Chapter 6
Complete Solutions of Elasticity

In this chapter general solutions of the homogeneous isotropic elastostatics and elastodynamics are discussed. The general solutions are related to both the displacement and stress governing equations, and emphasis is made on completeness of the solutions [See also Chap. 16].

6.1 Complete Solutions of Elastostatics

A vector field $\mathbf{u} = \mathbf{u}(\mathbf{x})$ on B that satisfies the displacement equation of equilibrium

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla(\text{div } \mathbf{u}) + \frac{\mathbf{b}}{\mu} = \mathbf{0} \tag{6.1}$$

is called *an elastic displacement field corresponding to* **b**.

Boussinesq-Papkovitch-Neuber (B-P-N) Solution. Let

$$\mathbf{u} = \psi - \frac{1}{4(1-\nu)}\nabla(\mathbf{x}\cdot\psi + \varphi) \tag{6.2}$$

where φ and ψ are fields on B that satisfy Poisson's equations

$$\nabla^2\psi = -\frac{1}{\mu}\mathbf{b} \tag{6.3}$$

and

$$\nabla^2\varphi = \frac{1}{\mu}\mathbf{b}\cdot\mathbf{x} \tag{6.4}$$

Then **u** is an elastic displacement field corresponding to **b**.

M. Reza Eslami et al., *Theory of Elasticity and Thermal Stresses*, Solid Mechanics and Its Applications 197, DOI: 10.1007/978-94-007-6356-2_6,

Boussinesq-Somigliana-Galerkin (B-S-G) Solution. Let $\mathbf{u}$ be a vector field given by

$$\mathbf{u} = \nabla^2 \mathbf{g} - \frac{1}{2(1-\nu)} \nabla (\text{div } \mathbf{g}) \tag{6.5}$$

where

$$\nabla^2 \nabla^2 \mathbf{g} = -\frac{1}{\mu} \mathbf{b} \tag{6.6}$$

Then $\mathbf{u}$ is an elastic displacement field corresponding to $\mathbf{b}$.

We say that a *representation for the displacement* $\mathbf{u}$ *expressed in terms of auxiliary functions is complete if these auxiliary functions exist for any* $\mathbf{u}$ *that satisfies the displacement equation of equilibrium* (6.1).

For B-P-N solution such auxiliary functions are the fields φ and ψ; while for B-S-G solution an auxiliary function is the field $\mathbf{g}$.

Completeness of B-P-N and B-S-G Solutions. Let $\mathbf{u}$ be a solution to the displacement equation of equilibrium with the body force $\mathbf{b}$. Then there exists a field $\mathbf{g}$ on B that satisfies Eqs. (6.5)–(6.6). Also, there exist fields φ and ψ that satisfy Eqs. (6.2)–(6.4).

B-P-N solution for axial symmetry. For an axially symmetric problem with $\mathbf{b} = \mathbf{0}$ in which $x_3 = z$ is the axis symmetry of a body, the displacement vector field $\mathbf{u} = \mathbf{u}(r, z)$ referred to the cylindrical coordinates (r, θ, z) takes the form

$$\mathbf{u} = \psi \mathbf{k} - \frac{1}{4(1-\nu)} \nabla (z\psi + \varphi) \tag{6.7}$$

where

$$z = \mathbf{x} \cdot \mathbf{k} \tag{6.8}$$

with $\mathbf{k}$ being a unit vector along the x_3 axis, and with scalar-valued harmonic functions $\varphi = \varphi(r, z)$ and $\psi = \psi(r, z)$. In components we obtain

$$\mathbf{u} = [u_r(r, z),\ 0,\ u_z(r, z)] \tag{6.9}$$

where

$$u_r = -\frac{1}{4(1-\nu)} \frac{\partial}{\partial r}(z\psi + \varphi) \tag{6.10}$$

$$u_z = \psi - \frac{1}{4(1-\nu)} \frac{\partial}{\partial z}(z\psi + \varphi) \tag{6.11}$$

and

$$\nabla^2 \varphi = 0, \quad \nabla^2 \psi = 0 \tag{6.12}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{6.13}$$

B-S-G solution for axial symmetry with $\mathbf{b} = \mathbf{0}$ and $\mathbf{g} = \chi\mathbf{k}$ is also called Love's solution.

$$\mathbf{u} = (\nabla^2\chi)\mathbf{k} - \frac{1}{2(1-\nu)}\nabla(\mathbf{k}\cdot\nabla\chi) \tag{6.14}$$

where

$$\nabla^2\nabla^2\chi = 0 \tag{6.15}$$

In cylindrical coordinates (r, θ, z)

$$u_r = -\frac{1}{2(1-\nu)}\frac{\partial^2}{\partial r\partial z}\chi \tag{6.16}$$

$$u_\theta = 0 \tag{6.17}$$

$$u_z = \frac{1}{2(1-\nu)}\left[2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\chi \tag{6.18}$$

6.2 Complete Solutions of Elastodynamics

The displacement equation of motion for a homogeneous isotropic elastic body takes the form

$$\Box_2^2\mathbf{u} + \left[\left(\frac{c_1}{c_2}\right)^2 - 1\right]\nabla(\text{div}\,\mathbf{u}) + \frac{\mathbf{b}}{\mu} = \mathbf{0} \tag{6.19}$$

where

$$\Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}, \quad \frac{1}{c_1^2} = \frac{\rho}{\lambda + 2\mu}, \quad \frac{1}{c_2^2} = \frac{\rho}{\mu} \tag{6.20}$$

The body force $\mathbf{b}$ is represented by Helmholtz's decomposition formula

$$\mathbf{b} = -\nabla h - \text{curl}\,\mathbf{k}, \quad \text{div}\,\mathbf{k} = 0 \tag{6.21}$$

A solution $\mathbf{u}$ on $\overline{B}\times[0, \infty)$ to Eq. (6.19) will be called an *elastic motion corresponding to* $\mathbf{b}$.

Green-Lame (G-L) Solution. Let

$$\mathbf{u} = \nabla\varphi + \text{curl}\,\psi \tag{6.22}$$

where φ and ψ satisfy, respectively, the equations

$$\Box_1^2\varphi = \frac{h}{\lambda + 2\mu} \tag{6.23}$$

and

$$\Box_2^2\psi = \frac{\mathbf{k}}{\mu} \tag{6.24}$$

Then $\mathbf{u}$ is an elastic motion corresponding to $\mathbf{b}$ given by Eq. (6.21). In Eq. (6.23)

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} \tag{6.25}$$

Cauchy-Kovalevski-Somigliana (C-K-S) Solution. Let

$$\mathbf{u} = \Box_1^2\mathbf{g} + \left(\frac{c_2^2}{c_1^2} - 1\right)\nabla(\text{div }\mathbf{g}) \tag{6.26}$$

where $\mathbf{g}$ satisfies the inhomogeneous biwave equation

$$\Box_1^2\Box_2^2\mathbf{g} = -\frac{\mathbf{b}}{\mu} \tag{6.27}$$

Then $\mathbf{u}$ is an elastic motion corresponding to $\mathbf{b}$.

Note. Both G-L and C-K-S solutions are complete.

6.3 Complete Stress Solution of Elastodynamics

The stress equation of motion for a homogeneous isotropic elastic body takes the form

$$\widehat{\nabla}(\text{div }\mathbf{S}) - \frac{\rho}{2\mu}\left[\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\text{tr }\ddot{\mathbf{S}})\mathbf{1}\right] = -\widehat{\nabla}\mathbf{b} \tag{6.28}$$

A solution $\mathbf{S}$ on $\bar{B} \times [0, \infty)$ to Eq. (6.28) will be called a *stress motion corresponding to* $\mathbf{b}$.

Stress Solution of Galerkin Type. Let

$$\mathbf{S} = \left[\left(\nabla\nabla - \nu\mathbf{1}\Box_2^2\right)\text{ tr }\mathbf{G} - 2(1-\nu)\Box_1^2\,\mathbf{G}\right] \tag{6.29}$$

where $\mathbf{G}$ is a symmetric second-order tensor field on $\bar{B} \times [0, \infty)$ that satisfies the equations

$$\Box_1^2\,\Box_2^2\,\mathbf{G} = \frac{1}{1-\nu}\widehat{\nabla}\mathbf{b} \tag{6.30}$$

and

$$\nabla^2 \mathbf{G} + \nabla\nabla(\text{tr}\,\mathbf{G}) - 2\widehat{\nabla}(\text{div}\,\mathbf{G}) = \mathbf{0} \tag{6.31}$$

Then $\mathbf{S}$ is a stress motion corresponding to $\mathbf{b}$, that is, $\mathbf{S}$ satisfies Eq. (6.28).

Completeness of the Stress Solution of Galerkin Type.

The stress solution of Galerkin type corresponding to homogeneous initial conditions is complete, that is, for any stress motion $\mathbf{S}$ corresponding to $\mathbf{b}$ there exists a second-order symmetric tensor field $\mathbf{G}$ such that Eqs. (6.29)–(6.31) are satisfied.

6.4 Problems and Solutions Related to Complete Solutions of Elasticity

Problem 6.1. The displacement $\mathbf{u} = \mathbf{u}(\mathbf{x}, \xi)$ at a point $\mathbf{x}$ due to a concentrated force $\boldsymbol{l}$ applied at a point ξ of a homogeneous isotropic infinite elastic body is given by $(\mathbf{x} \neq \xi)$

$$\mathbf{u}(\mathbf{x}, \xi) = \mathbf{U}(\mathbf{x}, \xi)\boldsymbol{l}$$

where

$$\mathbf{U}(\mathbf{x}, \xi) = \frac{1}{16\pi\mu(1-\nu)}\frac{1}{\mathrm{R}}\left[(3-4\nu)\mathbf{1} + \frac{(\mathbf{x}-\xi)\otimes(\mathbf{x}-\xi)}{\mathrm{R}^2}\right]$$

with

$$\mathrm{R} = |\mathbf{x} - \xi|$$

Use the stress-displacement relation to show that the associated stress $\mathbf{S} = \mathbf{S}(\mathbf{x}, \xi)$ takes the form

$$\mathbf{S}(\mathbf{x}, \xi) = -\frac{1}{8\pi(1-\nu)}\frac{1}{\mathrm{R}^3}\left\{\frac{3}{\mathrm{R}^2}[(\mathbf{x}-\xi)\cdot\boldsymbol{l}](\mathbf{x}-\xi)\otimes(\mathbf{x}-\xi)\right.$$
$$\left. +(1-2\nu)\{(\mathbf{x}-\xi)\otimes\boldsymbol{l} + \boldsymbol{l}\otimes(\mathbf{x}-\xi) - [(\mathbf{x}-\xi)\cdot\boldsymbol{l}]\,\mathbf{1}\}\right\}$$

Solution. The displacement $\mathbf{u}$ in components takes the form

$$u_i = U_{ik}\,\ell_k \tag{6.32}$$

where

$$U_{ik} = \frac{A}{2\mu}R^{-1}\left[(3-4\nu)\delta_{ik} + (x_i - \xi_i)(x_k - \xi_k)R^{-2}\right] \tag{6.33}$$

and

$$A = \frac{1}{8\pi(1-\nu)} \tag{6.34}$$

The stress tensor S_{ij} is computed from the stress-strain relation

$$S_{ij} = 2\mu\left(E_{ij} + \frac{\nu}{1-2\nu}E_{kk}\,\delta_{ij}\right) \tag{6.35}$$

where

$$E_{ij} = u_{(i,j)} = U_{(ik,j)}\,\ell_k \tag{6.36}$$

Calculating $U_{ik,j}$, by using Eq. (6.33), we obtain

$$\begin{aligned} U_{ik,j} = -\frac{A}{2\mu}R^{-3}\Big[&(3-4\nu)(x_j-\xi_j)\delta_{ki} + 3(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)R^{-2} \\ &-(x_k-\xi_k)\delta_{ij} - (x_i-\xi_i)\delta_{kj}\Big] \end{aligned} \tag{6.37}$$

