Chapter 8
Solutions to Particular Three-Dimensional Boundary Value Problems of Elastostatics

In this chapter the boundary value problems related to torsion of a prismatic bar bounded by a cylindrical lateral surface and by a pair of planes normal to the lateral surface, are discussed. It is assumed that a resultant torsion moment is applied at one of the bases while the other is subject to a warping and the lateral surface is stress free. In each of the problems an approximate three-dimensional formulation is reduced to a two-dimensional one for Laplace's or Poisson's equation on the cross section of the bar.

8.1 Torsion of Circular Bars

We consider a circular prismatic bar of length l and radius a referred to the Cartesian coordinates (x_1, x_2, x_3) in such a way that x_3 coincides with the axis of the bar, the bar is fixed at $x_3 = 0$ in the (x_1, x_2) plane, while at $x_3 = l$ a torsion moment M_3 is applied. This moment causes the bar to be twisted, and the generators of the circular cylinder deform into helical curves.

An elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ in the bar is approximated by $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$, where

$$\widetilde{u}_1 = -\alpha\, x_2\, x_3, \quad \widetilde{u}_2 = \alpha\, x_1\, x_3, \quad \widetilde{u}_3 = 0 \tag{8.1}$$

and α is the angle of twist per unit length along the x_3 axis

$$\begin{aligned} \widetilde{E}_{11} &= \widetilde{E}_{22} = \widetilde{E}_{33} = \widetilde{E}_{12} = 0 \\ \widetilde{E}_{23} &= \frac{1}{2}\alpha\, x_1, \quad \widetilde{E}_{31} = -\frac{1}{2}\alpha\, x_2 \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} \widetilde{S}_{11} &= \widetilde{S}_{22} = \widetilde{S}_{33} = \widetilde{S}_{12} = 0 \\ \widetilde{S}_{23} &= \mu\alpha\, x_1, \quad \widetilde{S}_{31} = -\mu\alpha\, x_2 \end{aligned} \tag{8.3}$$

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The torsion moment M_3 is

$$M_3 = \int_A (x_1 \widetilde{S}_{23} - x_2 \widetilde{S}_{31}) dx_1 dx_2 = \mu\alpha \int_A (x_1^2 + x_2^2) dx_1 dx_2 \equiv \mu\,\alpha\,J \tag{8.4}$$

where A is the area of the cross section : $A = \{(x_1, x_2) : \sqrt{x_1^2 + x_2^2} \le a\}$, and J is the polar moment of inertia of the cross section about its center. The product μJ is called the *torsional rigidity of the bar*. Also, since

$$n_\alpha = x_\alpha / a, \quad n_3 = 0 \quad \text{on} \quad \partial A \quad (\alpha = 1, 2) \tag{8.5}$$

therefore, because of Eq. (8.3)

$$\widetilde{S}_{ij} n_j = 0 \quad \text{on} \quad \partial A \quad (i, j = 1, 2, 3) \tag{8.6}$$

that is, the elastic state $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$ satisfies the homogeneous boundary conditions: (i) $\widetilde{\mathbf{u}} = \mathbf{0}$ on $x_3 = 0$, and (ii) $\widetilde{\mathbf{S}}\mathbf{n} = \mathbf{0}$ on the lateral surface $\partial A \times [0, l]$. A shear stress boundary condition on the plane $x_3 = l$ is replaced by application of the resultant moment M_3 on this plane.

