Chapter 2
Fundamentals of Linear Elasticity

In this chapter a number of concepts are introduced to describe a linear elastic body. In particular, the displacement vector, strain tensor, and stress tensor fields are introduced to define a linear elastic body which satisfies the strain-displacement relations, the equations of motion, and the constitutive relations. Also, the compatibility relations, the general solutions of elastostatics, and an alternative definition of the displacement field of elastodynamics are discussed. The stored energy of an elastic body, the positive definiteness and strong ellipticity of the elasticity fourth-order tensor, and the stress-strain-temperature relations for a thermoelastic body are also discussed.

2.1 Deformation of an Elastic Body

A material body B is defined as a set of elements $\mathbf{x}$, called particles, for which there is a one-to-one correspondence with the points of a region $\kappa(\mathrm{B})$ of a physical space; while a *deformation* of B is a map κ of B onto a region $\kappa(\mathrm{B})$ in E^3 with det $(\nabla\kappa) > 0$. The point $\kappa(\mathbf{x})$ is the place occupied by the particle $\mathbf{x}$ in the deformation κ, and

$$\mathbf{u}(\mathbf{x}) = \kappa(\mathbf{x}) - \mathbf{x} \tag{2.1}$$

is the *displacement* of $\mathbf{x}$.

If the mapping κ depends also on time $t \in [0, \infty)$, such a mapping defines a *motion* of B, and the displacement of $\mathbf{x}$ at time t is

$$\mathbf{u}(\mathbf{x}, \mathrm{t}) = \kappa(\mathbf{x}, t) - \mathbf{x} \tag{2.2}$$

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By the *deformation gradient* and the *displacement gradient* we mean the tensor fields $\mathbf{F} = \nabla\kappa$ and $\nabla\mathbf{u}$, respectively. A *finite strain tensor* $\mathbf{D}$ is defined by

$$\mathbf{D} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}) \tag{2.3}$$

or, equivalently, by

$$\mathbf{D} = \mathbf{E} + \frac{1}{2}(\nabla\mathbf{u})(\nabla\mathbf{u}^{\mathrm{T}}) \tag{2.4}$$

where

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) = \widehat{\nabla}\mathbf{u} \tag{2.5}$$

The tensor field $\mathbf{E}$ is called an *infinitesimal strain tensor*.

An *infinitesimal rigid displacement* of B is defined by

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{W}(\mathbf{x} - \mathbf{x}_0) \tag{2.6}$$

where $\mathbf{u}_0$, $\mathbf{x}_0$ are constant vectors and $\mathbf{W}$ is a skew constant tensor.

An *infinitesimal volume change* of B is defined by

$$\delta\mathrm{v}(\mathrm{B}) = \int_{\mathrm{B}} \mathrm{div}\,\mathbf{u}\,\mathrm{dv} \tag{2.7}$$

while

$$\mathrm{div}\,\mathbf{u} = \mathrm{tr}\,\mathbf{E} \tag{2.8}$$

represents a *dilatation field*.

If a deformation is not accompanied by a change of volume, that is, if $\delta\mathrm{v}(\mathrm{P}) = 0$ for every $\mathrm{P} \subset \mathrm{B}$, the displacement $\mathbf{u}$ is called *isochoric*.

Kirchhoff Theorem. If two displacement fields $\mathbf{u}_1$ and $\mathbf{u}_2$ correspond to the same strain field $\mathbf{E}$ then

$$\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{w} \tag{2.9}$$

where $\mathbf{w}$ is a rigid displacement field.

A *homogeneous displacement field* is defined by

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) \tag{2.10}$$

where $\mathbf{A}$ is an arbitrary constant tensor and $\mathbf{u}_0,\ \mathbf{x}_0$ are constant vectors. Clearly, if $\mathbf{A}$ is skew, (2.10) represents a rigid displacement, while for an arbitrary $\mathbf{A}$

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) + \mathbf{u}_2(\mathbf{x}) \tag{2.11}$$

where $\mathbf{u}_1(\mathbf{x})$ is a rigid displacement field and $\mathbf{u}_2(\mathbf{x})$ is a displacement field corresponding to the strain $\mathbf{E} = \text{sym}\,\mathbf{A}$. The displacement $\mathbf{u}_2(\mathbf{x})$ of the form

$$\mathbf{u}_2(\mathbf{x}) = \mathbf{E}\,(\mathbf{x} - \mathbf{x}_0) \tag{2.12}$$

corresponds to a *pure strain from* $\mathbf{x}_0$.

Let $\mathrm{e} > 0$ and let $\mathbf{n}$ be a unit vector. Then by substituting $\mathbf{E} = \mathrm{e}\,\mathbf{n} \otimes \mathbf{n}$ into (2.12) we obtain a *simple extension of amount* e in the direction $\mathbf{n}$; and by substituting $\mathbf{E} = \mathrm{e}\,\mathbf{1}$ into (2.12) we obtain *uniform dilatation of amount* e. Finally, let $g > 0$ and let $\mathbf{m}$ be a unit vector perpendicular to $\mathbf{n}$. Then substituting $\mathbf{E} = \mathrm{g}\,[\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}\,]$ into (2.12) we obtain a *simple shear of amount* g with respect to the pair $(\mathbf{m},\mathbf{n})$.

Decomposition of a strain tensor E into spherical and deviatoric tensors

$$\mathbf{E} = \mathbf{E}^{(\mathrm{s})} + \mathbf{E}^{(\mathrm{d})} \tag{2.13}$$

where

$$\mathbf{E}^{(\mathrm{s})} = \frac{1}{3}(\mathrm{tr}\,\mathbf{E})\,\mathbf{1} \tag{2.14}$$

is called a *spherical part* of $\mathbf{E}$, and $\mathbf{E}^{(\mathrm{d})} = \mathbf{E} - \mathbf{E}^{(\mathrm{s})}$ is called a *deviatoric part* of $\mathbf{E}$. Clearly,

$$\mathrm{tr}\,(\mathbf{E}^{(\mathrm{d})}) = 0 \tag{2.15}$$

2.2 Compatibility

Theorem Let $\mathrm{B} \subset E^3$ be simply connected. If $\mathbf{u}$ is a displacement field corresponding to a strain field $\mathbf{E}$ on B, that is, if

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \quad \text{on} \quad \mathrm{B} \tag{2.16}$$

then $\mathbf{E}$ satisfies the equations of compatibility

$$\text{curl curl}\,\mathbf{E} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{2.17}$$

Conversely, let $\mathbf{E}$ be a symmetric tensor field that satisfies the equations of compatibility (2.17), then there exists a displacement field $\mathbf{u}$ on B such that $\mathbf{u}$ and $\mathbf{E}$ satisfy (2.16).

In components the equations of compatibility (2.17) take the form

$$E_{ij,kl} + E_{kl,ij} - E_{ik,jl} - E_{jl,ik} = 0 \tag{2.18}$$

An alternative form of (2.17) reads

$$\nabla^2 \mathbf{E} + \nabla\nabla(\mathrm{tr}\,\mathbf{E}) - 2\widehat{\nabla}(\mathrm{div}\,\mathbf{E}) = \mathbf{0} \tag{2.19}$$

2.3 Motion and Equilibrium

Let S be a surface in B with unit normal $\mathbf{n}$. Let B be subject to a deformation, and let $\mathbf{s}_\mathrm{n} = \mathbf{s}_\mathrm{n}(\mathbf{x}, t)$ denote a force per unit area at $\mathbf{x}$ and for $t \geq 0$ exerted by a portion of B on the side S toward which $\mathbf{n}$ points on a portion of B on the other side of S. The force $\mathbf{s}_\mathrm{n}$ is called the *stress vector* at $(\mathbf{x}, t)$, while a second-order tensor field $\mathbf{S} = \mathbf{S}(\mathbf{x}, \mathrm{t})$ such that

$$\mathbf{S}\,\mathbf{n} = \mathbf{s}_\mathrm{n} \quad \text{on} \quad S \times [0, \infty) \tag{2.20}$$

is called a *time-dependent stress tensor field* on $S \times [0, \infty)$.

The equilibrium equations of elastostatics

$$\mathrm{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0} \tag{2.21}$$

$$\mathbf{S} = \mathbf{S}^\mathrm{T} \tag{2.22}$$

Equation (2.21) expresses the *balance of forces*, and Eq. (2.22) expresses the *balance of moments;* and $\mathbf{b}$ in (2.21) is the *body force vector.*

The Beltrami representation of S

$$\mathbf{S} = \mathrm{curl}\,\mathrm{curl}\,\mathbf{A} \tag{2.23}$$

where $\mathbf{A}$ is a symmetric tensor field, or

$$\mathbf{S} = -\nabla^2 \mathbf{G} + 2\widehat{\nabla}(\mathrm{div}\,\mathbf{G}) - (\mathrm{div}\,\mathrm{div}\,\mathbf{G})\,\mathbf{1} \tag{2.24}$$

where $\mathbf{G}$ is a symmetric tensor field.

Self-equilibrated stress field

If $\mathbf{S} = \mathbf{S}^\mathrm{T}$ on B, and

$$\int_\mathrm{S} \mathbf{S}\mathbf{n}\,\mathrm{da} = \mathbf{0} \tag{2.25}$$

$$\int_\mathrm{S} \mathbf{x} \times (\mathbf{S}\mathbf{n})\,\mathrm{da} = \mathbf{0} \tag{2.26}$$

for every closed surface S in B, then $\mathbf{S}$ is called a *self-equilibrated stress field.*

One can show that $\mathbf{S}$ given by (2.23) is a self-equilibrated stress field, and $\mathbf{S}$ *given by* (2.23) *is complete* in the sense that for any self-equilibrated $\mathbf{S}$ there is a symmetric tensor $\mathbf{A}$ such that (2.23) is satisfied.

The Beltrami-Schaefer representation of S

$$\mathbf{S} = \text{curl curl } \mathbf{A} + 2\widehat{\nabla}\mathbf{h} - (\text{div } \mathbf{h})\,\mathbf{1} \tag{2.27}$$

where **A** is a symmetric tensor field and **h** is a harmonic vector field on B.

2.4 Equations of Motion

$$\text{div } \mathbf{S} + \mathbf{b} = \rho\,\ddot{\mathbf{u}} \quad \text{on} \quad \text{B} \times [0, \infty) \tag{2.28}$$

where ρ is density and **b** is the body force vector field.

Kinetic energy of B for $t \geq 0$

$$\text{K(t)} = \frac{1}{2}\int_{\text{B}} \rho\,\dot{\mathbf{u}}^2 \text{dv} \tag{2.29}$$

Stress power of B for $t \geq 0$

$$\text{P(t)} = \int_{\text{B}} \mathbf{S} \cdot \dot{\mathbf{E}} \text{dv} \tag{2.30}$$

A dynamic process is identified with a triplet [**u**, **S**, **b**] that satisfies the equations of motion (2.28).

