Chapter 17
Plane Thermoelastic Problems

In this chapter the basic treatment of plane thermoelastic problems in a state of plane strain and a plane stress are recalled. Typical three methods for the solution of plane problems are presented: the thermal stress function method for both simply connected and multiply connected bodies, the complex variable method with use of the conformal mapping technique, and potential method for Navier's equations [See also Chap. 7].

17.1 Plane Strain and Plane Stress

The unified systems of the governing equations for both plane strain and plane stress are as follows:

The generalized Hooke's law is

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E^*}\left(\sigma_{xx} - \nu^*\sigma_{yy}\right) + \alpha^*\tau - c^* \\ \epsilon_{yy} &= \frac{1}{E^*}\left(\sigma_{yy} - \nu^*\sigma_{xx}\right) + \alpha^*\tau - c^* \\ \epsilon_{xy} &= \frac{1}{2G}\sigma_{xy}\end{aligned} \tag{17.1}$$

An alternative form

$$\begin{aligned}\sigma_{xx} &= (\lambda^* + 2\mu)\epsilon_{xx} + \lambda^*\epsilon_{yy} - \beta^*\tau \\ \sigma_{yy} &= (\lambda^* + 2\mu)\epsilon_{yy} + \lambda^*\epsilon_{xx} - \beta^*\tau \\ \sigma_{xy} &= 2\mu\epsilon_{xy}\end{aligned} \tag{17.1$'$}$$

where

$$E^* = \begin{cases} E' = \dfrac{E}{1-\nu^2} & \text{for plane strain} \\ E & \text{for plane stress}\end{cases}$$

M. Reza Eslami et al., *Theory of Elasticity and Thermal Stresses*, Solid Mechanics and Its Applications 197, DOI: 10.1007/978-94-007-6356-2_17,

$$\nu^* = \begin{cases} \nu' = \dfrac{\nu}{1-\nu} & \text{for plane strain} \\ \nu & \text{for plane stress} \end{cases}$$

$$\alpha^* = \begin{cases} \alpha' = (1+\nu)\alpha & \text{for plane strain} \\ \alpha & \text{for plane stress} \end{cases}$$

$$\lambda^* = \begin{cases} \lambda & \text{for plane strain} \\ \lambda' = \dfrac{2\mu\lambda}{\lambda+2\mu} & \text{for plane stress} \end{cases} \tag{17.2}$$

$$\beta^* = \begin{cases} \beta & \text{for plane strain} \\ \beta' = \dfrac{2\mu\beta}{\lambda+2\mu} & \text{for plane stress} \end{cases}$$

$$c^* = \begin{cases} \nu\epsilon_0 & \text{for plane strain} \\ 0 & \text{for plane stress} \end{cases}$$

The equilibrium equations in the absence of body forces are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \tag{17.3}$$

The compatibility equation is

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \tag{17.4}$$

Navier's equations are from Eqs. (7.25) and (7.35)

$$\begin{aligned} \mu\nabla^2 u_x + (\lambda^* + \mu)\frac{\partial e}{\partial x} - \beta^*\frac{\partial \tau}{\partial x} = 0 \\ \mu\nabla^2 u_y + (\lambda^* + \mu)\frac{\partial e}{\partial y} - \beta^*\frac{\partial \tau}{\partial y} = 0 \end{aligned} \tag{17.5}$$

where $e = \epsilon_{xx} + \epsilon_{yy} + c^*$.

The boundary conditions are

$$\sigma_{xx}l + \sigma_{yx}m = p_{nx}, \quad \sigma_{xy}l + \sigma_{yy}m = p_{ny} \tag{17.6}$$

Next, we show typical three analytical methods for the plane problem.