Torsion of Noncircular Prismatic Bars

A noncircular prismatic bar of length l is fixed at $x_3 = 0$ in the sense that the displacement components in the (x_1, x_2) plane vanish while the axial displacement is subject to a warping, and the other end $x_3 = l$ is twisted by a moment M_3; the lateral surface of the bar is stress free and no body forces are present. Therefore, an elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ in the bar is approximated by $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$ in which

$$\widetilde{u}_1 = -\alpha\, x_2\, x_3, \quad \widetilde{u}_2 = \alpha\, x_1\, x_3, \quad \widetilde{u}_3 = \alpha\, \psi(x_1,\, x_2) \tag{8.7}$$

where $\psi = \psi(x_1, x_2)$ is called a *warping function.* The strain-displacement relations, the equilibrium equations with zero body forces, and the constitutive stress-strain relations, respectively, associated with the displacements (8.7), take the forms

$$\begin{aligned} &\widetilde{E}_{11} = \widetilde{E}_{22} = \widetilde{E}_{33} = \widetilde{E}_{12} = 0 \\ &\widetilde{E}_{23} = \frac{1}{2}\alpha(\psi_{,2} + x_1), \quad \widetilde{E}_{31} = \frac{1}{2}\alpha\,(\psi_{,1} - x_2) \end{aligned} \tag{8.8}$$

$$\psi_{,11} + \psi_{,22} = 0 \tag{8.9}$$

and

$$\begin{aligned} &\widetilde{S}_{11} = \widetilde{S}_{22} = \widetilde{S}_{33} = \widetilde{S}_{12} = 0 \\ &\widetilde{S}_{23} = \mu\alpha(\psi_{,2} + x_1), \quad \widetilde{S}_{31} = \mu\alpha(\psi_{,1} - x_2) \end{aligned} \tag{8.10}$$

The torsion moment M_3 takes the form

$$M_3 = \int_A (x_1 \widetilde{S}_{23} - x_2 \widetilde{S}_{31}) dx_1 dx_2 = \alpha D \tag{8.11}$$

where

$$D = \mu \int_A (x_1^2 + x_2^2 + x_1 \psi_{,2} - x_2 \psi_{,1}) dx_1 dx_2 \tag{8.12}$$

is called the *torsional rigidity of the bar*.

The boundary conditions are satisfied in the following sense. The bases $x_3 = 0$ and $x_3 = l$ of the bar are the resultant force free, that is,

$$F_1 = \int_A \widetilde{S}_{31} dx_1 dx_2 = 0, \quad F_2 = \int_A \widetilde{S}_{32} dx_1 dx_2 = 0 \tag{8.13}$$

and the distribution of shear stresses on the base $x_3 = l$ is represented by the torsion moment M_3. To satisfy the stress free lateral surface boundary condition, we postulate that

$$\frac{\partial \psi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on} \quad \partial A \tag{8.14}$$

As a result, the torsion problem of a noncircular prismatic bar has been solved once a warping function $\psi = \psi(x_1, x_2)$ that satisfies the harmonic equation

$$\nabla^2 \psi = 0 \quad \text{on} \quad A \tag{8.15}$$

subject to the boundary condition

$$\frac{\partial \psi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on} \quad \partial A \tag{8.16}$$

has been found.

For example, for an elliptic bar with semi-axes a and b and with the center at the origin, we obtain

$$\psi(x_1, x_2) = \frac{b^2 - a^2}{b^2 + a^2} x_1 x_2 \tag{8.17}$$

and

$$D = \frac{\pi \mu a^3 b^3}{a^2 + b^2}, \quad M_3 = \alpha D \tag{8.18}$$

$$\widetilde{S}_{13} = -\frac{2M_3}{\pi a b^3} x_2, \quad \widetilde{S}_{13} = \frac{2M_3}{\pi a^3 b} x_1 \tag{8.19}$$

Prandtl's Stress Function

Prandtl's stress function $\phi = \phi(x_1, x_2)$ is defined in terms of the warping function $\psi = \psi(x_1, x_2)$ by the formulas

$$\begin{aligned}\phi_{,2} &= \mu\,\alpha(\psi_{,1} - x_2) = \widetilde{S}_{13} \\ \phi_{,1} &= -\mu\,\alpha(\psi_{,2} + x_1) = -\widetilde{S}_{23}\end{aligned} \tag{8.20}$$