Theorem An array of functions [**u**, **S**, **b**] is a dynamic process consistent with the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \overline{\text{B}} \tag{2.31}$$

if and only if

$$\text{i} * \text{div } \mathbf{S} + \mathbf{f} = \rho\,\mathbf{u} \quad \text{on} \quad \overline{\text{B}} \times [0, \infty) \tag{2.32}$$

where

$$\mathbf{f}(\mathbf{x}, t) = \text{i} * \mathbf{b}(\mathbf{x}, t) + \rho(\mathbf{x})\,[\mathbf{u}_0(\mathbf{x}) + t\,\dot{\mathbf{u}}_0(\mathbf{x})] \tag{2.33}$$

and

$$\text{i} = \text{i}(t) = t \tag{2.34}$$

The function $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ given by (2.33) is called *pseudo-body force field*.

Clearly, since $\rho > 0$, Eq. (2.32) provides an alternative definition of the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ related to the stress tensor $\mathbf{S} = \mathbf{S}(\mathbf{x},t)$.

2.5 Constitutive Relations

A body B is said to be *linearly elastic* if for every point $\mathbf{x} \in$ B there is a linear transformation $\mathbf{C}$ from the space of all symmetric tensors $\mathbf{E}$ into the space of all symmetric tensors $\mathbf{S}$, or

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] \tag{2.35}$$

In components

$$S_{ij} = C_{ijkl} E_{kl} \tag{2.36}$$

The tensor $\mathbf{C} = \mathbf{C}(\mathbf{x})$ is called the *elasticity tensor* field on B. It follows from Eq. (1.54) that

$$C_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{C}[(\mathbf{e}_k \otimes \mathbf{e}_l)] \tag{2.37}$$

and, since $\mathbf{S}$ and $\mathbf{E}$ are symmetric, we postulate that

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{2.38}$$

The elasticity tensor $\mathbf{C}$ is also assumed to be *invertible*, that means that a *restriction* of $\mathbf{C}$ to the space of all symmetric tensors is invertible. The elasticity tensor on the space of all tensors cannot be invertible since its value on every skew tensor is zero.

The *invertibility* of $\mathbf{C}$ means that there is a fourth-order tensor $\mathbf{K} = \mathbf{K}(\mathbf{x})$ such that

$$\mathbf{K} = \mathbf{C}^{-1} \tag{2.39}$$

Then equivalent form of (2.35) is

$$\mathbf{E} = \mathbf{K}[\mathbf{S}] \tag{2.40}$$

The tensor $\mathbf{K} = \mathbf{K}(\mathbf{x})$ is called the *compliance tensor*.

The fourth-order tensor $\mathbf{C}$ is *symmetric* if and only if

$$\mathbf{A} \cdot \mathbf{C}[\mathbf{B}] = \mathbf{B} \cdot \mathbf{C}[\mathbf{A}] \tag{2.41}$$

for any symmetric tensors $\mathbf{A}$ and $\mathbf{B}$.

In components the symmetry of $\mathbf{C}$ means that

$$C_{ijkl} = C_{klij} \tag{2.42}$$

The tensor $\mathbf{C}$ is *positive semi-definite* if

$$\mathbf{A} \cdot \mathbf{C}[\mathbf{A}] \geq 0 \tag{2.43}$$

for every symmetric tensor $\mathbf{A}$.

The tensor C is *positive definite* if

$$\mathbf{A} \cdot \mathsf{C}[\mathbf{A}] > 0 \tag{2.44}$$

for every symmetric nonzero tensor $\mathbf{A}$.

The compliance tensor K enjoys the properties similar to those of the elasticity tensor C [see, Eqs. (2.38) and (2.42)–(2.44)].

By an *anisotropic elastic body* we mean the body for which the tensor C possesses in general 21 different components.

2.6 Isotropic Elastic Body

For an *isotropic elastic body* the Eqs. (2.35) and (2.40), respectively, take the form

$$\mathbf{S} = 2\mu\,\mathbf{E} + \lambda\,(\mathrm{tr}\,\mathbf{E}) \tag{2.45}$$

and

$$\mathbf{E} = \frac{1}{2\mu}\left[\mathbf{S} - \frac{\lambda}{3\lambda + 2\mu}(\mathrm{tr}\,\mathbf{S})\ \mathbf{1}\right] \tag{2.46}$$

where λ and μ are Lamé moduli subject to the constitutive restrictions

$$\mu > 0, \quad 3\lambda + 2\mu > 0 \tag{2.47}$$

An alternative form of Eqs. (2.45) and (2.46), written in terms of Young's modulus E and Poisson's ratio ν, reads

$$\mathbf{S} = \frac{E}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\ \mathbf{1}\right] \tag{2.48}$$

$$\mathbf{E} = \frac{1}{E}\left[(1+\nu)\,\mathbf{S} - \nu\,(\mathrm{tr}\,\mathbf{S})\ \mathbf{1}\right] \tag{2.49}$$

where

$$E > 0 \quad \text{and} \quad -1 < \nu < 1/2 \tag{2.50}$$

Strain energy density of B

$$\mathrm{W}(\mathbf{E}) = \frac{1}{2}\mathbf{E} \cdot \mathsf{C}\,[\mathbf{E}] \tag{2.51}$$

Stress energy density of B

$$\widehat{\mathrm{W}}(\mathbf{S}) = \frac{1}{2}\mathbf{S} \cdot \mathsf{K}\,[\mathbf{S}] \tag{2.52}$$

The tensor C is said to be *strongly elliptic* if

$$\mathbf{A} \cdot \mathsf{C}[\mathbf{A}] > 0 \tag{2.53}$$

for every $\mathbf{A}$ of the form

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{b} \tag{2.54}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary nonzero vectors.

2.7 The Cauchy Relations

An anisotropic elastic body obeying, in addition to the symmetry relations (2.38) and (2.42), the restrictions

$$C_{ijkl} = C_{ikjl} \tag{2.55}$$

is said to be of the Cauchy type.

2.8 Constitutive Relations for a Thermoelastic Body

For an anisotropic body subject to an uneven heating the constitutive relations take the form

$$\mathbf{S} = \mathsf{C}\,[\mathbf{E}] + T\mathbf{M} \tag{2.56}$$

and

$$\mathbf{E} = \mathsf{K}\,[\mathbf{S}] + T\mathbf{A} \tag{2.57}$$

where

$$T = \theta - \theta_0, \quad \theta_0 > 0 \tag{2.58}$$

is a temperature change, $\mathbf{M} = \mathbf{M}^T$ is called the *stress-temperature tensor,* $\mathbf{A} = \mathbf{A}^T$ is called the *thermal expansion tensor*, θ is the *absolute temperature,* and θ_0 is a *reference temperature.*

Since relations (2.56) and (2.57) are equivalent

$$\mathsf{K} = \mathsf{C}^{-1} \quad \text{and} \quad \mathbf{A} = -\,\mathsf{K}[\mathbf{M}] \tag{2.59}$$

for an isotropic body

$$\mathbf{S} = 2\mu\,\mathbf{E} + \lambda\,(\mathrm{tr}\,\mathbf{E}) - (3\lambda + 2\mu)\alpha\,T\,\mathbf{1} \tag{2.60}$$

and

$$\mathbf{E} = \frac{1}{2\mu}\left[\mathbf{S} - \frac{\lambda}{3\lambda + 2\mu}(\mathrm{tr}\,\mathbf{S})\ \mathbf{1}\right] + \alpha\, T\, \mathbf{1} \tag{2.61}$$

where α is the *coefficient of thermal expansion*,
or

$$\mathbf{S} = \frac{E}{1+\nu}\left[\mathbf{E} + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\mathbf{E})\ \mathbf{1}\right] - \frac{E}{1-2\nu}\alpha\, T\, \mathbf{1} \tag{2.62}$$

and

$$\mathbf{E} = \frac{1}{E}\left[(1+\nu)\,\mathbf{S} - \nu\,(\mathrm{tr}\,\mathbf{S})\ \mathbf{1}\right] + \alpha\, T\, \mathbf{1} \tag{2.63}$$

2.9 Problems and Solutions Related to the Fundamentals of Linear Elasticity

Problem 2.1. Show that if $\mathbf{u}$ is a pure strain from $\mathbf{x}_0$, then $\mathbf{u}$ admits the decomposition

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \tag{2.64}$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are simple extensions in mutually perpendicular directions from $\mathbf{x}_0$.

Solution. Since $\mathbf{u}$ represents a pure strain from $\mathbf{x}_0$, $\mathbf{u}$ takes the form [see definition of $\mathbf{u}_2$ in (2.12)]

$$\mathbf{u} = \mathbf{E}(\mathbf{x} - \mathbf{x}_0) \tag{2.65}$$

where $\mathbf{E}$ is the strain tensor corresponding to $\mathbf{u}$. Now, by the decomposition spectral theorem [see Eq. (1.45) in which $\mathbf{T} = \mathbf{E}$ and $\lambda_i = e_i$]

$$\mathbf{E} = \sum_{i=1}^{3} e_i\ \mathbf{n}_i \otimes \mathbf{n}_i \tag{2.66}$$

where $\mathbf{n}_i$ is a principal direction corresponding to a principal value e_i of $\mathbf{E}$.

Substituting (2.66) into (2.65) we obtain

$$\mathbf{u} = \sum_{i=1}^{3} e_i(\mathbf{n}_i \otimes \mathbf{n}_i)(\mathbf{x} - \mathbf{x}_0) \tag{2.67}$$

Since for two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$

$$(\mathbf{a} \otimes \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} \tag{2.68}$$

therefore, Eq. (2.67) is equivalent to

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \tag{2.69}$$

where

$$\mathbf{u}_i = e_i[\mathbf{n}_i \cdot (\mathbf{x} - \mathbf{x}_0)]\mathbf{n}_i \text{ (no sum)} \tag{2.70}$$

Since $\mathbf{u}_i$ represents a simple extension of magnitude e_i in the direction of $\mathbf{n}_i$ [see Eq. (2.12)], and $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are orthogonal, Eq. (2.69) is equivalent to (2.64). This completes proof of (2.64).

Problem 2.2. Show that $\mathbf{u}$ in Problem 2.1 admits an alternative representation

$$\mathbf{u} = \mathbf{u}_\mathrm{d} + \mathbf{u}_\mathrm{c} \tag{2.71}$$

where $\mathbf{u}_\mathrm{d}$ is a uniform dilatation from $\mathbf{x}_0$, while $\mathbf{u}_\mathrm{c}$ is an isochoric pure strain from $\mathbf{x}_0$.

