Chapter 3
Formulation of Problems of Elasticity

In this chapter both the basic boundary value problems of elastostatics and initial-boundary value problems of elastodynamics are recalled; in particular, the mixed boundary value problems of isothermal and nonisothermal elastostatics, as well as the pure displacement and the pure stress problems of classical elastodynamics are discussed. The Betti reciprocal theorem of elastostatics and Graffi's reciprocal theorem of elastodynamics together with the uniqueness theorems are also presented. An emphasis is made on a pure stress initial-boundary value problem of incompatible elastodynamics in which a body possesses initially distributed defects. [See also Chap. 16.]

3.1 Boundary Value Problems of Elastostatics

Field Equations of Isothermal Elastostatics

The strain-displacement relation

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{3.1}$$

The equations of equilibrium

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.2}$$

The stress-strain relation

$$\mathbf{S} = \mathsf{C}\,[\mathbf{E}] \tag{3.3}$$

By eliminating $\mathbf{E}$ and $\mathbf{S}$ from Eqs. (3.1)–(3.3) we obtain the *displacement equation of equilibrium*

$$\operatorname{div}\,\mathsf{C}\,[\nabla\mathbf{u}] + \mathbf{b} = \mathbf{0} \tag{3.4}$$

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For a homogeneous isotropic body the displacement equation of equilibrium (3.4) reduces to

$$\mu\,\nabla^2\mathbf{u} + (\lambda + \mu)\,\nabla(\operatorname{div}\mathbf{u}) + \mathbf{b} = \mathbf{0} \tag{3.5}$$

or

$$\nabla^2\mathbf{u} + \frac{1}{1-2\nu}\nabla(\operatorname{div}\mathbf{u}) + \frac{\mathbf{b}}{\mu} = \mathbf{0} \tag{3.6}$$

or

$$(\lambda + 2\mu)\nabla(\operatorname{div}\mathbf{u}) - \mu\,\operatorname{curl}\operatorname{curl}\mathbf{u} + \mathbf{b} = \mathbf{0} \tag{3.7}$$

An equivalent form of the stress-strain relation (3.3) reads

$$\mathbf{E} = \mathsf{K}\,[\mathbf{S}] \tag{3.8}$$

Therefore, by eliminating **u** and **E** from Eqs. (3.1), (3.2), and (3.8) the *stress equations of equilibrium* are obtained

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.9}$$

$$\operatorname{curl}\operatorname{curl}\,\mathsf{K}[\mathbf{S}] = \mathbf{0} \tag{3.10}$$

For a homogeneous isotropic body, the stress equations of equilibrium (3.9)–(3.10) reduce to

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.11}$$

$$\nabla^2\mathbf{S} + \frac{1}{1+\nu}\nabla\nabla(\operatorname{tr}\mathbf{S}) + \frac{\nu}{1-\nu}(\operatorname{div}\mathbf{b})\mathbf{1} + 2\widehat{\nabla}\mathbf{b} = \mathbf{0} \tag{3.12}$$

Field Equations of nonisothermal Elastostatics

The strain-displacement relation

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{3.13}$$

The equations of equilibrium

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.14}$$

The stress-strain-temperature relation

$$\mathbf{S} = \mathsf{C}\,[\mathbf{E}] + T\,\mathbf{M} \tag{3.15}$$

or, the strain-stress-temperature relation

$$\mathbf{E} = \mathsf{K}\,[\mathbf{S}] + T\,\mathbf{A} \tag{3.16}$$

In Eqs. (3.15) T stands for a temperature change; while $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$ and $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ are the stress-temperature and thermal expansion tensors, respectively.

For an isotropic body Eqs. (3.15) and (3.16), respectively, take the form

$$\mathbf{S} = 2\mu\,\mathbf{E} + \lambda\,(\mathrm{tr}\,\mathbf{E})\,\mathbf{1} - (3\lambda + 2\mu)\,\alpha\,T\mathbf{1} \tag{3.17}$$

and

$$\mathbf{E} = \frac{1}{2\mu}\left[\mathbf{S} - \frac{\lambda}{3\lambda + 2\mu}(\mathrm{tr}\,\mathbf{S})\,\mathbf{1}\right] + \alpha\,T\,\mathbf{1} \tag{3.18}$$

By eliminating **E** and **S** from Eqs. (3.13)–(3.15) the *displacement-temperature equation of nonisothermal elastostatics* is obtained

$$\mathrm{div}\{\mathsf{C}\,[\nabla\mathbf{u}] + T\mathbf{M}\} + \mathbf{b} = \mathbf{0} \tag{3.19}$$

Also, by eliminating **u** and **E** from Eqs. (3.13), (3.14), and (3.16), the *stress-temperature equations of nonisothermal elastostatics* are obtained

$$\mathrm{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.20}$$

$$\mathrm{curl}\,\mathrm{curl}\,\{\mathsf{K}[\mathbf{S}] + T\mathbf{A}\} = \mathbf{0} \tag{3.21}$$

For an isotropic homogeneous body, Eqs. (3.19) and (3.20)–(3.21), respectively, take the form

$$\mu\,\nabla^2\mathbf{u} + (\lambda + \mu)\,\nabla(\mathrm{div}\,\mathbf{u}) - (3\lambda + 2\mu)\,\alpha\,\nabla\,T + \mathbf{b} = \mathbf{0} \tag{3.22}$$

and

$$\mathrm{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.23}$$

$$\nabla^2\mathbf{S} + \frac{1}{1+\nu}\nabla\nabla(\mathrm{tr}\,\mathbf{S}) + \frac{E\alpha}{1+\nu}\left(\nabla\nabla T + \frac{1+\nu}{1-\nu}\nabla^2 T\,\mathbf{1}\right) + \frac{\nu}{1-\nu}(\mathrm{div}\,\mathbf{b})\mathbf{1} + 2\widehat{\nabla}\mathbf{b} = \mathbf{0} \tag{3.24}$$

3.2 Concept of an Elastic State

An ordered array of functions $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is called an *elastic state* corresponding to the body force **b** if the functions **u**, **E**, and **S** satisfy the system of fundamental field equations (3.1)–(3.3) on B.

An *external force system* for s is defined as a pair $[\mathbf{b}, \mathbf{s}]$ where $\mathbf{s} = \mathbf{Sn}$ with $\mathbf{n}$ being an outward unit vector normal to $\partial \mathrm{B}$.

Theorem of work and energy. If $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is an elastic state corresponding to the external force system $[\mathbf{b}, \mathbf{s}]$ then

$$\int_{\partial \mathrm{B}} \mathbf{s} \cdot \mathbf{u} \, da + \int_{\mathrm{B}} \mathbf{b} \cdot \mathbf{u} \, dv = 2\mathrm{U}_{\mathrm{C}}(\mathbf{E}) \tag{3.25}$$

where $\mathrm{U}_{\mathrm{C}}(\mathbf{E})$ is the total strain energy of body B

$$\mathrm{U}_{\mathrm{C}}(\mathbf{E}) = \frac{1}{2} \int_{\mathrm{B}} \mathbf{E} \cdot \mathbf{C}\,[\mathbf{E}] \, dv \tag{3.26}$$

The Betti reciprocal theorem. Let the elasticity tensor $\mathbf{C}$ be symmetric, and let

$$s = [\mathbf{u}, \mathbf{E}, \mathbf{S}] \quad \text{and} \quad \widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}] \tag{3.27}$$

be elastic states corresponding to the external force systems $[\mathbf{b}, \mathbf{s}]$ and $[\widetilde{\mathbf{b}}, \widetilde{\mathbf{S}}]$, respectively. Then the following reciprocity relation holds

$$\int_{\partial \mathrm{B}} \mathbf{s} \cdot \widetilde{\mathbf{u}} \, da + \int_{\mathrm{B}} \mathbf{b} \cdot \widetilde{\mathbf{u}} \, dv = \int_{\partial \mathrm{B}} \widetilde{\mathbf{s}} \cdot \mathbf{u} \, da + \int_{\mathrm{B}} \widetilde{\mathbf{b}} \cdot \mathbf{u} \, dv = \int_{\mathrm{B}} \mathbf{S} \cdot \widetilde{\mathbf{E}} \, dv = \int_{\mathrm{B}} \widetilde{\mathbf{S}} \cdot \mathbf{E} \, dv \tag{3.28}$$

3.3 Concept of a Thermoelastic State

An ordered array of functions $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is called a *thermoelastic state* corresponding to an external force system $[\mathbf{b}, \mathbf{s}, T]$ if the functions $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{S}$ satisfy the field equations of thermoelastostatics (3.13)–(3.15) on B.

Thermoelastic reciprocal theorem. Let $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ and $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$ be thermoelastic states corresponding to the external force-temperature systems $[\mathbf{b}, \mathbf{s}, T]$ and $[\widetilde{\mathbf{b}}, \widetilde{\mathbf{s}}, \widetilde{T}]$, respectively. Then

$$\int_{\partial \mathrm{B}} \mathbf{s} \cdot \widetilde{\mathbf{u}} \, da + \int_{\mathrm{B}} \mathbf{b} \cdot \widetilde{\mathbf{u}} \, dv - \int_{\mathrm{B}} T\mathrm{M} \cdot \widetilde{\mathbf{E}} \, dv = \int_{\partial \mathrm{B}} \widetilde{\mathbf{s}} \cdot \mathbf{u} \, da + \int_{\mathrm{B}} \widetilde{\mathbf{b}} \cdot \mathbf{u} \, dv - \int_{\mathrm{B}} \widetilde{T}\mathbf{M} \cdot \mathbf{E} \, dv \tag{3.29}$$

3.4 Formulation of Boundary Value Problems

Mixed problems of elastostatics. By a mixed boundary value problem of elastostatics we mean the problem of finding an elastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ corresponding to a body force $\mathbf{b}$ and satisfying the boundary conditions: the displacement condition

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial \mathrm{B}_1 \tag{3.30}$$

and the traction condition

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B}_2 \tag{3.31}$$

where

$$\partial \mathrm{B}_1 \cup \partial \mathrm{B}_2 = \partial \mathrm{B}; \quad \partial \mathrm{B}_1 \cap \partial \mathrm{B}_2 = \varnothing \tag{3.32}$$

while $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{s}}$ are prescribed functions.

An elastic state s that satisfies the boundary conditions (3.30)–(3.31) is called a *solution to the mixed problem.*

If $\partial \mathrm{B}_2 = \varnothing$, the mixed problem becomes a *displacement boundary value problem*. If $\partial \mathrm{B}_1 = \varnothing$, the mixed problem becomes a *traction boundary value problem*.

A *displacement field corresponding to a solution to a mixed problem* is a vector field $\mathbf{u}$ with the property that there are symmetric tensor fields $\mathbf{E}$ and $\mathbf{S}$ such that $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a solution to the mixed problem.

A *stress field corresponding to a solution to a mixed problem* is a tensor field $\mathbf{S}$ with the property that there are $\mathbf{u}$ and $\mathbf{E}$ such that $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a solution to the mixed problem.

Mixed Problem in Terms of Displacements

A vector field $\mathbf{u}$ corresponds to a solution to the mixed problem if and only if

$$\operatorname{div} \mathbf{C}[\nabla \mathbf{u}] + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.33}$$

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial \mathrm{B}_1 \tag{3.34}$$

$$(\mathbf{C}[\nabla \mathbf{u}])\,\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B}_2 \tag{3.35}$$

Displacement Problem in Terms of Displacements

A vector field $\mathbf{u}$ corresponds to a solution to the displacement problem if and only if

$$\operatorname{div} \mathbf{C}[\nabla \mathbf{u}] + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.36}$$

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial \mathrm{B} \tag{3.37}$$

Traction Problem in Terms of Stresses

A tensor field **S** corresponds to a solution to the traction problem if and only if

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.38}$$

$$\operatorname{curl}\operatorname{curl} \mathsf{K}[\mathbf{S}] = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.39}$$

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B} \tag{3.40}$$

3.5 Uniqueness

Uniqueness Theorem for the Mixed Problem. If the elasticity tensor C is positive definite, then any two solutions of the mixed problem of elastostatics are equal to within a rigid displacement. If $\partial \mathrm{B}_1 \neq \varnothing$ then the rigid displacement vanishes.

3.6 Formulation of Problems of Nonisothermal Elastostatics

By a *mixed problem of nonisothermal elastostatics* we mean the problem of finding a thermoelastic state $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ that satisfies the field equations (3.13)–(3.15) on B subject to the boundary conditions (3.30)–(3.31).

