Chapter 4
Variational Formulation of Elastostatics

In this chapter the variational characterizations of a solution to a boundary value problem of elastostatics are recalled. They include the principle of minimum potential energy, the principle of minimum complementary energy, the Hu-Washizu principle, and the compatibility related principle for a traction problem. The variational principles are then used to solve typical problems of elastostatics.

4.1 Minimum Principles

To formulate the Principle of Minimum Potential Energy we recall the concept of the *strain energy,* of the *stress energy,* and of a *kinematically admissible state.*

By the *strain energy of a body B* we mean the integral

$$\mathrm{U_C}\{\mathbf{E}\} = \frac{1}{2}\int_{\mathrm{B}} \mathbf{E}\cdot\mathbf{C}[\mathbf{E}]\,dv \tag{4.1}$$

and by the *stress energy of a body* B we mean

$$\mathrm{U_K}\{\mathbf{S}\} = \frac{1}{2}\int_{\mathrm{B}} \mathbf{S}\cdot\mathbf{K}[\mathbf{S}]\,dv \tag{4.2}$$

Since $\mathbf{S} = \mathbf{C}[\mathbf{E}]$, therefore,

$$\mathrm{U_K}\{\mathbf{S}\} = \mathrm{U_C}\{\mathbf{E}\} \tag{4.3}$$

By a *kinematically admissible state* we mean a state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ that satisfies

(1) the strain-displacement relation

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \quad \text{on} \quad \mathrm{B} \tag{4.4}$$

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(2) the stress-strain relation

$$\mathbf{S} = \mathbf{C}\,[\mathbf{E}] \quad \text{on} \quad \mathrm{B} \tag{4.5}$$

(3) the displacement boundary condition

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial \mathrm{B}_1 \tag{4.6}$$

where $\widehat{\mathbf{u}}$ is prescribed on $\partial \mathrm{B}_1$.

The Principle of Minimum Potential Energy is related to a mixed boundary value problem of elastostatics [see Chap. 3 on Formulation of Problems of Elasticity].

The Principle of Minimum Potential Energy
Let R be the set of all kinematically admissible states. Define a functional $\mathrm{F} = \mathrm{F}\{.\}$ on R by

$$\mathrm{F}\{\mathrm{s}\} = \mathrm{U_C}\{\mathbf{E}\} - \int_{\mathrm{B}} \mathbf{b} \cdot \mathbf{u}\, dv - \int_{\partial \mathrm{B}_2} \widehat{\mathbf{s}} \cdot \mathbf{u}\, da \tag{4.7}$$

for every $s = [\mathbf{u},\ \mathbf{E},\ \mathbf{S}] \in \mathrm{R}$. Let s be a solution to the mixed problem of elastostatics. Then

$$\mathrm{F}\{\mathrm{s}\} \leq \mathrm{F}\{\tilde{\mathrm{s}}\} \quad \text{for every} \quad \tilde{\mathrm{s}} \in \mathrm{R} \tag{4.8}$$

and the equality holds true if s and $\tilde{\mathrm{s}}$ differ by a rigid displacement.

By letting $\mathbf{E} = \widehat{\nabla}\mathbf{u}$ in (4.7) an *alternative form of the Principle of Minimum Potential Energy* is obtained.

Let R_1 denote a set of displacement fields that satisfy the boundary conditions (4.6), and define a functional $\mathrm{F}_1\{.\}$ on R_1 by

$$\mathrm{F}_1\{\mathbf{u}\} = \frac{1}{2}\int_{\mathrm{B}} (\nabla\mathbf{u}) \cdot \mathbf{C}\,[\nabla\mathbf{u}]\, dv - \int_{\mathrm{B}} \mathbf{b} \cdot \mathbf{u}\, dv - \int_{\partial \mathrm{B}_2} \widehat{\mathbf{s}} \cdot \mathbf{u}\, da \quad \forall \mathbf{u} \in \mathrm{R}_1 \tag{4.9}$$

If $\mathbf{u}$ corresponds to a solution to the mixed problem, then

$$\mathrm{F}_1\{\mathbf{u}\} \leq \mathrm{F}_1\{\tilde{\mathbf{u}}\} \quad \forall \tilde{\mathbf{u}} \in \mathrm{R}_1 \tag{4.10}$$

To formulate the Principle of Minimum Complementary Energy, we introduce a concept of a *statically admissible stress field*. By such a field we mean a symmetric second-order tensor field $\mathbf{S}$ that satisfies

(1) the equation of equilibrium

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{4.11}$$

(2) the traction boundary condition

$$\mathbf{S}\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B}_2 \tag{4.12}$$

The Principle of Minimum Complementary Energy
Let P denote a set of all statically admissible stress fields, and let $\mathrm{G} = \mathrm{G}\{.\}$ be a functional on P defined by

$$\mathrm{G}\{\mathbf{S}\} = \mathrm{U_K}\{\mathbf{S}\} - \int_{\partial \mathrm{B}_1} \mathbf{s} \cdot \widehat{\mathbf{u}}\, da \quad \forall \mathbf{S} \in P \tag{4.13}$$

If $\mathbf{S}$ is a stress field corresponding to a solution to the mixed problem, then

$$\mathrm{G}\{\mathbf{S}\} \leq \mathrm{G}\{\tilde{\mathbf{S}}\} \quad \forall \tilde{\mathbf{S}} \in P \tag{4.14}$$

and the equality holds if $\mathbf{S} = \tilde{\mathbf{S}}$.

The Principle of Minimum Complementary Energy for Nonisothermal Elastostatics
The fundamental field equations of nonisothermal elastostatics may be written as

$$\mathbf{E} = \widehat{\nabla} \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}) \quad \text{on} \quad \mathrm{B} \tag{4.15}$$

$$\operatorname{div} \mathbf{S}' + \mathbf{b}' = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{4.16}$$

$$\mathbf{S}' = \mathbf{C}\,[\mathbf{E}] \quad \text{on} \quad \mathrm{B} \tag{4.17}$$

where

$$\mathbf{b}' = \mathbf{b} + \operatorname{div}(T\mathbf{M}) \tag{4.18}$$

$$\mathbf{S}' = \mathbf{S} - T\mathbf{M} \tag{4.19}$$

$$\mathbf{s}' \equiv \mathbf{S}'\mathbf{n} \tag{4.20}$$

The Principle of Minimum Complementary Energy of nonisothermal Elastostatics reads: Let P denote a set of all statically admissible stress fields, and let $\mathrm{G_T} = \mathrm{G_T}\{.\}$ be a functional on P defined by

$$\mathrm{G_T}\{\mathbf{S}\} = \mathrm{U_K}\{\mathbf{S}'\} - \int_{\partial \mathrm{B}_1} \mathbf{s}' \cdot \widehat{\mathbf{u}}\, da \quad \forall \mathbf{S} \in P \tag{4.21}$$

If $\mathbf{S}$ is a stress field corresponding to a solution to the mixed problem of nonisothermal elastostatics, then

$$\mathrm{G_T}\{\mathbf{S}\} \leq \mathrm{G_T}\{\tilde{\mathbf{S}}\} \quad \forall \tilde{\mathbf{S}} \in P \tag{4.22}$$

and the equality holds true if $\mathbf{S} = \tilde{\mathbf{S}}$.