Solution. We rewrite $\mathbf{E}$ of Problem 2.1 as

$$\mathbf{E} = \mathbf{E}^{(s)} + \mathbf{E}^{(d)} \tag{2.72}$$

where

$$\mathbf{E}^{(s)} = \frac{1}{3}\mathbf{1}(\mathrm{tr}\ \mathbf{E}), \quad \mathbf{E}^{(d)} = \mathbf{E} - \frac{1}{3}\mathbf{1}(\mathrm{tr}\ \mathbf{E}) \tag{2.73}$$

Then Eq. (2.65) of Problem 2.1 takes the form

$$\mathbf{u} = \mathbf{u}_d + \mathbf{u}_c \tag{2.74}$$

where

$$\mathbf{u}_d = \mathbf{E}^{(s)}(\mathbf{x} - \mathbf{x}_0) \tag{2.75}$$

$$\mathbf{u}_c = \mathbf{E}^{(d)}(\mathbf{x} - \mathbf{x}_0) \tag{2.76}$$

It follows from Eqs. (2.73) and (2.75) that $\mathbf{u}_d$ represents a uniform dilatation of magnitude $e = \frac{1}{3}(\mathrm{tr}\ \mathbf{E})$, while the condition $\mathrm{tr}\ \mathbf{E}^{(d)} = 0$ implies that $\mathbf{u}_c$ represents an isochoric pure strain. This completes solution to Problem 2.2.

Problem 2.3. Show that if $\mathbf{u}$ is a simple shear of amount γ with respect to the pair ($\mathbf{m}$, $\mathbf{n}$), where $\mathbf{m}$ and $\mathbf{n}$ are perpendicular unit vectors, then $\mathbf{u}$ admits the decomposition

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \tag{2.77}$$

where $\mathbf{u}_1$ is a simple extension of amount γ in the direction $\frac{1}{\sqrt{2}}(\mathbf{m} + \mathbf{n})$, and $\mathbf{u}_2$ is a simple extension of amount $-\gamma$ in the direction $\frac{1}{\sqrt{2}}(\mathbf{m} - \mathbf{n})$.

Solution. Since $\mathbf{u}$ represents a simple shear of amount γ with respect to $(\mathbf{m}, \mathbf{n})$, then the strain tensor corresponding to $\mathbf{u}$ takes the form [see the definition of a simple shear below Eq. (2.12)]

$$\mathbf{E} = \gamma\ (\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) \tag{2.78}$$

Let λ and $\mathbf{a}$ denote a principal value and a principal vector of $\mathbf{E}$, respectively. Then

$$\gamma\ (\mathbf{n} \cdot \mathbf{a})\mathbf{m} + \gamma(\mathbf{m} \cdot \mathbf{a})\mathbf{n} - \lambda\mathbf{a} = \mathbf{0} \tag{2.79}$$

It is easy to check that Eq. (2.79) has the three eigensolutions

$$\mathbf{a}_1 = \mathbf{m} \times \mathbf{n}, \qquad \lambda_1 = 0 \tag{2.80}$$

$$\mathbf{a}_2 = \frac{1}{\sqrt{2}}(\mathbf{m} + \mathbf{n}), \quad \lambda_2 = \gamma \tag{2.81}$$

$$\mathbf{a}_3 = \frac{1}{\sqrt{2}}(\mathbf{m} - \mathbf{n}), \quad \lambda_3 = -\gamma \tag{2.82}$$

Therefore, using the solution (2.67) of Problem 2.1 we find that Eq. (2.77) holds true. This completes solution of Problem 2.3.

Problem 2.4. Let $\mathbf{u}$ and $\mathbf{E}$ denote a displacement vector field and the corresponding strain tensor field defined on $\overline{\mathrm{B}}$. Show that the mean strain $\widehat{\mathbf{E}}(\mathrm{B})$ is represented by the surface integral

$$\widehat{\mathbf{E}}(\mathrm{B}) = \frac{1}{v(\mathrm{B})} \int_{\partial \mathrm{B}} \text{sym}\ (\mathbf{u} \otimes \mathbf{n})\, da \tag{2.83}$$

where $v(\mathrm{B})$ is the volume of B.

Solution. The mean strain $\widehat{\mathbf{E}}(B)$ is defined by

$$\widehat{\mathbf{E}}(B) = \frac{1}{v(B)} \int_{B} \mathbf{E}\, dv \tag{2.84}$$

In components we obtain

$$\widehat{E}_{ij}(B) = \frac{1}{v(B)} \int_{B} E_{ij}\, dv \tag{2.85}$$

Since

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.86}$$

therefore, by the divergence theorem,

$$\int_B E_{ij}\,dv = \frac{1}{2}\int_B (u_{i,j} + u_{j,i})dv = \frac{1}{2}\int_{\partial B} (u_i\,n_j + u_j\,n_i)da \tag{2.87}$$

Equations (2.87) and (2.85) imply that Eq. (2.83) holds true, and this completes solution to Problem 2.4.

Problem 2.5. Show that if $\mathbf{u} = \mathbf{0}$ on ∂B then

$$\int_{\mathrm{B}} (\nabla\mathbf{u})^2 dv \leq 2\int_{\mathrm{B}} |\mathbf{E}|^2\,dv \tag{2.88}$$

where $\mathbf{E}$ is the strain tensor field corresponding to a displacement field $\mathbf{u}$ on B.

Solution. We recall the relation

$$\nabla\mathbf{u} = \mathbf{E} + \mathbf{W} \tag{2.89}$$

where

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}) \tag{2.90}$$

and

$$\mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}) \tag{2.91}$$

Since $\mathbf{E}\cdot\mathbf{W} = 0$, Eq. (2.89) implies that

$$|\nabla\mathbf{u}|^2 = |\mathbf{E}|^2 + |\mathbf{W}|^2 \tag{2.92}$$

and it follows from Eqs. (2.90) and (2.91), respectively, that

$$|\mathbf{E}|^2 = \frac{1}{2}[(\nabla\mathbf{u})^2 + (\nabla\mathbf{u})\cdot(\nabla\mathbf{u}^{\mathrm{T}})] \tag{2.93}$$

and

$$|\mathbf{W}|^2 = \frac{1}{2}[(\nabla\mathbf{u})^2 - (\nabla\mathbf{u})\cdot(\nabla\mathbf{u}^{\mathrm{T}})] \tag{2.94}$$

Hence,

$$|\mathbf{E}|^2 - |\mathbf{W}|^2 = (\nabla\mathbf{u})\cdot(\nabla\mathbf{u}^{\mathrm{T}}) \tag{2.95}$$

Now

$$\begin{aligned}(\nabla\mathbf{u})\cdot(\nabla\mathbf{u}^{\mathrm{T}}) &= u_{i,j}\,u^T_{i,j} = u_{i,j}\,u_{j,i}\\ &= (u_{i,j}\,u_j)_{,i} - u_{i,ji}\,u_j\\ &= (u_{i,j}\,u_j)_{,i} - (u_{i,i}\,u_j)_{,j} + (u_{i,i})^2\end{aligned}$$

$$= (u_{j,i}\, u_i - u_{i,i}\, u_j)_{,j} + (u_{i,i})^2$$
$$= \text{div}[(\nabla \mathbf{u})\mathbf{u} - (\text{div}\, \mathbf{u})\mathbf{u}] + (\text{div}\, \mathbf{u})^2 \tag{2.96}$$

Therefore, integrating Eq. (2.96) over B, using the divergence theorem, and the homogeneous boundary condition: $\mathbf{u} = \mathbf{0}$ on ∂B, we obtain

$$\int_B (\nabla \mathbf{u}) \cdot (\nabla \mathbf{u}^{\mathrm{T}}) dv = \int_B (\text{div}\, \mathbf{u})^2 dv \tag{2.97}$$

Equations (2.95) and (2.97) imply that

$$\int_B (|\mathbf{E}|^2 - |\mathbf{W}|^2) dv = \int_B (\text{div}\, \mathbf{u})^2 dv \tag{2.98}$$

and it follows from Eq. (2.92) that

$$\int_B (|\mathbf{E}|^2 + |\mathbf{W}|^2) dv = \int_B |\nabla \mathbf{u}|^2 dv \tag{2.99}$$

Therefore, by adding Eqs. (2.98) and (2.99), we obtain

$$2 \int_B |\mathbf{E}|^2 dv = \int_B |\nabla \mathbf{u}|^2 dv + \int_B (\text{div}\, \mathbf{u})^2 dv \tag{2.100}$$

and Eq. (2.100) leads to the inequality

$$2 \int_B |\mathbf{E}|^2 dv \geq \int_B |\nabla \mathbf{u}|^2 dv \tag{2.101}$$

This completes solution of Problem 2.5.

Problem 2.6. (i) Let $\mathbf{E}$ be a strain tensor field on E^3 defined by the matrix

$$\mathbf{E} = \frac{N}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \tag{2.102}$$

where E, N, and ν are positive constants. Show that a solution $\mathbf{u}$ to the equation $\mathbf{E} = \widehat{\nabla} \mathbf{u}$ on E^3 subject to the condition $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ takes the form

$$\mathbf{u} = \left[\frac{N}{E} x_1,\ -\nu \frac{N}{E} x_2,\ -\nu \frac{N}{E} x_3 \right]^{\mathrm{T}} \tag{2.103}$$

(ii) Let $\mathbf{E}$ be a strain tensor field on E^3 defined by the matrix

$$\mathbf{E} = \frac{M}{EI} x_1 \begin{bmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.104}$$

where M, E, I, and ν are positive constants. Show that a solution $\mathbf{u}$ to the equation $\mathbf{E} = \widehat{\nabla}\mathbf{u}$ on E^3 subject to the condition $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ takes the form

$$\mathbf{u} = \frac{M}{EI} \left[\frac{1}{2}(x_3^2 + \nu x_1^2 - \nu x_2^2), \quad \nu x_1 x_2, \quad -x_1 x_3 \right]^{\mathrm{T}} \tag{2.105}$$

Solution. (i) Using (2.103) we find that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ and

$$\nabla\mathbf{u} = \frac{N}{E} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} \tag{2.106}$$

Since $\nabla\mathbf{u} = \nabla\mathbf{u}^{\mathrm{T}}$, the equation

$$\widehat{\nabla}\mathbf{u} = \mathbf{E} \tag{2.107}$$

in which $\mathbf{E}$ is given by (2.102) is identically satisfied. This completes a proof of (i).

