Chapter 9
Solutions to Particular Two-Dimensional Boundary Value Problems of Elastostatics

In this chapter a number of two-dimensional boundary value problems for a body under plane strain conditions or under generalized plane stress conditions are solved. The problems include: (i) a semispace subject to an internal concentrated body force, (ii) an elastic wedge subject to a concentrated load at its tip, and (iii) an infinite elastic strip subject to a discontinuous temperature field. To solve the problems a two-dimensional version of the Boussinesq-Papkovitch-Neuber solution as well as an Airy stress function method, are used.

9.1 The Two-Dimensional Version of Boussinesq-Papkovitch-Neuber Solution for a Body Under Plane Strain Conditions

An elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ corresponding to a body under plane strain conditions is described by the equations [see Eqs. 7.70 and 7.71 in Problem 7.1.]

$$u_\alpha = \psi_\alpha - \frac{1}{4(1-\nu)}(x_\gamma \psi_\gamma + \varphi)_{,\alpha} \tag{9.1}$$

where

$$\psi_{\alpha,\gamma\gamma} = -\frac{b_\alpha}{\mu} \tag{9.2}$$

and

$$\varphi_{,\gamma\gamma} = \frac{x_\gamma b_\gamma}{\mu} \tag{9.3}$$

The strains $E_{\alpha\beta}$ and stresses $S_{\alpha\beta}$, associated with u_α, are given, respectively, by

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$$E_{\alpha\beta} = \frac{1}{4(1-\nu)}[2(1-2\nu)\,\psi_{(\alpha,\beta)} - x_\gamma\,\psi_{\gamma,\alpha\beta} - \varphi_{,\alpha\beta}] \tag{9.4}$$

and

$$S_{\alpha\beta} = \frac{\mu}{4(1-\nu)}[2(1-2\nu)\,\psi_{(\alpha,\beta)} - x_\gamma\,\psi_{\gamma,\alpha\beta} + 2\nu\,\psi_{\gamma,\gamma}\delta_{\alpha\beta} - \varphi_{,\alpha\beta}] \tag{9.5}$$

If a concentrated force P_0 normal to the boundary of a semispace $|x_1| < \infty, \quad x_2 \geq 0$ is applied at the point $(x_1, x_2) = (0, 0)$, and suitable asymptotic conditions are imposed on $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ at infinity, then a suitable choice of the pair (φ, ψ_α) leads to the stress tensor $S_{\alpha\beta}$ in the form

$$S_{11} = -\frac{2P_0}{\pi\, r^4}x_1^2 x_2, \quad S_{22} = -\frac{2P_0}{\pi\, r^4}x_2^3, \quad S_{12} = -\frac{2P_0}{\pi\, r^4}x_1 x_2^2 \tag{9.6}$$

where

$$r = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2} \tag{9.7}$$

In polar coordinates (r, φ) related to the Cartesian coordinates (x_1, x_2) by

$$x_1 = r\, \cos\varphi, \quad x_2 = r\, \sin\varphi \tag{9.8}$$

we obtain

$$S_{rr} = -\frac{2P_0}{\pi\, r}\sin\,\varphi, \quad S_{\varphi\varphi} = S_{r\varphi} = 0 \tag{9.9}$$

Clearly, it follows from (9.6) and (9.9) that

$$|S| \to 0 \quad \text{as} \quad r \to \infty \tag{9.10}$$

Similarly, if a concentrated force T_0 tangent to the boundary of a semispace $|x_1| < \infty, x_2 \geq 0$ is applied at the point $(x_1, x_2) = (0, 0)$, and suitable asymptotic conditions are imposed on $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ at infinity, then a suitable choice of the pair (φ, ψ_α) leads to the stress tensor $S_{\alpha\beta}$ in the form

$$S_{11} = -\frac{2T_0}{\pi\, r^4}x_1^3, \quad S_{22} = -\frac{2T_0}{\pi\, r^4}x_1 x_2^2, \quad S_{12} = -\frac{2T_0}{\pi\, r^4}x_1^2 x_2 \tag{9.11}$$

In polar coordinates (r, φ) we obtain

$$S_{rr} = -\frac{2T_0}{\pi\, r}\cos\,\varphi, \quad S_{\varphi\varphi} = S_{r\varphi} = 0 \tag{9.12}$$

and it follows from Eqs. (9.11) and (9.12) that

$$|\mathbf{S}| \to 0 \quad \text{as} \quad r \to \infty \tag{9.13}$$

9.2 Problems and Solutions Related to Particular Two-Dimensional Boundary Value Problems of Elastostatics

Problem 9.1. Find an elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ corresponding to a concentrated body force in an interior of a homogeneous and isotropic semispace $|x_1| < \infty$, $x_2 \geq 0$, under plane strain conditions, when the boundary of semispace is stress free and the elastic state satisfies suitable asymptotic conditions at infinity.

Solution. We confine ourselves to the case when the semispace: $|x_1| < \infty$, $x_2 \geq 0$ with stress free boundary $x_2 = 0$ is subject to the body force of the form

$$b_\alpha = b_0\, \delta_{\alpha 2}\, \delta(x_1)\, \delta(x_2 - \xi_2) \tag{9.14}$$

where b_0 represents intensity of the force and $\xi_2 > 0$. This means that the semispace is subject to an internal force that is normal to its boundary and concentrated at the point $(0, \xi_2)$.

A solution $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ to the problem is to be found by using a restricted form of Boussinesq–Papkowitch–Neuber solution [see Eqs. (9.1)–(9.5) in which we let $\psi_1 = 0$, $\psi_2 = \psi$, $\varphi = \varphi$]

$$u_1 = -\frac{1}{4(1-\nu)}(x_2 \psi_{,1} + \varphi_{,1}) \tag{9.15}$$

$$u_2 = \frac{1}{4(1-\nu)}[(3-4\nu)\,\psi - x_2 \psi_{,2} - \varphi_{,2}] \tag{9.16}$$

where $\psi = \psi(x_1, x_2)$ and $\varphi = \varphi(x_1, x_2)$ satisfy Poisson's equations

$$\psi_{,rr} = -\frac{1}{\mu} b_2 \tag{9.17}$$

and

$$\varphi_{,rr} = \frac{1}{\mu} x_2 b_2 \tag{9.18}$$

The strain and stress fields are then given, respectively, by

$$E_{11} = \frac{1}{4(1-\nu)}[-x_2 \psi_{,11} - \varphi_{,11}] \tag{9.19}$$

$$E_{22} = \frac{1}{4(1-\nu)}[2(1-2\nu)\psi_{,2} - x_2 \psi_{,22} - \varphi_{,22}] \tag{9.20}$$

$$E_{12} = \frac{1}{4(1-\nu)}[(1-2\nu)\psi_{,1} - x_2 \psi_{,12} - \varphi_{,12}] \tag{9.21}$$

and

$$S_{11} = \frac{\mu}{2(1-\nu)} [-x_2\psi,_{11} + 2\nu\psi,_2 - \varphi,_{11}] \tag{9.22}$$

$$S_{22} = \frac{\mu}{2(1-\nu)} [2(1-\nu)\psi,_2 - x_2\psi,_{22} - \varphi,_{22}] \tag{9.23}$$

$$S_{12} = \frac{\mu}{2(1-\nu)} [(1-2\nu)\psi,_1 - x_2\psi,_{12} - \varphi,_{12}] \tag{9.24}$$

The boundary conditions take the form

$$S_{12}(x_1, 0) = S_{22}(x_1, 0) = 0 \quad \text{for } |x_1| < \infty \tag{9.25}$$

In addition, we assume suitable vanishing conditions at infinity, and suitable restrictions on $\mathbf{u}$ to obtain a unique solution to the problem.