Hence, taking the trace of (6.37) with respect to the indices i and j, we get

$$\begin{aligned} U_{ik,i} &= -\frac{A}{2\mu}R^{-3}[(3-4\nu)(x_k-\xi_k) + 3(x_k-\xi_k) - 3(x_k-\xi_k) - (x_k-\xi_k)] \\ &= -\frac{A}{2\mu}R^{-3} \times 2(1-2\nu)(x_k-\xi_k) \end{aligned} \tag{6.38}$$

Since

$$E_{ii} = U_{ik,i}\,\ell_k \tag{6.39}$$

therefore,

$$E_{ii} = E_{kk} = -\frac{A}{2\mu}R^{-3} \times 2(1-2\nu)(x_k-\xi_k)\ell_k \tag{6.40}$$

Also, by taking the symmetric part of (6.37) with respect to the indices i and j we obtain

$$\begin{aligned} U_{(ik,j)} = -\frac{A}{2\mu}R^{-3}\Big\{&(1-2\nu)[(x_j-\xi_j)\delta_{ki} + (x_i-\xi_i)\delta_{kj}] \\ &+3(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)R^{-2} - (x_k-\xi_k)\delta_{ij}\Big\} \end{aligned} \tag{6.41}$$

Hence, because of (6.36), we get

$$\begin{aligned} E_{ij} = -\frac{A}{2\mu}R^{-3}\Big\{&(1-2\nu)[(x_j-\xi_j)\ell_i+(x_i-\xi_i)\ell_j]+3(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)\ell_k R^{-2} \\ &-(x_k-\xi_k)\ell_k\,\delta_{ij}\Big\} \end{aligned} \tag{6.42}$$

Finally, substituting E_{ij} from (6.42) and E_{kk} from (6.40), respectively, into (6.35), we obtain

$$S_{ij} = -AR^{-3}\Big\{3R^{-2}(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)\ell_k \\ +(1-2\nu)[(x_i-\xi_i)\ell_j+(x_j-\xi_j)\ell_i-(x_k-\xi_k)\ell_k\delta_{ij}\Big\} \tag{6.43}$$

Equation. (6.43) is equivalent to the stress formula of Problem 6.1. This completes a solution to Problem 6.1.

Problem 6.2. The displacement equation of thermoelastostatics for a homogeneous isotropic body subject to a temperature change $T = T(\mathbf{x})$ takes the form

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla(\text{div }\mathbf{u}) - \frac{2+2\nu}{1-2\nu}\alpha\,\nabla T = \mathbf{0} \tag{6.44}$$

Let

$$\mathbf{u} = \psi - \frac{1}{4(1-\nu)}\nabla(\mathbf{x}\cdot\psi + \widehat{\varphi}) \tag{6.45}$$

where

$$\nabla^2\psi = \mathbf{0} \tag{6.46}$$

and

$$\nabla^2\widehat{\varphi} = -4(1+\nu)\alpha\,T \tag{6.47}$$

Show that $\mathbf{u}$ given by Eqs. (6.45) through (6.46) satisfies Eq. (6.44).

Solution. Eqs. (6.44)–(6.47) in components take the form

$$u_{i,kk} + \frac{1}{1-2\nu}u_{k,ki} - \frac{2+2\nu}{1-2\nu}\alpha\,T,_i = 0 \tag{6.48}$$

$$u_i = \psi_i - \frac{1}{4(1-\nu)}(x_a\psi_a + \widehat{\varphi}),_i \tag{6.49}$$

where

$$\psi_{i,aa} = 0 \tag{6.50}$$

and

$$\widehat{\varphi},_{kk} = -4(1+\nu)\alpha T \tag{6.51}$$

Taking the gradient of (6.49) we obtain

$$u_{i,k} = \psi_{i,k} - \frac{1}{4(1-\nu)}(x_a\psi_a + \widehat{\varphi}),_{ik} \tag{6.52}$$

Hence, from (6.50),

$$u_{i,kk} = -\frac{1}{4(1-\nu)}(x_a\psi_a + \widehat{\varphi}),_{ikk} \tag{6.53}$$

and

$$u_{k,k} = \psi_{k,k} - \frac{1}{4(1-\nu)}(x_a\psi_a + \widehat{\varphi}),_{kk} \tag{6.54}$$

$$u_{k,ki} = \left[\psi_{k,k} - \frac{1}{4(1-\nu)}(x_a\psi_a + \widehat{\varphi}),_{kk}\right],_i \tag{6.55}$$

Using the relation

$$(x_a\psi_a),_{kk} = 2\psi_{k,k} + x_a\psi_{a,kk} = 2\psi_{k,k} \tag{6.56}$$

we reduce (6.53) and (6.55) into

$$u_{i,kk} = -\frac{1}{4(1-\nu)}(2\psi_{k,k} + \widehat{\varphi},_{kk}),_i \tag{6.57}$$

and

$$u_{k,ki} = \left[\psi_{k,k} - \frac{1}{4(1-\nu)}(2\psi_{k,k} + \widehat{\varphi},_{kk})\right],_i \tag{6.58}$$

Therefore, substituting (6.57) and (6.58) into the LHS of (6.48) we obtain

$$\left\{-\frac{1}{4(1-\nu)}(2\psi_{k,k} + \widehat{\varphi},_{kk}) + \frac{1}{1-2\nu}\left[\psi_{k,k} - \frac{1}{4(1-\nu)}(2\psi_{k,k} + \widehat{\varphi},_{kk})\right] - \frac{2+2\nu}{1-2\nu}\alpha T\right\},_i$$
$$= -\frac{1}{2(1-2\nu)}\{\widehat{\varphi},_{kk} + 4(1+\nu)\alpha T\},_i \tag{6.59}$$

Equation (6.59) together with Eq. (6.51) imply that u_i given by (6.49) meets (6.48). This completes a solution to Problem 6.2.

Problem 6.3. The temperature change T of a homogeneous isotropic infinite elastic body is represented by

$$T(\mathbf{x}) = \widehat{T}\,\delta(\mathbf{x}) \tag{6.60}$$

where

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)\delta(x_3) \tag{6.61}$$

$\delta = \delta(x_i)$, $i = 1, 2, 3$, is a one dimensional Dirac delta function, and $\widehat{T}$ is a constant with the dimension $[\widehat{T}]$ = [Temperature × Volume]. Show that an elastic displacement $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and stress $\mathbf{S} = \mathbf{S}(\mathbf{x})$ corresponding to $T = T(\mathbf{x})$ are given by

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\frac{1+\nu}{1-\nu}\alpha\,\widehat{T}\,\nabla\,\frac{1}{|\mathbf{x}|} \tag{6.62}$$

and

$$\mathbf{S}(\mathbf{x}) = -\frac{\mu}{2\pi}\frac{1+\nu}{1-\nu}\alpha\,\widehat{T}\,(\nabla\nabla - \mathbf{1}\,\nabla^2)\,\frac{1}{|\mathbf{x}|} \tag{6.63}$$

Hint. Use the representation (6.61) through (6.63) of Problem 6.2 in which $\psi = \mathbf{0}$ and $T = \widehat{T}\,\delta(\mathbf{x})$. Also, note that

$$\mathbf{S} = -\frac{\mu}{2(1-\nu)}(\nabla\nabla - \mathbf{1}\,\nabla^2)\,\widehat{\varphi} \tag{6.64}$$

Solution. By letting $\psi = \mathbf{0}$ and

$$\widehat{\varphi} = -4(1-\nu)\phi \tag{6.65}$$

in Eqs. (6.61)–(6.63) of Problem 6.2 we obtain

$$u_i = \phi_{,i} \tag{6.66}$$

where

$$\nabla^2\phi = m\,T \tag{6.67}$$

and

$$m = \frac{1+\nu}{1-\nu}\alpha \tag{6.68}$$

Also,

$$S_{ij} = 2\mu(\phi_{,ij} - \nabla^2\phi\delta_{ij}) \tag{6.69}$$

If $T(\mathbf{x}) = \widehat{T}\delta(\mathbf{x})$, a solution to (6.67) in E^3 takes the form

$$\phi = -\frac{m\widehat{T}}{4\pi}\frac{1}{|\mathbf{x}|} \tag{6.70}$$

Substituting ϕ from (6.70) into (6.66) and (6.69), respectively, we obtain (6.62) and (6.63). This completes a solution to Problem 6.3.

Problem 6.4. A solution $\varphi = \varphi(\mathbf{x}, t)$ to the nonhomogeneous wave equation

$$\Box_0^2\varphi(\mathbf{x}, t) = -F(\mathbf{x}, t) \quad \text{on} \quad E^3 \times [0, \infty) \tag{6.71}$$

subject to the homogeneous initial conditions

$$\varphi(\mathbf{x}, 0) = 0, \quad \dot{\varphi}(\mathbf{x}, 0) = 0 \qquad \text{on} \qquad E^3 \tag{6.72}$$

takes the form

$$\varphi(\mathbf{x}, t) = \frac{1}{4\pi}\int\limits_{|\mathbf{x}-\xi|\le c\,t} \frac{F(\xi, t - |\mathbf{x}-\xi|\,/\,c)}{|\mathbf{x}-\xi|}\,dv(\xi) \quad \text{on} \quad E^3 \times [0, \infty) \tag{6.73}$$

Here

$$\Box_0^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{6.74}$$

Show that an equivalent form of Eq. (6.73) reads

$$\varphi(\mathbf{x}, t) = -\frac{c^2t^2}{4\pi}\int\limits_{|\xi|\leq 1}\frac{F[\mathbf{x} - c\,t\xi, (1 - |\xi|)t]}{|\xi|}\,dv(\xi) \quad \text{on} \quad E^3 \times [0, \infty) \tag{6.75}$$

Solution. Introduce the transformation of variables

$$\mathbf{x} - \xi = c\,t\zeta, \quad c\,t > 0 \tag{6.76}$$

Then

$$dv(\xi) = d\xi_1\, d\xi_2\, d\xi_3 = -c^3t^3 d\zeta_1 d\zeta_2 d\zeta_3 \tag{6.77}$$

and

$$dv(\xi) = -c^3t^3 dv(\zeta) \tag{6.78}$$

Since

$$|\mathbf{x} - \xi| = ct|\zeta| \leq ct \tag{6.79}$$

therefore,

$$|\zeta| \leq 1 \tag{6.80}$$

and the integral (6.73) reduces to (6.75). This completes a solution to Problem 6.4.

Problem 6.5. Let $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ be a solution to the stress equation of motion of a homogeneous anisotropic elastodynamics [see Eq. (3.51) in which ρ and K are constants]

$$\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \rho\,\mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \quad \text{on} \quad B \times [0, \infty) \tag{6.81}$$

subject to the initial conditions

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}_0(\mathrm{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in B \tag{6.82}$$

Here, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$, $\mathbf{S}_0 = \mathbf{S}_0(\mathbf{x})$, and $\dot{\mathbf{S}}_0 = \dot{\mathbf{S}}_0(\mathbf{x})$ are prescribed functions. Show that the compatibility condition

$$\operatorname{curl}\operatorname{curl}\mathbf{K}[\mathbf{S}] = 0 \quad \text{on} \quad B \times [0, \infty) \tag{6.83}$$

is satisfied if and only if there exists a vector field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ on $B \times [0, \infty)$ such that

$$\widehat{\nabla}\ddot{\mathbf{u}} = -\rho^{-1}\mathbf{B} \quad \text{on} \quad B \times [0, \infty) \tag{6.84}$$

and

$$\mathbf{S}_0(\mathbf{x}) = \mathbf{K}^{-1}\,[\widehat{\nabla}\mathbf{u}(\mathbf{x}, 0)], \quad \dot{\mathbf{S}}_0(\mathbf{x}) = \mathbf{K}^{-1}\,[\widehat{\nabla}\dot{\mathbf{u}}(\mathbf{x}, 0)] \quad \text{on} \quad B \tag{6.85}$$

Note that $\mathbf{B}$ in Eq. (6.81) represents an arbitrary second-order symmetric tensor field on $B \times [0, \infty)$, while $\mathbf{S}_0$ and $\dot{\mathbf{S}}_0$ in Eq. (6.82) stand for arbitrary second-order symmetric tensor fields on B.