(ii) Using (2.105) we obtain $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ and

$$\nabla\mathbf{u} = \frac{M}{EI} \begin{bmatrix} \nu x_1 & -\nu x_2 & x_3 \\ \nu x_2 & \nu x_1 & 0 \\ -x_3 & 0 & -x_1 \end{bmatrix} \tag{2.108}$$

Hence

$$\nabla\mathbf{u}^{\mathrm{T}} = \frac{M}{EI} \begin{bmatrix} \nu x_1 & \nu x_2 & -x_3 \\ -\nu x_2 & \nu x_1 & 0 \\ x_3 & 0 & -x_1 \end{bmatrix} \tag{2.109}$$

and

$$\widehat{\nabla}\mathbf{u} = \frac{M}{EI} \begin{bmatrix} \nu x_1 & 0 & 0 \\ 0 & \nu x_1 & 0 \\ 0 & 0 & -x_1 \end{bmatrix} \tag{2.110}$$

Equation (2.110) implies that $\mathbf{u}$ given by (2.105) satisfies the equation

$$\widehat{\nabla}\mathbf{u} = \mathbf{E} \tag{2.111}$$

where $\mathbf{E}$ is given by (2.104). This completes proof of (ii).

Problem 2.7. Given a stress tensor $\mathbf{S}$ at a point A, find: (i) the stress vector $\mathbf{s}$ on a plane through A parallel to the plane $\mathbf{n} \cdot \mathbf{x} - \mathrm{vt} = 0 \quad (|\mathbf{n}| = 1,\ \mathrm{v} > 0,\ \mathrm{t} \geq 0)$, (ii) the magnitude of $\mathbf{s}$, (iii) the angle between $\mathbf{s}$ and the normal to the plane, and (iv) the normal and tangential components of the stress vector $\mathbf{s}$.

Answers. (i) $\mathbf{s} = \mathbf{Sn}$; (ii) $|\mathbf{s}| = |\mathbf{Sn}|$; (iii) $\cos\ \theta = \mathbf{s} \cdot \mathbf{n}/\ |\mathbf{s}|$; (iv) $\mathbf{s} = \mathbf{s}_\mathrm{n} + \mathbf{s}_\tau$, where $\mathbf{s}_\mathrm{n} = (\mathbf{n} \cdot \mathbf{s})\, \mathbf{n}$ and $\mathbf{s}_\tau = \mathbf{n} \times (\mathbf{s} \times \mathbf{n})$.

Solution. Solution to Problem 2.7 is presented by the answers (i)–(iv).

Problem 2.8. Let $\{\mathbf{e}_i\}$ be an orthonormal basis for a stress tensor $\mathbf{S}$, and let $\{\mathbf{e}_i^*\}$ be an orthonormal basis formed by the eigenvectors of $\mathbf{S}$. Then a tensor $\mathbf{S}^*$ obtained from $\mathbf{S}$ by the transformation formula from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}_i^*\}$ takes the form

$$\mathbf{S}^* = \lambda_1\, \mathbf{e}_1^* \otimes \mathbf{e}_1^* + \lambda_2\, \mathbf{e}_2^* \otimes \mathbf{e}_2^* + \lambda_3\, \mathbf{e}_3^* \otimes \mathbf{e}_3^* \tag{2.112}$$

where λ_i is an eigenvalue of $\mathbf{S}$ corresponding to the eigenvector $\mathbf{e}_i^*$. Show that the function

$$\mathrm{g}(\mathbf{n}^*) = \left|\mathbf{s}_\tau^*\right| = \left|\mathbf{n}^* \times (\mathbf{S}^*\mathbf{n}^* \times \mathbf{n}^*)\right| \tag{2.113}$$

representing the tangent stress vector magnitude with regard to a plane with a normal $\mathbf{n}^*$ in the $\{\mathbf{e}_i^*\}$ basis, assumes the extreme values

$$\left|\mathbf{s}_\tau^*\right|_1 = \frac{1}{2}\,|\lambda_2 - \lambda_3| \tag{2.114}$$

$$\left|\mathbf{s}_\tau^*\right|_2 = \frac{1}{2}\,|\lambda_3 - \lambda_1| \tag{2.115}$$

and

$$\left|\mathbf{s}_\tau^*\right|_3 = \frac{1}{2}\,|\lambda_1 - \lambda_2| \tag{2.116}$$

at

$$\mathbf{n}_1^* = [0,\ \ \pm 1/\sqrt{2},\ \ \pm 1/\sqrt{2}]^\mathrm{T} \tag{2.117}$$

$$\mathbf{n}_2^* = [\ \pm 1/\sqrt{2},\ \ 0,\ \ \pm 1/\sqrt{2}]^\mathrm{T} \tag{2.118}$$

and

$$\mathbf{n}_3^* = [\ \pm 1/\sqrt{2},\ \ \pm 1/\sqrt{2},\ 0]^\mathrm{T} \tag{2.119}$$

respectively. Hence, if $\lambda_1 > \lambda_2 > \lambda_3$ then the largest tangential stress vector magnitude is

$$\left|\mathbf{s}_\tau^*\right|_2 = \frac{1}{2}\,|\lambda_3 - \lambda_1| \tag{2.120}$$

and this extreme vector acts on the plane that bisects the angle between $\mathbf{e}_1^*$ and $\mathbf{e}_3^*$.

Solution. It follows from (iv) of Problem 2.7 that

$$\mathbf{s}^* = \mathbf{s}_n^* + \mathbf{s}_\tau^* \tag{2.121}$$

where

$$\mathbf{s}^* = \mathbf{S}^*\mathbf{n}^* \tag{2.122}$$

and

$$\mathbf{s}_n^* = (\mathbf{s}^* \cdot \mathbf{n}^*)\mathbf{n}, \quad \mathbf{s}_\tau^* = \mathbf{n}^* \times (\mathbf{s}^* \times \mathbf{n}) \tag{2.123}$$

Using (2.112), (2.122) and (2.123), we obtain

$$\mathbf{s}^* = \lambda_1 n_1^* \mathbf{e}_1^* + \lambda_2 n_2^* \mathbf{e}_2^* + \lambda_3 n_3^* \mathbf{e}_3^* \tag{2.124}$$

and

$$\mathbf{s}^* \cdot \mathbf{n}^* = \lambda_1 (n_1^*)^2 + \lambda_2 (n_2^*)^2 + \lambda_3 (n_3^*)^2 \tag{2.125}$$

Since $\mathbf{s}_n^* \cdot \mathbf{s}_\tau^* = 0$, by squaring (2.121), we get

$$|\mathbf{s}^*|^2 = \left|\mathbf{s}_n^*\right|^2 + \left|\mathbf{s}_\tau^*\right|^2 \tag{2.126}$$

Now, introduce the function

$$\begin{aligned} f(\mathbf{n}^*) = \left|\mathbf{s}_\tau^*\right|^2 = |\mathbf{s}^*|^2 - \left|\mathbf{s}_n^*\right|^2 &= \lambda_1^2 (n_1^*)^2 + \lambda_2^2 (n_2^*)^2 + \lambda_3^2 (n_3^*)^2 \\ &- \left[\lambda_1 (n_1^*)^2 + \lambda_2 (n_2^*)^2 + \lambda_3^2 (n_3^*)^2\right]^2 \end{aligned} \tag{2.127}$$

If there is an extremum of $f = f(\mathbf{n}^*)$, treated as a function of $n_1^*,\ n_2^*$, and n_3^*, it is also an extremum of $g = g(\mathbf{n}^*) = \sqrt{f(\mathbf{n}^*)}$.

To find the extreme values of $f = f(\mathbf{n}^*)$ subject to the condition $|\mathbf{n}^*| = 1$ we solve the algebraic equation for $\mathbf{n}^*$

$$\frac{\partial}{\partial n_i^*}\left[f(\mathbf{n}^*) - t(|\mathbf{n}^*|^2 - 1)\right] = 0 \tag{2.128}$$

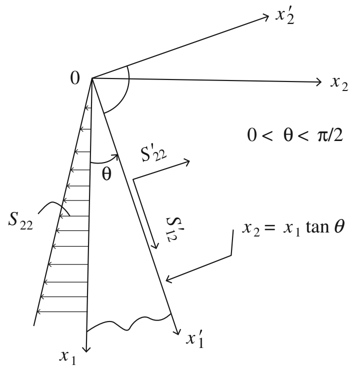
where t is a Lagrangian multiplier. In expanded form Eq. (2.128) takes the form

$$\left[\lambda_1^2 - 2\lambda_1(\mathbf{s}^* \cdot \mathbf{n}^*) - t\right] n_1^* = 0 \tag{2.129}$$

$$\left[\lambda_2^2 - 2\lambda_2(\mathbf{s}^* \cdot \mathbf{n}^*) - t\right] n_2^* = 0 \tag{2.130}$$

$$\left[\lambda_3^2 - 2\lambda_3(\mathbf{s}^* \cdot \mathbf{n}^*) - t\right] n_3^* = 0 \tag{2.131}$$

Fig. 2.1 The *wedge* region



where $(\mathbf{s}^* \cdot \mathbf{n}^*)$ is given by (2.125). It can be verified that the unit vectors $\mathbf{n}_1^*$, $\mathbf{n}_2^*$, and $\mathbf{n}_3^*$, given by Eqs. (2.117), (2.118), and (2.119), respectively, satisfy Eqs. (2.129)–(2.131) with $t = \lambda_2\lambda_3$. In addition, by substituting $\mathbf{n}_1^*$, $\mathbf{n}_2^*$, and $\mathbf{n}_3^*$ into (2.127), we obtain Eqs. (2.114), (2.115), and (2.116), respectively. Also, the vector $\mathbf{n}_2^*$ that is normal to the surface element on which the largest tangential stress vector $\left(\mathbf{s}_\tau^*\right)_2$ acts bisects the angle between $\mathbf{e}_1^*$ and $\mathbf{e}_3^*$. This completes solution of Problem 2.8.

Problem 2.9. Let $D = \{\mathbf{x} : \ x_1 \geq 0, \ \ x_1 \tan\theta \geq x_2 \geq 0\}$ be a two-dimensional wedge region shown in the Fig. 2.1, and let $S_{\alpha\beta} = S_{\alpha\beta}(\mathbf{x})$, $[\mathbf{x} = (x_1, x_2);\ \alpha, \beta = 1, 2]$ be a symmetric tensor field on D defined by

$$S_{11} = d\, x_2 + e\, x_1 - \rho\, g x_1, \quad S_{22} = -\gamma x_1, \quad S_{12} = S_{21} = -e\, x_2 \tag{2.132}$$

where d, e, g, ρ, and γ are constants $[g > 0,\ \ \rho > 0,\ \ \gamma > 0]$. (i) Show that

$$\text{div}\,\mathbf{S} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad D \tag{2.133}$$

where

$$\mathbf{b} = [\rho\, g,\ \ 0]^{\mathrm{T}} \quad \text{on} \quad D \tag{2.134}$$

(ii) Using the transformation formula from the system x_α to the system x_α' [see Eq. (1.157) in Problem 1.8] find the components $S_{\alpha\beta}'$ in terms of $S_{\alpha\beta}$, and show that

$$S_{12}' = 0 \quad \text{and} \quad S_{22}' = 0 \quad \text{for} \quad x_2 = x_1 \tan\theta \tag{2.135}$$

provided

$$e = \frac{\gamma}{\tan^2\theta}, \quad \text{and} \quad d = \frac{\rho\, g}{\tan\theta} - \frac{2\gamma}{\tan^3\theta} \tag{2.136}$$

(iii) Give diagrams of S_{11} and S_{12} over a horizontal section $x_1 = x_1^0 = \text{constant}$.
(iv) Give a diagram of S_{22} over the vertical section $x_2 = 0$.