Thermal stress function method

We introduce a thermal stress function χ related to the components of stress as follows

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{17.7}$$

The governing equation for the thermal stress function χ is

$$\nabla^4\chi = -\alpha^* E^* \nabla^2 \tau \tag{17.8}$$

where

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{17.9}$$

$$\alpha^* E^* = \begin{cases} \dfrac{\alpha E}{1-\nu} & \text{for plane strain} \\ \alpha E & \text{for plane stress} \end{cases} \tag{17.10}$$

The components of displacement can be expressed in the form

$$\begin{aligned} u_x &= \frac{1}{2G}\left[-\frac{\partial \chi}{\partial x} + \frac{1}{1+\nu^*}\frac{\partial \psi}{\partial y}\right] - c^* x \\ u_y &= \frac{1}{2G}\left[-\frac{\partial \chi}{\partial y} + \frac{1}{1+\nu^*}\frac{\partial \psi}{\partial x}\right] - c^* y \end{aligned} \tag{17.11}$$

where c^* is a constant and the function ψ satisfies the equation

$$\sigma_{xx} + \sigma_{yy} + \alpha^* E^* \tau = \nabla^2\chi + \alpha^* E^* \tau \equiv \frac{\partial^2 \psi}{\partial x \partial y} \tag{17.12}$$

in which

$$\frac{\partial^2}{\partial x \partial y}\nabla^2\psi = 0 \tag{17.13}$$

When the external force does not apply to the body, the boundary conditions of pure thermal stress problems are

$$\begin{aligned} &\chi(P) = C_1 x + C_2 y + C_3 \\ &\frac{\partial \chi(P)}{\partial n'} = C_1 \cos(n', x) + C_2 \cos(n', y) \end{aligned} \tag{17.14}$$

where n' denotes some direction which does not coincide with the direction of the contour, and C_1, C_2, and C_3 are arbitrary integration constants. The arbitrary integration constants C_1, C_2, and C_3 can be taken zero for a simply connected body. On the other hand, for a multiply connected body whose boundary consists of $m+1$ simply closed contours L_i $(i = 0, 1, \ldots, m)$, Eq. (17.14) can be rewritten as

$$\begin{aligned} &\chi(P_i) = C_{1i} x + C_{2i} y + C_{3i} \\ &\frac{\partial \chi(P_i)}{\partial n'} = C_{1i} \cos(n', x) + C_{2i} \cos(n', y) \end{aligned} \quad \text{on } L_i \ (i = 0, 1, \cdots, m) \tag{17.15}$$

where P_i is an arbitrary point on the i-th boundary contour L_i ($i = 0, 1, \ldots, m$), C_{1i}, C_{2i}, and C_{3i} are the integration constants on the boundary contour L_i ($i = 0, 1, \ldots, m$), and the integration constants on only one contour can be zero.

The conditions of single-valuedness of rotation and displacements in $(m+1)$-tuply connected body with traction free surfaces are

$$\oint_{L_i} \frac{\partial}{\partial n}(\nabla^2\chi + \alpha^* E^* \tau)\, ds = 0 \qquad (i = 1, \ldots, m) \tag{17.16}$$

$$\oint_{L_i} \left(x_1 \frac{\partial}{\partial s} - x_2 \frac{\partial}{\partial n} \right) (\nabla^2\chi + \alpha^* E^* \tau)\, ds = 0 \qquad (i = 1, \ldots, m) \tag{17.17}$$

$$\oint_{L_i} \left(x_1 \frac{\partial}{\partial n} + x_2 \frac{\partial}{\partial s} \right) (\nabla^2\chi + \alpha^* E^* \tau)\, ds = 0 \qquad (i = 1, \ldots, m) \tag{17.18}$$

The general solution of Eq. (17.8) for the thermal stress function χ may be expressed as the sum of the complementary solution χ_c and the particular solution χ_p

$$\chi = \chi_c + \chi_p \tag{17.19}$$

where the complementary solution χ_c and the particular solution χ_p are governed by

$$\nabla^4 \chi_c = 0 \tag{17.20}$$

$$\nabla^2 \chi_p = -\alpha^* E^* \tau \tag{17.21}$$

When the transient heat conduction equation with no heat generation is discussed, the particular solution χ_p is

$$\chi_p = -\alpha^* E^* \kappa \int_{t_r}^{t} \tau(x, y, t')\, dt' + \chi_{pr} + (t - t_r)\chi_{p0} \tag{17.22}$$

where t_r denotes the reference time, and χ_{pr} and χ_{p0} denote solutions of the following Poisson's and Laplace's equations, respectively

$$\nabla^2 \chi_{pr} = -\alpha^* E^* \tau_r, \quad \nabla^2 \chi_{p0} = 0 \tag{17.23}$$

in which τ_r denotes the temperature at the reference time t_r.