One can show that the boundary value problem for the warping function $\psi = \psi(x_1, x_2)$, described by Eqs. (8.15) and (8.16), is equivalent to finding a Prandtl's stress function $\phi = \phi(x_1, x_2)$ that satisfies Poisson's equation

$$\nabla^2\phi = -2\mu\,\alpha \quad \text{on} \quad A \tag{8.21}$$

subject to the homogeneous boundary condition

$$\phi = 0 \quad \text{on} \quad \partial A \tag{8.22}$$

while the torsion moment M_3 is calculated from the formula

$$M_3 = 2\int_A \phi(x_1, x_2)dx_1dx_2 \tag{8.23}$$

8.2 Problems and Solutions Related to Particular Three-Dimensional Boundary Value Problems of Elastostatics—Torsion Problems

Problem 8.1. Show that the warping function ψ = const solves the torsion problem of a circular bar.

Solution. By letting $\psi = 0$ in Eqs. (8.7)–(8.10) we obtain $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$, where $\widetilde{\mathbf{u}}$, $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{S}}$ are given by Eqs. (8.1)–(8.3), respectively, that describes a solution to the torsion problem of a circular bar.

Problem 8.2. Show that in the torsion problem of an elliptic bar, the resultant shear stress $\widetilde{S}_t$ at points on a given diameter of the ellipse is parallel to the tangent at the point of intersection of the diameter and the ellipse [see Fig. 8.1].

Solution. For an elliptic bar subject to a torsion moment M_3, the stresses $\widetilde{S}_{13}$ and $\widetilde{S}_{23}$, respectively, are given by [see Eqs. (8.19)]

$$\widetilde{S}_{13} = -\frac{2\,M_3}{\pi\,ab^3}x_2 \tag{8.24}$$

and

$$\widetilde{S}_{23} = \frac{2\,M_3}{\pi\, a^3 b} x_1 \tag{8.25}$$

The resultant shear stress magnitude is then computed from the formula

$$\widetilde{S}_t = \left(\widetilde{S}_{13}^2 + \widetilde{S}_{23}^2\right)^{1/2} = \frac{2\,M_3}{\pi\, ab}\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}\right)^{1/2} \tag{8.26}$$

Equations (8.24)–(8.26) hold true for any point of the elliptical cross section of the bar. In particular, for any such point, because of (8.24) and (8.25)

$$\frac{\widetilde{S}_{13}}{\widetilde{S}_{23}} = -\frac{a^2}{b^2}\frac{x_2}{x_1} \tag{8.27}$$

Therefore, the ratio $\widetilde{S}_{13}/\widetilde{S}_{23}$ is constant along the diameter of the ellipse shown in Fig. of Problem 8.2 represented by the equation

$$-\frac{a^2}{b^2}\frac{x_2}{x_1} = c = \text{const } (c > 0) \tag{8.28}$$

As a result, the resultant shear stress vector $\widetilde{\boldsymbol{\tau}} = \widetilde{S}_{13}\,\mathbf{e}_1 + \widetilde{S}_{23}\,\mathbf{e}_2$, where $\mathbf{e}_1 = (1,0)^T$, $\mathbf{e}_2 = (0,1)^T$, coincides with the tangent vector at the point of intersection of the diameter and the ellipse. Substituting x_2 from (8.28) into (8.26) we obtain

$$\widetilde{S}_t = \frac{2\,M_3}{\pi\, ab}\sqrt{1+c^2}\;\frac{|x_1|}{a^2} \tag{8.29}$$

This formula shows that for $x_1 > 0$ $\widetilde{S}_t$ is a linear function of x_1 along the diameter.

This completes a solution to Problem 8.2.

Problem 8.3. Show that the torsion moment in terms of Prandtl's stress function $\phi = \phi(x_1, x_2)$ is expressed by

$$M_3 = 2\int_A \phi(x_1, x_2)dx_1dx_2$$

Solution. A solution to this problem is obtained from Eqs. (8.11)–(8.12), (8.15)–(8.16), and (8.20)–(8.22).

