Chapter 5
Variational Principles of Elastodynamics

In this chapter both the classical Hamilton-Kirchhoff Principle and a convolutional variational principle of Gurtin's type that describes completely a solution to an initial-boundary value problem of elastodynamics are used to solve a number of typical problems of elastodynamics.

5.1 The Hamilton-Kirchhoff Principle

To formulate H-K principle we introduce a notion of *kinematically admissible process,* and by this we mean an admissible process that satisfies the strain-displacement relation, the stress-strain relation, and the displacement boundary condition.

(H-K) **The Hamilton-Kirchhoff Principle.** Let P denote the set of all kinematically admissible processes $p = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ on $\overline{\mathrm{B}} \times [0, \infty)$ satisfying the conditions

$$\mathbf{u}(\mathbf{x}, \mathrm{t}_1) = \mathbf{u}_1(\mathbf{x}), \quad \mathbf{u}(\mathbf{x}, \mathrm{t}_2) = \mathbf{u}_2(\mathbf{x}) \quad \text{on} \quad \overline{\mathrm{B}} \tag{5.1}$$

where t_1 and t_2 are two arbitrary points on the t-axis such that $0 \leq t_1 < t_2 < \infty$, and $\mathbf{u}_1(\mathbf{x})$ and $\mathbf{u}_2(\mathbf{x})$ are prescribed fields on $\overline{\mathrm{B}}$. Let $\mathsf{K} = \mathsf{K}\{p\}$ be the functional on P defined by

$$\mathsf{K}\{p\} = \int_{t_1}^{t_2} [\mathrm{F}(t) - \mathrm{K}(t)]\, dt \tag{5.2}$$

where

$$F(t) = \mathrm{U_C}\{\mathbf{E}\} - \int_{\mathrm{B}} \mathbf{b} \cdot \mathbf{u}\, dv - \int_{\partial \mathrm{B}_2} \hat{\mathbf{s}} \cdot \mathbf{u}\, da \tag{5.3}$$

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and

$$K(t) = \frac{1}{2} \int_{B} \rho\, \dot{\mathbf{u}}^2 dv \tag{5.4}$$

for every $p = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in P$. Then

$$\delta\, \mathsf{K}\{p\} = 0 \tag{5.5}$$

if and only if p satisfies the equation of motion and the traction boundary condition.

Clearly, in the (H-K) principle a displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathrm{t})$ needs to be prescribed at two points t_1 and t_2 of the time axis. If $t_1 = 0$, then $\mathbf{u}(\mathbf{x}, 0)$ may be identified with the initial value of the displacement vector in the formulation of an initial-boundary value problem, however, the value $\mathbf{u}(\mathbf{x}, \mathrm{t}_2)$ is not available in this formulation. This is the reason why the (H-K) principle can not be used to describe the initial-boundary value problem. A full variational characterization of an initial-boundary value problem of elastodynamics is due to Gurtin, and it has the form of a convolutional variational principle. The idea of a convolutional variational principle of elastodynamics is now explained using a traction initial-boundary value problem of incompatible elastodynamics. In such a problem we are to find a symmetric second-order tensor field $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ on $\overline{\mathrm{B}} \times [0, \infty)$ that satisfies the field equation

$$\hat{\nabla}[\rho^{-1}(\operatorname{div}\mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{5.6}$$

subject to the initial conditions

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}_0(\mathbf{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathrm{B} \tag{5.7}$$

and the boundary condition

$$\mathbf{s} = \mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on} \quad \partial\mathrm{B} \times [0, \infty) \tag{5.8}$$

Here $\mathbf{S}_0$ and $\dot{\mathbf{S}}_0$ are arbitrary symmetric tensor fields on B, and $\mathbf{B}$ is a prescribed symmetric second-order tensor field on $\overline{\mathrm{B}} \times [0, \infty)$. Moreover, ρ, $\mathbf{K}$, and $\hat{\mathbf{s}}$ have the same meaning as in classical elastodynamics.

First, we note that the problem is equivalent to the following one. Find a symmetric second-order tensor field on $\overline{\mathrm{B}} \times [0, \infty)$ that satisfies the integro-differential equation

$$\hat{\nabla}[\rho^{-1} t * (\operatorname{div}\mathbf{S})] - \mathbf{K}[\mathbf{S}] = -\hat{\mathbf{B}} \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{5.9}$$

subject to the boundary condition

$$\mathbf{s} = \mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on} \quad \partial\mathrm{B} \times [0, \infty) \tag{5.10}$$

where

$$\hat{\mathbf{B}} = t * \mathbf{B} + \mathbf{K}[\mathbf{S}_0 + \mathrm{t}\,\dot{\mathbf{S}}_0] \tag{5.11}$$

and $*$ stands for the convolution product, that is, for any two scalar functions $a = a(\mathbf{x}, t)$ and $b = b(\mathbf{x}, t)$

$$(a * b)(\mathbf{x}, t) = \int_0^t a(\mathbf{x}, \tau) b(\mathbf{x}, t - \tau)\, d\tau \tag{5.12}$$

Next, the convolutional variational principle is formulated for the problem described by Eqs. (5.9)–(5.10).

Principle of Incompatible Elastodynamics. Let N denote the set of all symmetric second-order tensor fields $\mathbf{S}$ on $\overline{\mathrm{B}} \times [0, \infty)$ that satisfy the traction boundary condition (5.8) $\equiv$ (5.10). Let $\mathrm{I}_t\{\mathbf{S}\}$ be the functional on N defined by

$$\mathrm{I}_t\{\mathbf{S}\} = \frac{1}{2} \int_{\mathrm{B}} \{ \rho^{-1} t * (\operatorname{div} \mathbf{S}) * (\operatorname{div} \mathbf{S}) + \mathbf{S} * \mathbf{K}[\mathbf{S}] - 2\,\mathbf{S} * \hat{\mathbf{B}} \}\, dv \tag{5.13}$$

Then

$$\delta\, \mathrm{I_t}\{\mathbf{S}\} = 0 \tag{5.14}$$

at a particular $\mathbf{S} \in \mathrm{N}$ if and only if $\mathbf{S}$ is a solution to the traction problem described by Eqs. (5.6)–(5.8).

Note. When the fields $\mathbf{B}$, $\mathbf{S}_0$, and $\dot{\mathbf{S}}_0$ are arbitrarily prescribed, the principle of incompatible elastodynamics may be useful in a study of elastic waves in bodies with various types of defects.