Mixed Thermoelastic Problem in Terms of Displacements

A vector field **u** corresponds to a solution to the mixed thermoelastic problem if and only if

$$\operatorname{div}\{\mathsf{C}\,[\nabla \mathbf{u}] + T\mathbf{M}\} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.41}$$

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial \mathrm{B}_1 \tag{3.42}$$

$$(\mathsf{C}\,[\nabla \mathbf{u}] + T\mathbf{M})\,\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B}_2 \tag{3.43}$$

Traction Thermoelastic Problem in Terms of Stresses

A tensor field **S** corresponds to a solution to the traction thermoelastic problem if and only if

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.44}$$

$$\operatorname{curl}\operatorname{curl}\{\mathsf{K}[\mathbf{S}] + T\mathbf{A}\} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.45}$$

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial \mathrm{B} \tag{3.46}$$

3.7 Initial-Boundary Value Problems of Elastodynamics

Field Equations of Isothermal Elastodynamics

The strain-displacement relation

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{3.47}$$

The equations of motion

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \rho\,\ddot{\mathbf{u}}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.48}$$

The stress-strain relation

$$\mathbf{S} = \mathbf{C}\,[\mathbf{E}] \tag{3.49}$$

or equivalently

$$\mathbf{E} = \mathbf{K}\,[\mathbf{S}] \tag{3.50}$$

By eliminating **E** and **S** from Eqs. (3.47)–(3.49) we obtain the *displacement equation of motion*

$$\operatorname{div}\,\mathbf{C}\,[\nabla\mathbf{u}] + \mathbf{b} = \rho\,\ddot{\mathbf{u}} \tag{3.51}$$

By eliminating **u** and **E** from Eqs. (3.47), (3.48), and (3.50) the *stress equation of motion* is obtained

$$\widehat{\nabla}[\rho^{-1}(\operatorname{div}\mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \tag{3.52}$$

where

$$\mathbf{B} = \widehat{\nabla}(\rho^{-1}\mathbf{b}) \tag{3.53}$$

For a homogeneous isotropic elastic body Eqs. (3.51) and (3.52), respectively, reduce to

$$\mu\,\nabla^2\mathbf{u} + (\lambda + \mu)\,\nabla(\operatorname{div}\mathbf{u}) + \mathbf{b} = \rho\,\ddot{\mathbf{u}} \tag{3.54}$$

and

$$\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \frac{\rho}{2\mu}\left[\ddot{\mathbf{S}} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\ddot{\mathbf{S}})\,\mathbf{1}\right] = -\widehat{\nabla}\,\mathbf{b} \tag{3.55}$$

Field Equations of nonisothermal Elastodynamics

The strain-displacement relation

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{3.56}$$

The equations of motion

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \rho\,\ddot{\mathbf{u}}, \quad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{3.57}$$

The stress-strain-temperature relation

$$\mathbf{S} = \mathbf{C}\,[\mathbf{E}] + T\,\mathbf{M} \tag{3.58}$$

or, the strain-stress-temperature relation

$$\mathbf{E} = \mathbf{K}\,[\mathbf{S}] + T\,\mathbf{A} \tag{3.59}$$

By eliminating **E** and **S** from Eqs. (3.56)–(3.58) we obtain the *displacement-temperature equation of motion*

$$\operatorname{div}(\mathbf{C}\,[\nabla\mathbf{u}] + T\mathbf{M}) + \mathbf{b} = \rho\,\ddot{\mathbf{u}} \tag{3.60}$$

By eliminating **u** and **E** from Eqs. (3.56), (3.57), and (3.59) the *stress-temperature equation of motion* is obtained

$$\widehat{\nabla}[\rho^{-1}(\operatorname{div}\mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\widetilde{\mathbf{B}} \tag{3.61}$$

where

$$\widetilde{\mathbf{B}} = \widehat{\nabla}(\rho^{-1}\mathbf{b}) + \ddot{T}\mathbf{A} \tag{3.62}$$

Here, **M** and **A** are the stress-temperature and the thermal expansion tensors, respectively.

3.8 Concept of an Elastic Process

An *elastic process* corresponding to a body force **b** is defined as an ordered set of functions $p = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ that complies with the fundamental system of field equations of isothermal elastodynamics (3.47)–(3.49).

An *external force system for p* is defined as a pair $[\mathbf{b}, \mathbf{s}]$ in which $\mathbf{s} = \mathbf{S}\mathbf{n}$.

Graffi's Reciprocal Theorem. Let $p = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be an elastic process corresponding to the external force system $[\mathbf{b}, \mathbf{s}]$ and to the initial data $[\mathbf{u}_0, \dot{\mathbf{u}}_0]$, Let $\widetilde{p} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$ be another elastic process corresponding to $[\widetilde{\mathbf{b}}, \widetilde{\mathbf{s}}]$ and $[\widetilde{\mathbf{u}}_0, \dot{\widetilde{\mathbf{u}}}_0]$. Then the following integral relations hold true

$$\mathrm{i} * \int_{\partial B} \mathbf{s} * \widetilde{\mathbf{u}}\, da + \int_{B} \mathbf{f} * \widetilde{\mathbf{u}}\, dv = \mathrm{i} * \int_{\partial B} \widetilde{\mathbf{s}} * \mathbf{u}\, da + \int_{B} \widetilde{\mathbf{f}} * \mathbf{u}\, dv \tag{3.63}$$

$$\int_{\partial B} \mathbf{s} * \widetilde{\mathbf{u}}\, da + \int_{B} \mathbf{b} * \widetilde{\mathbf{u}}\, dv + \int_{B} \rho\, (\mathbf{u}_0 \cdot \dot{\widetilde{\mathbf{u}}} + \dot{\mathbf{u}}_0 \cdot \widetilde{\mathbf{u}})\, dv$$
$$= \int_{\partial B} \widetilde{\mathbf{s}} * \mathbf{u}\, da + \int_{B} \widetilde{\mathbf{b}} * \mathbf{u}\, dv + \int_{B} \rho\, (\widetilde{\mathbf{u}}_0 \cdot \dot{\mathbf{u}} + \dot{\widetilde{\mathbf{u}}}_0 \cdot \mathbf{u})\, dv \tag{3.64}$$

Here

$$\mathrm{i} = \mathrm{i}(t) = t, \quad t \geq 0 \tag{3.65}$$

and $\mathbf{f}$ and $\widetilde{\mathbf{f}}$ are pseudo-body forces corresponding to $p = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ and $\widetilde{p} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$, respectively, defined by

$$\begin{aligned} \mathbf{f} &= \mathrm{i} * \mathbf{b}(\mathbf{x}, t) + \rho[\mathbf{u}_0(\mathbf{x}) + t\, \dot{\mathbf{u}}_0(\mathbf{x})] \\ \widetilde{\mathbf{f}} &= \mathrm{i} * \widetilde{\mathbf{b}}(\mathbf{x}, t) + \rho[\widetilde{\mathbf{u}}_0(\mathbf{x}) + t\, \dot{\widetilde{\mathbf{u}}}_0(\mathbf{x})] \end{aligned} \tag{3.66}$$

3.9 Formulation of Problems of Isothermal Elastodynamics

Mixed Problem in Terms of Displacements

A vector field $\mathbf{u}$ corresponds to a solution to a mixed problem of isothermal elastodynamics if and only if

$$\mathrm{div}\, \mathbf{C}\, [\nabla \mathbf{u}] + \mathbf{b} = \rho\, \ddot{\mathbf{u}} \quad \text{on} \quad B \times [0, \infty) \tag{3.67}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{on} \quad B \tag{3.68}$$

$$\mathbf{u} = \widehat{\mathbf{u}} \quad \text{on} \quad \partial B_1 \times [0, \infty) \tag{3.69}$$

$$(\mathbf{C}\, [\nabla \mathbf{u}]\,)\, \mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial B_2 \times [0, \infty) \tag{3.70}$$

Traction Problem in Terms of Stresses

A tensor field $\mathbf{S}$ corresponds to a solution to a traction problem of isothermal elastodynamics if and only if

$$\widehat{\nabla}[\rho^{-1}(\mathrm{div}\, \mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \quad \text{on} \quad B \times [0, \infty) \tag{3.71}$$

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{C}\, [\nabla \mathbf{u}_0], \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \mathbf{C}\, [\nabla \dot{\mathbf{u}}_0] \quad \text{on} \quad B \tag{3.72}$$

$$\mathbf{S}\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial B \times [0, \infty) \tag{3.73}$$

where

$$\mathbf{B} = \widehat{\nabla}(\rho^{-1}\mathbf{b}) \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{3.74}$$

Mixed Problem in Terms of Stresses

A tensor field $\mathbf{S}$ corresponds to a solution to a mixed problem of isothermal elastodynamics if and only if

$$\widehat{\nabla}[\rho^{-1}(\mathrm{i} * \operatorname{div} \mathbf{S} + \mathbf{f})] - \mathbf{K}[\mathbf{S}] = \mathbf{0} \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{3.75}$$

$$\rho^{-1}(\mathrm{i} * \operatorname{div} \mathbf{S} + \mathbf{f}) = \widehat{\mathbf{u}} \quad \text{on} \quad \partial\mathrm{B}_1 \times [0, \infty) \tag{3.76}$$

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial\mathrm{B}_2 \times [0, \infty) \tag{3.77}$$

where $\mathbf{f}$ is the pseudo-body force given by $(3.66)_1$.

Nonconventional Traction Problem in Terms of Stresses (Uniqueness). Let $\mathbf{S}$ be a solution to the following initial-boundary value problem. Find a symmetric second-order tensor field $\mathbf{S}$ on $\overline{\mathrm{B}} \times [0, \infty)$ that satisfies the field equation

$$\widehat{\nabla}[\rho^{-1}(\operatorname{div} \mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{F} \quad \text{on} \quad \mathrm{B} \times [0, \infty) \tag{3.78}$$

the initial conditions

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}_0(\mathbf{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \mathrm{B} \tag{3.79}$$

and the boundary condition

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial\mathrm{B} \times [0, \infty) \tag{3.80}$$

Here, $\mathbf{F}$ is an arbitrary symmetric second-order tensor field prescribed on $\overline{\mathrm{B}} \times [0, \infty)$, $\mathbf{S}_0$ and $\dot{\mathbf{S}}_0$ are prescribed symmetric tensor fields on B, and $\widehat{\mathbf{s}}$ is a prescribed vector field on $\partial\mathrm{B} \times [0, \infty)$. Then the problem described by Eqs. (3.78)–(3.80) has at most one solution (Uniqueness).

If $\mathbf{F} \neq \mathbf{B}$, where $\mathbf{B}$ is given by (3.74), $\mathbf{S}_0$ and $\dot{\mathbf{S}}_0$ are not given by Eqs. (3.72), then the problem (3.78)–(3.80) describes stress waves in an *elastic body with time-dependent continuously distributed defects.*

3.10 Problems and Solutions Related to the Formulation of Problems of Elasticity

Problem 3.1. For a homogeneous isotropic elastic body occupying a region $\mathrm{B} \subset E^3$ subject to zero body forces, the displacement equation of equilibrium takes the form [see Eq. (3.6) with $\mathbf{b} = \mathbf{0}$]

$$\nabla^2 \mathbf{u} + \frac{1}{1-2\nu} \nabla(\operatorname{div} \mathbf{u}) = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.81}$$

where ν is Poisson's ratio. Show that if $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is a solution to Eq. (3.81) then $\mathbf{u}$ also satisfies the equation

$$\nabla^2 \left[\mathbf{u} + \frac{\mathbf{x}}{2(1-2\nu)} (\operatorname{div} \mathbf{u}) \right] = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.82}$$

Solution. Equations (3.81) and (3.82) in components take the forms

$$u_{i,kk} + \frac{1}{1-2\nu} \mathrm{u}_{k,ki} = 0 \quad \text{on } B \tag{3.83}$$

and

$$\left[u_i + \frac{x_i}{2(1-2\nu)} u_{k,k} \right]_{,jj} = 0 \quad \text{on } B \tag{3.84}$$

respectively.

It follows from (3.83) that

$$u_{i,ikk} + \frac{1}{1-2\nu} u_{k,kii} = 0 \quad \text{on } B \tag{3.85}$$

or

$$\frac{2(1-\nu)}{1-2\nu} u_{i,ikk} = 0 \quad \text{on } B \tag{3.86}$$

Since $-1 < \nu < 1/2 < 1$ [see Eq. (2.50)]

$$\frac{2-2\nu}{1-2\nu} > 0 \tag{3.87}$$

and (3.86) implies that

$$u_{i,ikk} = 0 \quad \text{on } B \tag{3.88}$$

In addition

$$(x_i\, u_{k,k})_{,jj} = (\delta_{ij}\, u_{k,k} + x_i\, u_{k,kj})_{,j} = 2\delta_{ij}\, u_{k,kj} + x_i\, u_{k,kjj} \tag{3.89}$$

Hence, it follows from Eqs. (3.88) and (3.89) that

$$(x_i\, u_{k,k})_{,jj} = 2u_{k,ki} \tag{3.90}$$

Substituting (3.90) into (3.84) we obtain (3.83), and this completes solution of Problem 3.1.

Problem 3.2. An alternative form of Eq. (3.81) in Problem 3.1 reads [see Eq. (3.7) in which $\lambda = 2\mu\nu/(1-2\nu)$ and $\mathbf{b} = \mathbf{0}$]

$$\nabla(\operatorname{div}\mathbf{u}) - \frac{1-2\nu}{2-2\nu}\operatorname{curl}\operatorname{curl}\mathbf{u} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{3.91}$$

Show that if $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is a solution to Eq. (3.91) then

$$\int_{\mathrm{B}} \left[\operatorname{div}\mathbf{u})^2 + \frac{1-2\nu}{2-2\nu}(\operatorname{curl}\mathbf{u})^2 \right] \mathrm{dv} = \int_{\partial \mathrm{B}} \mathbf{u} \cdot \left[(\operatorname{div}\mathbf{u})\,\mathbf{n} + \frac{1-2\nu}{2-2\nu}(\operatorname{curl}\mathbf{u}) \times \mathbf{n} \right] \mathrm{da} \tag{3.92}$$

where $\mathbf{n}$ is the unit outward normal vector field on $\partial\mathrm{B}$.

Hint. Multiply Eq. (3.91) by $\mathbf{u}$ in the dot product sense, integrate the result over B, and use the divergence theorem.

Note. Since $-1 < \nu < 1/2$ [see Eq. (2.50)] then Eq. (3.92) implies that a displacement boundary value problem of homogeneous isotropic elastostatics may have at most one solution.