Solution. To show (i) we note that $S_{\alpha\beta} = S_{\alpha\beta}(x_1, x_2)$ given by Eq. (2.132) satisfies the equilibrium equation

$$S_{\alpha\beta,\beta} + b_\alpha = 0 \quad \text{on } D \tag{2.137}$$

since

$$S_{1\beta,\beta} = -\rho\, g, \quad S_{2\beta,\beta} = 0 \quad \text{on } D \tag{2.138}$$

for arbitrary constants d, e, g, ρ, and γ. To show (ii) we use the transformation formulas [see Eq. (1.157) in Problem 1.8]

$$S'_{11} = S_{11}\cos^2\theta + S_{12}\sin 2\theta + S_{22}\sin^2\theta \tag{2.139}$$

$$S'_{12} = \frac{1}{2}(S_{22} - S_{11})\sin 2\theta + S_{12}\ \cos 2\theta \tag{2.140}$$

$$S'_{22} = S_{11}\ \sin^2\theta - S_{12}\ \sin 2\theta + S_{22}\cos^2\theta \tag{2.141}$$

[see Fig. 2.1].

The components $S_{11},\ S_{12}$, and S_{22} taken on the line $x_2 = x_1\tan\theta$ assume the forms

$$S_{11}(x_1, x_1\tan\theta) = (d\tan\theta + e - \rho g)x_1 \tag{2.142}$$

$$S_{12}(x_1, x_1\tan\theta) = -(e\tan\theta)x_1 \tag{2.143}$$

$$S_{22}(x_1, x_1\tan\theta) = -\gamma\, x_1 \tag{2.144}$$

Therefore, substituting (2.142)–(2.144) into the RHS′ of (2.140) and (2.141), and equating the results to zero, we obtain the algebraic equations for the unknown constants e and d, provided γ and ρg are prescribed

$$e(\sin^2\theta + \tan\theta\ \sin 2\theta) + d\ \tan\theta\ \sin^2\theta = \gamma\ \cos^2\theta + \rho g\ \sin^2\theta \tag{2.145}$$

$$e(\sin 2\theta + 2\tan\theta\ \cos 2\theta) + d\ \tan\theta\ \sin 2\theta = -\gamma\ \sin 2\theta + \rho g\ \sin 2\theta \tag{2.146}$$

Dividing Eq. (2.145) by $\sin^2\theta$ and Eq. (2.146) by $\sin 2\theta$ and introducing the notation

$$\tan\theta = u \tag{2.147}$$

we obtain

$$\begin{aligned} 3e + du &= \frac{\gamma}{u^2} + \rho g \\ (2 - u^2)e + du &= -\gamma + \rho g \end{aligned} \tag{2.148}$$

It is easy to check that a unique solution (e, d) of Eqs. (2.148) takes the form (2.136) that is

$$e = \gamma/u^2, \quad d = \rho g/u - 2\gamma/u^3 \tag{2.149}$$

This completes proof of (ii).

Finally, when $x_1 = x_1^0 = \text{const}$, S_{11} and S_{12} are represented by straight lines on the planes (x_2, S_{11}) and (x_2, S_{12}), respectively, and S_{22} at $x_2 = 0$ is represented by a straight line passing through the origin 0 as shown in Fig. 2.1. This completes solution to Problem 2.9.

Problem 2.10. Let B denote a cylinder of length l and of arbitrary cross section, suspended from the upper end and subject to its own weight ρg. Then the stress tensor $\mathbf{S} = \mathbf{S}(\mathbf{x})$ on B takes the form

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho g x_3 \end{bmatrix} \tag{2.150}$$

since, in this case, the body force vector field is given by $\mathbf{b} = [0,\ 0,\ -\rho\, g]^{\mathrm{T}}$, and $\operatorname{div} \mathbf{S} + \mathbf{b} = \mathbf{0}$ on B. The stress vector $\mathbf{s}$ associated with $\mathbf{S}$ on ∂B has the following properties: $\mathbf{s} = [0, 0, \rho\, g\, l\,]^{\mathrm{T}}$ on the end plane $x_3 = l$; and $\mathbf{s} = \mathbf{0}$ on the plane $x_3 = 0$ and on the lateral surface of the cylinder since $\mathbf{n} = [n_1, n_2, 0]^{\mathrm{T}}$ on the surface. Assuming that the cylinder is made of a homogeneous isotropic elastic material, the associated strain tensor field $\mathbf{E}$ takes the form [see Eqs. (2.49)]

$$\mathbf{E} = \frac{\rho\, g\, x_3}{E} \begin{bmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.151}$$

where E and ν are Young's modulus and Poisson's ratio, respectively.

(i) Show that a solution $\mathbf{u}$ of the equation

$$\mathbf{E} = \widehat{\nabla} \mathbf{u} \quad \text{on} \quad \mathrm{B} \tag{2.152}$$

subject to the condition

$$\mathbf{u}(0, 0, l) = \mathbf{0} \tag{2.153}$$

takes the form

$$\mathbf{u} = \frac{\rho\, g}{E} \left[-\nu x_1 x_3,\quad -\nu x_2 x_3,\quad \frac{\nu}{2}(x_1^2 + x_2^2) + \frac{1}{2}(x_3^2 - l^2) \right]^{\mathrm{T}} \tag{2.154}$$

(ii) Plot $u_3 = u_3(0, 0, x_3)$ over the range $0 \le x_3 \le l$.

Solution. To solve the problem we use (2.154) and obtain

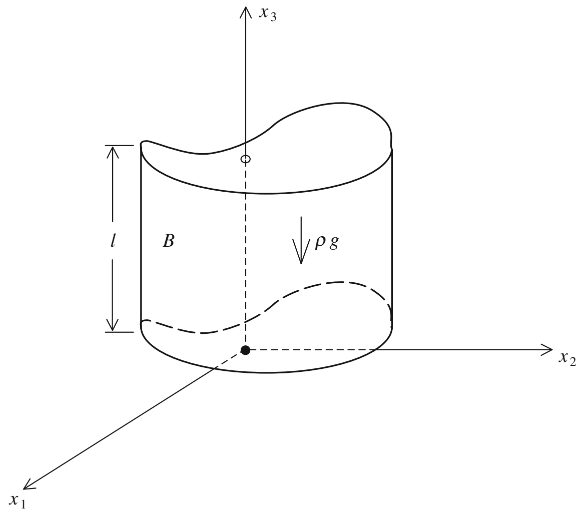


Fig. 2.2 The *cylinder* of arbitrary cross section

$$\nabla \mathbf{u} = \frac{\rho g}{E} \begin{bmatrix} -\nu x_3 & 0 & -\nu x_1 \\ 0 & -\nu x_3 & -\nu x_2 \\ \nu x_1 & \nu x_2 & x_3 \end{bmatrix} \tag{2.155}$$

and

$$\nabla \mathbf{u}^{\mathrm{T}} = \frac{\rho g}{E} \begin{bmatrix} -\nu x_3 & 0 & \nu x_1 \\ 0 & -\nu x_3 & \nu x_2 \\ -\nu x_1 & -\nu x_2 & x_3 \end{bmatrix} \tag{2.156}$$

Hence

$$\widehat{\nabla} \mathbf{u} = \frac{\rho g}{E} \begin{bmatrix} -\nu x_3 & 0 & 0 \\ 0 & -\nu x_3 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \tag{2.157}$$

Therefore, $\mathbf{u}$ given by (2.154) satisfies (2.152). Also, it is easy to prove that $\mathbf{u}$ satisfies (2.153). Finally, $u_3 = u_3(0, 0, x_3)$ is represented by a parabolic curve restricted to the interval $0 \leq x_3 \leq \ell$. This completes solution to Problem 2.10.

Problem 2.11. For a transversely isotropic elastic body each material point posses an axis of rotational symmetry, which means that the elastic properties are the same in any direction on any plane perpendicular to the axis, but they are different than those in the direction of the axis. If the x_3 axis coincides with the axis of symmetry, then the stress-strain relation for such a body takes the form

$$\begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{32} \\ S_{31} \\ S_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{32} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \tag{2.158}$$

where **S** and **E** are the stress and strain tensors, respectively, and five numerically independent moduli $c_{11}, c_{33}, c_{12}, c_{13},$ and c_{44} are related to the components C_{ijkl} of the fourth-order elasticity tensor **C** by [see Eq. (2.35)]

$$c_{11} = C_{1111}, \quad c_{12} = C_{1122}, \quad c_{13} = C_{1133}, \quad c_{33} = C_{3333}, \quad c_{44} = C_{1313} \tag{2.159}$$

Show that if the axis of symmetry of a transversely isotropic body coincides with the direction of an arbitrary unit vector **e**, then the stress-strain relation takes the form

$$\begin{aligned} \mathbf{S} = \mathbf{C}[\mathbf{E}] = {} & (c_{11}-c_{12})\mathbf{E} + \{c_{12}(\operatorname{tr}\mathbf{E}) - (c_{12}-c_{13})\mathbf{e}\cdot(\mathbf{E}\mathbf{e})\}\,\mathbf{1} \\ & - (c_{11}-c_{12}-2c_{44})\{\mathbf{e}\otimes(\mathbf{E}\mathbf{e}) + (\mathbf{E}\mathbf{e})\otimes\mathbf{e}\} \\ & - \{(c_{12}-c_{13})(\operatorname{tr}\mathbf{E}) - (c_{11}+c_{33}-2c_{13}-4c_{44})\mathbf{e}\cdot(\mathbf{E}\mathbf{e})\}\mathbf{e}\otimes\mathbf{e} \end{aligned} \tag{2.160}$$

Solution. For a transversely isotropic body in which the axis of symmetry coincides with an arbitrary unit vector **e**, the stress–strain relation takes the form[1]

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] \tag{2.161}$$

where **S** and **E** are the stress and strain tensors, respectively, and the elasticity tensor **C** is given by

$$\begin{aligned} \mathbf{C} = {} & c_{33}\mathbf{C}^{(1)} + (c_{11}+c_{12})\mathbf{C}^{(2)} + \sqrt{2}\,c_{13}(\mathbf{C}^{(3)} + \mathbf{C}^{(4)}) \\ & + (c_{11}-c_{12})\mathbf{C}^{(5)} + 2\,c_{44}\mathbf{C}^{(6)} \end{aligned} \tag{2.162}$$

In Eq. (2.162) the tensors $\mathbf{C}^{(a)}$, $a = 1, 2, 3, 4, 5, 6$, are defined by

$$\begin{aligned} & C^{(1)}_{ijkl} = A_{ij}\,A_{kl}, \quad C^{(2)}_{ijkl} = \frac{1}{2}\,B_{ij}\,B_{kl} \\ & C^{(3)}_{ijkl} = \frac{1}{\sqrt{2}}\,A_{ij}\,B_{kl}, \quad C^{(4)}_{ijkl} = \frac{1}{\sqrt{2}}\,B_{ij}\,A_{kl} \\ & C^{(5)}_{ijkl} = \frac{1}{2}(B_{ik}\,B_{jl} + B_{il}\,B_{jk} - B_{ij}\,B_{kl}) \\ & C^{(6)}_{ijkl} = \frac{1}{2}(A_{ik}\,B_{jl} + A_{il}\,B_{jk} + A_{jk}\,B_{il} + A_{jl}\,B_{ik}) \end{aligned} \tag{2.163}$$

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[1] See P. Chadwick, Proc. R. Soc. London, A **422**, p. 26 (1989).

where

$$A_{ij} = e_i\, e_j, \quad B_{ij} = \delta_{ij} - e_i\, e_j \tag{2.164}$$

and e_i are the components of $\mathbf{e}$ in the coordinates $\{x_i\}$.