To this end we look for a solution $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ in the form

$$s = s^{(0)} + s^{(1)} \tag{9.26}$$

where $s^{(0)} = [\mathbf{u}^{(0)}, \mathbf{E}^{(0)}, \mathbf{S}^{(0)}]$ is a solution for an infinite plane $|x_1| < \infty$, $|x_2| < \infty$ subject to the body force (9.14), and $s^{(1)} = [\mathbf{u}^{(1)}, \mathbf{E}^{(1)}, \mathbf{S}^{(1)}]$ is a solution for a semispace $|x_1| < \infty$, $x_2 \geq 0$ subject to the boundary conditions

$$S_{12}^{(1)}(x_1, 0) = -S_{12}^{(0)}(x_1, 0) \tag{9.27}$$

and

$$S_{22}^{(1)}(x_1, 0) = -S_{22}^{(0)}(x_1, 0) \tag{9.28}$$

This amounts to looking for a pair (ψ, φ) in the form

$$\psi = \psi^{(0)} + \psi^{(1)} \tag{9.29}$$

and

$$\varphi = \varphi^{(0)} + \varphi^{(1)} \tag{9.30}$$

where

$$\nabla^2\psi^{(0)} = -\frac{1}{\mu} b_2 \tag{9.31}$$

and

$$\nabla^2\varphi^{(0)} = \frac{1}{\mu} x_2 b_2 \tag{9.32}$$

and

$$\nabla^2\psi^{(1)} = 0, \quad \nabla^2\varphi^{(1)} = 0 \tag{9.33}$$

Substituting b_2 from (9.14) into (9.31) and (9.32), respectively, we obtain

$$\nabla^2 \psi^{(0)} = -\frac{b_0}{\mu} \delta(x_1)\delta(x_2 - \xi_2) \tag{9.34}$$

and

$$\nabla^2 \varphi^{(0)} = \frac{b_0 \xi_2}{\mu} \delta(x_1)\delta(x_2 - \xi_2) \tag{9.35}$$

where we used the identity

$$x_2\, \delta(x_2 - \xi_2) = \xi_2\, \delta(x_2 - \xi_2) \tag{9.36}$$

Equations (9.34) and (9.35) are to be satisfied for every $(x_1, x_2) \in E^2$ and for a fixed positive ξ_2. The unique solutions to Eqs. (9.34) and (9.35) are then given, respectively, by

$$\psi^{(0)} = -\frac{b_0}{2\pi\mu} \ln\left(\frac{r_1}{L}\right) \tag{9.37}$$

and

$$\varphi^{(0)} = \frac{b_0 \xi_2}{2\pi\mu} \ln\left(\frac{r_1}{L}\right) \tag{9.38}$$

where

$$r_1 = \sqrt{x_1^2 + (x_2 - \xi_2)^2} \tag{9.39}$$

and L is a positive constant of the length dimension. The solution $s^{(0)}$ is obtained by letting $\psi = \psi^{(0)}$ and $\varphi = \varphi^{(0)}$ into Eqs. (9.15)–(9.16), (9.19)–(9.21), and (9.22)–(9.24). Therefore, we obtain

$$u_1^{(0)} = \frac{b_0}{8\pi\mu(1-\nu)} (x_2 - \xi_2) \frac{\partial}{\partial x_1} \ln\left(\frac{r_1}{L}\right) \tag{9.40}$$

$$u_2^{(0)} = -\frac{b_0}{8\pi\mu(1-\nu)} \left[(3 - 4\nu) - (x_2 - \xi_2)\frac{\partial}{\partial x_2}\right] \ln\left(\frac{r_1}{L}\right) \tag{9.41}$$

and

$$E_{11}^{(0)} = \frac{b_0}{8\pi\mu(1-\nu)} (x_2 - \xi_2) \frac{\partial^2}{\partial x_1^2} \ln\left(\frac{r_1}{L}\right) \tag{9.42}$$

$$E_{22}^{(0)} = -\frac{b_0}{8\pi\mu(1-\nu)} \left[2(1 - 2\nu)\frac{\partial}{\partial x_2} - (x_2 - \xi_2)\frac{\partial^2}{\partial x_2^2}\right] \ln\left(\frac{r_1}{L}\right) \tag{9.43}$$

$$E_{12}^{(0)} = -\frac{b_0}{8\pi\mu(1-\nu)} \left[(1 - 2\nu)\frac{\partial}{\partial x_1} - (x_2 - \xi_2)\frac{\partial^2}{\partial x_1 \partial x_2}\right] \ln\left(\frac{r_1}{L}\right) \tag{9.44}$$

The stress components $S_{11}^{(0)}$, $S_{22}^{(0)}$, and $S_{12}^{(0)}$ are given by

$$S_{11}^{(0)} = -\frac{b_0}{4\pi(1-\nu)}\left[2\nu\frac{\partial}{\partial x_2} - (x_2-\xi_2)\frac{\partial^2}{\partial x_1^2}\right]\ln\left(\frac{r_1}{L}\right) \tag{9.45}$$

$$S_{22}^{(0)} = -\frac{b_0}{4\pi(1-\nu)}\left[2(1-\nu)\frac{\partial}{\partial x_2} - (x_2-\xi_2)\frac{\partial^2}{\partial x_2^2}\right]\ln\left(\frac{r_1}{L}\right) \tag{9.46}$$

$$S_{12}^{(0)} = -\frac{b_0}{4\pi(1-\nu)}\left[(1-2\nu)\frac{\partial}{\partial x_1} - (x_2-\xi_2)\frac{\partial^2}{\partial x_1 \partial x_2}\right]\ln\left(\frac{r_1}{L}\right) \tag{9.47}$$

The solution $s^{(1)}$ is to be found by letting $\psi = \psi^{(1)}$ and $\varphi = \varphi^{(1)}$ into Eqs.(9.15)–(9.16), (9.19)–(9.21), and (9.22)–(9.24), where $\psi^{(1)}$ and $\varphi^{(1)}$ satisfy Eqs. $(9.33)_1$ and $(9.33)_2$, respectively, for $|x_1| < \infty, x_2 > 0$, subject to suitable boundary conditions at $x_2 = 0$. A hint as to how $\psi^{(1)}$ and $\varphi^{(1)}$ could be found comes from the boundary conditions (9.27) and (9.28) written in terms of the pairs $(\psi^{(0)}, \varphi^{(0)})$ and $(\psi^{(1)}, \varphi^{(1)})$:

$$\left[(1-2\nu)\psi^{(1)} - \varphi,_2^{(1)}\right],_1(x_1,0) = -f(x_1) \tag{9.48}$$

$$\left[(2-2\nu)\psi^{(1)} - \varphi,_2^{(1)}\right],_2(x_1,0) = -g(x_1) \tag{9.49}$$

where

$$f(x_1) = \left[(1-2\nu)\psi^{(0)} - \varphi,_2^{(0)}\right],_1(x_1,0) \tag{9.50}$$

and

$$g(x_1) = \left[(2-2\nu)\psi^{(0)} - \varphi,_2^{(0)}\right],_2(x_1,0) \tag{9.51}$$

Since

$$\frac{\partial}{\partial x_1}\ln\left(\frac{r_1}{L}\right) = \int_0^\infty e^{-\alpha|x_2-\xi_2|}\sin\alpha x_1 d\alpha \tag{9.52}$$

and

$$\frac{\partial}{\partial x_2}\ln\left(\frac{r_1}{L}\right) = \int_0^\infty e^{-\alpha|x_2-\xi_2|}\cos\alpha x_1 d\alpha \tag{9.53}$$

therefore, because of (9.37) and (9.38) we obtain

$$\psi,_1^{(0)} = -\frac{b_0}{2\pi\mu}\int_0^\infty e^{-\alpha|x_2-\xi_2|}\sin\alpha x_1 d\alpha \tag{9.54}$$

and

$$\varphi,_{2}^{(0)} = \frac{b_0\xi_2}{2\pi\mu} \int_0^\infty e^{-\alpha|x_2-\xi_2|} \cos \alpha x_1 d\alpha \tag{9.55}$$

It follows from (9.55) that

$$\varphi,_{12}^{(0)} = -\frac{b_0\xi_2}{2\pi\mu} \int_0^\infty e^{-\alpha|x_2-\xi_2|}\alpha \sin \alpha x_1 d\alpha \tag{9.56}$$

and an extension of the RHS of (9.50) to include arbitrary point (x_1, x_2) reads

$$\begin{aligned} &\left[(1-2\nu)\psi,_{1}^{(0)} - \varphi,_{12}^{(0)}\right](x_1, x_2) \\ &\quad = -\frac{b_0}{2\pi\mu} \int_0^\infty e^{-\alpha|x_2-\xi_2|} \left[(1-2\nu) - \alpha\xi_2\right] \sin \alpha x_1 d\alpha \end{aligned} \tag{9.57}$$

Hence

$$f(x_1) = -\frac{b_0}{2\pi\mu} \int_0^\infty e^{-\alpha\xi_2} \left[(1-2\nu) - \alpha\xi_2\right] \sin \alpha x_1 d\alpha \tag{9.58}$$

Similarly, we obtain

$$\psi,_{2}^{(0)} = -\frac{b_0}{2\pi\mu} \int_0^\infty e^{-\alpha|x_2-\xi_2|} \cos \alpha x_1 d\alpha \tag{9.59}$$

and

$$\varphi,_{22}^{(0)} = \frac{b_0\xi_2}{2\pi\mu} \frac{\partial}{\partial x_2} \int_0^\infty e^{-\alpha|x_2-\xi_2|} \cos \alpha x_1 d\alpha \tag{9.60}$$