5.2 Problems and Solutions Related to Variational Principles of Elastodynamics

Problem 5.1. A symmetrical elastic beam of flexural rigidity EI, density ρ, and length L, is acted upon by: (i) the transverse force $F = F(x_1, t)$, (ii) the end shear forces V_0 and V_L, and (iii) the end bending moments M_0 and M_L shown in Fig. 5.1. The strain energy of the beam is

$$F(t) = \frac{1}{2} \int_0^L E\, I (u_2'')^2\, dx_1 \tag{5.15}$$

the kinetic energy of the beam is

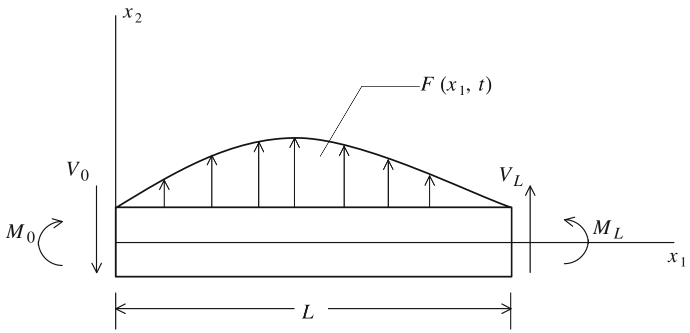

Fig. 5.1 The symmetrical beam

$$K(t) = \frac{1}{2}\int_0^L \rho\,(\dot{u}_2)^2\,dx_1 \tag{5.16}$$

and the energy of external forces is

$$\begin{aligned} V(t) = -\int_0^L F(x_1,t)\,u_2(x_1,t)\,dx_1 + V_0 u_2(0,t) \\ +\ M_0 u_2'(0,t) - V_L u_2(L,t) - M_L u_2'(L,t) \end{aligned} \tag{5.17}$$

where the prime denotes differentiation with respect to x_1. Let U be the set of functions $u_2 = u_2(x_1, t)$ that satisfies the conditions

$$u_2(x_1,t_1) = u(x_1), \quad u_2(x_1,t_2) = v(x_1) \tag{5.18}$$

where t_1 and t_2 are two arbitrary points on the t-axis such that $0 \le t_1 < t_2 < \infty$, and $u(x_1)$ and $v(x_1)$ are prescribed fields on $[0, L]$. Define a functional $\hat{K}\{u_2\}$ on U by

$$\hat{K}\{u_2\} = \int_{t_1}^{t_2} [F(t) + V(t) - K(t)]\,dt \tag{5.19}$$

Show that

$$\delta\hat{K}\{u_2\} = 0 \tag{5.20}$$

if and only if u_2 satisfies the equation of motion

$$(EIu_2'')'' + \rho\,\ddot{u}_2 = F \quad \text{on} \quad [0,L]\times[0,\infty) \tag{5.21}$$

and the boundary conditions

$$[(EI\,u_2'')']\,(0,t) = -V_0 \quad \text{on} \quad [0,\infty) \tag{5.22}$$

$$[(EI\,u_2'')]\,(0,t) = M_0 \quad \text{on} \quad [0,\infty) \tag{5.23}$$

$$[(EI\,u_2'')']\,(L,t) = -V_L \quad \text{on} \quad [0,\infty) \tag{5.24}$$

$$[(EI\,u_2'')]\,(L,t) = M_L \quad \text{on} \quad [0,\infty) \tag{5.25}$$

The field equaton (5.21) and the boundary conditions (5.22) through (5.25) describe flexural waves in the beam.

Solution. Introduce the notation

$$u_2(x_1,t) \equiv u(x,t) \tag{5.26}$$

Then the functional $\hat{K}\{u_2\}$ takes the form

$$\hat{K}\{u\} = \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^L dx\,[EI(u'')^2 - \rho(\dot{u})^2] + \int_{t_1}^{t_2}\left\{-\int_0^L Fu\,dx + V_0u(0,t) + M_0u'(0,t) - V_Lu(L,t) - M_Lu'(L,t)\right\}dt \tag{5.27}$$

Let $u \in U$ and $u + \omega\tilde{u} \in U$. Then

$$\tilde{u}(x,t_1) = \tilde{u}(x,t_2) = 0 \qquad x \in [0,L] \tag{5.28}$$

Computing $\delta\hat{K}\{u\}$ we obtain

$$\delta\hat{K}\{u\} = \frac{d}{d\omega}\,\hat{K}\{u+\omega\tilde{u}\}\bigg|_{\omega=0} = \int_{t_1}^{t_2} dt \int_0^L dx\,[EIu''\,\tilde{u}'' - \rho\,\dot{u}\,\dot{\tilde{u}}] + \int_{t_1}^{t_2} dt\left\{-\int_0^L F\tilde{u}\,dx + V_0\tilde{u}(0,t) + M_0\tilde{u}'(0,t) - V_L\tilde{u}(L,t) - M_L\tilde{u}'(L,t)\right\} \tag{5.29}$$

Next, note that integrating by parts we obtain

$$\int_0^L dx(EIu''\tilde{u}'') = (EIu'')\tilde{u}'\big|_{x=0}^{x=L} - \int_0^L dx(EIu'')'\tilde{u}'$$
$$= (EIu'')\tilde{u}'\big|_{x=0}^{x=L} - (EIu'')'\tilde{u}\big|_{x=0}^{x=L} + \int_0^L dx(EIu'')''\tilde{u} \quad (5.30)$$

and

$$-\int_{t_1}^{t_2} \rho\dot{u}\dot{\tilde{u}}\,dt = -\rho\dot{u}\tilde{u}\Big|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \rho\ddot{u}\tilde{u}\,dt \quad (5.31)$$

Hence, using the homogeneous conditions (5.28) we reduce (5.29) to the form

$$\delta\hat{K}\{u\} = \int_{t_1}^{t_2} dt \int_0^L dx\,[(EIu'')'' + \rho\ddot{u} - F]\tilde{u}(x,t)$$
$$+ \int_{t_1}^{t_2} dt\{[V_0 + (EIu'')'(0,t)]\tilde{u}(0,t) - [V_L + (EIu'')'(L,t)]\tilde{u}(L,t)$$
$$+ [M_0 - (EIu'')(0,t)]\tilde{u}'(0,t) - [M_L - (EIu'')(L,t)]\tilde{u}'(L,t)\} \quad (5.32)$$