Note. The functional $G_T = G_T\{.\}$ in Eq. (4.21) can be replaced by

$$G_T^*\{\mathbf{S}\} = U_K\{\mathbf{S}\} + \int_B T\mathbf{S}\cdot\mathbf{A}\,dv - \int_{\partial B_1} \mathbf{s}\cdot\widehat{\mathbf{u}}\,da \tag{4.23}$$

where $\mathbf{A}$ is the thermal expansion tensor.

4.2 The Rayleigh-Ritz Method

The functional $F_1 = F_1\{\mathbf{u}\}$ [see Eq. (4.9)] can be minimized by looking for $\mathbf{u}$ in an approximate form

$$\mathbf{u} \cong \mathbf{u}^{(N)} = \widehat{\mathbf{u}}^{(N)} + \sum_{k=1}^{N} a_k \mathbf{f}_k \quad \text{on} \quad \overline{B} \tag{4.24}$$

where $\widehat{\mathbf{u}}^{(N)}$ is a function on $\overline{B}$ such that

$$\widehat{\mathbf{u}}^{(N)} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial B_1 \tag{4.25}$$

and $\{\mathbf{f}_k\}$ stands for a set of functions on $\overline{B}$ such that

$$\mathbf{f}_k = \mathbf{0} \quad \text{on} \quad \partial B_1 \tag{4.26}$$

and a_k are unknown constants to be determined from the condition that $F_1 = F_1\{\mathbf{u}^{(N)}\} \equiv \varphi(a_1, a_2, a_3, \ldots, a_N)$ attains a minimum, that is, from the conditions

$$\frac{\partial \varphi}{\partial a_i}(a_1, a_2, a_3, \ldots, a_N) = 0 \quad i = 1, 2, 3, \ldots, N \tag{4.27}$$

One can show that Eqs. (4.27) represent a linear nonhomogeneous system of algebraic equations for which there is a unique solution $(a_1, a_2, a_3, \ldots, a_N)$.

Similarly, if $\partial B_1 = \varnothing$, the functional $G = G\{.\}$ [see Eq. (4.13)] can be minimized by letting $\mathbf{S}$ in the form

$$\mathbf{S} \cong \mathbf{S}^{(N)} = \widehat{\mathbf{S}}^{(N)} + \sum_{k=1}^{N} a_k \mathbf{S}_k \quad \text{on} \quad \overline{B} \tag{4.28}$$

where $\widehat{\mathbf{S}}^{(N)}$ is selected in such a way that

$$\operatorname{div}\widehat{\mathbf{S}}^{(N)} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad B \tag{4.29}$$

and

$$\widehat{\mathbf{S}}^{(N)}\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial B \tag{4.30}$$

while $\mathbf{S}_k$ are to satisfy the equations

$$\text{div}\, \mathbf{S}_k = \mathbf{0} \quad \text{on} \quad B \tag{4.31}$$

and

$$\mathbf{S}_k \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial B \tag{4.32}$$

The unknown coefficients a_k are obtained by solving the linear algebraic equations

$$\frac{\partial \psi}{\partial\, a_i}(a_1, a_2, a_3, \ldots, a_N) = 0 \quad i = 1, 2, 3, \ldots, N \tag{4.33}$$

where

$$\psi(a_1, a_2, a_3, \ldots, a_N) \equiv G\{\mathbf{S}^{(N)}\} \tag{4.34}$$

The method of minimizing $F_1 = F_1\{\mathbf{u}\}$ and $G = G\{\mathbf{S}\}$ by postulating $\mathbf{u}$ and $\mathbf{S}$ by formulas (4.24) and (4.28), respectively, is called the *Rayleigh-Ritz Method.*

4.3 Variational Principles

Let $H\{s\}$ be a functional on A, where A is a set of admissible states $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$. By the *first variation of* $H\{s\}$ we mean the number

$$\delta_{\tilde{s}} H\{s\} = \left. \frac{d}{d\omega} H\{s + \omega\, \tilde{s}\} \right|_{\omega=0} \tag{4.35}$$

where s and $\tilde{s} \in \mathsf{A}$, and $s + \omega\, \tilde{s} \in \mathsf{A}$ for every scalar ω, and we say that

$$\delta_{\tilde{s}} H\{s\} \equiv \delta\, H\{s\} = 0 \tag{4.36}$$

if $\delta_{\tilde{s}} H\{s\}$ exists and equals zero for any $\tilde{s}$ consistent with the relation $s + \omega\, \tilde{s} \in \mathsf{A}$.

Hu-Washizu Principle
Let A denote the set of all admissible states of elastostatics, and let $H\{s\}$ be the functional on A defined by

$$H\{s\} = U_C\{\mathbf{E}\} - \int_B \mathbf{S} \cdot \mathbf{E}\, dv - \int_B (\text{div}\, \mathbf{S} + \mathbf{b}) \cdot \mathbf{u}\, dv + \int_{\partial B_1} \mathbf{s} \cdot \widehat{\mathbf{u}}\, da + \int_{\partial B_2} (\mathbf{s} - \widehat{\mathbf{s}}) \cdot \mathbf{u}\, da$$
$$\forall\, s = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in \mathsf{A} \tag{4.37}$$

Then

$$\delta\, \mathrm{H}\{\mathrm{s}\} = 0 \tag{4.38}$$

if and only if s is a solution to the mixed problem.

Note 1. If the set A in Hu-Washizu Principle is restricted to the set of all kinematically admissible states R [see the Principle of Minimum Potential Energy] then Hu-Washizu Principle reduces to that of Minimum Potential Energy.

Hellinger-Reissner Principle
Let A_1 denote the set of all admissible states that satisfy the strain-displacement relation, and let $\mathrm{H}_1 = \mathrm{H}_1\{\mathrm{s}\}$ be the functional on A_1 defined by

$$\mathrm{H}_1\{\mathrm{s}\} = \mathrm{U}_\mathrm{K}\{\mathbf{S}\} - \int\limits_{\mathrm{B}} \mathbf{S}\cdot\mathbf{E}\, dv + \int\limits_{\mathrm{B}} \mathbf{b}\cdot\mathbf{u}\, dv + \int\limits_{\partial \mathrm{B}_1} \mathbf{s}\cdot(\mathbf{u}-\widehat{\mathbf{u}})\, da + \int\limits_{\partial \mathrm{B}_2} \widehat{\mathbf{s}}\cdot\mathbf{u}\, da$$

$$\forall\, \mathrm{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in \mathsf{A}_1 \tag{4.39}$$

Then

$$\delta\, \mathrm{H}_1\{\mathrm{s}\} = 0 \tag{4.40}$$

if and only if s is a solution to the mixed problem.

Note 2. By restricting A_1 to the set $\mathsf{A}_2 = \mathsf{A}_1 \cap \mathrm{P}$, where P is the set of all statically admissible states, we reduce Hellinger-Reissner Principle to that of the Principle of Minimum Complementary Energy.

4.4 Compatibility-Related Principle

Consider a traction problem for a body B subject to an external load $[\mathbf{b}, \widehat{\mathbf{s}}]$. Let Q denote the set of all admissible states that satisfy the equation of equilibrium, the stress-strain relations, and the traction boundary condition; and let I{.} be the functional on Q defined by

$$\mathrm{I}\{\mathrm{s}\} = \mathrm{U}_\mathrm{K}\{\mathbf{S}\} = \frac{1}{2}\int\limits_{\mathrm{B}} \mathbf{S}\cdot\mathbf{K}[\mathbf{S}]\, dv \quad \forall\, \mathrm{s} = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \in \mathrm{Q} \tag{4.41}$$

Then

$$\delta\, \mathrm{I}\{\mathrm{s}\} = 0 \tag{4.42}$$

if and only if s is a solution to the mixed problem.