Using (2.163) and (2.164), we obtain

$$C^{(1)}_{ijkl}\, E_{kl} = e_i\, e_j\, e_k\, e_l\, E_{kl} \tag{2.165}$$

or in direct notation

$$\mathbf{C}^{(1)}[\mathbf{E}] = [\mathbf{e} \cdot (\mathbf{E}\mathbf{e})]\mathbf{e} \otimes \mathbf{e} \tag{2.166}$$

Similarly, by (2.163) and (2.164), we get

$$C^{(2)}_{ijkl}\, E_{kl} = \frac{1}{2}(\delta_{ij} - e_i\, e_j)(\delta_{kl} - e_k\, e_l)E_{kl} \tag{2.167}$$

or

$$\mathbf{C}^{(2)}[\mathbf{E}] = \frac{1}{2}(\mathbf{1} - \mathbf{e} \otimes \mathbf{e})[\text{tr}\,\mathbf{E} - \mathbf{e} \cdot (\mathbf{E}\mathbf{e})] \tag{2.168}$$

Also, using (2.163) and (2.164), we obtain

$$\sqrt{2}\left(C^{(3)}_{ijkl} + C^{(4)}_{ijkl}\right)E_{kl} = e_i\, e_j(\delta_{kl} - e_k\, e_l)E_{kl} + (\delta_{ij} - e_i\, e_j)e_k\, e_l\, E_{kl} \tag{2.169}$$

or

$$\sqrt{2}(\mathbf{C}^{(3)}[\mathbf{E}] + \mathbf{C}^{(4)}[\mathbf{E}]) = [\mathbf{e} \cdot (\mathbf{E}\mathbf{e})]\mathbf{1} + [\text{tr}\,\mathbf{E} - 2\mathbf{e} \cdot (\mathbf{E}\mathbf{e})]\mathbf{e} \otimes \mathbf{e} \tag{2.170}$$

and

$$\begin{aligned} C^{(5)}_{ijkl}\, E_{kl} = {} & E_{ij} - e_i\, e_k\, E_{jk} - e_j\, e_k\, E_{ik} + e_i\, e_j\, e_k\, e_l\, E_{kl} \\ & - \frac{1}{2}(\delta_{ij} - e_i\, e_j)(E_{kk} - e_k\, e_l\, E_{kl}) \end{aligned} \tag{2.171}$$

or

$$\begin{aligned} \mathbf{C}^{(5)}[\mathbf{E}] = {} & \mathbf{E} - \mathbf{e} \otimes (\mathbf{E}\mathbf{e}) - (\mathbf{E}\mathbf{e}) \otimes \mathbf{e} + \mathbf{e} \otimes \mathbf{e}(\mathbf{e} \cdot \mathbf{E}\mathbf{e}) \\ & - \frac{1}{2}(\mathbf{1} - \mathbf{e} \otimes \mathbf{e})(\text{tr}\,\mathbf{E} - \mathbf{e} \cdot \mathbf{E}\mathbf{e}) \end{aligned} \tag{2.172}$$

and

$$C^{(6)}_{ijkl}\, E_{kl} = e_i\, E_{jk}\, e_k + E_{ik}\, e_k\, e_j - 2e_i\, e_j\, e_k\, e_l\, E_{kl} \tag{2.173}$$

or

$$\mathbf{C}^{(6)}[\mathbf{E}] = \mathbf{e} \otimes \mathbf{E}\mathbf{e} + (\mathbf{E}\mathbf{e}) \otimes \mathbf{e} - 2\mathbf{e} \otimes \mathbf{e}[\mathbf{e} \cdot (\mathbf{E}\mathbf{e})] \tag{2.174}$$

Therefore, substituting C from (2.162) into (2.161) and using (2.166), (2.168), (2.170), (2.172), and (2.174), we obtain (2.160). Note that the representation (2.160) coincides with (2.158) if $\mathbf{e} = (0, 0, 1)$. This can be proved by substituting $\mathbf{e} = (0, 0, 1)$ into (2.160).

Problem 2.12. Show that the stress-strain relation (2.160) in Problem 2.11 is invertible provided

$$c \equiv (c_{11} + c_{12})c_{33} - 2c_{13}^2 > 0, \quad c_{11} > |c_{12}|, \quad c_{44} > 0 \tag{2.175}$$

and that the strain-stress relation reads

$$\begin{aligned}
\mathbf{E} = \mathsf{K}[\mathbf{S}] = {} & (c_{11} - c_{12})^{-1}\mathbf{S} + \frac{1}{2}\Big[\big\{c^{-1}c_{33} - (c_{11} - c_{12})^{-1}\big\}(\operatorname{tr}\mathbf{S}) \\
& - \big\{c^{-1}(c_{33} + 2c_{13}) - (c_{11} - c_{12})^{-1}\big\}\,\mathbf{e}\cdot(\mathbf{Se})\Big]\,\mathbf{1} \\
& - \left\{(c_{11} - c_{12})^{-1} - \frac{1}{2}c_{44}^{-1}\right\}\{\mathbf{e}\otimes(\mathbf{Se}) + (\mathbf{Se})\otimes\mathbf{e}\} \\
& - \frac{1}{2}\Big[\{c^{-1}(c_{33} + 2c_{13}) - (c_{11} - c_{12})^{-1}\}(\operatorname{tr}\mathbf{S}) \\
& - \{c^{-1}(2\,c_{11} + c_{33} + 2c_{12} + 4c_{13}) + (c_{11} - c_{12})^{-1} - 2c_{44}^{-1}\}\,\mathbf{e}\cdot(\mathbf{Se})\Big]\,\mathbf{e}\otimes\mathbf{e}
\end{aligned} \tag{2.176}$$

Solution. To show that (2.160) in Problem 2.11 is invertible if the inequalities (2.175) are satisfied, and the inverted formula takes the form (2.176), consider the fourth-order tensor

$$\mathsf{A} = a_1\mathsf{C}^{(1)} + a_2\mathsf{C}^{(2)} + a_3(\mathsf{C}^{(3)} + \mathsf{C}^{(4)}) + a_5\mathsf{C}^{(5)} + a_6\mathsf{C}^{(6)} \tag{2.177}$$

where $a_1,\ a_2,\ a_3,\ a_5$, and a_6 are scalars that satisfy the inequalities

$$a_1 > 0, \quad a_2 > 0, \quad a_1a_2 - a_3^2 > 0, \quad a_5 > 0, \quad a_6 > 0 \tag{2.178}$$

and $\mathsf{C}^{(a)}$ $(a = 1, 2, 3, 4, 5, 6)$ are the fourth-order tensors defined by (2.163) and (2.164) in Problem 2.11.

Then, there is A^{-1} in the form

$$\mathsf{A}^{-1} = \left(a_1a_2 - a_3^2\right)^{-1}\{a_2\mathsf{C}^{(1)} + a_1\mathsf{C}^{(2)} - a_3(\mathsf{C}^{(3)} + \mathsf{C}^{(4)})\} + a_5^{-1}\mathsf{C}^{(5)} + a_6^{-1}\mathsf{C}^{(6)} \tag{2.179}$$

such that

$$\mathsf{A}\mathsf{A}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{1} \tag{2.180}$$

where $\mathsf{1}$ is the fourth-order identity tensor with components

$$I_{ijkl} = \frac{1}{2}(\delta_{ik}\,\delta_{jl} + \delta_{jk}\,\delta_{il}) \tag{2.181}$$

To prove (2.180) we use (2.163) and (2.164) of Problem 2.11 to obtain the 6×6 tensor matrix

$$[\mathbf{C}^{(a)}\ \mathbf{C}^{(b)}] = \begin{bmatrix} \mathbf{C}^{(1)} & \mathbf{0} & \mathbf{C}^{(3)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{(2)} & \mathbf{0} & \mathbf{C}^{(4)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{(3)} & \mathbf{0} & \mathbf{C}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}^{(4)} & \mathbf{0} & \mathbf{C}^{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}^{(5)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}^{(6)} \end{bmatrix} \tag{2.182}$$

as well as the identity

$$\mathbf{C}^{(1)} + \mathbf{C}^{(2)} + \mathbf{C}^{(5)} + \mathbf{C}^{(6)} = \mathbf{1} \tag{2.183}$$

Calculating the tensor $\mathbf{A}\,\mathbf{A}^{-1}$, by using Eqs. (2.177), (2.179), (2.182), and (2.183) we obtain

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{1} \tag{2.184}$$

Similarly, using Eqs. (2.177), (2.179), (2.182), and (2.183) we get

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{1} \tag{2.185}$$

Now, by letting

$$\begin{aligned} &a_1 = c_{33}, \quad a_2 = c_{11} + c_{12}, \quad a_3 = \sqrt{2}\,c_{13} \\ &a_5 = c_{11} - c_{12}, \quad a_6 = 2c_{44} \end{aligned} \tag{2.186}$$

in Eq. (2.177) we obtain $\mathbf{A} = \mathbf{C}$, and the inequalities (2.178) reduce to those of (2.175). Also, Eq. (2.179) reduces to

$$\begin{aligned} \mathbf{C}^{-1} = \mathbf{K} = c^{-1}\{(c_{11} + c_{12})\mathbf{C}^{(1)} + c_{33}\mathbf{C}^{(2)} - \sqrt{2}\,c_{13}(\mathbf{C}^{(3)} + \mathbf{C}^{(4)})\} \\ + (c_{11} - c_{12})^{-1}\mathbf{C}^{(5)} + 2^{-1}c_{44}^{-1}\mathbf{C}^{(6)} \end{aligned} \tag{2.187}$$

where

$$c = (c_{11} + c_{12})c_{33} - 2c_{13}^2 > 0 \tag{2.188}$$

Therefore, the strain–stress relation reads

$$\begin{aligned} \mathbf{E} = \mathbf{K}[\mathbf{S}] = c^{-1}\{(c_{11} + c_{12})\mathbf{C}^{(1)}[\mathbf{S}] + c_{33}\mathbf{C}^{(2)}[\mathbf{S}] - \sqrt{2}\,c_{13}(\mathbf{C}^{(3)}[\mathbf{S}] + \mathbf{C}^{(4)}[\mathbf{S}])\} \\ + (c_{11} - c_{12})^{-1}\mathbf{C}^{(5)}[\mathbf{S}] + 2^{-1}c_{44}^{-1}\mathbf{C}^{(6)}[\mathbf{S}] \end{aligned} \tag{2.189}$$

Finally, if Eqs. (2.166), (2.168), (2.170), (2.172), and (2.174) of Problem 2.11 in which $\mathbf{E}$ is replaced by $\mathbf{S}$ are taken into account, Eq. (2.189) reduces to (2.176).