Now, if $u = u(x,t)$ satisfies (5.21)–(5.25) then $\delta\hat{K}\{u\} = 0$. Conversely, if $\delta\hat{K}\{u\} = 0$ then selecting $\tilde{u} = \tilde{u}(x,t)$ in such a way that $\tilde{u} = \tilde{u}(x,t)$ is an arbitrary smooth function on $[0,L] \times [t_1,t_2]$ and such that $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$ on $[t_1,t_2]$ and $\tilde{u}'(0,t) = \tilde{u}'(L,t) = 0$ on $[t_1,t_2]$, from Eq. (5.32) we obtain

$$\int_{t_1}^{t_2}\int_0^L [(EIu'')'' + \rho\ddot{u} - F]\tilde{u}\,dtdx = 0 \quad (5.33)$$

and by the Fundamental Lemma of the calculus of variations we obtain

$$(EIu'')'' + \rho\ddot{u} = F \quad (5.34)$$

Next, by selecting $\tilde{u} = \tilde{u}(x,t)$ in such a way that $\tilde{u}$ is an arbitrary smooth function on $[0,L]\times[t_1,t_2]$ that complies with the conditions $\tilde{u}(0,t) \neq 0$ on $[t_1,t_2]$, $\tilde{u}(L,t) = 0$, $\tilde{u}'(0,t) = \tilde{u}'(L,t) = 0$ on $[t_1,t_2]$, and by using (5.32) and (5.34), we obtain

$$\int_{t_1}^{t_2} [V_0 + (EIu'')'(0,t)]\tilde{u}(0,t) = 0 \quad (5.35)$$

This together with the Fundamental Lemma of calculus of variations yields

$$(EIu'')'(0,t) = -V_0 \tag{5.36}$$

Next, by selecting $\tilde{u}$ to be an arbitrary smooth function on $[0, L] \times [t_1, t_2]$ that satisfies the conditions $\tilde{u}(L,t) \neq 0$ on $[t_1, t_2]$, $\tilde{u}'(0,t) = 0$, and $\tilde{u}'(L,t) = 0$ on $[t_1, t_2]$, we find from Eqs. (5.34), (5.36), and (5.32) that

$$\int_{t_1}^{t_2} [V_L + (EIu'')'(L,t)]\tilde{u}(L,t)dt = 0 \tag{5.37}$$

Equation (5.37) together with the Fundamental Lemma of calculus of variations imply that

$$(EIu'')'(L,t) = -V_L \tag{5.38}$$

Next, by selecting $\tilde{u}$ to be an arbitrary smooth function on $[0, L] \times [t_1, t_2]$ that meets the conditions $\tilde{u}'(0,t) \neq 0$ on $[t_1, t_2]$, and $\tilde{u}'(L,t) = 0$ on $[t_1, t_2]$, by virtue of Eqs. (5.34), (5.36), (5.38), and (5.32), we obtain

$$\int_{t_1}^{t_2} [M_0 - (EIu'')(0,t)]\tilde{u}'(0,t)dt = 0 \tag{5.39}$$

This together with the Fundamental Lemma of calculus of variations yields

$$(EIu'')(0,t) = M_0 \tag{5.40}$$

Finally, by letting $\tilde{u}$ to be an arbitrary smooth function on $[0, L] \times [t_1, t_2]$ and such that $\tilde{u}'(L,t) \neq 0$, from Eqs. (5.34), (5.36), (5.38), (5.40), and (5.32) we obtain

$$\int_{t_1}^{t_2} [M_L - (EIu'')(L,t)]\tilde{u}'(L,t) = 0 \tag{5.41}$$

Equation (5.41) together with the Fundamental Lemma of calculus of variations yields

$$(EIu'')(L,t) = M_L \tag{5.42}$$

This completes a solution to Problem 5.1.

Problem 5.2. A thin elastic membrane of uniform area density $\hat{\rho}$ is stretched to a uniform tension $\hat{T}$ over a region C_0 of the x_1, x_2 plane. The membrane is subject to a vertical load $f = f(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$ and the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \dot{u}(\mathbf{x}, 0) = \dot{u}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in C_0$$

where $u = u(\mathbf{x}, t)$ is a vertical deflection of the membrane on $\overline{C}_0 \times [0, \infty)$, and $u_0(\mathbf{x})$ and $\dot{u}_0(\mathbf{x})$ are prescribed functions on C_0. Also, $u = u(\mathbf{x}, t)$ on $\partial C_0 \times [0, \infty)$ is represented by a given function $g = g(\mathbf{x}, t)$. The strain energy of the membrane is

$$F(t) = \frac{\hat{T}}{2} \int\limits_{C_0} u_{,\alpha} u_{,\alpha} \, da \tag{5.43}$$

The kinetic energy of the membrane is

$$K(t) = \frac{\hat{\rho}}{2} \int\limits_{C_0} (\dot{u})^2 \, da \tag{5.44}$$

The external load energy is

$$V(t) = -\int\limits_{C_0} f \, u \, da \tag{5.45}$$

Let U be the set of functions $u = u(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$ that satisfy the conditions

$$u(\mathbf{x}, t_1) = a(\mathbf{x}), \quad u(\mathbf{x}, t_2) = b(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in C_0 \tag{5.46}$$

and

$$u(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.47}$$

where t_1 and t_2 have the same meaning as in Problem 5.1, and $a(\mathbf{x})$ and $b(\mathbf{x})$ are prescribed functions on C_0. Define a functional $\hat{K}\{.\}$ on U by

$$\hat{K}\{u\} = \int\limits_{t_1}^{t_2} [F(t) + V(t) - K(t)] \, dt \tag{5.48}$$

Show that the condition

$$\delta \, \hat{K}\{u\} = 0 \quad \text{on} \quad U \tag{5.49}$$

implies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u = -\frac{f}{\hat{T}} \quad \text{on} \quad C_0 \times [0, \infty) \tag{5.50}$$

where

$$c = \sqrt{\frac{\hat{T}}{\hat{\rho}}} \tag{5.51}$$

Note that $[\hat{T}] = [\text{Force} \times L^{-1}],\ [\hat{\rho}] = [\text{Density} \times L],\ [c] = [LT^{-1}]$, where L and T are the length and time units, respectively.