Solution. A solution to Problem 6.5 is based on the following

Lemma. A symmetric tensor field $\mathbf{E}$ on $B \times [0, \infty)$ satisfies the condition

$$\text{curl curl } \mathbf{E} = \mathbf{0} \quad \text{on } B \times [0, \infty) \tag{6.86}$$

if and only if there is a vector field $\mathbf{u}$ on $B \times [0, \infty)$ such that

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} \quad \text{on } B \times (0, \infty) \tag{6.87}$$

Proof of Lemma. The proof is split into two parts
(i) (6.87) ⇒ (6.86), (ii) (6.86) ⇒ (6.87).

To show (i) we substitute (6.87) into the LHS of Eq. (6.86) and find that Eq. (6.86) holds true. To show (ii), we note that Eq. (6.86) implies that there is a vector field $\mathbf{a}$ such that

$$\text{curl } \mathbf{E} = \nabla\mathbf{a} \tag{6.88}$$

Since, by Helmholtz's theorem, there are a scalar field φ and a vector field $\mathbf{b}$ such that

$$\mathbf{a} = \nabla\varphi + \text{curl } \mathbf{b}, \quad \text{div } \mathbf{b} = 0 \tag{6.89}$$

Equation (6.88) can be written as

$$\text{curl } \mathbf{E} = \nabla\nabla\varphi + \nabla \text{ curl } \mathbf{b} \tag{6.90}$$

or

$$\varepsilon_{iab}\, E_{jb,a} = \varphi_{,ij} + \varepsilon_{iab}\, b_{b,aj} \tag{6.91}$$

By taking the trace of (6.91), it is, by letting $i = j$ in (6.91), we obtain

$$\varphi_{,ii} = 0 \quad \text{or} \quad \text{div } \nabla\varphi = 0 \tag{6.92}$$

This implies that there is a vector field $\mathbf{c}$ on $B \times [0, \infty)$ such that

$$\nabla\varphi = \text{curl } \mathbf{c} \tag{6.93}$$

Substituting (6.93) into (6.90) we obtain

$$\text{curl } \mathbf{E} = \nabla \text{ curl } (\mathbf{c} + \mathbf{b}) \tag{6.94}$$

Since for any vector $\mathbf{v}$

$$\nabla \text{ curl } \mathbf{v} = \text{curl } (\nabla \mathbf{v}^T) \tag{6.95}$$

or

$$\varepsilon_{iab}\, v_{b,aj} = \varepsilon_{iab}(v_{j,b})^T_{,a} \tag{6.96}$$

therefore, Eqs. (6.94) and (6.95) imply that

$$\text{curl } (\mathbf{E} - \nabla \mathbf{v}^T) = \mathbf{0} \tag{6.97}$$

where

$$\mathbf{v} = \mathbf{b} + \mathbf{c} \tag{6.98}$$

Next, it follows from Eq. (6.97) that there is a vector field $\mathbf{e}$ on $B \times [0, \infty)$ such that

$$\mathbf{E} - \nabla \mathbf{v}^T = \nabla \mathbf{e} \tag{6.99}$$

By taking the transpose of (6.99) and using the symmetry of $\mathbf{E}$ ($\mathbf{E} = \mathbf{E}^T$) we obtain

$$\mathbf{E} - \nabla \mathbf{v} = \nabla \mathbf{e}^T \tag{6.100}$$

By adding Eqs. (6.99) and (6.100) we get

$$\mathbf{E} = \widehat{\nabla}(\mathbf{v} + \mathbf{e}) \tag{6.101}$$

Hence, if we let

$$\mathbf{u} = \mathbf{v} + \mathbf{e} \tag{6.102}$$

in Eq. (6.101) we obtain (6.87). This shows (ii), and proof of Lemma is complete.

To show that the compatibility condition (6.83) is satisfied if and only if there exists a vector field $\mathbf{u}$ on $B \times [0, \infty)$ such that (6.84) and (6.85) hold true, we note that Eqs. (6.81) and (6.82) are satisfied if and only if

$$\mathbf{K}[\mathbf{S}] = \mathbf{K}[\mathbf{S}_0 + t\dot{\mathbf{S}}_0] + t * \rho^{-1}(\widehat{\nabla}\text{div}\, \mathbf{S} + \mathbf{B}) \tag{6.103}$$

Applying curl curl to this equation we obtain

$$\text{curl curl } \mathbf{K}[\mathbf{S}] = \text{curl curl } \left\{ t * \rho^{-1}\mathbf{B} + \mathbf{K}[\mathbf{S}_0 + t\dot{\mathbf{S}}_0] \right\} \tag{6.104}$$

Hence, the compatibility condition (6.83) is equivalent to

$$\text{curl curl } \{t * \rho^{-1}\mathbf{B} + \mathbf{K}[\mathbf{S}_0 + t\dot{\mathbf{S}}_0]\} = \mathbf{0} \tag{6.105}$$

Using the Lemma we find that Eq. (6.105) holds true if and only if there is a vector field $\mathbf{u}$ such that

$$t * \rho^{-1}\mathbf{B} + \mathbf{K}[\mathbf{S}_0 + t\dot{\mathbf{S}}_0] = \widehat{\nabla}\mathbf{u} \tag{6.106}$$

If $\mathbf{B}$, $\mathbf{S}_0$, and $\dot{\mathbf{S}}_0$ are given by (6.84)–(6.85), then Eq. (6.106) is identically satisfied. Conversely, by differentiating twice (6.106) with respect to time we obtain (6.84). Also, by differentiating (6.106) with respect to time and letting $t = 0$ we obtain

$$\dot{\mathbf{S}}_0 = \mathbf{K}^{-1}[\widehat{\nabla}\dot{\mathbf{u}}(\mathbf{x}, 0)] \tag{6.107}$$

Finally, by letting $t = 0$ in (6.106) we get

$$\mathbf{S}_0 = \mathbf{K}^{-1}[\widehat{\nabla}\mathbf{u}(\mathbf{x}, 0)] \tag{6.108}$$

This completes a solution to Problem 6.5.

Problem 6.6. Consider a homogeneous isotropic elastic body occupying a region B. Let $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ be a tensor field defined by

$$\mathbf{S}(\mathbf{x}, t) = \left[\left(\nabla\nabla - \nu\mathbf{1}\Box_2^2\right) \operatorname{tr} \chi - 2(1-\nu)\Box_1^2\chi\right] \quad \text{on} \quad \overline{B} \times [0, \infty) \tag{6.109}$$

where $\chi = \chi(\mathbf{x}, t)$ is a symmetric second-order tensor field that satisfies the equations

$$\Box_1^2\Box_2^2\chi = \frac{1}{1-\nu}\widehat{\nabla}\mathbf{b} \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{6.110}$$

and

$$\nabla^2\chi + \nabla\nabla(\operatorname{tr}\chi) - 2\widehat{\nabla}(\operatorname{div}\chi) = \mathbf{0} \quad \text{on} \quad B \times [0, \infty) \tag{6.111}$$

Show that $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ satisfies the stress equation of motion [see Eq. (6.28)]

$$\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \frac{\rho}{2\mu}\left[\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\,\mathbf{1}\right] = -\widehat{\nabla}\mathbf{b} \quad \text{on} \quad B \times [0, \infty) \tag{6.112}$$

Note. The stress field $\mathbf{S}$ in the form of Eqs. (6.109) through (6.111) is a tensor solution of the homogeneous isotropic elastodynamics of the Galerkin type. To show this we let $\chi = -[\mu/(1-\nu)]\widehat{\nabla}\mathbf{g}$, where $\mathbf{g}$ is the Galerkin vector satisfying Eq. (6.27). Then Eqs. (6.110 and 6.111) are satisfied identically, and Eq. (6.109) reduces to

$$\mathbf{S} = \mu\left[2\Box_1^2\widehat{\nabla}\mathbf{g} - \frac{1}{1-\nu}\nabla\nabla(\operatorname{div}\mathbf{g}) + \frac{\nu}{1-\nu}\mathbf{1}\,\Box_2^2(\operatorname{div}\mathbf{g})\right] \tag{6.113}$$

The stress field $\mathbf{S}$ given by (6.111) corresponds to a solution of C-K-S [or Galerkin] type defined by Eqs. (6.26)–(6.27).

Solution. Eqs. (6.109)–(6.111), respectively, in components take the form

$$S_{ij} = \chi_{aa,ij} - \nu \Box_2^2 \chi_{aa}\, \delta_{ij} - 2(1-\nu)\Box_1^2 \chi_{ij} \tag{6.114}$$

$$\Box_1^2 \Box_2^2 \chi_{ij} = \tfrac{1}{1-\nu} b_{(i,j)} \tag{6.115}$$

and

$$\chi_{ij,aa} + \chi_{aa,ij} - \chi_{ik,kj} - \chi_{jk,ki} = 0 \tag{6.116}$$

The stress equation of motion (6.112) is rewritten as

$$S_{(ik,kj)} - \frac{\rho}{2\mu}\left(\ddot{S}_{ij} - \frac{\nu}{1+\nu}\ddot{S}_{kk}\, \delta_{ij}\right) = -b_{(i,j)} \tag{6.117}$$

In Eqs. (6.114)–(6.115)

$$\Box_1^2 = \nabla^2 - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}, \quad \Box_2^2 = \nabla^2 - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} \tag{6.118}$$

and

$$\frac{1}{c_2^2} = \frac{\rho}{\mu}, \quad \frac{1}{c_1^2} = \frac{1}{c_2^2}\frac{1-2\nu}{2-2\nu} \tag{6.119}$$

By taking the trace of (6.114) we obtain

$$S_{aa} = -(1+\nu)\, \Box_2^2\, \chi_{aa} \tag{6.120}$$

Hence, an alternative form of (6.114) reads

$$S_{ij} - \frac{\nu}{1+\nu} S_{kk}\, \delta_{ij} = \chi_{aa,ij} - 2(1-\nu)\Box_1^2\, \chi_{ij} \tag{6.121}$$

Next, using (6.114) we obtain

$$S_{(ik,kj)} = \chi_{aa,kkij} - \nu \Box_2^2\, \chi_{aa,ij} - 2(1-\nu)\Box_1^2\, \chi_{(ik,kj)} \tag{6.122}$$

Since, from (6.116)

$$2\chi_{(ik,kj)} = \chi_{ij,aa} + \chi_{aa,ij} \tag{6.123}$$

therefore, (6.122) can be written as

$$S_{(ik,kj)} = \nabla^2 \chi_{aa,ij} - \nu \Box_2^2\, \chi_{aa,ij} - (1-\nu)\Box_1^2\, \chi_{aa,ij} - (1-\nu)\Box_1^2\, \chi_{ij,aa} \tag{6.124}$$

or

$$S_{(ik,kj)} = \frac{1}{2c_2^2}\ddot{\chi}_{aa,ij} - (1-\nu)\Box_1^2\, \chi_{ij,aa} \tag{6.125}$$

Substituting (6.121) and (6.125) into the LHS of (6.117) we obtain

$$S_{(ik,kj)} - \frac{1}{2c_2^2}\left(\ddot{S}_{ij} - \frac{\nu}{1+\nu}\ddot{S}_{kk}\delta_{ij}\right) = \frac{1}{2c_2^2}\,\ddot{\chi}_{aa,ij} - (1-\nu)\Box_1^2\,\chi_{ij,aa}$$
$$- \frac{1}{2c_2^2}\left[\ddot{\chi}_{aa,ij} - 2(1-\nu)\Box_1^2\,\ddot{\chi}_{ij}\right] = -(1-\nu)\Box_1^2\,\Box_2^2\,\chi_{ij} \tag{6.126}$$

Since χ_{ij} satisfies Eq. (6.115), therefore, by virtue of (6.126), S_{ij} given by (6.114) meets (6.117). This completes a solution to Problem 6.6.