A proof of the above variational principles is based on the *Fundamental Lemma of Calculus of Variations* which states that for every smooth function $\tilde{\mathrm{g}} = \tilde{\mathrm{g}}(\mathbf{x})$ on

$\overline{\mathrm{B}}$ that vanishes near $\partial\mathrm{B}$, and for a fixed continuous function $\mathrm{f} = \mathrm{f}(\mathbf{x})$ on $\overline{\mathrm{B}}$, the condition $\int_{\mathrm{B}} \mathrm{f}(\mathbf{x})\tilde{\mathrm{g}}(\mathbf{x})\, dv(\mathbf{x}) = 0$ is equivalent to $\mathrm{f}(\mathbf{x}) = 0$ on $\overline{\mathrm{B}}$.

4.5 Problems and Solutions Related to Variational Formulation of Elastostatics

Problem 4.1. Consider a generalized plane stress traction problem of homogeneous isotropic elastostatics for a region C_0 of (x_1, x_2) plane (see Sect. 7). For such a problem the stress energy is represented by the integral

$$\overline{\mathrm{U}}_{\mathrm{K}}\{\overline{\mathbf{S}}\} = \frac{1}{2}\int_{\mathrm{C}_0} \overline{\mathbf{S}} \cdot \mathbf{K}[\overline{\mathbf{S}}]\, da \tag{4.43}$$

where $\overline{\mathbf{S}}$ is the stress tensor corresponding to a solution $\overline{s} = [\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{S}}]$ of the traction problem, and

$$\overline{\mathbf{E}} = \mathbf{K}[\overline{\mathbf{S}}] = \frac{1}{2\mu}\left[\overline{\mathbf{S}} - \frac{\nu}{1+\nu}(\mathrm{tr}\, \overline{\mathbf{S}})\, \mathbf{1}\right] \quad \text{on} \quad \mathrm{C}_0 \tag{4.44}$$

$$\mathrm{div}\, \overline{\mathbf{S}} + \overline{\mathbf{b}} = \mathbf{0} \quad \text{on} \quad \mathrm{C}_0 \tag{4.45}$$

$$\overline{\mathbf{E}} = \widehat{\nabla}\overline{\mathbf{u}} \quad \text{on} \quad \mathrm{C}_0 \tag{4.46}$$

and

$$\overline{\mathbf{S}}\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial\mathrm{C}_0 \tag{4.47}$$

Let $\overline{\mathrm{Q}}$ denote the set of all admissible states that satisfy Eq. (4.44) through (4.47) except for Eq. (4.46). Define the functional $\overline{I}\{.\}$ on $\overline{\mathrm{Q}}$ by

$$\overline{I}\{\overline{s}\} = \mathrm{U}_{\mathrm{K}}\{\overline{\mathbf{S}}\} \quad \text{for every } \overline{s} \in \overline{\mathrm{Q}} \tag{4.48}$$

Show that

$$\delta\, \overline{I}\{\overline{s}\} = 0 \tag{4.49}$$

if and only if $\bar{\mathbf{s}}$ is a solution to the traction problem.

Hint: The proof is similar to that of the compatibility-related principle of Sect. 4.4. First, we note that if $\bar{\mathrm{s}} \in \overline{\mathrm{Q}}$ and $\tilde{\mathrm{s}} \in \overline{\mathrm{Q}}$ then $\bar{\mathrm{s}} + \omega\tilde{\mathrm{s}} \in \overline{\mathrm{Q}}$ for every scalar ω, and

$$\delta\, \overline{I}\{\bar{\mathrm{s}}\} = \int_{\mathrm{C}_0} \tilde{\mathbf{S}} \cdot \overline{\mathbf{E}}\, da \tag{4.50}$$

Next, by letting

$$\tilde{S}_{\alpha\beta} = \varepsilon_{\alpha\gamma 3}\, \varepsilon_{\beta\delta 3}\, \tilde{F}_{,\gamma\delta} \tag{4.51}$$

where $\tilde{F}$ is an Airy stress function such that $\tilde{F}$, $\tilde{F}_{,\alpha}$, and $\tilde{F}_{,\alpha\beta}$ $(\alpha, \beta = 1, 2)$ vanish near $\partial\, \mathrm{C}_0$, w find that

$$\delta\, \overline{I}\{\overline{\mathrm{s}}\} = \int_{\mathrm{C}_0} \tilde{F}\, \varepsilon_{\alpha\gamma 3}\, \varepsilon_{\beta\delta 3}\, \overline{E}_{\alpha\beta,\gamma\delta}\, da \tag{4.52}$$

The proof then follows from (4.52).

Solution. We are to show that

(A) If $\overline{s}$ is a solution to the traction problem then

$$\delta \overline{I}(\overline{s}) = 0 \tag{4.53}$$

and

(B) If

$$\delta \overline{I}(\overline{s}) = 0 \quad \text{for } \overline{s} \in \overline{Q} \tag{4.54}$$

then $\overline{s}$ is a solution to the traction problem.

Proof of (A). Using (4.52) we obtain

$$\delta \overline{I}(\overline{s}) = \int_{C_0} \tilde{F} \varepsilon_{\alpha\gamma 3}\, \varepsilon_{\beta\delta 3}\, \overline{E}_{\alpha\beta,\gamma\delta}\, da \tag{4.55}$$

Since $\overline{s} = [\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{S}}]$ is a solution to the fraction problem, Eqs. (4.44)–(4.47) are satisfied, and in particular

$$\overline{E}_{\alpha\beta} = \overline{u}_{(\alpha,\beta)} \tag{4.56}$$

Substituting (4.56) into the RHS of (4.55) we obtain (4.53), and this completes proof of (A).

Proof of (B). We assume that

$$\delta \overline{I}(\overline{s}) = 0 \quad \text{for } \overline{s} \in \overline{Q} \tag{4.57}$$

or

$$\int_{C_0} \tilde{F} \varepsilon_{\alpha\gamma 3}\, \varepsilon_{\beta\delta 3}\, \overline{E}_{\alpha\beta,\gamma\delta}\, da = 0 \tag{4.58}$$

where $\tilde{F}$ is an arbitrary function on C_0 that vanishes near ∂C_0, and $\overline{E}_{\alpha\beta}$ is a symmetric second order tensor field on C_0 that complies with Eqs. (4.44), (4.45), and (4.47). It follows from (4.58) and the Fundamental Lemma of calculus of variations that

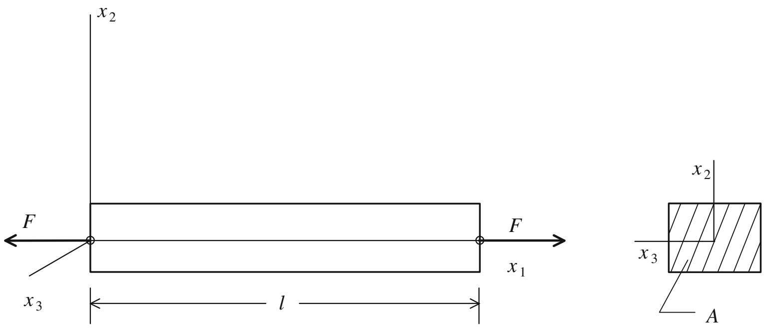


Fig. 4.1 The prismatic bar in simple tension

$$\varepsilon_{\alpha\gamma 3}\, \varepsilon_{\beta\delta 3}\, \overline{E}_{\alpha\beta,\gamma\delta} = 0 \quad \text{on} \quad C_0 \tag{4.59}$$

This implies that there is $\overline{u}_\alpha$ such that

$$\overline{E}_{\alpha\beta} = \overline{u}_{(\alpha,\beta)} \tag{4.60}$$

As a result $\overline{s} = [\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{S}}]$ satisfies Eqs. (4.44)–(4.47), that is, $\overline{s}$ is a solution to the traction problem. This completes proof of (B).