Complex variable method

The biharmonic function χ_c governed by Eq. (17.20) can be represented by two complex functions $\varphi(z)$ and $\psi_1(z)$ as follows

$$\chi_c = \frac{1}{2} \left[\bar{z}\varphi(z) + z\overline{\varphi(z)} + \psi_1(z) + \overline{\psi_1(z)} \right] \tag{17.24}$$

where the upper bar denotes its conjugate complex function

$$\bar{z} = x - iy, \quad \overline{\varphi(z)} = p - iq \tag{17.25}$$

where $i^2 = -1$. Hence, the thermal stress function χ can be represented by two complex functions and the particular solution χ_p

$$\chi = \chi_c + \chi_p = \frac{1}{2}\left[\bar{z}\varphi(z) + z\overline{\varphi(z)} + \psi_1(z) + \overline{\psi_1(z)}\right] + \chi_p \tag{17.26}$$

The plane thermal stresses are given by

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4\mathrm{Re}\left[\varphi'(z)\right] - \alpha^* E^* \tau \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2\left[\bar{z}\varphi''(z) + \psi'(z)\right] + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^2 \chi_p \end{aligned} \tag{17.27}$$

where $\psi(z) \equiv \psi_1'(z)$, and the complex functions $\varphi(z)$ and $\psi(z)$ are called the complex stress functions.

The components of displacement are

$$u_x + iu_y = \frac{1}{2G}\left[\frac{3-\nu^*}{1+\nu^*}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \left(\frac{\partial \chi_p}{\partial x} + i\frac{\partial \chi_p}{\partial y}\right)\right] - c^*(x+iy) \tag{17.28}$$

The boundary condition for the pure thermoelastic problem without traction is

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = -\left(\frac{\partial \chi_p}{\partial x} + i\frac{\partial \chi_p}{\partial y}\right) + C \tag{17.29}$$

The resultant moment M about the origin of the coordinate system is

$$M = \mathrm{Re}\Big[\psi_1(z) - z\psi_1'(z) - z\bar{z}\varphi'(z)\Big]_A^P - \left[x\frac{\partial \chi_p}{\partial x} + y\frac{\partial \chi_p}{\partial y} - \chi_p\right]_A^P \tag{17.30}$$

Let us translate a given region S in the complex z-plane into a region Σ in the complex ζ-plane by use of the conformal mapping function $\omega(\zeta)$

$$z = x + iy = \omega(\zeta), \quad \zeta = \xi + i\eta = \rho e^{i\theta} \tag{17.31}$$

A curvilinear coordinate system (ρ, θ) consists of curves $\rho =$ constant and radii $\theta =$ constant. The components (u_ρ, u_θ) of displacement vector $\boldsymbol{u}$ in the ζ-plane referred to a curvilinear coordinate system (ρ, θ) can be expressed by the components (u_x, u_y) of displacement vector $\boldsymbol{u}$ in the z-plane referred to a Cartesian coordinate system (x, y)

$$u_\rho + iu_\theta = e^{-i\alpha}(u_x + iu_y) \tag{17.32}$$

where α denotes an angle between x axis and ρ axis. The components of stress in plane problems referred to a curvilinear coordinate system (ρ, θ) can be expressed by the components referred to a Cartesian coordinate system (x, y) as follows

$$\begin{aligned} \sigma_{\rho\rho} + \sigma_{\theta\theta} &= \sigma_{xx} + \sigma_{yy} \\ \sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\theta} &= e^{2i\alpha}(\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}) \end{aligned} \tag{17.33}$$

With the conformal mapping function $\omega(\zeta)$, Eq. (17.32) with $c^* = 0$ becomes

$$\begin{aligned} u_\rho + iu_\theta = \frac{1}{2G}\frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\Big[\frac{3-\nu^*}{1+\nu^*}\phi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\Psi(\zeta)} \\ - \frac{\zeta}{\rho}\frac{1}{\overline{\omega'(\zeta)}}\Big(\frac{\partial\chi_p}{\partial\rho} + i\frac{1}{\rho}\frac{\partial\chi_p}{\partial\theta}\Big)\Big] \end{aligned} \tag{17.34}$$