For $0 \le x_2 < \xi_2$ Eq. (9.60) takes the form

$$\varphi,_{22}^{(0)} = \frac{b_0\xi_2}{2\pi\mu} \int_0^\infty e^{-\alpha(\xi_2-x_2)}\alpha \cos \alpha x_1 d\alpha \tag{9.61}$$

Hence, using (9.59) and (9.61) we reduce (9.51) to the form

$$g(x_1) = -\frac{b_0}{2\pi\mu} \int_0^\infty e^{-\alpha\xi_2}[(2-2\nu) + \alpha\xi_2] \cos\alpha x_1 d\alpha \tag{9.62}$$

Since

$$\nabla^2[e^{-\alpha x_2} \cos\alpha x_1] = 0 \tag{9.63}$$

an inspection of Eqs. (9.33) and of the boundary conditions (9.48) and (9.49) in which f and g are given by the integrals (9.58) and (9.62), respectively, leads to the integral form of $\psi^{(1)}$ and $\varphi^{(1)}$ for $|x_1| < \infty$, $x_2 > 0$:

$$\psi^{(1)}(x_1, x_2) = \int_0^\infty A(\alpha) e^{-\alpha x_2} \cos\alpha x_1 d\alpha \tag{9.64}$$

and

$$\varphi^{(1)}(x_1, x_2) = \int_0^\infty B(\alpha) e^{-\alpha x_2} \cos\alpha x_1 d\alpha \tag{9.65}$$

where $A(\alpha)$ and $B(\alpha)$ are arbitrary functions on $[0, \infty)$ to be selected in such a way that the boundary conditions (9.48) and (9.49) are satisfied. For the partial derivatives of $\psi^{(1)}$ and $\varphi^{(1)}$ that come into the boundary conditions (9.48) and (9.49) we obtain

$$\psi,_1^{(1)} = -\int_0^\infty A(\alpha) e^{-\alpha x_2} \alpha \sin\alpha x_1 d\alpha \tag{9.66}$$

$$\varphi,_2^{(1)} = -\int_0^\infty B(\alpha) e^{-\alpha x_2} \alpha \cos\alpha x_1 d\alpha \tag{9.67}$$

$$\varphi,_{21}^{(1)} = \int_0^\infty B(\alpha) e^{-\alpha x_2} \alpha^2 \sin\alpha x_1 d\alpha \tag{9.68}$$

$$\psi,_2^{(1)} = -\int_0^\infty A(\alpha) e^{-\alpha x_2} \alpha \cos\alpha x_1 d\alpha \tag{9.69}$$

$$\varphi,_{22}^{(1)} = \int_0^\infty B(\alpha) e^{-\alpha x_2} \alpha^2 \cos\alpha x_1 d\alpha \tag{9.70}$$

Therefore, substituting (9.66) and (9.68) into (9.48), and (9.69) and (9.70) into (9.49), and using f and g in the forms (9.58) and (9.62), respectively, we find that the functions $A = A(\alpha)$ and $B = B(\alpha)$ must satisfy the linear algebraic equations

$$(1-2\nu)A+\alpha B=-\frac{b_0}{2\pi\mu}e^{-\alpha\xi_2}\frac{1}{\alpha}[1-2\nu-\alpha\xi_2]$$
$$(2-2\nu)A+\alpha B=-\frac{b_0}{2\pi\mu}e^{-\alpha\xi_2}\frac{1}{\alpha}[2-2\nu+\alpha\xi_2] \tag{9.71}$$

and the only solution (A, B) to (9.71) takes the form

$$A=-\frac{b_0}{2\pi\mu}e^{-\alpha\xi_2}\frac{(1+2\alpha\xi_2)}{\alpha} \tag{9.72}$$
$$B=\frac{b_0\xi_2}{2\pi\mu}e^{-\alpha\xi_2}\frac{(3-4\nu)}{\alpha} \tag{9.73}$$

It follows from Eqs. (9.64)–(9.65) and (9.72)–(9.73) that the integral representations of $\psi^{(1)}$ and $\varphi^{(1)}$ are divergent, however, all partial derivatives of $\psi^{(1)}$ and $\varphi^{(1)}$ are represented by the convergent integrals. This implies that the integral representations of $\mathbf{E}^{(1)}$ and $\mathbf{S}^{(1)}$ are convergent. In the following we are to obtain first the integral forms of $\mathbf{E}^{(1)}$ and $\mathbf{S}^{(1)}$, as well as of $u_1^{(1)}$ and $u_{2,1}^{(1)}$, and next the integral representation of $u_{2,1}^{(1)}$ is used to recover $u_2^{(1)}$ by integration. Note that an alternative form of Eqs. (9.19)–(9.21) and (9.22)–(9.24), respectively, taken at $\psi=\psi^{(1)}$ and $\varphi=\varphi^{(1)}$, reads

$$E_{11}^{(1)}+E_{22}^{(1)}=\frac{1-2\nu}{2-2\nu}\psi,_2^{(1)} \tag{9.74}$$

$$E_{11}^{(1)}-E_{22}^{(1)}=-\frac{1-2\nu}{2-2\nu}\psi,_2^{(1)}-\frac{1}{4(1-\nu)}\left[x_2\left(\psi,_{11}^{(1)}-\psi,_{22}^{(1)}\right)+\varphi,_{11}^{(1)}-\varphi,_{22}^{(1)}\right] \tag{9.75}$$
$$E_{12}^{(1)}=\frac{1-2\nu}{4(1-\nu)}\psi,_1^{(1)}-\frac{1}{4(1-\nu)}\left(x_2\psi,_{12}^{(1)}+\varphi,_{12}^{(1)}\right) \tag{9.76}$$

and

$$S_{11}^{(1)}+S_{22}^{(1)}=\frac{\mu}{1-\nu}\psi,_2^{(1)} \tag{9.77}$$

$$S_{11}^{(1)}-S_{22}^{(1)}=-\mu\frac{1-2\nu}{1-\nu}\psi,_2^{(1)}-\frac{\mu}{2(1-\nu)}\left[x_2\left(\psi,_{11}^{(1)}-\psi,_{22}^{(1)}\right)+\varphi,_{11}^{(1)}-\varphi,_{22}^{(1)}\right] \tag{9.78}$$
$$S_{12}^{(1)}=\frac{\mu(1-2\nu)}{2(1-\nu)}\psi,_1^{(1)}-\frac{\mu}{2(1-\nu)}\left(x_2\psi,_{12}^{(1)}+\varphi,_{12}^{(1)}\right) \tag{9.79}$$

Also, it follows from (9.15) and (9.16), respectively, taken at $\psi=\psi^{(1)}$ and $\varphi=\varphi^{(1)}$ that

$$u_1^{(1)}=-\frac{1}{4(1-\nu)}\left(x_2\psi,_1^{(1)}+\varphi,_1^{(1)}\right) \tag{9.80}$$

and

$$u^{(1)}_{2,1} = \frac{1}{4(1-\nu)}\left[(3-4\nu)\psi,^{(1)}_{1} - x_2\psi,^{(1)}_{12} - \varphi,^{(1)}_{12}\right] \tag{9.81}$$

Therefore, substituting $\psi^{(1)}$ and $\varphi^{(1)}$ from (9.64) and (9.65), respectively, where A and B are given by (9.72) and (9.73), respectively into Eqs. (9.74)–(9.81), we obtain

$$E^{(1)}_{11} + E^{(1)}_{22} = \frac{b_0}{2\pi\mu}\frac{1-2\nu}{2-2\nu}\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\cos\alpha x_1 d\alpha \tag{9.82}$$

$$E^{(1)}_{11} - E^{(1)}_{22} = -\frac{b_0}{2\pi\mu}\frac{1-2\nu}{2-2\nu}\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\cos\alpha x_1 d\alpha$$
$$-\frac{b_0}{4\pi\mu(1-\nu)}\left\{\int_0^\infty e^{-\alpha(x_2+\xi_2)}\alpha[x_2(1+2\alpha\xi_2)-\xi_2(3-4\nu)]\cos\alpha x_1 d\alpha\right\} \tag{9.83}$$

$$E^{(1)}_{12} = \frac{b_0}{2\pi\mu}\frac{1-2\nu}{4(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\sin\alpha x_1 d\alpha$$
$$+\frac{b_0}{2\pi\mu}\frac{1}{4(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}[x_2-(3-4\nu)\xi_2+2\alpha\xi_2 x_2]\times\alpha\sin\alpha x_1 d\alpha \tag{9.84}$$

and

$$S^{(1)}_{11} + S^{(1)}_{22} = \frac{b_0}{2\pi(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\cos\alpha x_1 d\alpha \tag{9.85}$$

$$S^{(1)}_{11} - S^{(1)}_{22} = -\frac{b_0(1-2\nu)}{2\pi(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\cos\alpha x_1 d\alpha$$
$$-\frac{b_0}{2\pi(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}\alpha[x_2-(3-4\nu)\xi_2+2\alpha\xi_2 x_2]\cos\alpha x_1 d\alpha \tag{9.86}$$

$$S_{12}^{(1)} = \frac{b_0(1-2\nu)}{4\pi(1-\nu)} \int_0^\infty e^{-\alpha(x_2+\xi_2)}(1+2\alpha\xi_2)\sin\alpha x_1 d\alpha + \frac{b_0}{4\pi(1-\nu)} \int_0^\infty e^{-\alpha(x_2+\xi_2)}[x_2-(3-4\nu)\xi_2+2\alpha\xi_2 x_2]\alpha \sin\alpha x_1 d\alpha \tag{9.87}$$