Problem 6.7. Let **S** be the tensor solution of homogeneous isotropic elastodynamics of Problem 6.6 corresponding to homogeneous initial conditions. Show that the solution is complete, that is, there exists a second-order symmetric tensor field χ such that Eqs. (6.109) through (6.111) of Problem 6.6 are satisfied.

Solution. To solve Problem 6.7 we prove the two Lemmas.

Lemma 1. Let **S** satisfy the field equation

$$\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \frac{1}{2c_2^2}\left(\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\mathbf{1}\right) = -\widehat{\nabla}\mathbf{b} \quad \text{on } B\times[0,\infty) \tag{6.127}$$

subject to the homogeneous initial conditions

$$\mathbf{S}(\mathbf{x},0) = \mathbf{0}, \quad \dot{\mathbf{S}}(\mathbf{x},0) = 0 \quad \text{on } B \tag{6.128}$$

Then

$$\text{curl}\,\text{curl}\,\mathbf{E} = \mathbf{0} \quad \text{on } B\times[0,\infty) \tag{6.129}$$

where

$$\mathbf{E} = \frac{1}{2\mu}\left(\mathbf{S} - \frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{S})\mathbf{1}\right) \tag{6.130}$$

Lemma 2. Let **S** satisfy the hypotheses of Lemma 1, and let $\tilde{\mathbf{S}}$ be a continuation of **S** on $E^3\times[0,\infty)$ such that

$$\text{curl}\,\text{curl}\,\tilde{\mathbf{E}} = \mathbf{0} \quad \text{on } E^3\times[0,\infty) \tag{6.131}$$

where

$$\tilde{\mathbf{E}} = \frac{1}{2\mu}\left(\tilde{\mathbf{S}} - \frac{\nu}{1+\nu}(\operatorname{tr}\tilde{\mathbf{S}})\mathbf{1}\right) \quad \text{on } E^3\times[0,\infty) \tag{6.132}$$

Define a second-order tensor field χ on $\bar{B}\times[0,\infty)$ such that

$$-2(1-\nu)\chi = 2\mu\,\frac{c_1^2t^2}{4\pi}\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\tilde{\mathbf{E}}(\mathbf{x}-c_1t\xi,(1-|\xi|)t) + \frac{2\mu}{1-2\nu}\,\frac{c_1^2c_2^2t^4}{16\pi^2}\nabla\nabla$$

$$\times \int_{|\xi|\le 1} \frac{dv(\xi)}{|\xi|} \int_{|\eta|\le 1} \frac{dv(\eta)}{|\eta|} (\text{tr}\, \tilde{\mathbf{E}})$$
$$\times\, (\mathbf{x} - c_1 t\xi - c_2 t\eta,\ (1-|\xi|)(1-|\eta|)t) \tag{6.133}$$

Then

$$\Box_2^2(\text{tr}\,\chi) = -\frac{2\mu}{1-2\nu}(\text{tr}\,\mathbf{E}) \tag{6.134}$$

Notes

(1) Equations (6.129) and (6.131), respectively, are equivalent to

$$2\widehat{\nabla}(\text{div}\,\mathbf{S}) - \nabla^2\mathbf{S} + \frac{1}{1+\nu}(\nu\mathbf{1}\nabla^2 - \nabla\nabla)(\text{tr}\,\mathbf{S}) = \mathbf{0} \tag{6.135}$$

and

$$2\widehat{\nabla}(\text{div}\,\tilde{\mathbf{S}}) - \nabla^2\tilde{\mathbf{S}} + \frac{1}{1+\nu}(\nu\mathbf{1}\nabla^2 - \nabla\nabla)(\text{tr}\,\tilde{\mathbf{S}}) = \mathbf{0} \tag{6.136}$$

To prove that (6.135) $\Leftrightarrow$ (6.129) we use the identity [see Problem 1.12, Eq. (1.204) in which **S** is replaced by **E**]

$$\text{curl curl}\,\mathbf{E} = 2\widehat{\nabla}(\text{div}\,\mathbf{E}) - \nabla^2\mathbf{E} - \nabla\nabla\,(\text{tr}\,\mathbf{E}) + \mathbf{1}[\nabla^2(\text{tr}\,\mathbf{E}) - \text{div div}\,\mathbf{E}] \tag{6.137}$$

Equation (6.129) implies that the LHS of (6.137) vanishes which written in components means

$$E_{ik,kj} + E_{jk,ki} - E_{ij,kk} - E_{aa,ij} + \delta_{ij}(E_{aa,bb} - E_{ab,ab}) = 0 \tag{6.138}$$

By letting $i = j$ in (6.138) we obtain

$$2E_{ik,ki} - 2E_{aa,bb} + 3(E_{aa,bb} - E_{ab,ab}) = 0 \tag{6.139}$$

or

$$E_{aa,bb} - E_{ab,ab} = 0 \tag{6.140}$$

Hence (6.129) is equivalent to

$$2\widehat{\nabla}(\text{div}\,\mathbf{E}) - \nabla^2\mathbf{E} - \nabla\nabla(\text{tr}\,\mathbf{E}) = \mathbf{0} \tag{6.141}$$

Substituting **E** from (6.130) into (6.141), and using the relations

$$\text{div}\,\mathbf{E} = \frac{1}{2\mu}\left[\text{div}\,\mathbf{S} - \frac{\nu}{1+\nu}\nabla(\text{tr}\,\mathbf{S})\right] \tag{6.142}$$

$$(\text{tr}\,\mathbf{E}) = \frac{1}{2\mu}\,\frac{1-2\nu}{1+\nu}(\text{tr}\,\mathbf{S}) \tag{6.143}$$

$$2\widehat{\nabla}\nabla(\text{tr}\,\mathbf{S}) = 2\nabla\nabla(\text{tr}\,\mathbf{S}) \tag{6.144}$$

we obtain

$$2\widehat{\nabla}(\text{div}\,\mathbf{S}) - \nabla^2\mathbf{S} - \frac{2\nu}{1+\nu}\,\nabla\nabla(\text{tr}\,\mathbf{S}) + \frac{\nu}{1+\nu}\mathbf{1}\nabla^2(\text{tr}\,\mathbf{S}) - \frac{1-2\nu}{1+\nu}\nabla\nabla(\text{tr}\,\mathbf{S})$$

$$= 2\widehat{\nabla}(\text{div}\,\mathbf{S}) - \nabla^2\mathbf{S} + \frac{1}{1+\nu}[\nu\mathbf{1}\nabla^2(\text{tr}\,\mathbf{S}) - \nabla\nabla(\text{tr}\,\mathbf{S})] = \mathbf{0} \tag{6.145}$$

Therefore, (6.129) ⇔ (6.135) ⇔ (6.145).

(2) An alternative form of (6.133) reads

$$-2(1-\nu)\chi = \frac{c_1^2 t^2}{4\pi}\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\left[\tilde{\mathbf{S}} - \frac{\nu}{1+\nu}(\text{tr}\,\tilde{\mathbf{S}})\mathbf{1}\right](\mathbf{x} - c_1 t\xi,\ t(1-|\xi|))$$

$$+ \frac{1}{1+\nu}\nabla\nabla\left(\frac{c_1^2 c_2^2 t^4}{16\pi^2}\right)\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}(\text{tr}\,\tilde{\mathbf{S}})$$

$$\times\,(\mathbf{x} - c_1 t\xi - c_2 t\eta,\ t(1-|\xi|)(1-|\eta|)) \tag{6.146}$$

To prove that (6.133) ⇔ (6.146) we substitute $\tilde{\mathbf{E}}$ from (6.132) into (6.133) and obtain (6.146).

(3) A solution $\widehat{\varphi} = \widehat{\varphi}(\mathbf{x}, t)$ of the biwave equation

$$\square_1^2\,\square_2^2\widehat{\varphi} = f \quad \text{on } \bar{B}\times[0,\infty) \tag{6.147}$$

subject to the homogeneous initial conditions

$$\widehat{\varphi}^{(k)}(\mathbf{x}, 0) = 0 \quad \text{on } B, \quad k = 0, 1, 2, 3 \tag{6.148}$$

takes the form of iterated retarded potential

$$\widehat{\varphi}(\mathbf{x}, t) = \frac{c_1^2 c_2^2 t^4}{16\pi^2}\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}f(\mathbf{x} - c_1 t\xi - c_2 t\eta,\ t(1-|\xi|)(1-|\eta|)) \tag{6.149}$$

To show that $\widehat{\varphi}$ given by (6.149) satisfies Eq. (6.147), note that because of the solution to Problem 6.4, a solution to the equation

$$\square_1^2 u = f \quad \text{on } B\times[0,\infty) \tag{6.150}$$

subject to the homogeneous initial conditions

$$u(\mathbf{x}, 0) = 0, \ \dot{u}(\mathbf{x}, 0) = 0 \quad \text{on } B \tag{6.151}$$

takes the form

$$u(\mathbf{x}, t) = \frac{c_1^2 t^2}{4\pi} \int\limits_{|\xi| \le 1} \frac{dv(\xi)}{|\xi|} f(\mathbf{x} - c_1 t \xi, \ t(1 - |\xi|)) \tag{6.152}$$

Similarly, a solution to the equation

$$\square_2^2 \widehat{\varphi} = u \quad \text{on } B \times [0, \infty) \tag{6.153}$$

in which u is prescribed, and subject to the conditions

$$\widehat{\varphi}(\mathbf{x}, 0) = 0, \quad \dot{\widehat{\varphi}}(\mathbf{x}, 0) = 0 \quad \text{on } B \tag{6.154}$$

takes the form

$$\widehat{\varphi}(\mathbf{x}, t) = \frac{c_2^2 t^2}{4\pi} \int\limits_{|\eta| \le 1} \frac{dv(\eta)}{|\eta|} u(\mathbf{x} - c_2 t \eta, \ t(1 - |\eta|)) \tag{6.155}$$

Note that substituting u from (6.153) into (6.150) we obtain

$$\square_1^2 \, \square_2^2 \, \widehat{\varphi} = f \tag{6.156}$$

and it follows from (6.152) that

$$\begin{aligned} &u(\mathbf{x} - c_2 t \eta, \ t(1 - |\eta|)) \\ &= \frac{c_1^2 t^2}{4\pi} \int\limits_{|\xi| \le 1} \frac{dv(\xi)}{|\xi|} \times f(\mathbf{x} - c_1 t \xi - c_2 t \eta, \ t(1 - |\xi|)(1 - |\eta|)) \end{aligned} \tag{6.157}$$

Therefore, substituting (6.157) into (6.155) we find that a solution of (6.156) takes the form (6.149). Also, by differentiating (6.149) with respect to time we obtain

$$\widehat{\varphi}(\mathbf{x}, 0) = \dot{\widehat{\varphi}}(\mathbf{x}, 0) = \ddot{\widehat{\varphi}}(\mathbf{x}, 0) = \dddot{\widehat{\varphi}}(\mathbf{x}, 0) = 0 \tag{6.158}$$

This completes the proof that $\widehat{\varphi}$ given by (6.149) satisfies (6.147) and (6.148).