Problem 4.2. Consider an elastic prismatic bar in simple tension shown in Fig. 4.1. The stress energy of the bar takes the form

$$\mathrm{U_K}\{\mathbf{S}\} = \int_0^l \left(\int_A \frac{1}{2E} S_{11}^2 \, da \right) dx_1 = \frac{1}{2E} \int_0^l \left(\frac{F}{A}\right)^2 A\, dx = \frac{F^2 l}{2EA} \tag{4.61}$$

where A is the cross section of the bar, and E denotes Young's modulus.

The strain energy of the bar is obtained from

$$\mathrm{U_C}\{\mathbf{E}\} = \mathrm{U_K}\{\mathbf{S}\} = \frac{EAe^2}{2l} \tag{4.62}$$

where e is an elongation of the bar produced by the force $F = AEE_{11} = AEe/l$. The elastic state of the bar is then represented by

$$s = [u_1, E_{11}, S_{11}] = [e, e/l, F/A] \tag{4.63}$$

(i) Define a potential energy of the bar as $\widehat{\mathrm{F}}\{\mathrm{s}\} \equiv \varphi(e)$ and show that the relation

$$\delta\varphi(e) = 0 \tag{4.64}$$

is equivalent to the condition

$$\frac{\partial U_{\mathrm{C}}}{\partial e} = F \tag{4.65}$$

(ii) Define a complementary energy of the bar as $\widehat{\mathrm{G}}\{\mathrm{s}\} \equiv \psi(F)$ and show that the condition

$$\delta\psi(F) = 0 \tag{4.66}$$

is equivalent to the equation

$$\frac{\partial U_{\mathrm{K}}}{\partial F} = e \tag{4.67}$$

Hint: The functions $\varphi = \varphi(e)$ and $\psi = \psi(F)$ are given by

$$\varphi(e) = \frac{EA}{2l}e^2 - Fe$$

and

$$\psi(F) = \frac{l}{2EA}F^2 - Fe$$

respectively.

Note: Equations (4.65) and (4.67) constitute the *Castigliano theorem.*

Solution. The potential energy of the bar is given by

$$\varphi(e) = U_c(e) - Fe \tag{4.68}$$

where

$$U_c(e) = \frac{EAe^2}{2l} \tag{4.69}$$

Hence, the relation

$$\delta\varphi(e) = \varphi'(e) = 0 \tag{4.70}$$

takes the form

$$\frac{\partial U_c}{\partial e} = F \tag{4.71}$$

Equations (4.69) and (4.71) imply that

$$F = \frac{EAe}{l} \tag{4.72}$$

which is consistent with the definition of F. This shows that (i) holds true. To prove (ii) we define the complementary energy of the bar as

$$\psi(F) = U_k(F) - Fe \tag{4.73}$$

where

$$U_k(F) = \frac{F^2 l}{2EA} \tag{4.74}$$

and from the relation

$$\delta\psi(F) = \psi'(F) = 0 \tag{4.75}$$

we obtain

$$\frac{\partial U_k}{\partial F} = e \tag{4.76}$$

Equations (4.74) and (4.76) imply that

$$e = \frac{Fl}{EA} \tag{4.77}$$

which is consistent with the definition of e. This shows that (ii) holds true. Hence, a solution to Problem 4.2 is complete.

Problem 4.3. The complementary energy of a cantilever beam loaded at the end by force P takes the form (see Fig. 4.2)

$$\begin{aligned}\psi(P) &= \frac{1}{2E}\int_B S_{11}^2\,dv - Pu_2(l)\\ &= \frac{1}{2E}\int_0^l \left\{\int_A \frac{M^2(x_1)}{I^2} x_2^2\,dA\right\} dx_1 - Pu_2(l)\end{aligned} \tag{4.78}$$

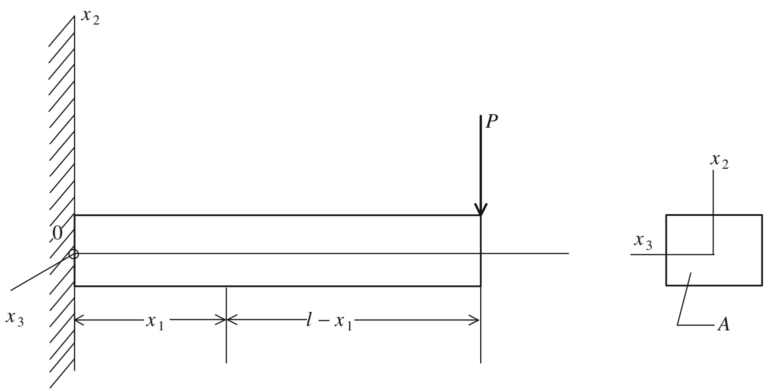


Fig. 4.2 The cantilever beam loaded at the end

where $M = M(x_1)$ and I stand for the bending moment and the moment of inertia of the area A with respect to the x_3 axis, respectively, given by

$$M(x_1) = P(l - x_1), \quad I = \int_A x_2^2 \, da \tag{4.79}$$

Use the minimum complementary energy principle for the cantilever beam in the form

$$\delta\psi(P) = 0 \tag{4.80}$$

to show that the magnitude of deflection at the end of the beam is

$$u_2(l) = \frac{Pl^3}{3EI} \tag{4.81}$$

Solution. Substituting $M = M(x_1)$ and I from (4.79) into (4.78) and performing the integration we obtain.

$$\psi(P) = \frac{P^2}{2EI}\frac{l^3}{3} - P\, u_2(l) \tag{4.82}$$

Finally, using the minimum complementary energy principle

$$\delta\psi(P) = \psi'(P) = 0 \tag{4.83}$$

we arrive at (4.81), and this completes a solution to Problem 4.3.