The stress fields (17.33) are expressed by

$$\begin{aligned} \sigma_{\rho\rho} + \sigma_{\theta\theta} &= 4\,\mathrm{Re}\Big[\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big] - \alpha^* E^* \tau \\ \sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\theta} &= \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\Big\{\overline{\omega(\zeta)}\phi(\zeta)\Big[\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big]' + \Psi'(\zeta)\Big\} \\ &\quad + \frac{4\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\Big\{\frac{\partial^2\chi_p}{\partial\zeta^2}\frac{1}{\omega'(\zeta)} - \frac{\partial\chi_p}{\partial\zeta}\frac{\omega''(\zeta)}{[\omega'(\zeta)]^2}\Big\} \end{aligned} \tag{17.35}$$

Potential method

Navier's equations (17.5) can be rewritten as

$$\begin{aligned} \mu\nabla^2 u_x + (\lambda^* + \mu)\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) - \beta^*\frac{\partial\tau}{\partial x} = 0 \\ \mu\nabla^2 u_y + (\lambda^* + \mu)\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2}\right) - \beta^*\frac{\partial\tau}{\partial y} = 0 \end{aligned} \tag{17.36}$$

The general solutions of Navier's equations (17.36) for the plane problem can be expressed as the sum of the complementary solutions u_x^c and u_y^c, and the particular solutions u_x^p and u_y^p

$$u_x = u_x^c + u_x^p, \quad u_y = u_y^c + u_y^p \tag{17.37}$$

The particular solutions u_x^p and u_y^p can be expressed in terms of Goodier's thermoelastic potential Φ as follows:

$$u_x^p = \Phi_{,x}, \quad u_y^p = \Phi_{,y} \tag{17.38}$$

Φ must satisfy the equation as

$$\nabla^2 \Phi = K\tau \tag{17.39}$$

where

$$K = \frac{\beta^*}{\lambda^* + 2\mu} = (1 + \nu^*)\alpha^* \tag{17.40}$$

The complementary solutions u_x^c and u_y^c of Navier's equations (17.36) are expressed by two plane harmonic functions.

$$u_x^c = \frac{3 - \nu^*}{1 + \nu^*}\phi_1 - x\frac{\partial \phi_1}{\partial x} - y\frac{\partial \phi_2}{\partial x}, \quad u_y^c = \frac{3 - \nu^*}{1 + \nu^*}\phi_2 - x\frac{\partial \phi_1}{\partial y} - y\frac{\partial \phi_2}{\partial y} \tag{17.41}$$

where two functions ϕ_1 and ϕ_2 are harmonic

$$\nabla^2 \phi_1 = 0, \quad \nabla^2 \phi_2 = 0 \tag{17.42}$$

17.2 Problems and Solutions Related to Plane Thermoelastic Problems

Problem 17.1. Derive the governing equation for χ to be expressed by Eq. (17.8).

Solution. From Eq. (17.7) the thermal stress function χ is defined by

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{17.43}$$

The equilibrium equations (17.3) are automatically satisfied by use of the thermal stress function χ. The compatibility equation is from Eq. (17.4)

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \tag{17.44}$$

Using Hooke's law, and substituting the thermal stress function χ into Eq. (17.44), we obtain

$$\begin{aligned}
&\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \\
&\quad = \frac{\partial^2}{\partial y^2}\left[\frac{1}{E^*}\left(\sigma_{xx} - \nu^* \sigma_{yy}\right) + \alpha^* \tau - c^*\right] \\
&\qquad + \frac{\partial^2}{\partial x^2}\left[\frac{1}{E^*}\left(\sigma_{yy} - \nu^* \sigma_{xx}\right) + \alpha^* \tau - c^*\right] - 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{2G}\sigma_{xy}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^2}{\partial y^2}\Big[\frac{1}{E^*}\Big(\frac{\partial^2 \chi}{\partial y^2} - \nu^* \frac{\partial^2 \chi}{\partial x^2}\Big) + \alpha^* \tau - c^*\Big] \\
&\quad + \frac{\partial^2}{\partial x^2}\Big[\frac{1}{E^*}\Big(\frac{\partial^2 \chi}{\partial x^2} - \nu^* \frac{\partial^2 \chi}{\partial y^2}\Big) + \alpha^* \tau - c^*\Big] \\
&\quad + 2\frac{\partial^2}{\partial x \partial y}\Big(\frac{1+\nu^*}{E^*}\frac{\partial^2 \chi}{\partial x \partial y}\Big) \\
&= \frac{1}{E^*}\Big[\frac{\partial^4 \chi}{\partial x^4} + 2\frac{\partial^4 \chi}{\partial x^2 \partial y^2} + \frac{\partial^4 \chi}{\partial y^4} - \alpha^* E^* \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\tau\Big] = 0 \qquad (17.45)
\end{aligned}
$$