Proof of Lemma 1. Applying the operator curl curl to Eq. (6.127) and using the relation

$$\text{curl curl } \widehat{\nabla}(\text{div } \mathbf{S} + \mathbf{b}) = \mathbf{0} \tag{6.159}$$

we obtain

$$\text{curl curl } \ddot{\mathbf{E}} = \mathbf{0} \tag{6.160}$$

where $\mathbf{E}$ is given by Eq. (6.130).

From (6.128) and (6.130)

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{0}, \quad \dot{\mathbf{E}}(\mathbf{x}, 0) = \mathbf{0} \tag{6.161}$$

Therefore, integrating (6.160) twice with respect to time, and using (6.161), we arrive at Eq. (6.129). This completes the proof of Lemma 1.

Proof of Lemma 2. Introduce $\tilde{\varphi} = \tilde{\varphi}(\mathbf{x}, t)$ on $E^3 \times [0, \infty)$ by the formula

$$\tilde{\varphi}(\mathbf{x}, t) = -\frac{c_2^2 t^2}{4\pi} \frac{1}{1+\nu} \int\limits_{|\xi| \le 1} \frac{dv(\xi)}{|\xi|} (\text{tr } \tilde{\mathbf{S}})(\mathbf{x} - c_2 t \xi, t(1 - |\xi|)) \tag{6.162}$$

and let $\tilde{\chi} = \tilde{\chi}(\mathbf{x}, t)$ be an extension of χ on $E^3 \times [0, \infty)$.

Then

$$\left.\begin{array}{ll} \tilde{\varphi}(\mathbf{x}, 0) = 0, & \dot{\tilde{\varphi}}(\mathbf{x}, 0) = 0 \\ \tilde{\chi}(\mathbf{x}, 0) = 0, & \dot{\tilde{\chi}}(\mathbf{x}, 0) = 0 \end{array}\right. \quad \text{on } E^3 \tag{6.163}$$

and

$$\Box_2^2 \, \tilde{\varphi} = -\frac{1}{1+\nu} (\text{tr } \tilde{\mathbf{S}}) \quad \text{on } E^3 \times [0, \infty) \tag{6.164}$$

Also, using Note 3, and applying the wave operator $\Box_1^2$ to Eq. (6.146), extended to $E^3 \times [0, \infty)$, we obtain

$$\begin{aligned} -2(1-\nu)\Box_1^2 \, \tilde{\chi} = {} & \left[\tilde{\mathbf{S}} - \frac{\nu}{1+\nu} (\text{tr } \tilde{\mathbf{S}}) \mathbf{1} \right] + \nabla\nabla \frac{c_2^2 t^2}{4\pi} \frac{1}{1+\nu} \\ & \times \int\limits_{|\eta| \le 1} \frac{dv(\eta)}{|\eta|} (\text{tr } \tilde{\mathbf{S}})(\mathbf{x} - c_2 t \eta, \; t(1 - |\eta|)) \end{aligned} \tag{6.165}$$

Taking the trace of (6.165) and using definition of $\tilde{\varphi}$ [see (6.162)] we get

$$-\,2(1-\nu)\Box_1^2 (\text{tr } \tilde{\chi}) = \frac{1-2\nu}{1+\nu} (\text{tr } \tilde{\mathbf{S}}) - \nabla^2 \tilde{\varphi} \tag{6.166}$$

Next, multiplying (6.164) by $(1 - 2\nu)$, and using the identity

$$(1-2\nu)\Box_2^2 = 2(1-\nu)\Box_1^2 - \nabla^2 \tag{6.167}$$

we obtain

$$2(1-\nu)\Box_1^2\tilde{\varphi} = -\frac{1-2\nu}{1+\nu}(\text{tr}\,\tilde{\mathbf{S}}) + \nabla^2\tilde{\varphi} \tag{6.168}$$

By addition of Eqs. (6.166) and (6.168) we get

$$\Box_1^2(\tilde{\varphi} - \text{tr}\,\tilde{\chi}) = 0 \tag{6.169}$$

Since, in view of (6.163),

$$(\tilde{\varphi} - \text{tr}\,\tilde{\chi})(\mathbf{x}, 0) = 0 \quad \text{on } E^3 \tag{6.170}$$

and

$$(\dot{\tilde{\varphi}} - \text{tr}\,\dot{\tilde{\chi}})(\mathbf{x}, 0) = 0 \quad \text{on } E^3 \tag{6.171}$$

therefore, it follows from (6.169) that

$$\tilde{\varphi} = \text{tr}\,\tilde{\chi} \quad \text{on } E^3 \times [0, \infty) \tag{6.172}$$

Substituting $\tilde{\varphi}$ from (6.172) into (6.162) and applying the operator $\nabla\nabla$ we get

$$\nabla\nabla(\text{tr}\,\tilde{\chi}) = -\frac{c_2^2 t^2}{4\pi}\,\frac{1}{1+\nu}\,\nabla\nabla \int\limits_{|\xi|\leq 1} \frac{dv(\xi)}{|\xi|}(\text{tr}\,\tilde{\mathbf{S}})(\mathbf{x} - c_2 t\xi, t(1-|\xi|)) \tag{6.173}$$

Also, substituting $\tilde{\varphi}$ from (6.172) into (6.164) we obtain

$$\Box_2^2(\text{tr}\,\tilde{\chi}) = -\frac{1}{1+\nu}(\text{tr}\,\tilde{\mathbf{S}}) \tag{6.174}$$

Since, because of (6.132),

$$(\text{tr}\,\tilde{\mathbf{E}}) = \frac{1}{2\mu}\,\frac{1-2\nu}{1+\nu}(\text{tr}\,\tilde{\mathbf{S}}) \tag{6.175}$$

Equation (6.174) is equivalent to (6.134). A restriction of (6.134) to $\bar{B} \times [0, \infty)$ leads to

$$\Box_2^2(\text{tr}\,\chi) = -\frac{1}{1+\nu}(\text{tr}\,\mathbf{S}) \tag{6.176}$$

This completes proof of Lemma 2.

Solution to Problem 6.7. We are to show that χ introduced by Lemma 2 [see Eq. (6.133)] satisfies Eqs. (6.109)–(6.111) of Problem 6.6.

By Lemma 2 [see also Eq. (6.176)]

$$\Box_2^2(\text{tr}\,\chi) = -\frac{1}{1+\nu}(\text{tr}\,\mathbf{S}) \tag{6.177}$$

An equivalent form to Eq. (6.177) reads

$$\operatorname{tr}\chi = -\frac{c_2^2 t^2}{4\pi}\frac{1}{1+\nu}\int\limits_{|\xi|\leq 1}\frac{dv(\xi)}{|\xi|}(\operatorname{tr}\mathbf{S})(\mathbf{x}-c_2 t\xi, t(1-|\xi|)) \tag{6.178}$$

By applying the operator $\nabla\nabla$ to this equation we obtain

$$\nabla\nabla(\operatorname{tr}\chi) = -\frac{c_2^2 t^2}{4\pi}\nabla\nabla\frac{1}{1+\nu}\int\limits_{|\xi|\leq 1}\frac{dv(\xi)}{|\xi|}(\operatorname{tr}\mathbf{S})(\mathbf{x}-c_2 t\xi,\ t(1-|\xi|)) \tag{6.179}$$

It follows from Eq. (6.165), restricted to $\bar{B}\times[0,\ \infty)$, and from Eq. (6.179) that

$$-2(1-\nu)\Box_1^2\chi = \mathbf{S} - \frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{S})\mathbf{1} - \nabla\nabla(\operatorname{tr}\chi) \tag{6.180}$$

Also, from Eqs. (6.177) and (6.180), we obtain

$$\mathbf{S} = \nabla\nabla(\operatorname{tr}\chi) - \mathbf{1}\nu\Box_2^2(\operatorname{tr}\chi) - 2(1-\nu)\Box_1^2\chi \tag{6.181}$$

Therefore, χ introduced by Lemma 2 satisfies Eq. (6.109) of Problem 6.6.

Next, applying the operator $\Box_1^2\Box_2^2$ to Eq. (6.146) we obtain

$$\begin{aligned}-2(1-\nu)\Box_1^2\Box_2^2\chi &= \Box_2^2\left(\mathbf{S} - \frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{S})\mathbf{1}\right) + \frac{1}{1+\nu}\nabla\nabla(\operatorname{tr}\mathbf{S}) \\ &= \frac{1}{1+\nu}\nabla\nabla(\operatorname{tr}\mathbf{S}) + \nabla^2\mathbf{S} - \frac{\nu}{1+\nu}\nabla^2(\operatorname{tr}\mathbf{S})\mathbf{1} \\ &\quad - \frac{1}{c_2^2}\left(\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\mathbf{1}\right)\end{aligned} \tag{6.182}$$

It follows from (6.145) that

$$\nabla^2\mathbf{S} + \frac{1}{1+\nu}\left[\nabla\nabla(\operatorname{tr}\mathbf{S}) - \nu\mathbf{1}\nabla^2(\operatorname{tr}\mathbf{S})\right] = 2\widehat{\nabla}(\operatorname{div}\mathbf{S}) \tag{6.183}$$

Therefore, Eq. (6.182) is reduced to

$$-2(1-\nu)\Box_1^2\Box_2^2\chi = 2\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \frac{1}{c_2^2}\left(\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\mathbf{1}\right) \tag{6.184}$$

and, since $\mathbf{S}$ is a solution to Eq. (6.127), we obtain

$$\Box_1^2\Box_2^2\chi = \frac{1}{1-\nu}\widehat{\nabla}\mathbf{b} \tag{6.185}$$

This shows that χ introduced by Lemma 2 satisfies Eq. (6.110) of Problem 6.6. Finally, introduce the notation

$$\psi = -2(1-\nu)\chi \tag{6.186}$$

then using Eqs. (6.146) and (6.178) we obtain

$$\begin{aligned}
\left(\nabla^2 - 2\widehat{\nabla}\text{div} + \nabla\nabla\text{tr}\right)\ \psi = {} & \frac{c_1^2 t^2}{4\pi} \int\limits_{|\xi|\le 1} \frac{dv(\xi)}{|\xi|}\Big\{\nabla^2\tilde{\mathbf{S}} - 2\widehat{\nabla}(\text{div}\,\tilde{\mathbf{S}}) + \frac{1}{1+\nu}\Big[\nabla\nabla(\text{tr}\,\tilde{\mathbf{S}}) \\
& -\nu\mathbf{1}\nabla^2(\text{tr}\,\tilde{\mathbf{S}})\Big]\Big\}\times(\mathbf{x}-c_1 t\xi,\, t(1-|\xi|)) \\
& - \frac{c_1^2 t^2}{4\pi}\frac{(1-2\nu)}{1+\nu}\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\nabla\nabla(\text{tr}\,\tilde{\mathbf{S}})(\mathbf{x}-c_1 t\xi,\, t(1-|\xi|)) \\
& - \nabla\nabla\nabla^2\left(\frac{c_1^2 c_2^2 t^4}{16\pi^2}\right)\frac{1}{1+\nu}\int\limits_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}(\text{tr}\,\tilde{\mathbf{S}}) \\
& \times(\mathbf{x} - c_1 t\xi - c_2 t\eta,\ t(1-|\xi|)(1-|\eta|)) \\
& + \nabla\nabla\frac{2(1-\nu)}{1+\nu}\frac{c_2^2 t^2}{4\pi}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}(\text{tr}\,\tilde{\mathbf{S}}) \\
& \times(\mathbf{x} - c_2 t\eta,\ t(1-|\eta|))
\end{aligned} \tag{6.187}$$