Problem 2.13. Prove that the inequalities (2.175) in Problem 2.12 are necessary and sufficient conditions for the elasticity tensor C (compliance tensor K) to be positive definite. This means that the strain energy density (stress energy density) of a transversely isotropic body is positive definite if and only if the inequalities (2.175) in Problem 2.12 hold true.

Solution. Define the fourth-order tensor H by

$$\mathsf{H} = \sqrt{a_1} a_1' \left(a_1'^{\,2} + a_3^2 \right)^{-1/2} \mathsf{C}^{(1)} + \sqrt{a_2} a_2' \left(a_2'^{\,2} + a_3^2 \right)^{-1/2} \mathsf{C}^{(2)} + a_3 (a_1' + a_2')^{-1/2} (\mathsf{C}^{(3)} + \mathsf{C}^{(4)}) + \sqrt{a_5}\, \mathsf{C}^{(5)} + \sqrt{a_6}\, \mathsf{C}^{(6)} \tag{2.190}$$

where

$$a_1' = a_1 + (a_1\, a_2 - a_3)^{1/2}, \quad a_2' = a_2 + (a_1\, a_2 - a_3)^{1/2} \tag{2.191}$$

and $a_1,\ a_2,\ a_3,\ a_5$, and a_6 satisfy the inequalities (2.178) of Problem 2.12.

Then using the matrix equation (2.182) of Problem 2.12 we obtain

$$\mathsf{H}\,\mathsf{H} = \mathsf{A} \tag{2.192}$$

where A is the fourth-order tensor given by (2.177) of Problem 2.12. Hence, we get

$$\mathsf{A}[\mathbf{E}] = \mathsf{H}(\mathsf{H}\,[\mathbf{E}]) \quad \forall\, \mathbf{E} = \mathbf{E}^{\mathrm{T}} \neq \mathbf{0} \tag{2.193}$$

and

$$\mathbf{E} \cdot \mathsf{A}[\mathbf{E}] = \mathbf{E} \cdot \mathsf{H}(\mathsf{H}\,[\mathbf{E}]) \tag{2.194}$$

or

$$\mathbf{E} \cdot \mathsf{A}[\mathbf{E}] = (\mathsf{H}\,[\mathbf{E}]) \cdot (\mathsf{H}\,[\mathbf{E}]) \tag{2.195}$$

Since

$$(\mathsf{H}\,[\mathbf{E}]) \cdot (\mathsf{H}\,[\mathbf{E}]) > 0 \quad \forall\, \mathbf{E} = \mathbf{E}^{\mathrm{T}} \neq \mathbf{0} \tag{2.196}$$

therefore, expressing a_1, a_2, a_3, a_5, and a_6 in terms of $c_{11}, c_{12}, c_{13}, c_{33}$, and c_{44} [see Eqs. (2.186) of Problem 2.12], we conclude that if (2.175) of Problem 2.12 is satisfied then

$$\mathbf{E} \cdot \mathsf{C}\,[\mathbf{E}] > 0 \quad \forall\, \mathbf{E} = \mathbf{E}^{\mathrm{T}} \neq \mathbf{0} \tag{2.197}$$

that is, the elasticity tensor C is positive definite.

To prove that (2.197) implies (2.175) of Problem 2.12, that is, that (2.175) of Problem 2.12 are also necessary conditions, we take advantage of the fact that $\mathbf{E}$

in (2.197) is an arbitrary second-order symmetric tensor, and select the following choices

$$\mathbf{E}_1 = \alpha \mathbf{e} \otimes \mathbf{e} + \frac{\beta}{\sqrt{2}}(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) \tag{2.198}$$

$$\mathbf{E}_2 = \mathbf{e} \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{e} \tag{2.199}$$

$$\mathbf{E}_3 = \mathbf{c} \otimes (\mathbf{e} \times \mathbf{c}) + (\mathbf{e} \times \mathbf{c}) \otimes \mathbf{c} \tag{2.200}$$

where α and β are the real numbers, and $\mathbf{c}$ is an arbitrary unit vector orthogonal to $\mathbf{e}$.

Then, substituting (2.198), (2.199), and (2.200) into (2.197), respectively, we obtain

$$\mathbf{E}_1 \cdot \mathsf{C}\,[\mathbf{E}_1] = c_{33}\alpha^2 + 2\sqrt{2}\, c_{13}\, \alpha\beta + (c_{11} + c_{12})\beta^2 \tag{2.201}$$

$$\mathbf{E}_2 \cdot \mathsf{C}\,[\mathbf{E}_2] = 4c_{44} \tag{2.202}$$

$$\mathbf{E}_3 \cdot \mathsf{C}\,[\mathbf{E}_3] = 2\,(c_{11} - c_{12}) \tag{2.203}$$

Since the RHS of (2.201) is positive for non-vanishing numbers α and β, we obtain

$$\Delta = 8c_{13}^2\,\alpha^2 - 4(c_{11} + c_{12})c_{33}\alpha^2 < 0 \tag{2.204}$$

or

$$c = (c_{11} + c_{12})c_{33} - 2c_{13}^2 > 0 \tag{2.205}$$

and

$$c_{11} + c_{12} > 0, \quad c_{33} > 0 \tag{2.206}$$

Also, Eqs. (2.202) and (2.203) together with the positiveness of C imply that

$$c_{44} > 0, \quad c_{11} - c_{12} > 0 \tag{2.207}$$

Since the inequalities (2.205)–(2.207) are equivalent to the inequalities (2.175) of Problem 2.12, the solution to Problem 2.13 is complete.

Problem 2.14. Consider a plane $\mathbf{n} \cdot \mathbf{x} - \mathrm{v}\mathrm{t} = 0 \quad (|\mathbf{n}| = 1,\ \mathrm{v} > 0,\ \mathrm{t} \geq 0)$. Let $\mathbf{S}$ be the stress tensor obtained form Eq. (2.160) of Problem 2.11 in which $0 < \mathbf{e} \cdot \mathbf{n} < 1$, and the strain tensor $\mathbf{E}$ is defined by

$$\mathbf{E} = \mathrm{sym}\,(\mathbf{n} \otimes \mathbf{a}) \tag{2.208}$$

where $\mathbf{a}$ is an arbitrary vector orthogonal to $\mathbf{n}$. Let $\mathbf{S}^{\perp}$ and $\mathbf{S}^{||}$ represent the normal and tangential parts of $\mathbf{S}$ with respect to the plane [see Problem 1.4 in which $\mathbf{T}$ is replaced by $\mathbf{S}$]. Show that

$$\mathbf{S} = (c_{11} - c_{12})\,\mathrm{sym}\,(\mathbf{n} \otimes \mathbf{a}) - (c_{11} - c_{12} - 2c_{44})\,\cos\theta\,\mathrm{sym}\,(\mathbf{e} \otimes \mathbf{a}) \quad (2.209)$$

$$\mathbf{S}^{\perp} = [(c_{11} - c_{12})\,\sin^2\theta + 2c_{44}\cos^2\theta\,]\,\mathrm{sym}\,(\mathbf{n} \otimes \mathbf{a}) \quad (2.210)$$

$$\mathbf{S}^{||} = (c_{11} - c_{12} - 2c_{44})\,\cos\theta\,\mathrm{sym}\,[(\mathbf{n}\cos\theta - \mathbf{e}) \otimes \mathbf{a}] \quad (2.211)$$

where

$$\cos\theta = \mathbf{e} \cdot \mathbf{n}, \qquad 0 < \theta < \pi/2 \quad (2.212)$$

Solution. If we let $\mathbf{E} = \mathrm{sym}(\mathbf{n} \otimes \mathbf{a})$ into Eq. (2.160) of Problem 2.11, we obtain (2.209). To obtain (2.210) and (2.211) we use the formulas:

$$\mathbf{S}^{\perp} = 2\,\mathrm{sym}(\mathbf{n} \otimes \mathbf{S}\mathbf{n}) - (\mathbf{n} \cdot \mathbf{S}\mathbf{n})\mathbf{n} \otimes \mathbf{n} \quad (2.213)$$

and

$$\mathbf{S}^{\|} = (\mathbf{1} - \mathbf{n} \otimes \mathbf{n})[\mathbf{S}(\mathbf{1} - \mathbf{n} \otimes \mathbf{n})] \quad (2.214)$$

By substituting $\mathbf{S}$ from Eq. (2.209) into Eqs. (2.213) and (2.214), we obtain (2.210) and (2.211), respectively. This completes solution to Problem 2.14.