Solution. In components Eq. (3.91) takes the form

$$u_{k,ki} - \frac{1-2\nu}{2-2\nu}\varepsilon_{iab}\,\varepsilon_{bcd}\,u_{d,ca} = 0 \tag{3.93}$$

Since

$$u_i\,u_{k,ki} = (u_i\,u_{k,k})_{,i} - (u_{i,i})^2 \tag{3.94}$$

$$u_i\,\varepsilon_{iab}\,\varepsilon_{bcd}\,u_{d,ca} = \varepsilon_{iab}\,\varepsilon_{bcd}[(u_{d,c}\,u_i)_{,a} - u_{d,c}\,u_{i,a}] \tag{3.95}$$

and

$$\varepsilon_{iab}\,\varepsilon_{bcd}\,u_{d,c}\,u_{i,a} = -(\varepsilon_{bai}\,u_{i,a})(\varepsilon_{bcd}\,u_{d,c}) \tag{3.96}$$

therefore, multiplying (3.93) by u_i we obtain

$$(u_{i,i})^2 + \frac{1-2\nu}{2-2\nu}(\varepsilon_{bai}\,u_{i,a})(\varepsilon_{bcd}\,u_{d,c}) = \left(u_{k,k}\,u_a + \frac{1-2\nu}{2-2\nu}\varepsilon_{iab}\,\varepsilon_{bcd}\,u_{d,c}\,u_i \right)_{,a} \tag{3.97}$$

Finally, integrating (3.97) over B and using the divergence theorem we obtain

$$\int_B \left[(u_{i,i})^2 + \frac{1-2\nu}{2-2\nu} (\varepsilon_{bai}\, u_{i,a})(\varepsilon_{bcd}\, u_{d,c}) \right] dv$$
$$= \int_{\partial B} u_a \left[u_{k,k}\, n_a + \frac{1-2\nu}{2-2\nu} \varepsilon_{abi} (\varepsilon_{bcd}\, u_{d,c}) n_i \right] da \tag{3.98}$$

Equation (3.98) is equivalent to (3.92), and this completes solution of Problem 3.2.

Problem 3.3. Show that for a homogeneous isotropic infinite elastic body subject to a temperature change $T = T(\mathbf{x})$ its volume change is represented by the formula

$$\operatorname{tr} \mathbf{E}(\mathbf{x}) = \frac{1+\nu}{1-\nu} \alpha\, T(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in E^3 \tag{3.99}$$

where ν and α denote Poisson's ratio and coefficient of thermal expansion, respectively.

Hint. Apply the reciprocal relation (3.28) to the external force-temperature systems $[\mathbf{b}, \mathbf{s}, T] = [\mathbf{0}, \mathbf{0}, T]$ and $[\widetilde{\mathbf{b}}, \widetilde{\mathbf{s}}, \widetilde{T}] = [\mathbf{0}, \mathbf{0}, \delta(\mathbf{x} - \xi)]$ on E^3. Also note that for an isotropic body $\mathbf{M} = -(3\lambda + 2\mu)\alpha\, \mathbf{1}$ and $\operatorname{tr} \widetilde{\mathbf{E}} = [(3\lambda + 2\mu)\,/(\lambda + 2\mu)]\alpha\, \widetilde{T}$.

Solution. Let $s = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ and $\widetilde{s} = [\widetilde{\mathbf{u}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{S}}]$ be the thermoelastic states produced on E^3 by the external thermomechanical loads $[\mathbf{b}, \mathbf{s}, T] = [\mathbf{0}, \mathbf{0}, T(\mathbf{x})]$ and $[\widetilde{\mathbf{b}}, \widetilde{\mathbf{s}}, \widetilde{T}] = [\mathbf{0}, \mathbf{0}, \delta(\mathbf{x} - \xi)]$, respectively. Applying Eq. (3.29) to the states s and $\widetilde{S}$ we obtain

$$\int_{E^3} T\mathbf{M} \cdot \widetilde{\mathbf{E}}\, dv = \int_{E^3} \widetilde{T}\mathbf{M} \cdot \mathbf{E}\, dv \tag{3.100}$$

For a homogeneous isotropic thermoelastic solid

$$\mathbf{M} = -(3\lambda + 2\mu)\alpha\mathbf{1} \tag{3.101}$$

Therefore,

$$\mathbf{M} \cdot \widetilde{\mathbf{E}} = -(3\lambda + 2\mu)\alpha(\operatorname{tr} \widetilde{\mathbf{E}}) \tag{3.102}$$

and

$$\mathbf{M} \cdot \mathbf{E} = -(3\lambda + 2\mu)\alpha(\operatorname{tr} \mathbf{E}) \tag{3.103}$$

where λ and μ are Lamé constants. Substituting (3.102) and (3.103) into (3.100) we get

$$\int_{E^3} T(\operatorname{tr} \widetilde{\mathbf{E}}) dv = \int_{E^3} \widetilde{T}(\operatorname{tr} \mathbf{E}) dv \tag{3.104}$$

Since $\widetilde{s}$ is the thermoelastic state produced by the temperature $\widetilde{T}$ on E^3, $\widetilde{\mathbf{u}}$ takes the form

$$\widetilde{\mathbf{u}} = \nabla \widetilde{\phi} \tag{3.105}$$

where the thermoelastic potential $\widetilde{\phi}$ satisfies Poisson's equation

$$\nabla^2 \widetilde{\phi} = \frac{1+\nu}{1-\nu} \alpha \widetilde{T} \tag{3.106}$$

Hence

$$\operatorname{tr} \widetilde{\mathbf{E}} = \nabla^2 \widetilde{\phi} = \frac{1+\nu}{1-\nu} \alpha \widetilde{T} \tag{3.107}$$

Therefore, substituting (3.107) into (3.104) we obtain

$$\frac{1+\nu}{1-\nu} \alpha \int_{E^3} T(\xi) \delta(\mathbf{x} - \xi) dv(\xi) = \int_{E^3} \delta(\mathbf{x} - \xi) [\operatorname{tr} \mathbf{E}(\xi)] dv(\xi) \tag{3.108}$$

or using the filtrating property of the delta function we arrive at Eq. (3.99). This completes solution of Problem 3.3.

Problem 3.4. Assume T_0 to be a constant temperature, and let a_i (i = 1, 2, 3) be positive constants of the length dimension. Show that for a homogeneous isotropic infinite elastic body subject to the temperature change

$$\begin{aligned} T(\mathbf{x}) = T_0[H(x_1 + a_1) - H(x_1 - a_1)] \times [H(x_2 + a_2) - H(x_2 - a_2)] \\ \times [H(x_3 + a_3) - H(x_3 - a_3)] \end{aligned} \tag{3.109}$$

where $H = H(x)$ denotes the Heaviside function defined by: $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$; the stress components S_{ij} are represented by the formulas

$$\begin{aligned} S_{ij}(\mathbf{x}) = \mathrm{A}_0 \int_{-\mathrm{a}_1}^{\mathrm{a}_1} d\xi_1 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 \right]^{-1/2} \\ + 4\pi \delta_{ij} \mathrm{A}_0 \int_{-\mathrm{a}_1}^{\mathrm{a}_1} d\xi_1 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \, \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \end{aligned} \tag{3.110}$$

where

$$\mathrm{A}_0 = -\frac{\mu}{2\pi} \frac{1+\nu}{1-\nu} \alpha \, T_0 \tag{3.111}$$

Also, show that the integrals on the RHS of Eq. (3.110) can be calculated in terms of elementary functions, and for the exterior of the parallelepiped

$$|x_1| \leq a_1, \quad |x_2| \leq a_2 \quad |x_3| \leq a_3 \tag{3.112}$$

we obtain

$$S_{12} = A_0 \ln \left[\frac{(x_3 + a_3 + r_{+1.+2.-3})}{(x_3 - a_3 + r_{+1.+2.+3})} \frac{(x_3 - a_3 + r_{+1.-2.+3})}{(x_3 + a_3 + r_{+1.-2.-3})} \times \frac{(x_3 - a_3 + r_{-1.+2.+3})}{(x_3 + a_3 + r_{-1.+2.-3})} \frac{(x_3 + a_3 + r_{-1.-2.-3})}{(x_3 - a_3 + r_{-1.-2.+3})} \right] \tag{3.113}$$

$$S_{23} = A_0 \ln \left[\frac{(x_1 + a_1 + r_{-1.+2.+3})}{(x_1 - a_1 + r_{+1.+2.+3})} \frac{(x_1 - a_1 + r_{+1.+2.-3})}{(x_1 + a_1 + r_{-1.+2.-3})} \times \frac{(x_1 - a_1 + r_{+1.-2.+3})}{(x_1 + a_1 + r_{-1.-2.+3})} \frac{(x_1 + a_1 + r_{-1.-2.-3})}{(x_1 - a_1 + r_{+1.-2.-3})} \right] \tag{3.114}$$

$$S_{31} = A_0 \ln \left[\frac{(x_2 + a_2 + r_{+1.-2.+3})}{(x_2 - a_2 + r_{+1.+2.+3})} \frac{(x_2 - a_2 + r_{-1.+2.+3})}{(x_2 + a_2 + r_{-1.-2.+3})} \times \frac{(x_2 - a_2 + r_{+1.+2.-3})}{(x_2 + a_2 + r_{+1.-2.-3})} \frac{(x_2 + a_2 + r_{-1.-2.-3})}{(x_2 - a_2 + r_{-1.+2.-3})} \right] \tag{3.115}$$

and

$$\begin{aligned} S_{11} = A_0 \Bigg[& \tan^{-1}\left(\frac{x_2 + a_2}{x_1 - a_1}\frac{x_3 + a_3}{r_{+1.-2.-3}}\right) - \tan^{-1}\left(\frac{x_2 + a_2}{x_1 - a_1}\frac{x_3 - a_3}{r_{+1.-2.+3}}\right) \\ & - \tan^{-1}\left(\frac{x_2 - a_2}{x_1 - a_1}\frac{x_3 + a_3}{r_{+1.+2.-3}}\right) + \tan^{-1}\left(\frac{x_2 - a_2}{x_1 - a_1}\frac{x_3 - a_3}{r_{+1.+2.+3}}\right) \\ & - \tan^{-1}\left(\frac{x_2 + a_2}{x_1 + a_1}\frac{x_3 + a_3}{r_{-1.-2.-3}}\right) + \tan^{-1}\left(\frac{x_2 + a_2}{x_1 + a_1}\frac{x_3 - a_3}{r_{-1.-2.+3}}\right) \\ & + \tan^{-1}\left(\frac{x_2 - a_2}{x_1 + a_1}\frac{x_3 + a_3}{r_{-1.+2.-3}}\right) - \tan^{-1}\left(\frac{x_2 - a_2}{x_1 + a_1}\frac{x_3 - a_3}{r_{-1.+2.+3}}\right) \Bigg] \end{aligned} \tag{3.116}$$

$$\begin{aligned} S_{22} = A_0 \Bigg[& \tan^{-1}\left(\frac{x_3 + a_3}{x_2 - a_2}\frac{x_1 + a_1}{r_{-1.+2.-3}}\right) - \tan^{-1}\left(\frac{x_3 + a_3}{x_2 - a_2}\frac{x_1 - a_1}{r_{+1.+2.-3}}\right) \\ & - \tan^{-1}\left(\frac{x_3 - a_3}{x_2 - a_2}\frac{x_1 + a_1}{r_{-1.+2.+3}}\right) + \tan^{-1}\left(\frac{x_3 - a_3}{x_2 - a_2}\frac{x_1 - a_1}{r_{+1.+2.+3}}\right) \\ & - \tan^{-1}\left(\frac{x_3 + a_3}{x_2 + a_2}\frac{x_1 + a_1}{r_{-1.-2.-3}}\right) + \tan^{-1}\left(\frac{x_3 + a_3}{x_2 + a_2}\frac{x_1 - a_1}{r_{+1.-2.-3}}\right) \\ & + \tan^{-1}\left(\frac{x_3 - a_3}{x_2 + a_2}\frac{x_1 + a_1}{r_{-1.-2.+3}}\right) - \tan^{-1}\left(\frac{x_3 - a_3}{x_2 + a_2}\frac{x_1 - a_1}{r_{+1.-2.+3}}\right) \Bigg] \end{aligned} \tag{3.117}$$

$$S_{33} = A_0 \Bigg[\tan^{-1}\left(\frac{x_1+a_1}{x_3-a_3}\frac{x_2+a_2}{r_{-1.-2.+3}}\right) - \tan^{-1}\left(\frac{x_1+a_1}{x_3-a_3}\frac{x_2-a_2}{r_{-1.+2.+3}}\right)$$
$$- \tan^{-1}\left(\frac{x_1-a_1}{x_3-a_3}\frac{x_2+a_2}{r_{+1.-2.-3}}\right) + \tan^{-1}\left(\frac{x_1-a_1}{x_3-a_3}\frac{x_2-a_2}{r_{+1.+2.+3}}\right)$$
$$- \tan^{-1}\left(\frac{x_1+a_1}{x_3+a_3}\frac{x_2+a_2}{r_{-1.-2.-3}}\right) + \tan^{-1}\left(\frac{x_1+a_1}{x_3+a_3}\frac{x_2-a_2}{r_{-1.+2.-3}}\right)$$
$$+ \tan^{-1}\left(\frac{x_1-a_1}{x_3+a_3}\frac{x_2+a_2}{r_{+1.-2.-3}}\right) - \tan^{-1}\left(\frac{x_1-a_1}{x_3+a_3}\frac{x_2-a_2}{r_{+1.+2.-3}}\right)\Bigg] \tag{3.118}$$

where

$$r_{\pm 1.\pm 2.\pm 3} = [(x_1 \mp a_1)^2 + (x_2 \mp a_2)^2 + (x_3 \mp a_3)^2]^{1/2} \tag{3.119}$$

