). Until now, our studies did not directly address this issue, but deeper studies in this area are definitely needed, both with young and older pupils.

(2) Our studies focused predominantly on communication with/about number. Of course this is only one of the developments that may be conducive to children's mathematical literacy. Space is an equally interesting and important mathematical object that is accessible for young children, and that can be constructed and explored on the basis of symbolic representations. Both the invention of effective language codes to refer to space characteristics, and communication *about* space concepts may be an applicable distinction here as well. In addition to communication about number, space can also be a good stepping stone for the commencement of serious mathematising with young children (see Lehrer and Pritchard 2002).

(3) The research discussed above is based on observational studies that provide a valid basis for new ideas about young children's mathematical communication and introduction into mathematical literacy. Further qualitative research, however, is needed to deepen our understanding of the nature of children's mathematising as a way of orientation in a situation. Moreover, more in-depth analysis is necessary to reveal the structure of the communicatively constructed mathematical narratives (its topics and organised predicates—see van Oers 2006, for further elaboration of this approach), and to relate them to conceptual understanding. Moreover, further large scale quasi-experimental research is needed to test the relationships and to strengthen the evidence base. Some of this research has already been carried out (see van Oers 2010; Poland et al. 2009) but needs to be continued.

(4) The role of the teacher is not articulated clearly in the research presented here. Nevertheless, the teacher's role and her/his abilities to communicate properly with children for the broad enhancement of children's mathematical literacy, is essential. More research into the teachers' proficiency in mathematical communication and in bridging the demands of the mathematical community and the pupils' personal constructions, is needed here for a deeper insight in pupils' developing possibilities to communicate with/about number. A related issue regards the content and aims of the curriculum: how can teachers combine children's communication about number with the 'official' curriculum requirements that teachers also have to achieve? As Fijma (2012) has demonstrated, it is important in a play-based curriculum that the teacher has the curriculum requirements for mathematical learning always in mind, when collaborating in children's projects. With these contents in mind she should decide in advance which of these should be explored and practiced with children in the context of their playful activities. Our classroom experiences show that a mathematically proficient teacher can often quite easily identify the mathematical dimensions in children's play and turn these meaningfully into objects for further study and refinement which make sense for the children and their engagement in play. When this results in improved mastery or understanding, this contributes to the children's ability to participate in the sociocultural practice that they are imitating (like in the example of the post office and the mail man above). A basic pedagogical prerequisite here is that the teacher really believes in children's potential to mathematise with their own notions of quantity, space, relation etc., and to communicate about them with others. Communicating about number (or any other mathematical object) is not to be seen as an incidental special activity to be practiced now and then, but should be consistently embedded across the curriculum, in order to foster pupils' mathematical orientation in the world to a maximal extent.

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A Framework for Examining Technologies and Early Mathematics Learning

Kristy Goodwin and Kate Highfield

Background

Over the past decades there has been increased impetus to use technology in early childhood learning settings (Clements and Sarama 2002; Edwards 2005; Haugland 1997, Plowman and Stephen 2003, 2005; Yelland 2010). In addition, there is a wealth of new technologies and interactive multimedia and technology resources available for mathematics teaching and learning. However, research is yet to articulate and substantiate their use and impact on student learning (Highfield and Goodwin 2008).

In mathematics learning visual representations are essential for communicating ideas and concepts (Goldin and Kaput 1996) and new technologies offer new affordances for representation (Highfield and Mulligan 2007; Moyer et al. 2005). Advances in interactive multimedia and manipulable technologies provide learners with the opportunity to view and manipulate dynamic media and share external representations with ease. In mathematics, studies have established that computers provide "unique opportunities for learning" (Clements 2002, p. 174) and provide "greater scope to facilitate numeracy skills in young children." (Kilderry and Yelland 2005, p. 113).

Over the last decade there has been an exponential growth in the educational multimedia market, with a plethora of interactive technologies available for mathematics learning and teaching such as interactive whiteboards, educational software, iPads and robotics. However, as outlined above, the ubiquitous application of interactive representations in mathematics has not been well supported by a corpus of

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research to substantiate their effectiveness, particularly in early mathematics learning. There has been an assumed sense of superiority of interactive technologies, without a corresponding corpus of evidence supporting their cognitive value (Scaife and Rogers 1996).

In considering screen-based resources (interactive multimedia), it is unknown as to whether different pedagogical designs evoke qualitative differences in the kinds of representations students internalise. Little is known as to what students extrapolate from various dynamic, interactive on-screen representations. With some multimedia, learners often have to coordinate multiple and diverse representations, placing various demands on their cognitive infrastructure. This does not necessarily lead to better learning and may actually hinder students' learning outcomes. Screen embellishments and animations may also impose unnecessary additional cognitive demands on the learner. With other multimedia forms, the onus is on the students to develop their own multimedia representations, which requires a significant cognitive investment on each learner's behalf. Students may not engage with the representations in ways conducive to learning but instead, they may engage in superficial processing (Rogers and Scaife 1998).

Further, the impact of different multimedia designs on learning remains largely un-researched and this problem is further pronounced with young learners, where there is even less research. A systematic examination of the potential affordances and impact of the available mathematical multimedia on young students' learning is required to identify various multimedia attributes for mathematics learning. It is widely accepted that humans have a limited working memory (Baddeley 1986), thus instructional representations must be designed with the goal of reducing extraneous cognitive load. Multimedia design principles must be commensurate with how learners perceive and interpret the information presented to them on-screen.

The past decade has seen an increasing body of research on the application of screen-based technologies for mathematics teaching and learning (Clements and Sarama 2009; Heid 2005; Plowman and Stephen 2005; Zevenbergen and Lerman 2008). However, a significant proportion of that research examines screen-based tools (Highfield and Goodwin 2008) and the same depth of research is not present in mathematics learning with techno-toys. This means that in addition to the concerns outlined with interactive multimedia there are an increasing range of alternate technologies, such as simple robotics and techno-toys that provide alternate experiences with technologies. These additional tools may provide unique opportunities for mathematics learning, or indeed may add to the complexity of the technological and pedagogical landscape for young mathematics learners.

One specific example of an alternate technology is simple robotics. To date the research available on the role of programmable toys in mathematical development is restricted and often focuses on older children, with limited studies investigating their role in young children's mathematical learning (Janka 2008; O'Meara 2011; Stoeckelmayr et al. 2011). Additionally, it appears that this limited research has not been disseminated in such a way as to impact upon the professional practice of early childhood educators (Clements and Sarama 2004; Edwards 2005; Waters 2004).

It is clear that further research is needed to investigate the impact of technology in mathematics learning, with a focus on a broad range of technologies including screen-based interactive multimedia and manipulable toys such as simple robotics.

Studies on Early Mathematics Learning with Technology This section provides an overview of four studies conducted by the two authors (Highfield and Goodwin) as part of early research (Highfield and Mulligan 2007), their PhD theses (Goodwin 2009; Highfield 2012) and current research project (Goodwin and Highfield 2012). Each of these studies examined key technologies appropriate for early mathematics learning. Goodwin (2009) and Highfield (2012) both focus on the use of children's representation as evidence of mathematics learning. This section presents a brief overview of these studies, with exemplars from these projects provided.

Goodwin's (2009) study investigated young students' (aged five to eight years) use of a variety of interactive multimedia to develop their concepts of fractions. A classification scheme and taxonomy of interactive multimedia was established. Three classes of multimedia were grouped according to the ways in which the students interact with the representations: (i) instructive multimedia; (ii) manipulable multimedia; and (iii) constructive multimedia. With a specific focus on the impact and affordances of the three different types of multimedia on young students' concept image of fractions, the study also focused on how learners at the extremes of mathematical achievement used and responded to the multimedia.

Goodwin's (2009) design-based research study amalgamated a constructivist teaching experiment and a case study approach. The study was comprised of two iterations, involving a total of 86 students from three Kindergarten (the first year of formal schooling) classes and a Year One (the second year of formal schooling) class. Both iterations examined the influence of an intervention employing the three different types of interactive multimedia previously listed. Iteration One involved one Kindergarten and one Year One class who participated in a four-week intervention and constituted a pilot study for the next iteration. Iteration Two involved two Kindergarten classes: an intervention class and a comparison class, in which a 12-week intervention was implemented. Data sources included students' drawings collected before, during and after the intervention, a multimedia fraction assessment administered before and after the intervention, digital screen and audio recordings of students' computer work and video-stimulated recall interviews to ascertain students' recall of the multimedia content. Case study data from four students in each intervention class (two low-achieving and two high-achieving students) included digital screen recordings and video-stimulated recall interviews. A mixed method approach (Creswell and Clark 2007) to analysis was adopted, incorporating both qualitative and quantitative approaches. Innovative data analysis and reporting techniques were utilised to provide rich and authentic data to support the themes related to the impact and affordances of the interactive multimedia. Data analysis involved coding screen recordings and interview data using *Studiocde* software. Triangulating case study data from the analysis of post-lesson drawings, screen recordings and video-stimulated recall interviews provided a more complete description of phenomena and promoted greater reliability.

Results from Goodwin's (2009) study indicated substantial improvements in the intervention students' drawings and multimedia fraction assessments. All students in Iteration One showed improvements in terms of their concept image of a fraction, as projected at the post-intervention assessment point. In Iteration Two, the intervention students showed more advanced and sophisticated concept images of fractions than the non-intervention sample at three assessment points (pre-, during- and post-intervention). In both iterations the students' concept image of fractions developed between the pre- and post-intervention assessment points, becoming more sophisticated in terms of the level of structure, mathematical concepts and use of symbol notation. Intervention children, who used interactive technologies, could successfully depict multiple representations and showed evidence of advanced mathematical ideas such as non-unit fractions and equivalent fractions and counter examples after the intervention. Many intervention students' concept images also included alternative, 'non-schooled' depictions of fractions and increased use of formal symbol notation ah the post-intervention assessment point.

Goodwin's (2009) analysis of the intervention data documented differences between the three types of interactive multimedia in terms of the concept images projected. Analysis of post-lesson drawings suggested that the students demonstrated the most developed and advanced representations after using manipulable multimedia. There was a higher incidence of students' recalling idiosyncratic, superfluous and non-mathematical details and displaying 'crowded' images after using instructive multimedia and fewer, less developed representations generated when using constructive multimedia.

Throughout this work, case study data corroborated findings from the intervention data that suggested that manipulable multimedia had the greatest impact on students' concept image. Each classification of multimedia offered distinct affordances in terms of the frequency of the representations the students observed or created, the ease of experimentation with the representation and the levels of student engagement. The importance of the provision of instant feedback and evidence of multimedia design principles were also reflected in the case study data.

A standardised mathematics assessment, *I Can Do Maths* (Doig and de Lemos 2000), was administered to before the intervention to identify 'high-' and low-achievers'. This enabled the researcher to determine if high- and low-achievers used and responded to interactive multimedia in different ways. Differences were also noted between how the low- and high-achievers used the multimedia and recalled what had been presented. The low-achievers had a greater tendency to focus on the superfluous and surface details embedded in the multimedia resulting in superficial processing of the multimedia. In contrast, the high-achievers were adept at selecting the salient information from the multimedia to construct effective mental models.

The second study, by Highfield (2012) examined a manipulable form of technology: simple robotic toys (Bee-bots and Pro-bots). This work was pertinent, given the ubiquity of young children's engagement in technology and consistent research focus on screen-based tools. These programmable toys offer tangible interactions and provide opportunity for young learners to engage in a range of mathematical concepts and processes as they input, execute and reflect upon programs. This study focused on young children's engagement, representation and dynamic manipulation of tools as they engaged with these toys in play and teacher directed tasks.

Highfield's (2012) study followed 31 children, aged three to seven years as they engaged in a twelve-week program in their classroom environment. Children from two contexts participated, a prior-to-school setting and a nearby primary school, with three groups of children: Three-year olds, Four-year-olds and a Year One class. Each group of children completed five phases of the study, including pre- and post-interviews, a training session at the Macquarie ICT Innovations Centre, and a sequence of teaching and learning episodes.

This work also drew on design-study methodology (Gravemeijer and van Eerde 2009) and adopted multiple layers of analysis, with children's mathematics learning examined through video data of classroom engagement and play and through drawn representations. Video data were analysed to explore children's use of gesture, action, dialogue and representations of programming. A multi-faceted theoretical approach exemplified the interconnection between the development of semiotic systems, incorporating speech, gesture, embodied action and representation of dynamic concepts.

Highfield's research highlighted the affordances of simple programmable toys in mathematics learning and problem solving. Data indicated that children explored a range of mathematical concepts and processes including number, unit iteration, estimation, angle and geometry concepts. Further, children engaged in meta-cognitive processes integrating planning, prediction, observation, reflection and revision as components of problem-solving. Children's strategy use in these tasks, such as acting out with the toy and using symbols and gesture, provided insight into emergent mathematical thinking.

A third study examined screen-based resources with a specific focus on virtual manipulatives, such as those available through the National Library of Virtual Manipulatives (accessible through http://nlvm.usu.edu/). Highfield and Mulligan's work (2007) examined how web-based tools provide a unique representational opportunity, creating a dynamic, virtual representation of a concrete material. This small-scale study explored virtual manipulatives and open-ended drawing software as tools in mathematical patterning with pre-school children.

This research was conducted as a constructivist teaching experiment, (Hunting et al. 1996) with three dyads of preschool children, aged between four and five years. Integrating elements of design study, this approach allowed for teaching episodes to be constructed and scaffolded systematically, with revisions occurring based on children's progress and engagement with pattern-eliciting tasks. Each dyad was assigned to one of three learning modalities using: concrete materials (such as blocks, counters, animal pictures, stamps, paint, pencils); a combination of concrete materials, dynamic interactive software (Kidpix) and virtual manipulatives (virtual Pattern Blocks); or, dynamic interactive software (Kidpix) and virtual manipulatives (Pattern Blocks). Once allocated a modality, children completed six, 40-minute teaching and learning episodes, conducted by the researcher over a 4-week period. Children engaged in pattern-eliciting tasks such as making wrapping paper, with tasks and resources matched so that comparison between traditional modalities and technological tools was possible.

Within this project Highfield and Mulligan (2007) demonstrated young children's ability to develop skills in simple patterning over the four-week period, with no significant differences evident when comparing children's patterning skills while using traditional materials or technological tools. Data did however indicate that children were motivated to engage with the dynamic interactive software and virtual manipulatives when patterning. Extended engagement with technology meant that children using technology were more likely to experiment with representations, creating an increased number of patterns and transformations when compared with children using concrete materials. In addition, the technological tools enabled increased representational detail and accuracy.

The final study presented within this chapter was conducted by both authors, Goodwin and Highfield (2012). This work examined Apps for learning, and was pertinent given the increasing popularity of touch devices such as iPods and iPads. At present there is a preponderance of Apps for these devices that are designed for young children and are marketed as 'educational'. Currently, there appears to be no review process involved in classifying Apps as 'educational' and as a result many Apps are strategically placed by developers in the lucrative 'Education' section of the iTunes App Store by developers. However, despite the plethora of Apps currently available for young children, research has failed to keep pace with the growth in this technology, with limited systematic analysis of educational Apps and those designed specifically for young children. This research project outlined a content analysis of the paid Apps that are currently available in the 'Education' section of the iTunes App Store. The findings of this study provided key information for both parents, teachers and App developers in the selection, use and design of Apps.

Within this study Goodwin and Highfield (2012) conducted an analysis of the "Top Ten" paid Apps located in the 'Education' section of the iTunes App store at four different points in time (six-monthly intervals) from April 2010 to October 2011. Data were obtained for three countries: United States of America, United Kingdom and Australia and Apps ($n = 360$) were coded using the following characteristics: age, subject area and classification of pedagogic design. In 2012 (Highfield and Goodwin under review) these data were revised to include two additional collection points, increasing the analysis to be over six intervals (April 2010 to October 2012).

In findings that were similar to Shuler's work (2012), Goodwin and Highfield's (2012) analysis, revised for this chapter, found that 29 % of the top ten Apps were designed for toddlers, 24 % for elementary children 13 % for secondary education. This study aggregates data for all three countries and shows a large proportion (34 %) were classified as 'Multi-age' with the App classified as suitable for a wide age range of students such as preschool and elementary children. Classification by content presented demonstrated the areas of Literacy (21 %) and Science (19 %) as the most common subject areas represented in Apps analysed, with Apps addressing multiple curriculum areas (such as numeracy and literacy) representing 18 % of the content. While many Apps embed mathematical processes, such as scoring and problem solving in game play, Apps that focus on this key area appear under-represented in the 'Education' section, with only 15 % focusing specifically

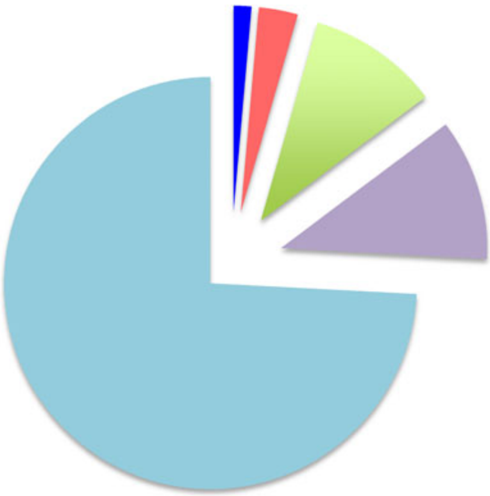

Fig. 1 An analysis of the "education" section of the App store, classifying Apps by pedagogic design (April 2010 to October 2012)

on mathematical content. Creative Arts are also limited in the 'Education' classification with only 6 % of Apps classified as focusing on this curriculum area.

Data analysis classifying Apps by pedagogic design affords pertinent data, highlighting a predominance of "instructive" Apps, with 85 % of 'Educational' Apps classified as Instructive, or as combining Instructive and Manipulable design pedagogies. Of particular note is the limited presence of Constructive Apps in the 'Education' classification. Here only 4 % of Apps were classified as Constructive or Constructive/Manipulable. Figure 1 provides a visual overview of analysis of popular 'education' Apps by pedagogic design.

Additional analysis indicates that these Apps are available in other sections of the App Store such as 'Apps for Kids' or 'Entertainment', rather than classified as 'Education'. This classification is intriguing and perhaps implies a diminished understanding of the educational potential of open-ended learning and representational tools.

Re-framing Current Research Using Analysis by Pedagogic Design

While each of these studies present could be seen to outline disparate examples of current research each can be re-framed as having unique affordances for mathematics learning when re-conceptualized in light of their pedagogic design. The following section outlines current classifications of educational technologies and then outlines Goodwin's (2009) classification of educational technologies.

Numerous authors (Clements and Nastasi 1992; Handal and Herrington 2003; Hosein et al. 2008; Hoyles and Noss 2003; Sarama 2003) have presented taxonomies that classify various types of educational software. However, there appears little consensus as to the most appropriate classification scheme. This is further compounded by the fact that many of the classification schemes and taxonomies become irrelevant as the technologies they were describing developed, become more complex or were superseded by technological developments.

Previous classification schemes have not taken into account how different tools encode and display mathematical ideas in different representational forms. Thus, most of the existing taxonomies and classification schemes have focused on the functionality of the software in terms of what the learner can do with it (Kurz et al. 2005). Students interact with different multimedia representations in distinctive ways to make sense of and integrate the representations into their cognitive infrastructure (Sedig 2004). In fact, there are no known frameworks that systematically analyse the way in which multimedia representations are designed and how their design impacts on students' understanding of the representations.

Whilst multimedia tools have shown the potential to improve mathematics learning (Atkinson 2005; Sedig et al. 2003; Clements et al. 2008), there does not exist any systematic way of classifying how learners engage with mathematical, multimedia representations. Scaife and Rogers (1996) highlight the paucity of research on the cognitive value of representations, especially those contained within multimedia applications. Given that different types of multimedia exist, as the previous classification schemes have identified (Handal and Herrington 2003; Kurz et al. 2005), a prescriptive taxonomy would help to identify how learners interact with and respond to different multimedia. Scaife and Rogers (1996) call for a systematic approach to evaluate the merits of different types of on-screen representations, with an explicit focus on how students cognitively interact with them. This would assist multimedia designers develop appropriate interaction techniques and design characteristics in future products. It would also enable teachers to design appropriate learning activities to complement learning experiences with multimedia.

A Classification of Interactive, Mathematics Multimedia for Young Learners

The classification scheme presented in this chapter specifically describes, classifies and seeks to evaluate mathematical multimedia, with the particular focus of analysing the instructional design considerations in relation to the way the representations are presented to the learner. The genesis of this scheme was to overcome limitations of previous taxonomies, by describing the unique affordances of different interactive multimedia. Whilst this evaluative framework was established to analyse the available multimedia specifically designed for young learners, the framework could be equally applied to multimedia designed for older learners and, possibly, disciplines other than mathematics.

Three broad classifications of interactive multimedia are proposed, as exemplified by Fig. 2: instructive, manipulable and constructive multimedia. This scheme

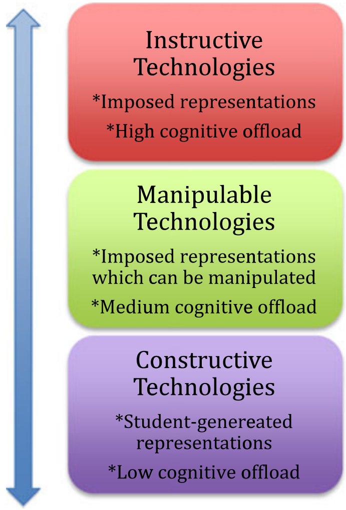

Fig. 2 A continuum of the pedagogic design of interactive technologies

extends the theoretical perspectives in the field of learning with interactive multimedia, by presenting a framework that can be applied to a range of digital technologies and interactive media. The classification scheme is based on the design features of the interactive multimedia, with a particular focus on the learner's locus of control over the representations presented on screen. The classification scheme also considers the type and level of cognitive demand and interactions afforded by the multimedia. The lines of demarcation between each of the classifications presented in Fig. 2 are not fixed. The classification scheme does not suggest that one design approach is superior to another as each particular representational mode has unique utilitarian functions that may be suitable at different stages of the learning cycle. Exemplars, arising from the aforementioned studies, are presented for each of these classifications and are detailed in the following sections.

1. Instructive Technologies At the top of the continuum in Fig. 2, are applications that are classified as instructive. These applications are based on a behaviourist theory of learning that assumes that knowledge can be directly transmitted to the learner. Such applications rely on reward, repetition, regular review and feedback loops and contingent increments of difficulty to teach various skills (Atkins 1993). Representations of concepts are essentially imposed on the learner. These tools promote procedural learning and are based on the philosophical assumption that knowledge can be presented symbolically and learned in a linear fashion. The learners perceive messages encoded in the medium and sometimes interact with the technology (Jonassen 1994). A fundamental tenant of this type of software is that an "expert", the designer constructs the screen representations that are presented to the student. Software applications adhering to this classification, base their learning experiences on a stimulus-response-reinforcement model. Students are required to master and replicate knowledge through closed, pre-programmed learning tasks, usually using the stimulus-response format: the software is designed to compare student input with a pre-determined answer.

Drill-and-practice CD-ROMs are a prime example of instructive multimedia. These CD-ROMs elicit homogeneous responses from users via imposed tasks. The market for such educational CD-ROMs expanded rapidly in the 1990s and has recently stagnated because of the ease through which materials can be now disseminated on the Internet (Buckingham 2007). However, educational CD-ROMs are still a popular choice amongst educators and parents, particularly with younger learners where there is a prevalence of age-related CD-ROMs designed to meet curriculum standards. Described as "shovelware" (Buckingham 2007, p. 129), educational drill-and-practice CD-ROMs have been criticised for their attempts to "jazz up the curriculum with a superficial gloss of kid-friendly digital culture" (Buckingham 2007, p. 136). Interactivity is often superficial, limited to animated objects that can be activated by the learner clicking on an icon or reactive interactivity that results from the learner entering a correct pre-determined response.

In relation to the cognitive investment required by the learner, instructive multimedia generally demand the least amount of the learner's cognitive energy of the three classifications. Typically, the students assume a passive role when using instructive multimedia as they do not have to expend much mental effort to process the information conveyed on-screen. Interactivity is often restricted to surface level interactivity (Aldrich et al. 1998; Evans 2007; Inkpen 2001; Sedig and Liang 2008; Triona and Klahr 2003) such as clicking or dragging a correct response.

Exemplar of Instructive Technology One example used within Goodwin's (2009) study of instructive technology is Galaxy Kids. Maths: CD-ROM (Published by Sunshine Multimedia 2005). Differences were noted between how the low- and high-achievers used the various multimedia and recalled what had been presented. When using instructive technologies the low-achievers had a greater tendency to focus on the superfluous and surface details embedded in the multimedia, resulting in superficial processing of the multimedia. The inclusion of idiosyncratic details, such as actions and attributes of the on-screen character, referred to as an 'animated pedagogical agent' (APA) in children's representations were most frequent after using instructive technologies. In contrast, the high-achievers were adept at selecting the salient information from the multimedia to construct effective mental models of fractions, in this instance.

As exemplified by Figs. 3 and 4, the same student responded differently to various interactive multimedia. After the instructive technology "Hydroslide" (Galaxy Kids Maths CD-ROM), the student's post-lesson drawing (Fig. 3), included non-mathematical attributes such as the water slide and the APAs called "Digits". However, the same student's post-lesson drawing lacked evidence of an awareness of equal partitioning, despite this being the instructional focus of "Hydroslide". In contrast, the same student, at a later point in the intervention, completed the drawing shown in Fig. 4. This drawing was completed after the student had used the manipulable technology "Fraction Fiddle: Tool". Figure 4 reflects an understanding of equal-partitioning, formal symbol notation and a basic depiction of equivalent

Fig. 3 A student's post-lesson drawing of "Hydroslide" (instructive multimedia)

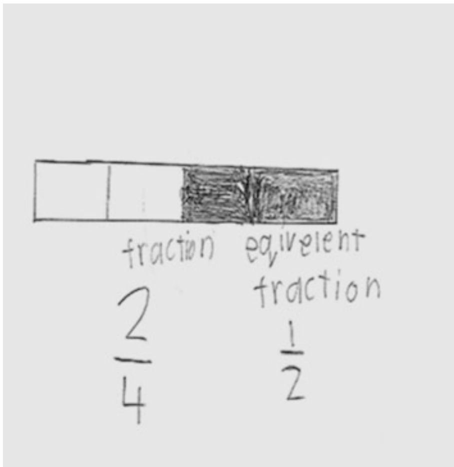

Fig. 4 A student's post-lesson drawing of "Fraction Fiddle: Tool" DLO (manipulable multimedia)

fractions. It is important to acknowledge that the two multimedia tasks described above, were focusing on two different concepts, which prevents any direct comparisons from being made. However, it appears that the manipulable technology, which was devoid of superfluous and irrelevant embellishments may have supported the learners' conceptual understanding. In contrast, the instructive technology, with its highly contextualised representations and the inclusion of animations, sound effects and characters, was less successful in supporting the development of fraction concepts.

The authors posited that the animations and characters contained within the instructive technology, may have detracted the learners' attention away from the embedded mathematical concepts within the CD-ROM. The learners, particularly the low-achieving students were hindered in their recall of mathematical features as their cognitive resources were directed towards processing non-essential information. These findings support the previous work of Mayer and Moreno (2002) and

Mayer (2001) who have also shown that the redundant use of embellishments compromises students' working memory and adversely impacts on their cognitive load.