Solution. The functional $\hat{K} = \hat{K}\{u\}$ takes the form

$$\hat{K}\{u\} = \int_{t_1}^{t_2} dt \int_{C_0} \left(\frac{\widehat{T}}{2} u_{,\alpha}\, u_{,\alpha} - \frac{\hat{\rho}}{2} \dot{u}^2 - f u \right) da$$
$$\text{for every} \qquad u \in U \tag{5.52}$$

Let $u \in U$ and $u + \omega\tilde{u} \in U$. Then

$$\tilde{u}(\mathbf{x}, t_1) = \tilde{u}(\mathbf{x}, t_2) = 0 \quad \text{for } \mathbf{x} \in C_0 \tag{5.53}$$

and

$$\tilde{u}(\mathbf{x}, t) = 0 \quad \text{on } \partial C_0 \times [0, \infty) \tag{5.54}$$

Computing $\delta \widehat{K}\{u\}$ we obtain

$$\begin{aligned} \delta \widehat{K}\{u\} &= \frac{d}{d\omega} \widehat{K}\{u + \omega\tilde{u}\}|_{\omega=0} \\ &= \int_{t_1}^{t_2} dt \int_{C_0} (\widehat{T} u_{,\alpha}\, \tilde{u}_{,\alpha} - \hat{\rho}\dot{u}\dot{\tilde{u}} - f\tilde{u}) da \end{aligned} \tag{5.55}$$

Since

$$u_{,\alpha}\, \tilde{u}_{,\alpha} = (u_{,\alpha}\, \tilde{u})_{,\alpha} - u_{,\alpha\alpha}\, \tilde{u} \tag{5.56}$$

and

$$\dot{u}\dot{\tilde{u}} = (\dot{u}\tilde{u})^{\cdot} - \ddot{u}\tilde{u} \tag{5.57}$$

therefore, using the divergence theorem and the homogeneous conditions (5.53) and (5.54), we reduce (5.55) into the form

$$\delta \widehat{K}\{u\} = \int_{t_1}^{t_2} dt \int_{C_0} (-\widehat{T} u_{,\alpha\alpha} + \hat{\rho}\ddot{u} - f)\tilde{u}\, da \tag{5.58}$$

Hence, the condition

$$\delta \widehat{K}\{u\} = 0 \quad \text{on } U \tag{5.59}$$

together with the Fundamental Lemma of calculus of variations imply Eq. (5.50). This completes a solution to Problem 5.2.

Problem 5.3. Transverse waves propagating in a thin elastic membrane are described by the field equation (see Problem 5.2.)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u = -\frac{f}{\hat{T}} \quad \text{on} \quad C_0 \times [0, \infty) \tag{5.60}$$

the initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \dot{u}(\mathbf{x}, 0) = \dot{u}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in C_0 \tag{5.61}$$

and the boundary condition

$$u(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.62}$$

Let $\hat{U}$ be a set of functions $u = u(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$ that satisfy the boundary condition (5.62). Define a functional $\mathcal{F}_t\{.\}$ on $\hat{U}$ in such a way that

$$\delta F_t\{u\} = 0 \tag{5.63}$$

if and only if $u = u(\mathbf{x}, t)$ is a solution to the initial-boundary value problem (5.60) through (5.62).

Solution. By transforming the initial-boundary value problem (5.60)–(5.62) to an equivalent integro-differential boundary-value problem in a way similar to that of the Principle of Incompatible Elastodynamics [see Eqs. (5.6)–(5.12)] we find that the functional $\mathcal{F}_t\{u\}$ on $\hat{U}$ takes the form

$$\mathcal{F}_t\{u\} = \frac{1}{2}\int_{C_0} (i * u_{,\alpha} * u_{,\alpha} + \frac{1}{c^2} u * u - 2g * u) da \tag{5.64}$$

where

$$i = i(t) = t \tag{5.65}$$

and

$$g = i * \frac{f}{\widehat{T}} + \frac{1}{c^2}(u_0 + t\dot{u}_0) \tag{5.66}$$

The associated variational principle reads:

$$\delta\mathcal{F}_t\{u\} = 0 \quad \text{on } \hat{U} \tag{5.67}$$

if and only if u is a solution to the initial-boundary value problem (5.60)–(5.62). This completes a solution to Problem 5.3.

Problem 5.4. A homogeneous isotropic thin elastic plate defined over a region C_0 of the x_1, x_2 plane, and clamped on its boundary ∂C_0, is subject to a transverse load $p = p(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$. The strain energy of the plate is

$$F(t) = \frac{D}{2} \int_{C_0} (\nabla^2 w)^2 \, da \tag{5.68}$$

The kinetic energy of the plate is

$$K(t) = \frac{\hat{\rho}}{2} \int_{C_0} (\dot{w})^2 \, da \tag{5.69}$$

The external energy is

$$V(t) = -\int_{C_0} p \, w \, da \tag{5.70}$$

Here, $w = w(\mathbf{x}, t)$ is a transverse deflection of the plate on $C_0 \times [0, \infty)$, D is the bending rigidity of the plate ($[D] = [\text{Force} \times \text{Length}]$), and $\hat{\rho}$ is the area density of the plate ($[\hat{\rho}] = [\text{Density} \times \text{Length}]$).

Let W be the set of functions $w = w(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$ that satisfy the conditions

$$w(\mathbf{x}, t_1) = a(\mathbf{x}), \quad w(\mathbf{x}, t_2) = b(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in C_0 \tag{5.71}$$

and

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.72}$$

where t_1, t_2, $a(\mathbf{x})$ and $b(\mathbf{x})$ have the same meaning as in Problem 5.2, and $\partial/\partial n$ is the normal derivative on ∂C_0. Define a functional $\hat{K}\{.\}$ on W by

$$\hat{K}\{w\} = \int_{t_1}^{t_2} [F(t) + V(t) - K(t)] \, dt \tag{5.73}$$

Show that

$$\delta \hat{K}\{w\} = 0 \quad \text{on} \quad W \tag{5.74}$$

if and only if $w = w(\mathbf{x}, t)$ satisfies the differential equation

$$\nabla^2 \nabla^2 w + \frac{\hat{\rho}}{D} \frac{\partial^2 w}{\partial t^2} = \frac{p}{D} \quad \text{on} \quad C_0 \times [0, \infty) \tag{5.75}$$

and the boundary conditions

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.76}$$

Solution. The functional $\hat{K} = \hat{K}\{w\}$ on W takes the form

$$\widehat{K}\{w\} = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{C_0} [D(\nabla^2 w)^2 - \hat{\rho}\dot{w}^2 - 2pw]da \tag{5.77}$$

Let $w \in W$, $w + \omega\tilde{w} \in W$. Then

$$\tilde{w}(\mathbf{x}, t_1) = \tilde{w}(\mathbf{x}, t_2) = 0 \quad \text{for } \mathbf{x} \in C_0 \tag{5.78}$$

and

$$\tilde{w} = 0, \quad \frac{\partial \tilde{w}}{\partial n} = 0 \quad \text{on } \partial C_0 \times [0, \infty) \tag{5.79}$$

Hence, we obtain

$$\begin{aligned} \delta\widehat{K}\{w\} &= \frac{d}{d\omega}\widehat{K}\{w + \omega\tilde{w}\}|_{\omega=0} \\ &= \int_{t_1}^{t_2} dt \int_{C_0} [D(\nabla^2 w)(\nabla^2 \tilde{w}) - \hat{\rho}\dot{w}\dot{\tilde{w}} - p\tilde{w}]da \end{aligned} \tag{5.80}$$