Note that Eq. (3.114) follows from Eq. (3.113) by the transformation of indices

$$1 \to 2, \quad 2 \to 3, \quad 3 \to 1$$

and Eq. (3.115) follows from Eq. (3.114) by the transformation of indices

$$2 \to 3, \quad 3 \to 1, \quad 1 \to 2$$

Also, Eq. (3.117) follows from Eq. (3.116) by the transformation of indices

$$1 \to 2, \quad 2 \to 3, \quad 3 \to 1$$

and Eq. (3.118) follows from Eq. (3.117) by the transformation of indices

$$2 \to 3, \quad 3 \to 1, \quad 1 \to 2.$$

Hint. To find S_{12} use the formula

$$\int \frac{du}{\sqrt{u^2+a^2}} = \ln\left(u + \sqrt{u^2+a^2}\right) \tag{3.120}$$

and to calculate S_{11} take advantage of the formulas

$$\int \frac{du}{(\sqrt{u^2+a^2})^3} = \frac{1}{a^2}\frac{u}{\sqrt{u^2+a^2}} \tag{3.121}$$

and

$$\int \frac{du}{(u^2+b^2)\sqrt{u^2+a^2}} = \frac{1}{b\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{u\sqrt{a^2-b^2}}{b\sqrt{u^2+a^2}}\right) \tag{3.122}$$

where a and b are constants subject to the conditions

$$a \neq 0, \quad b \neq 0, \quad |a| > |b| \tag{3.123}$$

Solution. To show (3.110) we note that

$$S_{ij}(\mathbf{x}) = \int_{-a_1}^{a_1} d\xi_1 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \, S^*_{ij}(\mathbf{x}\xi) \tag{3.124}$$

where

$$S^*_{ij}(\mathbf{x}, \xi) = 2\mu \left(\phi^*_{,ij} - \delta_{ij} \phi^*_{,kk} \right) \tag{3.125}$$

and

$$\phi^*_{,kk} = \frac{1+\nu}{1-\nu} \alpha \, \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \tag{3.126}$$

Since

$$\phi^*(\mathbf{x}, \xi) = -\frac{1}{4\pi} \frac{1+\nu}{1-\nu} \alpha \frac{1}{|\mathbf{x} - \xi|}$$

therefore

$$S^*_{ij}(\mathbf{x}\,\xi) = A_0 \left\{ \frac{\partial^2}{\partial \xi_i \partial \xi_j} \frac{1}{|\mathbf{x} - \xi|} + 4\pi \delta_{ij} \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \delta(x_3 - \xi_3) \right\}$$

where A_0 is given by (3.111). Hence, substituting S^*_{ij} into Eq. (3.124) we obtain (3.110).

To show (3.113)–(3.118) we note that for the exterior of the parallelepiped

$$|x_1| \leq a_1, \quad |x_2| \leq a_2, \quad |x_3| \leq a_3 \tag{3.127}$$

Equation (3.110) reduces to

$$\begin{aligned} S_{ij}(\mathbf{x}) = A_0 & \int_{-a_1}^{a_1} d\xi_1 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \\ & \times \frac{\partial^2}{\partial \xi_i \, \partial \xi_j} [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \\ & i, j = 1, \ 2, \ 3. \end{aligned} \tag{3.128}$$

Letting $i = 1, \ j = 2$ in (3.128) we obtain

$$S_{12}(\mathbf{x}) = A_0 \int_{-a_3}^{a_3} d\xi_3 \int_{-a_2}^{a_2} d\xi_2 \, \frac{\partial}{\partial \xi_2} \times \Big\{ [(x_1 - a_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{-1/2} - [(x_1 + a_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \Big\} \tag{3.129}$$

or

$$\begin{aligned} S_{12}(\mathbf{x}) = A_0 \int_{-a_3}^{a_3} d\xi_3 \times \Big\{ & [(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \\ & - [(x_1 - a_1)^2 + (x_2 + a_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \\ & - [(x_1 + a_1)^2 + (x_2 - a_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \\ & + [(x_1 + a_1)^2 + (x_2 + a_2)^2 + (x_3 - \xi_3)^2]^{-1/2} \Big\} \end{aligned} \tag{3.130}$$

Since for every $b > 0$

$$\int_{-a_3}^{a_3} [b^2 + (x_3 - \xi_3)^2]^{-1/2} \, d\xi_3 = \int_{x_3 - a_3}^{x_3 + a_3} (b^2 + u^2)^{-1/2} \, du \tag{3.131}$$

and by virtue of (3.121)

$$\int (b^2 + u^2)^{-1/2} \, du = \ln \left(u + \sqrt{u^2 + b^2} \right) \tag{3.132}$$

if follows from (3.130) that

$$\begin{aligned} S_{12} = A_0 \ln \Bigg\{ & \frac{(x_3 + a_3 + r_{+1.+2.-3})}{(x_3 - a_3 + r_{+1.+2.+3})} \, \frac{(x_3 - a_3 + r_{+1.-2.+3})}{(x_3 + a_3 + r_{+1.-2.-3})} \\ & \times \frac{(x_3 - a_3 + r_{-1.+2.+3})}{(x_3 + a_3 + r_{-1.+2.-3})} \, \frac{(x_3 + a_3 + r_{-1.-2.-3})}{(x_3 - a_3 + r_{-1.-2.+3})} \Bigg\} \end{aligned} \tag{3.133}$$

where $r_{\pm1.\pm2.\pm3}$ is defined by (3.119). This completes proof of (3.113). The components S_{23} and S_{31}, respectively, are obtained from (3.113) by the transformation of the indices

$$1 \to 2, \quad 2 \to 3, \quad 3 \to 1 \tag{3.134}$$

and

$$2 \to 3, \quad 3 \to 1, \quad 1 \to 2 \tag{3.135}$$

By letting $i = 1,\ j = 1$ in (3.128) we obtain

$$S_{11}(\mathbf{x}) = A_0 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \int_{-a_1}^{a_1} d\xi_1 \frac{\partial^2}{\partial \xi_1^2} \times [(x_2-\xi_2)^2 + (x_3-\xi_3)^2 + (x_1-\xi_1)^2]^{-1/2} \tag{3.136}$$

Since

$$\frac{\partial}{\partial \xi_1} [(x_1-\xi_1)^2 + \alpha^2]^{-1/2} = (x_1-\xi_1)[(x_1-\xi_1)^2 + \alpha^2]^{-3/2} \tag{3.137}$$

where

$$\alpha^2 = (x_2-\xi_2)^2 + (x_3-\xi_3)^2 \tag{3.138}$$

therefore Eq. (3.136) takes the form

$$S_{11}(\mathbf{x}) = A_0 \int_{-a_2}^{a_2} d\xi_2 \int_{-a_3}^{a_3} d\xi_3 \times \left\{ \frac{(x_1-a_1)}{[(x_1-a_1)^2+\alpha^2]^{3/2}} - \frac{(x_1+a_1)}{[(x_1+a_1)^2+\alpha^2]^{3/2}} \right\} \tag{3.139}$$

Now, because of (3.121),

$$\begin{aligned}\int_{-a_3}^{a_3} d\xi_3 \frac{1}{[(x_1-a_1)^2+\alpha^2]^{3/2}} &= \int_{x_3-a_3}^{x_3+a_3} \frac{du}{[(x_1-a_1)^2+(x_2-\xi_2)^2+u^2]^{3/2}} \\ &= \frac{1}{(x_1-a_1)^2+(x_2-\xi_2)^2} \left\{ \frac{(x_3+a_3)}{[(x_1-a_1)^2+(x_2-\xi_2)^2+(x_3+a_3)^2]^{1/2}} \right. \\ &\qquad \left. - \frac{(x_3-a_3)}{[(x_1-a_1)^2+(x_2-\xi_2)^2+(x_3-a_3)^2]^{1/2}} \right\}\end{aligned} \tag{3.140}$$

Also, using (3.122), we obtain

$$\begin{aligned}&\int_{-a_2}^{a_2} d\xi_2 \frac{1}{(x_2-\xi_2)^2+(x_1-a_1)^2} \frac{1}{[(x_1-a_1)^2+(x_3+a_3)^2+(x_2-\xi_2)^2]^{1/2}} \\ &= \int_{-a_2}^{a_2} d\xi_2 \frac{1}{(x_2-\xi_2)^2+b^2} \frac{1}{[(x_2-\xi_2)^2+a^2]^{1/2}} = \int_{x_2-a_2}^{x_2+a_2} \frac{1}{u^2+b^2} \frac{1}{\sqrt{u^2+a^2}} du \\ &= \frac{1}{b\sqrt{a^2-b^2}} \left\{ \tan^{-1} \frac{u\sqrt{a^2-b^2}}{b\sqrt{u^2+a^2}} \right\}_{u=x_2-a_2}^{u=x_2+a_2}\end{aligned} \tag{3.141}$$

where

$$b^2 = (x_1 - a_1)^2, \quad a^2 = (x_1 - a_1)^2 + (x_3 + a_3)^2 \tag{3.142}$$

By letting $x_1 > a_1$ and $x_3 > -a_3$ we receive

$$b = x_1 - a_1, \quad \sqrt{a^2 - b^2} = x_3 + a_3 \tag{3.143}$$

and reduce Eq. (3.141) to

$$\int_{-a_2}^{a_2} d\xi_2 \frac{1}{(x_2 - \xi_2)^2 + b^2} \frac{1}{[(x_2 - \xi_2)^2 + a^2]^{1/2}}$$
$$= \frac{1}{(x_1 - a_1)(x_3 + a_3)} \left\{ \tan^{-1} \frac{x_2 + a_2}{x_1 - a_1} \frac{x_3 + a_3}{r_{+1.-2.-3}} - \tan^{-1} \frac{x_2 - a_2}{x_1 - a_1} \frac{x_3 + a_3}{r_{+1.+2.-3}} \right\} \tag{3.144}$$

It follows from Eq. (3.139) that

$$\begin{aligned} S_{11}(\mathbf{x}) = A_0 \int_{-a_2}^{a_2} d\xi_2 \, (x_1 - a_1) \, & \frac{1}{(x_1 - a_1)^2 + (x_2 - \xi_2)^2} \\ \times & \left\{ \frac{(x_3 + a_3)}{[(x_2 - \xi_2)^2 + (x_1 - a_1)^2 + (x_3 + a_3)^2]^{1/2}} \right. \\ & \left. - \frac{(x_3 - a_3)}{[(x_2 - \xi_2)^2 + (x_1 - a_1)^2 + (x_3 - a_3)^2]^{1/2}} \right\} \\ - A_0 \int_{-a_2}^{a_2} d\xi_2 \, (x_1 + a_1) \, & \frac{1}{(x_1 + a_1)^2 + \left(x_2 - \xi_2^2\right)^2} \\ \times & \left\{ \frac{(x_3 + a_3)}{[(x_2 - \xi_2)^2 + (x_1 + a_1)^2 + (x_3 + a_3)^2]^{1/2}} \right. \\ & \left. - \frac{(x_3 - a_3)}{[(x_2 - \xi_2)^2 + (x_1 + a_1)^2 + (x_3 - a_3)^2]^{1/2}} \right\} \end{aligned} \tag{3.145}$$

Therefore, using Eq. (3.141) as well as equations obtained from Eq. (3.141) by suitable choice of a and b, we obtain (3.116).

The components S_{22} and S_{33} are obtained from Eq. (3.116) by suitable transformation of indices.

Problem 3.5. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathrm{t})$ be a solution of the vector equation

$$\nabla^2 \mathbf{u} - \frac{1}{c^2} \frac{\partial^2 \mathbf{u}}{\partial t^2} = -\frac{\mathbf{f}}{c^2} \quad \text{on} \quad \mathrm{B} \times (0, \infty) \tag{3.146}$$

subject to the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \overline{\mathrm{B}} \tag{3.147}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is a prescribed vector field on $\overline{\mathrm{B}} \times [0, \infty)$; and $\mathbf{u}_0(\mathbf{x})$ and $\dot{\mathbf{u}}_0(\mathbf{x})$ are prescribed vector fields on $\overline{\mathrm{B}}$; and $c > 0$.