Problem 8.4. Show that Prandtl's stress function $\phi = \phi(x_1, x_2)$ given by

$$\phi(x_1,x_2) = \frac{32\mu\,\alpha\, a_1^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^3}\left[1 - \frac{\cosh\left(\frac{n\pi\, x_2}{2a_1}\right)}{\cosh\left(\frac{n\pi\, a_2}{2a_1}\right)}\right]\cos\left(\frac{n\pi\, x_1}{2a_1}\right)$$

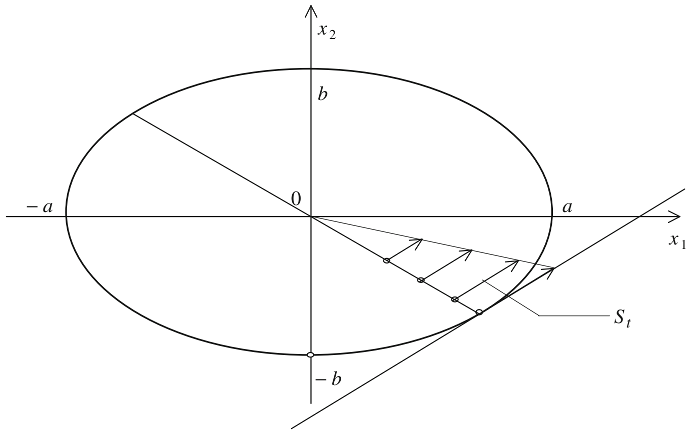

Fig. 8.1 The cross section of an elliptic bar in torsion

solves the torsion problem of a bar with the rectangular cross section: $|x_1| \leq a_1$, $|x_2| \leq a_2$. Also, show that in this case the torsion moment

$$M_3 = 2 \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \phi(x_1, x_2) dx_1 dx_2 = \mu \, \alpha (2a_1)^3 (2a_2) \, k^*$$

where

$$k^* = \frac{1}{3} \left[1 - \frac{192}{\pi^5} \left(\frac{a_1}{a_2} \right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh \left(\frac{n\pi \, a_2}{2a_1} \right) \right]$$

Solution. For the rectangular cross section $C_0 : \ |x_1| \leq a_1, \ |x_2| \leq a_2$, Prandtl's stress function $\phi = \phi(x_1, x_2)$ satisfies Poisson's equation

$$\nabla^2 \phi = -2\mu\alpha \quad \text{on } C_0 \tag{8.30}$$

subject to the homogeneous boundary condition

$$\phi = 0 \quad \text{on } \partial C_0 \tag{8.31}$$

Since

$$\cos \left(\frac{n\pi}{2} \right) = 0 \quad \text{for } n = 1, 3, 5, \ldots \tag{8.32}$$

therefore, $\phi = \phi(x_1, x_2)$ given by

$$\phi(x_1, x_2) = \frac{32\,\mu\alpha\,a_1^2}{\pi^3} \times \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^3} \left[1 - \frac{ch\left(\frac{n\pi x_2}{2a_1}\right)}{ch\left(\frac{n\pi a_2}{a_1}\right)}\right] \cos\left(\frac{n\pi x_1}{2a_1}\right) \tag{8.33}$$

satisfies the homogeneous boundary condition (8.31).

In addition, applying ∇^2 to (8.33) and using the identity

$$\nabla^2 \left[\cos\left(\frac{n\pi x_1}{2a_1}\right) ch\left(\frac{n\pi x_2}{2a_1}\right)\right] = 0 \tag{8.34}$$

we obtain

$$\nabla^2\phi = -\frac{8\,\mu\alpha}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x_1}{2a_1}\right)}{n} \tag{8.35}$$

Hence, ϕ given by (8.33) satisfies (8.30) if the function 1 on $|x_1| \leq a_1$ can be represented by the Fourier's series

$$1 = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x_1}{2a_1}\right)}{n} \qquad |x_1| \leq a_1 \tag{8.36}$$