Therefore, the governing equation for thermal stress function χ is

$$\nabla^4 \chi = -\alpha^* E^* \nabla^2 \tau \qquad \text{(Answer)}$$

where

$$\alpha^* E^* = \begin{cases} (1+\nu)\alpha \dfrac{E}{1-\nu^2} = \dfrac{\alpha E}{1-\nu} & \text{for plane strain} \\ \alpha E & \text{for plane stress} \end{cases} \qquad (17.46)$$

Problem 17.2. Prove that the arbitrary integration constants C_1, C_2, and C_3 in Eq. (17.14) may be taken as zero for a simply connected body.

Solution. We take

$$\chi = \chi^* + C_1 x + C_2 y + C_3 \qquad (17.47)$$

Substitution of Eq. (17.47) into Eqs. (17.8), (17.7) and (17.14) gives the governing equation

$$\nabla^4 \chi^* = -\alpha^* E^* \nabla^2 \tau \qquad (17.48)$$

the stresses

$$\sigma_{xx} = \frac{\partial^2 \chi^*}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \chi^*}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi^*}{\partial x \partial y} \qquad (17.49)$$

and the boundary conditions

$$\chi^*(P) = 0, \quad \frac{\partial \chi^*(P)}{\partial n'} = 0 \quad \text{on } L \qquad (17.50)$$

Since the function $C_1 x + C_2 y + C_3$ does not appear in the governing Eq. (17.48), the stresses (17.49) and the boundary conditions (17.50), the integration constants can be taken zero for the simply connected body.

Problem 17.3. Prove that the integration constants C_{1i}, C_{2i}, and C_{3i} on the boundary contour L_i $(i = 0, 1, \ldots, m)$ in Eq. (17.15) can be taken zero on only one contour.

Solution. We take

$$\chi = \chi^* + C_{10} x + C_{20} y + C_{30} \qquad (17.51)$$

Substitution of Eq. (17.51) into Eqs. (17.8), (17.7) and (17.15) gives the governing equation

$$\nabla^4 \chi^* = -\alpha^* E^* \nabla^2 \tau \tag{17.52}$$

the stresses

$$\sigma_{xx} = \frac{\partial^2 \chi^*}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \chi^*}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \chi^*}{\partial x \partial y} \tag{17.53}$$

and the boundary conditions

$$\begin{aligned}
&\chi^*(P_0) = 0, \quad \frac{\partial \chi^*(P_0)}{\partial n'} = 0 \quad \text{on } L_0 \\
&\chi^*(P_i) = (C_{1i} - C_{10})x + (C_{2i} - C_{20})y + (C_{3i} - C_{30}) \\
&\frac{\partial \chi^*(P_i)}{\partial n'} = (C_{1i} - C_{10})\cos(n', x) + (C_{2i} - C_{20})\cos(n', y) \\
&\qquad\qquad\qquad\qquad \text{on } L_i (i = 1, 2, \cdots, n)
\end{aligned} \tag{17.54}$$

If we put

$$C_{1i}^* = C_{1i} - C_{10}, \quad C_{2i}^* = C_{2i} - C_{20}, \quad C_{3i}^* = C_{3i} - C_{30} \tag{17.55}$$

equations (17.54) reduce to

$$\begin{aligned}
&\chi^*(P_0) = 0, \quad \frac{\partial \chi^*(P_0)}{\partial n'} = 0 \quad \text{on } L_0 \\
&\chi^*(P_i) = C_{1i}^* x + C_{2i}^* y + C_{3i}^* \\
&\frac{\partial \chi^*(P_i)}{\partial n'} = C_{1i}^* \cos(n', x) + C_{2i}^* \cos(n', y) \quad \text{on } L_i (i = 1, 2, \cdots, n)
\end{aligned} \tag{17.56}$$

Taking Eqs. (17.52), (17.53) and (17.56) into consideration, the integration constants on only one contour can be taken zero.

Problem 17.4. Prove that the thermal stress is not produced in a strip with thickness l, when the steady temperature distribution without the internal heat generation is treated.