Also, let $\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}(\mathbf{x}, t)$ be a solution of the vector equation

$$\nabla^2 \widetilde{\mathbf{u}} - \frac{1}{c^2} \frac{\partial^2 \widetilde{\mathbf{u}}}{\partial t^2} = -\frac{\widetilde{\mathbf{f}}}{c^2} \quad \text{on} \quad \mathrm{B} \times (0, \infty) \tag{3.148}$$

subject to the initial conditions

$$\widetilde{\mathbf{u}}(\mathbf{x}, 0) = \widetilde{\mathbf{u}}_0(\mathbf{x}), \quad \dot{\widetilde{\mathbf{u}}}(\mathbf{x}, 0) = \dot{\widetilde{\mathbf{u}}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \overline{\mathrm{B}} \tag{3.149}$$

where $\widetilde{\mathbf{f}} = \widetilde{\mathbf{f}}(\mathbf{x}, t) \neq \mathbf{f}(\mathbf{x}, \mathrm{t})$, $\widetilde{\mathbf{u}}_0(\mathbf{x}) \neq \mathbf{u}_0(\mathbf{x})$, and $\dot{\widetilde{\mathbf{u}}}_0(\mathbf{x}) \neq \dot{\mathbf{u}}_0(\mathbf{x})$ are prescribed functions on $\overline{\mathrm{B}} \times [0, \infty)$, $\overline{\mathrm{B}}$, and $\overline{\mathrm{B}}$, respectively. Show that the following reciprocal relation holds true

$$\frac{1}{c^2} \int_{\mathrm{B}} (\mathbf{u} * \widetilde{\mathbf{f}} + \mathbf{u} \cdot \dot{\widetilde{\mathbf{u}}}_0 + \dot{\mathbf{u}} \cdot \widetilde{\mathbf{u}}_0)\, dv + \int_{\partial \mathrm{B}} \mathbf{u} * \frac{\partial \widetilde{\mathbf{u}}}{\partial \mathrm{n}} da$$
$$= \frac{1}{c^2} \int_{\mathrm{B}} (\widetilde{\mathbf{u}} * \mathbf{f} + \widetilde{\mathbf{u}} \cdot \dot{\mathbf{u}}_0 + \dot{\widetilde{\mathbf{u}}} \cdot \mathbf{u}_0)\, dv + \int_{\partial \mathrm{B}} \widetilde{\mathbf{u}} * \frac{\partial \mathbf{u}}{\partial \mathrm{n}} da \tag{3.150}$$

where $*$ represents the *inner convolutional product*, that is, for any two vector fields $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$ and $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ on $\overline{\mathrm{B}} \times [0, \infty)$

$$\mathbf{a} * \mathbf{b} = \int_0^t \mathbf{a}(\mathbf{x}, t - \tau) \cdot \mathbf{b}(\mathbf{x}, \tau)\, d\tau \tag{3.151}$$

Solution. Let $\overline{f}(\mathbf{x}, p)$ denote the Laplace transform of a function $f = f(\mathbf{x}, t)$ defined by

$$Lf \equiv \overline{f}(\mathbf{x}, p) = \int_0^\infty e^{-pt} f(\mathbf{x}, t) dt \tag{3.152}$$

Then

$$\overline{\dot{f}(\mathbf{x}, t)} = p\, \overline{f}(\mathbf{x}, p) - f(\mathbf{x}, 0) \tag{3.153}$$

$$\overline{\ddot{f}(\mathbf{x}, t)} = p^2\, \overline{f}(\mathbf{x}, p) - \dot{f}(\mathbf{x}, 0) - p f(\mathbf{x}, 0) \tag{3.154}$$

Now, Eqs. (3.146) and (3.147) in components take the forms

$$u_{i,kk} - \frac{1}{c^2}\ddot{u}_i = -\frac{f_i}{c^2} \tag{3.155}$$

and

$$u_i(\mathbf{x}, 0) = u_{0i}(\mathbf{x}),\ \dot{u}_i(\mathbf{x}, 0) = \dot{u}_{0i}(\mathbf{x}) \tag{3.156}$$

Taking the Laplace transform of Eq. (3.155) and using (3.156) we obtain

$$\overline{u}_{i,kk} - \frac{1}{c^2}(p^2\overline{u_i} - \dot{u}_{0i} - p\,u_{0i}) = -\frac{\overline{f_i}}{c^2} \tag{3.157}$$

Similarly, Eqs. (3.148) and (3.149) imply that

$$\overline{\widetilde{u}}_{i,kk} - \frac{1}{c^2}(p^2\overline{\widetilde{u}}_i - \dot{\widetilde{u}}_{0i} - p\,\widetilde{u}_{0i}) = \frac{\overline{\widetilde{f}}_i}{c^2} \tag{3.158}$$

Multiplying (3.157) by $\overline{\widetilde{u}_i}$ and (3.158) by $\overline{u_i}$, respectively, we obtain

$$\overline{\widetilde{u}}_i\,\overline{u}_{i,kk} - \frac{1}{c^2}(p^2\overline{\widetilde{u}}_i\overline{u}_i - \overline{\widetilde{u}}_i\dot{u}_{0i} - p\,\overline{\widetilde{u}}_i u_{0i}) = -\frac{\overline{f_i}\,\overline{\widetilde{u}}_i}{c^2} \tag{3.159}$$

$$\overline{u}_i\,\overline{\widetilde{u}}_{i,kk} - \frac{1}{c^2}(p^2\overline{u}_i\overline{\widetilde{u}}_i - \overline{u}_i\dot{\widetilde{u}}_{0i} - p\,\overline{u}_i\widetilde{u}_{0i}) = -\frac{\overline{\widetilde{f}}_i\,\overline{u}_i}{c^2} \tag{3.160}$$

Since

$$\overline{\widetilde{u}}_i\overline{u}_{i,kk} = (\overline{\widetilde{u}}_i\overline{u}_{i,k})_{,k} - \overline{\widetilde{u}}_{i,k}\overline{u}_{i,k} \tag{3.161}$$

and

$$\overline{u}_i\overline{\widetilde{u}}_{i,kk} = (\overline{u}_i\overline{\widetilde{u}}_{i,k})_{,k} - \overline{u}_{i,k}\overline{\widetilde{u}}_{i,k} \tag{3.162}$$

therefore, subtracting (3.160) from (3.159), and using the divergence theorem we obtain

$$\int_{\partial B}(\overline{\widetilde{u}}_i\overline{u}_{i,k}\,n_k - \overline{u}_i\overline{\widetilde{u}}_{i,k}\,n_k)da + \frac{1}{c^2}\int_B(\dot{u}_{0i}\overline{\widetilde{u}}_i + p\,u_{0i}\overline{\widetilde{u}}_i - \dot{\widetilde{u}}_{0i}\overline{u}_i - p\widetilde{u}_{0i}\overline{u}_i)dv$$
$$= -\frac{1}{c^2}\int_B(\overline{f_i}\overline{\widetilde{u}}_i - \overline{\widetilde{f}}_i\overline{u}_i)dv \tag{3.163}$$

or

$$\int_{\partial B}\overline{u}_i\overline{\widetilde{u}}_{i,k}n_k\,da + \frac{1}{c^2}\int_B[\overline{u}_i\overline{\widetilde{f}}_i + \overline{u}_i\dot{\widetilde{u}}_{0i} + (p\overline{u}_i - u_{0i})\widetilde{u}_{0i} + u_{0i}\widetilde{u}_{0i}]dv$$

$$= \int\limits_{\partial B} \overline{\overline{u}}_i \overline{u}_{i,k}\, n_k\, da + \frac{1}{c^2} \int\limits_{B} [\overline{\overline{u}}_i \overline{f_i} + \overline{\overline{u}}_i \dot{u}_{0i} + (p\overline{\overline{u}}_i - \widetilde{u}_{0i}) u_{0i} + \widetilde{u}_{0i} u_{0i}] dv \tag{3.164}$$

Using the formula

$$L^{-1}(\overline{f}\,\overline{g}) = f * g \tag{3.165}$$

and applying the operator L^{-1} to Eq. (3.164) we arrive at Eq. (3.150). This completes solution of Problem 3.5.

Problem 3.6. Let $\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}(\mathbf{x}, \mathrm{t})$ be a symmetric second-order tensor field that satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) \widetilde{\mathbf{U}} = -\frac{\widetilde{\mathbf{F}}}{c^2} \quad \text{on} \quad \mathrm{B} \times (0, \infty) \tag{3.166}$$

subject to the initial conditions

$$\widetilde{\mathbf{U}}(\mathbf{x}, 0) = \widetilde{\mathbf{U}}_0(\mathbf{x}), \quad \dot{\widetilde{\mathbf{U}}}(\mathbf{x}, 0) = \dot{\widetilde{\mathbf{U}}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \overline{\mathrm{B}} \tag{3.167}$$

where $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}(\mathbf{x}, t)$, $\widetilde{\mathbf{U}}_0(\mathbf{x})$, and $\dot{\widetilde{\mathbf{U}}}_0(\mathbf{x})$ are prescribed functions on $\overline{\mathrm{B}} \times [0, \infty)$, $\overline{\mathrm{B}}$, and $\overline{\mathrm{B}}$, respectively. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be a solution to Eq. (3.166) and (3.167) of Problem 3.5. Show that

$$\begin{aligned} &\frac{1}{c^2} \int\limits_{\mathrm{B}} (\widetilde{\mathbf{F}} * \mathbf{u} + \dot{\widetilde{\mathbf{U}}}_0 \mathbf{u} + \widetilde{\mathbf{U}}_0 \dot{\mathbf{u}})\, dv + \int\limits_{\partial \mathrm{B}} \frac{\partial \widetilde{\mathbf{U}}}{\partial \mathrm{n}} * \mathbf{u}\, da \\ &= \frac{1}{c^2} \int\limits_{\mathrm{B}} (\widetilde{\mathbf{U}} * \mathbf{f} + \dot{\widetilde{\mathbf{U}}} \mathbf{u}_0 + \widetilde{\mathbf{U}} \dot{\mathbf{u}}_0)\, dv + \int\limits_{\partial \mathrm{B}} \widetilde{\mathbf{U}} * \frac{\partial \mathbf{u}}{\partial \mathrm{n}} da \end{aligned} \tag{3.168}$$

where for any tensor field $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ on $\overline{\mathrm{B}} \times [0, \infty)$ and for any vector field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ on $\overline{\mathrm{B}} \times [0, \infty)$

$$\mathbf{T} * \mathbf{v} = \int\limits_0^t \mathbf{T}(\mathbf{x}, t - \tau) \mathbf{v}(\mathbf{x}, \tau)\, d\tau \tag{3.169}$$

Solution. Equation (3.150) of Problem 3.5 in components takes the form

$$\begin{aligned} &\frac{1}{c^2} \int\limits_{B} (\widetilde{f_i} * u_i + \dot{\widetilde{u}}_{0i} u_i + \widetilde{u}_{0i} \dot{u}_i) dv + \int\limits_{\partial B} \frac{\partial \widetilde{u}_i}{\partial n} * u_i da \\ &= \frac{1}{c^2} \int\limits_{B} (f_i * \widetilde{u}_i + \dot{u}_{0i} \widetilde{u}_i + u_{0i} \dot{\widetilde{u}}_i) dv + \int\limits_{\partial B} \frac{\partial u_i}{\partial n} * \widetilde{u}_i da \end{aligned} \tag{3.170}$$

It follows from the formulation of Problems 3.5 and 3.6 that Eq. (3.170) holds also true if for a fixed index j we let

$$\widetilde{u}_i = \widetilde{U}_{ij}, \quad \widetilde{u}_{0i} = \widetilde{U}_{0ij}, \quad \dot{\widetilde{u}}_{0i} = \dot{\widetilde{U}}_{0ij}, \quad \widetilde{f}_i = \widetilde{F}_{ij} \tag{3.171}$$

Therefore, substituting (3.171) into (3.170) we obtain

$$\begin{aligned} &\frac{1}{c^2}\int\limits_B (\widetilde{F}_{ij} * u_i + \dot{\widetilde{U}}_{0ij} u_i + \widetilde{U}_{0ij} \dot{u}_i) dv + \int\limits_B \frac{\partial \widetilde{U}_{ij}}{\partial n} * u_i da \\ &= \frac{1}{c^2}\int\limits_B (f_i * \widetilde{U}_{ij} + \dot{u}_{0i}\, \widetilde{U}_{ij} + u_{0i}\, \dot{\widetilde{U}}_{ij}) dv + \int\limits_B \frac{\partial u_i}{\partial n} * \widetilde{U}_{ij}\, da \end{aligned} \tag{3.172}$$

Finally, the symmetry of tensors $\widetilde{U}_{ij}$, $\widetilde{F}_{ij}$, $\widetilde{U}_{0ij}$, and $\dot{\widetilde{U}}_{0ij}$, as well as the relation

$$a * b = b * a \tag{3.173}$$

valid for arbitrary functions $a = a(\mathbf{x}, t)$ and $b = b(\mathbf{x}, t)$, imply that Eq. (3.172) is equivalent to Eq. (3.168). This completes solution to Problem 3.6.

Problem 3.7. Let $\mathbf{G} = \mathbf{G}(\mathbf{x}, \xi; t)$ be a symmetric second-order tensor field that satisfies the wave equation

$$\Box_0^2 \mathbf{G} = -\mathbf{1}\delta(\mathbf{x} - \xi)\delta(t) \quad \text{for} \quad \mathbf{x} \in E^3, \xi \in E^3, \ t > 0 \tag{3.174}$$

subject to the homogeneous initial conditions

$$\mathbf{G}(\mathbf{x}, \xi; 0) = \mathbf{0}, \quad \dot{\mathbf{G}}(\mathbf{x}, \xi; 0) = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in E^3, \xi \in E^3 \tag{3.175}$$

where

$$\Box_0^2 = \frac{\partial^2}{\partial x_k \partial x_k} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad (k = 1, 2, 3) \tag{3.176}$$

Show that a solution $\mathbf{u}$ to Eqs. (3.146) and (3.147) of Problem 3.5 admits the integral representation

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) = &\frac{1}{c^2}\int\limits_{\mathrm{B}} (\mathbf{G} * \mathbf{f} + \dot{\mathbf{G}}\mathbf{u}_0 + \mathbf{G}\dot{\mathbf{u}}_0)\, dv(\xi) \\ &+ \int\limits_{\partial \mathrm{B}} \left(\mathbf{G} * \frac{\partial \mathbf{u}}{\partial \mathrm{n}} - \frac{\partial \mathbf{G}}{\partial \mathrm{n}} * \mathbf{u} \right) da(\xi) \end{aligned} \tag{3.177}$$

Hint. Apply the reciprocal relation (3.168) of Problem 3.6 in which $\widetilde{\mathbf{F}}/c^2 = \mathbf{1}\delta(\mathbf{x} - \xi)\delta(t)$ and $\widetilde{\mathbf{U}} = \mathbf{G}(\mathbf{x}, \xi; t)$.