Screen recordings from Goodwin's (2009) study also exemplified how young learners sought instant feedback provided by the APA in the instructive technology. Many of the students' verbalisations recorded whilst using the CD-ROM indicated that the students were noting the frequency of correct responses as demonstrated by the APA dancing or providing a 'thumbs-up' gesture. One student stated, "That's so cool. I got Number Cruncher [APA] to the net level. He's free. He can escape the dungeon." This particular child was focused on achieving the result of enabling the APA to complete the assigned task, but there was no discussion about the embedded mathematical content, which involved placing half the number of rocks into a container to catapult the APA to another level within the dungeon. In the video-stimulated recall interview this child was unable to explain what they had learned in the previous lesson, other than to recall how they had helped Number Cruncher.

Although this chapter only provides one example additional data in Goodwin's thesis (2009) enables the inference that the exclusive use of instructive technologies may not afford optimal mathematical learning for young students. Young learners need to identify the salient ideas and key mathematical concepts contained within instructive technologies through rich discourse with their peers and teachers after using these types of resources. It is imperative that teachers do not falsely assume that young children have mastered the mathematical content embedded in the instructive technology especially if there are other distracting elements.

2. Manipulable Technologies The second classification of software and interactive technology is termed manipulable technologies (Fig. 2). This type of manipulable technology allows for guided student discovery and experimentation, but within a pre-determined representational context. The symbolic and iconic images are often presented to the student, but these can be instantiated and altered on the screen by user input. Whilst the representations are pre-imposed on the student by an "expert", students have an opportunity to manipulate the representations and test new configurations and ideas. The availability of manipulative variables allows learners to interact with and gain meaning from the interactive tools. In this sense, the computer acts as a "hypothesis testing learning environment" (Kong and Kwong 2003, p. 138). The student must interpret and purposefully interact with the screen representations. These programs are more sensitive to students' partially formed mathematical responses and may allow for the development of alternative representations as they mediate the cognitive capabilities of the learner (Hoyles and Noss 2003).

The cognitive effort elicited by manipulable technologies is greater than applications classified as instructive multimedia, but possibly less than those tools within the constructive multimedia category. Manipulable multimedia may reduce the amount of cognitive effort required to generate a representation while allowing the learner to direct their cognitive energy and conscious attention towards understanding and internalising the mathematical representations on screen.

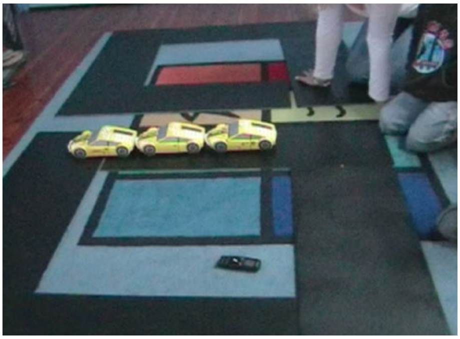

Fig. 5 Teacher using Pro-bots to measure the track in twin road task

Exemplar of Manipulable Technology Simple robotics present an example of manipulable technologies. Used throughout Highfield's (2012) work these tools offer a limited range of programming possibilities. In programming the robot the child must understand the available movements, the programming interface and then must enter a program. Children then often observe and reflect on the program, revising their attempts in a cyclic process. Here the manipulable tools are seen as promoting opportunity for reflection and revision of thoughts. Multiple semiotic systems used in processing and then representing movement provide insight into children's understanding.

In this example the children (aged four years) worked to program simple robotic toys (Pro-bots) around square roads. Pro-bots use a simple user interface described in Highfield (2010) to enter and execute programs of movement. The task outlined in this example was one of many (outlined in Highfield 2010). Here to move the Pro-bot around the square path the children were required to input four steps on each side, then a turn, repeating this to complete the square. As a class the children watched as their teacher measured the road using the Pro-bots as a unit of measure, as can be seen in Fig. 5. Following this the children worked in pairs to solve the problem and successfully move the toy around the path.

The children also used chalk to indicate step length, drawing symbols on the pathway. One child began by using chalk to mark many steps (Fig. 6). After discussing how many steps he needed and re-programming the toy the child revised his problem-solving strategy (Fig. 7).

The boy used a symbol system to plan his pathway with the robot. His initial use of tally marks was modified to use arrows that are adapted from (or resemble in some way) the arrows seen on the Pro-bot itself. This task presented an opportunity for the children to estimate, measure and program the Pro-bot to move around a square track using the pre-set steps on the toy. This presented an opportunity for the children to demonstrate more planning and problem-solving. By planning their actions the children engaged with geometric concepts, such as the attributes of a square, including four sides of equal length and corners at a 90° angle. Further, the children engaged with these concepts concurrently with dynamic concepts, such as the robot rotating 90°, and each side requiring four steps.

Fig. 6 The child's initial representation of many steps for the road task

Fig. 7 The child's second representation using arrows to indicate steps for the road task

3. Constructive Technologies At the other extreme of the continuum (Fig. 2) is constructive multimedia. As the name suggests, such software is based on contemporary adoptions of constructivist approaches of teaching and learning and provides learners with the opportunity to generate their own mathematical representations. These types of software are based on the assumption that technology can be used as "cognitive learning tools" which can be employed to facilitate learning and support the thinking processes of learners (Jonassen 1994, p. 62). Hence, the technology functions as an expressive tool. In the current classification scheme, the term 'constructive multimedia' refers to technologies that allow learners to create multimedia artefacts. Hence, the learner constructs the representations using multimedia tools.

This type of technology provides opportunities for students' intuitive understandings to be made explicit. The learner uses the available digital tools inherent in the software to construct mathematical representations. Hence, these tools engage learners in meaningful learning activities that support critical and reflective thinking about concepts. These tools assist in providing insights into students' conceptions and provide unique opportunities for mathematical modelling and expression (Noss and Hoyles 1996). The software, in this instance, amplifies the students' learning, making explicit their mental models and levels of conceptual understanding. Modifiable graphics enable students to easily create their own multimedia representations not possible with inert media (Clements 1999). Further, many of these tools allow representation and, as young learners can save and re-visit these tools, may also promote reflection on learning.

Constructive multimedia programs demand a significant cognitive investment on the learner's behalf, as the onus is on them to generate the representation. As a result there is a low level of cognitive offloading, as the technology assumes some of the cognitive load for the learner. Effectively using these tools to convey conceptual understandings requires more sophisticated cognitive skills and a significant cognitive investment on the learners' behalf than more instructive multimedia. It is possible that learners may expend too much mental effort manipulating and selecting the digital authoring tools and thus, may detract from their learning.

Exemplars of Constructive Technology Constructive Technology Exemplar—2Simple software. In Goodwin's (2009) and Highfield's (2007) study the participants also engaged with constructive technologies. One example of this (arising from Goodwin 2009) was 2Create a Story (2 Simple Software 2006) used to create a multimedia fraction story. When using this tool, the onus was on the learner to construct the representation, as there were no representational models provided, as there were with the instructive and manipulable technologies.

The constructive technologies provided two key affordances for young learners: (i) they could externalise their thinking; and (ii) they could compensate for their developing fine motor and literacy skills. Using 2Create a Story (2 Simple Software 2006), the Kindergarten students were able to create a digital artifact with their own representations, symbol notation, and verbally explain their drawing. The computer mouse, in conjunction with the on-screen drawing tools, enabled the young learners to easily create a digital artifact that was indicative of their understanding of fractions. They were able to experiment and manipulate representations (they were unable to do this with the instructive technology used but were easily able to do this with manipulable technology). This ensured that their conceptual understanding of fractions was not constrained by their fine motor and/or literacy development.

The open-ended design of the constructive technology allowed for students to depict 'counter examples' of fractions, as shown in Fig. 8. Counter examples are described as representations that challenge conceptual understanding (conflict), to show why some conjectures and representations are false (Liz et al. 2006). In this study, counter examples were considered to be students' intentional depiction of an incorrect representation of a fraction, with an accompanying icon or

Fig. 8 A screen capture of a student's depiction of a counter example, using "2Create a Story" (constructive multimedia)

comment to signal that the representation was incorrect. Counter examples were also considered to indicate understandings of advanced fraction concepts. Figure 8 is an example of a counter example. The student formed the notion that half of an object needed to be two equal-sized pieces and had applied this idea to partitioning a rocket ship. There was no other multimedia activity, used throughout the research study, where a rocket ship was used to depict a half. Hence, the constructive technology allowed the child to demonstrate this sophisticated understanding of fractions, in a way not possible with other types of technology.

Similar findings were seen in Highfield and Mulligan's (2007) research, where constructive technologies enable ease of representation, representation of sophisticated concepts and prolonged engagement. Here these open-ended tools facilitated increased engagement in mathematical thinking and opportunity for more advanced representation.

Discussion and Conclusions

Whilst there is growth in the availability of technological infrastructure and interactive multimedia for early mathematics learning, there is a dearth of research exploring their effectiveness. Existing literature has called for further research to examine the impact of new technologies on young students' mathematics learning (Clements and Sarama 2002, 2004; Highfield and Goodwin 2008). The studies reported in this chapter have supported and extended current research by revealing that interactive multimedia has a substantial impact on young students' development of basic mathematical concepts. In addition, these studies provide evidence that different multimedia offer unique affordances for learners, in terms of their unique design attributes.

The studies presented in this chapter also challenge the widespread belief that young students are incapable of dealing with complex mathematical concepts.

Rather, the findings support previous research that young students are capable of dealing with powerful mathematical ideas (Ginsburg et al. 1999; Perry and Dockett 2008). These studies highlight the representational opportunities that technologies provide. Further, they highlight the dynamic presentation of information and the dynamic manipulation of materials as providing access to advanced mathematical concepts.

The cumulative data from these studies highlight the potential benefits of interactive technologies in early mathematics learning. New representational opportunities, afforded by interactive multimedia and digital technologies allow young students to explore and manipulate mathematical ideas in ways not previously conceived with more traditional teaching approaches and concrete materials. In turn, young children are able to explore more complex and advanced concepts than those proposed by traditional curricula. Goodwin's (2009) comparative analysis outlined substantial benefits of *manipulable* and *constructive* interactive technologies in early fraction learning. Students' representations after using *constructive* and *manipulable* interactive technologies in Goodwin's (2009) work showed more advanced and sophisticated concept images of fractions when compared to a traditional curriculum. Further, the use of these interactive tools enabled students to depict multiple representations and reflect and revisit work (Goodwin 2009; Highfield and Mulligan 2007). Within each of these examples children's active cognitive engagement enabled them to explore sophisticated mathematical content. Here technologies enabled mathematics learning beyond what is frequently encountered in traditional curriculum. In addition, Highfield (2012) demonstrated the potential affordances associated with simple programmable toys for problem solving, spatial and geometric concepts. These robotic toys provide a further example of *manipulable* technologies as a non-screen based tool.

Whilst dynamic, on-screen representations provide unique opportunities for young learners in terms of developing mathematical concepts, further research needs to explore how the pedagogic design of interactive technologies impacts on their potential to support young children's learning. As Goodwin's (2009) study exemplified, the inclusion of superfluous details such as animations and extraneous sound effects, as are typically included in *instructive* multimedia, place demands on the students' cognitive load. Students' attentional resources are often diverted to processing the redundant information included in screen embellishments and not on the embedded mathematical content. This limitation was more evident with low-achieving students, than high-achieving students, as they have a tendency to focus on the superfluous inclusions, hampering their understanding of the mathematical content. Hence, a closer examination of the pedagogic design of multimedia needs to consider its impact on young students' cognitive load.

Examinations of technologies for early mathematics learning, when presented within this framework, highlight the need for a range of pedagogic designs. Further critical analysis of learning afforded by the differing technological designs is needed to inform teacher pedagogic decisions. This has particular implications for teaching practice, where teachers must consider the pedagogic aim of their lesson sequence, prior to selecting technological based resources to support these goals. For example *instructive* technologies may provide opportunity to develop fluency in mathematical computation (e.g. factors to ten). Using these technologies the child could be presented with different combinations of numbers and be asked to provide a correct answer, with the real-time feedback enabling children to practice skills. Alternately, *constructive* technologies may enable learners to represent multiple alternate pathways of learning such as documenting strategies for addition rather than practicing pre-set tasks. Here, the teacher's purposeful choice of technology would need to be carefully aligned with their pedagogic goal.

Goodwin and Highfield's (2012) work outlines the dominance of *instructive* design for young children in new technologies, such as iPad Apps. Here again, judicious and purposeful selection of tools for specific mathematics learning is needed. These findings question the assumption that technology and interactive multimedia are always beneficial for learning (Goodwin and Highfield 2012).

Significant implications for future research arise from these studies, with further work investigating each of these pedagogic designs needed to effectively examine their potential affordances for young mathematics learners. Given that this age group is laying essential foundations for future mathematics learning it is imperative that the research agenda focuses on optimal technology use the early years. Further, dissemination of this research to teachers is needed, with additional research examining teacher pedagogic decisions also needed.

The studies reported in this chapter have assumed that students' language (used in interviews) and drawn representations are evidence of their learning. However, future studies utilising new data collection technologies such as digital brain imaging would be advantageous in examining the cognitive processes of students using interactive technologies. In addition, given the significant growth in touch technologies, such as iPads, further research is required to confirm whether the findings outlined in these studies are replicated with these new devices.

Implications for Teaching and Learning

The technological landscape is changing rapidly and new devices, applications and software are constantly evolving. As such, teachers need ongoing access to professional learning. Initial teacher qualifications alone are not sufficient for this technological society and need to be complemented by further opportunities for learning. Professional learning sessions need to have a dual focus: (i) they need to develop teachers' familiarity with various technologies (technological knowledge) and (ii) they must also focus on how to embed these technologies in sound pedagogical frameworks (pedagogical knowledge).

A consistent finding from both the Goodwin (2009) and Highfield (2012) studies relates to how young students find it difficult to interpret and process extraneous information contained within multimedia representations. Therefore, teachers must implement explicit strategies to ensure that young students develop the ability to locate the salient aspects within multimedia representations and avoid focusing on non-essential aspects. Structured follow-up questions and/or activities may assist the students, particularly the low-achievers, focus their attention on the mathematical aspects contained within the representation.

Using a design study approach, the Goodwin (2009) and Highfield (2012) studies also revealed how the design of the lessons in the interventions was an effective format when using technologies with young children. Common teaching practice often focuses on isolated and stand-alone use of technology, with a brief introductory session focusing on the technical and procedural aspects of the technological tool, followed by individual, pair-work, or small-group use of the multimedia. There is an emphasis on task completion, with little opportunity to discuss the students' learning. However, in these studies, discussion sessions were an essential component of the lesson sequence as it enabled the students to share their discoveries and showcase their work and seek peer assistance for difficulties. Teachers should ensure that a plenary, sharing component always follows individual or group use of multimedia.

The findings of the current study have exemplified differences in the way high- and low-achieving students use and respond to different multimedia and interactive technologies. It is paramount that teachers consider the students' prior knowledge when using any technology to align pedagogical approaches with students' needs. Hence, the impact and affordances are different for students at the extremes of achievement. This is not to suggest that *instructive* multimedia should not be used with low-achieving students. Instead, it is imperative that teachers ensure that after using *instructive* and *constructive multimedia* that plenary sessions are conducted to focus students' attention on the mathematical aspects of the multimedia. Alternatively, teachers can assign tasks for learners to complete during or after using interactive multimedia, to ensure that students focus on the intended learning in the multimedia. This is sound pedagogic practice that would benefit both high- and low-achieving students.

Implications for Further Research

There is a dearth of research that explores how young children use and respond to various technologies. Given that there has been an exponential growth in this sector, in terms of the availability of these resources for young learners, there is a dire need for more research to be concentrated in this area. The studies presented in this chapter provide evidence to indicate that young children's early mathematical learning can be enhanced through the use of various technologies, but they have also suggested that the design of the technology can have an adverse effect on learners.

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The Role of Picture Books in Young Children's Mathematics Learning

Marja Van den Heuvel-Panhuizen and Iliada Elia

Picture Books as a Didactical Tool

Many mathematics curricula recommend that early mathematics education should endorse a broad range of mathematics covering the "big math ideas" in areas such as number and operations, geometry (shape and space), measurement, and patterns, including problem solving and reasoning within these mathematical areas (Board of Studies NSW 2006; Clements and Sarama 2009; Clements et al. 2004; Hunting et al. 2012; NAEYC and NCTM 2002; NCTM 2000; Sarama and Clements 2009; Seo and Ginsburg 2004; Van den Heuvel-Panhuizen 2008; Van den Heuvel-Panhuizen and Buys 2008). Teaching mathematical concepts and processes can be successfully done as early as kindergarten or even in the prekindergarten years (Ginsburg and Amit 2008). Of course, there is a difference in the methods of teaching young children and older children. A major reason for this dissimilarity is that preschool or kindergarten is for many children between the ages of three and six, the first place to attend an institutional educational setting. This means that the early childhood period involves the transition from informal learning in the family setting to the formal learning in school. Therefore, in the early years of education it is essential for the learning of mathematics to be connected to their everyday experiences. Moreover, like it is the case for students at any age, the learning of mathematics should make sense to them. A didactical tool which has the potential to provide children with an appealing context is children's literature; it makes the problems, situations and questions that children encounter in the story meaningful to them (Columba et al. 2005; Moyer 2000; Whitin and Wilde 1992).

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On the basis of the Vygotskian and action-psychological approach to learning (Van Oers 1996), the personal and cultural development of a person is enhanced only when learning is meaningful. On the one hand, meaningfulness of the learning process in mathematics refers to the process of acquiring mathematics as an activity involving meanings that are historically developed and approved. On the other hand, the learning of mathematics as a meaningful activity encompasses the process of incorporating personal sense to the actions, techniques and outcomes included in mathematics (Van Oers 1996). Both kinds of meaning in the learning of mathematics could be encompassed by reading children picture books, that is, books consisting of text and pictures, in which pictures have a fundamental role in full communication and understanding (Nikolajeva and Scott 2000).

Learning mathematics takes place when children are given the opportunity to reconstruct mathematical objects in a meaningful way. To accomplish this, children need to be assisted by representatives of the community (Van Oers 1996), or—as Vygotsky would have called them—'more knowledgeable others' (McLaughlin et al. 2005), such as parents and teachers. However, this idea of more knowledgeable others can also be extended by giving picture books the role of 'more knowledgeable material' (Van den Heuvel-Panhuizen and Van den Boogaard 2008) because the books can guide the learner toward higher levels of proficiency. Picture books can be regarded as a community agent conveying culturally developed mathematical meanings. Furthermore, Lovitt and Clarke (1992) pointed out that picture books can offer cognitive hooks to explore mathematical concepts and skills. That is, through their interaction with picture books, children may be enabled to encounter problematic situations, ask their own questions, search for answers, consider different points of view, exchange views with others and incorporate their own findings to existing knowledge. In other words, the use of picture books in mathematics teaching gives children the opportunity to construct their learning (using similar processes as those of scientists), by attaching personal meaning to the mathematical objects involved in the books and thereby gain a mathematical understanding.

Reading picture books can have a dual function in the mathematics teaching and learning process. Firstly, it can be an informal and spontaneous activity that children engage in, especially when they are 'reading' a book by themselves during free play. Secondly, it can be a goal-directed activity, organized and directed by the teacher. Given the meaningfulness of reading picture books in the learning process, the use of picture books in either way in mathematics teaching enables the teacher to open the scope to mathematical concepts which are not belonging to the traditionally approved curriculum for young learners such as reading a graph (see Fig. 1), understanding a cross-section, and measuring a long hair tail laid down in a spiral form (see for further elaborations of these examples, Van den Heuvel-Panhuizen et al. 2009). Moreover, reading picture books can be an activity that motivates children to participate and in which they are able to participate based on their available competencies.

The connection between reading picture books and early childhood education has a long history. It dates from 1652 when Comenius published his picture book Orbis Pictus for assisting children to make impressions in the mind (Schickedanz

Fig. 1 Page 3 of the picture book *The surprise* [De verrassing] (Van Ommen 2003)

1995) through the visual images included in the book. In line with Comenius's ideas, the importance of pictures in books for learning has also been supported by recent studies which have shown that picture books' pictures are the focal points of mathematical interaction while reading picture books to young children (Anderson et al. 2005).

By means of their visual images in combination with the text, picture books can contribute to initial stages of interpreting and using representations and in this way support the development of mathematical understanding. According to Van Oers (1996, p. 109), "the improvement of mathematics education by innovations in the early school years must be based on a general introduction to semiotic activity that can be accomplished by giving these children (from 4 to 7 years of age) assistance and opportunities for practice with the activity of forming, exchanging, and negotiating all kinds of meaning within everyday practices".

In the next sections we will elaborate on how picture books can enhance mathematical learning in the kindergarten years. The findings reported are mainly derived from a research program carried out in the PICO-ma project (PIcture books and COncept development MAthematics) in the Netherlands.

Learning-Supportive Characteristics of Picture Books

There is evidence that different books vary in the amounts and kinds of mathematics-related utterances they evoke in children (Anderson et al. 2005). This means that some picture books might have more power than others to provide children an environment in which they can learn mathematics. To gain more knowledge about the

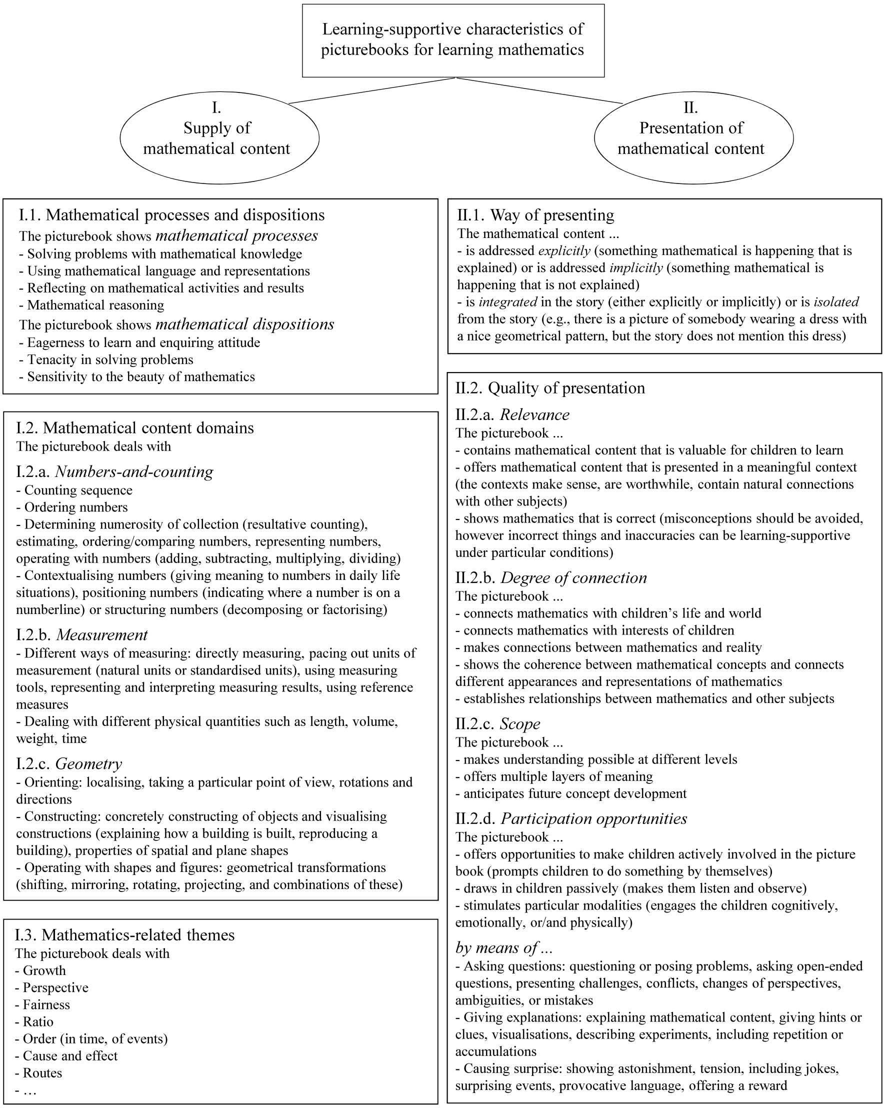

Fig. 2 Framework of learning-supportive characteristics of picture books for learning mathematics; from Van den Heuvel-Panhuizen and Elia (2012, p. 34)

characteristics picture books can have to contribute to the initiation and further development of mathematical understanding in young children we made an inventory of the learning-supportive characteristics of picture books (a full description of this study can be found in Van den Heuvel-Panhuizen and Elia 2012) resulting in the framework shown in Fig. 2.

The framework has two main parts. Part I incorporates the mathematics that is addressed in a picture book and Part II focuses on the way in which this mathematics is brought up in the book.

Part I is based on the fact that a picture book needs to include some mathematical content so as to offer children a setting in which they can learn mathematics. Mathematical content is approached here in a broad sense. In addition to the usual topics, such as numbers-and-counting, measurement, and geometry, we also consider mathematical processes and dispositions, and mathematics-related themes as mathematical content. The themes include phenomena and situations children know from daily life, in which mathematics play a role, such as growth, patterns, fairness, and cause and effect.

Part II describes how the mathematics is presented in a picture book. We found that a distinction can be made between the way of presentation and the quality of presentation. The way of presentation indicates whether the mathematics is addressed explicitly or implicitly, and whether the mathematics is incorporated in a story or presented in an isolated way, which in itself does not say whether it is learning-supportive or not. Mathematics addressed implicitly (e.g. a nice mathematical pattern on the fabric of a character's clothing) can be equally inspiring as mathematics that forms the heart of the story (e.g. the main character is measuring something). In contrast to the way of presentation, the quality of presentation has a more direct relation to whether or not the picture book contributes to the mathematical development of children. The quality of presentation includes relevance, degree of connection, scope, and participation opportunities. Relevance refers to whether mathematics in children's literature is worthy of being learned, is presented in meaningful contexts and is correct. The next component of quality of presentation concerns the degree to which connections are realized between mathematical concepts and the interests of children, the real world, other mathematical concepts, and other subject areas. The scope of the mathematical content encompasses making understanding possible at different levels, offering multiple layers of meaning and anticipating future concept development. Finally, participation opportunities entail offering children opportunities for being involved cognitively, emotionally, or physically by means of asking questions, giving explanations, and causing surprise.