In addition, Eqs. (9.80) and (9.81), respectively, imply that

$$u_1^{(1)} = -\frac{b_0}{8\pi\mu(1-\nu)} \int_0^\infty e^{-\alpha(x_2+\xi_2)}[x_2-(3-4\nu)\xi_2+2\alpha\xi_2 x_2]\sin\alpha x_1 d\alpha \tag{9.88}$$

and

$$u_{2,1}^{(1)} = \frac{b_0}{8\pi\mu(1-\nu)} \int_0^\infty e^{-\alpha(x_2+\xi_2)}\{3-4\nu+[(3-4\nu)\xi_2+x_2]\alpha+2\xi_2 x_2\alpha^2\} \times \sin\alpha x_1 d\alpha \tag{9.89}$$

It follows from Eqs. (9.82)–(9.89), respectively, that $\mathbf{E}^{(1)}$, $\mathbf{S}^{(1)}$, $u_1^{(1)}$, and $u_{2,1}^{(1)}$ are represented by the convergent integrals for any point of the semispace: $|x_1|<\infty$, $x_2\geq 0$. In addition, by using the formulas [see (9.52) and (9.53)]

$$\int_0^\infty e^{-\alpha u}\cos\alpha x_1 d\alpha = \frac{\partial}{\partial u}\ln\left(\frac{R}{L}\right) \tag{9.90}$$

$$\int_0^\infty e^{-\alpha u}\sin\alpha x_1 d\alpha = \frac{\partial}{\partial x_1}\ln\left(\frac{R}{L}\right) \tag{9.91}$$

and the formulas obtained from (9.90) and (9.91) by differentiation

$$\int_0^\infty e^{-\alpha u}\alpha\sin\alpha x_1 d\alpha = -\frac{\partial^2}{\partial x_1\partial u}\ln\left(\frac{R}{L}\right) \tag{9.92}$$

$$\int_0^\infty e^{-\alpha u}\alpha^2\sin\alpha x_1 d\alpha = \frac{\partial^3}{\partial x_1\partial u^2}\left[\ln\left(\frac{R}{L}\right)\right] \tag{9.93}$$

$$\int_0^\infty e^{-\alpha u}\alpha\cos\alpha x_1 d\alpha = -\frac{\partial^2}{\partial u^2}\left[\ln\left(\frac{R}{L}\right)\right] \tag{9.94}$$

$$\int_0^\infty e^{-\alpha u}\alpha^2\cos\alpha x_1 d\alpha = \frac{\partial^3}{\partial u^3}\left[\ln\left(\frac{R}{L}\right)\right] \tag{9.95}$$

where

$$R = \sqrt{x_1^2 + u^2}, \quad u > 0 \tag{9.96}$$

the fields $\mathbf{E}^{(1)}$, $\mathbf{S}^{(1)}$, $u_1^{(1)}$, and $u_{2,1}^{(1)}$ can be obtained in terms of elementary functions.

For example, by using (9.91) and (9.92), the closed form of $u_1^{(1)}$ is obtained

$$u_1^{(1)} = -\frac{b_0}{8\pi\mu(1-\nu)}\left\{[x_2 - (3-4\nu)\xi_2]\frac{\partial}{\partial x_1}\left[\ln\left(\frac{r_2}{L}\right)\right]\right.$$
$$\left. -2x_2\xi_2\frac{\partial^2}{\partial x_1 \partial x_2}\left[\ln\left(\frac{r_2}{L}\right)\right]\right\} \tag{9.97}$$

where

$$r_2 = \sqrt{x_1^2 + (x_2 + \xi_2)^2} \tag{9.98}$$

To obtain a closed-form of $u_2^{(1)}$ we integrate (9.89) with respect to x_1 over the interval $[0, x_1]$ and obtain

$$u_2^{(1)}(x_1, x_2) - u_2^{(1)}(0, x_2) = \frac{b_0}{8\pi\mu(1-\nu)}\int_0^\infty e^{-\alpha(x_2+\xi_2)}$$
$$\times \{3 - 4\nu + [(3-4\nu)\xi_2 + x_2]\alpha + 2\xi_2 x_2 \alpha^2\}$$
$$\times \frac{1 - \cos\alpha x_1}{\alpha} d\alpha \tag{9.99}$$

By letting

$$u_2^{(1)}(0, x_2) = 0 \quad \text{for } x_2 > 0 \tag{9.100}$$

Equation (9.99) can be written as

$$u_2^{(1)} = \frac{b_0}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\int_0^\infty e^{-\alpha(x_2+\xi_2)}\frac{(1-\cos\alpha x_1)}{\alpha}d\alpha\right.$$
$$+ [(3-4\nu)\xi_2 + x_2]\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1-\cos\alpha x_1)d\alpha$$
$$\left. + 2\xi_2 x_2 \int_0^\infty e^{-\alpha(x_2+\xi_2)}\alpha(1-\cos\alpha x_1)d\alpha\right\} \tag{9.101}$$

By integrating (9.91) with respect to x_1 we obtain

$$\int_0^\infty e^{-\alpha u}\frac{1-\cos\alpha x_1}{\alpha}d\alpha = \ln\left(\frac{R}{u}\right) \tag{9.102}$$

Hence

$$\int_0^\infty e^{-\alpha(x_2+\xi_2)}\frac{1-\cos\alpha x_1}{\alpha}d\alpha = \ln\frac{r_2}{x_2+\xi_2} \tag{9.103}$$

and by differentiation of (9.103) with respect to x_2, we obtain

$$\int_0^\infty e^{-\alpha(x_2+\xi_2)}(1-\cos\alpha x_1)d\alpha = -\frac{\partial}{\partial x_2}\ln\left(\frac{r_2}{x_2+\xi_2}\right) \tag{9.104}$$

and

$$\int_0^\infty e^{-\alpha(x_2+\xi_2)}\alpha(1-\cos\alpha x_1)d\alpha = \frac{\partial^2}{\partial x_2^2}\ln\left(\frac{r_2}{x_2+\xi_2}\right) \tag{9.105}$$

Finally, substituting (9.103), (9.104), and (9.105) into (9.101) we obtain $u_2^{(1)}$ in the form

$$u_2^{(1)} = \frac{b_0}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\ln\left(\frac{r_2}{x_2+\xi_2}\right) - [(3-4\nu)\xi_2 + x_2]\frac{\partial}{\partial x_2}\ln\left(\frac{r_2}{x_2+\xi_2}\right) + 2\xi_2 x_2\frac{\partial^2}{\partial x_2^2}\ln\left(\frac{r_2}{x_2+\xi_2}\right)\right\} \tag{9.106}$$

This completes a solution to Problem 9.1 in which the semispace is subject to an internal force that is normal to its boundary and concentrated at the point $(0, \xi_2)$.

In a similar way a solution to Prob. 9.1 in which the semispace is subject to a force that is parallel to its boundary and concentrated at $(0, \xi_2)$, may be obtained.

Let $U_{\alpha 2}$ and $U_{\alpha 1}$, respectively, denote the displacement of the semispace corresponding to the unit normal and parallel forces at $(0, \xi_2)$, and let $\boldsymbol{l} = (l_1, l_2)$ be an arbitrary force concentrated at $(0, \xi_2)$. Then the displacement u_α corresponding to a solution to Problem 9.1 in which the semispace is subject to the concentrated force $\boldsymbol{l}$ at $(0, \xi_2)$ takes the form

$$u_\alpha = U_{\alpha\beta} l_\beta \tag{9.107}$$

This completes a solution to Problem 9.1 in which the semispace with stress free boundary is subject to a concentrated force $\boldsymbol{l}$ at $(0, \xi_2)$.