Since [see (6.167)]

$$\nabla^2 = 2(1-\nu)\Box_1^2 - (1-2\nu)\Box_2^2 \tag{6.188}$$

therefore

$$\begin{aligned}
& \nabla^2\left(\frac{c_1^2 c_2^2 t^4}{16\pi^4}\right)\int_{|\xi|\le 1}\frac{dv(\xi)}{|\xi|}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}(\text{tr}\,\tilde{\mathbf{S}})(\mathbf{x} - c_1 t\xi - c_2 t\eta,\ t(1-|\xi|)(1-|\eta|)) \\
& \quad = 2(1-\nu)\frac{c_2^2 t^2}{4\pi}\int\limits_{|\eta|\le 1}\frac{dv(\eta)}{|\eta|}(\text{tr}\,\tilde{\mathbf{S}})(\mathbf{x} - c_2 t\eta,\ t(1-|\eta|)) \\
& \qquad - (1-2\nu)\frac{c_1^2 t^2}{4\pi}\int\limits_{|\xi|<1}\frac{dv(\xi)}{|\xi|}(\text{tr}\,\tilde{\mathbf{S}})(\mathbf{x} - c_1 t\xi,\ t(1-|\xi|))
\end{aligned} \tag{6.189}$$

Substituting (6.189) into the RHS of (6.187) we obtain

$$\nabla^2\psi - 2\widehat{\nabla}(\text{div}\,\psi) + \nabla\nabla(\text{tr}\,\psi) = \frac{c_1^2 t^2}{4\pi}\int\limits_{|\xi|\leq 1}\frac{dv(\xi)}{|\xi|}\left\{\nabla^2\,\tilde{\mathbf{S}} - 2\widehat{\nabla}(\text{div}\,\tilde{\mathbf{S}}) + \frac{1}{1+\nu}\right.$$
$$\left.\times\,[\nabla\nabla(\text{tr}\,\tilde{\mathbf{S}}) - \nu\mathbf{1}\nabla^2(\text{tr}\,\tilde{\mathbf{S}})]\right\}$$
$$\times\,(\mathbf{x} - c_1 t\xi, t(1-|\xi|)) \tag{6.190}$$

Because of (6.136) the integrand on the RHS of (6.190) vanishes. Therefore, if follows from Eqs. (6.186) and (6.190) that

$$\nabla^2\chi - 2\widehat{\nabla}(\text{div}\,\chi) + \nabla\nabla(\text{tr}\,\chi) = 0 \tag{6.191}$$

This means that χ introduced by Lemma 2 satisfies Eq. (6.111) of Problem 6.6. Therefore, χ meets Eqs. (6.109)–(6.111), and a solution to Problem 6.7 is complete.

Problem 6.8. Consider the stress equation of motion

$$\widehat{\nabla}(\text{div}\,\mathbf{S}) - \frac{\rho}{2\mu}\left[\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\text{tr}\,\ddot{\mathbf{S}})\,\mathbf{1}\right] = -\widehat{\nabla}\mathbf{b} \quad \text{on} \quad B\times[0,\infty) \tag{6.192}$$

subject to the initial conditions

$$\mathbf{S}(\mathbf{x},0) = \mathbf{S}_0(\mathbf{x}), \quad \ddot{\mathbf{S}}(\mathbf{x},0) = \ddot{\mathbf{S}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in B \tag{6.193}$$

where $\mathbf{b}$, $\mathbf{S}_0$, and $\dot{\mathbf{S}}_0$ are prescribed functions. Define a scalar field $\alpha = \alpha(\mathbf{x}, t)$ and a vector field $\beta = \beta(\mathbf{x}, t)$ by

$$\alpha(\mathbf{x},t) = \frac{1}{4\pi c_1^2}\,\text{div}\int\limits_B \frac{\gamma(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}dv(\mathbf{y}) \tag{6.194}$$

and

$$\beta(\mathbf{x},t) = -\frac{1}{4\pi c_2^2}\,\text{curl}\int\limits_B \frac{\gamma(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}dv(\mathbf{y}) \tag{6.195}$$

where

$$\gamma(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) + \text{div}\,[S_0(\mathbf{x}) + t\,\dot{\mathbf{S}}_0(\mathbf{x})] \tag{6.196}$$

Let ϕ and ω satisfy the equations

$$\Box_1^2\phi = \alpha \quad \text{on} \quad B\times[0,\infty) \tag{6.197}$$

and

$$\Box_2^2\omega = \beta, \quad \text{div}\,\omega = 0 \quad \text{on} \quad B\times[0,\infty) \tag{6.198}$$

subject to the homogeneous initial conditions

$$\begin{matrix} \phi(\mathbf{x},0) = \dot{\phi}(\mathbf{x},0) = 0 \\ \omega(\mathbf{x},0) = \dot{\omega}(\mathbf{x},0) = 0 \end{matrix} \text{ on } B \tag{6.199}$$

Let

$$\mathbf{S}(\mathbf{x},t) = \mathbf{S}_0(\mathbf{x}) + t\,\dot{\mathbf{S}}_0(\mathbf{x}) + 2c_2^2\,[\nabla\nabla\phi + \widehat{\nabla}(\text{curl}\,\boldsymbol{\omega})] + \left(c_1^2 - 2c_2^2\right)\nabla^2\phi\,\mathbf{1} \tag{6.200}$$

Show that $\mathbf{S}$ satisfies Eqs. (6.192) and (6.193).

Note. The solution (i), in which ϕ and $\boldsymbol{\omega}$ satisfy Equations (6.197) through (6.199), represents a tensor solution of homogeneous isotropic elastodynamics of the Lame-type [see Eqs. (6.22)–(6.24)].

Solution. Let $\phi = \phi(\mathbf{x},t)$ and $\omega = \omega(\mathbf{x},t)$ satisfy the equations

$$\Box_1^2\phi = \alpha \quad \text{on } B \times [0,\infty) \tag{6.201}$$

$$\Box_2^2\omega = \beta, \quad \text{div}\,\omega = 0 \quad \text{on } B \times [0,\infty) \tag{6.202}$$

subject to the homogeneous initial conditions

$$\phi(\mathbf{x},0) = 0, \quad \dot{\phi}(\mathbf{x},0) = 0 \quad \text{on } B \tag{6.203}$$

$$\omega(\mathbf{x},0) = \mathbf{0}, \quad \dot{\omega}(\mathbf{x},0) = \mathbf{0} \quad \text{on } B \tag{6.204}$$

where α and β are defined by

$$\alpha(\mathbf{x},t) = +\frac{1}{4\pi c_1^2}\,\text{div} \int_B \frac{\gamma(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\,dv(\mathbf{y}) \tag{6.205}$$

$$\beta(\mathbf{x},t) = -\frac{1}{4\pi c_2^2}\,\text{curl} \int_B \frac{\gamma(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\,dv(\mathbf{y}) \tag{6.206}$$

and

$$\gamma(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) + \text{div}[\mathbf{S}_0(\mathbf{x}) + t\,\dot{\mathbf{S}}_0(\mathbf{x})] \tag{6.207}$$

Define $\mathbf{S} = \mathbf{S}(\mathbf{x},t)$ on $\bar{B} \times [0,\infty)$ by

$$\mathbf{S}(\mathbf{x},t) = \mathbf{S}_0(\mathbf{x}) + t\,\dot{\mathbf{S}}_0(\mathbf{x}) + 2c_2^2\,\widehat{\nabla}\,(\nabla\phi + \text{curl}\,\omega) + \left(c_1^2 - 2c_2^2\right)\nabla^2\phi\mathbf{1} \tag{6.208}$$

we are to show that

$$\widehat{\nabla}(\text{div}\,\mathbf{S}) - \frac{1}{2c_2^2}\left(\ddot{\mathbf{S}} - \frac{\nu}{1+\nu}(\text{tr}\,\ddot{\mathbf{S}})\mathbf{1}\right) = -\widehat{\nabla}\,\mathbf{b} \quad \text{on } B \times [0,\infty) \tag{6.209}$$

and

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}_0(\mathbf{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(\mathbf{x}) \quad \text{on } B. \tag{6.210}$$

To this end we note first that due to the homogeneous initial conditions (6.203) and (6.204), $\mathbf{S}$ given by (6.208) meets the nonhomogeneous initial conditions (6.210). To show that $\mathbf{S}$ satisfies the field Eq. (6.209) we make the following steps.

Applying the identity

$$\text{curl curl } \mathbf{u} = \nabla \text{ div } \mathbf{u} - \nabla^2 \mathbf{u} \tag{6.211}$$

to the field

$$\mathbf{u}(\mathbf{x}, t) = -\frac{1}{4\pi} \int_B \frac{\gamma(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|} dv(\mathbf{y}) \tag{6.212}$$

that satisfies Poisson's equation

$$\nabla^2 \mathbf{u} = \gamma \tag{6.213}$$

and using the definitions of α and β [see Eqs. (6.205) and (6.206)] we obtain

$$-c_1^2 \, \nabla\alpha - c_2^2 \text{ curl } \beta = \gamma \tag{6.214}$$

Next, by using the relations

$$\frac{1}{c_2^2} = \frac{\rho}{\mu}, \quad \frac{1}{c_1^2} = \frac{1}{c_2^2} \frac{1 - 2\nu}{2 - 2\nu} \tag{6.215}$$

and differentiating (6.208) twice with respect to time we obtain

$$\ddot{\mathbf{S}} = 2\, c_2^2 \left[\widehat{\nabla}(\nabla\, \ddot{\phi} + \text{curl } \ddot{\omega}) + \frac{\nu}{1 - 2\nu} \; \nabla^2 \ddot{\phi} \; \mathbf{1} \right] \tag{6.216}$$

Since

$$\text{tr } \widehat{\nabla} \; (\nabla \; \ddot{\phi}) = \nabla^2 \ddot{\phi}, \quad \text{tr } \widehat{\nabla}(\text{curl } \ddot{\omega}) = 0 \tag{6.217}$$

therefore, taking the trace of (6.216) we get

$$\text{tr } \ddot{\mathbf{S}} = c_2^2 \, \frac{2(1 + \nu)}{1 - 2\nu} \; \nabla^2 \ddot{\phi} \tag{6.218}$$

and it follows from (6.216) and (6.218) that

$$\frac{1}{2\mu} \left[\ddot{\mathbf{S}} - \frac{\nu}{1 + \nu} (\text{tr } \ddot{\mathbf{S}}) \mathbf{1} \right] = \frac{1}{\rho} \widehat{\nabla}(\nabla \ddot{\phi} + \text{curl } \ddot{\omega}) \tag{6.219}$$

In addition, if follows from (6.208) that

$$\operatorname{div}\mathbf{S}+\mathbf{b}=\operatorname{div}\left\{\mathbf{S}_0+t\,\dot{\mathbf{S}}_0+2c_2^2\,\widehat{\nabla}(\nabla\phi+\operatorname{curl}\omega)+(c_1^2-2c_2^2)\nabla^2\phi\mathbf{1}\right\}+\mathbf{b} \tag{6.220}$$

$$=\operatorname{div}(\mathbf{S}_0+t\,\dot{\mathbf{S}}_0)+\mathbf{b}(\mathbf{x},t)+\left[c_2^2\,\nabla^2+(c_1^2-c_2^2)\nabla\operatorname{div}\right](\nabla\phi+\operatorname{curl}\omega) \tag{6.221}$$

Hence, in view of (6.207) and (6.214) we get

$$\operatorname{div}\mathbf{S}+\mathbf{b}=-c_1^2\,\nabla\alpha-c_2^2\operatorname{curl}\beta+\nabla^2\,(c_1^2\nabla\,\phi+c_2^2\operatorname{curl}\omega) \tag{6.222}$$

Finally, applying the operator $\widehat{\nabla}$ to (6.222) we obtain

$$\begin{aligned}\widehat{\nabla}(\operatorname{div}\mathbf{S}+\mathbf{b})&=\widehat{\nabla}[c_1^2\nabla(\nabla^2\phi-\alpha)+c_2^2\operatorname{curl}(\nabla^2\omega-\beta)]\\&=\widehat{\nabla}\left[c_1^2\nabla\left(\Box_1^2\phi-\alpha+c_1^{-2}\ddot{\phi}\right)+c_2^2\operatorname{curl}\left(\Box_2^2\omega-\beta+c_2^{-2}\ddot{\boldsymbol{\omega}}\right)\right]\end{aligned} \tag{6.223}$$

Since ϕ and ω satisfy Eqs. (6.201) and (6.202), respectively, therefore dividing (6.223) by ρ and taking into account (6.219) we find that $\mathbf{S}=\mathbf{S}(\mathbf{x},t)$ satisfies the stress equation of motion in the form

$$\rho^{-1}\widehat{\nabla}\,(\operatorname{div}\mathbf{S}+\mathbf{b})-\frac{1}{2\mu}\left[\ddot{\mathbf{S}}-\frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\mathbf{1}\right]=0 \tag{6.224}$$

This completes solution to Problem 6.8.