Problem 2.15. Show that for a transversely isotropic elastic body the stress energy density

$$\widehat{\mathrm{W}}(\mathbf{S}) = \frac{1}{2}\mathbf{S} \cdot \mathbf{K}[\mathbf{S}] \quad (2.215)$$

corresponding to the stress tensor given by Eq. (2.209) of Problem 2.14 takes the form

$$\widehat{\mathrm{W}}(\mathbf{S}) = \frac{1}{2}\mathbf{a}^2\left[\frac{1}{2}(c_{11} - c_{12})\,\sin^2\theta + c_{44}\cos^2\theta\right] \quad (2.216)$$

Also, show that

$$\widehat{\mathrm{W}}(\mathbf{S}) = \widehat{\mathrm{W}}(\mathbf{S}^{\perp}) - \widehat{\mathrm{W}}(\mathbf{S}^{||}) \quad (2.217)$$

where $\widehat{\mathrm{W}}(\mathbf{S}^{\perp})$ and $\widehat{\mathrm{W}}(\mathbf{S}^{||})$ represent the "normal" and "tangential" stress energies, respectively, given by

$$\widehat{\mathrm{W}}(\mathbf{S}^{\perp}) = \frac{1}{2}\mathbf{S}^{\perp} \cdot \mathbf{K}[\mathbf{S}^{\perp}] = \widehat{\mathrm{W}}(\mathbf{S})\left[1 + \frac{1}{8}c_{44}^{-1}(c_{11} - c_{12})^{-1}(c_{11} - c_{12} - 2c_{44})^2\sin^2 2\theta\right] \quad (2.218)$$

and

$$\widehat{\mathrm{W}}(\mathbf{S}^{||}) = \frac{1}{2}\mathbf{S}^{||} \cdot \mathbf{K}[\mathbf{S}^{||}] = \widehat{\mathrm{W}}(\mathbf{S})\left[\frac{1}{8}c_{44}^{-1}(c_{11} - c_{12})^{-1}(c_{11} - c_{12} - 2c_{44})^2\sin^2 2\theta\right] \quad (2.219)$$

Here, $\mathbf{S}^{\perp}$ and $\mathbf{S}^{||}$ are given by Eqs. (2.210) and (2.211), respectively, of Problem 2.14.

Solution. If we substitute **S** from Eq. (2.209) of Problem 2.14 into Eq. (2.215) we arrive at (2.216). Next, using Eqs. (2.208) and (2.210) of Problem 2.14 we obtain (2.218); and using Eqs. (2.208) and (2.211) of Problem 2.14, we get Eq. (2.219). Finally, subtracting (2.219) from (2.218) we obtain Eq. (2.217). This completes solution to Problem 2.15.

Problem 2.16. Let $\varphi(\theta) = \widehat{\mathrm{W}}(\mathbf{S}^{||})/\widehat{\mathrm{W}}(\mathbf{S}^{\perp})$, where $\widehat{\mathrm{W}}(\mathbf{S}^{\perp})$ and $\widehat{\mathrm{W}}(\mathbf{S}^{||})$ denote the normal and tangential stress energy densities, respectively, of Problem 2.15. Show that

$$\max_{\theta \in [0, \pi/2]} [\varphi(\theta)] = \varphi(\pi/4) = \frac{\mathrm{A}^2}{1 + \mathrm{A}^2} \tag{2.220}$$

where

$$\mathrm{A} = \frac{1}{2\sqrt{2}} \frac{|c_{11} - c_{12} - 2c_{44}|}{(c_{11} - c_{12})^{1/2} c_{44}^{1/2}} \tag{2.221}$$

Note. When the body is isotropic we have

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{12} = c_{13} = \lambda, \quad c_{44} = \mu \tag{2.222}$$

where λ and μ are the Lamé material constants. In this case Eq. (2.221) reduces to $\mathrm{A} = 0$, which means that for an isotropic body the tangential stress energy corresponding to the stress (2.211) of Problem 2.14 vanishes.

Solution. Note that, by using Eqs. (2.218) and (2.219) of Problem 2.15, we obtain

$$\varphi(\theta) = \widehat{W}(\mathbf{S}^{\|})/\widehat{W}(\mathbf{S}^{\perp}) = \frac{A^2 \sin^2 2\theta}{(1 + A^2 \sin^2 2\theta)} \tag{2.223}$$

where A is given by Eq. (2.221). Equation (2.223) implies (2.220), and this completes solution of Problem 2.16.

Problem 2.17. Let **u**, **E**, and **S** denote the displacement vector, strain tensor, and stress tensor fields, respectively, corresponding to a body force **b** and a temperature change T. Suppose that the fields **u**, **E**, and **S** satisfy the equations

$$\mathbf{E} = \widehat{\nabla}\mathbf{u} \quad \text{on} \quad \mathrm{B} \tag{2.224}$$

$$\operatorname{div} \mathbf{S} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{2.225}$$

$$\mathbf{S} = \mathbf{C}[\mathbf{E}] + T\mathbf{M} \quad \text{on} \quad \mathrm{B} \tag{2.226}$$

where B is a bounded domain in E^3; while **C** and **M** denote the elasticity and stress-temperature tensors, respectively, independent of $\mathbf{x} \in \overline{\mathrm{B}}$. Also, suppose that an alternative equation to Eq. (2.226) reads

$$\mathbf{E} = \mathbf{K}[\mathbf{S}] + T\mathbf{A} \quad \text{on} \quad \mathrm{B} \tag{2.227}$$

where K and $\mathbf{A}$ represent the compliance and thermal expansion tensors, respectively. Let $\widehat{\mathrm{f}} = \widehat{\mathrm{f}}\,(\mathrm{B})$ denote the mean value of a function $\mathrm{f} = \mathrm{f}(\mathbf{x})$ on B

$$\widehat{\mathrm{f}}\,(\mathrm{B}) = \frac{1}{v(\mathrm{B})} \int\limits_{\mathrm{B}} \mathrm{f}(\mathbf{x})\, dv \tag{2.228}$$

where $v(\mathrm{B})$ stands for the volume of B. Show that

$$\widehat{\mathbf{E}}\,(\mathrm{B}) = \frac{1}{v(\mathrm{B})} \int\limits_{\partial\mathrm{B}} \mathrm{sym}\,(\mathbf{u} \otimes \mathbf{n}) da \tag{2.229}$$

and

$$\widehat{\mathbf{S}}\,(\mathrm{B}) = \frac{1}{v(\mathrm{B})} \left[\int\limits_{\partial\mathrm{B}} \mathrm{sym}\,(\mathbf{x} \otimes \mathbf{Sn})\; da + \int\limits_{\mathrm{B}} \mathbf{sym}\,(\mathbf{u} \otimes \mathbf{b})\;\; dv \right] \tag{2.230}$$

where $\mathbf{n}$ is the outward unit normal on ∂B. Also, show that

$$\widehat{\mathbf{E}}(\mathrm{B}) = \mathsf{K}[\widehat{\mathbf{S}}(\mathrm{B})] + \widehat{T}(\mathrm{B})\mathbf{A} \tag{2.231}$$

and

$$\widehat{\mathbf{S}}(\mathrm{B}) = \mathsf{C}[\widehat{\mathbf{E}}(\mathrm{B})] + \widehat{T}(\mathrm{B})\mathbf{M} \tag{2.232}$$

Solution. Equation (2.229) is identical with Eq. (2.83) of Problem 2.4. Therefore, a proof of (2.229) is the same as that of (2.83) of Problem 2.4. To show (2.230), we note that Eq. (2.225) implies the tensorial equation

$$\mathbf{x} \otimes \mathrm{div}\; \mathbf{S} + \mathbf{x} \otimes \mathbf{b} = \mathbf{0} \tag{2.233}$$

or in components

$$x_i\; S_{jk,k} + x_i\, b_j = 0 \tag{2.234}$$

An equivalent form of Eq. (2.234) reads

$$(x_i\; S_{jk}),_k - S_{ji} + x_i\, b_j = 0 \tag{2.235}$$

Integrating Eq. (2.235) over B and using the divergence theorem we obtain

$$\int\limits_{\partial B} x_i\; S_{jk}\, n_k\, da - \int\limits_{B} S_{ji}\, dv + \int\limits_{B} x_i\; b_j\, dv = 0 \tag{2.236}$$

Finally, taking into account the symmetry of S_{ij}, and applying the operator sym to Eq. (2.236); and dividing (2.236) by $v(\mathrm{B})$, we obtain (2.230). Also applying the mean value operator to Eqs. (2.227) and (2.226), we obtain (2.231) and (2.232), respectively; since the fourth-order tensors C and K, and the second-order tensors $\mathbf{M}$ and $\mathbf{A}$ are independent of $\mathbf{x}$. This completes solution of Problem 2.17.

Problem 2.18. The volume change $\delta v(\mathrm{B})$ associated with the fields $\mathbf{u}$, $\mathbf{E}$, and $\mathbf{S}$ in Problem 2.17 is defined by [see Eqs. (2.7)–(2.8)]

$$\delta v(\mathrm{B}) = v(\mathrm{B})\,\mathrm{tr}\,\widehat{\mathbf{E}}(\mathrm{B}) \tag{2.237}$$

Show that

$$\text{(i)} \qquad \delta v(\mathrm{B}) = 0, \quad \widehat{\mathbf{S}}(\mathrm{B}) = \widehat{T}(\mathrm{B})\,\mathbf{M} \quad \text{if} \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\mathrm{B} \tag{2.238}$$

and

$$\text{(ii)} \qquad \widehat{\mathbf{S}}(\mathrm{B}) = \mathbf{0}, \quad \widehat{\mathbf{E}}(\mathrm{B}) = \widehat{T}(\mathrm{B})\mathbf{A}, \quad \delta v(\mathrm{B}) = v(\mathrm{B})\widehat{T}(\mathrm{B})\,\mathrm{tr}\,\mathbf{A}$$
$$\text{if} \quad \mathbf{Sn} = \mathbf{0} \quad \text{on} \quad \partial\mathrm{B} \quad \text{and} \quad \mathbf{b} = \mathbf{0} \quad \text{on} \quad \mathrm{B} \tag{2.239}$$

Note. Equations (2.239) imply that the volume change $\delta v(\mathrm{B})$ of a homogeneous isotropic thermoelastic body with zero stress vector on $\partial\mathrm{B}$ and zero body force vector on B subject to a temperature change T on B is given by

$$\delta v(\mathrm{B}) = 3\,\alpha\,\widehat{T}(\mathrm{B})\,v(\mathrm{B}) \tag{2.240}$$

where α is the coefficient of linear thermal expansion of the body.

Solution. If $\mathbf{u} = \mathbf{0}$ on ∂B then it follows from Eq. (2.229) of Problem 2.17 that $\widehat{\mathbf{E}}(B) = \mathbf{0}$. This together with Eq. (2.232) of Problem 2.17 and Eq. (2.237) implies (i).

To show (ii) we note that if $\mathbf{Sn} = \mathbf{0}$ on ∂B and $\mathbf{b} = \mathbf{0}$ on $\overline{B}$ then, by virtue of (2.230) of Problem 2.17 we obtain

$$\widehat{\mathbf{S}}(B) = \mathbf{0} \tag{2.241}$$

Hence, using (2.231) of Problem 2.17 we get

$$\widehat{\mathbf{E}}(B) = \widehat{T}(B)\mathbf{A} \tag{2.242}$$

Finally, taking the trace of (2.234) and using (2.237) we obtain

$$\delta v(B) = v(B)\ \widehat{T}(B)\mathrm{tr}\ \mathbf{A} \tag{2.243}$$

This completes proof of (ii). The result (2.240) follows from the fact that in a homogeneous isotropic body

$$\mathrm{tr}\ \mathbf{A} = 3\alpha \tag{2.244}$$

This completes solution of Problem 2.18.