Problem 4.4. An elastic beam which is clamped at one end and simply supported at the other end is loaded at an internal point $x_1 = \xi$ by force P (see Fig. 4.3)

The potential energy of the beam, treated as a functional depending on a deflection of the beam $u_2 = u_2(x_1)$, takes the form

$$\varphi\{u_2\} = \frac{EI}{2}\int_0^l \left(\frac{d^2u_2}{dx_1^2}\right)^2 dx_1 - Pu_2(\xi) \tag{4.84}$$

and $u_2 \in \tilde{P} = \{u_2 = u_2(x_1) : u_2(0) = u_2'(0) = 0; \quad u_2(l) = u_2''(l) = 0\}$. Let $u_2 = u_2(x_1)$ be a solution of the equation

$$EI\,\frac{d^4u_2}{dx_1^4} = P\delta(x_1 - \xi) \quad \text{for} \quad 0 < x_1 < l \tag{4.85}$$

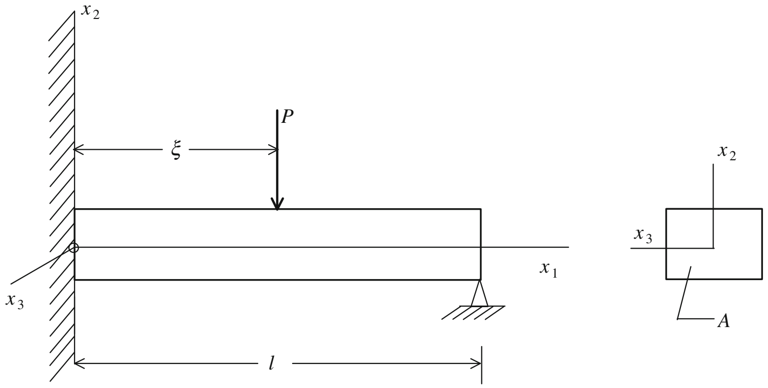


Fig. 4.3 The beam clamped at one end and simply supported at the other end

subject to the conditions

$$u_2(0) = u_2'(0) = 0; \quad u_2(l) = u_2''(l) = 0 \tag{4.86}$$

Show that

$$\delta\varphi\{u_2\} = 0 \tag{4.87}$$

if and only if u_2 is a solution to the boundary value problem (4.85)–(4.86).

Solution. Since

$$\delta\varphi\{u_2\} = \frac{d}{d\omega}\varphi\{u_2 + \omega\tilde{u}_2\}\bigg|_{\omega=0} \tag{4.88}$$

where

$$\tilde{u}_2(0) = \tilde{u}_2'(0) = 0 \tag{4.89}$$

and

$$\tilde{u}_2(l) = \tilde{u}_2''(l) = 0 \tag{4.90}$$

therefore, Eq. (4.88) takes the form

$$\delta\varphi\{u_2\} = EI\int_0^l u_2''(x)\tilde{u}_2''(x)dx - P\,\tilde{u}_2(\xi) \tag{4.91}$$

Integrating by parts we obtain

$$\int_0^l u_2''(x)\tilde{u}_2''(x)dx = u_2''(x)\tilde{u}_2'(x)\Big|_{x=0}^{x=l} - u_2'''(x)\tilde{u}_2(x)\Big|_{x=0}^{x=l} + \int_0^l u_2^{(4)}(x)\tilde{u}_2(x)dx \tag{4.92}$$

Since $u_2 \in \tilde{P}$ and $\tilde{u}_2 \in \tilde{P}$, Eq. (4.92) reduces to

$$\int_0^l u_2''(x)\tilde{u}_2''(x)dx = \int_0^l u_2^{(4)}(x)\tilde{u}_2(x)dx \tag{4.93}$$

and Eq. (4.91) takes the form

$$\delta\varphi\{u_2\} = \int_0^l \left[EI\, u_2^{(4)}(x) - P\delta(x-\xi)\right]\tilde{u}_2(x)dx \tag{4.94}$$

Equation (4.94) together with the Fundamental Lemma of calculus of variations imply that Eq. (4.87) is satisfied if and only if u_2 is a solution to problem (4.85)–(4.86). And this completes a solution to Problem 4.4.

Problem 4.5. Use the Rayleigh-Ritz method to show that an approximate deflection of the beam of Problem 4.4 takes the form ($x_1 = x$)

$$u_2(x) = -\mathrm{c}l^3\left(\frac{x}{l}\right)^2\left(1-\frac{x}{l}\right)\left(1-\frac{2}{3}\frac{x}{l}\right) \tag{4.95}$$

where

$$\mathrm{c} = -\frac{5}{4}\frac{P}{EI}\left(\frac{\xi}{l}\right)^2\left(1-\frac{\xi}{l}\right)\left(1-\frac{2}{3}\frac{\xi}{l}\right) \tag{4.96}$$

Also, show that for $\xi = l/2$ we obtain

$$u_2(l/2) = 0.0086\frac{l^3 P}{EI} \tag{4.97}$$

Solution. Note that $u_2 = u_2(x)$ given by Eq. (4.95) can be written in the form

$$u_2(x) = -cl^3 f\left(\frac{x}{l}\right) \tag{4.98}$$

where

$$f\left(\frac{x}{l}\right) = \left(\frac{x}{l}\right)^2\left(1-\frac{x}{l}\right)\left(1-\frac{2}{3}\frac{x}{l}\right) \tag{4.99}$$

and

$$f(0) = f'(0) = f(1) = f''(1) = 0 \tag{4.100}$$

Hence

$$u_2(x) \in \tilde{P} \tag{4.101}$$

where $\tilde{P}$ is the domain of the functional $\varphi\{u_2\}$ from Problem 4.4, and substituting (4.98) into Eq. (4.95) of Problem 4.4 we obtain

$$\varphi\{u_2\} = l^3 \left\{ \frac{EI}{2} c^2 \int_0^1 [f''(u)]^2 du + P\, c\, f\left(\frac{\xi}{l}\right) \right\} \equiv \psi(c) \tag{4.102}$$

The condition

$$\delta\varphi\{u_2\} = \psi'(c) = 0 \tag{4.103}$$

is satisfied if and only if

$$c \int_0^1 (f'')^2 du = -\frac{P}{EI}\, f\left(\frac{\xi}{l}\right) \tag{4.104}$$

Since

$$f''(u) = 2(4u^2 - 5u + 1) \tag{4.105}$$

and

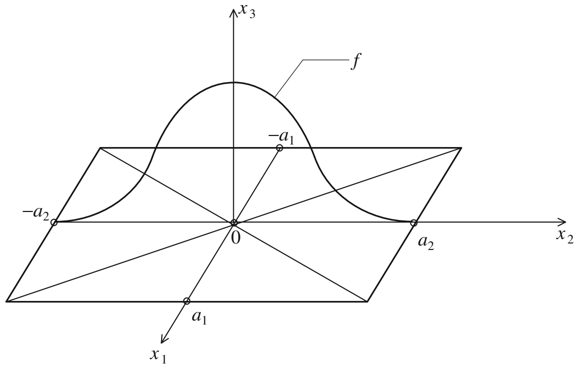
$$\int_0^1 (f'')^2 du = \frac{4}{5} \tag{4.106}$$

it follows from Eq. (4.104) that c is given by Eq. (4.96). Finally, by letting $x = l/2$ and $\xi = l/2$ in Eqs. (4.95) and (4.96), respectively, we obtain (4.97). This completes a solution to Problem 4.5.