Problem 9.2. Find an elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ corresponding to a concentrated body force in an interior of a homogeneous and isotropic semispace $|x_1| < \infty$,

$x_2 \geq 0$, under plane strain conditions, when the boundary of semispace is clamped and the elastic state vanishes at infinity.

Solution. Let the semispace $|x_1| < \infty$, $x_2 \geq 0$ with a clamped boundary $x_2 = 0$ be subject to the body force

$$b_\alpha = b_0 \delta_{\alpha 2} \delta(x_1) \delta(x_2 - \xi_2) \tag{9.108}$$

An elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ corresponding to (9.108) and satisfying the boundary conditions

$$u_1(x_1, 0) = u_2(x_1, 0) = 0 \quad |x_1| < \infty \tag{9.109}$$

and suitable vanishing conditions at infinity may be found in a way similar to that of Problem 9.1. To this end we use Eqs. (9.109)–(9.119) of Problem 9.1 to obtain **u**, **E**, and **S**, respectively.

In particular, u_1 and u_2 are to be found from the equations

$$u_1 = -\frac{1}{4(1-\nu)}(x_2 \psi_{,1} + \varphi_{,1}) \tag{9.110}$$

$$u_2 = \frac{1}{4(1-\nu)}[(3-4\nu)\psi - x_2 \psi_{,2} - \varphi_{,2}] \tag{9.111}$$

where ψ and φ satisfy Poisson's equations

$$\nabla^2 \psi = -\frac{1}{\mu} b_2 \tag{9.112}$$

and

$$\nabla^2 \varphi = \frac{1}{\mu} x_2 b_2 \tag{9.113}$$

To find a pair (ψ, φ) that generates (u_1, u_2) by Eqs. (9.110)–(9.111) in such a way that Eqs. (9.109) are satisfied, we let

$$\psi = \psi^{(0)} + \psi^{(1)} \quad |x_1| < \infty, x_2 \geq 0 \tag{9.114}$$

and

$$\varphi = \varphi^{(0)} + \varphi^{(1)} \quad |x_1| < \infty, x_2 \geq 0 \tag{9.115}$$

where

$$\nabla^2 \psi^{(0)} = -\frac{1}{\mu} b_2 \quad |x_1| < \infty, x_2 \geq 0 \tag{9.116}$$

and

$$\nabla^2 \varphi^{(0)} = \frac{1}{\mu} x_2 b_2 \quad |x_1| < \infty, x_2 \geq 0 \tag{9.117}$$

and

$$\nabla^2 \psi^{(1)} = 0, \quad \nabla^2 \varphi^{(1)} = 0 \tag{9.118}$$

and the harmonic functions $\psi^{(1)}$ and $\varphi^{(1)}$ defined for $|x_1| < \infty, \quad x_2 \geq 0$ are selected in such a way that u_1 and u_2 vanish at $x_2 = 0$. To obtain a pair $(\psi^{(0)}, \varphi^{(0)})$ we extend Eqs. (9.116)–(9.117) to the whole plane E^2 in which a normal force of intensity b_0 is concentrated at $(0, \xi_2)$ and a normal force of intensity—b_0 is concentrated at $(0, -\xi_2)$. This amounts to solving the equations

$$\nabla^2 \psi^{(0)} = -\frac{b_0}{\mu}\delta(x_1)[\delta(x_2 - \xi_2) - \delta(x_2 + \xi_2)] \tag{9.119}$$

and

$$\nabla^2 \varphi^{(0)} = \frac{1}{\mu} b_0 x_2 [\delta(x_2 - \xi_2) - \delta(x_2 + \xi_2)] \quad \text{for } |x_1| < \infty, \ |x_2| < \infty \tag{9.120}$$

Note that a restriction of Eqs. (9.119)–(9.120) to the semispace $|x_1| < \infty, x_2 \geq 0$ leads to Eqs. (9.116)–(9.117), and an extension of $(\psi^{(0)}, \varphi^{(0)})$ is denoted in the same way as its restriction.

Since

$$x_2 \delta(x_2 - \xi_2) = \xi_2 \delta(x_2 - \xi_2) \tag{9.121}$$

then

$$-x_2 \delta(x_2 - \xi_2) = -\xi_2 \delta(x_2 - \xi_2) \tag{9.122}$$

and replacing ξ_2 by $-\xi_2$ in (9.122) we get

$$-x_2 \delta(x_2 + \xi_2) = \xi_2 \delta(x_2 + \xi_2) \tag{9.123}$$

Hence, Eq. (9.120) can be written as

$$\nabla^2 \varphi^{(0)} = \frac{b_0 \xi_2}{\mu}[\delta(x_2 - \xi_2) + \delta(x_2 + \xi_2)] \quad \text{for } |x_1| < \infty, |x_2| < \infty \tag{9.124}$$

Proceeding in a way similar to that of solving Eqs. (9.129) and (9.130) of Problem 9.1, from Eqs. (9.119) and (9.124), respectively, we obtain

$$\psi^{(0)} = -\frac{b_0}{2\pi\mu}\left[\ln\left(\frac{r_1}{L}\right) - \ln\left(\frac{r_2}{L}\right)\right] \tag{9.125}$$

and

$$\varphi^{(0)} = \frac{b_0 \xi_2}{2\pi\mu}\left[\ln\left(\frac{r_1}{L}\right) + \ln\left(\frac{r_2}{L}\right)\right] \tag{9.126}$$

where

$$r_{1.2} = \sqrt{x_1^2 + (x_2 \mp \xi_2)^2} \tag{9.127}$$

It follows from (9.125)–(9.127) that

$$\psi^{(0)}(x_1, 0) = 0, \quad |x_1| < \infty \tag{9.128}$$

and

$$\varphi^{(0)}(x_1, 0) = \frac{b_0 \xi_2}{\pi \mu} \ln\left(\frac{r_0}{L}\right), \quad |x_1| < \infty \tag{9.129}$$

where

$$r_0 = \sqrt{x_1^2 + \xi_2^2} \tag{9.130}$$

Also, using (9.110) and (9.111) in which $\psi = \psi^{(0)}$ and $\varphi = \varphi^{(0)}$, and letting $x_2 = 0$ we obtain

$$u_1^{(0)}(x_1, 0) = -\frac{b_0 \xi_2}{4\pi \mu (1-\nu)} \frac{\partial}{\partial x_1} \ln\left(\frac{r_0}{L}\right) \tag{9.131}$$

and

$$u_2^{(0)}(x_1, 0) = 0 \tag{9.132}$$

The displacements $u_1^{(1)}$ and $u_2^{(1)}$ are represented by

$$u_1^{(1)}(x_1, x_2) = -\frac{1}{4(1-\nu)} \left(x_2 \psi,_1^{(1)} + \varphi,_1^{(1)} \right) \tag{9.133}$$

$$u_2^{(1)}(x_1, x_2) = \frac{1}{4(1-\nu)} \left[(3-4\nu)\psi^{(1)} - x_2 \psi,_2^{(1)} - \varphi,_2^{(1)} \right] \tag{9.134}$$

where $\psi^{(1)}$ and $\varphi^{(1)}$ are harmonic on the semispace $|x_1| < \infty, x_2 > 0$.

It is easy to check that the function $\varphi^{(1)} = \varphi^{(1)}(x_1, x_2)$ given by

$$\varphi^{(1)}(x_1, x_2) = -\frac{b_0 \xi_2}{\pi \mu} \ln\left(\frac{r_2}{L}\right) \tag{9.135}$$

satisfies the Laplace's equation

$$\nabla^2 \varphi^{(1)} = 0 \quad \text{for } |x_1| < \infty,\ x_2 > 0 \tag{9.136}$$

and complies with the boundary condition

$$u_1(x_1, 0) = u_1^{(0)}(x_1, 0) + u_2^{(1)}(x_1, 0) = 0 \tag{9.137}$$

To find $\psi^{(1)}$, note that because of (9.132) and (9.134), $\psi^{(1)}$ must satisfy the Laplace equation for $|x_1| < \infty, x_2 > 0$ subject to the boundary condition

$$(3-4\nu)\psi^{(1)}(x_1,0) = \varphi,_2^{(1)}(x_1,0) = -\frac{b_0\xi_2}{\pi\mu}\frac{\partial}{\partial x_2}\ln\left(\frac{r_2}{L}\right)\bigg|_{x_2=0}$$
$$= -\frac{b_0\xi_2}{\pi\mu}\frac{\partial}{\partial \xi_2}\ln\left(\frac{r_0}{L}\right) \tag{9.138}$$