Children's Spontaneous Mathematics-Related Utterances

According to McLaughlin et al. (2005), cognitive engagement, that is, the interaction between student and instructional content during a learning situation, is an essential condition for learning. Similarly, we can assume that picture books also need to engage children cognitively in order to support their learning of mathematics. To get insight in the cognitive engagement and particularly the mathematical thinking that is evoked when young children are read a picture book we carried out two studies in which we investigated the students' utterances that emerged during book reading sessions (Elia et al. 2010; Van den Heuvel-Panhuizen and Van den

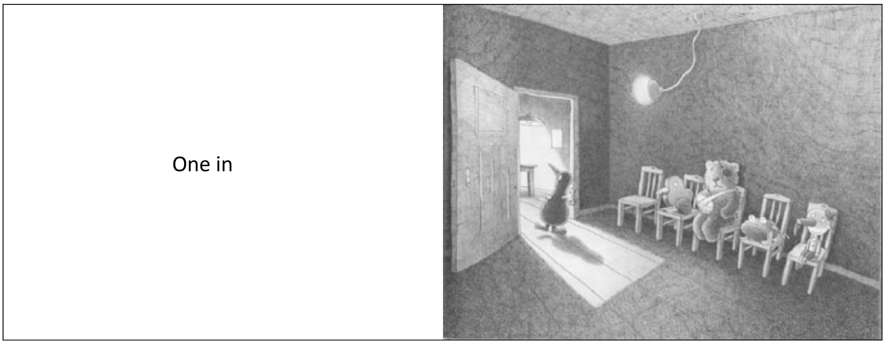

Fig. 3 Page 3 of the picture book *Vijfde zijn* [Being Fifth] (Jandl and Junge 2000)

Boogaard 2008). We regarded these overt reactions as a reflection of the children's cognitive engagement with the content of the picture book. In fact, the utterances (see examples of verbatim utterances displayed in italic format in Table 1) gave us access to the children's mental processing which could be mathematics related (domain-specific utterances) or not (general utterances).

The focus in these studies was on the picture books themselves and not necessarily on how utterances were prompted by a reader. Of course, it is not easy to isolate the influence of the characteristics of the book itself from the influence caused by the characteristics of the reading of the book on the children's cognitive engagement. To avoid interference of these two influences we gave the readers explicit guidelines for how to read a book. They just had to verbalize the plain text displayed in the book without any further prompting or questioning. In essence, these studies intended to explore the child's thinking that takes place, when he or she hears the story and sees the pictures.

In the study by Van den Heuvel-Panhuizen and Van den Boogaard (2008) four 5-year-old children were each individually read a specific book by one of the authors without any questioning and probing. The children who participated in the study were in the second year of kindergarten (K2). Thus, they had no formal instruction in mathematics or reading and they could not read independently. The children's scores on a mathematics test and an oral language test were at about average compared with their classmates' scores.

The book under investigation, *Vijfde zijn* [Being Fifth] (Jandl and Junge 2000), is a trade book of high literary quality—indicated by the number of awards won by the book (see Van den Heuvel-Panhuizen and Van den Boogaard 2008). Moreover, judging from what the reviews of the book said, it can be concluded that the book was not written with the intent to teach children mathematical concepts. The story is about a doctor's waiting room in which five broken toys are waiting for their turn. The toys go into the room behind the door one by one (see Fig. 3). When each toy comes out of this room it is repaired and the next toy goes in. The fifth (and last) one to go in the room is the wooden puppet with the broken nose. Only then is it revealed that the brightly lit room is a doctor's office.

In the reading session, the general rule was to give the children many opportunities to react. Therefore, the children were invited in advance to say what happens in the pictures every time the reader turned a page. Of course not all cases led to utterances. Some pages elicited more utterances than others. To keep the influence of the reader as constant as possible across the reading sessions, we set up reading guidelines that excluded spontaneous assistance by the reader. In addition, a reading scenario was developed that explained how each page should be presented. The reading sessions were carried out by the same reader strictly according to the guidelines and the reading scenario. The book was only read once to each child.

A detailed coding scheme was developed (see details about it in Van den Heuvel-Panhuizen and Van den Boogaard 2008) to understand and classify children's utterances that might reflect cognitive engagement. In general, the smallest possible meaningful grammatical part of a response was considered as the unit of analysis. These parts mostly contain a finite verb or a verb phrase, but sometimes they only contain a subject or an object, or even merely a sigh or an exclamation.

In total, the four children produced 432 utterances spread over a total of 22 pages, front cover, back cover and end papers included. An overview of the types of utterances, each illustrated by an example, is given in Table 1.

All four children showed cognitive engagement when they were read the picture book. About half of the utterances were mathematics-related and all four children of the study were found to contribute to this result. Across the children the mathematics-related utterances were about equally distributed over the pages of the book, indicating that the book as a whole has the potential to evoke mathematical thinking.

The children's mathematics-related utterances were distinguished into two different types with respect to their content: the spatial orientation-related utterances and the number-related utterances. The spatial orientation-related utterances (31 % of all utterances) exceeded the number-related utterances (14 %). Of this latter type, most utterances referred to resultative counting, "how many there are". A closer look at all the utterances that reflect resultative counting revealed that in a number of cases the children were structuring numbers, including composing and decomposing. For example, when describing a picture in which the five toys are sitting in the waiting room, a child said "two are looking at the ceiling, and three are watching television". Within the spatial orientation-related utterances, the children spontaneously took the waiting room perspective instead of the doctor's office perspective that is taken by the author of the book. As a result, there was a discrepancy between the children's utterances and the text. Interestingly, three of the children changed their waiting room perspective one or more times into the doctor's office perspective.

In conclusion, the book *Vijfde zijn* provided the children with a meaningful context in which they could actively construct mathematical knowledge about number and spatial orientation. However, while in this study, the role of the reader was minimized, the interaction with knowledgeable others which is considered a crucial element of the learning process was put in a different perspective. Instead of having verbal interaction with an adult, which is mostly associated with that other, this study made a reasonable case for extending the concept of the knowledgeable

Table 1 Categories of utterances and examples

General qualification of utterances

01. Description static: *And the door is open*[a]

02. Description static comparison: *Again hundred chairs!*

03. Description dynamic stationary: *He looked at the lamp*

04. Description dynamic stationary comparison: *And this one looked there again*

05. Description dynamic relocation: *Now the frog goes in*

06. Description dynamic relocation comparison: *He comes back again*

07. Posing question: *Why does he have a sticking plaster?*

08. Assumption story line: *Four will be leaving*

09. Assumption other: *Maybe they are waiting*

10. Explanation own utterance: (After saying that puppet cries:) … *because he is alone*

11. Explanation other: (After text "Hello doctor":) *That is the waiting … the waiting … at the doctor's …*

12. Giving opinion: *Nice book!*

13. Commenting text: (After text "Door open. One out":) *Out is go outside*

14. Commenting picture: (Pointing to "noses" in doctor's office:) … *this doesn't actually belong at the doctor's*

15. Repeating text: (After text "One in":) *One in*

16. Correction: (After saying that "this one will come":) … *no, this one will go out*

17. Self reflection own utterance: (After said something:) … *that is what I think*

18. Self reflection other: (While trying to describe how the bear is looking*:) I cannot see it very well*

19. Contemplation: (When wondering whether the duck has a little string or a little stick:) *Hm …*

20. External reference other story: *It looks like Pinocchio*, …

21. External reference other: *My little sister is outside now*

22. Unclear utterance

Domain-specific qualification of utterances—number-related

N1. Resultative counting: *Three people are sitting here* (13)[b]

N2. Using all/everyone: … *because everyone is gone* (10)

N3. Using none/nobody: *Maybe, when this one is gone … then nobody is there* (08)

N4. Using some: *And some are sitting* … (01)

N5. Using ordinal numbers: *Fifth?* (07)

Domain-specific qualification of utterances—spatial orientation-related

S1. WRP[c] (Ladybird is coming into waiting room:) … *and a ladybird is coming* (05)

S2. WRP+adjunct: (Pointing to penguin and doctor's office:) *And then this one wants to go there* (13)

S3. DOP[d] (After text "Door open. One out", while pointing to ladybird:) *This one is out* (13)

S4. DOP+adjunct: (After text "Door open. One out", while pointing to doctor's office:) *At that, one out* (13)

S5. Describing direction: *And the lamp is almost upwards* (01)

[a] The verbatim utterances are in *italic format*

[b] The number between parentheses refers to the general utterance involved. Note that a general utterance can be number-related as well as spatial orientation-related

[c] WRP = Waiting room perspective

[d] DOP = Doctor's office perspective

other by including knowledgeable material, which a picture book can be. In other words, this study's findings suggest that just by telling and illustrating an appealing story, picture book authors unintentionally offer children a rich environment for mathematical thinking. This important conclusion motivated us to continue our explorations of the power of pictures in picture books for evoking kindergartners' mathematics-related thinking.

The Role of the Pictures in a Picture Book

Pictures are an indispensible component of picture books. They have a major role in telling the story by serving different functions. Thus, to gain a deeper understanding of how picture books can support the learning of mathematics we decided to set up a study (Elia et al. 2010) to investigate the role of the pictures included in a mathematics-related picture book on young children's spontaneous mathematical cognitive activity when they are read such a picture book.

According to Theodoulou et al. (2004) and Elia et al. (2007) pictures may serve different functions in arithmetic problem solving: decorative, representational, organizational and informational. Decorative pictures just accompany the problem without providing information that is relevant to the mathematical content of the problem. Representational pictures depict a part or the entire mathematical content of the problem. They are not essential though for the understanding or the solution of the problem. These pictures can facilitate the understanding of the meaning of the problem and its solution, but in fact, because they are not essential for understanding the problem, they can be ignored. Organizational pictures give directions for organizing the problem's mathematical information for making drawings or written work that may support finding a solution. Like the representational pictures, they are not necessary for the solution of the problem. Informational pictures provide information that is essential for the solution of the problem. That is, they represent visually the mathematical content of the problem often with groups of elements that may frame the counting process.

Pictures in picture books usually depict what is described in the text, serving a representational function, but may even go beyond this role by adding further details. Through the interplay of text and image, which have different content, meaning can be generated (Sipe 1998). Therefore, pictures in picture books may also have an informational function. In picture books that contain mathematical content, pictures can include also components which may support the understanding of this mathematical content. Generally pictorial mathematics-related components can have a representational or an informational function. Mathematics-related components, which have a representational function show, for example, the collection of a number of objects which is described in the text, whereas mathematics-related components, which have an informational function, depict numerical information which is not included in the text.

For the study (Elia et al. 2010) in which we explored the role of the pictures, we used the book *Six brave little monkeys in the jungle* (O'Leary 2005). In contrast to

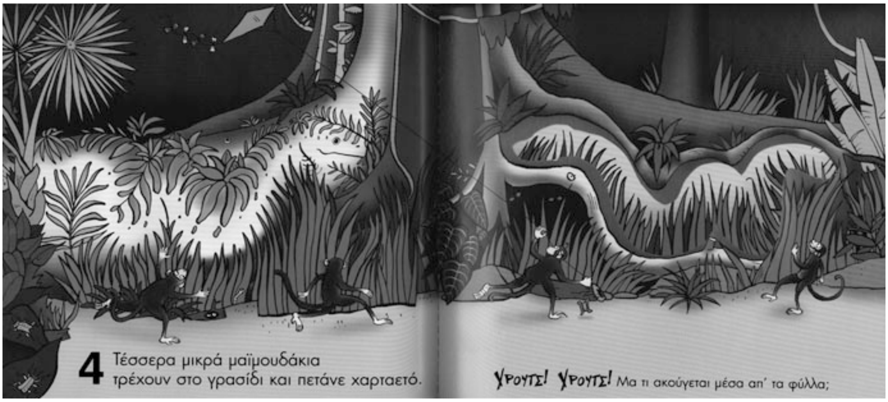

Fig. 4 Page 5 of *Six brave little monkeys in the jungle* (O'Leary 2005): *Text left*: 4 little monkeys are running on the grass and are flying the kite. *Text right*: Hrutz! Hrutz! But what is that sound through the leaves?

Vijfde zijn [Being fifth] discussed in the previous section and read to four children in the Netherlands, the picture book about the six monkeys is written for the purpose of teaching mathematics. At the back cover of the book it is stated that it can be used in the teaching of counting backwards. Four 5-year-old children from Cyprus were read the book individually without any probing. The story is about six monkeys that live in the jungle. In every page there is a hidden jungle animal that scares the playful monkeys and consequently a monkey disappears. In the end, the monkey that is left comes up with an idea so as to get back at the scary animals.

All of the pictures of the book have both story-related and mathematics-related components with either a representational or an informational function. Sometimes the story-related and the mathematics-related components have both the same function and sometimes their function differs. For example, the two types of components of the picture in page 5 (including the left side as well as the right side) have both a representational function. The story-related components illustrate a part of the text, namely, that the monkeys are running on the grass and are flying the kite (see Fig. 4). The mathematics-related components represent visually the numerical information that is described in the text with a group of four monkeys.

The story-related and the mathematics-related components of pictures can also have both an informational function. The story-related components of these pictures offer supplementary information to the content of the story-related text. For example, in page 2 (see Fig. 5), the story-related components of the picture reveal the cause of the decrease of the monkeys—an attack by a wild animal such as a leopard—which is not given in the text "Watch out! The jungle is dangerous!" The mathematics-related components of the picture in page 2 provide information about the monkey that goes away (see the tail of the monkey that is running away on the bottom left part of page 2) and the number of the monkeys that are still there (see right side of page 2).

Fig. 5 Page 2 of *Six brave little monkeys in the jungle* (O'Leary 2005): *Text left*: Grrr!!! *Text right*: Watch out! The jungle is dangerous!

The analysis of the children's utterances showed that the book as a whole had the potential for cognitively engaging children. All four children demonstrated cognitive engagement which resulted in general utterances as well as in mathematics-related utterances. Despite the fact that the book was written for the purpose of teaching mathematics and explicitly displayed mathematics through numbers and number symbols, we found mathematics-related utterances accounted for only 27 % of the total number of utterances. This suggests that picture books, which have been written for didactical purposes may not evoke mathematics-related thinking as effectively as might be assumed.

Children's domain-specific mathematics-related utterances, in this study (Elia et al. 2010), fell into three categories: number-related, spatial-topological and measurement-related. The spatial-topological utterances, which included specifying position, topological relations, recognition of shapes or figures and using the terms "here" and "there", were the most frequently found. The measurement-related utterances, which involved references to the size of objects and time, could be most rarely identified in children's reactions.

Most of the number-related utterances had to do with determining the number of a collection of objects (how many there are). The main ways children used to achieve this were subitizing and counting, which are fundamental and powerful skills in the development of children's understanding of numbers (Baroody 1987; Clements and Sarama 2009). Counting backwards, however, which was the explicit focus of the picture book, was not detected in children's reactions. Only one child noticed that the number of monkeys altered every time, but without making explicit that every time a monkey left, the number decreased by one. Furthermore, children tended to compare the collections across pages by recognizing that the collection of the current page is different from or smaller than the collection of the previous page. This indicates that the picture book itself motivated the children to use counting in a meaningful way and make inferences based on their counting, that is, to compare collections of objects appearing in the pictures. Establishing the numerosity of a collection by subitizing, was another important process that the children explicitly used. In our study, the pictures of the book stimulated children to recognize that groups are composed of smaller groups (i.e., 4 is 2 and 2). This contributes to the development of knowledge of number relationships which provide an early basis for addition and subtraction (Fuson 1992). Children's number related utterances included also the recognition of numerical symbols and the use of words referring to a quantity of objects such as some, many, all and none/nobody. In sum, the picture book used in this study elicited various ideas that are basic and important in the development of the understanding of number (Clements and Sarama 2009).

The pictures with a representational function were found to evoke mathematical thinking to a greater extent than the pictures with an informational function. Mathematics-related components with a representational function evoked a greater amount of utterances in all three categories: number-related, spatial-topological, and measurement-related. The components with a representational function provide another 'description' that is additional to the text, whereas in the case of the components with an informational function, the mathematical information can be acquired only from the picture, as the content of the text does not give the whole information. This result suggests that combining text and pictures of a similar content has a greater power to mathematically engage children than combining text and pictures of different content. According to a number of researchers (Mayer 2001; Schnotz 2005) pictures and text of coherent or semantically related content facilitate mental model construction, whereas learning only from a diagram (or a picture) is quite difficult, particularly for novices (Kalyuga et al. 2000).

After finishing the study we were left with the question whether the children would have generated more utterances if they were prompted to do so. For example, probing by the reader may further support children's mathematical thinking in the pictures in which the components have an informational function in relation to the mathematical content of the text. In order to get more insight on how to read mathematics-related picture books to young children so that children's engagement with mathematics is enhanced, a further study was set up as described in the next section.

How Picture Books Can be Read to Elicit Children's Thinking

Additionally to picture books reading in which children listen to an adult in a passive way, picture books can also be read in a dialogic way in which children are active participants when they are read a picture book. This latter style of book reading is developed by Whitehurst and his colleagues (Arnold et al. 1994; Whitehurst et al. 1988) for parental book reading and reading in day care centers. This dialogic book reading implies that "the adult is encouraged to ask open-ended questions and to avoid yes/no or pointing questions. For example, the adult might say, 'What is Eeyore doing?' or 'You tell me about this page' instead of 'Is Eeyore lying down?'"

(Whitehurst et al. 1994, p. 680). Although studies have provided evidence for the efficacy of dialogic reading for expressive language development in preschool children (Hargrave and Sénéchal 2000) and vocabulary development of children up to the end of grade 2 (Whitehurst et al. 1999), we chose to adapt this dialogic book reading approach by asking the teachers, involved in our project, not to ask too many questions. The reasons for this were the following. On the one hand, we wanted to use in a certain degree—similarly as in the two previous studies—the own power of the picture books to elicit mathematical thinking in children. To let the books do the work, we requested the teachers to maintain a reserved attitude and not to take each aspect of the story as a starting point for an extended class discussion, since lengthy or frequent intermissions could break the flow of being in the story and consequently diminish the story's own power to contribute to the mathematical development of the children. On the other hand, we tried to enhance the cognitive involvement of the children by asking the teachers to act as a role model of cognitive engagement or as a person who provokes discussion with the children. We suggested the teachers to react to the story and pictures in the picture books by asking oneself questions, playing dumb, and showing inquiring expressions. In the next section we give a short classroom vignette of each way of reacting to illustrate the children's mathematical thinking that is elicited by this teacher behavior. The observations are from one teacher in an inner-city school in a large city in the Netherlands. Most children at this school have an at-risk background. The teacher read the picture books to a small group of six of her children who are in K2 (5- to 6-year-olds).

Asking Oneself a Question

The picture book that is read is called *22 Wezen* [22 Orphans] (Veldkamp and Hopman 1999). It is about twenty-two parentless children who live in an orphanage. On pages 5 and 6 (see Fig. 6) the stern lady principal takes the children to bed. In the dormitory, eight double-decker beds are visible. The lady principal's huge body blocks the view on the other beds.

Classroom Vignette

Teacher:	[The teacher reads the text in the picture book and continues.] Yes, and she's so afraid of an accident that she sends them all to bed. But I wonder, are there enough beds for everyone?
All children:	[All children react.] Yes… I think so… No… [Children start immediately counting and pointing at the beds.]
Teacher:	Wait, if we all count at once, you'll be confused. Could you start again? Wait, we'll all take our turns to count, right?
N:	I can't see anything.
Teacher:	Then you need to sit right, [Name of N]. [Name of S], what do you think, is there enough room?

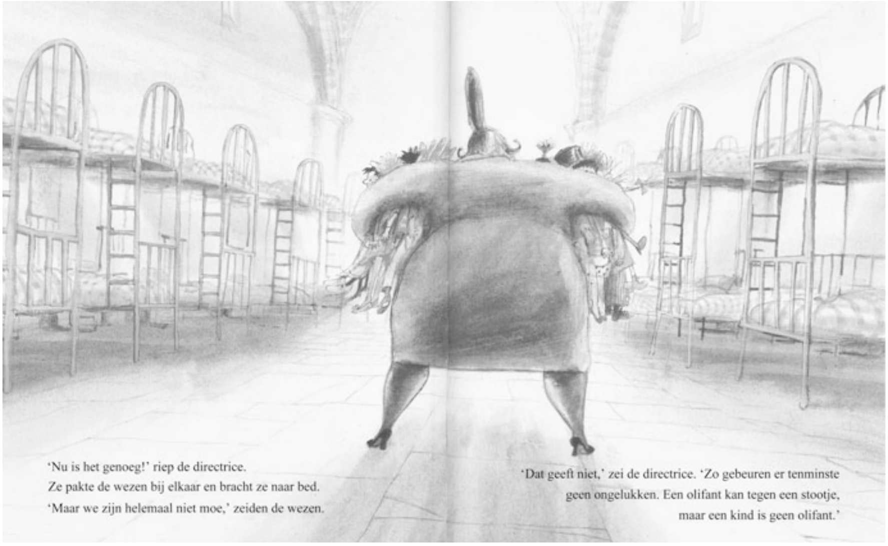

Fig. 6 Pages 5 and 6 of *22 Wezen* [22 Orphans] (Veldkamp and Hopman 1999). Text page 5: "That is enough!" the lady principal shouted. She collected the orphans and put them to bed. "But we are not at all tired," the orphans said. Text page 6: "it doesn't matter," the lady principal said. At least no accidents happen this way. An elephant can stand rough handling, but a child is not an elephant"

S:	No.
Teacher:	You don't think so. What were you doing just then? What were you going to count?
S:	The beds.
Teacher:	You were going to count the beds. Well, go ahead.
S:	[Counts the bed one by one mumbling and at the same time points out in the air the beds in the book] 16!
Teacher:	16. But how do you get that man... I only see... 1, 2, 3, 4, 5, 6, 7, 8 beds.
S:	Because it, because they are double...
Teacher:	Oh, they're double-decker beds.
Y:	[Starts counting and points with her spread-out index finger and middle finger at the beds in the book] 2, 4, 6, 8, 10, 12 [points at the beds, behind the lady principal]. There have to be some here.
Teacher:	There have to be, yes!
Y:	12, 14, 16, [points at where she thinks the beds are, child 6 also points them out] 17... [with some help from the teacher] 18, 20, 22!
Teacher:	Yes, that's it! You did that very well. [Name of S] was counting like this [teacher points with her index finger at the beds in the book], but you can also count in steps of two, right?

This classroom vignette makes clear that the question the teacher asked herself (lines 2–3) elicited the children to start to count (line 5 and further). The way the children took over the question suggests that this question became also a question for the children themselves. Moreover, the structure of the double-decker beds stimulated children to count in two's (line 21).

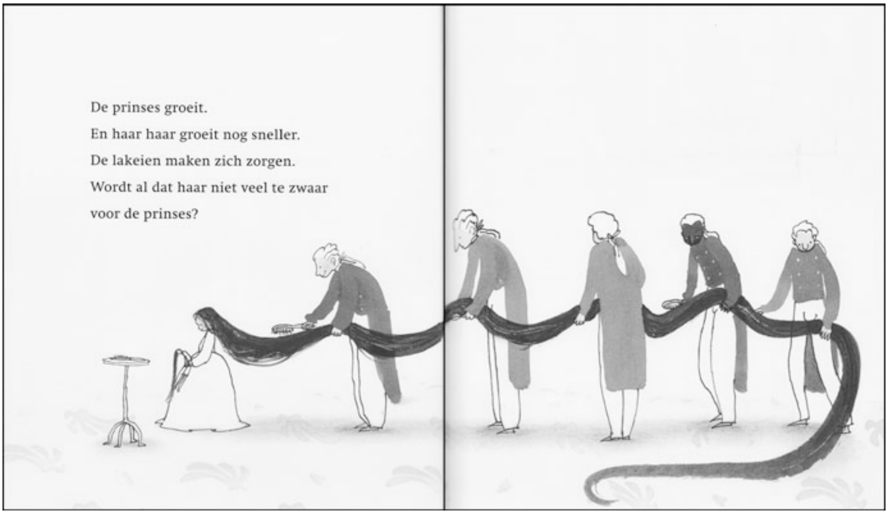

Fig. 7 Pages 3 and 4 of the book De prinses met de lange haren [The princess with the long hair] (Van Haeringen 1999). Text page 3: The princess grows. And her hair grows even faster. The lackeys are worried. Is the hair not becoming too heavy for the princess?

Playing Dumb

Another possible way to make the children cognitively active is feigning ignorance. An example of such 'playing dumb' behavior of the teacher was evident in the previous classroom vignette (see line 18). After child S counted 16 beds, the teacher reacted with surprise, and counted 8 beds, as if she did not see that each bed contained two mattresses.

A second example of 'playing dumb' to elicit children's mathematical thinking became apparent in the next classroom vignette. The reading session is with the picture book *De prinses met de lange haren* [The princess with the long hair] (Van Haeringen 1999) (see Fig. 7).

Classroom Vignette

Teacher:	[The teacher reads the complete text.] In the book it says: her hair grows even faster than the princess. How can one tell?
V:	Because her hair is super long now.
Teacher:	And the princess?
V:	Small.
Teacher:	She is not that tall, hey?
J:	She was a baby, right? Now her hair is grown and then she had got long hair.
Teacher:	She has got long hair, yes.
V:	Five men got to hold it.
	[V raises four fingers.]

Teacher:	[The teacher asks the children to speak one by one for they all speak at the same time. Then she continues.]
	The princess has grown just such a bit, right?
	[The teacher points to the princess from top to toe.]
K:	Up till here.
	[K points to a part of the hair that has the same length as the princess.]
B:	She is this big.
	[B walks to the book and measures the height of the princess with the fingertip of her index fingers.]
	One, two, three, four, five, six, seven.
Teacher:	Seven what?
All children:	Seven meters!
Teacher:	Seven meters?
	So you have fingers of one meter?
B:	No!
K:	[K measures seven times his fingertip on the hair of the princess and says.]
	Up till here.
Teacher:	So she has grown seven fingers, that is how tall she is.
	[In the meantime J tries to measures the whole length of the hair with her fingertip.]
Teacher:	You have very thin fingers.
	Shall I go and see with my fingers how tall she is?
	[The teacher measures the princess with her index fingers.]
All children:	One, two, three, four, five.
Teacher:	With me she is just five fingers tall.
V:	Because you have thick fingers.
Teacher:	Yes, my fingers are much thicker.
J:	Look at my little finger.
	[J measures the princess with the tip of her little finger.]
	One, two...

The picture book and the classroom discussion about the princess with the long hair seemed to offer opportunities to deal with the concept of measurement with very young children differently than mathematics textbooks generally suggest. The picture book does not reflect a smooth building-up of the different aspects of measuring; starting with easy situations and gradually including more complex situations. Instead, in the context of reading this picture book, children are engaged in measuring something that is bent, possibly before they have done so with measuring straight lines. Moreover, they can be engaged in dealing with growth at different speeds (that of the hair compared to that of the girl) before the children have, we assume, a good understanding of the growth of objects on their own.

Moreover, when discussing the length of the hair, the children had different ways of representing the length, such as the amount of time it takes to follow the hair with your finger or to walk the length in the gym where the book reading took place, and the number of fingertips needed to measure the hair.

Further, the feigned ignorance of the teacher (see line 65) revealed that the children understood the relationship between the unit size and the number of iterations quite well, while other researchers have observed that this key measurement principle is undeveloped in many children. For example, Grant and Kline (2003, p. 52) described a first-grade class where a difference in the unit of measurement (children's

Fig. 8 Pages 3 and 4 of the book *Feodoor heeft zeven zussen* [Feodoor has seven sisters] (Huiberts and Posthuma 2006) Text page 3: At night before he goes to sleep, he doesn't get just one kiss. No, his seven sisters give him, altogether twenty-one kisses. Fourteen arms around him, and he is wrapped up well from head to foot. Then, he is read six stories and one poem. Finally, seven fingers reach for the light-switch

feet lengths) led to a dispute over the actual measure of a distance; "a significant number of students thought that smaller feet would lead to a smaller measure". This is an interesting contrast to the children in our study, who realized that the teacher's thick fingers would lead to a smaller number of 'counts'. It is rather likely that the teacher's playing dumb behavior ("With me she is just five fingers tall"), triggered the children's idea of fairness in measurement, i.e., they felt the necessity to have a similar unit.