To show (8.36) we multiply (8.36) by $\cos\left(\frac{k\pi x_1}{2a_1}\right)$ and integrate over $|x_1| \leq a_1$, and obtain

$$\int_{-a_1}^{a_1} \cos\left(\frac{k\pi x_1}{2a_1}\right) dx_1 = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi}{2}\right)}{n} \times \int_{-a_1}^{a_1} \cos\left(\frac{k\pi x_1}{2a_1}\right) \cos\left(\frac{n\pi x_1}{2a_1}\right) dx_1 \tag{8.37}$$

Since

$$\int_{-a_1}^{a_1} \cos\left(\frac{k\pi x_1}{2a_1}\right) \cos\left(\frac{n\pi x_1}{2a_1}\right) dx_1 = a_1 \delta_{kn} \quad \text{for } n, k = 1, 3, 5, \ldots \tag{8.38}$$

and

$$\int_{-a_1}^{a_1} \cos\left(\frac{k\pi x_1}{2a_1}\right) dx_1 = \frac{4a_1}{k\pi} \sin\left(\frac{k\pi}{2}\right) \tag{8.39}$$

therefore, Eq. (8.37) is an identity. This proves that the expansion (8.36) holds true, and as a result ϕ given by (8.33) solves the torsion problem of a bar with the rectangular cross section.

To calculate the torsion moment we use the formula

$$M_3 = 2 \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \phi(x_1, x_2) dx_1 dx_2 \tag{8.40}$$

Substituting ϕ from (8.33) into (8.40) we obtain

$$\begin{aligned} M_3 &= \frac{64\,\mu\alpha a_1^2}{\pi^3} \times \int_{-a_1}^{a} \int_{-a_2}^{a_2} \sum_{n=1,3,5,\ldots} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^3} \left[1 - \frac{\operatorname{ch}\left(\frac{n\pi x_2}{2a_1}\right)}{\operatorname{ch}\left(\frac{n\pi a_2}{2a_1}\right)} \right] \cos\left(\frac{n\pi x_1}{2a_1}\right) dx_1 dx_2 \\ &= \frac{32\,\mu\alpha(2a_1)^3(2a_2)}{\pi^4} \sum_{n=1,3,5,\ldots} \frac{1}{n^4} - \frac{64\,\mu\alpha(2a_1)^4}{\pi^5} \sum_{n=1,3,5,\ldots} \frac{1}{n^5} \tanh\left(\frac{n\pi a_2}{2a_1}\right) \end{aligned} \tag{8.41}$$

Since

$$\sum_{n=1,3,5,\ldots} \frac{1}{n^4} = \frac{\pi^4}{96} \tag{8.42}$$

therefore, substituting (8.42) into (8.41) we obtain

$$M_3 = \frac{1}{3}\,\mu\alpha(2a_1)^3 2a_2 \times \left[1 - \frac{192}{\pi^5} \frac{a_1}{a_2} \sum_{n=1,3,5,\ldots} \frac{1}{n^5} \tanh\left(\frac{n\pi a_2}{2a_1}\right) \right] \tag{8.43}$$

This completes a solution to Problem 8.4.

Problem 8.5. Show that Prandtl's stress function

$$\phi(r,\theta) = \frac{\mu\alpha}{2}(r^2 - b^2)\left(\frac{2a\cos\theta}{r} - 1\right)$$

defined over the region

$$0 < b \le r \le 2a - b, \quad -\cos^{-1}\left(\frac{b}{2a}\right) \le \theta \le \cos^{-1}\left(\frac{b}{2a}\right)$$

solves the torsion problem of the circular shaft with a circular groove shown in Fig. 8.2; in particular, find the stresses $\widetilde{S}_{13}$ and $\widetilde{S}_{23}$ on the boundary of the shaft.