Problem 4.6. The potential energy of a rectangular thin elastic membrane fixed at its boundary and subject to a vertical load $f = f(x_1, x_2)$ is

$$I\{u\} = \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} \left(\frac{T_0}{2} u_{,\alpha} u_{,\alpha} - f\, u \right) dx_1 dx_2 \tag{4.107}$$

Fig. 4.4 The thin membrane fixed at its boundary



where $u \in \widehat{P}$, and

$$\widehat{P} = \{u = u(x_1, x_2) \;:\; u(\pm a_1, x_2) = 0 \quad \text{for} \quad |x_2| < a_2; \\ u(x_1, \pm a_2) = 0 \quad \text{for} \quad |x_1| < a_1\} \tag{4.108}$$

Here, $u = u(x_1, x_2)$ is a deflection of the membrane in the x_3 direction, and T_0 is a uniform tension of the membrane (see Fig. 4.4). Let the load function $f = f(x_1, x_2)$ be represented by the series

$$f(x_1, x_2) = \sum_{m,n=1}^{\infty} f_{mn} \sin \frac{m\pi(x_1 - a_1)}{2a_1} \sin \frac{n\pi(x_2 - a_2)}{2a_2} \tag{4.109}$$

Use the Rayleigh-Ritz method to show that the functional $I\{u\}$ attains a minimum over $\widehat{P}$ at

$$u(x_1, x_2) = \sum_{m,n=1}^{\infty} u_{mn} \sin \frac{m\pi(x_1 - a_1)}{2a_1} \sin \frac{n\pi(x_2 - a_2)}{2a_2} \tag{4.110}$$

where

$$u_{mn} = \frac{1}{T_0} \frac{f_{mn}}{[(m\pi/2a_1)^2 + (n\pi/2a_2)^2]} \quad m, n = 1, 2, 3, \ldots \tag{4.111}$$

Solution. Let C_0 stand for the interior of rectangular region

$$C_0 = \{(x_1, \; x_2) : |x_1| < a_1, \; |x_2| < a_2\} \tag{4.112}$$

and let ∂C_0 denote its boundary.

Then

$$\widehat{P} = \{u : u = 0 \quad \text{on} \quad \partial C_0\} \tag{4.113}$$

let $u \in \widehat{P}$ and $u + \omega\tilde{u} \in \widehat{P}$, where ω is a scalar. Then

$$\tilde{u} \in \widehat{P}, \text{ that is, } \quad \tilde{u} = 0 \quad \text{on } \partial C_0 \tag{4.114}$$

Computing the first variation of $I\{u\}$ we obtain

$$\delta I\{u\} = \frac{d}{d\omega} I\{u + \omega\tilde{u}\}|_{\omega=0} = \int_{C_0} (T_0 u_{,\alpha} \ \tilde{u}_{,\alpha} - f\tilde{u}) da \tag{4.115}$$

Since

$$u_{,\alpha} \ \tilde{u}_{,\alpha} = (u_{,\alpha} \ \tilde{u})_{,\alpha} - u_{,\alpha\alpha} \ \tilde{u} \tag{4.116}$$

therefore, using the divergence theorem, from Eqs. (4.115) and (4.116) we obtain

$$\delta I\{u\} = -\int_{C_0} (T_0 u_{,\alpha\alpha} + f)\tilde{u} \, da \tag{4.117}$$

and

$$\delta I\{u\} = 0 \quad \text{for every } u \in \widehat{P} \tag{4.118}$$

if and only if $u = u(x_1, x_2)$ is a solution to the boundary value problem

$$u_{,\alpha\alpha} = -\frac{1}{T_0} f \quad \text{on } \ C_0 \tag{4.119}$$

$$u = 0 \quad \text{on } \ \partial C_0 \tag{4.120}$$

Therefore, the Rayleigh Ritz method applied to the functional $I = I\{u\}$ leads to a solution of problem (4.119)–(4.120). It is easy to show, by substituting (4.110) into Eq. (4.119), that $u = u(x_1, x_2)$ given by (4.110) is a solution to problem (4.119)–(4.120).

To obtain the formula (4.110) by the Rayleigh Ritz method we look for $u = u(x_1, x_2)$ that minimizes $I\{u\}$ in the form

$$u(x_1, x_2) = \sum_{mn} c_{mn} \varphi_m(x_1) \ \psi_n(x_2) \tag{4.121}$$

where

$$\varphi_m(x_1) = \sin \frac{m\pi(x_1 - a_1)}{2a_1} \tag{4.122}$$

and

$$\psi_n(x_2) = \sin \frac{n\pi(x_2 - a_2)}{2a_2} \tag{4.123}$$

Substituting u from (4.121) into (4.107) and using f given by (4.109) we obtain

$$I\{u\} \equiv F(c_{mn}) = \int_{-a_1}^{a_1} dx_1 \int_{-a_2}^{a_2} dx_2 \left\{ \frac{T_0}{2} \left[\sum_{mn} c_{mn} \varphi_m'(x_1)\, \psi_n(x_2) \right]^2 \right.$$
$$+ \frac{T_0}{2} \left[\sum_{mn} c_{mn} \varphi_m(x_1)\, \psi_n'(x_2) \right]^2 - \left[\sum_{mn} c_{mn} \varphi_m(x_1)\, \psi_n(x_2) \right]$$
$$\left. \times \left[\sum_{pq} f_{pq} \varphi_p(x_1)\, \psi_q(x_2) \right] \right\} \tag{4.124}$$

The conditions

$$\frac{\partial F}{\partial c_{mn}} = 0 \quad m, n = 1, 2, \ldots \tag{4.125}$$

together with the orthogonality relations

$$\frac{1}{a_1} \int_{-a_1}^{a_1} \varphi_m(x_1)\, \varphi_k(x_1) dx_1 = \delta_{mk} \tag{4.126}$$

$$\frac{1}{a_2} \int_{-a_2}^{a_2} \psi_m(x_2)\, \psi_k(x_2) dx_2 = \delta_{mk} \tag{4.127}$$

lead to the simple algebraic equation for c_{mn}

$$T_0 c_{mn} [(m\pi/2a_1)^2 + (n\pi/2a_2)^2] - f_{mn} = 0 \tag{4.128}$$

Therefore, $c_{mn} = u_{mn}$, where u_{mn} is given by (4.111). This completes a solution to Problem 4.6.

Problem 4.7. Use the solution obtained in Problem 4.6 to find the deflection of a square membrane of side a that is held fixed at its boundary and is vertically loaded by a load f of the form

$$f(x_1, x_2) = f_0[H(x_1 + \varepsilon) - H(x_1 - \varepsilon)][H(x_2 + \varepsilon) - H(x_2 - \varepsilon)] \tag{4.129}$$

where $H = H(x)$ is the Heaviside function, and f_0 and ε are positive constants ($0 < \varepsilon < a$). Also, compute a deflection of the square membrane at its center when $\varepsilon = a/8$.

Solution. Let f be a function represented by the double series [see (4.109) of Problem 4.6]

$$f(x_1, x_2) = \sum_{mn} f_{mn} \varphi_m(x_1)\, \psi_n(x_2) \tag{4.130}$$

where φ_m and ψ_n are given by Eqs. (4.122) and (4.123), respectively, of Problem 4.6. Using the orthogonality conditions (4.126) and (4.127) of Problem 4.6, we find that

$$f_{mn} = \frac{1}{a_1 a_2} \int_{-a_1}^{a_1} dx_1 \int_{-a_2}^{a_2} dx_2\ f(x_1, x_2)\ \varphi_m(x_1)\ \psi_n(x_2) \tag{4.131}$$

For a square membrane of side a

$$a_1 = a_2 = a \tag{4.132}$$

and

$$\varphi_m(x_1) = \sin \frac{m\pi(x_1 - a)}{2a} \tag{4.133}$$

$$\psi_n(x_2) = \sin \frac{n\pi(x_2 - a)}{2a} \tag{4.134}$$

Substituting f from (4.129) into (4.131) we obtain

$$f_{mn} = \frac{f_0}{a^2} \int_{-\varepsilon}^{\varepsilon} dx_1 \int_{-\varepsilon}^{\varepsilon} dx_2\ \varphi_m(x_1)\psi_n(x_2) \tag{4.135}$$

$$= \frac{16}{\pi^2} f_0 \frac{1}{mn} \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{m\pi}{2}\ \frac{\varepsilon}{a}\right) \times \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{2}\ \frac{\varepsilon}{a}\right) \tag{4.136}$$

Therefore, for a load f of the form (4.129) the deflection of the membrane is given by

$$u(x_1, x_2) = \sum_{m,n=1}^{\infty} u_{mn}\varphi_m(x_1)\ \varphi_n(x_2) \tag{4.137}$$

where

$$u_{mn} = \frac{1}{T_0}\ \frac{4a^2}{\pi^2}\ \frac{f_{mn}}{m^2 + n^2} \tag{4.138}$$

and f_{mn} is given by (4.136).