Solution. To solve this problem we let in Eq. (3.168) of Problem 3.6 the following

$$\widetilde{\mathbf{U}} = \mathbf{G}(\mathbf{x},\ \xi;\ t), \quad \widetilde{\mathbf{U}}_0 = \mathbf{G}(\mathbf{x},\ \xi;\ 0) = \mathbf{0}$$
$$\dot{\widetilde{\mathbf{U}}}_0 = \dot{\mathbf{G}}(\mathbf{x},\ \xi;\ 0) = \mathbf{0}, \quad \widetilde{\mathbf{F}}/c^2 = \mathbf{1}\delta(\mathbf{x}-\xi)\,\delta(t) \tag{3.178}$$

and

$$dv = dv(\xi), \quad da = da(\xi) \tag{3.179}$$

and obtain

$$\int_B [\mathbf{1}\,\delta(\mathbf{x}-\xi)\delta(t)] * \mathbf{u}(\xi;\ t)dv(\xi) + \int_{\partial B} \left(\frac{\partial \mathbf{G}}{\partial n}\right) * \mathbf{u}\, da(\xi)$$
$$= \frac{1}{c^2}\int_B (\mathbf{G} * \mathbf{f} + \dot{\mathbf{G}}\,\mathbf{u}_0 + \mathbf{G}\,\dot{\mathbf{u}}_0)dv(\xi) + \int_{\partial B} \mathbf{G} * \frac{\partial \mathbf{u}}{\partial n} da(\xi) \tag{3.180}$$

Finally, using the filtrating property of the delta function we find that (3.180) is equivalent to (3.177). This completes solution to Problem 3.7.

Problem 3.8. Show that a unique solution to Eqs. (3.174) and (3.175) of Problem 3.7 takes the form

$$\mathbf{G}(\mathbf{x}, \xi; t) = \frac{1}{4\pi\,|\mathbf{x}-\xi|}\delta\left(t - \frac{|\mathbf{x}-\xi|}{c}\right)\mathbf{1} \tag{3.181}$$

and, hence, reduce Eq. (3.177) from Problem 3.7 to the Poisson-Kirchhoff integral representation

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{4\pi\, c^2}\int_{\mathrm{B}} \frac{\mathbf{f}(\xi, t - |\mathbf{x}-\xi|\ /c)}{|\mathbf{x}-\xi|}\, dv(\xi) + \frac{\partial}{\partial t}[t\, M_{\mathbf{x},ct}(\mathbf{u}_0)] + t\, M_{\mathbf{x},ct}(\dot{\mathbf{u}}_0)$$
$$+ \frac{1}{4\pi}\int_{\partial \mathrm{B}} \left\{ \frac{1}{|\mathbf{x}-\xi|}\frac{\partial \mathbf{u}}{\partial \mathrm{n}}(\xi, t - |\mathbf{x}-\xi|\ /c) - \mathbf{u}(\xi, t - |\mathbf{x}-\xi|\ /c)\frac{\partial}{\partial \mathrm{n}}\frac{1}{|\mathbf{x}-\xi|}\right.$$
$$\left. + \frac{1}{c\ |\mathbf{x}-\xi|}\left[\frac{\partial}{\partial \mathrm{n}}|\mathbf{x}-\xi|\right]\left[\frac{\partial \mathbf{u}}{\partial t}(\xi, t - |\mathbf{x}-\xi|\ /c)\right]\right\} da(\xi) \tag{3.182}$$

where for any vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$ on $\mathrm{B} \subset E^3$ the symbol $\mathrm{M}_{\mathbf{x},ct}(\mathbf{v})$ represents the mean value of $\mathbf{v}$ over the spherical surface with a center at $\mathbf{x}$ and of radius ct, that is,

$$\mathrm{M}_{\mathbf{x},ct}(\mathbf{v}) = \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} d\theta \ \sin\theta$$
$$\times\, \mathbf{v}(\mathrm{x}_1 + ct\ \sin\theta\ \cos\varphi, \mathrm{x}_2 + ct\ \sin\theta\ \sin\varphi, x_3 + ct\ \cos\theta) \quad (3.183)$$

and we adopt the convention that all relevant quantities vanish for negative time arguments.

Note. If $\mathrm{B} = E^3$ and $\mathbf{f} = \mathbf{0}$ on $E^3 \times [0, \infty)$ then Eq. (3.182) reduces to the form

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial}{\partial t}[t\ M_{\mathbf{x},ct}(\mathbf{u}_0)] + t\ M_{\mathbf{x},\mathrm{ct}}(\dot{\mathbf{u}}_0) \quad (3.184)$$

Solution. Equation (3.174) of Problem 3.7 takes the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{G} = -\mathbf{1}\,\delta(\mathbf{x} - \xi)\,\delta(t) \quad (3.185)$$

Applying the Laplace transform to this equation and using the homogeneous initial conditions (3.175) of Problem 3.7 we obtain

$$\left[\nabla^2 - \left(\frac{p}{c}\right)^2\right]\overline{\mathbf{G}} = -\mathbf{1}\,\delta(\mathbf{x} - \xi) \quad (3.186)$$

where

$$\overline{\mathbf{G}} = \overline{\mathbf{G}}(\mathbf{x}, \xi;\ p) = \int_0^{\infty} e^{-pt}\mathbf{G}(\mathbf{x}, \xi, t)dt \quad (3.187)$$

The only solution to Eq. (3.186) in E^3 that vanishes as $|\mathbf{x}| \to \infty$, $|\xi| < \infty$, takes the form $(p > 0)$

$$\overline{\mathbf{G}} = \frac{1}{4\pi}\,\frac{1}{|\mathbf{x} - \xi|}\,e^{-\frac{p}{c}|\mathbf{x}-\xi|}\mathbf{1} \quad (3.188)$$

Hence, applying the operator L^{-1} to (3.188) we obtain

$$\mathbf{G}(\mathbf{x},\ \xi;\ t) = \frac{1}{4\pi\,|\mathbf{x} - \xi|}\,\delta\left(t - \frac{1}{c}|\mathbf{x} - \xi|\right)\mathbf{1} \quad (3.189)$$

This completes proof of (3.181). To show that (3.182) holds true, we split (3.177) of Problem 3.7 into the sum

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^{(1)}(\mathbf{x}, t) + \mathbf{u}^{(2)}(\mathbf{x}, t) + \mathbf{u}^{(3)}(\mathbf{x}, t) \quad (3.190)$$

where

$$\mathbf{u}^{(1)}(\mathbf{x}, t) = \frac{1}{c^2} \int_B \mathbf{G} * \mathbf{f} \, dv(\xi) \tag{3.191}$$

$$\mathbf{u}^{(2)}(\mathbf{x}, t) = \int_{\partial B} \left(\mathbf{G} * \frac{\partial \mathbf{u}}{\partial n} - \frac{\partial \mathbf{G}}{\partial n} * \mathbf{u} \right) da(\xi) \tag{3.192}$$

$$\mathbf{u}^{(3)}(\mathbf{x}, t) = \frac{1}{c^2} \int_B (\dot{\mathbf{G}} \, \mathbf{u}_0 + \mathbf{G} \, \dot{\mathbf{u}}_0) \, dv(\xi) \tag{3.193}$$

If $\mathbf{G}$ from Eq. (3.189) is substituted into Eq. (3.191) we obtain

$$\mathbf{u}^{(1)}(\mathbf{x}, t) = \frac{1}{4\pi c^2} \int_B dv(\xi) \int_0^t d\tau \frac{1}{|\mathbf{x} - \xi|} \mathbf{f}(\xi, t - \tau) \, \delta \left(\tau - \frac{1}{c} |\mathbf{x} - \xi| \right) \tag{3.194}$$

Using the filtrating property of the delta function

$$\int_0^t \delta(\tau - t_0) \, g(t - \tau) d\tau = g(t - t_0) H(t - t_0) \tag{3.195}$$

where $g = g(t)$ is an arbitrary function and $H = H(t)$ is the Heaviside function

$$H(t) = \begin{Bmatrix} 1 & t > 0 \\ 0 & t < 0 \end{Bmatrix} \tag{3.196}$$

we reduce Eq. (3.194) to the form

$$\mathbf{u}^{(1)}(\mathbf{x}, t) = \frac{1}{4\pi c^2} \int_B \frac{\mathbf{f}\left(\xi, t - \frac{R}{c}\right)}{R} \, H\left(t - \frac{R}{c}\right) dv(\xi) \tag{3.197}$$

where

$$R = |\mathbf{x} - \xi| \tag{3.198}$$

The function $\mathbf{u}^{(1)}(\mathbf{x}, t)$ given by (3.197) is identical to the first integral on the RHS of Eq. (3.182) when the convention that $\mathbf{f}(\mathbf{x}, t)$ vanishes for $t \leq 0$ is adopted.

An alternative form of (3.197) reads

$$\mathbf{u}^{(1)}(\mathbf{x}, t) = \frac{1}{4\pi c^2} \int_{B \cap S(\mathbf{x}, ct)} \frac{1}{|\mathbf{x} - \xi|} \mathbf{f}\left(\xi, t - \frac{1}{c} |\mathbf{x} - \xi| \right) dv(\xi) \tag{3.199}$$

where

$$S(\mathbf{x}, ct) = \{\xi : |\xi - \mathbf{x}| < ct\}.$$

To show that $\mathbf{u}^{(2)}(\mathbf{x}, t)$ is identical to the last integral on the RHS of (3.182), we apply the Laplace transform to Eq. (3.192), use Eq. (3.188), and obtain

$$\overline{\mathbf{u}}^{(2)}(\mathbf{x}, p) = \frac{1}{4\pi} \int_{\partial B} \left\{ \frac{1}{R} e^{-\frac{p}{c}R} \frac{\partial \overline{\mathbf{u}}}{\partial n} - \left[\frac{\partial}{\partial n} \left(\frac{1}{R} \right) e^{-\frac{p}{c}R} - \frac{1}{cR} \frac{\partial R}{\partial n} e^{-\frac{p}{c}R} p \right] \overline{\mathbf{u}} \right\} da(\xi) \tag{3.200}$$

Applying the inverse Laplace transform to Eq. (3.200) we obtain

$$\mathbf{u}^{(2)}(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\partial B} \left\{ \frac{1}{R} \frac{\partial \mathbf{u}}{\partial n} \left(\xi, t - \frac{R}{c} \right) - \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \mathbf{u} \left(\xi, t - \frac{R}{c} \right) + \frac{1}{cR} \frac{\partial R}{\partial n} \dot{\mathbf{u}} \left(\xi, t - \frac{R}{c} \right) \right\} H \left(t - \frac{R}{c} \right) da(\xi) \tag{3.201}$$

The function $\mathbf{u}^{(2)}$ given by (3.201) is equivalent to the last integral on the RHS of (3.182), if the convention that $\mathbf{u}(\mathbf{x}, t) = \mathbf{0}$ for $t \le 0$ is adopted. An equivalent form of (3.201) reads

$$\mathbf{u}^{(2)}(\mathbf{x}, t) = \frac{1}{4\pi} \int_{\partial B \cap S(\mathbf{x}, ct)} \left\{ \frac{1}{R} \frac{\partial \mathbf{u}}{\partial n} \left(\xi, t - \frac{R}{c} \right) - \frac{\partial}{\partial n} \left(\frac{1}{R} \right) \mathbf{u} \left(\xi, t - \frac{R}{c} \right) + \frac{1}{cR} \frac{\partial R}{\partial n} \dot{\mathbf{u}} \left(\xi, t - \frac{R}{c} \right) \right\} da(\xi) \tag{3.202}$$

To show that $\mathbf{u}^{(3)}(\mathbf{x}, t)$ given by (3.193) is equal to a sum of the second and third terms on the RHS of (3.182), consider the integral

$$\mathbf{h}(\mathbf{x}, t) = \frac{1}{c^2} \int_B \mathbf{G}\, \dot{\mathbf{u}}_0 \, dv(\xi) \tag{3.203}$$

Since

$$\frac{1}{R} \delta \left(t - \frac{R}{c} \right) = \frac{tc^2}{R^2} \delta(R - ct) \tag{3.204}$$

therefore, an alternative form of $\mathbf{G}$ given by (3.189) reads

$$\mathbf{G}(\mathbf{x}, \xi, t) = \frac{tc^2}{4\pi R^2} \delta(R - ct) \mathbf{1} \tag{3.205}$$

and the function $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$ takes the form

$$\mathbf{h}(\mathbf{x}, t) = \frac{t}{4\pi} \int_B \frac{\dot{\mathbf{u}}_0(\xi)}{|\mathbf{x} - \xi|^2} \, \delta(|\mathbf{x} - \xi| - ct) dv(\xi) \tag{3.206}$$

Next, we let

$$\mathbf{R} = \xi - \mathbf{x} \tag{3.207}$$

and introduce the spherical coordinates (R, φ, θ) with a center at $\mathbf{x}$

$$\begin{aligned} R_1 &= R \cos \varphi \, \sin \theta \\ R_2 &= R \sin \varphi \, \sin \theta \\ R_3 &= R \cos \theta \end{aligned} \tag{3.208}$$

$$0 \le R < \infty, \quad 0 \le \varphi \le 2\pi, \quad 0 \le \theta < \pi \tag{3.209}$$

Then the integral (3.206) takes the form

$$\mathbf{h}(\mathbf{x}, t) = \frac{t}{4\pi} \int_{B^*} \frac{\dot{\mathbf{u}}_0(\mathbf{R} + \mathbf{x})}{R^2} \delta(R - ct) dv(\mathbf{R}) \tag{3.210}$$

where

$$dv(\mathbf{R}) = R^2 \sin \theta \, d\varphi \, d\theta \, dR \tag{3.211}$$

and

$$B^* = \{(R, \varphi, \theta) : R_a < R < R_b; \; 0 \le \varphi \le 2\pi, 0 \le \theta \le \pi\} \tag{3.212}$$

The domain B^* is a mapping of B under the transformation defined by Eqs. (3.207) and (3.208), and R_a and R_b are uniquely defined nonnegative numbers. Hence, Eq. (3.210) can also be written as

$$\begin{aligned} \mathbf{h}(\mathbf{x}, t) = \frac{t}{4\pi} \int_{R_a}^{R_b} dR \, \delta(R - ct) \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \\ \times \dot{\mathbf{u}}_0(x_1 + R \cos \varphi \sin \theta, \; x_2 + R \sin \varphi \sin \theta, \; x_3 + R \cos \theta) \end{aligned} \tag{3.213}$$

or

$$\begin{aligned} \mathbf{h}(\mathbf{x}, t) = \frac{t}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \\ \times \dot{\mathbf{u}}_0(x_1 + ct \cos \varphi \sin \theta, \; x_2 + ct \sin \varphi \sin \theta, \; x_3 + ct \cos \theta) \end{aligned} \tag{3.214}$$

or

$$\mathbf{h}(\mathbf{x}, t) = t \, \mathbf{M}_{\mathbf{x},ct}(\dot{\mathbf{u}}_0) \tag{3.215}$$

where $\mathbf{M}_{\mathbf{x},ct}(\dot{\mathbf{u}}_0)$ is defined by (3.183). Finally, if we note that

$$\mathbf{g}(\mathbf{x},t) \overset{\text{df}}{=} \frac{1}{c^2} \int_B \dot{\mathbf{G}}\mathbf{u}_0 \, dv(\xi) \tag{3.216}$$

can be written as

$$\mathbf{g}(\mathbf{x},t) = \frac{\partial}{\partial t} \left\{ \frac{1}{c^2} \int_B \mathbf{G}\mathbf{u}_0 \, dv(\xi) \right\} \tag{3.217}$$

then computing the integral on the RHS of (3.217) in a way similar to that of the integral $\mathbf{h} = \mathbf{h}(\mathbf{x},t)$, and taking into account Eq. (3.193) we obtain

$$\mathbf{u}^{(3)}(\mathbf{x},t) = \frac{\partial}{\partial t} \left[t\mathbf{M}_{\mathbf{x},ct}(\mathbf{u}_0) \right] + t\mathbf{M}_{\mathbf{x},ct}(\dot{\mathbf{u}}_0) \tag{3.218}$$

This completes proof of (3.182).