Hint. Use the polar coordinates

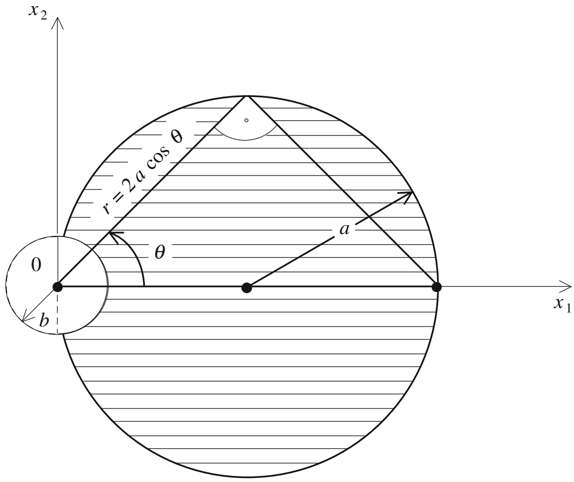

Fig. 8.2 The cross section of a circular bar with a circular groove

$$x_1 = r\ \cos\theta, \quad x_2 = r\ \sin\theta$$

Solution. First, we note that the function

$$\phi(r,\theta) = \frac{\mu\alpha}{2}(r^2 - b^2)\left(\frac{2a\cos\theta}{r} - 1\right) \tag{8.44}$$

vanishes on the boundary of the circular shaft with a circular groove shown in Fig. of Problem 8.5.

Next, using ∇^2 in the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{8.45}$$

and the equations

$$\nabla^2(r\cos\ \theta) = \nabla^2(r^{-1}\cos\theta) = 0 \tag{8.46}$$

and applying ∇^2 to (8.44) we obtain

$$\nabla^2\phi = -2\,\mu\alpha \tag{8.47}$$

Therefore, ϕ solves the torsion problem of the circular shaft with a circular groove. In particular, the stresses $\widetilde{S}_{13}$ and $\widetilde{S}_{23}$ are computed from the formulas [see Eqs. (8.20)]

$$\widetilde{S}_{13} = \phi_{,2}, \quad \widetilde{S}_{23} = -\phi_{,1} \tag{8.48}$$

Substituting ϕ from (8.44) into (8.48), and using the polar coordinates, we obtain

$$\widetilde{S}_{13} = \mu\alpha x_2 \left(2a\, x_1 \frac{b^2}{r^4} - 1 \right) \tag{8.49}$$

and

$$\widetilde{S}_{23} = -\mu\alpha \left[a \left(1 - \frac{b^2}{r^2} \right) - x_1 + 2a\, x_1^2 \frac{b^2}{r^4} \right] \tag{8.50}$$

In Eqs. (8.49) and (8.50)

$$x_1 = r\cos\theta \quad x_2 = r\sin\theta \tag{8.51}$$

By letting $r = b$ in (8.49) and (8.50) we get

$$\widetilde{S}_{13}|_{r=b} = \mu\alpha(2a\cos\theta - b)\sin\theta \tag{8.52}$$

$$\widetilde{S}_{23}|_{r=b} = -\mu\alpha(2a\cos\theta - b)\cos\theta \tag{8.53}$$

Hence, the resultant shear stress magnitude for $r = b$ takes the form

$$\widetilde{S}_t = \left(\widetilde{S}_{13}^2 + \widetilde{S}_{23}^2 \right)^{1/2} = \mu\alpha(2a\cos\theta - b) \tag{8.54}$$

Since

$$\frac{\partial \widetilde{S}_t}{\partial\theta} = 0, \quad \frac{\partial^2 \widetilde{S}_t}{\partial\theta^2} < 0 \quad \text{at } \theta = 0 \tag{8.55}$$

the function $\widetilde{S}_t = \widetilde{S}_t(\theta)$ attains a maximum at $\theta = 0$. Hence, the resultant shear stress attains a maximum at the point $(x_1, x_2) = (b, 0)$ and

$$\widetilde{S}_t(\theta = 0) = \mu\alpha(2a - b) \tag{8.56}$$

If $b \to 0$, the RHS of (8.56) $\to 2\mu\alpha a$. Hence, for a small groove radius the maximum resultant shear stress doubles that of a bar with a circular cross section [see Eq. (8.3)].

This completes a solution to Problem 8.5.