Letting $x_1 = 0$ and $x_2 = 0$ in (4.137) we obtain

$$u(0,0) = \frac{64a^2}{\pi^4} \frac{f_0}{T_0} \times \sum_{m,n=1}^{\infty} \frac{1}{mn(m^2 + n^2)} \sin^2 \frac{m\pi}{2} \sin^2 \frac{m\pi}{2} \left(\frac{\varepsilon}{a}\right) \times \sin^2 \frac{n\pi}{2} \sin \frac{n\pi}{2} \left(\frac{\varepsilon}{a}\right) \tag{4.139}$$

Since

$$\sin^2 \frac{m\pi}{2} = \frac{1}{2}(1 - \cos m\pi) = \frac{1 - (-)^m}{2} \tag{4.140}$$

and

$$\sin^2 \frac{n\pi}{2} = \frac{1}{2}(1 - \cos n\pi) = \frac{1 - (-)^n}{2} \tag{4.141}$$

therefore, (4.139) can be written as

$$u(0,0) = \frac{64a^2}{\pi^4} \frac{f_0}{T_0} \times \sum_{m,n=1,3,5,\ldots}^{\infty} \frac{1}{mn(m^2+n^2)} \sin \frac{m\pi}{2} \left(\frac{\varepsilon}{a}\right) \sin \frac{n\pi}{2} \left(\frac{\varepsilon}{a}\right) \tag{4.142}$$

Using the orthogonality relations

$$\int_0^1 \sin m\pi\zeta \ \sin n\pi\zeta \ d\zeta = \frac{1}{2} \delta_{mn} \tag{4.143}$$

it is easy to show that

$$\frac{\pi}{4n^2} \left[1 - \frac{\cos\text{ h} \left[\frac{n\pi}{2}(1-2\zeta)\right]}{\cos\text{ h} \frac{n\pi}{2}} \right] = \sum_{m=1,3,5,\ldots}^{\infty} \frac{\sin m\pi\zeta}{m(m^2+n^2)} \quad \text{for } 0 < \zeta < 1 \tag{4.144}$$

Since

$$\varepsilon < 2a$$

therefore, letting $\zeta = \varepsilon/2a < 1$ into (4.144) we reduce the double series (4.142) to the single one

$$u(0,0) = \frac{16a^2}{\pi^3} \frac{f_0}{T_0} \times \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin \left(\frac{n\pi}{2} \frac{\varepsilon}{a}\right) \left[1 - \frac{\cos\text{ h} \frac{n\pi}{2} \left(1 - \frac{\varepsilon}{a}\right)}{\cosh \frac{n\pi}{2}} \right] \tag{4.145}$$

Finally, letting $\varepsilon/a = 1/8$ in (4.145) we get

$$u(0,0) = \frac{16a^2}{\pi^3} \frac{f_0}{T_0} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{16} \times \left[1 - \frac{\cos\text{ h} \left(\frac{7}{16} n\pi\right)}{\cos\text{ h} \left(\frac{1}{2} n\pi\right)} \right] \tag{4.146}$$

This completes a solution to Problem 4.7.

Problem 4.8. The potential energy of a rectangular thin elastic plate that is simply supported along all the edges and is vertically loaded by a force P at a point (ξ_1, ξ_2)

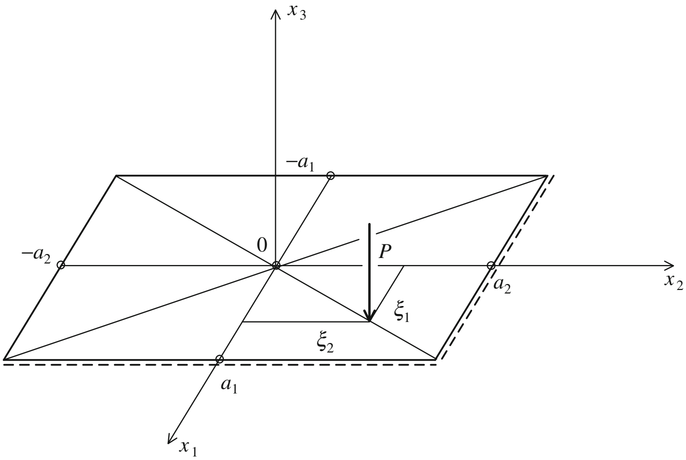


Fig. 4.5 The *rectangular* thin plate simply supported along all edges

takes the form

$$\widehat{I}\{w\} = \frac{1}{2} D \int_{-a_1}^{a_1} \int_{-a_2}^{a_2} (\nabla^2 w)^2 dx_1 dx_2 - P\, w(\xi_1, \xi_2) \tag{4.147}$$

where $w \in \tilde{P}$, and

$$\begin{aligned} \tilde{P} = \{w = w(x_1, x_2) : \; & w(\pm a_1, x_2) = 0, \quad \nabla^2 w(\pm a_1, x_2) = 0 \quad \text{for } |x_2| < a_2; \\ & w(x_1, \pm a_2) = 0, \quad \nabla^2 w(x_1, \pm a_2) = 0 \quad \text{for } |x_1| < a_1\} \end{aligned} \tag{4.148}$$

Here $w = w(x_1, x_2)$ is a deflection of the plate, and D is the bending rigidity of the plate (see Fig. 4.5).

Show that a minimum of the functional $\widehat{I}\{.\}$ over $\tilde{P}$ is attained at a function $w = w(x_1, x_2)$ represented by the series

$$w(x_1, x_2) = \sum_{m,n=1}^{\infty} w_{mn} \sin \frac{m\pi(x_1 - a_1)}{2a_1} \sin \frac{n\pi(x_2 - a_2)}{2a_2} \tag{4.149}$$

where

$$w_{mn} = \frac{P}{Da_1a_2} \frac{\sin \dfrac{m\pi}{2a_1}(\xi_1 - a_1) \sin \dfrac{n\pi}{2a_2}(\xi_2 - a_2)}{[(m\pi/2a_1)^2 + (n\pi/2a_2)^2]^2} \quad m, n = 1, 2, 3, \ldots \tag{4.150}$$