Since

$$\frac{\partial}{\partial \xi_2}\ln\left(\frac{r_0}{L}\right) = \int_0^\infty e^{-\alpha\xi_2}\cos\alpha x_1 d\alpha \tag{9.139}$$

therefore, (9.138) takes the form

$$(3-4\nu)\psi^{(1)}(x_1,0) = -\frac{b_0\xi_2}{\pi\mu}\int_0^\infty e^{-\alpha\xi_2}\cos\alpha x_1 d\alpha \tag{9.140}$$

and we find that for any point of the semispace $|x_1| < \infty, x_2 \geq 0$

$$\psi^{(1)}(x_1,x_2) = -\frac{b_0\xi_2}{\pi\mu(3-4\nu)}\frac{\partial}{\partial x_2}\ln\left(\frac{r_2}{L}\right) \tag{9.141}$$

As a result, because of (9.114) and (9.115), (9.125) and (9.126), and (9.135) and (9.141), we obtain

$$\psi(x_1,x_2) = -\frac{b_0}{2\pi\mu(3-4\nu)}\left[(3-4\nu)\ln\left(\frac{r_1}{r_2}\right) + 2\xi_2\frac{\partial}{\partial x_2}\ln\left(\frac{r_2}{L}\right)\right] \tag{9.142}$$

and

$$\varphi(x_1,x_2) = \frac{b_0\xi_2}{2\pi\mu}\ln\left(\frac{r_1}{r_2}\right) \tag{9.143}$$

Next, substituting ψ and φ from Eqs. (9.142) and (9.143), respectively, to Eqs. (9.110)–(9.111), we obtain a closed-form displacement vector **u** corresponding to the solution $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$. The associated fields **E** and **S**, respectively, are obtained by substituting ψ and φ from (9.142) and (9.143), into Eqs. (9.113)–(9.115) and (9.116)–(9.119) of Problem 9.1.

This completes a solution to Problem 9.2 when the concentrated body force is normal to the clamped boundary $x_2 = 0$.

If the concentrated body force is parallel to the clamped boundary $x_2 = 0$, Problem 9.2 may be solved in a similar way. When the body force is arbitrarily oriented with regard to the clamped boundary $x_2 = 0$, a solution to Problem 9.2 is a linear combination of the solutions corresponding to the normal and parallel directions of the concentrated body force. This completes a solution to Problem 9.2.

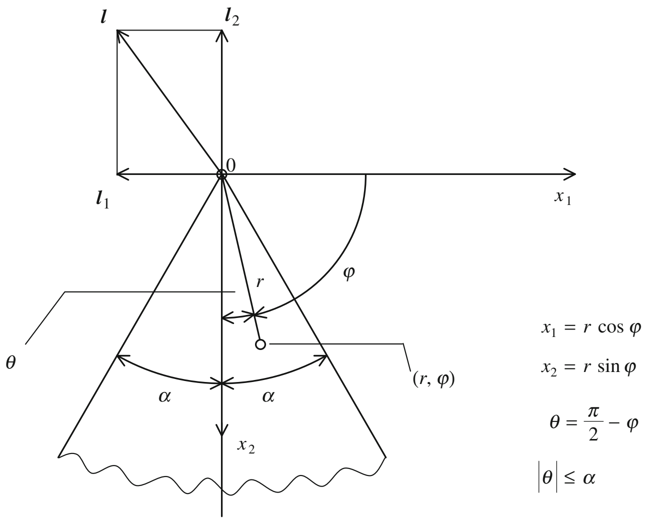

Fig. 9.1 The infinite wedge loaded by a concentrated force

Problem 9.3. Suppose that a homogeneous isotropic infinite elastic wedge, subjected to generalized plane stress conditions, is loaded in its plane by a concentrated force $\boldsymbol{l}$ applied at its tip (see Fig. 9.1)

$$x_1 = r\ \cos\varphi, \quad \theta = \frac{\pi}{2} - \varphi$$
$$x_2 = r\ \sin\varphi, \quad |\theta| \le \alpha$$

Show that the stress components $\overline{S}_{rr}, \overline{S}_{r\varphi},$ and $\overline{S}_{\varphi\varphi}$ corresponding to the force $\boldsymbol{l}$ and vanishing at infinity take the form

$$\overline{S}_{rr}(r,\varphi) = \frac{2l_1}{r}\frac{\cos\varphi}{(2\alpha - \sin 2\alpha)} + \frac{2l_2}{r}\frac{\sin\varphi}{(2\alpha + \sin 2\alpha)}$$
$$\overline{S}_{r\varphi}(r,\varphi) = \overline{S}_{\varphi\varphi}(r,\varphi) = 0$$

for every $0 < r < \infty$., $\pi/2 - \alpha \le \varphi \le \pi/2 + \alpha$. Note that $l_1 < 0$ and $l_2 < 0$, and $\left|\overline{S}_{rr}\right| \to \infty$ for $\alpha \to 0$ and $r > 0$.

Solution. A solution to this problem is to be given in the two cases

(i) $l_1 \ne 0, \quad l_2 = 0$
(ii) $l_1 = 0, \quad l_2 \ne 0$

The case (i). The stress components $S^{||}_{rr}$, $S^{||}_{r\varphi}$, and $S^{||}_{\varphi\varphi}$ in a semi-infinite disk $|x_1| < \infty, x_2 \geq 0$ subject to a tangent concentrated force T_0 at (0, 0) take the form [see Eq. (9.12)]

$$S^{||}_{rr}(r,\varphi) = -\frac{2T_0}{\pi r}\cos\varphi = -\frac{2T_0}{\pi r}\sin\theta \tag{9.144}$$

$$S^{||}_{r\varphi}(r,\varphi) = S^{||}_{\varphi\varphi}(r,\varphi) = 0 \tag{9.145}$$

where $0 \leq r < \infty, 0 \leq \varphi < 2\pi$.

A restriction of Eqs. (9.144) and (9.145) to the wedge region shown in the Figure provides a solution to Problem 9.3 in case (i) if a resultant tangent force at the tip of the wedge is equal to l_1, that is, if for every $r \geq 0$

$$\int_{-\alpha}^{\alpha} S^{||}_{rr}\, r\sin\theta d\theta = l_1 \tag{9.146}$$

Substituting $S^{||}_{rr}$ from (9.144) into (9.146) we obtain

$$-\frac{2T_0}{\pi}\times 2\int_0^{\alpha}\sin^2\theta\, d\theta = -\frac{2T_0}{\pi}\int_0^{\alpha}(1-\cos 2\theta)\, d\theta = -\frac{T_0}{\pi}(2\alpha - \sin 2\alpha) = l_1 \tag{9.147}$$

Hence

$$T_0 = -\frac{l_1\pi}{2\alpha - \sin 2\alpha} \tag{9.148}$$

and substituting T_0 from (9.148) into (9.144) we obtain

$$S^{*}_{rr}(r,\varphi) = \frac{2l_1\cos\varphi}{r(2\alpha - \sin 2\alpha)} \tag{9.149}$$

$$S^{*}_{r\varphi}(r,\varphi) = 0, \quad S^{*}_{\varphi\varphi}(r,\varphi) = 0 \quad \text{for } 0 < r < \infty, \left|\varphi - \frac{\pi}{2}\right| \leq \alpha \tag{9.150}$$

The stress components S^{*}_{rr}, $S^{*}_{r\varphi}$, and $S^{*}_{\varphi\varphi}$ represent a solution to Problem 9.3 in case (i).

The case (ii). The stress components $S^{\perp}_{rr}$, $S^{\perp}_{r\varphi}$, and $S^{\perp}_{\varphi\varphi}$ in a semi-infinite disk $|x_1| < \infty, x_2 \geq 0$ subject to a normal force P_0 concentrated at (0, 0) take the form [see Eqs. (9.9)]

$$S^{\perp}_{rr}(r,\varphi) = -\frac{2P_0}{\pi r}\sin\varphi = -\frac{2P_0}{\pi r}\cos\theta \tag{9.151}$$

$$S^{\perp}_{r\varphi}(r,\varphi) = S^{\perp}_{\varphi\varphi}(r,\varphi) = 0 \tag{9.152}$$

for every $0 < r < \infty,\ 0 < \varphi \leq 2\pi$.