Since

$$\begin{aligned} &(\nabla^2 w)(\nabla^2 \tilde{w}) = w_{,\alpha\alpha}\ \tilde{w}_{,\beta\beta} = (w_{,\alpha\alpha}\ \tilde{w}_{,\beta})_{,\beta} \\ &-w_{,\alpha\alpha\beta}\ \tilde{w}_{,\beta} = (w_{,\alpha\alpha}\tilde{w}_{,\beta} - w_{,\alpha\alpha\beta}\tilde{w})_{,\beta} + w_{,\alpha\alpha\beta\beta}\tilde{w} \end{aligned} \tag{5.81}$$

and

$$\dot{w}\dot{\tilde{w}} = (\dot{w}\tilde{w})^{\cdot} - \ddot{w}\tilde{w} \tag{5.82}$$

therefore, using the divergence theorem as well as the homogeneous conditions (5.78) and (5.79), we reduce (5.80) to the form

$$\delta\widehat{K}\{w\} = \int_{t_1}^{t_2} dt \int_{C_0} (D\nabla^4 w + \hat{\rho}\ddot{w} - p)\tilde{w}da \tag{5.83}$$

Hence, by virtue of the Fundamental Lemma of calculus of variations

$$\delta \widehat{K}\{w\} = 0 \quad \text{on } W \tag{5.84}$$

if and only if w satisfies the differential equation

$$\nabla^2\nabla^2 w + \frac{\hat{\rho}}{D}\frac{\partial^2 w}{\partial t^2} = \frac{p}{D} \quad \text{on } C_0 \times [0, \infty) \tag{5.85}$$

and the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial C_0 \times [0, \infty) \tag{5.86}$$

This completes a solution to Problem 5.4.

Problem 5.5. Transverse waves propagating in a clamped thin elastic plate are described by the equations (see Problem 5.4)

$$\nabla^2\nabla^2 w + \frac{\hat{\rho}}{D}\frac{\partial^2 w}{\partial t^2} = \frac{p}{D} \quad \text{on} \quad C_0 \times [0, \infty) \tag{5.87}$$

$$w(\mathbf{x}, 0) = w_0(\mathbf{x}), \quad \dot{w}(\mathbf{x}, 0) = \dot{w}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in C_0 \tag{5.88}$$

and

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.89}$$

where $w_0(\mathbf{x})$ and $\dot{w}_0(\mathbf{x})$ are prescribed functions on C_0. Let W^* denote the set of functions $w = w(\mathbf{x}, t)$ that satisfy the homogeneous boundary conditions (5.89). Find a functional $\hat{\mathcal{F}}_t\{.\}$ on W^* with the property that

$$\delta\, \hat{\mathcal{F}}_t\{w\} = 0 \quad \text{on} \quad W^* \tag{5.90}$$

if and only if w is a solution to the initial-boundary value problem (5.87) through (5.89).

Solution. First, we note that the initial-boundary value problem (5.87)–(5.89) is equivalent to the following boundary-value problem. Find $w = w(\mathbf{x}, t)$ on $C_0 \times [0, \infty)$ that satisfies the integro-differential equation.

$$i * \nabla^4 w + \frac{1}{c^2} w = h \quad \text{on} \quad C_0 \times [0, \infty) \tag{5.91}$$

subject to the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial C_0 \times [0, \infty) \tag{5.92}$$

Here,

$$i = i(t) = t, \quad h(\mathbf{x}, t) = i * \frac{p}{D} + \frac{1}{c^2}(w_0 + t\dot{w}_0),$$
$$\text{and} \quad \frac{1}{c^2} = \frac{\hat{\rho}}{D} \tag{5.93}$$

Next, we define a functional $\hat{\mathcal{F}}_t \{w\}$ on W^* by

$$\hat{\mathcal{F}}_t\{w\} = \frac{1}{2}\int_{C_0} \left(i * \nabla^2 w * \nabla^2 w + \frac{1}{c^2} w * w - 2h * w \right) da \tag{5.94}$$

By computing $\delta\hat{\mathcal{F}}_t \{w\}$, we obtain

$$\delta\hat{\mathcal{F}}_t\{w\} = \int_{C_0} \left(i * \nabla^4 w + \frac{1}{c^2} w - h \right) * \tilde{w} da \tag{5.95}$$

where $\tilde{w}$ is an arbitrary smooth function on C_0 such that

$$\tilde{w} = \frac{\partial \tilde{w}}{\partial n} = 0 \quad \text{on } \partial C_0 \times [0, \infty) \tag{5.96}$$

Therefore, using the Fundamental Lemma of calculus of variations, it follows from Eq. (5.95) that the condition

$$\delta\hat{\mathcal{F}}_t\{w\} = 0 \quad \text{on } W^* \tag{5.97}$$

holds true if and only if w is a solution to the initial-boundary value problem (5.87)–(5.89). This completes a solution to Problem 5.5.

Problem 5.6. Free longitudinal vibrations of an elastic bar are defined as solutions of the form

$$u(x, t) = \phi(x)\ \sin(\omega t + \gamma) \tag{5.98}$$

to the homogeneous wave equation

$$\frac{\partial}{\partial x}\left(E \frac{\partial u}{\partial x} \right) - \rho \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{on} \quad [0, L] \times [0, \infty) \tag{5.99}$$

subject to the homogeneous boundary conditions

$$u(0, t) = u(L, t) = 0 \quad \text{on} \quad [0, \infty) \tag{5.100}$$

or

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \quad \text{on} \quad [0,\infty) \tag{5.101}$$