Just Showing an Inquiring Expression

The next book is *Feodoor heeft zeven zussen* [Feodoor has seven sisters] (Huiberts and Posthuma 2006) (see Fig. 8).

Classroom Vignette

Teacher:	[The teacher reads the text till "altogether twenty-one kisses". Then, the teacher stops and shows an inquiring expression by raising her eyebrows.]
All children:	[All children react together; look at each other; reactions are mumbled.]
Teacher:	Twenty-one kisses!
E:	[Starts counting while tapping her cheek] Three, four, five.
Y:	On two sides.
All children:	[All children react to what Y says; only the word 'two' can be made out.]
M:	[Says something inaudible to the teacher.]
Y:	...plus thirteen?
Teacher:	No, he received twenty-one kisses, and you just said [she looks at Y] he gets a kiss on each side from every sister, right [teacher points at the first sister in the picture in the book] because you were already starting to count. You said two...

All children: [The teacher points at the second sister] Four.
All children: [The teacher points at the third sister] Six.
All children: [The teacher points at the fourth sister] Eight.
All children: [The teacher points at the fifth sister] Ten.
All children: [The teacher points at the sixth sister] Twelve.
All children: [The teacher points at the seventh sister; children hesitate]
Y: [Starts, doesn't finish the word] Thir…
Teacher: F…
All children: Fourteen.
Teacher: Fourteen, but then it's not right. They say twenty-one kisses.
E: Okay, then it's here, here and here [points at her own face to show where the kisses are placed; one on the left cheek, one on the right cheek, and one on the forehead.]
Teacher: Oh! Yes, maybe he is doing that. A kiss here [teacher points out the kisses on her own face]… they kiss him here, here and one on his forehead. How many for each sister?
E: [Inaudible.]
Teacher: Well, let's see if that is correct. [Teacher taps her finger on the first sister for each number] One, two, three.
All children [Start counting along out loud.]
Teacher: [Taps her finger on the first sister] Four, five, six.
All children: [Count along out loud, while the teacher taps her finger on the sister concerned] Seven, eight, nine.
E: [Counts along with the teacher with her fingers] Ten, eleven, twelve, … thirteen, fourteen, fifteen, … sixteen, seventeen, eighteen.
Teacher and children: Nineteen, twenty, twenty-one.
Teacher: Hey, that's right.
E: That's here, here and here [points out the places on her face].
Teacher: [Looks around the table] He got twenty-one kisses, from his seven sisters.
N: … Kisses.
Teacher: Yes, kisses [points out the places on her face].
E: Thought well about it… here and here?
Teacher: Yes, from each sister he gets…
All children: Three.
Teacher: Three kisses.

This classroom vignette again illustrates that asking questions is not the only means teachers have available to get the children actively involved in the mathematics-related events included in picture books. The teacher's wondering about the twenty-one kisses (see lines 71–72) raised also questions in the mind of the children and prompted them to react (see line 73) and check the number. After first starting with one kiss on each cheek, which resulted in 14 kisses, child E came with the solution to have a kiss on both cheeks and one on the forehead (see lines 94–96). This is a perfect solution for not getting confused. In all it is very remarkable that these kindergartners who never have been taught multiplication tables—not to mention the multiplication table of seven—could handle these large numbers. The story of the seven sisters provides a viable context for the number fourteen and twenty-one, from which the children may benefit in their further learning of mathematics. Whether or in what ways reading picture books to young children contributes to their mathematical performance is the topic of the next section.

Effectiveness of Picture Book Reading on Kindergartners' Performance in Mathematics

In our research program, an experimental study (pre-test–post-test control group design) was set up in order to find out whether an intervention involving picture book reading could contribute to the development of kindergartners' mathematical understanding (Van den Heuvel-Panhuizen et al. 2013). Another major focus of the particular study was to investigate the relationship between the intervention effect and characteristics of kindergartners, including kindergarten year, age, gender, home language, socioeconomic status (SES), urbanization level of the location of the school the children attended, mathematics and language ability.

Set up of the Experiment

In total 384 children participated in our study: 199 children from nine classes in the experimental group and 185 from nine classes in a comparable control group. The experimental group consisted of 106 boys and 93 girls, 84 of whom were in kindergarten year 1 (K1) and 115 in kindergarten year 2 (K2). The average age at the time of the pretest was 5 years and 3 months. The control group consisted of 95 boys and 90 girls, of whom 66 were in K1 and 119 in K2. The average age at the time of the pretest was 5 years and 4 months.

During three months, the children of the experimental group were read a collection of picture books in which mathematical topics (i.e. number, measurement, or geometry) are unintentionally addressed by the authors of the books. Within these mathematical domains we focused respectively on numbers and number relations, growth, and perspective. Altogether, eight books were used within each domain. The total of 24 picture books were selected on the basis of the learning-supportive characteristics (see section "*Learning-Supportive Characteristics of Picture Books*") included in the framework that was developed as a result of a literature review and an expert consultation (Van den Heuvel-Panhuizen and Elia 2012).

Reading guidelines were developed with a focus on each book's own power to promote the children's mathematical thinking. Teacher behavior like (1) asking oneself a question out loud about the mathematics, (2) playing dumb, and (3) just showing an inquiring expression was suggested in the reading keys, the written guidelines that were developed for this study. In advance of the intervention, the teachers in the experimental group received training on the picture book program, which entailed researchers explaining to them the set-up of the reading sessions and how to use the reading keys by showing them illustrative video-recordings of the pilot sessions. During the three-month intervention, the teachers in the experimental group read two picture books in class per week. After the book reading sessions, the books were at the children's disposal during free play. In the experimental classes, the intervention book reading program replaced the regular book reading. The control group classes only did their regular book reading which could incidentally include

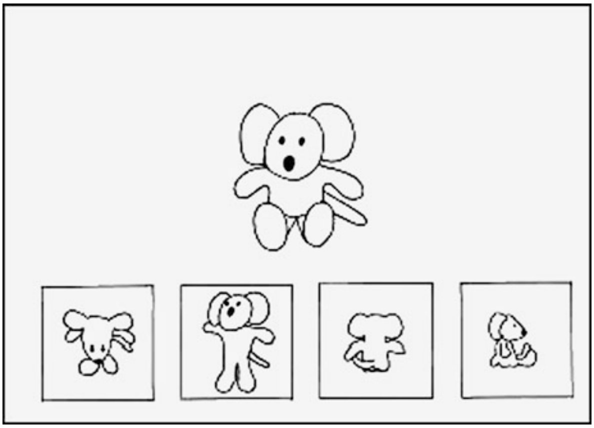

Fig. 9 PICO test item on perspective. Test instruction read to the children: "There is Mouse. How would Mouse look if you looked down on him like a bird? Underline the way Mouse looks from above"

some books that were also used in the experimental group. However, the logs of the teachers did not show us that this was the case. Both the experimental group and the control group followed their regular mathematics program. The latter group was not informed about the real purpose of the study, but were told that the test data were collected to describe the children's development in mathematics.

A project test called the PICO test was developed and used to assess children's mathematics performance before and after the intervention. The PICO test items consist of multiple-choice questions on arithmetic (number and number relations), measurement (length with emphasis on growth), and geometry (perspective). Figure 9 shows an example of a test item on perspective.

Results

To investigate the intervention effect on the mathematical understanding of the sample as a whole we performed a regression analysis where the gain score (PICO posttest score minus PICO pretest score) was used as the dependent variable and the intervention as the independent variable (Model 1). In order to find estimates of the intervention effect with the least bias, another regression analysis was applied, in which the various variables representing children's characteristics (kindergarten year, age, gender, home language, SES, urbanization level of school location, mathematics ability, and language ability) were included (Model 2). Both models revealed a significant intervention effect (Model 1: $B = .91$, $p = .01$; Model 2: $B = .77$, $p = .03$) with the explained variance increasing from $R^2 = .02$ in Model 1 to $R^2 = .04$ in Model 2. The effect sizes, as defined by Cohen (1988), were calculated for each model in order to investigate the size of the general intervention effect. We found small effect sizes. For Model 1 the effect size was $d = .16$ (meaning that the difference between the pretest and the posttest was .16 times the standard deviation of the pretest scores) and for Model 2 the effect size was $d = .13$. For the gain score in the control group we found an effect size of $d = .59$, which means that the influence of the intervention amounted to be 27 % (.16/.59 = .27) larger than this effect size in the control group. In Model 2, the Cohen's d was .13, indicating an increase in effect size of 22 %. This finding supports the assumption that picture book reading can yield significant learning outcomes in early years mathematics.

To investigate the influence of the intervention on the gain scores in the different subgroups, we carried out regression analyses in each of the subgroups which were based on the covariates. In some of the subgroups we found a significant intervention effect. This was the case in the K2 subgroup, the subgroup with the older children, the subgroup with Dutch as home language, the subgroup with the higher SES, the subgroup of children who attended a school in a small town, the subgroup with the lowest mathematics ability and the subgroup with the lowest language ability and the subgroup with the highest language ability. In the subgroup of girls we found a significant and relatively strong intervention effect ($B = 1.37$, $p = .01$, $d = .24$), whereas for the boys there was not a significant intervention effect ($B = .49$, $p = .16$, $d = .08$).

After carrying out regression analyses in each of the subgroups separately, we examined whether the found intervention effects differed between the subgroups. This analysis turned out that there were no differential intervention effects. This applied also to the effects found in the two gender subgroups ($B = .88$, $p = .12$). Yet, this differential intervention effect for gender was not of a negligible size. Moreover, the effect size found for girls ($d = .24$) was three times as large as that for boys ($d = .08$).

In sum, a major conclusion of the above study about the effectiveness of picture book reading is that reading picture books can support children's mathematical understanding and therefore, according to us, should have a significant place in the kindergarten curriculum. Such a picture book reading program seems to be effective for a wide range of children in a whole-class setting, including children of different ages, socio-economic backgrounds, language and mathematical abilities. The particularly positive results of reading picture books for the mathematical development of girls is another advantage that may help girls have a better start in mathematics when they enter first grade (Carr and Davis 2001; Penner and Paret 2008).

Final Remarks

When Robert Hunting and Lyn English recently gave an interview on the Australian RadioNational,[1] the interviewer started with the following question: "We are encouraged to read to children as early as possible but how can we encourage the early learning of mathematics?" Our answer would be: Do the same for mathematics, read them picture books. At least, among other things, this is one way to encourage the

[1] The interview was broadcasted by the Australian RadioNational on the 12th of March 2012; http://www.abc.net.au/radionational/programs/lifematters/young-children-and-mathematics/3895004.

early learning of mathematics. As we have discussed in this chapter, there are several approaches of using picture books for this aim. One approach is just reading the books to the children without the reader giving prompts. In this approach it is the own power of the picture books that elicits mathematical thinking in children. Our study has shown that just reading the books can make the children cognitively active and can lead to mathematics-related utterances. Another approach is a focused way of dialogic reading, which means a way of reading in which the power of the picture books and cognitive involvement of the children is enhanced by having the reader as a role model of cognitive engagement or as a person who provokes discussion with the children that brings them to mathematical reasoning as well.

A further approach is adding mathematical activities to the picture book reading. In this last option the book reading is followed by story-related (mathematical) activities in class. This approach has not yet been investigated by us, but examples of it can be found in studies by Jennings et al. (1992), Hong (1996), Young-Loveridge (2004), and Casey et al. (2008).

Apart from these three goal-directed ways of picture book reading in which an adult reads the book, children can also 'read' picture books by themselves during free play. We wonder whether this would also give them support in developing mathematical understanding. Further explorations are necessary in this self-contained learning.

Another issue that needs further research is the role of the teacher and what is necessary to fulfill this role. First of all, we think that teachers should recognize picture books as a didactical tool in mathematics education for young children. Secondly, they should be able to see the mathematics in picture books of high literary quality even if these books have not been written for teaching mathematical concepts. Thirdly, if the two foregoing points are reached, we think that teachers will have possibilities to contribute to children's mathematical thinking in a way that might be attractive for the children as well as for the teachers themselves. However, as said before, more research is needed at these points.

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Improving Numeracy Outcomes for Young Australian Indigenous Children

Marina M. Papic

Introduction

This chapter provides research-based evidence on successful approaches to improving learning opportunities, particularly in early numeracy, for young Australian Indigenous children. A series of studies (Papic and Mulligan 2007; Papic et al. 2011) focused on developing children's early algebraic and mathematical reasoning skills and teachers' pedagogical and mathematical content knowledge (Hill et al. 2008) through on going, supportive professional development will be presented. Two initial studies informed the development of the *Patterns and Early Algebra* (PEAP) *Professional Development* (PD) *Program*, an Australian Research Council Linkages project[1] 2011–2013. PEAP PD advances young children's patterning, early algebraic and mathematical thinking skills, working towards the broader goal of closing the gap in numeracy achievement for Indigenous children in rural and regional early childhood settings.

National statistics highlight the unacceptable levels of disadvantage faced by Indigenous Australians in living standards, life-expectancy, education, health and employment (Australian Government 2009). Australian Indigenous children aged 0–14 years make up 39 % of the Australian Indigenous population. According to the National Report on Schooling in Australia (Ministerial Council on Education, Employment, Training and Youth Affairs, MCEETYA 2008), literacy and numeracy results for Indigenous students are consistently below the national average, especially in remote areas; only forty-seven percent of Indigenous Australian children in year 7 are achieving results at the benchmark for numeracy (p. 29). In the early

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childhood years Indigenous children are less likely to participate in preschool programs than non-Indigenous children and they have higher rates of absenteeism in primary school (Frigo et al. 2004). Poor educational outcomes of Indigenous children in later years of schooling are indicative of inadequate educational progress in the early years of schooling (Adams 1998, p. 8).

For the first time in our history, the Australian Government has set specific targets to address Indigenous disadvantage (Australian Government 2009). Suitable educational opportunities not just in literacy and numeracy but also in developing skills and attitudes for lifelong learning play a critical role in achieving these targets, particularly in the early years. Children's learning experiences in the early years play a crucial role in setting foundations for lifelong learning (Clements and Sarama 2007). The development and evaluation of appropriate programs to promote literacy and numeracy for Indigenous children must be aligned with initiatives to support professionals within their communities (Perso 2009). Raising professional and community expectations of young Indigenous learners and access to effective professional development programs and resources are critical to achieving this goal (Meaney et al. 2008).

Despite the government agenda to address Indigenous disadvantage and close the gap in numeracy achievements, there is a paucity of focused and longitudinal research into the educational outcomes for Indigenous students. Perry and Dockett's (2004) review calls for studies of successful approaches to the mathematics education of young Indigenous students and Mellor and Corrigan (2004) argue for more qualitative and case study research along with more rigorous evaluation of indicators by quantified research. "Changes to practice in the field of Indigenous education are required because we really do not know enough about improving Indigenous students' learning outcomes". We also don't know enough about the best way to support Indigenous professionals and build their pedagogical content knowledge. Furthermore, there is lack of research on numeracy assessment tools to assess the learning outcomes of young Indigenous students and even less evidence of effective numeracy programs for Indigenous children. Current research focuses on building new initiatives/programs that develop mathematical reasoning, skills in justification and argumentation, and abilities to identify similarity and difference and express generality which are critical to later mathematical achievement.

This chapter is structured as follows. An overview of the importance of the early years' mathematical development is presented with an emphasis on teaching and learning for Indigenous children. The research on patterning and early algebraic thinking supports the focus of patterning in early numeracy programs such as PEAP (Papic 2009). Key findings from the two studies that informed the development of the current PEAP project are presented. A case study of one of the early childhood centres involved in the current longitudinal study is then highlighted. A focus on the development of children's patterning and early algebraic thinking skills is included here along with exemplars of the development of teachers' pedagogical and mathematical content knowledge.

Mathematics in the Early Years

Internationally, educational policy and practice is being increasingly directed towards improving early childhood education with new programs designed to promote mathematics learning e.g. *Building Blocks Project* (Clements and Sarama 2007), *Curious Minds* (van Nes and de Lange 2007) and the *Pattern and Structure Project* (Mulligan and Mitchelmore 2009; Mulligan et al. 2006). It has been found that high quality, developmentally appropriate early childhood programs can produce both short- and long-term positive effects on children's cognitive and social development (Barnett et al. 2007). In mathematics education, recent research indicates that early intervention can prevent later learning difficulties (Clements and Sarama 2007; Doig et al. 2003; VanDerHeyden et al. 2006). "Without active intervention it seems likely that children with little mathematical knowledge at the beginning of formal schooling will remain low achievers throughout their primary years and probably beyond" (Aubrey et al. 2006, p. 44). The quality, scope and depth of both the teaching and assessment of early mathematics are now regarded as critical to future success in the subject (Stevenson and Stigler 1992; Young-Loveridge et al. 1998).

Research on mathematics learning has often been restricted to an analysis of children's developmental levels of single concepts such as counting, but has not provided insight into common underlying processes that develop mathematical thinking, reasoning and early generalization. "Given opportunities to engage in mathematical experiences that promote emergent generalization, children are capable of abstracting complex patterns before they start formal schooling. The crucial components are exposure to a variety of patterns in differing modes and orientations, and scaffolding by an adult to justify and transfer these patterns to other media" (Papic et al. 2011, p. 263). The challenge is for early childhood educators to integrate "patterning and structural relationships in mathematics so that a more holistic outcome may be achieved much earlier than previously expected" (Papic et al. 2011, p. 264).

Early Mathematics Teaching and Learning for Indigenous Children

Several Australian research projects have focused on improving mathematics learning opportunities for Indigenous children and supporting Indigenous early childhood professionals (Adams 1998; Meaney et al. 2008; Warren et al. 2008). Recent research focused on improving mathematics learning in Indigenous communities identifies the importance of learning through hands-on experiences that best supports young Indigenous students to engage with mathematical ideas (e.g. Cooper et al. 2006). Recent policy documents on early years' education highlight the need for educators to maintain high expectations of the capabilities of children (Australian Government Department of Education, Employment and Workplace Relations, DEEWR 2009). Raising teacher and community expectations of young Indigenous children will contribute to improving early numeracy. The current study builds on recent research focused on pedagogy that supports Indigenous students' learning and enhances students' engagement with mathematical learning opportunities (Warren and de Vries 2010, p. 4).

Initiatives in the preschool and early years of formal schooling such as a current program for Indigenous children in Queensland (Warren et al. 2008) and the Count Me In Too Indigenous program (NSW Department of Education and Training 2001) exemplify systematic attempts to provide effective mathematics programs suitable for Indigenous contexts.

The importance of community consultation has been highlighted in developing culturally-appropriate teaching and learning experiences, resources and professional support in mathematics (Howard and Perry 2007). The *Maths in Indigenous Contexts Project*, conducted in both rural and urban settings, applied community consultation guidelines to increase teachers' understanding of the needs and cultures in which Aboriginal students live. There were substantial benefits for both the teachers and the community. Similarly the current four year *Make It Count Project* (Australian Association of Mathematics Teachers 2010) aims to identify practices that will focus on the school as a unit of change with contribution from the entire school and wider community (Perso 2009).

In a recent Australian Government project *Mathematical Thinking of Preschool Children in Rural and Regional Australia: Research and Practice*, 19 Indigenous early childhood professionals described their understandings and beliefs about young children's learning, including mathematics which they saw as inextricably linked to Indigenous peoples' place in their community (Papic et al. 2010). The larger study (Hunting et al. 2008) revealed that 64 preschool practitioners were able to identify and provide convincing examples of both incidental and planned mathematical activities across a breadth of content strands, including number and operations, measurement, geometry, data collection, and fundamental classifying and ordering activities (Hunting et al. 2008). However, they were less aware of the mathematical *processes* such as problem solving, explaining and interpreting, inherent in many of the activities preschool children typically engage in on a daily basis. The development of such processes is critical to a deep understanding of mathematics in subsequent years of formal schooling (Papic et al. 2011) and is often dependent on a teacher's knowledge of both subject matter and pedagogical content specific to teaching mathematics.

An understanding of how concepts and ideas relate provides a foundation for *pedagogical content knowledge* that enables teachers to make ideas accessible to others (Sullivan et al. 2009). "The ability of preschool practitioners to plan developmentally appropriate experiences that foster the advancement of mathematical concepts and processes of young children is dependent on a complex combination of both mathematical subject matter and pedagogical content knowledge" (Bobis et al. 2010). However, to enhance the quality of mathematical teaching in the early years, professional development initiatives must build not only the pedagogical content knowledge of teachers but also address the beliefs, attitudes and dispositions teachers bring to the classroom (Ball et al. 2008).

Mathematical Patterning and Early Algebraic Thinking

"Algebra can be viewed as a symbolical language that enables us to express relationships and generalizations, usually involving numbers, and use them in order to solve problems without the extensive numerical computations that might otherwise be necessary (e.g., when using trial and error)", (Papic et al. 2011, p. 239). The central idea here is that of a generalization, that is, a relationship that holds over an entire class of values and not only in isolated instances (Dörfler 1991). "From this perspective, finding and using generalizations may be considered as *algebraic thinking*" (Papic et al. 2011, p. 239). The roots of algebraic thinking therefore lie in detecting sameness and difference, in making distinctions, in classifying and labelling, or simply in algorithm seeking (Mason 1996).

Mason and his colleagues (Mason et al. 2007, 2005, 2009), have argued for a focus on generalization in the early years. Mason holds the view "that students come to school with natural powers of generalization and abilities to express generality, and that the development of algebraic reasoning is, in large part, a matter of tapping into those naturally occurring capacities for didactic purposes" (Lins and Kaput 2004, p. 54).

A pattern is a type of generalization, in that it involves a relationship that is "everywhere the same". It is perhaps on this basis that some researchers have claimed that early algebraic thinking develops from the ability to see and represent patterns in early childhood (Mason et al. 2005). Others have claimed that the integration of patterning in early mathematics learning is critical to the abstraction of mathematical ideas and relationships, and the development of mathematical reasoning in young children (English 2004; Mulligan and Mitchelmore 2009; Papic et al. 2011). The integration of patterning in early mathematics learning can promote the development of mathematical modelling and heuristic strategies in problem-solving contexts. In working with young Aboriginal children it is useful to consider the patterns and symbols children may frequently be exposed to: patterns in traditional and contemporary art works (iconic dots and concentric circles), cave paintings and rock art; patterns and symbols that express Dreamtime stories; and weather patterns.

At a fundamental level, "patterning is an essential skill in early mathematics learning, particularly in the development of spatial awareness, sequencing and ordering, comparison and classification. This includes the ability to identify and describe attributes of objects and similarities and differences between them" (Papic 2007, p. 8). Patterning is critical to the development of key mathematical concepts and processes such as counting and multiplicative thinking. Recent research with young children has shown that the early development of pattern and structure positively influences, mathematical achievement overall and provides a stronger foundation for algebraic thinking (Papic et al. 2011).

Patterns and Early Algebra Studies

Two main studies informed the development of the *Patterns and Early Algebra* (PEAP) *Professional Development* (PD) *Program*.

Initial Study

The initial study focused on the development of patterning strategies in 53 children from two preschools (Papic and Mulligan 2007). One preschool implemented a 6-month intervention focussing on Repeating and Spatial patterns. Three early childhood educators participated in two three hour initial training sessions and were then provided with ongoing weekly support where the researcher visited the centre one day a week and held an additional weekly meeting with staff. The weekly visits provided valuable opportunities for team teaching and modelling of teaching strategies, questioning skills and recording of observations of children's patterning development. The additional meeting provided a platform for staff and the researcher to evaluate the week, share successes and any challenges and plan for the following week's experiences. The second preschool implemented their regular preschool program with no additional professional development or support from the researcher.

An interview-based Early Mathematical Patterning Assessment (EMPA) (Papic et al. 2011) was developed and administered pre- and post-intervention, and again following the first year of formal schooling. The intervention group outperformed the comparison group across a wide range of patterning tasks at the post and follow-up assessments. Intervention children demonstrated greater understanding of patterns as repeated units and spatial relationships. In contrast, most of the comparison group treated repeating patterns as alternating items and rarely recognised simple geometrical patterns. Intervention children were able to justify various patterns and transfer patterns to different media. The Intervention had drawn children's attention to structure. The early childhood educators "repeatedly encouraged children to look for structural similarities and differences between the given pattern and their copy of it" (Papic et al. 2011, p. 261). One year after the Intervention, the Intervention children continued to outperform the non-Intervention children on patterning tasks including growing triangular and square number patterns which neither group had been exposed to during the Intervention or the first year of formal schooling. The Intervention children also outperformed the non-Intervention children on a standard numeracy assessment, Schedule of Early Number Assessment SENA 1 (NSW Department of Education and Training 2001) at the end of the first year of formal schooling.

The Intervention included individual or small group sessions with each child once a week. An instruction framework based on patterning tasks guided the sessions (Papic et al. 2011). Children copied and drew patterns, identified the unit of repeat and the number of repetitions, were encouraged to describe similarities and differences between patterns and explain their strategies and thinking. Children were also encouraged to identify the unit of repeat in various repeating patterns and generalize this to create other patterns using various materials, still containing the same pattern structure (e.g. ABC, ABBA, ABCD). Teachers were also supported to "Patternise" their regular preschool program where they incorporated rich patterning experiences within their daily planning and implementation. A component of the Intervention was to document children's mathematical thinking and pattering in free play.

A critical factor of the Intervention was developing early childhood educators' pedagogical and content knowledge. This enabled them to incorporate patterns, problem solving and mathematical language and concepts into their teaching. The educators felt that they had gained new ideas on how to include patterning experiences into the daily curriculum:

> We now have a heightened awareness of the importance of patterning and spatial structure tasks in the early childhood setting and their importance in providing foundations for mathematical development. We are more aware of children initiating complicated pattern making in free play and by documenting these experiences provide appropriate experiences for children to explore patterns further at their own level. (*Teacher C1*)

Follow up Study

The Intervention and the approach to professional development from the initial study were refined based on the feedback from early childhood educators. This included a simpler and shorter interview-based Early Mathematical Patterning Assessment (EMPA), revision and inclusion of an additional level on the Instructional Frameworks, the inclusion of examples of documentation in the teacher Professional Development Package and a shorter implementation time. The *PEAP PD Program* (Papic 2009) was designed to be used in the follow up study throughout the training period and implementation of the program.

In the follow up study,[2] two long day care centres which operate a preschool program were randomly selected from a list of centres that contacted the researcher over a 24 month period requesting more information on the initial study. A number of these centres were Aboriginal Children's services therefore one Aboriginal service and one mainstream service was randomly selected. Across the two centres a total of 64 children and nine early childhood educators were involved in the study. One training day was conducted at each of the centres and three support visits throughout the 10 week program implementation. The professional development focused on building early childhood educators' understandings of mathematical and pedagogical content knowledge including an understanding of different types of patterns, early algebraic thinking and approaches to developing children's mathematical thinking including problem solving tasks, seeing similarity and difference, questioning, communicating, justifying, reasoning and generalizing. This was done through analysis of video footage, team teaching, modelling teaching strategies, analysis of teachers' planning, observations and documentation of children, sharing of teachers' reflective journals and focus group discussions in week 5 and week 10 of program implementation. Focus groups identified the importance of building teacher's pedagogical and content knowledge:

> We wouldn't have been able to achieve this through our normal routine and our normal program. As capable as we know children are and we are we wouldn't have thought children

[2]Funded by Macquarie University.