Problem 3.9. Let $\mathbf{G}^* = \mathbf{G}^*(\mathbf{x},\xi;t)$ be a solution to the initial-boundary value problem:

$$\Box_0^2 \mathbf{G}^* = -\mathbf{1}\delta(\mathbf{x}-\xi)\delta(t) \quad \text{for} \quad \mathbf{x}, \ \ \xi \in \mathrm{B}, \quad t > 0 \tag{3.219}$$

$$\mathbf{G}^*(\mathbf{x},\xi;0) = \mathbf{0}, \quad \dot{\mathbf{G}}^*(\mathbf{x},\xi;0) = \mathbf{0} \quad \text{for} \quad \mathbf{x},\xi \in \mathrm{B} \tag{3.220}$$

and

$$\mathbf{G}^*(\mathbf{x},\xi;t) = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \partial\mathrm{B}, \quad t > 0, \quad \xi \in \overline{\mathrm{B}} \tag{3.221}$$

and let $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ be a solution to the initial-boundary value problem

$$\Box_0^2 \mathbf{u} = -\frac{\mathbf{f}}{c^2} \quad \text{on} \quad \mathrm{B} \times (0,\infty) \tag{3.222}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x},0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \overline{\mathrm{B}} \tag{3.223}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t) \quad \text{on} \quad \partial\mathrm{B} \times [0,\infty) \tag{3.224}$$

where the functions $\mathbf{f}$, $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, and $\mathbf{g}$ are prescribed. Use the representation formula (3.177) of Problem 3.7 to show that

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{c^2} \int_{\mathrm{B}} (\mathbf{G}^* * \mathbf{f} + \dot{\mathbf{G}}^* \mathbf{u}_0 + \mathbf{G}^* \dot{\mathbf{u}}_0) \, dv(\xi) - \int_{\partial\mathrm{B}} \left(\frac{\partial \mathbf{G}^*}{\partial \mathrm{n}} * \mathbf{g} \right) da(\xi) \tag{3.225}$$

Solution. The representation formula (3.177) of Problem 3.7 reads

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{c^2} \int\limits_{B} (\mathbf{G} * \mathbf{f} + \dot{\mathbf{G}}\mathbf{u}_0 + \mathbf{G}\dot{\mathbf{u}}_0)\, dv(\xi) + \int\limits_{\partial B} \left(\mathbf{G} * \frac{\partial \mathbf{u}}{\partial n} - \frac{\partial \mathbf{G}}{\partial n} * \mathbf{u} \right) da(\xi) \tag{3.226}$$

where $\mathbf{G}$ satisfies Eqs. (3.174) and (3.175) of Problem 3.7, and $\mathbf{u}$ satisfies Eqs. (3.146) and (3.147) of Problem 3.5.

By letting $\mathbf{G} = \mathbf{G}^*$ in Eq. (3.226) and using the boundary conditions (3.221) and (3.224) we obtain (3.225).

This completes a solution to Problem 3.9.

Problem 3.10. A tensor field $\mathbf{S}$ corresponds to the solution of a traction problem of classical elastodynamics if and only if

$$\widehat{\nabla}[\rho^{-1}(\operatorname{div} \mathbf{S})] - \mathbf{K}[\ddot{\mathbf{S}}] = -\mathbf{B} \quad \text{on} \quad B \times [0, \infty) \tag{3.227}$$

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}^{(0)}(\mathbf{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}^{(0)}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in B \tag{3.228}$$

$$\mathbf{Sn} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial B \times [0, \infty) \tag{3.229}$$

[see Eqs. (3.71)–(3.73)] in which $\mathbf{B}$ is expressed in terms of a body force $\mathbf{b}$, and $\mathbf{S}^{(0)}$ and $\dot{\mathbf{S}}^{(0)}$ are defined in terms of two vector fields. A tensor field $\mathbf{S}$ corresponding to an external load $[\mathbf{B}, \mathbf{S}^{(0)}, \dot{\mathbf{S}}^{(0)}, \widehat{\mathbf{s}}]$ is said to be of a σ-type if $\mathbf{S}$ satisfies Eqs. (3.227) through (3.229) with an arbitrary symmetric second-order tensor field $\mathbf{B}$ and arbitrary symmetric initial tensor fields $\mathbf{S}^{(0)}$ and $\dot{\mathbf{S}}^{(0)}$, not necessarily related to the data of classic elastodynamics. Show that if $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are two different tensorial fields of σ-type corresponding to the external loads $[\mathbf{B}, \mathbf{S}^{(0)}, \dot{\mathbf{S}}^{(0)}, \widehat{\mathbf{s}}]$ and $[\widetilde{\mathbf{B}}, \widetilde{\mathbf{S}}^{(0)}, \dot{\widetilde{\mathbf{S}}}^{(0)}, \widehat{\widetilde{\mathbf{s}}}]$, respectively, then the following reciprocal relation holds true

$$\begin{aligned} &\int\limits_{B} \left\{ \widetilde{\mathbf{B}} * \mathbf{S} + \widetilde{\mathbf{S}}^{(0)} \cdot \mathbf{K}[\dot{\mathbf{S}}] + \dot{\widetilde{\mathbf{S}}}^{(0)} \cdot \mathbf{K}[\mathbf{S}] \right\} dv + \int\limits_{\partial B} \rho^{-1} (\operatorname{div} \widetilde{\mathbf{S}}) * (\mathbf{Sn})\, da \\ &= \int\limits_{B} \left\{ \mathbf{B} * \widetilde{\mathbf{S}} + \mathbf{S}^{(0)} \cdot \mathbf{K}[\dot{\widetilde{\mathbf{S}}}] + \dot{\mathbf{S}}^{(0)} \cdot \mathbf{K}[\widetilde{\mathbf{S}}] \right\} dv + \int\limits_{\partial B} \rho^{-1} (\operatorname{div} \mathbf{S}) * (\widetilde{\mathbf{S}}\mathbf{n})\, da \end{aligned} \tag{3.230}$$

Solution. The tensor fields S_{ij} and $\widetilde{S}_{ij}$, respectively, satisfy the equations

$$(\rho^{-1} S_{(ik,k),j}) - K_{ijkl}\, \ddot{S}_{kl} = -B_{ij} \quad \text{on } B \times (0, \infty) \tag{3.231}$$

$$S_{ij}(\mathbf{x}, 0) = S_{ij}^{(0)}(\mathbf{x}), \quad \dot{S}_{ij}(\mathbf{x}, 0) = \dot{S}_{ij}^{(0)}(\mathbf{x}) \quad \text{on } B \tag{3.232}$$

$$S_{ij} n_j = \widehat{s}_i \quad \text{on } \partial B \times (0, \infty) \tag{3.233}$$

and

$$(\rho^{-1}\widetilde{S}_{(ik,k),j}) - K_{ijkl}\,\ddot{\widetilde{S}}_{kl} = -\widetilde{B}_{ij} \quad \text{on } B \times (0, \infty) \tag{3.234}$$

$$\widetilde{S}_{ij}(\mathbf{x}, 0) = \widetilde{S}^{(0)}_{ij}(\mathbf{x}), \quad \dot{\widetilde{S}}_{ij}(\mathbf{x}, 0) = \dot{\widetilde{S}}^{(0)}_{ij}(\mathbf{x}) \quad \text{on } B \tag{3.235}$$

$$\widetilde{S}_{ij}n_j = \widehat{\widetilde{S}}_i \quad \text{on } \partial B \times (0, \infty) \tag{3.236}$$

Applying the Laplace transform to Eqs. (3.231) and (3.234), and using the initial conditions (3.232) and (3.235), respectively, we obtain

$$(\rho^{-1}\overline{S}_{(ik,k),j}) - K_{ijkl}\left(p^2\overline{S}_{kl} - \dot{S}^{(0)}_{kl} - pS^{(0)}_{kl}\right) = -\overline{B_{ij}} \tag{3.237}$$

and

$$(\rho^{-1}\overline{\widetilde{S}}_{(ik,k),j}) - K_{ijkl}\left(p^2\overline{\widetilde{S}}_{kl} - \dot{\widetilde{S}}^{(0)}_{kl} - p\widetilde{S}^{(0)}_{kl}\right) = -\overline{\widetilde{B}_{ij}} \tag{3.238}$$

Next, multiplying (3.237) by $\overline{\widetilde{S}}_{ij}$ and (3.238) by $-\overline{S}_{ij}$, and adding up the results we obtain

$$\begin{aligned}\overline{\widetilde{S}}_{ij}(\rho^{-1}\overline{S}_{ik,k})_{,j} - \overline{\widetilde{S}}_{ij}\;K_{ijkl}\left(p^2\overline{S}_{kl} - \dot{S}^{(0)}_{kl} - pS^{(0)}_{kl}\right) - \overline{S}_{ij}(\rho^{-1}\overline{\widetilde{S}}_{ik,k})_{,j}\\ + \overline{S}_{ij}\;K_{ijkl}\left(p^2\overline{\widetilde{S}}_{kl} - \dot{\widetilde{S}}^{(0)}_{kl} - p\widetilde{S}^{(0)}_{kl}\right) + \overline{B}_{ij}\;\overline{\widetilde{S}}_{ij} - \overline{\widetilde{B}}_{ij}\;\overline{S}_{ij} = 0\end{aligned} \tag{3.239}$$

Since

$$\overline{\widetilde{S}}_{ij}(\rho^{-1}\overline{S}_{ik,k})_{,j} - \overline{S}_{ij}(\rho^{-1}\overline{\widetilde{S}}_{ik,k})_{,j} = (\overline{\widetilde{S}}_{ij}\rho^{-1}\overline{S}_{ik,k} - \overline{S}_{ij}\rho^{-1}\overline{\widetilde{S}}_{ik,k})_{,j} \tag{3.240}$$

and

$$K_{ijkl}\;\overline{\widetilde{S}}_{ij}\;\overline{S}_{kl} = K_{ijkl}\;\overline{S}_{ij}\;\overline{\widetilde{S}}_{kl} \tag{3.241}$$

therefore, by integrating (3.239) over B and using the divergence theorem, we obtain

$$\begin{aligned}\int_B \left(\overline{B}_{ij}\;\overline{\widetilde{S}}_{ij} - \overline{\widetilde{B}}_{ij}\;\overline{S}_{ij}\right) dv(\xi) + \int_B K_{ijkl}\Big[\overline{\widetilde{S}}_{ij}\;\dot{S}^{(0)}_{kl} + \left(p\overline{\widetilde{S}}_{ij} - \widetilde{S}^{(0)}_{ij}\right) S^{(0)}_{kl} + \widetilde{S}^{0}_{ij}\;S^{(0)}_{kl}\\ - \overline{S}_{ij}\;\dot{\widetilde{S}}^{(0)}_{kl} - \left(p\overline{S}_{ij} - S^{(0)}_{ij}\right)\widetilde{S}^{(0)}_{kl} - S^{(0)}_{ij}\widetilde{S}^{(0)}_{kl}\Big]dv\\ + \int_{\partial B} \rho^{-1}\left(\overline{\widetilde{S}}_{ij}\;\overline{S}_{ik,k} - \overline{S}_{ij}\;\overline{\widetilde{S}}_{ik,k}\right) n_j da(\xi) = 0\end{aligned} \tag{3.242}$$

Finally, applying the inverse Laplace transform to (3.242), using the convolution theorem

$$\overline{f * g} = \overline{f}\overline{g} \tag{3.243}$$

as well as the relation

$$K_{ijkl}\, \widetilde{S}^{(0)}_{ij} S^{(0)}_{kl} = K_{ijkl}\, S^{(0)}_{ij} \widetilde{S}^{(0)}_{kl} \tag{3.244}$$

we obtain

$$\int_B (\mathbf{B} * \widetilde{\mathbf{S}} - \widetilde{\mathbf{B}} * \mathbf{S}) dv(\xi) + \int_B \left\{ \widetilde{\mathbf{S}} \cdot \mathbf{K}[\dot{\mathbf{S}}^{(0)}] + \dot{\widetilde{\mathbf{S}}} \cdot \mathbf{K}[\mathbf{S}^{(0)}] - \mathbf{S} \cdot \mathbf{K}[\dot{\widetilde{\mathbf{S}}}^{(0)}] - \dot{\mathbf{S}} \cdot \mathbf{K}[\widetilde{\mathbf{S}}^{(0)}] \right\} dv(\xi)$$
$$+ \int_B \rho^{-1} \left[(\text{div}\, \mathbf{S}) * (\widetilde{\mathbf{S}}\mathbf{n}) - (\text{div}\, \widetilde{\mathbf{S}}) * (\mathbf{S}\mathbf{n}) \right] da(\xi) = 0 \tag{3.245}$$

Since $\mathbf{K}$ is symmetric

$$\mathbf{A} \cdot \mathbf{K}[\mathbf{B}] = \mathbf{B} \cdot \mathbf{K}[\mathbf{A}] \quad \forall\, \mathbf{A} \text{ and } \mathbf{B} \tag{3.246}$$

therefore, Eq. (3.245) is equivalent to Eq. (3.230), and this completes a solution to Problem 3.10.