Problem 6.9. Let $\mathbf{S}$ be a symmetric second-order tensor field on $B\times[0,\infty)$ that satisfies the stress equation of motion

$$\widehat{\nabla}(\operatorname{div}\mathbf{S})-\frac{\rho}{2\mu}\left[\ddot{\mathbf{S}}-\frac{\nu}{1+\nu}(\operatorname{tr}\ddot{\mathbf{S}})\,\mathbf{1}\right]=-\widehat{\nabla}\mathbf{b}\quad\text{on}\quad B\times[0,\infty) \tag{6.225}$$

subject to the initial conditions

$$\mathbf{S}(\mathbf{x},0)=\mathbf{S}_0(\mathbf{x}),\quad\dot{\mathbf{S}}(\mathbf{x},0)=\dot{\mathbf{S}}_0(\mathbf{x})\quad\text{for}\quad\mathbf{x}\in B \tag{6.226}$$

Show that there are a scalar field $\phi=\phi(\mathbf{x},t)$ and a vector field $\omega=\omega(\mathbf{x},t)$ such that

$$\mathbf{S}=\mathbf{S}_0+t\,\dot{\mathbf{S}}_0+2c_2^2\,\left[\nabla\nabla\phi+\widehat{\nabla}(\operatorname{curl}\omega)\right]+(c_1^2-2c_2^2)\nabla^2\phi\,\mathbf{1} \tag{6.227}$$

$$\Box_1^2\phi=\alpha,\quad\phi(\mathbf{x},0)=\dot{\phi}(\mathbf{x},0)=0 \tag{6.228}$$

$$\Box_2^2 \omega = \beta, \quad \operatorname{div} \omega = 0, \quad \omega(\mathbf{x}, 0) = \dot{\omega}(\mathbf{x}, 0) = \mathbf{0} \tag{6.229}$$

where the fields α and β are given by Eqs. (6.227) and (6.228), respectively, of Problem 6.8.

Note. Solution to Problem 6.9 implies that the tensor solution of Lame-type [Eqs. (6.227) through (6.229)] is complete.

Solution. In this problem the fields $\mathbf{S}_0$, $\dot{\mathbf{S}}_0$, and $\mathbf{b}$ are prescribed. Therefore, the fields α and β, given by Eqs. (6.234) and (6.235), respectively, of Problem 6.8 are given.

Let $\phi^{(0)}$ and $\omega^{(0)}$ be solutions to the equations [see Eqs. (6.230) and (6.231) of Problem 6.8]

$$\Box_1^2 \phi^{(0)} = \alpha \quad \text{on } B \times [0, \infty) \tag{6.230}$$

$$\Box_2^2 \omega^{(0)} = \beta, \quad \operatorname{div} \omega = 0 \quad \text{on } B \times [0, \infty) \tag{6.231}$$

subject to the homogeneous initial conditions

$$\phi^{(0)}(\mathbf{x}, 0) = 0, \qquad \dot{\phi}^{(0)}(\mathbf{x}, 0) = 0 \tag{6.232}$$

$$\omega^{(0)}(\mathbf{x}, 0) = \mathbf{0}, \qquad \dot{\omega}^{(0)}(\mathbf{x}, 0) = 0 \tag{6.233}$$

For example, $\phi^{(0)}$ and $\omega^{(0)}$ can be taken in the form of retarded potentials. Then, it follows from the solution to Problem 6.8 that the tensor field $\mathbf{S}^{(0)}$ defined by

$$\mathbf{S}^{(0)}(\mathbf{x}, t) = \mathbf{S}_0(\mathbf{x}) + t\dot{\mathbf{S}}_0(\mathbf{x}) + 2c_2^2 \widehat{\nabla} \left(\nabla \phi^{(0)} + \operatorname{curl} \omega^{(0)}\right) + \left(c_1^2 - 2c_2^2\right) \nabla^2 \phi^{(0)} \mathbf{1} \tag{6.234}$$

satisfies the equation

$$\widehat{\nabla} \left(\operatorname{div} \mathbf{S}^{(0)} + \mathbf{b}\right) - \frac{\rho}{2\mu} \left[\ddot{\mathbf{S}}^{(0)} - \frac{\nu}{1+\nu} (\operatorname{tr} \ddot{\mathbf{S}}^{(0)}) \mathbf{1}\right] = \mathbf{0} \tag{6.235}$$

In addition, because of (6.232) and (6.233)

$$\mathbf{S}^{(0)}(\mathbf{x}, 0) = \mathbf{S}_0(\mathbf{x}), \quad \dot{\mathbf{S}}^{(0)}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(x) \tag{6.236}$$

Introduce the notation

$$\mathbf{R} = \mathbf{S} - \mathbf{S}^{(0)} \tag{6.237}$$

where $\mathbf{S}$ satisfies Eqs. (6.225) and (6.226) of the Problem 6.9. Then $\mathbf{R}$ satisfies the equations

$$\widehat{\nabla} (\operatorname{div} \mathbf{R}) - \frac{\rho}{2\mu} \left[\ddot{\mathbf{R}} - \frac{\nu}{1+\nu} (\operatorname{tr} \ddot{\mathbf{R}}) \mathbf{1}\right] = \mathbf{0} \tag{6.238}$$

and

$$\mathbf{R}(\mathbf{x}, 0) = \mathbf{0}, \quad \dot{\mathbf{R}}(\mathbf{x}, 0) = \mathbf{0} \tag{6.239}$$

An equivalent form of (6.238) reads

$$\ddot{\mathbf{R}} = \rho^{-1}\left\{2\mu\widehat{\nabla}(\text{div}\,\mathbf{R}) + \frac{2\mu\nu}{1-2\nu} \times [\text{div}(\text{div}\,\mathbf{R})]\mathbf{1}\right\} \tag{6.240}$$

In components (6.240) reads

$$\ddot{R}_{ij} = \rho^{-1}\left\{\mu(R_{ik,kj} + R_{jk,ki}) + \frac{2\mu\nu}{1-2\nu}R_{ab,ab}\,\delta_{ij}\right\} \tag{6.241}$$

Hence

$$\ddot{R}_{ia,a} = \rho^{-1}\left\{\mu(R_{ik,kaa} + R_{ak,kia}) + \frac{2\mu\nu}{1-2\nu}R_{mn,mni}\right\} \tag{6.242}$$

Taking into account the relations

$$\nabla^2 = \nabla\,\text{div} - \text{curl curl} \tag{6.243}$$

$$c_2^2 = \frac{\mu}{\rho}, \quad c_1^2 = \frac{2(1-\nu)}{1-2\nu}c_2^2 \tag{6.244}$$

and using direct notation, we reduce (6.242) to the form

$$\text{div}\,\ddot{\mathbf{R}} = \left(c_1^2\,\nabla\,\text{div} - c_2^2\,\text{curl curl}\right)\,\text{div}\,\mathbf{R} \tag{6.245}$$

Let ξ and $\mathbf{r}$ be defined by

$$\xi = c_1^2\,\text{div div}\,\mathbf{R} * t \tag{6.246}$$

$$\mathbf{r} = -c_2^2\,\text{curl div}\,\mathbf{R} * t \tag{6.247}$$

where $*$ represents the convolution product. Then

$$\ddot{\xi} = c_1^2\,\text{div div}\,\mathbf{R} \tag{6.248}$$

$$\xi(\mathbf{x}, 0) = 0, \quad \dot{\xi}(\mathbf{x}, 0) = 0 \tag{6.249}$$

and

$$\ddot{\mathbf{r}} = -c_2^2\,\text{curl div}\,\mathbf{R} \tag{6.250}$$

$$\mathbf{r}(\mathbf{x}, 0) = 0, \quad \dot{\mathbf{r}}(\mathbf{x}, 0) = 0 \tag{6.251}$$

$$\text{div}\,\mathbf{r}(\mathbf{x}, t) = 0 \tag{6.252}$$

It follows from (6.239) and (6.245)–(6.251) that

$$\operatorname{div} \mathbf{R} = \nabla\xi + \operatorname{curl} \mathbf{r} \tag{6.253}$$

By taking the div operator on (6.253) we obtain

$$\operatorname{div}\operatorname{div} \mathbf{R} = \nabla^2\xi \tag{6.254}$$

and applying the curl to (6.253) we get

$$\operatorname{curl}\operatorname{div} \mathbf{R} = \operatorname{curl}\operatorname{curl} \mathbf{r} \tag{6.255}$$

Hence, in view of (6.243) and (6.252)

$$\operatorname{curl}\operatorname{div} \mathbf{R} = -\nabla^2\mathbf{r} \tag{6.256}$$

Also, because of Eqs. (6.248) and (6.254) we obtain

$$\Box_1^2\xi = 0 \tag{6.257}$$

and using Eqs. (6.250) and (6.256) we obtain

$$\Box_2^2\mathbf{r} = \mathbf{0} \tag{6.258}$$

Substituting div $\mathbf{R}$ from (6.253) into (6.240) we obtain

$$\ddot{\mathbf{R}} = \frac{\mu}{\rho}\left\{2\widehat{\nabla} + \mathbf{1}\frac{2\nu}{1-2\nu}\operatorname{div}\right\}(\nabla\xi + \operatorname{curl}\mathbf{r}) \tag{6.259}$$

Let

$$\phi^{(1)} = \xi * t, \quad \omega^{(1)} = \mathbf{r} * t \tag{6.260}$$

Then

$$\ddot{\phi}^{(1)} = \xi, \quad \phi^{(1)}(\mathbf{x},0) = \dot{\phi}^{(1)}(\mathbf{x},0) = 0 \tag{6.261}$$

and

$$\ddot{\omega}^{(1)} = \mathbf{r}, \quad \omega^{(1)}(\mathbf{x},0) = \dot{\omega}^{(1)}(\mathbf{x},0) = \mathbf{0}, \quad \operatorname{div}\omega^{(1)} = 0 \tag{6.262}$$