Fig. 1 Game: Estimate 6!

are able to do what they did ... We weren't thinking like that about mathematics let alone getting the children to think like that. (*Teacher B1*)

My mathematical knowledge has really grown. I thought I was working on patterns with children but now I know they weren't really patterns. (*Teacher B2*)

I wasn't really good at maths but I feel this has helped me ... and it has built my self-confidence. (*Teacher B3*)

The program has helped us with the language ... how to get the children to explain their thinking ... Before we started this program I would have thought that counting up to high numbers would have been really important to go to school, like counting to 100 wow! But I would never have thought of all the other maths skills ... subitising, classifying, ordinal numbers, and developing mathematical thinking is really important, asking children to explain their thinking, look at what's the same, what's different, looking for patterns ... (*Teacher B2*)

Educators also identify the importance of the assessment tool, the learning framework and using culturally appropriate materials (see Fig. 1).

At first I felt bad about putting children on a level based on my judgement of assessing them on a couple of tasks ... because we don't normally assess children that way but then what I appreciated then out of that process was that we had a really close look at where children were and then using the Frameworks we get to see this amazing development and this amazing journey that they have gone on ... it was a great way to see where they were and where they were going and it helped us direct them as well ... Without that assessment tool we would not have looked at the children's skills and thinking that closely ... The program itself has given us the strategies to really assess the children at their right level and then engage them in mathematical thinking and learning then to develop them further ... given us the skills to develop them further in this way that is culturally empowering, that is why kids like [child's name] are pumped because they can identify with it and feel real proud ... in that number line game she beams when she plays that game because she has a connection with it (see Fig. 1). (*Teacher B1*)

The teachers were confident in extending children's learning and taking advantage of opportunities to develop children's mathematical thinking and skills in justifying their thinking and strategies: "we are now better equipped to extend the learning because we now have the skills ... we are able to capitalise on opportunities" (Teacher B1). The excerpt below exemplifies how Teacher B1 extended a patterning experience with a child who created a cyclic pattern of camels.

Fig. 2 Adding a unit of repeat to a complex repeating cyclic pattern

Teacher: What's the pattern that you made?
Child T11: Big purple, little purple, little yellow, big green. It's three times.

The teacher creates another repeat and asks the child, "If I want my camels to join up in the circle with your camels but I still want to keep the pattern repeating where will I join in?" The child recognises that there is an error in the size of the green camel the teacher has created and explains to her that she needs to swap it with a large green camel. The child then confidently adds the unit of repeat to the rest of his pattern of camels (see Fig. 2). The teacher continues to extend the child's learning by asking the child where the pattern begins. The child identifies the large purple camel. The teacher then asks the child if the pattern could begin somewhere else. The child points to a large green camel.

Teacher: Okay, what's the pattern then?
Child T11: Big green, big purple, little purple, little yellow. *The child points to each big green camel saying* one time, two times, three, times, four times *to show the number of repetitions*. The child clearly has an understanding of units of repeat as he does not say the whole unit of repeat just the first item in each repetition.

The teacher then asked the child to identify another starting point and other possible units of repeat. The teacher and the child explored the two different directions the pattern could go, clockwise and anticlockwise and the different unit of repeats this would create.

Teacher: How come you can start the pattern from anywhere?
Child: Because it's a circle.

The teacher continued to extend the experience by creating problem solving situations for the child to solve. The teacher asked the child to close his eyes and said she would take one camel away from the circle. The teacher removed the yellow camel and rejoined the circle then asked the child to open his eyes and identify the

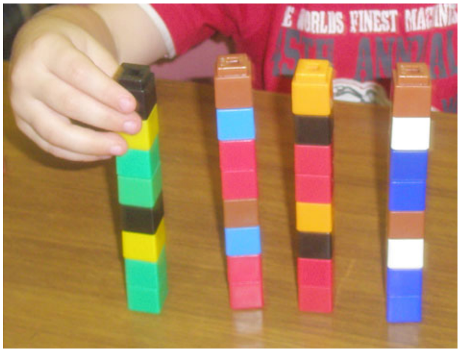


Fig. 3 Complex repeating patterns (AABC repetition)

missing camel. The child looked careful at the circle and then replied “I think I have to count everyone to see which one is missing”. The child goes on to count the green camels only, “one green, two green, three green, four green”, then the purple camels “one big purple, two big purple, three big purple, four big purple”, the child suddenly has a big smile on his face and says, “I know which one’s missing, it’s one of the yellow”. The teacher asked him how he knew. The child responded “because there’s only three”. The child then placed the yellow camel in the correct spot back in the circle.

These examples show how the development of the teacher’s pedagogical and content knowledge gave her the skills to extend the child’s mathematical thinking through problem solving and questioning. It is imperative that teachers ‘listen to’ their children not ‘listen for’ an anticipated response. In having a genuine interest in what children do, teachers can extend children’s learning through appropriate and contextual dialogue rather than getting them to say and do ‘what is expected’.

Teachers also developed their own skills in assisting children to abstract the unit of repeat from various repeating patterns and generalize the pattern structure. The following excerpt shows dialogue between the teacher and a child working with complex repeating patterns. The child created a repeating pattern (AABC—blue, blue, white, brown repetition) and the teacher created another block tower using different colours but containing the same pattern structure (AABC—red, red, black, orange). The teacher asked the child to look at the two towers (Fig. 3).

Teacher: What is the same between this tower and that tower?
Child: This one has blue, blue and this one has red, red.
Teacher: What’s the same about that?
Child: Because they both have two same blocks.
Teacher: What else is the same?
Child: Cause it’s repeated two times in both.

The teacher asked the child to identify what else was the same between the two towers. The child then identified the different colours in the third and fourth position of the unit of repeat and with the teacher’s assistance was able to verbalise this.

Teacher: Two the same and . . .
Child: One different and another different one.

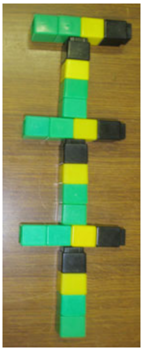


Fig. 4 Hopscotch pattern created with an AABC repetition

The teacher went on to ask the child to create another tower that has the same pattern "but it doesn't have to have the same colours". The child went on to create another AABC pattern: red, red, blue, brown repeated twice (Fig. 3).

Teacher: So what was the pattern?
Child: Red, red, blue, brown, one time, red, red, blue, brown, two times.
Teacher: So, we made a pattern using two blocks that are the same colour.
Child: *The child cut in and said* and two blocks of different colours.
Teacher: Do you think you can do it again? This time don't use red.
Child: Okay, green, green … then yellow, black. Green, green, yellow, black (see Fig. 3).
Teacher: So we have four towers. What is the pattern in all these towers? So in every tower we used …
Child: The same colour twice then a different colour twice.
Teacher: So, in each pattern we used four blocks but we only used …
Child: Three.
Teacher: Three what?
Child: Three colours.

The teacher then asked the child to break up the towers to show the units of repeat and asked the child again what was the same about each tower pattern? The teacher prompted the child to identify that the structure was the same and that each unit of repeat was repeated two times. The teacher then used one of the units of repeat to encourage the child to make a hopscotch pattern (see Fig. 4) and assisted the child to use the mathematical language of vertical and horizontal when explaining his pattern.

Teacher: Which way are you putting that?
Child: Vertical. And the next one is horizontal.

When the child completed the pattern the teacher asked the child "What is the pattern?"

Child: Vertical, horizontal, vertical, horizontal, vertical, horizontal.
Teacher: What does it look like from my end?
Child: Horizontal, vertical, horizontal, vertical, horizontal, vertical.

In these examples the child was able to express relationships between the towers by looking at similarities and differences in the repeating patterns and with teacher scaffolding went on to make simple generalizations about the unit of repeat. Generalizations at this level can be referred to as pre-algebraic thinking (Papic et al. 2011).

Current Study: Improving Numeracy Outcomes for Young Australian Indigenous Children Through the PEAP PD Program

A three-year longitudinal study (2011–2013) is currently being conducted in rural and regional communities across NSW. The project consists of a design study integrating a quasi-experimental approach and adopts a social-constructivist approach to learning. The PEAP PD Program is being implemented in 13 Aboriginal Childcare Services or Preschools where approximately 85 % of staff and children are Indigenous and two privately operated services with a high percentage of enrolments from Aboriginal families (50–60 %). Seven services participated in 2011–2012 and the remaining eight will participate in 2012–2013. Approximately 60 early childhood educators and 240 4–5 year old children are involved in the study.

The PEAP program is implemented in collaboration with Indigenous and non-Indigenous early childhood educators working within the services. Through ongoing professional development and support from staff from Gowrie[3] Indigenous Professional Support Unit (IPSU) educators build pedagogical content knowledge and understanding of mathematical content. Children will be tracked into Kindergarten (NSW first year of formal schooling) and their mathematical development will be assessed using the standardised Kindergarten assessment *Best Start Numeracy* (NSW Department of Education and Training 2009). Additional information will be gained from the children's Kindergarten teachers through semi-structured interviews to identify the teachers' perceptions of the children in terms of mathematical knowledge and confidence in the classroom to engage, question, communicate and justify their thinking during mathematical experiences.

The study builds on the previous studies outlined in this chapter and the work of Warren and colleagues that "draws from and adapts relevant mainstream research about young students' numeracy learning and endeavours to situate these findings in local settings where Indigenous cultural practices are recognised and respected" (Warren et al. 2009, p. 46). However, the proposed project adopts a community-based approach to learning by utilising existing positive relationships developed between researchers and professionals within the selected early childhood centres.

[3] ARC Linkage industry partner.

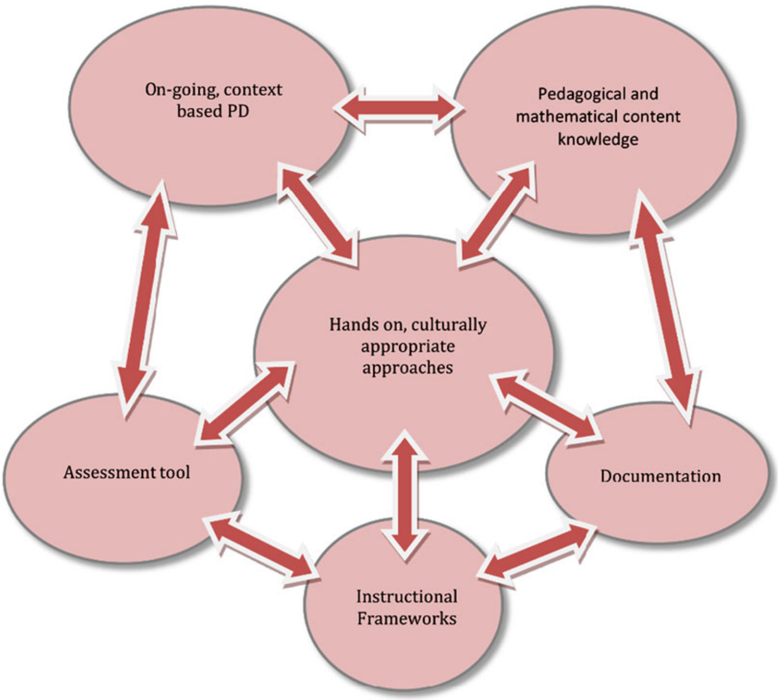


Fig. 5 Six critical components of the Patterns and Early Algebra Professional Development Program

The study aims to:

1. Advance pedagogical content knowledge and mathematical subject knowledge of early childhood educators working in Aboriginal Children's Services that are closely tied to the local Indigenous communities.
2. Support teachers to implement an early numeracy program that explicitly aims to develop mathematical thinking and reasoning.

Feedback from Teachers involved in the previous studies identified six key factors (see Fig. 5) as critical components of the PEAP PD Program:

1. Development of pedagogical and mathematical content knowledge of educators
2. On-going, context based professional development
3. Assessment tool to map children's mathematical capabilities
4. Instructional Frameworks to guide teaching and learning
5. Hands on, culturally appropriate approaches to early numeracy
6. Documentation of children's mathematical explorations and thinking

Development of Pedagogical and Mathematical Content Knowledge of Educators

Each of the services received three days initial training. This comprised building understanding of mathematical and pedagogical content knowledge, including an understanding of different types of patterns, early algebraic thinking and approaches to developing children's mathematical thinking e.g. problem solving tasks, seeing similarity and difference, questioning, communicating, justifying, reasoning and generalizing. Staff worked through a training package (Papic 2009) and engaged in practical patterning and early algebra experiences. Implementation of the interview based assessment tool, *Early Mathematical Patterning Assessment* (*EMPA*), familiarity with the *Frameworks of Assessment and Learning* and approaches to documentation were key components of the training sessions. Early childhood educators were also introduced to NING™ (Glam Media Technologies), an online platform that allows participants of the study to create their own social network. The *PEAP social networking website* allows early childhood educators to communicate both with researchers and network with other early childhood services implementing the PEAP program. NING™ also provides early childhood educators with a platform to clarify concerns, ask questions, share experiences and view additional resources supplied by researchers at all times of the day.

On-Going, Context Based Professional Development

Building on the three days of training, educators at each centre receive nine additional visits. One member of the research team, this includes staff from Gowrie IPSU, make a whole day visit. Each visit focuses on a particular area of development. This includes:

(a) supporting teachers to plan activities and opportunities that incorporate or promote rich patterning experiences and that build on children's interests and explorations,
(b) modelling numeracy teaching experiences,
(c) working with staff to document children engaged in numeracy through their free play using the National Early Years Learning Framework (DEEWR 2009),
(d) documenting children's mathematical thinking and language,
(e) working with staff as they implement the structured patterning activities with individual children focusing on developing children's mathematical thinking and reasoning skills through conversation with children, questioning and extending children's learning, and
(f) integrating technology into teaching and learning, providing opportunities for children to explore patterning and other mathematical concepts and processes.

An additional two days of support is provided to all services the year after implementation, focusing on an area of development identified by the early childhood educators in each centre. This support is different for each service and designed in partnership with the staff.

Assessment Tool to Map Children's Mathematical Capabilities

An interview based assessment tool *Early Mathematical Patterning Assessment* (*EMPA*), was revised from earlier studies (Papic et al. 2011). EMPA (see Table 1) assessed children's pattern recognition and problem solving strategies. EMPA was administered by the teachers to children on a one-to-one basis at the start of the implementation of the program. Each assessment took approximately 15 minutes to administer. A variety of tasks assessed children's facility with repeating and spatial patterns and included copying, drawing and continuing patterns and identifying the number of dots or objects in various spatial arrangements. Children's responses were initially coded for accuracy then their various strategies for each task were identified (see Table 2). These strategies were then classified into four increasing levels of sophistication (see Table 3), "focusing on the structure of the representation and the use of a unit of repeat" (Papic et al. 2011, p. 247).

An Instructional Framework to Guide Teaching and Learning

Instructional Frameworks incorporating pattern-eliciting tasks guided individual teaching over the 12 week period (see Tables 4 and 5). The Frameworks provided critical opportunities for developing early algebraic and mathematical thinking through sequential problem-solving patterning activities. Tasks focus on repeating (Table 4) and spatial (Table 5) patterns. Development levels reflect Mulligan and colleagues' analysis on levels of pattern and structure with first-graders (Mulligan et al. 2004).

Through the pattern-eliciting tasks children were encouraged to identify similarity and differences within patterns and between patterns, abstract the unit of repeat and the number of repetitions, create same patterns structures (e.g. ABC) using different materials (e.g. blocks, shapes), justify their thinking, and view patterns from different orientations. Children's performance on the EMPA was used to identify their level on each of the Frameworks.

Hands on, Culturally Appropriate Approaches to Early Numeracy

The importance of hands-on experiences has been highlighted in recent research literature as a critical approach to engaging young Indigenous children with mathematical ideas (Cooper et al. 2006). All components of the current program incorporated hands on experiences that are culturally appropriate. These included using cultural appropriate materials, games that integrated Aboriginal designs and artwork, Indigenous stories and cooking experiences and cultural events.

Table 1 Early Mathematical Patterning Assessment (EMPA)

Tower patterns			
TP1	Children are shown a tower pattern of 6 blocks:  (*Blocks are joined to make a vertical tower*)	1. Copy with blocks 2. Record with colour pencils 3. Continue pattern	1. *Look at my tower. Make a tower pattern exactly like mine, with the same colour and the same number of blocks, in the same position as mine.* 2. *Draw the tower pattern using the colour pencils. Make your drawing look exactly like my tower, same colour and number of blocks in the same positions as mine.* 3. *Look at my tower pattern. What do you think the next blocks on my tower would be if I continued the pattern? Put them on for me.*
TP2	Children are shown a tower pattern of 6 blocks:  screened screened (*Blocks are joined to make a vertical tower*)	1. Identify screened element 2. Record with colour pencils	1. *Look at my tower pattern. I've got cardboard hiding two of the blocks* (3*rd and* 4*th block is screened from children's view*). *What is the colour of the hidden blocks*? 2. *Draw this tower showing all the blocks including the hidden blocks* (2 *blocks remain screened during this task*).
TP3	Children are shown a tower pattern of 6 blocks:  screened (*Blocks are joined to make a vertical tower*)	1. Identify screened block	1. *Look at my tower pattern. I've hidden one of the blocks* (5*th block is screened from children's view*). *What colour is the hidden block*?
TP4	Children are shown a tower pattern of 6 blocks:  (*Blocks are joined to make a vertical tower*)	1. Copy tower from memory 2. Draw tower from memory	1. *Look at my tower*. (*Now screen whole tower*). *Make a tower pattern exactly like mine, with the same colour and the same number of blocks, in the same positions as mine.* (*Remove tower made by child*) 2. *Have another look at my tower* (*now screen whole tower*) *draw the tower pattern using the colour pencils. Make your drawing look exactly the same as my tower.*

Table 1 (Continued)

Spatial patterns			
S1	3 counters, then 5, then 1, then 4, then 2 are placed in front of the children	1. Count to determine how many	1. *How many counters are there*? (3) (*Repeat for* 5, 1, 4 *and* 2)
S2	*This activity is done only if the child can count some objects and determine 'how many'—see pre-activity S1.* Children are shown various patterns for a brief period then asked to identify how many in the pattern (see below for patterns presented)	1. Identify regular dot patterns 2. Identify grid dot patterns 3. Identify stair case block patterns (*concrete*) 4. Identify random collections (irregular dot patterns)	1. *I'm going to show you this card/these blocks very quickly*. (*Say silently* 1000, 2000) *How many dots/blocks were there*?

Table 2 Early Mathematical Patterning Assessment (EMPA) recording sheet

TP1	1. Copy with blocks	1. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
	2. Record with colour pencils	2. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
	3. Continue pattern	3. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
TP2	1. Identify screened element	1. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
	2. Record with colour pencils	2. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
TP3	1. Identify screened block	1. Yes/No ____________________
TP4	1. Copy tower from memory	1. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
	2. Draw tower from memory	2. Yes/No ____________________ ☐ colour ☐ number ☐ pattern
S1	1. Counting to determine how many	3 5 1 4 2
S2	1. Identify regular dot patterns	3 5 4 6
	2. Identify grid dot patterns	5 3 4
	3. Identify vertical stair case block patterns	5 3
	4. Identifying random collections	5 4 6 3

Table 3 Early Mathematical Patterning Assessment: Classification of children's strategies

Performance on Tower pattern assessment tasks	Level	Stage
Children had great difficulty completing Tower pattern tasks: gave no response or, solution strategies for drawn tower representations were either scribbles or markings where no units were evident.	1	*Pre-structural* (*PR*)
Solution strategies for tower representations were not represented in a row or column or, if they were presented in a row or column incorrect number and colour of blocks were evident. Solution strategies frequently inconsistent and do not show repetition of pattern elements.	2	*Emergent* (*E*)
Solution strategies for tower representations were presented in a row or column with at least one property (colour, number or unit of repeat) in most of their representation.	3	*Structural* (*S*)
Solution strategies for tower representations were predominantly accurate. Patterns were represented in a row or column with the correct properties: colour, number and pattern element.	4	*Advanced structural* 1 (*AS1*)

Performance on Spatial pattern assessment tasks	Level	Stage
Cannot count to six by ones.	1	*Pre-emergent* (*PE*)
Recognises regular dot patterns for very small numbers such as 2 or 3, however, used unitary counting for larger numbers.	2	*Emergent* (*E*)
Instantly recognised regular dice patterns 1–6 and irregular dot patterns for small numbers such as 2 and 3, however, used unitary counting of the irregular dot pattern for larger numbers.	3	*Perceptual* (*P*)
Instantly recognise regular dice patterns and irregular dot patterns 1–6.	4	*Conceptual* (*C*)

Documentation of Children's Mathematical Explorations and Thinking

Documenting children's learning is a critical component of the assessment of children's learning where educators "collect rich and meaningful information that depicts children's learning in context, describes their progress and identifies their strengths, skills and understanding" (DEEWR 2009, p. 17). Documenting children's mathematical learning also involves the recording of children's problem solving strategies, mathematical thinking, explanations, generalizations and use of mathematical language. This not only showcases children's current understandings and knowledge but provides a powerful tool for educators to plan appropriate follow up learning experiences and opportunities. The professional development provided to early childhood educators during the weekly support visits focused on building teachers' skills and knowledge in documenting children's mathematical explorations and learning in free play and intentional teaching situations. Educators were

Table 4 Repeating pattern Instructional Framework to guide teaching and learning

Level 1—*Pre-structural*

1. Copying 2-block tower with blocks
2. Copying 2 block tower
   (a) Constructing with coloured tiles
   (b) Drawing with Textas
3. Designing own 2-block tower
4. Drawing 2-block tower from memory

Level 2—*Emergent*

5. Copying 4-block ABAB tower (2 colours × 2)
6. Drawing 4-block ABAB tower (2 colours × 2) by copying using textas
7. Designing own 4-block ABAB tower (2 colours × 2)
8. Drawing 4-block ABAB tower (2 colours × 2) from memory
9. Continuing tower pattern to make 6-block tower (2 colours × 3)

Level 3—*Structural*

10. Copying 6-block tower ABABAB (2 colours × 3)
11. Drawing 6-block tower ABABAB (2 colours × 3) by copying using textas
12. Designing own 6-block tower ABABAB (2 colours × 3)
13. Drawing 6-block tower ABABAB (2 colours × 3) from memory
14. Continuing tower pattern to make 8-block tower (2 colours × 4)
15. Finding the missing block or error e.g. RBRBBR RBRB_B

Level 4—*Advanced structural 1*

16. Copying block towers with 3 colour repetitions (ABC) × 2 × 3 and × 4
17. Drawing by copying block towers with 3 colour repetitions (ABC) × 2 × 3 and × 4 using textas
18. Designing own block towers with 3 colour repetitions (ABC) × 2 × 3 and × 4
19. Drawing from memory block towers with 3 colour repetitions (ABC) × 2 × 3 and × 4 using textas
20. Continuing tower pattern e.g. RBGRBG_ _ _ YBOYBOYBO_ _ _
21. Finding the missing block or error e.g. RBGRBGRBR RBGRBG_BG
22. Continuing various complex single variable patterns e.g. ABBABBABB_ _ _, ABBCABBCABBC_ _ _ _
23. Copying various complex single variable repetitions from memory
24. Designing various complex single variable repetitions
25. Drawing various complex single variable repetitions from memory using textas

Level 5—*Advanced structural 2*

26. Designing own multi-variable repetitions
27–30. Recognising, continuing and creating repeating border patterns
   27. Identify the various unit of repeat in border patterns
   28. Continuing a given border pattern
   29. Creating a border pattern around a picture
   30. identifying the missing item in the border pattern
31. Recognising, copying and drawing hopscotch patterns from different orientations.
32. Creating hopscotch patterns

Table 5 Spatial pattern Instructional Framework to guide teaching and learning

Level 1—*Pre-emergent*	Level 2—*Emergent*	Level 3—*Perceptual*	Level 4—*Conceptual*
Rote counts forwards and backwards 1–6	Subitises regular dot patterns:	Subitises irregular dot patterns:	Subitises dot patterns 1–10:
Counts up to 6 items	❖ 1, 2, 3	❖ 1, 2, 3	❖ 1, 2, 3, 10
Identifies numerals: 1, 2, 3, 4, 5, 6	❖ 4, 5, 6	❖ 4, 5, 6	❖ 4, 5, 6 ❖ 7, 8, 9

supported to use mathematical language and terminology in their documentation, analyse this information and then plan rich follow up mathematical activities and environments.

Initial Findings: A Case Study

This section provides a snapshot of one centre's implementation of the PEAP PD program in 2011. An Aboriginal Children's Service west of Sydney participated in the training and 12 week implementation of the PEAP Program. Three staff members worked in the 3–5 year old room; one University trained, one Diploma trained and one with a Certificate Three. Fifteen children, commencing formal schooling the following year, participated for the entire duration of the program.

Children were assessed using the EMPA. Figure 6 shows the tower pattern tasks each child completed over the 12 week period. All children had at least the ten recommended sessions except for Rosie, Tania and Tatian who had 7, 8 and 9 respectively. All 15 children showed improvement with 87 % ($n = 13$) of children working at Level 3 or above at the completion of the program. At Level 3 (Tasks 10–15) children worked with 6 block towers with AB repetitions. Teachers encouraged children to look for similarities and differences between the towers, to identify the unit of repeat and the number of times the pattern element is repeated, to explain their solution strategies for remembering the tower from memory. Teachers asked children to compare their drawings with their constructions to determine whether they focused on the same elements of colour, number and order and asked children to explain their pattern.

At Level 4 (Tasks 16–25) children were also asked to abstract the unit of repeat and generalize the pattern using other materials. Children were encouraged to explain why the patterns were the same e.g. Ivan: "It's like this one, four colours, three times". Children were also encouraged to make different units of repeat with the same items. Figure 7 shows the pattern Tania made with two blues and a red that she spun using the 'splash' dice. The teacher encouraged Tania to use the same colours but make different units of repeat. Figure 8 shows the additional two patterns Tania created. Tania went on to make the same patterns using counters and she concluded that no more patterns could be made with two blues and one red.