Here, ω is a circular frequency of vibrations, γ is a dimensionless constant, and $\phi = \phi(x)$ is an unknown function that complies with Eqs. (5.99) and (5.100), or Eqs. (5.99) and (5.101). Substituting $u = u(\mathbf{x}, t)$ from Eq. (5.98) into (5.99) through (5.101) we obtain

$$\frac{d}{dx}\left(E\frac{d\phi}{dx}\right) + \lambda\phi = 0 \quad \text{on} \quad [0, L] \tag{5.102}$$

$$\phi(0) = \phi(L) = 0 \tag{5.103}$$

or

$$\phi'(0) = \phi'(L) = 0 \tag{5.104}$$

where the prime stands for derivative with respect to x, and

$$\lambda = \rho\omega^2 \tag{5.105}$$

Therefore, introduction of (5.98) into (5.99) through (5.101) results in an eigenproblem in which an eigenfunction $\phi = \phi(x)$ corresponding to an eigenvalue λ is to be found. An eigenproblem that covers both boundary conditions (5.100) and (5.101) can be written as

$$\frac{d}{dx}\left(E\frac{d\phi}{dx}\right) + \lambda\phi = 0 \quad \text{on} \quad [0, L] \tag{5.106}$$

$$\phi'(0) - \alpha\,\phi(0) = 0, \quad \phi'(L) + \beta\,\phi(L) = 0 \tag{5.107}$$

where $|\alpha| + |\beta| > 0$. Let U be the set of functions $\phi = \phi(x)$ on $[0, L]$ that satisfy the boundary conditions (5.107). Define a functional $\pi\{.\}$ on U by

$$\pi\{\phi\} = \frac{1}{2}\int_0^L \left[E\left(\frac{d\phi}{dx}\right)^2 - \lambda\phi^2\right] dx + \frac{1}{2}\alpha\,E(0)\;[\phi(0)]^2 + \frac{1}{2}\beta\,E(L)\;[\phi(L)]^2 \tag{5.108}$$

Show that

$$\delta\,\pi\{\phi\} = 0 \quad \text{over} \quad U \tag{5.109}$$

if and only if $\phi = \phi(x)$ is an eigenfunction corresponding to an eigenvalue λ in the eigenproblem (5.106) and (5.107).

Solution. Let $\phi \in U$ and $\phi + \omega\,\tilde{\phi} \in U$. Then

$$\tilde{\phi}'(0) - \alpha\tilde{\phi}(0) = 0, \quad \tilde{\phi}'(L) + \beta\tilde{\phi}(L) = 0 \tag{5.110}$$

and

$$\pi\{\phi+\omega\tilde{\phi}\} = \frac{1}{2}\int_0^L [E(\phi'+\omega\tilde{\phi}')^2 - \lambda(\phi+\omega\tilde{\phi})^2]dx + \frac{1}{2}\alpha E(0)[\phi(0)+\omega\tilde{\phi}(0)]^2 + \frac{1}{2}\beta E(L)[\phi(L)+\omega\tilde{\phi}(L)]^2 \quad (5.111)$$

Hence, we obtain

$$\delta\pi\{\phi\} = \frac{d}{d\omega}\pi\{\phi+\omega\tilde{\phi}\}\Big|_{\omega=0} = \int_0^L [E\phi'\tilde{\phi}' - \lambda\phi\tilde{\phi}]\,dx + \alpha E(0)\,\phi(0)\,\tilde{\phi}(0) + \beta E(L)\,\phi(L)\,\tilde{\phi}(L) \quad (5.112)$$

Since

$$\int_0^L E\phi'\tilde{\phi}'dx = E\phi'\tilde{\phi}\Big|_{x=0}^{x=L} - \int_0^L (E\phi')'\tilde{\phi}dx \quad (5.113)$$

therefore, Eq. (5.112) takes the form

$$\delta\pi\{\phi\} = -\int_0^L [(E\phi')' + \lambda\phi]\tilde{\phi}dx - E(0)[\phi'(0) - \alpha\phi(0)]\,\tilde{\phi}(0) + E(L)[\phi'(L) + \beta\phi(L)]\tilde{\phi}(L) \quad (5.114)$$

Now, if $\phi = \phi(x)$ is an eigenfunction corresponding to an eigenvalue λ in the problem (5.106)–(5.107), then by virtue of (5.114) $\delta\pi\{\phi\} = 0$ over U. Conversely, if $\delta\pi\{\phi\} = 0$ then selecting $\tilde{\phi} = \tilde{\phi}(x)$ to be a smooth function on $[0, L]$ such that $\tilde{\phi}(0) = \tilde{\phi}(L) = 0$, and using the Fundamental Lemma of calculus of variations, we obtain

$$(E\phi')' + \lambda\phi = 0 \quad \text{on} \quad [0, L] \quad (5.115)$$

Next, if $\delta\pi\{\phi\} = 0$ then selecting $\tilde{\phi} = \tilde{\phi}(x)$ to be a smooth function on $[0, L]$ and such that $\tilde{\phi}(L) = 0$, and $\tilde{\phi}(0) \neq 0$, by virtue of (5.115), we obtain

$$E(0)[\phi'(0) - \alpha\phi(0)]\tilde{\phi}(0) = 0 \quad (5.116)$$

Since

$$E(0) > 0 \quad (5.117)$$

Equation (5.116) implies that $\phi = \phi(x)$ satisfies the boundary condition

$$\phi'(0) - \alpha\phi(0) = 0 \tag{5.118}$$

Finally, if $\delta\pi\{\phi\} = 0$ then selecting $\tilde{\phi}$ to be a smooth function on $[0, L]$ and such that $\tilde{\phi}(L) \neq 0$, by virtue of (5.115) and (5.118), we obtain

$$E(L)[\phi'(L) + \beta\phi(L)]\tilde{\phi}(L) = 0 \tag{5.119}$$

Since $E(L) > 0$, Eq. (5.119) implies that

$$\phi'(L) + \beta\phi(L) = 0 \tag{5.120}$$

This shows that if Eq. (5.110) holds true then (ϕ, λ) is an eigenpair for the problem (5.106)–(5.107). This completes a solution to Problem 5.6.

Problem 5.7. Free lateral vibrations of an elastic bar clamped at the end $x = 0$ and supported by a spring of stiffness k at the end $x = L$ are defined as solutions of the form

$$u(x, t) = \phi(x)\, \sin(\omega t + \gamma) \tag{5.121}$$

to the equation [see Problem 5.1, Eq. (5.127) in which $u_2 = u$, and $F = 0$]

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) + \rho\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{on} \quad [0, L] \times [0, \infty) \tag{5.122}$$

subject to the boundary conditions

$$u(0, t) = u'(0, t) = 0 \quad \text{on} \quad [0, \infty) \tag{5.123}$$

$$u''(L, t) = 0, \quad (EI\,u'')'(L, t) - k\,u(L, t) = 0 \quad \text{on} \quad [0, \infty) \tag{5.124}$$

Let $\rho = \text{const}$, and $\lambda = \rho\,\omega^2$. Then the associated eigenproblem reads

$$(EI\,\phi'')'' - \lambda\phi = 0 \quad \text{on} \quad [0, L] \tag{5.125}$$

$$\phi(0) = \phi'(0) = 0 \tag{5.126}$$

$$\phi''(L) = 0, \quad (EI\phi'')'(L) - k\phi(L) = 0 \tag{5.127}$$

Let V denote the set of functions $\phi = \phi(x)$ on $[0, L]$ that satisfy the boundary conditions (5.126) and (5.127). Define a functional $\pi\{.\}$ on V by

$$\pi\{\phi\} = \frac{1}{2}\int_0^L EI\,(\phi'')^2 dx + \frac{1}{2}k[\phi(L)]^2 - \frac{\lambda}{2}\int_0^L \phi^2 dx \tag{5.128}$$

Show that

$$\delta\,\pi\{\phi\} = 0 \quad \text{over} \quad V \tag{5.129}$$

if and only if (λ, ϕ) is a solution to the eigenproblem (5.125) through (5.127).