Problem 3.11. Let $S^{(kl)}_{ij} = S^{(kl)}_{ij}(\mathbf{x}, \xi; t)$ be a solution of the following equation

$$(\rho^{-1} S^{(kl)}_{(ik,k),j}) - K_{ijpq} \ddot{S}^{(kl)}_{pq} = 0$$
$$\text{for} \quad \mathbf{x} \in E^3, \quad \xi \in E^3; \quad \text{i, j, k, l} = 1, 2, 3 \tag{3.247}$$

subject to the initial conditions

$$S^{(kl)}_{ij}(\mathbf{x}, \xi; 0) = 0, \quad \dot{S}^{(kl)}_{ij}(\mathbf{x}, \xi; 0) = C_{ijkl}\delta(\mathbf{x} - \xi)$$
$$\text{for} \quad \mathbf{x} \in E^3, \quad \xi \in E^3; \quad \text{i, j, k, l} = 1, 2, 3 \tag{3.248}$$

where K_{ijkl} denotes the components of the compliance tensor $\mathbf{K}$, and C_{ijkl} stands for the components of elasticity tensor $\mathbf{C}$, that is,

$$C_{ijkl} K_{klmn} = \delta_{(im}\delta_{nj)} \tag{3.249}$$

Let $S_{ij} = S_{ij}(\mathbf{x}, t)$ be a solution of the equation

$$(\rho^{-1} S_{(ik,k),j}) - K_{ijkl} \ddot{S}_{kl} = 0 \quad \text{for} \quad \mathbf{x} \in \text{B}, \quad t > 0 \tag{3.250}$$

subject to the homogeneous initial conditions

$$S_{ij}(\mathbf{x}, 0) = 0, \quad \dot{S}_{ij}(\mathbf{x}, 0) = 0 \quad \text{for} \quad \mathbf{x} \in \text{B} \tag{3.251}$$

and the boundary condition

$$S_{ij} n_j = \widehat{s}_i \quad \text{on} \quad \partial\text{B} \times [0, \infty) \tag{3.252}$$

Use the reciprocal relation (3.230) of Problem 3.10 to show that

$$S_{kl}(\mathbf{x},t) = \int_{\partial B} \rho^{-1} \left(S_{im,m} * S_{ij}^{(kl)} n_j - \widehat{s}_i * S_{im,m}^{(kl)} \right) da(\xi) \tag{3.253}$$

Note. Equation (3.253) provides a solution to the traction initial-boundary value problem of classical elastodynamics if the field $S_{im,m}$ on $\partial B \times [0,\infty)$ is found from an associated integral equation on $\partial B \times [0,\infty)$. The idea of solving a traction problem of elastodynamics in terms of displacements through an associated boundary integral equation is due to V.D. Kupradze.

Solution. Note that for a fixed pair (k,l) $\widetilde{S}_{ij} = S_{ij}^{(kl)}(\mathbf{x}, \xi; t)$ is a tensor field of σ-type corresponding to the data: $\widetilde{B}_{ij} = 0$, $\widehat{\widetilde{s}}_i \neq 0$, $\widetilde{S}_{ij}^{(0)} = 0$, and $\dot{\widetilde{S}}_{ij}^{(0)} = C_{ijkl}\delta(\mathbf{x} - \xi)$; and $S_{ij} = S_{ij}(\mathbf{x},t)$ is a tensor field of σ-type corresponding to the data: $B_{ij} = 0$, $\widehat{s}_i \neq 0$, $S_{ij}^{(0)} = 0$, and $\dot{S}_{ij}^{(0)} = 0$. Therefore, using the reciprocal relation (3.230) of Problem 3.10. in which $\widetilde{S}_{ij} = S_{ij}^{(kl)}(\mathbf{x}, \xi; t)$ and $S_{ij} = S_{ij}(\mathbf{x},t)$, we obtain

$$\int_B \dot{\widetilde{S}}_{ij}^{(0)} K_{ijpq}\, S_{pq} dv(\xi) = \int_{\partial B} \rho^{-1} (S_{im,m} * \widetilde{S}_{ij} n_j - \widehat{s}_i * \widetilde{S}_{im,m})\, da(\xi) \tag{3.254}$$

where

$$\dot{\widetilde{S}}_{ij}^{(0)} = \dot{S}_{ij}^{(kl)}(\mathbf{x}, \xi; 0) = C_{ijkl}\, \delta(\mathbf{x} - \xi) \tag{3.255}$$

$$\widetilde{S}_{ij} n_j = S_{ij}^{(kl)}(\mathbf{x}, \xi; t) n_j(\xi) \tag{3.256}$$

$$\widetilde{S}_{im,m} = S_{im,m}^{(kl)}(\mathbf{x}, \xi; t) \tag{3.257}$$

Since

$$\begin{aligned}\dot{\widetilde{S}}_{ij}^{(0)} K_{ijpq}\, S_{pq} &= C_{ijkl}\, K_{ijpq}\, S_{pq}\, \delta(\mathbf{x} - \xi) = C_{klij}\, K_{ijpq}\, S_{pq}\, \delta(\mathbf{x} - \xi) \\ &= \delta_{(kp}\, \delta_{ql)}\, S_{pq}\, \delta(\mathbf{x} - \xi) = S_{kl}(\xi,t)\delta(\mathbf{x} - \xi)\end{aligned} \tag{3.258}$$

therefore, Eq. (3.254) takes the form

$$\int_B S_{kl}(\xi,t)\delta(\mathbf{x} - \xi) dv(\xi) = \int_{\partial B} \rho^{-1} \left(S_{im,m} * S_{ij}^{(kl)} n_j - \widehat{s}_i * S_{im,m}^{(kl)} \right) da(\xi) \tag{3.259}$$

Finally, using the filtrating property of the delta function we obtain (3.253). This completes a solution to Problem 3.11.

Problem 3.12. Consider the pure stress initial-boundary value problem of linear elastodynamics for a homogeneous isotropic *incompressible* elastic body B [see Eq. (3.55) with $\mu > 0$ and $\lambda \to \infty$]. Find a tensor field $\mathbf{S} = \mathbf{S}(\mathbf{x}, t)$ on $B \times [0, \infty)$ that satisfies the equation

$$\widehat{\nabla}(\operatorname{div}\mathbf{S}) - \frac{\rho}{2\mu}\left[\ddot{\mathbf{S}} - \frac{1}{3}(\operatorname{tr}\ddot{\mathbf{S}})\,\mathbf{1}\right] = -\widehat{\nabla}\mathbf{b} \quad \text{on} \quad B \times [0, \infty) \tag{3.260}$$

subject to the initial conditions

$$\mathbf{S}(\mathbf{x}, 0) = \mathbf{S}_0(\mathbf{x}), \quad \dot{\mathbf{S}}(\mathbf{x}, 0) = \dot{\mathbf{S}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in B \tag{3.261}$$

and the traction boundary condition

$$\mathbf{S}\mathbf{n} = \widehat{\mathbf{s}} \quad \text{on} \quad \partial B \times [0, \infty) \tag{3.262}$$

Here, $\mathbf{b}, \widehat{\mathbf{s}}, \mathbf{S}_0$, and $\dot{\mathbf{S}}_0$, are prescribed functions ($\mu > 0$, $\rho > 0$). Show that the problem (3.260) through (3.262) may have at most one solution.

Solution. We are to show that the field equation

$$S_{(ik,kj)} - \frac{\rho}{2\mu}\left(\ddot{S}_{ij} - \frac{1}{3}\ddot{S}_{kk}\,\delta_{ij}\right) = 0 \quad \text{on } B \times [0, \infty) \tag{3.263}$$

subject to the homogeneous initial conditions

$$S_{ij}(\mathbf{x}, 0) = 0, \quad \dot{S}_{ij}(\mathbf{x}, 0) = 0 \quad \text{on } B \tag{3.264}$$

and the homogeneous traction boundary condition

$$S_{ij}\, n_j = 0 \quad \text{on } \partial B \times [0, \infty) \tag{3.265}$$

imply that

$$S_{ij} = 0 \quad \text{on } \overline{B} \times [0, \infty) \tag{3.266}$$

To this end we multiply (3.263) by $\dot{S}_{ij}$ and obtain

$$S_{(ik,kj)}\,\dot{S}_{ij} - \frac{\rho}{2\mu}\left(\ddot{S}_{ij}\,\dot{S}_{ij} - \frac{1}{3}\ddot{S}_{kk}\,\dot{S}_{ii}\right) = 0 \tag{3.267}$$

Since

$$S_{(ik,kj)}\,\dot{S}_{ij} = S_{ik,kj}\,\dot{S}_{ij} = (S_{ik,k}\,\dot{S}_{ij}),_j - S_{ik,k}\,\dot{S}_{ij,j} \tag{3.268}$$

therefore (3.267) can be written in the form

$$(S_{ik,k}\,\dot{S}_{ij})_{,j} - S_{ik,k}\,\dot{S}_{ij,j} - \frac{\rho}{2\mu}\left[\frac{1}{2}\frac{\partial}{\partial t}(\dot{S}_{ij}\,\dot{S}_{ij}) - \frac{1}{3}\frac{1}{2}\frac{\partial}{\partial t}(\dot{S}_{aa})^2\right] = 0 \tag{3.269}$$

or

$$(S_{ik,k}\,\dot{S}_{ij})_{,j} - \frac{1}{2}\frac{\partial}{\partial t}(S_{ik,k}\,S_{ij,j}) - \frac{\rho}{2\mu}\left[\frac{1}{2}\frac{\partial}{\partial t}(\dot{S}_{ij}\,\dot{S}_{ij}) - \frac{1}{3}\frac{1}{2}\frac{\partial}{\partial t}(\dot{S}_{aa})^2\right] = 0 \tag{3.270}$$

Integrating Eq. (3.270) over the cartesian product $B \times [0, t]$, using the divergence theorem, the homogeneous initial conditions (3.264) as well as the boundary condition

$$\dot{S}_{ij}\,n_j = 0 \quad \text{on } \partial B \times [0, \infty) \tag{3.271}$$

obtained by differentiation of (3.265) with respect to time, we obtain

$$\int_B \left\{S_{ik,k}\,S_{ij,j} + \frac{\rho}{2\mu}\left[\dot{S}_{ij}\,\dot{S}_{ij} - \frac{1}{3}(\dot{S}_{aa})^2\right]\right\} dv = 0 \tag{3.272}$$

Since

$$\dot{S}_{ij} = \dot{S}^{(d)}_{ij} + \frac{1}{3}\dot{S}_{aa}\,\delta_{ij} \tag{3.273}$$

where

$$\dot{S}^{(d)}_{ij} = \dot{S}_{ij} - \frac{1}{3}\dot{S}_{aa}\,\delta_{ij} \tag{3.274}$$

and

$$\dot{S}_{ij}\,\dot{S}_{ij} = \dot{S}^{(d)}_{ij}\dot{S}^{(d)}_{ij} + \frac{1}{3}(\dot{S}_{aa})^2 \tag{3.275}$$

therefore, Eq. (3.272) can be written as

$$\int_B \left(S_{ik,k}\,S_{ij,j} + \frac{\rho}{2\mu}\dot{S}^{(d)}_{ij}\,\dot{S}^{(d)}_{ij}\right) dv = 0 \tag{3.276}$$

Equation (3.276) implies that

$$S_{ik,k} = 0, \quad \dot{S}^{(d)}_{ij} = 0 \quad \text{on } \overline{B} \times [0, \infty) \tag{3.277}$$

Equation $(3.277)_2$ together with the homogeneous initial conditions $(3.264)_1$ imply that

$$S_{ij} = \frac{1}{3}S_{aa}\,\delta_{ij} \quad \text{on } \overline{B} \times [0, \infty) \tag{3.278}$$

Equations (3.278) and $(3.277)_1$ imply that

$$S_{aa,i} = 0 \quad \text{on } \overline{B} \times [0, \infty) \tag{3.279}$$

which is equivalent to

$$S_{aa}(\mathbf{x}, t) = c(t) \quad \text{on } \overline{B} \times [0, \infty) \tag{3.280}$$

where $c = c(t)$ is an arbitrary function of time.

Finally, Eqs. (3.265), (3.278), and (3.280) lead to

$$c(t)n_i(\mathbf{x}) = 0 \quad \text{on } \partial B \times [0, \infty) \tag{3.281}$$

Since $|n_i\, n_i| = 1$ on ∂B, therefore Eq. (3.281) implies that

$$|c(t)| = 0 \tag{3.282}$$

Equation (3.282) together with Eq. (3.278) implies Eq. (3.266), and this completes a solution to Problem 3.12.