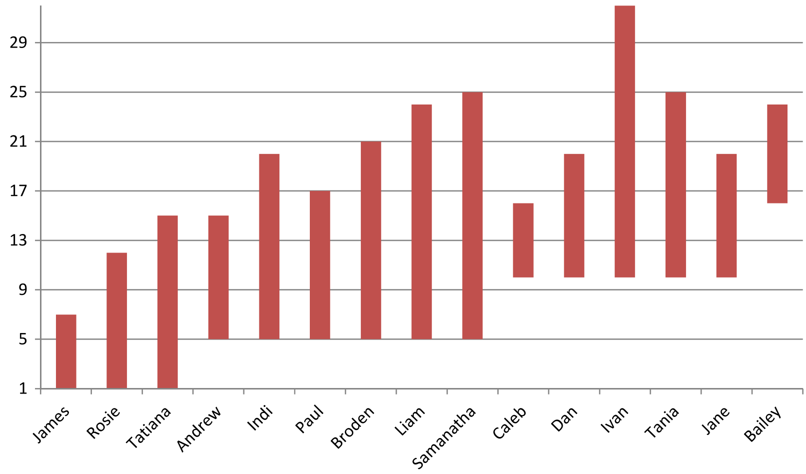
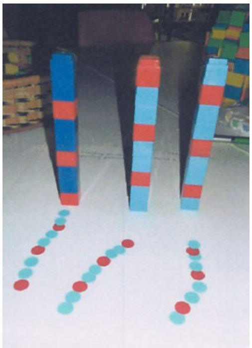


Fig. 6 Tower pattern tasks completed by children

Fig. 7 Pattern made with two blues and one red

Fig. 8 Additional units of repeat made with two blues and one red



Three of the children (20 %) were working through Level 5 Tasks (26–32) at the completion of the program. At Level 5 children were working with complex patterns

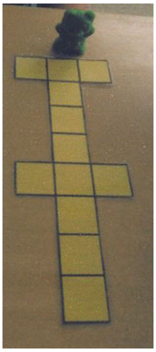


Fig. 9 Hopscotch pattern presented to children

Fig. 10 Drawing hopscotch from two different perspectives

using a variety of materials. Patterns were presented in different orientations such as cyclic patterns (see Fig. 2) and hopscotch patterns (see Fig. 9). Figure 9 shows a hopscotch pattern Ivan was presented with. The teddy represents the starting point of the hopscotch. Ivan drew the hopscotch from the teddy's perspective (Fig. 10, left drawing) and then visualised and drew what the hopscotch would look like from the opposite direction (Fig. 10 right drawing).

Ivan could also create his own complex hopscotch pattern (Fig. 11). Ivan went on to explain his pattern: "See this is a pattern of 'I' for Isaac, three green across, three blue up and three yellow across . . . two times".

Children also participated in six spatial pattern sessions. Tasks focused on regular and irregular dot patterns. Children worked through a series of activities at their level, determined by the EMPA, until they showed competency at that level (see Table 4). Activities that explore spatial patterns are critical in developing pattern and structure in mathematical representation. They develop skills in visualising numbers and number combinations and develop essential skills for mental computation (addition, subtraction and multiplication). Tasks included games that were culturally appropriate. For example children played "Bush Tucker" (Fig. 12). Children roll the dice to 'catch' the corresponding number of witchetty grubs for their 'bush tucker bag'. This task enables the children to practice skills in subitising, counting, addition and subtraction. Children also played "Echidna spikes" (Fig. 13). Each child

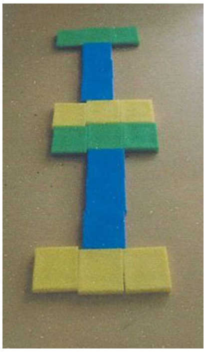


Fig. 11 Ivan's hopscotch pattern

Fig. 12 Game: Bush Tucker!

Fig. 13 Game: Echidna Spikes!

received an echidna playing board and took turns rolling a die, collecting a corresponding number of sticks (all the same colour) and placing them on the spikes of the echidna. On the next roll children repeated the process however, this time chose a different colour (adapted from NSW Department of Education and Training 2001). When all the spikes on the echidna were covered children were encouraged to share their combination of ten with their friends.

Fig. 14 Dot puzzles

Teacher W2: What does your echidna look like?
Alec: It has three blue spikes, three green spikes, and four purple spikes. That's 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 spikes!

Children also played with various puzzles, containing regular and irregular dot patterns (e.g. Fig. 14).

Nine children (60 %) commenced on Level 1 on the Spatial Pattern Instructional Framework (Table 5). Four children (27 %) commenced on Level 2 and two children (13 %) commenced on Level 3. At the end of the six sessions, six children (40 %) were still working on developing skills in subitising regular dot patterns 1–6 (Level 2), five children (33 %) were exploring irregular dot patterns 1–6 (Level 3) and the remaining 4 children (27 %) were working through activities that included spatial patterns 1–10 (Level 4).

At the focus group sessions all three early childhood educators acknowledge they had grown in their mathematical content and pedagogical knowledge:

Teacher W1: My mathematical knowledge has grown dramatically.
Teacher W2: I think we've all achieved something and it's not just the children, it's the staff as well.
Teacher W3: I think the confidence in the children as well as their ability to learn without realising what they're doing and the confidence in ourselves as well. I'm more willing now to stop and let them try without intervening and asking them more questions about what they're doing whereas before I'd just jump in and help them whereas I don't anymore.

The teachers understanding of the importance of patterns and its connection with other mathematical concepts and ideas was developing over the 12 week implementation period.

Teacher W1: Now that these kids can pattern they can do any maths. … I can clearly see now that this is what's needed. Because from this they are learning colours they are learning shapes they are learning "more", they are learning subtraction they are learning "lots of" and "groups" and it's got every mathematical concept they will ever need! (Week 6 of implementation)

The staff identified that some of the parents noticed the development in their children:

In block corner today Tania and Jane were building with the ‘Kids Connects’, and they both realised that they can make bay blades, when the other children seen this they all wanted to make some so Liam, Ivan and Broden all came to make some. Jane and Tania asked Aunty J if they could keep their together so that they could play with them again tomorrow. Tania also decided that while in this area today she was going to build a pyramid.

She started by placing 11 small blocks in a straight line on top of the block cupboard. To the top of this she added 10 small blocks, evenly spaced between each of blocks on the bottom row. She continued to follow this pattern decreasing the number of blocks by one for each row that she added. The last row that she added to the pyramid had two blocks in it, she then walked over to Aunty K and said “Look what I made.” Liam heard what Tania had said and looked at her pyramid. He then said to Tania “You need one more block on top.” To which Tania replied “I can’t find one.” Liam then helped Tania to find another block for the top of the pyramid. Once the block was added Liam stated “That’s better”. Tania then worked with Liam to decorate the pyramid with people and different small blocks. *EYLF LO 4: Children are confident and involved learners. Tania is able to create and use representation to organise and communicate mathematical ideas and concepts.*

Fig. 15 Teacher’s documentation of mathematical exploration in free play

Teacher W1: Just little comments they’ll say, oh they’ve been doing heaps of counting at home and I’ll get the file—so that’s why they’ve been doing that. So it’s sort of clicking with them because they’re going home and they’re saying things and they’re counting. Broden’s mum said he’s been going home and lining things up and patterning things and telling her how to do things … He said he’s been doing times tables and she’s asked him how do you know that and he said because Aunty J’s been showing me. And she thought we were doing actual times tables and I showed her his portfolio and I said no we’re doing it with the blocks and patterns.

The teachers’ development showed in their planning where they were more confident to document the mathematical explorations and learning of the children. They were using mathematical language in their documentation and linking it appropriately to the Early Years Learning Framework (see Fig. 15).

The early childhood educators communicated that the children’s problem solving skills had developed. “They have learnt the art of problem solving” (Teacher W3). Educators were more conscious of the resources they were putting out throughout the day, considering whether they would provide opportunities for mathematical exploration, development and problem solving. The educators were also a lot more aware of the mathematics in children’s play, drawings and paintings. The teachers also highlighted that the children's confidence had grown and they were using mathematical language and having more conversations.

Teacher W2: The way they talk is better and the conversation they have with you whilst doing the work is just like wow, where did that come from ... they understand it!

Teacher W3: They are a lot more confident! Not just doing that [the program] but, out in the room as a whole.

Discussion and Concluding Points

The current study, Patterns and Early Algebra (PEAP), Professional Development (PD) Program, has advanced the pedagogical content knowledge and mathematical subject knowledge of early childhood educators working in the 15 Early Childhood Services and Preschools. Through on-going, context based, and supportive professional development early childhood educators involved in the three year project implemented an early numeracy program that explicitly aimed to develop children's mathematical thinking and reasoning.

The three studies outlined in this chapter provide empirical evidence that young children can develop sophisticated pattern concepts and skills and that children prior to formal schooling can abstract, generalize and explain patterns and pattern structures. They can view patterns from different orientations and use various materials to create complex linear, cyclic and 3D patterns. Teachers recorded the development in children's mathematical language and thinking. They documented episodes of children problem solving, reasoning and generalizing their thinking. These skills are critical for long term mathematical growth and development (Papic et al. 2009, 2011) however, they can only be effectively achieved if teachers are given appropriate support to plan and implement rich mathematical tasks and environments.

If we are to improve numeracy outcomes for young Indigenous children it is imperative that programs are "culturally empowering" (Teacher B1) both for the children and the staff. The impact of such a program on children's numeracy skills and mathematical confidence and competence will be explored when children's Kindergarten Best Start (NSW Department of Education and Training 2009) results are collected and analysed along with interview data collected from the Kindergarten teachers who are teaching the children the year after the PEAP Program implementation. While the PEAP PD project has not been completed, it has empowered the early childhood professionals involved in the first two years of the project: "I have been teaching for over twenty years ... this [PEAP] has reopened my eyes to teaching. It has given me that love back of teaching" (Teacher W1).

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Enhancing Teacher Professional Development for Early Years Mathematics Teachers Working in Disadvantaged Contexts

Elizabeth Warren and Janine Quine

Introduction

The imperative to build strong foundations for mathematical understanding in early childhood settings catering for students from disadvantaged backgrounds is widely acknowledged. It is now well recognised that young children enter early childhood settings with substantive intuitive knowledge about mathematics and this can serve as a base for developing formal mathematical thinking (Carpenter et al. 2003). In addition, young children are capable of engaging with challenging mathematical concepts (e.g., Balfanz et al. 2003). There is also strong evidence that an understanding of mathematics at an early age impacts on later mathematical achievement (Aubrey et al. 2006). These strong foundations are particularly crucial for students from disadvantaged backgrounds. While we recognise that many outside school factors contribute to disadvantaged students being unsuccessful, quality learning is known to be strongly associated with quality teaching (Hattie 2009; Smart et al. 2008). This chapter shares the results of the first year of our longitudinal study situated in the first three years of schooling in some of the most disadvantaged contexts in Queensland, the geographically second largest state of Australia. Its particular focus is drawing implications for professional development for all teachers working in disadvantaged contexts.

Background

Students who are most at risk are often from disadvantaged backgrounds. In a large study with 20,000 students across a range of age groups Denton and West (2002)

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found that by the end of the first year of schooling, a very significant gap in the understanding of mathematical concepts existed between students from high income families and low income families, with the latter gaining little in the first year of school. They also reported that students who possess little mathematical knowledge at the commencement of formal schooling remain low achievers throughout the elementary years and beyond. Given that students from low income families usually come to formal schooling with the same basic readiness to learn as compared with the more advantaged students (Denton and West 2002), the quality of education that occurs in the first years of schooling is crucial to bridging the gap between the two groups.

Disadvantaged Contexts

Disadvantaged students predominantly come from neighbourhoods that exhibit financial disadvantage. In addition to socioeconomic disadvantage, the poorest neighbourhoods tend to have higher rates of social isolation, unemployment, racial isolation, and financial dependence (Sampson 2000). These types of neighbourhoods are a worldwide phenomena, existing in urban, rural and remote contexts. Minority groups such as African Americans, Latinos, Indigenous, and Pacifica groups tend to be particularly vulnerable to neighbourhood disadvantage as they are more likely to cluster in financially disadvantaged areas (Catsambis and Beveridge 2001; Massey and Denton 1993).

Schools in disadvantaged neighbourhoods share a variety of common traits. First, these schools tend to be situated at the lowest levels on a variety of performance measures (e.g., National and International tests of literacy and numeracy performance, and attendance, retention and future employment). Second, these schools have high staff turnover. They experience difficulties in attracting and retaining high quality teachers, and the teachers that they do attract tend to be inexperienced and lack a commitment to teaching in these contexts (Heslop 2011; Mills and Gale 2010; Roberts 2005; Sharplin 2008). Third, these schools commonly possess is poor management and poor professional practice (Lupton 2004). This is exacerbated by the large staff turnover, including classroom teachers and personnel in management positions such as the principal and staff responsible for curriculum development and implementation (Lyons et al. 2006).

Thus, maximising the mathematical achievement of students living in these contexts is not simply about addressing financial disadvantage. While financial disadvantage as the prime predictor of student success and school readiness is well documented (Smart et al. 2008), what students bring to the table (Hattie 2003, p. 1) in conjunction with what teachers know, do and care about (Hattie 2003, p. 2) are the main two dimensions that make a difference. Hattie (2003), in a synthesis of over 500,000 studies from 1992–1999 claims that 50 % of gains in student achievement were related to students' prior understanding of mathematical concepts and their attitude towards learning mathematics. Thirty percent was related to teachers' characteristics. These included the types of knowledge teachers possess, their understanding of the curriculum and its interrelatedness, and their ability to take ownership of their lessons by changing, combining and adding to them according to students' needs and goals.

Teaching Mathematics in These Contexts

Teachers in disadvantaged contexts, who are often inexperienced in both teaching and working in these contexts, encounter many extraneous difficulties that impact on their teaching. Studies have shown that very few feel prepared academically, culturally or professionally by their pre-service education to effectively teach disadvantaged students (Lyons et al. 2006; MCEECDYA 2011; Mills and Gale 2003; White and Reid 2008). Due to high staff turnover, there is often a paucity of experienced teachers to act as mentors. With fewer experienced teachers to mentor beginning teachers, the professional journeys of those starting out can be fraught with obstacles. This is further aggravated in rural and remote schools where they are also geographically and socially isolated. While geographical and social isolation is a very difficult dimension for young urban trained teachers, professional isolation from a pedagogical perspective presents a number of significant problems in terms of effectively engaging and teaching students in mathematics (Cresswell and Underwood 2003; Lyons et al. 2006; Munns et al. 2008). Due to these deficiencies, many teachers are unable to create highly effective instructional programs (Kent 2004; Lyons et al. 2006). Ensuing behavioural problems and poor learning outcomes of their students are often seen as being beyond the teacher's control (Jones 2009). A common pedagogical response can be highly structured classrooms, repetitive learning, reliance on simple achievable worksheets, less time given to teaching, and lowered expectations (Hewitson 2007; Munns et al. 2008).

Providing support for the development of high quality teachers (expert teachers) is the most important agenda schools can adopt to raise student achievement (Hattie 2009; Smith and Gillespie 2007; Timperley 2008; Villegas-Reimers 2003; Webster-Wright 2009). Characteristics of expert teachers are not necessarily related to experience or to their own subject matter knowledge. In a study involving 90 elementary school teachers in the UK, Askew (2008) conjectured that the strength of support provided by 'experts' within the school environment positively impacts on student achievement. The dimensions of this support were the involvement of 'experts' with strong mathematical backgrounds, and in depth knowledge about the psychology of learning and the pedagogy of elementary mathematics. Expert teachers are more focused on solving the learning problems exhibited by individual students in their classroom, and can anticipate, plan and improvise as required by the situation. Their primary attention is on student learning in terms of the affective domain and the quality of their achievement (Hattie 2003). They know what to teach, and how to structure and organise this in the context of their particular students and circumstances. Research on teacher professional learning (PL) and professional development (PD) consistently demonstrates the powerful influence PD can have in assisting teachers to become experts (Hattie 2003). Thus, the professional development support that occurs in disadvantaged contexts is a key to student improvement.

The particular research questions addressed in this chapter are:

1. What elements of professional development enhance early years teaching in schools situated in disadvantaged contexts?
2. How do these elements impact on teaching and learning in these contexts?
3. How does geographical location influence their effectiveness?

Representation Oral language and Engagement in Mathematics (RoleM) is a four-year longitudinal study that follows a large sample of students living in disadvantaged contexts as they progress through the first four years of school. The conceptualization of the professional development model used in the study is informed by literature pertaining to (a) effective professional development practice, (b) theories of learning, and (c) efficacious resources that support at risk students' learning. This chapter focuses on the results of the first year of the study.

Research Informing Professional Development Practice

Five overarching principles drawn from the literature on effective professional development (what is given to the teacher in professional development sessions) and professional learning (what the teacher learns as the result of the professional development) underpinned the conceptualization of the model used in RoleM. Briefly, these professional development requirements need to:

1. Be more than one off events;
2. Emphasise teacher knowledge that is known to improve student learning;
3. Be situated in the context of the participant's classroom;
4. Be clearly linked to desirable and achievable student outcomes; and
5. Provide resources (both physical and in terms of support) that assist teachers' professional learning and teaching.

While professional development has often been viewed as a panacea to all teaching and learning problems that occur in the classroom (Clement and Vandenberghe 2000; Webster-Wright 2009), evidence strongly suggests professional learning is continuing to have a greater impact on effecting teacher change (Gardner 1996). Continuing professional development (CPD) experiences that facilitate teacher understanding of the impact changing pedagogical practices have on student responses and learning, result in improved student outcomes and greater teacher responsibility (Timperley 2008). Teacher change is even more evident when CPD includes a focus on classroom practicalities (Porter et al. 2000).

Second, PD emphasising general teachers' knowledge and teaching competencies known to improve student learning, requires teachers to reconsider their current practices. Curriculum updates, new pedagogical and content knowledge, and theories of student learning and teaching are potential catalysts for change (Garet et al.

2001). In contradiction to a number of studies relating to teachers' subject matter knowledge, teachers perceive that their instructional practices do change when they have greater knowledge of the subject they are teaching (Garet et al. 2001). In addition, providing information and activities that are directly linked to 'valued student outcomes', is critical for the overall success of CPD (Porter et al. 2000; Timperley 2008). The results of research demonstrate that unless this occurs, teachers are unlikely to make a difference to their practices or to student learning (Black and William 1998; Timperley 2008).

Third, professional development is more meaningful to teachers when it is situated within the context of their workplace (Gravani 2007; Murrell 2001; Webster-Wright 2009). The notion of PL is that it is active, social and related to practice (Webster-Wright 2009). Teachers only become active learners when they become active investigators of their own teaching (Shulman 2004). Continual professional learning (CPL) begins as teachers interact with the knowledge they have acquired through CPD and its implementation in the classroom (Gravani 2007; Jarvis and Parker 2005; Murrell 2001). Learning continues as teachers engage in a cyclical process of continual observation and reflection on what is occurring within the teaching and learning environment, and applying and testing their ideas in practice (Kolb 1984; Webster-Wright 2009); the depth and authenticity of the learning is always defined by the teacher.

Fourth, the most significant changes in teacher beliefs and attitudes occur when teachers have multiple opportunities to absorb new information, put it into practice and observe improved student learning outcomes (Guskey 1988; Timperley 2008). Thus, for authentic teacher change to occur teachers need to experience successful classroom implementation of new ideas presented at PD and take time to reflect on student learning. Collaboration with PD facilitators is an essential dimension for facilitating this as these 'experts' provide the necessary scaffolding and support teachers require to implement new pedagogical and content knowledge (Darling-Hammond 1997). Regular on-site visits, by the PD facilitators, allow the teacher as learner to observe instructional strategies modelled in the classroom by the facilitator and to then practice these with extensive support and feedback (Elmore 1996; Joyce and Showers 1995).

Finally, resourcing has an impact on teachers' capacity to effectively teach mathematics, that is, teaching evidenced by improved student learning. Often classroom contexts consisting of marginalised students from low socio-economic, different cultural backgrounds or isolated regions are under resourced (Clements 2004; Cresswell and Underwood 2003; Lyons et al. 2006). Teachers in these contexts lack access to foundational mathematical experiences, such as, high quality mathematic resources and high quality mentor teachers (Lyons et al. 2006). However, research demonstrates that while teachers want and need practical resources these must be linked to specific curricular objectives (Rogers et al. 2006). Once again this amplifies the importance of teachers seeing everything they do in terms of being connected to 'valued student outcomes' (Timperley 2008).

There are a number of measurable outcomes utilised to determine how effective the professional development has been. These outcomes fall into three broad categories. The first pertains to the teachers' affective domain (Guskey 2003). This is hinged on the premise that if teachers enjoy the professional development session they are more likely to implement the ideas and activities in their classrooms (Salpeter 2003). The second is associated with measurable gains in students' achievement (Kent 2004). This is underpinned by the notion that the implementation of the ideas presented at the professional development will result in greater learning outcomes for students. Hence, for professional development to be considered effective, positive changes in students' outcomes should occur. Finally, effective professional development is seen as resulting in changed teacher behaviour, especially in terms of their classroom practice. This is related to the finding that teachers' classroom practices and students' background have a similar effect on student learning outcomes (Wenglinsky 2002). All of these have implications for how professional development is delivered and how this delivery is measured.

Theories of Learning Informing Professional Development Practice

In the Vygotskian socio-cultural perspective, learning is a contextualised and holistic experience. Thus professional learning happens over a long time and is dependent on the interaction that occurs between the learner, the context, and what is learned (Gravani 2007; Jarvis and Parker 2005; Murrell 2001). Integral to continued professional learning is the notion of the Zone of Proximal Development (ZPD) (Vygotsky 1978). ZPD is defined as an individual's potential capacity for development through the assistance of a more knowing person (Vygotsky 1978). The significance of ZPD is that it determines the lower and upper bounds of the zone within which PD instruction and teacher learning should be directed. In the lower bounds, formal PD sessions provide important information that teachers need to know about mathematical content, changes in the curriculum, innovative teaching strategies, and using resources effectively. However, instruction is only efficacious when it goes beyond the notion of simply assisting a person to acquire a particular set of skills or knowledge. Such instruction enables learners to extend themselves through active engagement, exploration and investigation of teaching and learning concepts and activities. In the upper bounds of the ZPD, the 'more knowing person', or 'expert', provides support for teachers through mentoring and scaffolding as these teachers are guided towards competent and accomplished practices (Brockbank and McGill 2006). A purported result of such a model is that the learner is better placed to independently implement innovative pedagogical practices across all curriculum areas after the 'expert' has withdrawn.

The nature and quality of teacher's reflection influence the depth and scope of learning as much as that of the learner's capability (Phillips 2008; Wells 1999). Thus extensive reflection when combined with action, transforms experience into learning (Schon 1983). Teacher reflection serves both an instrumental and a critical function (van Manen 1977). The former encourages teachers to reflect on teaching and learning problems that arise in their classrooms, and formulate practical plans that may solve the problem. Reflection as a critical function provides cognitive and affective insights that can challenge assumptions teachers hold about such things as: the nature of teaching and themselves as teacher, and their students' ability as learners in mathematics (van Manen 1977). As Dewey stated, genuine thinking only occurs 'when there is a tendency to doubt' (as cited in Garrison 2006, p. 3). With on-going support, teachers and 'experts' become co-constructors of knowledge moving within and beyond each others' ZPD.

Research Informing the Development of Resources

The underlying premise that drove the development of the resources provided by RoleM reflected the principles of equitable teaching and student learning. With regard to equitable teaching, briefly, these require ensuring that the resources are: conceptually orientated, open-ended to cater for the differential that exists in student's ability, of high cognitive demand, and are culturally appropriate (Boaler and Staples 2008; Freedman et al. 2005). In addition, resource design was considered a high priority aimed at: attending to how students work, encouraging students to work together, and promoting students effort rather than achievement. As many of the students in this study are Australian Indigenous students the literature pertaining to supporting Australian Indigenous student learning also informed the development of the resources. The resultant framework recognised that these students as learners are (a) imaginable, (b) contextual, (c) kinesthetic, (d) cooperative, and (e) person orientated (Nichol and Robinson 2000; Nichol 2008). In addition these students learn through teachers modelling ideas. The resources also encapsulated:

– *Learning pathways*—Providing a gradual progression along a learning path, with the teacher first modelling what is required, followed by small group work and finally working on an individual basis
– *Integrated experiences*—Involving listening, reading, writing, recording, and speaking about concepts to enhance transference of skills
– *Focused teaching*—Encouraging direct or explicit teaching in conjunction with modelling and giving clear explanations of experiences and setting high expectations
– *Multi-representational*—Using and linking a wide variety of mathematical representations, including number lines, charts, concrete and symbolic
– *Language building*—Encouraging students to move between home language, mathematical language, and Standard Australian English (SAE) as they communicate their mathematical learning, and
– *Making connections*—Making connections within mathematics and with the home and the community.

(Frigo and Simpson 2001; Warren et al. 2009).

These resources were provided to teachers in the form of curriculum documents, learning activities (in both written and digital format), concrete resources (e.g., counters, number ladders, grids), digital resources (e.g., activity sheets, blackline masters) and assessment items (in both written and digital format).

The RoleM Professional Development Model

The model utilised in the RoleM professional development is a socio-constructivist professional development model based on the theories of Vygotsky (1978). It was constructed on the principle that learning is cyclical, consisting of four distinct components; Knowing Person; Collaborative Planning; Collaborative Implementation; and Collaborative Sharing. The RoleM (PD) model was built on the Transformative Teaching Early Years Mathematics Model (TTEYM) created to assist Year 1 teachers to develop and implement Patterns and Algebra activities in their classroom (Warren 2009). It involved teachers in self-reflection as they trial approaches and resources in their classrooms to improve the quality of their teaching practice. It was based on the view that teachers have the ability to improve their practice by trialing 'proven' effective learning experiences, and through continuous cycles of on-the-job reflections and discussions with experts from the field (Castle and Aichele 1994). Figure 1 presents the key components of the RoleM professional development model together with the key focus of each.

As indicated in the model, a cycle of reflect, plan, implement, and share/evaluate was used by teachers during the project. Table 1 presents a summary of the phases together with the activity that occurred at each phase.

Methods

While claims can be made about teacher change from teacher self reported interviews or classroom observations, one of the measurable dimensions of effective professional development is the impact it has had on student learning (Kent 2004; Timperley 2008). Hence, in this chapter we include data relating to the learning outcomes of the students that these teachers taught mathematics to using the RoleM materials.

Sample

The 15 schools that participated in RoleM are from three distinct contexts, metropolitan/provincial, remote, and very remote. These three categories reflect their geographical location. The metropolitan/provincial schools are within close proximity to either the capital city or a city with a population of over 25,000. Remote schools are in geographical locations considered to be spatially distant from

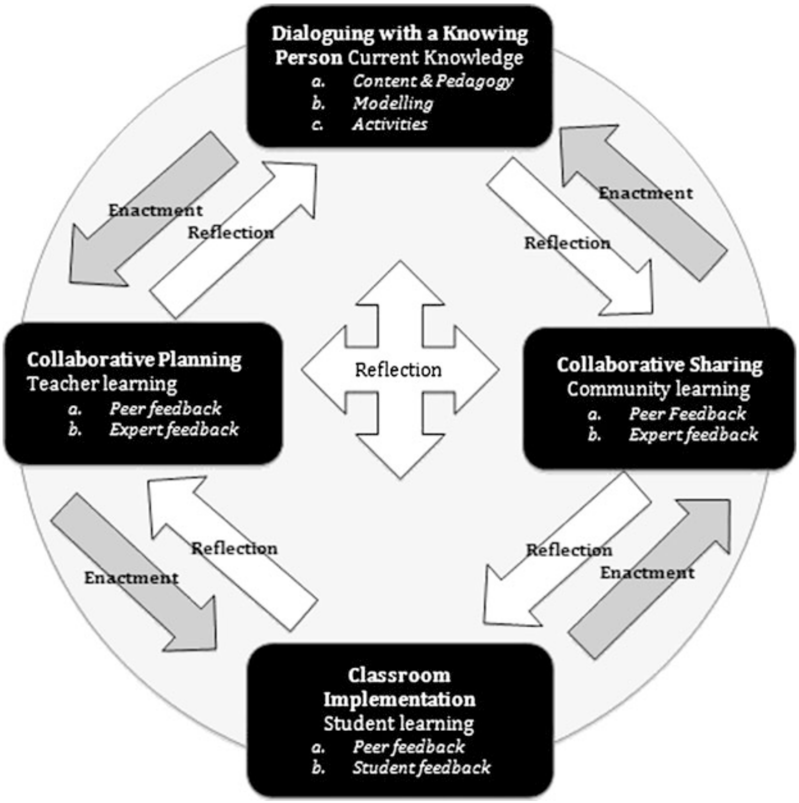

Fig. 1 The RoleM professional development model

metropolitan and provincial centres. Very remote schools are in a geographical location that is spatially very distant from provincial and metropolitan areas (Jones 2004). Queensland, the state in which all 15 schools are located, is the second largest state in Australia and has the third largest population.