Solution. Let $\phi \in V$ and $\phi + \omega\tilde{\phi} \in V$. Then

$$\tilde{\phi}(0) = 0, \quad \tilde{\phi}'(0) = 0 \tag{5.130}$$

Computing the first variation of the functional $\pi\{\phi\}$ given by (5.128), we obtain

$$\begin{aligned}\delta\pi\{\phi\} &= \frac{d}{d\omega}\pi\;\{\phi + \omega\tilde{\phi}\}\Big|_{\omega=0} \\ &= \int_0^L (EI\,\phi''\tilde{\phi}'' - \lambda\phi\tilde{\phi})dx + k\phi(L)\,\tilde{\phi}(L)\end{aligned} \tag{5.131}$$

Since

$$\int_0^L EI\,\phi''\tilde{\phi}''dx = (EI\,\phi'')\tilde{\phi}'\Big|_{x=0}^{x=L} - (EI\,\phi'')'\tilde{\phi}\Big|_{x=0}^{x=L} + \int_0^L (EI\,\phi'')''\,\tilde{\phi}dx \tag{5.132}$$

therefore, using (5.130) we reduce (5.131) into the form

$$\delta\pi\{\phi\} = \int_0^L [(EI\,\phi'')'' - \lambda\phi]\tilde{\phi}dx + (EI\,\phi'')(L)\tilde{\phi}'(L) - [(EI\,\phi'')'(L) - k\phi(L)]\tilde{\phi}(L) \tag{5.133}$$

Now, if (λ, ϕ) is a solution to the eigenproblem (5.125)–(5.127), then $\delta\pi\{\phi\} = 0$. Conversely, if $\delta\pi\{\phi\} = 0$ over V, then selecting $\tilde{\phi}$ to be an arbitrary smooth function on $[0, L]$ such that $\tilde{\phi}(x) \not\equiv 0$ for $x \in (0, L)$, $\tilde{\phi}'(L) = 0$, $\tilde{\phi}(L) = 0$, we obtain

$$\int_0^L [(EI\,\phi'')'' - \lambda\phi]\,\tilde{\phi}\,dx = 0 \tag{5.134}$$

Equation (5.134) together with the Fundamental Lemma of calculus of variations implies

$$(EI\,\phi'')'' - \lambda\phi = 0 \quad \text{on} \quad [0, L] \tag{5.135}$$

Next, by selecting $\tilde{\phi}$ on $[0, L]$ in such a way that

$$\tilde{\phi}'(L) \neq 0, \quad \tilde{\phi}(L) = 0 \tag{5.136}$$

we find that the condition $\delta\pi\{\phi\} = 0$ and Eq. (5.135) imply that

$$(EI\,\phi'')(L) = 0 \tag{5.137}$$

Since

$$E(L) > 0, \quad I(L) > 0 \tag{5.138}$$

we obtain

$$\phi''(L) = 0 \tag{5.139}$$

Finally, by selecting $\tilde{\phi}$ on $[0, L]$ in such a way that

$$\tilde{\phi}(L) \neq 0 \tag{5.140}$$

we conclude that the condition $\delta\pi\{\phi\} = 0$ together with Eqs. (5.135), and (5.139) lead to the boundary condition

$$(EI\,\phi'')'(L) - k\,\phi(L) = 0 \tag{5.141}$$

This completes a solution to Problem 5.7.

Problem 5.8. Show that the eigenvalues $\lambda_{\rm i}$ and the eigenfunctions $\phi_i = \phi_i(x)$ for the longitudinal vibrations of a uniform elastic bar having one end clamped and the other end free are given by the relations

$$\omega_{\rm i} = \sqrt{\frac{\lambda_{\rm i}}{\rho}} = \frac{(2i-1)}{2L}\sqrt{\frac{E}{\rho}}$$

$$\phi_{\rm i}(x) = \sin\frac{(2i-1)\pi\,x}{2L}, \quad i = 1, 2, 3, \ldots, 0 \leq x \leq L$$

(see Problem 5.6).

Solution. For an elastic bar that is clamped at $x = 0$ and free at $x = L$ the eigenproblem reads

$$E\phi''(x) + \lambda\phi(x) = 0 \qquad x \in [0, L] \tag{5.142}$$

$$\phi(0) = 0, \quad \phi'(L) = 0 \tag{5.143}$$

where

$$\lambda = \omega^2\rho \tag{5.144}$$

There is an infinite sequence of eigensolutions (λ_i, ϕ_i) to the problem (5.142)–(5.143) of the form

$$\lambda_i = \frac{(2i-1)^2\pi^2}{4L^2}E \tag{5.145}$$

$$\phi_i(x) = \sin\frac{(2i-1)\pi x}{2L}, \quad i = 1, 2, 3, \ldots \tag{5.146}$$

This can be shown by substituting (5.145) and (5.146) into (5.142), and by showing that $\phi_i(x)$ satisfies (5.143). By combining (5.144) and (5.145) we obtain

$$\omega_i \equiv \sqrt{\frac{\lambda_i}{\rho}} = \frac{(2i-1)\pi}{2L}\sqrt{\frac{E}{\rho}} \tag{5.147}$$

This completes a solution to Problem 5.8.

Problem 5.9. Show that the eigenvalues λ_i and the eigenfunctions $\phi_i = \phi_i(x)$ for the lateral vibrations of a uniform, simply supported elastic beam are given by the relations

$$\omega_i = \sqrt{\frac{\lambda_i}{\rho}} = \frac{\pi^2 i^2}{L^2}\sqrt{\frac{EI}{\rho}}$$

$$\phi_i(x) = \sin\frac{i\pi x}{L}, \quad i = 1, 2, 3\ldots, 0 \le x \le L$$

(see Problem 5.1).