Similarly as in the case (i), a restriction of Eqs. (9.151)–(9.152) to the wedge region provides a solution to Prob. 9.3 in case (ii) if a resultant normal force at the tip of wedge is equal to l_2, that is, if for every $r \geq 0$

$$\int_{-\alpha}^{\alpha} S_{rr}^{\perp}\, r \cos\theta d\theta = l_2 \tag{9.153}$$

Substituting $S_{rr}^{\perp}$ from (9.151) into (9.153) and integrating, we obtain

$$-\frac{2P_0}{\pi}\int_{0}^{\alpha}(1+\cos 2\theta)d\theta = l_2 \tag{9.154}$$

or

$$P_0 = -\frac{l_2\pi}{2\alpha + \sin 2\alpha} \tag{9.155}$$

Finally, substituting P_0 from (9.155) into (9.151) we obtain

$$S_{rr}^{**}(r,\varphi) = \frac{2l_2 \sin\varphi}{r(2\alpha + \sin 2\alpha)} \tag{9.156}$$

$$S_{r\varphi}^{**}(r,\varphi) = S_{\varphi\varphi}^{**}(r,\varphi) = 0 \tag{9.157}$$

for every $0 < r < \infty,\ \left|\frac{\pi}{2} - \varphi\right| \leq \alpha$.

The stress components $S_{rr}^{**},\ S_{r\varphi}^{**}$, and $S_{\varphi\varphi}^{**}$, represent a solution to Prob. 9.3 in case (ii).

A solution to Problem 9.3 takes the form

$$\overline{S}_{rr} = S_{rr}^{*} + S_{rr}^{**},\quad \overline{S}_{r\varphi} = \overline{S}_{\varphi\varphi} = 0 \tag{9.158}$$

This completes a solution to Problem 9.3

Problem 9.4. Show that for a homogeneous isotropic infinite elastic wedge under generalized plane stress conditions loaded by a concentrated moment M at its tip (see Fig. 9.2) the stress components $\overline{S}_{rr},\ \overline{S}_{r\varphi},$ and $\overline{S}_{\varphi\varphi}$ vanishing at infinity take the form

$$\overline{S}_{rr}(\mathrm{r},\varphi) = \frac{2M}{r^2}\frac{\sin(2\varphi - \alpha)}{\sin\alpha - \alpha\cos\alpha}$$

$$\overline{S}_{\mathrm{r}\varphi}(\mathrm{r},\varphi) = -\frac{M}{r^2}\frac{\cos(2\varphi-\alpha) - \cos\alpha}{\sin\alpha - \alpha\cos\alpha}$$

$$\overline{S}_{\varphi\varphi}(\mathrm{r},\varphi) = 0 \quad \text{for every} \quad r > 0,\quad 0 < \varphi < \alpha$$

where

$$M = -r \int_0^{\alpha} (\overline{S}_{r\varphi} r) \, \mathrm{d}\varphi$$

Note that the stress components $\overline{S}_{rr}$ and $\overline{S}_{r\varphi}$ become unbounded for $\alpha = \alpha^*$, where α^* is the only root of the equation

$$\sin \alpha^* - \alpha^* \cos \alpha^* = 0$$

that is, for $\alpha^* = 257.4°$. Hence, the solution makes sense for an elastic wedge that obeys the condition $0 < \alpha < \alpha^*$.

Solution. The stress components $\overline{S}_{rr}$, $\overline{S}_{r\varphi}$, and $\overline{S}_{\varphi\varphi}$ produced in the wedge by a moment M at its tip (see Figure) are to be found using an Airy stress function $\overline{F} = \overline{F}(r, \varphi)$

$$\overline{S}_{rr} = \left(\nabla^2 - \frac{\partial^2}{\partial r^2} \right) \overline{F} \tag{9.159}$$

$$\overline{S}_{\varphi\varphi} = \frac{\partial^2}{\partial r^2} \overline{F} \tag{9.160}$$

$$\overline{S}_{r\varphi} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \overline{F}}{\partial \varphi} \right) \tag{9.161}$$

where

$$\nabla^2 \nabla^2 \overline{F} = 0 \tag{9.162}$$

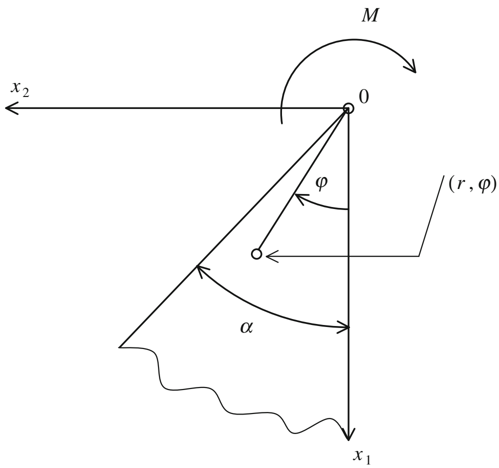

Fig. 9.2 The infinite wedge loaded by a concentrated moment

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \tag{9.163}$$

By inspection of (9.160) and (9.161), and of the definition of moment M, we conclude that a biharmonic function $\overline{F} = \overline{F}(\varphi)$ is to solve the problem. In the following we are to show that a solution to Problem 9.4 is obtained if $\bar{F}$ takes the form

$$\overline{F}(r, \varphi) = c_1(2\varphi - \alpha) + c_2 \sin(2\varphi - \alpha) \tag{9.164}$$

where c_1 and c_2 are constants to be determined from the stress free boundary conditions at $\varphi = 0$ and $\varphi = \alpha$ for $r > 0$, and from the definition of M:

$$M = -r \int_0^\alpha \overline{S}_{r\varphi}\, r d\varphi \tag{9.165}$$

Note that $\overline{F}$ given by (9.164) satisfies Eq. (9.162) since

$$\nabla^2\, \nabla^2 \overline{F} = c_2 \nabla^2 [r^{-2} \sin(2\varphi - \alpha)] = 0 \quad \text{for } 0 < r < \infty,\ 0 \le \varphi \le \alpha \tag{9.166}$$

Substituting $\overline{F}$ from (9.164) into (9.159), (9.160), and (9.161), respectively, we obtain

$$\overline{S}_{rr} = -4c_2 \frac{\sin(2\varphi - \alpha)}{r^2} \tag{9.167}$$

$$\overline{S}_{\varphi\varphi} = 0 \tag{9.168}$$

$$\overline{S}_{r\varphi} = \frac{2}{r^2}\, [c_1 + c_2 \cos(2\varphi - \alpha)] \tag{9.169}$$

Also note that the boundary conditions

$$\overline{S}_{r\varphi} = 0 \quad \text{at } \varphi = 0 \text{ and } \varphi = \alpha \tag{9.170}$$

are satisfied if

$$c_1 = -c_2 \cos\alpha \tag{9.171}$$

Therefore, substituting (9.171) into (9.164) and (9.169), respectively, we obtain

$$\overline{F} = c_2\, [\sin(2\varphi - \alpha) - (2\varphi - \alpha)\cos\alpha] \tag{9.172}$$

and

$$\overline{S}_{r\varphi} = \frac{2c_2}{r^2}\, [\cos(2\varphi - \alpha) - \cos\alpha] \tag{9.173}$$

Hence, substituting (9.173) into (9.165) we obtain

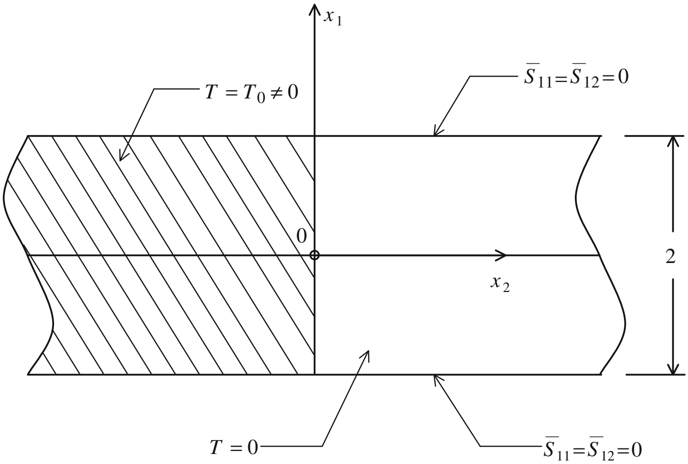

Fig. 9.3 The infinite strip

$$M = -2c_2 \int_0^{\alpha} [\cos(2\varphi - \alpha) - \cos\alpha] d\varphi \tag{9.174}$$

or

$$c_2 = -\frac{M}{2(\sin\alpha - \alpha\cos\alpha)} \tag{9.175}$$

Finally, by substituting c_2 from (9.175) into (9.167) and (9.173), respectively, we obtain the required result. This completes a solution to Problem 9.4.