The commonality that binds these schools is that students enrolled are often from disadvantaged backgrounds, with the very remote schools having the most disadvantaged students. The majority of participating schools also have large populations of Indigenous Australian students. In the instance of the metropolitan context, the balance consists of students from other ethnic backgrounds, including Vietnam, and the Pacific Islands. All the teachers were non-Indigenous, and many had minimal teaching experience (up to 3 years), or were new graduates. Schools in these contexts experience difficulties attracting staff, and often staff stay for short periods. These positions are not well sought after by the teaching profession, with positions in remote and very remote locations being a last choice for many (Heslop 2011). Table 2 presents the schools with the number of students (Indigenous and non-Indigenous) and teachers from each geographical location that participated in the research.

Table 1 Summary of phases that occurred throughout first year of RoleM

Phase	Activity	Timeframe
Dialoguing with experts—professional development day	Discussing student mathematical learning and research based effective pedagogy; modelling the use of resources to support improved student outcomes; and sharing effective learning experiences	Three times throughout the year
Collaborative planning	Interviews: Teachers reflecting on professional development day in terms of their pedagogy, mathematical understandings and student learning, and making decisions based on the needs of their students within their particular contexts	Three weeks after every professional development day
Classroom implementation	On-site visits: Teachers and visiting experts together implementing effective learning experiences in their classroom	Week 3–10 of each term
Collaborative sharing	Teachers sharing with visiting experts, and each other, examples of student learning, adaptations of existing learning experiences, and new learning experiences that they had developed for their particular contexts	Ongoing throughout the year

Table 2 Number of schools, teachers and students at each school location

School location	No. of schools	No. of teachers	Number of students	
			Indigenous	Non-Indigenous
Metropolitan/ provincial	6	13	72	171
Remote	4	8	62	43
Very remote	5	10	96	4
Total	15	31	230	218

All students were in either Preparatory (Prep) or Year 1, the first two years of formal schooling in Queensland. Their age range was from 4 year 6 months to 5 years 6 months. Thus the sample comprised 448 students and their 31 teachers.

For the vast majority of participating students, including the Australian Indigenous students, Standard Australian English is not their first language. In Queensland, unlike other states in Australia there is a paucity of people speaking an Aboriginal language. Indigenous students in Queensland tend to speak Aboriginal English. Aboriginal English (AE) is a dialectical form of English. The form and structure of this language exhibit many of the speech patterns of standard English in addition to words originating from Aboriginal languages (Eades 1995; Williams 1988). For the other participating students' their first language aligned with their ethnic background.

Instrument Development and Data Collection

Teachers

Semi-structured interviews: A semi-structured interview is often described as 'a conversation with a purpose' (Smith et al. 2010, p. 57). It provides an interactive space that permits participants to use their own words to tell their own story. It also allows for the collection of large quantities of rich data over a period of time. Smith et al. (2010) make the point that the focus of participants' conversation in the initial interview is guided by questions determined by the researchers. However, it is from the initial interview that participants' concerns become evident. As such, future interviews can further investigate those concerns.

Three semi-structured interviews occurred throughout the first year of RoleM. Four themes around change were embedded in each teacher interview: mathematical knowledge and understandings; perception of student learning and abilities, pedagogical practices, and the use of mathematical oral language. Every teacher was required to reflect on and articulate what changes had occurred, how they occurred or if changes had occurred at all. Before each interview, researchers discussed the types of interview questions that would be appropriate to determine a fuller understanding of each theme.

Three interviews were conducted throughout the year. These interviews occurred three weeks after the on-site visits by researchers. Interview questions were emailed to participants prior to the interview allowing them the opportunity to prepare their responses. Interviews were of 30 minutes duration and conducted by telephone at a convenient time for participants. All interviews were audio-recorded for later transcription.

Students

Two diagnostic tests were developed to assess students' understanding of key mathematical concepts as they entered the Prep year. The first test focused on ascertaining students' understanding of the many words used in mathematics. The language of mathematics is complex and many of the words used do not reflect the meaning associated with them in Standard Australian English. For example, more and less are commonly used in everyday contexts to compare two groups. But in mathematics these words are also used to describe an increase or decrease in a group.

The words chosen for this diagnostic test were based on the results of our past research conducted over a two-year period (Warren and deVries 2009). Originally words were selected from the mathematics dimension of the Boehm Test, a standardised oral language test (Boehm 1971). Over a two-year period, adjustments were made according to the ease or difficulty students experienced with the selected words, and the new word lists introduced to the current mathematics syllabus. The end result was a bank of 30 questions consisting of words that students (both Indigenous and non-Indigenous) experience difficulty in understanding (and words

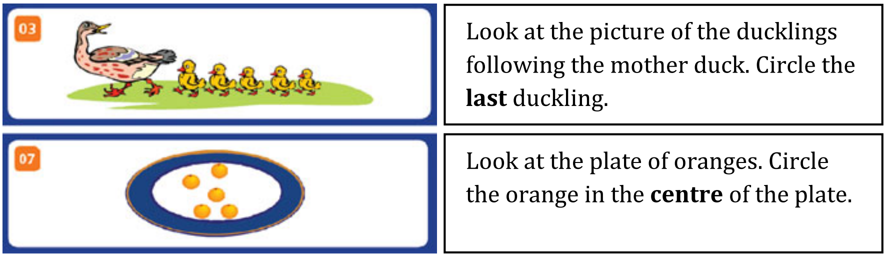

Fig. 2 Selection of two items from the language test

that underpin communication within mathematical contexts). Figure 2 presents a selection of words from this test together with the instruction that was read out aloud.

An analysis of existing validated diagnostic tests for example, Diagnostic Mathematics Tasks (DMT) (Schleiger and Gough 2001) and I Can Do Maths (Doig and de Lemos 2000), informed the type and style of question asked. A mapping of the concepts utilised in published tests indicated that there were gaps in these tests' mathematical content and that there was a reliance on students being able to give written responses. Thus, the decision was made to develop our own mathematics test. The structure of the diagnostic mathematics test reflected the structure of previous Australian national and state tests (the National Assessment Program—Literacy and Numeracy (NAPLAN) and Queensland's Year 3, 5 and 7 tests). The structure also incorporated findings from our past experience in testing young students' understanding of mathematical concepts (Warren and deVries 2009). In our previous research we used a one on one interview to ascertain Indigenous and non-Indigenous students' understanding of mathematics. This style of testing proved to be problematic. It was time consuming and was reliant on the 'skills' of the interviewer. How the interviewer asked the questions and the gestures and facial cues used as they interacted with the students influenced the results. To deal with the issues of time and reliability we moved to a whole class testing format that mirrored that used in NAPLAN. Over a three-year period we have refined the test so that it is easy to administer and involves minimal writing. Figure 3 presents three items from the test.

To ensure consistency in data collection, members of the RoleM team administered all pre and post tests. The reliability of the data was strengthened by all members of the team participating in a workshop on how to deliver the test. In addition, each test was accompanied by explicit written instructions.

Data Analysis

Teachers

The data were analysed using a grounded methodological approach. All interviews were transcribed and the text analysed in an attempt to identify the teachers' self

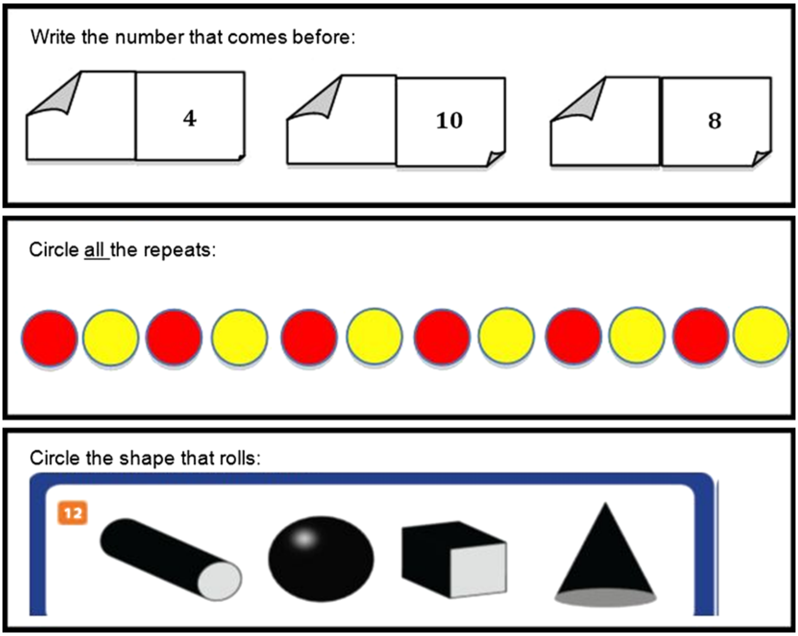

Fig. 3 Three items from the mathematics test

reported beliefs and practices. This process uses an Open-coding approach to break down the data into distinct units of meaning. From the interview data key words and phrases emerged. A fundamental feature of grounded theory is the application of the constant comparative method. This involves comparing like with like, to look for emerging patterns and themes. This process facilitates the identification of concepts and incorporates a progression from merely describing what is happening in the data to explaining the relationship between and across incidents. In this study, the constant comparative method involved examining various subsets of responses from teachers. Finally, a more holistic approach was adopted. This requires a more sophisticated coding technique that is commonly referred to as axial coding and involves the process of abstraction onto a theoretical level (Glaser and Strauss 1967). Axial coding is the appreciation of concepts in terms of their dynamic interrelationships, and should form the basis for the construction of the theory.

Researchers conducted independent member checks. Each researcher read each transcript, identified and sorted the emergent themes into sub-themes and compared the data across interviews. Consensus was reached concerning the nature of each theme and sub-theme, which was accompanied by supporting evidence from the transcripts. Where disagreement existed, researchers returned to the raw data gathering excerpts to support particular stances until there was consensus. Data fell into three broad themes: the RoleM (PD) model, teacher knowledge and understandings, and student learning. However, as the year progressed the sub-themes changed. Each participating teacher did not mention all the sub-themes in their interview. To give insights into how important of each sub-theme was to the cohort of participants, a tally was kept of the number of teachers who mentioned each sub-theme in their interview. The more frequently the theme was mentioned across the cohort of participants they more important it was deemed to be.

Students

All the responses for the pre and post-tests (language and mathematics) were coded and were initially entered into a spreadsheet. This spreadsheet was then analysed to ensure that the data was accurate. Finally the data were transferred to a statistical package for further analysis.

Results

Teachers

The data are presented under the three main themes identified from the analysis, namely the RoleM (PD) model, teacher knowledge and understandings, and student learning. Under each theme a number of subthemes emerged across the year. Tables 3, 4 and 5 present the subthemes for each theme. In addition an example from a teacher's transcript for each subtheme is included together with the percentage frequency of teachers who referred to each of these subthemes across the first year of RoleM.

By November, 83 % acknowledged how valuable the RoleM model was in providing more effective ways to enhance their teaching and their students' learning, with the same percentage stating that this had resulted in a change to their pedagogical practices. Many had also begun to transfer the knowledge they were gaining from their Professional learning to other curriculum areas. Fundamental to these changes were the materials themselves. Making mathematics learning more engaging and fun through structured play-based learning had been instrumental to this change.

Three of the sub-themes referred to the use and importance of mathematics language. There was a noted move in the conversation over the data collection period. Initially teachers were focused on their lack of confidence and the students' lack of confidence in using mathematical language. As the year progressed they became increasingly more conscious of and confident in using the mathematical language as they discussed mathematical concepts with their students. Teachers' attention also turned to their increase in mathematical and pedagogical knowledge as a result of their participation in the first year of RoleM.

Changes in pedagogical practices and their consistent use of oral language were recognised by teachers as having a corresponding positive impact on students' engagement and learning. By the end of the year, 93 % of teachers stated that their students were excited to participate in maths lessons, more confident to 'have-a-go'

Table 3 RoleM (PD) model with the five subthemes and frequency of agreement to each

Sub-theme	Example from transcript	Percentage frequency of agreement		
		Interview 1 ($n = 31$)	Interview 2 ($n = 31$)	Interview 3 ($n = 29$)
Resources useful and engaging	*I enjoyed being able to do the activities… to hear the language we would use*	90 %	–	–
On-site visits enhanced teachers' confidence and knowledge	*Yes it clarified some stuff for me. Was good to get the feedback*	78 %	71 %	–
PD provided direction and confidence to teach mathematics	*Gave me direction. It is all there you can't go wrong*	66 %	71 %	–
PD provided effective ways of teaching and learning	*It has made my teaching of mathematics easier and am covering everything*	–	–	83 %
Transferring effective pedagogy learnt through RoleM to other curriculum areas	*There is so much more I can now do. So many more ways to do things*	–	–	39 %

and were increasingly using maths in everyday situations. This is a considerable jump from the second interview where 66 % commented positively on student engagement and 58 % on student confidence. Teachers were increasingly seeing their own pedagogical practices as instrumental in improving student learning outcomes, rather than focusing on their students' lack of proficiency in SAE as the barrier. As a result, 83 % of teachers have stated that they now had higher expectations of their students' abilities.

The next section summarises the results for the participating students.

Students

The sample comprised 448 students from the first two years of school (Preparatory and Year 1) with an average age of 5.76 years. Table 6 presents the number of non-Indigenous and Indigenous students in each year level.

As indicated in Table 1, the majority of the Indigenous students were attending the remote or very remote schools, while the majority of non-Indigenous students attended the metropolitan/provincial schools. The Preparatory year is a non-compulsory year that has recently been introduced into the Australian Curriculum. A formal curriculum is presently being developed and introduced to this level of

Table 4 Teaching and learning with the five subthemes and frequency of agreement to each

Sub-theme	Example from transcript	Percentage frequency of agreement		
		Interview 1 ($n = 31$)	Interview 2 ($n = 31$)	Interview 3 ($n = 29$)
Not confident in using maths language	*I don't feel very confident using it…*	48 %	–	–
Becoming more aware of the importance of maths language	*Learning to know what language is important*	48 %	–	–
More conscious of how and when to use maths language	*Use a lot more mathematical language now. It's more explicit and more different terms*	–	65 %	79 %
Maths knowledge and confidence has increased	*It has changed it and it has improved. There is so much more I can now do, so many ways to do things*	–	58 %	83 %
Teaching of mathematics continues to improve	*It's more hands on and given me an alternative to worksheets*	–	–	83 %

Table 5 Student learning with the three subthemes and frequency of agreement to each

Sub-theme	Example from transcript	Percentage frequency of agreement		
		Interview 1 ($n = 31$)	Interview 2 ($n = 31$)	Interview 3 ($n = 29$)
Language barrier impacts on student learning	*If they don't have SAE they can't participate because that's the language that I teach in*	81 %	–	–
High level of student engagement	*They don't really notice that they are doing maths, they are enjoying it*	–	66 %	–
Students were more confident with maths and using maths language	*They seem more relaxed with it… I am seeing the same language being used in different ways in their free play. This is the area they enjoy most*	–	58 %	93 %
Expectations of students' capabilities continue to improve	*I have much higher expectations because they are achieving so well*	–	–	83 %

Table 6 Number of non-Indigenous and Indigenous students in each year level

Year level	Students		Total
	Indigenous	Non-Indigenous	
Prep	124	143	267
Year 1	106	75	181
Totals (448)	230	218	448

Table 7 Maths paired t-test results for the three geographical locations

Group	Pre-test mean	Post-test mean	t-score	p value
Metropolitan/provincial ($n = 243$)	8.74	15.98	28.81	.000**
Remote ($n = 105$)	9.96	14.30	9.82	.000**
Very remote ($n = 100$)	7.32	12.83	13.65	.000**

Note: **Statistically significant $p < .005$

Table 8 Language paired t-test results for each geographical location

Location	Pre-test mean	Post-test mean	t-score	p value
Metropolitan/provincial ($n = 243$)	15.48	21.54	17.28	.000**
Remote ($n = 105$)	17.52	22.03	8.91	.000**
Very remote ($n = 100$)	12.42	17.49	9.95	.000**

Note: **Statistically significant $p < .005$

schooling. Thus the participating Preparatory and Year 1 students had had limited contact with mathematical activities prior to the commencement of RoleM. Hence, for the purpose of this paper the Preparatory and Year 1 cohort of students have been combined at each location.

To ascertain the impact that the intervention had on students' learning, paired t tests were performed, comparing students' pre and post scores for the mathematics test and the language test. In order to examine the impact the RoleM PD had in differing geographical contexts, the results are presented according to each geographical location. Tables 7 and 8 summarise the results of the paired t tests for mathematics and language respectively.

The results of Tables 7 and 8 indicate that there was a statistically significant increase in students' scores for the mathematics and language tests in all three geographical locations.

Discussion and Conclusions

Principles of Professional Development Practice

All five overarching principles identified in the literature on professional development proved important, however, as the year progressed the level of importance that teachers gave to each changed. At the commencement of the year, teachers' initial feedback indicated that the provision of the resources with their clear directions of what they needed to do was a crucial element for successfully implementing teaching and learning episodes in mathematics. By including these resources the following issues were simultaneously addressed: (a) under-resourced classrooms, a feature that is common in many disadvantaged schools (Clements 2004; Cresswell and Underwood 2003); (b) ill-prepared teachers who work in these contexts (Lyons et al. 2006); and, (c) under-confidence that many early childhood teachers exhibit when teaching mathematics (Aubrey et al. 2006). In addition the principle of situating the PD in the context of the participants' classroom was also critical. Much of the discussion that occurred during this first visit focused on solving the learning problems identified by the classroom teachers prior to the visit and ensuring that students were engaged in the learning, both of which are dimensions of 'expert' teaching (Askew 2008; Hattie 2003).

While the emphasis on teacher knowledge was integral to all aspects of the RoleM PD model, how this impacted over the year of the project suggests that changing teachers' understanding of mathematics is complex. Their professional learning journey began with a strong emphasis on gaining confidence to teach mathematics (Interview 1). It seemed that until they had gained some confidence they were unable to reconceptualize their pedagogical practices or their mathematical content knowledge. Without this increase in confidence, the potentiality of new pedagogical and content knowledge and theories of students' learning and teachings as a catalyst for change is lost (Garet et al. 2001). It was not until Interview 3 that the focus of the feedback moved to sharing their increased understanding of effective pedagogy for teaching mathematics and transferring this to other subject areas. In Interview 3 they also shared their improvements in mathematical knowledge, especially with regard to maths language. We would also suggest that it is imperative to provide simple engaging activities linked to 'valued student outcomes' (Timperley 2008) for teachers to trial during the PD.

The language of mathematics was an ongoing concern for all teachers throughout the year. Given that Standard English was not their students' preferred language of communication, this is understandable. As the year progressed teachers' confidence in using mathematical language increased and became more explicit. We conjecture that an initial focus on language used in the classroom may be a way of engaging early years teachers in these contexts for meaningful change. But as emphasised in the RoleM learning activities provided, an exploration of language needs to be in conjunction with the use of a range of mathematical representations embedded in mathematical activities. The advantages of this initial focus on language are that

(a) teachers perceive it as a clear barrier to student learning (Interview 1), (b) improved student outcomes are clearly evident (Black and William 1998), (c) an increased awareness of mathematical language is linked to a greater knowledge about the mathematics they are teaching (Garet et al. 2001), and (d) for early years teachers who are under confident in teaching mathematics but are confident in teaching literacy, this is somewhat familiar territory (Gresham 2007). As evidenced by the teachers' responses in Interview 3 it was only after their students exhibited confidence in using mathematical language that teachers began to share that their achievement expectations for students had increased (Interview 3).

The Effectiveness of the RoleM PD

The RoleM PD has proven to be highly effective on all three measurable dimensions of effectiveness: teachers' affective domain (Guskey 2003); gains in student achievement (Kent 2004); and, changed teacher behaviour (Wenglinsky 2002).

The results of our research indicate that teachers' affective domain is important to teacher change. However, it is not simply about teachers' enjoyment at the professional development session (Salpeter 2003). The premise that enjoyment results in implementation of the ideas presented in the PD session is too simplistic and flawed; enjoyment is not enough. Our results suggest that an increase in teacher confidence in disadvantaged contexts is crucial to success. Many of these teachers are beginning teachers who often feel unprepared academically, culturally and professionally to teach in these contexts (Lyons et al. 2006; Mills and Gale 2003). In addition there is often a paucity of experienced teachers to act as their mentors. We suggest that a measure of PD effectiveness includes a teacher confidence dimension, especially a dimension that measures known aspects of expert teaching, namely, confidence in their ability to anticipate, plan and improvise as required, confidence in their mathematical background, and confidence in their understanding of how these students in these contexts best learn (Askew 2008).

The quantitative data presented in this paper clearly show a significant improvement in student achievement in the first year of RoleM. This improvement occurred in all three geographical locations. In Queensland the early years of education have been problematic, especially with regard to numeracy outcomes. In a large study investigating learning experiences and teaching practices prior to Year 1, Thorpe et al. (2004) found that over the year of their study many students made negative progress in their understanding of basic numeracy concepts. Their sample included students from both advantaged and disadvantaged contexts, and from all geographical locations. Thus, the improvement these students exhibited in the first year of RoleM further evidence that teachers can make a difference to student learning outcomes (Hattie 2003).

Finally, teachers exhibited measurable change, especially in terms of their classroom practice and their perceptions of their students. By the end of the year they were moving to a more 'hands on' visual approach to teaching mathematics; an approach that is more conducive to learning for many students in these contexts

(Nichol 2008). Also, they were now setting higher educational outcomes for their students. The learning outcomes for their students were no longer perceived as beyond their control (Jones 2009). There was no longer a reliance on simple achievable worksheets or a lowering of their expectations for their students (Hewitson 2007; Munns et al. 2008).

Implications and Recommendations

Based on the data and our own reflections, we argue that professional development occurs in three cycles: Beginning, Middle and Final. Each builds on the previous cycle and as we move through the cycles there is a change of emphasis on particular aspects of teaching and learning mathematics.

Beginning Cycle Focus: Building Teachers' Confidence

Many teachers in disadvantaged contexts are often unable to confidently provide effective mathematical learning opportunities that enhance student learning. Building teacher confidence is thus the initial focus of Professional Development. This occurs through teachers:

1. *Developing knowledge of the language of mathematics and its use orally.* This helps teachers to bridge the gap between SAE, AE and the language of mathematics. Emphasising teachers' use of oral language in conjunction with rich mathematical representations ensures that both teachers and students develop a shared mathematical register. This is essential for students to actively engage with, comprehend, and communicate in mathematics.
2. *Developing an understanding of how to effectively use proven mathematical learning experiences.* Many teachers use resources without ever fully cognitively engaging students in mathematics. Trialing proved experiences in the classroom context assists teachers to feel more confident in teaching mathematics.
3. *Being given support in their classrooms with on-site visits by expert teachers.* These visits need to be responsive to teachers' specific requests. Teachers' confidence builds when they can see that their students' are engaged in learning mathematics and experiencing success.

Middle Cycle Focus: Building Students' Confidence

As teachers become more confident in teaching mathematics, and are effectively utilising proven resources to support student learning, the emphasis is directed towards developing student confidence. This occurs through teachers:

1. *Gaining more general mathematics pedagogical knowledge.* An important dimension of building students' confidence is learning how to confidently use differing ways of teaching mathematics. This assists teachers to cater for a range of learning styles within these mathematics classrooms, and helps them to gain a deeper understanding of how students learn.
2. *Gaining a deeper understanding of how to differentiate learning.* Providing all students with a feeling of success helps them to gain confidence in their own ability to do mathematics. Being able to differentiate learning activities begins to address the wide range of students' abilities that exists in many disadvantaged communities and allows all students to experience success.

Final Cycle Focus: Increasing Expectations for Their Students

As teachers gain an understanding of the capability of their students as learners, the emphasis is directed to setting higher learning expectations for their students. This occurs through teachers:

1. *Gaining a deeper understanding of mathematical content knowledge.* In order to set higher expectations for their students teachers need to have a deeper knowledge of mathematical concepts, conceptual frameworks, and learning trajectories. This assists teachers to understand the hierarchical nature of mathematics, how understanding builds on prior understanding, and provides ways to use this knowledge to target student's learning.
2. *Gaining a deeper understanding of mathematical pedagogical knowledge.* The ability to anticipate, plan and improvise is at the heart of teaching mathematics effectively. Having the ability to be flexible in how mathematics can be taught includes understanding the different ways that mathematical concepts can be represented and how these representations are connected. It also includes knowledge about what makes a subject cognitively easy or difficult for student learning, as well as knowledge of misconceptions and pre-conceptions students may hold about mathematical concepts.

Although many teachers who work in disadvantaged contexts lack the academic, cultural and professional knowledge, confidence and experience to maximise student learning, providing a well designed professional development model can play a significant role in effecting teacher change. Professional development opportunities that are ongoing, contextualised, emphasise teacher content and pedagogical knowledge, are clearly linked to achievable student outcomes and provide effective resources are more likely to enhance professional learning. Building teacher confidence in their ability to positively influence student learning and engagement in mathematics is the first step to improving student learning outcomes and raising expectations. If early years students from disadvantaged contexts are to experience future achievement in mathematics, then it is essential that they are taught by quality teachers who can build strong mathematical foundations for them.

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Mathematical Modeling at the Intersection of Elementary Mathematics, Art, and Engineering Education

Heidi A. Diefes-Dux, Lindsay Whittenberg, and Roxanne McKee

Interdisciplinary Art Education

Art and Mathematics

Leveraging art education to support the learning of mathematics, and other subjects, is not new (e.g., reading: Bookbinder 1975; science: Kohl and Potter 1993). Take a walk down any elementary hallway and one will find child created tessellations, a la Escher, on display. Certainly, rich mathematical structures, such as tessellations, platonic solids and polyhedra, the golden ratio, and symmetry and patterns, can be explored through a study of physical settings and artworks, as demonstrated in a course entitled Mathematics in Art and Architecture offered at the National University of Singapore (Aslaksen n.d.).

The study of and participation in art is seen as something that children enjoy; therefore, it can provide motivation for learning mathematics. Forseth (1980) found in a study of fourth graders that "the use of the art activities seems to help create a more favorable predisposition toward learning mathematics without any impairment to achievement in math" (p. 21). The work of Edens and Potter (2007) suggests that the "art room may be a context for developing students' spatial understanding, an

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ability associated with artistic as well as mathematical ability" (p. 294). To take advantage of complementary learning opportunities, Bickley-Green (1995) proposed steps to integrating art and mathematics curricula, starting with the identification of congruent content based on literature in mathematics and art.