Solution. For a uniform, simply supported beam with the lateral vibrations, the eigenproblem takes the form

$$EI\phi^{(4)} - \lambda\phi = 0 \quad \text{on} \quad [0, L] \tag{5.148}$$

$$\phi(0) = \phi''(0) = 0, \quad \phi(L) = \phi''(L) = 0 \tag{5.149}$$

where

$$\lambda = \omega^2\rho \tag{5.150}$$

There is an infinite sequence of eigensolutions (λ_i, ϕ_i) to the problem (5.148)–(5.149) of the form

$$\lambda_i = EI\left(\frac{i\pi}{L}\right)^4 \tag{5.151}$$

$$\phi_i(x) = \sin\frac{i\pi x}{L} \quad i = 1, 2, 3, \ldots \tag{5.152}$$

To prove that (λ_i, ϕ_i) given by (5.151)–(5.152) satisfies Eqs. (5.148)–(5.149), we note that

$$\phi_i''(x) = -\left(\frac{i\pi}{L}\right)^2 \phi_i(x) \tag{5.153}$$

and

$$\phi_i^{(4)}(x) = \left(\frac{i\pi}{L}\right)^4 \phi_i(x) \tag{5.154}$$

Substituting (5.151) and (5.152) into (5.148) and using (5.154) we find that $\phi_i = \phi_i(x)$ satisfies Eq. (5.148) on $[0, L]$. Also, it follows from Eqs. (5.152) and (5.153) that the boundary conditions (5.149) are satisfied; and Eqs. (5.150) and (5.151) imply that

$$\omega_i = \sqrt{\frac{\lambda i}{\rho}} = \frac{\pi^2 i^2}{L^2}\sqrt{\frac{EI}{\rho}} \tag{5.155}$$

These steps complete a solution to Problem 5.9.

Problem 5.10. Show that the eigenvalues λ_{mn} and the eigenfunctions $\phi_{mn} = \phi_{mn}(x)$ for the transversal vibrations of a rectangular elastic membrane: $0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2$, that is clamped on its boundary, are given by

$$\omega_{mn} = \sqrt{\frac{\lambda_{mn}}{\hat{\rho}}} = \pi\sqrt{\frac{\hat{T}}{\hat{\rho}}\left(\frac{m^2}{a_1^2} + \frac{n^2}{a_2^2}\right)}$$

$$\phi_{mn}(x_1, x_2) = \sin\frac{m\pi x_1}{a_1}\sin\frac{n\pi x_2}{a_2},$$
$$m, n = 1, 2, 3, \ldots, 0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2$$

(See Problem 5.2).

Solution. Let C_0 denote the rectangular region

$$0 < x_1 < a_1, \quad 0 < x_2 < a_2 \tag{5.156}$$

and let ∂C_0 be its boundary. Then the associated eigenproblem reads. Find an eigenpair (λ, ϕ) such that

$$\hat{T}\,\nabla^2\phi + \lambda\phi = 0 \quad \text{on } C_0 \tag{5.157}$$

and

$$\phi = 0 \quad \text{on } \partial C_0 \tag{5.158}$$

where

$$\lambda = \omega^2 \hat{\rho} \tag{5.159}$$

There is an infinite number of eigenpairs $(\lambda_{mn}, \phi_{mn}),\ m, n = 1, 2, 3, \ldots$ that satisfy Eqs. (5.157) and (5.158), and they are given by Equation

$$\lambda_{mn} = \pi^2 \hat{T} \left(\frac{m^2}{a_1^2} + \frac{n^2}{a_2^2} \right) \tag{5.160}$$

$$\phi_{mn}(x_1, x_2) = \sin\left(\frac{m\pi x_1}{a_1}\right) \sin\left(\frac{n\pi x_2}{a_2}\right) \tag{5.161}$$

This can be proved by substituting (5.152) and (5.153) into (5.149) and (5.150).

Also, the eigenvalues λ_{mn} generate the eigenfrequencies ω_{mn} by the formulas

$$\omega_{mn} = \sqrt{\frac{\lambda_{mn}}{\hat{\rho}}} = \pi \sqrt{\frac{\hat{T}}{\hat{\rho}}} \sqrt{\left(\frac{m^2}{a_1^2}\right) + \left(\frac{n^2}{a_2^2}\right)} \tag{5.162}$$

This completes a solution to Problem 5.10.

Problem 5.11. Show that the eigenvalues λ_{mn} and the eigenfunctions $\phi_{mn} = \phi_{mn}(x_1, x_2)$ for the transversal vibrations of a thin elastic rectangular plate: $0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2$, that is simply supported on its boundary are given by the relations

$$\omega_{mn} = \sqrt{\frac{\lambda_{mn}}{\hat{\rho}}} = \pi^2 \left(\frac{m^2}{a_1^2} + \frac{n^2}{a_2^2} \right) \sqrt{\frac{D}{\hat{\rho}}}$$

$$\phi_{mn}(x_1, x_2) = \sin \frac{m\,\pi\,x_1}{a_1} \sin \frac{n\,\pi\,x_2}{a_2},$$

$$m, n = 1, 2, 3, \ldots, 0 \le x_1 \le a_1,\ 0 \le x_2 \le a_2$$

(See Problem 5.4).

Solution. The eigenproblem associated with the transversal vibrations of a thin elastic rectangular plate that is simply supported on its boundary, reads [see Eq. (5.85) of Problem 5.4]

$$D\, \nabla^2 \nabla^2\, \phi - \lambda \phi = 0 \quad \text{on } C_0 \tag{5.163}$$

$$\phi = \nabla^2 \phi = 0 \quad \text{on } \partial C_0 \tag{5.164}$$

where

$$\lambda = \omega^2 \hat{\rho} \tag{5.165}$$

and C_0 and ∂C_0 are the same as in Problem 5.10.

There are an infinite number of eigenpairs $(\lambda_{mn}, \phi_{mn})$ that satisfy Eqs. (5.163) and (5.164), and the eigenpairs are given by

$$\lambda_{mn} = \pi^4 D \left(\frac{m^2}{a_1^2} + \frac{n^2}{a_2^2} \right)^2 \tag{5.166}$$

$$\phi_{mn}(x_1, x_2) = \sin\left(\frac{m\pi x_1}{a_1}\right) \sin\left(\frac{n\pi x_2}{a_2}\right) \quad m, n = 1, 2, 3, \ldots \tag{5.167}$$

This is proved by substituting (5.166) and (5.167) into (5.163) and (5.164).

Also, by using (5.165) the eigenfrequencies ω_{mn} are obtained

$$\omega_{mn} = \sqrt{\frac{\lambda_{mn}}{\hat{\rho}}} = \pi^2 \left(\frac{m^2}{a_1^2} + \frac{n^2}{a_2^2} \right) \sqrt{\frac{D}{\hat{\rho}}} \tag{5.168}$$

This completes a solution to Problem 5.11.