Integrating (6.259) twice with respect to time and taking into account the homogeneous initial conditions (6.239), (6.261) and (6.262) we obtain

$$\mathbf{R} = \rho^{-1}\mu\left(2\widehat{\nabla} + \mathbf{1}\frac{2\nu}{1-2\nu}\operatorname{div}\right)\left(\nabla\phi^{(1)} + \operatorname{curl}\omega^{(1)}\right) \tag{6.263}$$

where, because of (6.257) and (6.258), $(6.261)_1$ and $(6.262)_1$,

$$\square_1^2 \ddot{\phi}^{(1)} = 0 \tag{6.264}$$

and

$$\square_2^2 \ddot{\omega}^{(1)} = \mathbf{0} \tag{6.265}$$

Since, in view of (6.249), (6.251), $(6.261)_2$ and $(6.262)_2$

$$\phi^{(1)}(\mathbf{x},0) = \dot{\phi}^{(1)}(\mathbf{x},0) = \ddot{\phi}^{(1)}(\mathbf{x},0) = \dddot{\phi}^{(1)}(\mathbf{x},0) = 0 \tag{6.266}$$

$$\omega^{(1)}(\mathbf{x},0) = \dot{\omega}^{(1)}(\mathbf{x},0) = \ddot{\omega}^{(1)}(\mathbf{x},0) = \dddot{\omega}^{(1)}(\mathbf{x},0) = 0 \tag{6.267}$$

therefore, integrating twice Eqs. (6.264) and (6.265), with respect to time, we obtain

$$\square_1^2\, \phi^{(1)} = 0 \tag{6.268}$$

and

$$\square_1^2\, \omega^{(1)} = \mathbf{0} \tag{6.269}$$

Also, note that an alternative form of (6.263) reads

$$\mathbf{R} = 2c_2^2 \widehat{\nabla} \left(\nabla \phi^{(1)} + \operatorname{curl} \omega^{(1)} \right) + \left(c_1^2 - 2c_2^2 \right) \left(\nabla^2 \phi^{(1)} \right) \mathbf{1} \tag{6.270}$$

Therefore, because of (6.234), (6.237), and (6.270)

$$\mathbf{S} = \mathbf{S}_0 + \mathbf{R} = \mathbf{S}_0(\mathbf{x}) + t\dot{\mathbf{S}}_0(\mathbf{x}) + 2c_2^2 \widehat{\nabla} \left[\nabla \left(\phi^{(0)} + \phi^{(1)} \right) + \operatorname{curl} \left(\omega^{(0)} + \omega^{(1)} \right) \right] + \left(c_1^2 - 2c_2^2 \right) \left[\nabla^2 \left(\phi^{(0)} + \phi^{(1)} \right) \right] \mathbf{1} \tag{6.271}$$

The fields ϕ and ω are defined by

$$\phi = \phi^{(0)} + \phi^{(1)}, \quad \omega = \omega^{(0)} + \omega^{(1)} \tag{6.272}$$

where $(\phi^{(0)},\ \omega^{(0)})$ and $(\phi^{(1)},\ \omega^{(1)})$ have been defined before.

If the definition (6.272) of ϕ and ω is taken into account, Eq. (6.271) reduces to Eq. (6.225) of Problem 6.9.

Finally, if we note that the pair $(\phi^{(0)},\ \omega^{(0)})$ satisfies Eqs. (6.230)–(6.233) and the pair $(\phi^{(1)},\ \omega^{(1)})$ satisfies Eqs. (6.261)–(6.262), and (6.268)–(6.269), we find that the pair $(\phi,\ \omega)$ meets Eqs. (6.228)–(6.229). This completes a solution to Problem 6.9.

Problem 6.10. Consider the stress equation of motion in the form

$$\widehat{\nabla}(\operatorname{div} \mathbf{S}) - \rho\, \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \quad \text{on} \quad B \times [0, \infty) \tag{6.273}$$

subject to the homogeneous initial conditions

$$\mathbf{S}(\mathbf{x},0)=\mathbf{0},\quad \dot{\mathbf{S}}(\mathbf{x},\mathbf{0})=\mathbf{0}\quad \text{for}\quad \mathbf{x}\in B \tag{6.274}$$

where

$$\mathsf{K}[\mathbf{S}]=\frac{1}{2\mu}\left[\mathbf{S}-\frac{\nu}{1+\nu}(\operatorname{tr}\mathbf{S})\,\mathbf{1}\right] \tag{6.275}$$

and $\mathbf{B}=\mathbf{B}(\mathbf{x},t)$ is an arbitrary symmetric second-order tensor field on $B\times[0,\infty)$. Define a vector field $\mathbf{v}=\mathbf{v}(\mathbf{x},t)$ by

$$\mathbf{v}(\mathbf{x},t)=-\frac{c_2^2t^2}{4\pi}\int\limits_{|\xi|\le 1}\frac{\mathbf{f}[\mathbf{x}-c_2t\xi,\ (1-|\xi|)t]}{|\xi|}\,dv(\xi) \tag{6.276}$$

where

$$\mathbf{f}(\mathbf{x},t)=\left\{\left(\frac{c_1^2}{c_2^2}-1\right)\nabla g+\frac{1}{\rho\,c_2^2}\operatorname{div}\mathsf{K}^{-1}[\mathbf{B}]\right\}(\mathbf{x},t) \tag{6.277}$$

$$\mathrm{g}(\mathbf{x},t)=-\frac{c_1^2t^2}{4\pi}\int\limits_{|\xi|\le 1}\frac{\mathrm{h}[\mathbf{x}-c_1t\xi,\ (1-|\xi|)t]}{|\xi|}\,dv(\xi) \tag{6.278}$$

and

$$\mathrm{h}(\mathbf{x},t)=\frac{1}{\rho\,c_1^2}\operatorname{div}\operatorname{div}\mathsf{K}^{-1}[\mathbf{B}](\mathbf{x},t) \tag{6.279}$$

Let

$$\mathbf{S}(\mathbf{x},t)=\frac{1}{\rho}\mathsf{K}^{-1}[\widehat{\nabla}\mathbf{v}+\mathbf{B}]*t \tag{6.280}$$

Show that $\mathbf{S}$ satisfies Eqs. (6.273) and (6.274).
Hint. Use the result of Problem 6.4 that the function

$$\varphi(\mathbf{x},t)=-\frac{c^2t^2}{4\pi}\int\limits_{|\xi|\le 1}\frac{F[\mathbf{x}-ct\xi,(1-|\xi|)t]}{|\xi|}\,dv(\xi)\quad \text{on}\quad E^3\times[0,\infty) \tag{6.281}$$

satisfies the inhomogeneous wave equation

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi=-F\quad \text{on}\quad E^3\times[0,\infty) \tag{6.282}$$

subject to the homogeneous initial conditions

$$\varphi(\mathbf{x},0)=\dot{\varphi}(\mathbf{x},0)=0 \tag{6.283}$$

Solution. To show that $\mathbf{S}$ given by

$$\mathbf{S}(\mathbf{x}, t) = \rho^{-1}\mathsf{K}^{-1}[\widehat{\nabla}\mathbf{v} + \mathbf{B}] * t \tag{6.284}$$

satisfies Eqs. (6.273)–(6.274), we note that

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{0}, \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \mathbf{0} \tag{6.285}$$

Hence, **S** given by (6.284), satisfies Eqs. (6.274). To show that **S** given by (6.284) satisfies (6.273) we substitute **S** given by (6.284) to (6.273) and obtain

$$\widehat{\nabla}(\text{div }\mathbf{S}) - \rho\mathsf{K}[\ddot{\mathbf{S}}] = \widehat{\nabla}(\text{div }\mathbf{S}) - \widehat{\nabla}\mathbf{v} - \mathbf{B} = -\mathbf{B} \tag{6.286}$$

In the following we prove that

$$\mathbf{v} = \text{div }\mathbf{S} \tag{6.287}$$

This implies that **S** given by (6.284) meets (6.273). To this end we note that from Eqs. (6.276)–(6.277) we obtain

$$\Box_2^2\mathbf{v} = -\mathbf{f} = -\left\{\left(\frac{c_1^2}{c_2^2} - 1\right)\nabla g + \frac{1}{\rho c_2^2}\text{div }\mathsf{K}^{-1}[\mathbf{B}]\right\} \tag{6.288}$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{0}, \quad \dot{\mathbf{v}}(\mathbf{x}, 0) = \mathbf{0} \tag{6.289}$$

Also, Eqs. (6.278)–(6.280) imply that

$$\Box_1^2 g = -h = -\frac{1}{\rho c_1^2}\text{div div }\mathsf{K}^{-1}[\mathbf{B}] \tag{6.290}$$

$$g(\mathbf{x}, 0) = \dot{g}(\mathbf{x}, 0) = 0 \tag{6.291}$$

By taking the div operator of (6.288) we get

$$\Box_2^2\text{div }\mathbf{v} = -\left\{\left(\frac{c_1^2}{c_2^2} - 1\right)\nabla^2 g + \frac{1}{\rho c_2^2}\text{ div div }\mathsf{K}^{-1}[\mathbf{B}]\right\} \tag{6.292}$$

By eliminating div div $\mathsf{K}^{-1}[\mathbf{B}]$ from Eqs. (6.290) and (6.292), we obtain

$$\begin{aligned}\Box_2^2(\text{div }\mathbf{v}) &= -\left\{\left(\frac{c_1^2}{c_2^2} - 1\right)\nabla^2 g - \frac{c_1^2}{c_2^2}\Box_1^2 g\right\} \\ &= -\left\{-\nabla^2 g + \frac{1}{c_2^2}\ddot{g}\right\} = +\Box_2^2 g\end{aligned} \tag{6.293}$$

Hence

$$\Box_2^2(\text{div }\mathbf{v} - g) = 0 \tag{6.294}$$

Since, $\mathbf{v}$ and g satisfy the homogeneous initial conditions (6.289) and (6.291), respectively, Eq. (6.294) implies that

$$\text{div}\ \mathbf{v} = g \tag{6.295}$$

Substituting g from (6.295) into the RHS of (6.288), we obtain

$$\ddot{\mathbf{v}} = c_2^2 \nabla^2 \mathbf{v} + \left(c_1^2 - c_2^2\right) \nabla \text{div}\ \mathbf{v} + \rho^{-1} \text{div}\ \mathsf{K}^{-1}[\mathbf{B}] \tag{6.296}$$

Since

$$\nabla^2 \mathbf{v} = 2\,\text{div}\ (\widehat{\nabla} \mathbf{v}) - \nabla \text{div}\ \mathbf{v} \tag{6.297}$$

and

$$\nabla \text{div}\ \mathbf{v} = \text{div}\ [\mathbf{1}\ \text{tr}\ (\widehat{\nabla} \mathbf{v})] \tag{6.298}$$

therefore, Eq. (6.296) can be written as

$$\ddot{\mathbf{v}} = \text{div} \left\{2c_2^2 (\widehat{\nabla} \mathbf{v}) + \left(c_1^2 - 2c_2^2\right) \mathbf{1}\ \text{tr}\ (\widehat{\nabla} \mathbf{v}) + \rho^{-1} \mathsf{K}^{-1}[\mathbf{B}]\right\} \tag{6.299}$$

or, in view of (6.284),

$$\ddot{\mathbf{v}} = \text{div}\ \ddot{\mathbf{S}} \tag{6.300}$$

Integrating (6.300) with respect to time twice, and using the homogeneous initial conditions for $\mathbf{v}$ and $\mathbf{S}$, given by Eqs. (6.285) and (6.289), respectively, we arrive at Eq. (6.287). This completes a solution to Problem 6.10.

Note that the solution to Problem 6.10 provides an effective solution of the incompatible elastodynamics when $\mathbf{B}$ represents a space-time distribution of defects on $B \times [0, \infty)$.