Art and Engineering

The obvious link between art education and engineering education lies with creative thinking. At the university level, this space was explored through the development of an interdisciplinary project for environmental engineering students and art students related to sustainability and food within the local community (Costantino et al. 2010). Shuster (2008) talks more broadly about the value of liberal and fine arts education for preparing engineers as it "brings us into more direct confrontation with the creative process, with ambiguous concepts and data, and with more diverse avenues of perception" (p. 98).

Another link between art education and engineering education is the methods and mediums artists use to create their expressions. In the K-12 setting, this connection was brought to bear at the prototyping stage of design in a grade 6–12 informal curriculum related to animatronic toy design and manufacture (Mativo and Sirinterlikci 2004; Sirinterlikci and Mativo 2005). Here the art emphasis was on sketching and sculpting and the use of art materials and manual techniques (Mativo and Sirinterlikci 2004).

Engineering has also been seen as way to contribute to art. For example, Roman (2010) created an exercise in which students design a sculpting material using both common everyday materials and more advanced materials. The first part of this exercise involves an activity almost akin to reverse engineering—students must explore existing art to learn about materials used in sculpting.

So the intersection between art education and engineering education can be both a give and take. From art, young children can learn creative thinking and methods for expressing their ideas. From engineering, young children can learn the engineering design process. Here we consider the potential for art education learning objectives to help young children learn about engineering while promoting the understanding of mathematical concepts.

Model-Eliciting Activities

A Model-Eliciting Activity is an authentic, client-driven, open-ended mathematical modeling problem. A MEA adheres to six design principles (Lesh et al. 1993, 2000); these have been modified for engineering education purposes (Diefes-Dux et al. 2008).

- Model Construction: The activity requires students to create a mathematical model that addresses the needs of a given client.

- Reality: The activity is set in a realistic engineering situation that requires the use of a mathematical model to solve the problem.
- Generalizability: The activity requires that the model students create be sharable, re-usable, and modifiable. This means that the model must be useful to someone other than the creator of the model.
- Self-Evaluation: The criterion for "goodness of response" is embedded in the activity so that students are put in a position to self-evaluate their work.
- Model Documentation: The activity requires that the model be documented in some fashion.
- Effective Prototype: The solving of the MEA provides a memorable experience students can draw on when they encounter other structurally similar situations.

In finding a solution to a MEA, students must mathematize a real world situation, document their work, and strive for generalizability (i.e. a solution that is share-able with others, reusable in similar situations, and modifiable for analogous situations). The problem itself must provide a means for students to test their solution. The elements, processes and relationships (Lesh and Clarke 2000) used in the students' resulting models provide insight into students' thinking—understanding of concepts and achievement of learning objectives. Among the challenges to using MEAs in elementary settings are young children's lack of sufficient language development to communicate their models in writing and their limited exposure to and development of step-wise procedures.

Activity Development and Implementation

The *Sticker* MEA was used as an introduction to MEAs in a week-long teacher professional development workshop. In this MEA, students develop a procedure (model) for the client company Wacky Stickers n'Stuff. The procedure (model) must maximize the number of stickers of given shape and dimensions that can be cut from a specified size sheet of stock paper. During implementation, the shape (e.g. squares and triangles) and dimensions of the sticker and the size of the stock paper change. After each change, students test their procedures (models) to see if they still work; that is, they test for reusability and modifiability. Then, they revise their models. Students also exchange their models, allowing peers to apply them to a test case shape and stock paper and provide feedback; this is a test for share-ability.

This activity reminded two participating elementary art teachers from different buildings in the same district of another activity seen in another professional development workshop—a Draw-A-Monster Activity (e.g. Monster Exchange http://www.monsterexchange.org/). A combination of the MEA and the Draw-A-Monster Activity was seen as a means of addressing a third grade art education learning objective while engaging students in engineering thinking and processes. Specifically, the art learning objective was related to differentiating geometric and organic (free-from) shapes. This classroom learning object falls within the Texas Essential Knowledge and Skills (TEKS) focused on "(1) Perception" in which “the student develops and organizes ideas from the environment” and “The student is expected to ... (B) identify art elements such as color, texture, form, line, space, ...” (TEKS 117.11. Art Grade 3 http://ritter.tea.state.tx.us/rules/tac/chapter117/ch117a.pdf).

The resulting Draw-A-Monster activity is described below. It took more than 5 weeks to implement as students attended art class only once a week and there were coordination considerations with other activities and between the two art teachers.

Day 1 began by telling the children that they have 3 minutes to draw a monster. After they finished, the art teacher asked the children, “Who thinks that they could describe their monster to me so well that I could draw the exact same monster without ever looking at your paper?” An eager volunteer came up to the front of the room for a demonstration. The child was instructed to face the class and describe their monster to the teacher using their words, starting at the head and work their way down the monster’s body. As the child described the monster, the teacher drew her interpretation of the description on the document camera so that the whole class could be engaged in the process. The volunteer was not allowed to see what the teacher was drawing. The student used phrases like “draw a pig nose”, or “draw monster teeth”. After the description was complete, the teacher allowed the volunteer to look at the projection to see if the teacher-made drawing matched the original drawing. At this point, everyone had a good laugh! A discussion about the difficulty of this assignment then followed. The question, “What could have made this easier?” was asked. The objective was to help the children realize that by limiting the number of colors and using only geometric shapes in their drawings, their monster becomes much easier to describe.

On Day 2, after reviewing the previous day’s findings, the children were told to draw a monster that they would be able to describe in words to another classmate. The children were constrained to using only 5 colors and encouraged to only use geometric shapes. The difference between geometric and organic shapes was again explained. The children were given 5 minutes to draw their monster (Original drawing). As soon as they are done, they wrote directions for someone else to use to draw their monster. The children were given the remainder of the period ($\sim$ 30 minutes) to complete their directions. Those unable to finish, took their assignment back to their homeroom to complete and bring back next time.

The children’s models are in the descriptions of their geometric figures. The purpose of a given model is to enable others to recreate a figure. So, the geometric figures are the objects, the relations are statements about orientation and location, and the operations or rules are the descriptions of the properties of the figures to be drawn.

On Day 3, the children traded their directions with someone else in class. They were given 20 minutes to draw the monster (Peer Review 1 drawing). At this point, they were asked to make corrections in marker on each other’s written papers. They corrected anything from grammar to punctuation. They also identified areas in the directions that “don’t make sense” by circling the confusing text. When they finished drawing each other’s monsters to the best of their abilities, they return the directions with the corrections to the owner along with their peer-review monster drawing. The children then looked over their papers to see what corrections needed to be made to their final drafts. The rest of period was spent writing a second draft. Again, those unable to finish, took their assignment back to their homeroom to complete and bring back next time. Once all final drafts were complete, the two art teachers exchanged their class work.

Day 4 started with each student receiving directions for how to draw a monster, written by another third grader at a different campus. The children were given 20 minutes to draw the monster to the best of their ability (Peer Review 2 drawing). If they ran into problems along the way, they were told to do the best that they could since the person that wrote the directions was not there to answer their questions. They again used a marker to circle parts of the directions that were unclear.

Finally, on Day 5, the children received their own drawings and directions from the other school along with the peer review drawings done by the other children. The children reviewed their papers to find out if their writing was successful—the more the original drawing and peer-review drawings match, the better the student did at describing their monster. The activity concluded with a discussion of the importance of this project. The children addressed questions like, "What were the difficulties?", "What would you do differently now that you have done it once?", and "What would they do differently now that they have done it once?"

This Draw-A-Monster activity meets five of the six design principles of an MEA. The children must, at some level, mathematize their art (model construction). The use of formal geometric vocabulary and spatial reasoning skills provide a memorable experience to draw upon (effective prototype). There are built-in criteria for the children to use to test and revise their current ways of thinking (self-assessment)—the children actually create their own test case data set which is their monster drawing. Through the procedure writing (model documentation), the children create a share-able and re-usable (generalizable) product. The only missing principle is a realistic context to drive the need for the drawing and drawing instructions (reality).

Child Thinking Revealed

Four pieces of student work are used here to demonstrate child thinking that informs the next implementation. These four were selected for the variety, yet representativeness of the work produced by the children.

Child A's work represents a reasonable attempt to (1) use geometric shapes in the Original drawing (Fig. 1a) and (2) respond to the peer review comment and the Peer Review 1 drawing (Fig. 1b). Child A wrote:

> My Monster clawed. Begin by drawing a red circle head and a red circle oval body also a 2 blue triangle horns. The face is a circle nose and a circle mouth and 2 teeth inside and 2 black eyes also 2 red rectangle arms and 3 purple claws in each arm. 2 red legs and 2 purple claws. [Child A, Draft 1]

The peer review comment asked for "more" with arrows pointing to the words about the nose and mouth. Child A revised the instructions:

> My Monster clawed. Begin by drawing a red circle head and a red circle oval body also a 2 blue triangle horns. The face have a blue purplish nose and a black circle mouth and 2 sharp teeth inside, 2 black eye, and two more finish touch is 2 red rectangle arm and 3 purple claws in each arm and 2 red legs and 2 purple claws. [Child A, Draft 2]

Child A addresses the request for information about the nose and mouth by providing a color for each. However, Child A does not seem to attend to errors in the Peer Review 1 drawing concerning the eyes, nose, and the orientation of the two teeth. Peer Review 2 drawing (Fig. 1c) shows a different interpretation of claws, highlighting the problem with using organic shapes in drawing instructions.

Child B drew a more complex monster and did not stick with geometric shapes (Fig. 2a). This child instructed:

> My Monster Fly Ant. On the face, it is shaped like a super hero mask with one eye in the middle, and spiky hair. He also has 2 circles on the back, on top of the circles are wings. 2 wings. And a wormy red mouth. and a curvey neck, under it's back is a rectangle with 9 webbed feet. 3 with triangles, one with a triangle, 3 with squares, and one with half rectangle. [Child B, Draft 1]

The use of organic shapes and the fact that the monster is not similar to a human in form resulted in the Child B having to write more description of the shapes and more about the spatial relationships among the shapes than appeared in many other student works.

Following the initial in-class peer review, Child B saw the Peer Review 1 drawing (Fig. 2b) and one comment on Draft 1. The comment, marked at the first reference to "wings", said "more what shape?" Peer Review 1 drawing had some features that were remarkably close to the Original, like the head and hair. Child B revised his instructions to say:

> My Monster Fly Ant. My monster is facing the right. 2 circles on the back. and the face is shaped like a super hero mask. The super hero mask there is one eye in the middle. The neck is shaped like a cresent. And 4 feet are pointy. and one is squigly. Also one has 3 squares. One the back, it has wings. Like the tails on batman's batmobile. with a squirmy mouth. [Child B, Draft 2]

From Draft 2, it is apparent that Child B attended to both differences between the Original drawing and the Peer Review 1 drawing and the peer reviewer comment. In the Peer Review 1 drawing, the monster was facing forward rather than to the left, had a different neck, and had feet that were all the same. Child B addressed these issues along with the comment about the shape of the wings. Unfortunately, Child B lost some detail from Draft 1—the relative position of the shapes and the number of wings. Peer Review 2 drawing (Fig. 2c) shows the difficulty that this peer reviewer had with interpreting the Draft 2 instructions.

Child C, like Child A, used geometric shapes to make the Original monster drawing (Fig. 3a).

> My Monster Speedy. begin by drawing a big black olvol. draw 2 trianglels at the bottom. draw a circle at the top of the oval then draw circles for the eyes and then draw an oval mouth then draw triangles for the teeth. for the hands draw a big triangle on both sides of the oval. on the head draw a line connetct triangles and put circles in the triangles. Put a big

Fig. 1a Draw-a-Monster drawings for Child A—Original

Fig. 1b Draw-a-Monster drawings for Child A—Peer Review 1

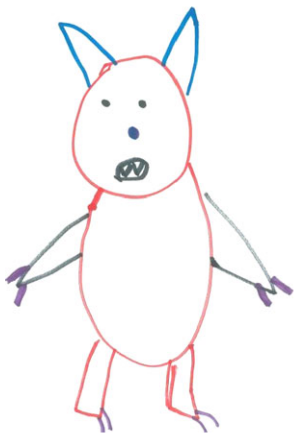

Fig. 1c Draw-a-Monster drawings for Child A—Peer Review 2

rectangol in the middle of the oval and then draw three starts in the middle of the rectangal. Then draw a circle in the middle of the eye and the name is speedy. And when you did that you are done [Child C, Draft 1]

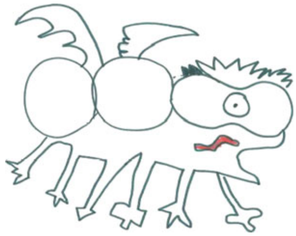

Fig. 2a Draw-a-Monster drawings for Child B—Original

Fig. 2b Draw-a-Monster drawings for Child B—Peer Review 1

Fig. 2c Draw-a-Monster drawings for Child B—Peer Review 2

Child C lost detail while attended to the struggles of the in-class peer to complete most of the features of the drawing (Fig. 3b) by writing a step-wise procedure.
My Monster Speedy

1. Begin by drawing a big black ocal then draw 2 triangels at the bottom of the oval.
2. Draw a circle at the top of the oval and draw 2 green eyes in the circle then in the oval draw triangle teeth after that draw intonas [antennas].
3. Draw a big triangle on the side of the oval to get arms. In the middle of the oval draw a ractangal that is blue and has stars in it.
4. If it looks good you are done!!! [Child C, Draft 2]

But to an even greater extent than Child B, Child C lost many procedural details. For instance, Child C introduced antennas—an organic shape—rather than describe how to draw the antennas with geometric shapes. This loss of detail is noticeable in the Peer Review 2 drawing (Fig. 3c).

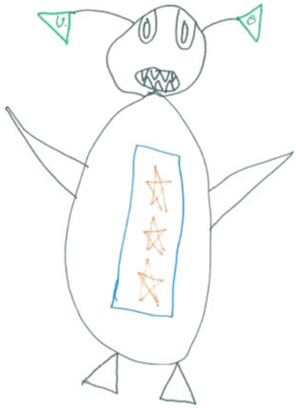

Fig. 3a Draw-a-Monster work for Child C—Original

Fig. 3b Draw-a-Monster work for Child C—Peer Review 1

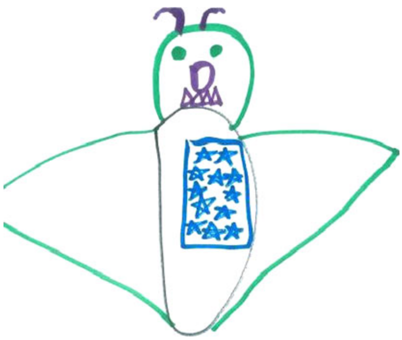

Fig. 3c Draw-a-Monster work for Child C—Peer Review 2

Finally, Child D had an incomplete in-class peer review (Fig. 4b) which caused little substantive revision. Child D's instructions for the original monster drawing (Fig. 4a) were:

> My Monster Craze. Begin by drawing a big blue, square head. Draw green eyes rectangle mouth orange triangle teeth and red intenas [antennas]. Draw a purple rectangle neck. At the two points draw a blue triangle body with a pattern of red and purple stripes. Draw two orange rectangle arms with red triangle claws. draw orange line legs all around orange blob legs. [Child D, Draft 1]

The second draft read:

> My Monster Craze. Start with a blue rectangle head with green eyes a rectangle mouth with 4 orange teeth on the top and bottom. on the top add to red antennas. Put a purple rectangle

> [?] neck. At the two points draw a purple triangle body with a pattern of red and purple diagonal stripes. Draw orange rectangle arms with red triangle claws. Draw orange line legs with orange blob feet. [Child D, Draft 2]

More problems with the instructions were revealed by the second round peer reviewer (Fig. 4c). Certainly, issues with the size and location of the neck and orientation of the teeth are more noticeable.

This selection of student work reveals that the children continued to include organic shapes in their drawings after the Day 1 discussion concerning the potential difficulties of describing them. This informs revision to the activity with regards to the achievement of the art education learning objective concerning the differentiation of geometric and organic shapes. Students could list all of the shapes they are using in their drawing before they write their first set of instructions. Then they could revise their drawing to eliminate any organic shapes that they find. Alternatively, students could rework their monster drawing and their instructions to ease problems they detect with the continued use of organic shapes.

Opportunities for Greater Learning

While the art teachers expressed satisfaction with children's achievement of the art learning objective, they were overwhelmed with the time commitment to the writing activity. A debriefing of the activity revealed opportunities to (1) connect to the grade-level classroom for mathematics and language arts instruction and (2) strengthen the engineering component of the activity. In fact, this activity provides a wealth of opportunities to not only teach or reinforce the art concept but bolster learning experiences in mathematics, language arts, and engineering if the teaching of this activity were spread across the art teacher and the classroom teacher. Teaching in this way would maximize the learning potential of the activity and allow teacher expertise to be applied in appropriate places. Teaching in this way would also shorten the overall duration of the activity but strengthen its use during the shorter overall implementation period as some components would be taught in the art room and others in the regular classroom. Further, this would enable the activity to serve as a precursor to more mathematically complex MEAs as the children would have had the experience of writing a procedure, participating in peer review, and revising their solutions and ways of thinking about a problem in context.

Mathematics and Language Arts Connections

There is a great need to advance all students' abilities to both use mathematics and write in mathematical terms with fluency when solving problems. Engaging young children in age-appropriate activities that begin to build these skills prepare them to solve increasingly complex and open-ended problems.

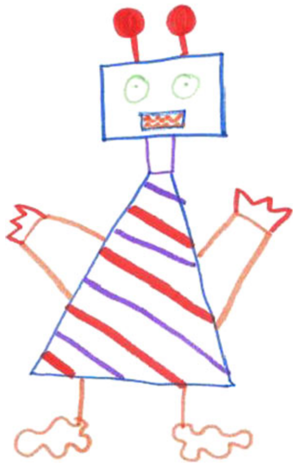

Fig. 4a Draw-a-Monster drawings for Child D—Original

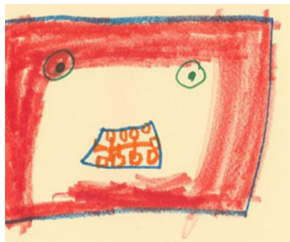

Fig. 4b Draw-a-Monster drawings for Child D—Peer Review 1

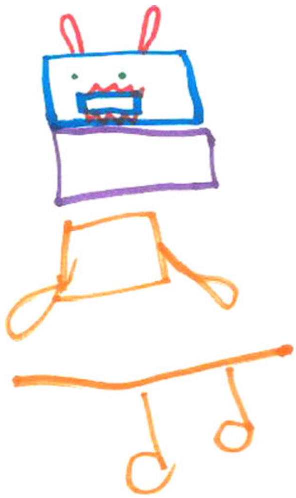

Fig. 4c Draw-a-Monster drawings for Child D—Peer Review 2

The classroom teacher could take responsibility for the mathematics and language arts connections. This activity connects to various Texas Essential Knowledge and Skills (TEKS) for Mathematics as stated in Chapter 111 (http://ritter.tea.state.tx.

Table 1 Texas Essential Knowledge and Skills (TEKS) for Mathematics applicable to the Draw-a-Monster Activity

Kindergarten
(7) Geometry and spatial reasoning. The student describes the relative positions of objects. The student is expected to: (A) describe one object in relation to another using informal language such as over, under, above, and below; and (B) place an object in a specified position.
(8) Geometry and spatial reasoning. The student uses attributes to determine how objects are alike and different. The student is expected to: (A) describe and identify an object by its attributes using informal language; (B) compare two objects based on their attributes; and
(9) Geometry and spatial reasoning. The student recognizes attributes of two- and three-dimensional geometric figures. The student is expected to: (C) describe, identify, and compare circles, triangles, rectangles, and squares (a special type of rectangle).
Grade 1
(6) Geometry and spatial reasoning. The student uses attributes to identify two- and three-dimensional geometric figures. The student compares and contrasts two- and three-dimensional geometric figures or both. The student is expected to: (A) describe and identify two-dimensional geometric figures, including circles, triangles, rectangles, and squares (a special type of rectangle);
Grade 3
(8) Geometry and spatial reasoning. The student uses formal geometric vocabulary. The student is expected to identify, classify, and describe two- and three-dimensional geometric figures by their attributes. The student compares two-dimensional figures, three-dimensional figures, or both by their attributes using formal geometry vocabulary.
(9) Geometry and spatial reasoning. The student recognizes congruence and symmetry. The student is expected to: (A) identify congruent two-dimensional figures; (B) create two-dimensional figures with lines of symmetry using concrete models and technology; and (C) identify lines of symmetry in two-dimensional geometric figures.

us/rules/tac/chapter111/ch111a.html) of the Texas Administrative Code (TAC) (Texas Education Agency n.d.). As a grade three activity, this activity reinforces TEKS for Mathematics from Kindergarten and Grade 1 TEKS and addresses two Grade 3 TEKS (Table 1). Students are expected to use formal language to describe geometric shapes (TEKS 3–8) and their relative position (TEKS K-7A) in their Draw-A-Monster procedures. In this implementation, there was very little

Table 2 Texas Essential Knowledge and Skills (TEKS) for English Language Arts and Reading applicable to the Draw-a-Monster Activity

Grade 3

(17) Writing/Writing Process. Students use elements of the writing process (planning, drafting, revising, editing, and publishing) to compose text.

(A) plan a first draft by selecting a genre appropriate for conveying the intended meaning to an audience and generating ideas through a range of strategies (e.g., brainstorming, graphic organizers, logs, journals);
(B) develop drafts by categorizing ideas and organizing them into paragraphs;
(C) revise drafts for coherence, organization, use of simple and compound sentences, and audience;
(D) edit drafts for grammar, mechanics, and spelling using a teacher-developed rubric; and
(E) publish written work for a specific audience.

(20) Writing/Expository and Procedural Texts. Students write expository and procedural or work-related texts to communicate ideas and information to specific audiences for specific purposes.

(22) Oral and Written Conventions/Conventions. Students understand the function of and use the conventions of academic language when speaking and writing. Students continue to apply earlier standards with greater complexity. Students are expected to:

(A) use and understand the function of [...] parts of speech in the context of reading, writing, and speaking
(B) use the complete subject and the complete predicate in a sentence; and
(C) use complete simple and compound sentences with correct subject-verb agreement.

description of the relative position or orientation of shapes or lines. This seemed to be due to a reliance on the known location and relative position of human body parts and an assumption that monsters will not be a great departure from the human form. In fact, many of the monsters drawn by the children were human-like in form. So, either the teacher would need to emphasize the writing about the relative position of shapes or the context of the activity would need to be altered to more naturally address this issue. During peer review, peers place shapes in specified positions according to the children's procedures (TEKS K-7B). Following peer review, the children should actively identify differences between the actual and the intended shapes and positions in the drawings. The shapes employed (TEKS K-9, 1–6) could be manipulated through the problem constraints. Congruency and symmetry of shapes (TEKS 3–9) could be woven into children's procedure writing or discussions around how to interpret peer review drawings.

This activity also connects to various TEKS for English Language Arts and Reading as stated in Chapter 110 (http://ritter.tea.state.tx.us/rules/tac/chapter110/ch110a.html) of the TAC (Texas Education Agency n.d.); connections to Grade 3 TEKS are shown in Table 2. Certainly, the writing of the students, as exemplified in samples A through D above, would benefit from classroom teacher attention. The art teachers for this implementation noted that the children were not very motivated to write well during art class. The writing type for this activity is expository and procedural

(TEKS 20). As is, this activity informally walks students through a writing process that includes drafting and revision (TEKS 17A-D). A more formal execution of the process would help students with issues of organization and writing conventions (TEKS 22). The intended audience (TEKS 17E) in the current activity is peers, though that could be reconsidered (this is discussed more in regards to embedding more engineering).

Engineering Connections

Making connections to engineering is about identifying engineering practices embedded in activities and exploiting them for student learning purposes, and identifying ways to re-contextualize activities to provide young children with a window on the "what and where" of engineering. There are a number of opportunities to highlight engineering processes already present in the Draw-a-Monster activity:

- Engineers work within constraints. In this activity, the students are limited to geometric shapes and five colors. These could be manipulated in various ways for different learning outcomes, such as the use of shapes that are being added to students' vocabulary.
- Engineers write procedures that others must be able to implement. Take for example design engineers at Vermeer (http://www2.vermeer.com/vermeer/NA/en/N/), a designer and manufacturer of very large complex construction equipment. They must write assembly instructions for their manufacturers to assemble the equipment and instructional manuals for users of their equipment. So, an ability to write clearly is important in engineering.
- Engineers test their product with intended users (Anderson 2012). Since peers are the faux audience in this activity, they test the procedures (which are the products). Teachers could clarify when the students are acting in the roles of artists versus engineers versus product users.
- Engineers engage in peer review, giving critical feedback or input on designed products. IEEE members "seek, accept, and offer honest criticism of technical work [and] acknowledge and correct errors" (Institute of Electrical and Electronics Engineers 2012)

The giving, receiving, and attending to feedback is critical in engineering as it leads to improved products and processes; the feedback process should also lead to increased student learning and improved work products. At the university level, the giving, receiving, and attending to feedback during open-ended mathematical modeling is difficult for students (Diefes-Dux and Verleger 2009; Carnes et al. 2011; Fry et al. 2011), possibly due to their lack of preparatory experiences with feedback. The peer review process in the Draw-a-Monster activity is an excellent venue for engaging young children more deeply in this engineering practice, thus preparing them for future activities in which the gathering of feedback is employed. The feedback process in the Draw-a-Monster activity could be augmented by formalizing elements of the peer review process. First, having two peer reviews of the first draft of the instructions might reveal to young children (1) problems that consistently appear when their procedures are applied and (2) different problems that appear due to different interpretations of the procedures. Having a second peer reviewer might also overcome the problem of having one marginal peer review. Second, the receiving of and attending to feedback process could be strengthened. The children, in this implementation, attended to a mix of peer written and drawn feedback, often undermining the differences that could be found in the peer drawings. In the future students could more formally list the difference they find between their original drawing and the peer review drawing before editing their instructions. This would improve the number of things students attempt to remedy in their revision. It was however seen that during the revision process, there was shifting attention to details. There were some gains from attending to feedback, but these were often accompanied by some losses from the original draft. Lesh and Doerr (2003) refer to this as unstable ways of thinking; “facts and observations that are salient at one moment may be forgotten a moment later when somewhat different perspectives are adopted” (p. 26). The loss of detail might be alleviated by having students compare their first and second drafts.

Further, with revision, the MEA reality principle could be better met by setting the activity in a more authentic engineering context. For instance, the context could be established wherein the students’ instructions could be a precursor to having a computer replicate artwork and they are limited to the shapes that come with a basic computer drawing package. This would enable both the art and classroom teachers to talk about one or more types of engineering (e.g. computer science or computer engineering) and technologies created by engineers (e.g. software and hardware). Teachers could even begin talk about hybrid education paths that merge art, mathematics, and engineering (e.g. Fischer 2002). Such a context would also enable teachers to connect this activity to their technology standards for using computer tools.

Summary and Conclusions

A Draw-a-Monster activity was combined with design principles for MEAs to achieve an art learning objective. Young children’s achievement of the learning objective was highlighted in four child cases. Opportunities to better connect this activity with mathematics, language arts, and engineering were discussed. Overall this activity provides a child-friendly introduction to aspects of open-ended mathematical modeling problems and could introduce young children to a number of aspects of the work of engineers. These student experiences should lay the foundation for working in teams on engineering activities with greater mathematical complexity.

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