Hint: Use the series representation of the concentrated load P

$$
\begin{aligned}
&P\delta(x_1-\xi_1)\delta(x_2-\xi_2)\\
&\quad=\frac{P}{a_1a_2}\sum_{m,n=1}^{\infty}\sin\frac{m\pi}{2a_1}(\xi_1-a_1)\sin\frac{n\pi}{2a_2}(\xi_2-a_2)\sin\frac{m\pi}{2a_1}(x_1-a_1)\\
&\qquad\times\sin\frac{n\pi}{2a_2}(x_2-a_2)\\
&\qquad\text{for every } |x_1|<a_1,\quad |x_2|<a_2,\quad |\xi_1|<a_1,\quad |\xi_2|<a_2.
\end{aligned}
\tag{4.151}
$$

Solution. Let $w\in\tilde{P}$ and $\tilde{w}\in\tilde{P}$. Then $w+\omega\tilde{w}\in\tilde{P}$, and the first variation of $\widehat{I}\{w\}s$ takes the form

$$
\begin{aligned}
\delta\widehat{I}\{w\}=&\frac{d}{d\omega}\,\widehat{I}\,\{w+\omega\tilde{w}\}_{|\omega=0}\\
=&D\int_{-a_1}^{a_1}dx_1\int_{-a_2}^{a_2}dx_2(\nabla^2w)(\nabla^2\tilde{w})-P\tilde{w}(\xi)
\end{aligned}
\tag{4.152}
$$

Let C_0 be an interior of the rectangular region, and let ∂C_0 denote its boundary. Then Eq. (4.152) can be written as

$$
\delta\widehat{I}\{w\}=D\int_{C_0}w_{,\alpha\alpha}\;\tilde{w}_{,\beta\beta}\;da-P\tilde{w}(\xi)
\tag{4.153}
$$

Since

$$
\begin{aligned}
w_{,\alpha\alpha}\;\tilde{w}_{,\beta\beta}&=(w_{,\alpha\alpha}\;\tilde{w}_{,\beta}\,)_{,\beta}-w_{,\alpha\alpha\beta}\;\tilde{w}_{,\beta}\\
&=(w_{,\alpha\alpha}\;\tilde{w}_{,\beta}-w_{,\alpha\alpha\beta}\;\tilde{w})_{,\beta}+w_{,\alpha\alpha\beta\beta}\;\tilde{w}
\end{aligned}
\tag{4.154}
$$

therefore, integrating (4.154) over C_0, using the divergence theorem, and the relations

$$
w_{,\alpha\alpha}=0,\quad \tilde{w}=0\quad\text{on }\partial C_0
\tag{4.155}
$$

we reduce (4.153) to the form

$$
\delta\widehat{I}\{w\}=\int_{C_0}[D\nabla^4w-P\delta(\mathbf{x}-\xi)]\tilde{w}(\xi)da
\tag{4.156}
$$

A minimum of the functional $\widehat{I}\{w\}$ over $\tilde{P}$ is attained at w that satisfies the field equation

$$
\nabla^4w=\frac{P}{D}\delta(\mathbf{x}-\xi)\quad\text{on }C_0
\tag{4.157}
$$

subject to the homogeneous b conditions

$$w = 0, \quad \nabla^2 w = 0 \quad \text{on } \partial C_0 \tag{4.158}$$

To obtain a solution to problem (4.157)–(4.158) we use the representation of $\delta(\mathbf{x}-\xi)$

$$\delta(\mathbf{x} - \xi) = \frac{1}{a_1 a_2} \sum_{m,n=1}^{\infty} \varphi_m(x_1)\, \varphi_m(\xi_1)\, \psi_n(x_2)\, \psi_n(\xi_2) \tag{4.159}$$

where $\varphi_m(x_1)$ and $\psi_n(x_2)$, respectively, are given by Eqs. (4.122) and (4.123) of Problem 4.6 Since

$$\nabla^2 \varphi_m(x_1)\, \psi_n(x_2) = -\left[\left(\frac{m\pi}{2a_1}\right)^2 + \left(\frac{n\pi}{2a_2}\right)^2\right] \varphi_m(x_1)\, \psi_n(x_2) \tag{4.160}$$

therefore, by looking for a solution of Eq. (4.157) in the form

$$w(x_1, x_2) = \sum_{m,n=1}^{\infty} w_{mn}\, \varphi_m(x_1)\, \psi_n(x_2) \tag{4.161}$$

and substituting (4.159) and (4.161) into (4.157) we find that

$$w_{mn} \left[\left(\frac{m\pi}{2a_1}\right)^2 + \left(\frac{n\pi}{2a_2}\right)^2\right]^2 = \frac{P}{D a_1 a_2}\, \varphi_m(\xi_1)\, \psi_n(\xi_2) \tag{4.162}$$

This completes a solution to Problem 4.8.

Problem 4.9. Show that the central deflection of a square plate of side a that is simply supported along all the edges, and is loaded by a force P at its center, takes the form

$$w(0,0) \approx 0.0459 \frac{Pa^2}{D} \tag{4.163}$$

Hint: Use the result obtained in Problem 4.8 when $\xi_1 = \xi_2 = 0$, $x_1 = x_2 = 0$, $a_1 = a_2 = a$

$$w(0,0) = \frac{16 P a^2}{D \pi^4} \sum_{m,n=1}^{\infty} \frac{1}{[(2m-1)^2 + (2n-1)^2]^2} \tag{4.164}$$

Also, by taking advantage of the formula

$$\sum_{m=1}^{\infty} \frac{1}{[(2m-1)^2 + x^2]^2} = \frac{\pi}{8x^3}\left(\tan\text{h}\frac{\pi x}{2} - \frac{\pi x}{2}\sec\text{h}^2\frac{\pi x}{2}\right) \quad \text{for every} \quad x > 0 \tag{4.165}$$

which is obtained by differentiating with respect to x the formula

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2} = \frac{\pi}{4x}\tan\text{h}\frac{\pi x}{2} \tag{4.166}$$

we reduce Eq. (4.164) to the simple form

$$w(0,0) = \frac{2Pa^2}{D\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[\tan\text{h}\frac{\pi}{2}(2n-1) - \frac{\pi}{2}(2n-1)\sec\text{h}^2\frac{\pi}{2}(2n-1)\right] \tag{4.167}$$

The result (4.163) then follows by truncating the series (4.167).

Solution. By letting $a_1 = a_2 = a,\ x_1 = x_2 = 0,\ \xi_1 = \xi_2 = 0$ in Eq. (4.165) of Problem 4.8 we obtain

$$w(0,0) = \sum_{m,n=1}^{\infty} w_{mn}\ \sin\left(\frac{m\pi}{2}\right)\ \sin\left(\frac{n\pi}{2}\right) \tag{4.168}$$

where

$$w_{mn} = \frac{P}{Da^2}\ \frac{\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{(m^2\pi^2/4a^2 + n^2\pi^2/4a^2)^2} \tag{4.169}$$

Hence

$$w(0,0) = \frac{16Pa^2}{D\pi^4}\sum_{m,n=1}^{\infty}\frac{\sin^2\left(\frac{m\pi}{2}\right)\sin^2\left(\frac{n\pi}{2}\right)}{(m^2+n^2)^2} \tag{4.170}$$

or

$$w(0,0) = \frac{16Pa^2}{D\pi^4}\sum_{m,n=1}^{\infty}\frac{1}{[(2m-1)^2 + (2n-1)^2]^2} \tag{4.171}$$

which is equivalent to Eq. (4.164).

Finally, using (4.165) with $x = 2n - 1$, we reduce (4.171) to the single series formula (4.167). This completes solution to Problem 4.9.