Problem 9.5. Consider a homogeneous isotropic infinite elastic strip under generalized plane stress conditions: $|x_1| \leq 1,\ |x_2| < \infty$ subject to the temperature field of the form

$$\overline{T}(x_1, x_2) = T_0[1 - H(x_2)] \tag{9.176}$$

where T_0 is a constant temperature and $H = H(x)$ is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 1/2 & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{9.177}$$

Note that in this case, we complemented the definition of the Heaviside function by specifying its value at $x = 0$ (Fig. 9.3).

Show that the stress tensor field $\overline{S} = \overline{S}(x_1, x_2)$ corresponding to the discontinuous temperature (9.176) is represented by the sum

$$\overline{S} = \overline{S}^{(1)} + \overline{S}^{(2)} \tag{9.178}$$

where

$$\overline{S}_{11}^{(1)} = -E\alpha T_0[1 - H(x_2)], \quad \overline{S}_{22}^{(1)} = \overline{S}_{12}^{(1)} = 0 \tag{9.179}$$

and

$$\overline{S}_{11}^{(2)} = F_{,22}, \quad \overline{S}_{22}^{(2)} = F_{,11}, \quad \overline{S}_{12}^{(2)} = -F_{,12} \tag{9.180}$$

where the biharmonic function $F = F(x_1, x_2)$ is given by

$$F(x_1, x_2) = E\alpha T_0 \left[\frac{x_2^2}{4} + \frac{2}{\pi} \int_0^\infty (A \cosh \beta x_1 + B \beta x_1 \sinh \beta x_1) \frac{\sin \beta x_2}{\beta^3} d\beta \right] \tag{9.181}$$

$$A = \frac{\sinh \beta + \beta \cosh \beta}{\sinh 2\beta + 2\beta}, \quad B = -\frac{\sinh \beta}{\sinh 2\beta + 2\beta} \tag{9.182}$$

Hint. Note that

$$\begin{aligned} \overline{S}_{11}^{(2)}(\pm 1, x_2) &= -\overline{S}_{11}^{(1)}(\pm 1, x_2) \\ &= \frac{E\alpha T_0}{2} \left(1 - \frac{2}{\pi} \int_0^\infty \frac{\sin \beta x_2}{\beta} d\beta \right) \quad \text{for} \quad |x_2| < \infty \end{aligned}$$

Solution. To solve the problem we recall the integral representation of the Heaviside step function

$$H(x_2) = \left\{ \begin{array}{ll} 0 \text{ for } & x_2 < 0 \\ \frac{1}{2} \text{ for } & x_2 = 0 \\ 1 \text{ for } & x_2 > 0 \end{array} \right\} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \beta x_2}{\beta} d\beta \tag{9.183}$$

The temperature $\overline{T}(x_1, x_2)$ on the strip $|x_1| \leq 1$, $|x_2| < \infty$ is then represented by

$$\overline{T}(x_1, x_2) = T_0 \, [1 - H(x_2)] = \frac{T_0}{2} \left[1 - \frac{2}{\pi} \int_0^\infty \frac{\sin \beta x_2}{\beta} d\beta \right] \tag{9.184}$$

We are to find a stress tensor $\overline{\mathbf{S}}$ corresponding to a solution $s = [\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{S}}]$ of a problem of thermo-elastostatics for the strip subject to the discontinuous temperature (9.184) when the strip boundaries $x_1 = 1$ and $x_1 = -1$ are stress free, that is, when

$$\overline{S}_{11}(\pm 1, x_2) = \overline{S}_{12}(\pm 1, x_2) = 0 \quad \text{for } |x_2| < \infty \tag{9.185}$$

To this end we look for s in the form

$$s = s_1 + s_2 \tag{9.186}$$

where $s_1 = [\mathbf{u}^{(1)}, \mathbf{E}^{(1)}, \mathbf{S}^{(1)}]$ is generated from a displacement potential $\overline{\phi} = \overline{\phi}(x_2)$ by the formulas

$$\mathbf{u}^{(1)} = \nabla\overline{\phi} \tag{9.187}$$

$$\mathbf{E}^{(1)} = \nabla\nabla\overline{\phi} \tag{9.188}$$

$$\mathbf{S}^{(1)} = 2\mu(\nabla\nabla\overline{\phi} - \nabla^2\overline{\phi}\mathbf{1}) \tag{9.189}$$

in which $\overline{\phi}$ satisfies Poisson's equation

$$\nabla^2\overline{\phi} = m_0\overline{T}, \quad m_0 = (1+\nu)\alpha \tag{9.190}$$

where $\overline{T}$ is given by Eq. (9.184) and $s_2 = [\mathbf{u}^{(2)}, \mathbf{E}^{(2)}, \mathbf{S}^{(2)}]$ is a solution of isothermal elastostatics for the strip that complies with the boundary conditions

$$\begin{aligned} S_{11}^{(2)}(\pm 1, x_2) &= -S_{11}^{(1)}(\pm 1, x_2) \\ S_{12}^{(2)}(\pm 1, x_2) &= -S_{12}^{(1)}(\pm 1, x_2) \end{aligned} \tag{9.191}$$

The stress components $S_{11}^{(2)}$, $S_{22}^{(2)}$, and $S_{12}^{(2)}$ are to be computed from an Airy stress function $\overline{F} = \overline{F}(x_1, x_2)$ by the formulas

$$S_{11}^{(2)} = \overline{F}_{,22}\,,\ S_{22}^{(2)} = \overline{F}_{,11}\,,\ S_{12}^{(2)} = -\overline{F}_{,12} \tag{9.192}$$

$$\nabla^2\nabla^2\overline{F} = 0 \tag{9.193}$$

Since $\overline{\phi} = \overline{\phi}(x_2)$, it follows from Eqs. (9.189) and (9.190) that the stress components $S_{11}^{(1)}$, $S_{22}^{(1)}$, and $S_{12}^{(1)}$ are given by

$$S_{11}^{(1)} = -2\mu m_0\overline{T} = -E\alpha T_0[1 - H(x_2)] \tag{9.194}$$

$$S_{22}^{(1)} = S_{12}^{(1)} = 0, \quad |x_1| \le 1,\ |x_2| < \infty \tag{9.195}$$

and the problem is reduced to that of finding a biharmonic function $\overline{F} = \overline{F}(x_1, x_2)$ on the strip region: $|x_1| \le 1$, $|x_2| < \infty$ that complies with the boundary conditions

$$\overline{F}_{,22}\,(\pm 1, x_2) = 2\mu m_0\overline{T} \tag{9.196}$$

and

$$\overline{F}_{,12}\,(\pm 1, x_2) = 0 \tag{9.197}$$

An alternative form of (9.196) reads

$$\overline{F}_{,22}(\pm 1, x_2) = \frac{E\alpha T_0}{2}\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin \beta x_2}{\beta} d\beta\right] \tag{9.198}$$

Since

$$\nabla^2(\cos h\beta x_1 \ \sin \beta x_2) = 0 \tag{9.199}$$

$$\nabla^2\nabla^2(\beta x_1 \ \sinh \beta x_1 \ \sin \beta x_2) = 0 \tag{9.200}$$

and

$$\overline{F}(x_1, x_2) = \overline{F}(-x_1, x_2) \tag{9.201}$$

the function $\overline{F}$ is postulated in the form [see Eq. 9.181 of Problem 9.5]

$$\overline{F}(x_1, x_2) = E\alpha T_0\left[\frac{x_2^2}{4} + \frac{2}{\pi}\int_0^\infty (A\cosh \beta x_1 + B\beta x_1 \sinh \beta x_1)\,\frac{\sin \beta x_2}{\beta^3} d\beta\right] \tag{9.202}$$

where A and B are arbitrary functions on $[0, \infty)$ that make the integral (9.202) to converge for $|x_1| \leq 1$, $|x_2| < \infty$.

Substituting (9.202) into the boundary conditions (9.197) and (9.198), respectively, we obtain

$$A \sinh \beta + B(\sinh \beta + \beta \cosh \beta) = 0$$

and

$$A \cosh \beta + B\beta \sinh \beta = 1/2 \tag{9.203}$$

A unique solution to Eqs. (9.203) takes the form

$$A = \frac{\sinh \beta + \beta \cosh \beta}{\sinh 2\beta + 2\beta} \tag{9.204}$$

$$B = -\frac{\sinh \beta}{\sinh 2\beta + 2\beta} \tag{9.205}$$

Hence substituting $\overline{F} = \overline{F}(x_1, x_2)$ given by (9.202) in which A and B are given by (9.204) and (9.205), respectively, into (9.192) we obtain the integral representation of the stress tensor $\mathbf{S}^{(2)}$. A solution to Problem 9.5 is obtained in the form

$$\overline{\mathbf{S}} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)} \tag{9.206}$$

where $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$ are given by Eqs. (9.194)–(9.195) and (9.192), respectively.