Solution. The heat conduction equation without internal heat generation is

$$\nabla^2 T = 0 \tag{17.57}$$

The thermal stress function χ satisfies the equation

$$\nabla^4 \chi = -\alpha^* E^* \nabla^2 \tau \tag{17.58}$$

where $\tau = T - T_0$. From Eqs. (17.57) and (17.58) we get

$$\nabla^4 \chi = 0 \tag{17.59}$$

The general solution of Eq. (17.59) is

$$\chi = A_0 + A_1 x + B_1 y + A_2 x^2 + B_2 y^2 + C_2 xy + A_3 x^3 + B_3 y^3 + C_3 x^2 y + D_3 x y^2 \tag{17.60}$$

Thermal stresses are

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial^2 y} = 2B_2 + 6B_3 y + 2D_3 x$$
$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} = 2A_2 + 6A_3 x + 2C_3 y$$
$$\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} = -C_2 - 2C_3 x - 2D_3 y \tag{17.61}$$

The boundary conditions are

$$\sigma_{xx} = 0\,, \quad \sigma_{xy} = 0 \quad \text{on } x = 0,\ l \tag{17.62}$$

The unknown coefficients are determined from Eq. (17.62) as

$$B_2 = 0, \quad B_3 = 0, \quad D_3 = 0\,, \quad C_2 = 0, \quad C_3 = 0 \tag{17.63}$$

From the condition of $\lim_{y\to\infty} \sigma_{yy} = 0$, we get

$$A_2 = 0, \quad A_3 = 0 \tag{17.64}$$

Then, the thermal stress is not produced in the strip.

Problem 17.5. Find the displacements in a strip when a steady temperature is given by

$$T = T_a + (T_b - T_a)\frac{x}{l} \tag{17.65}$$

Solution. Thermal stress is not produced in a strip, since the temperature given by Eq. (17.65) is the steady temperature without internal heat generation. As the thermal stress is not produced in a strip, a harmonic function ψ expressed by Eq. (17.12) reduces to

$$\frac{\partial^2 \psi}{\partial x \partial y} = \nabla^2 \chi + \alpha^* E^* \tau = \alpha^* E^* \tau = \alpha^* E^* \Big[T_a - T_0 + (T_b - T_a)\frac{x}{l} \Big] \tag{17.66}$$

where T_0 denotes the initial temperature. The integration of Eq. (17.66) gives

$$\psi = A + Bx + Cy + \alpha^* E^* \Big[(T_a - T_0)xy + (T_b - T_a)\frac{x^2 y}{2l} \Big] \tag{17.67}$$

The displacements (17.11) with no thermal stress reduce to

$$u_x = \frac{1}{2G(1+\nu^*)}\frac{\partial\psi}{\partial y} - c^* x, \quad u_y = \frac{1}{2G(1+\nu^*)}\frac{\partial\psi}{\partial x} - c^* y \tag{17.68}$$

Substitution of Eq. (17.67) into Eq. (17.68) gives

$$\begin{aligned} u_x &= \frac{C}{E^*} - c^* x + \alpha^*\Big[(T_a - T_0)x + (T_b - T_a)\frac{x^2}{2l}\Big] \\ u_y &= \frac{B}{E^*} - c^* y + \alpha^*\Big[(T_a - T_0)y + (T_b - T_a)\frac{xy}{l}\Big] \end{aligned} \qquad \text{(Answer)}$$

Problem 17.6. Derive the solutions of Eq. (17.20) in a Cartesian coordinate system.

Solution. First, we consider the solutions of Laplace's equation in a Cartesian coordinate system by use of the method of separation of variables. Laplace's equation is

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)h(x, y) = 0 \tag{17.69}$$

We assume that the harmonic function can be expressed by the product of two unknown functions, each of which has only one variable

$$h(x, y) = f(r)g(y) \tag{17.70}$$

Substitution of Eq. (17.70) into Eq. (17.69) gives

$$\frac{d^2 f(x)}{dx^2} + a^2 f(x) = 0, \quad \frac{d^2 g(y)}{dy^2} - a^2 g(y) = 0 \tag{17.71}$$

or

$$\frac{d^2 f(x)}{dx^2} - a^2 f(x) = 0, \quad \frac{d^2 g(y)}{dy^2} + a^2 g(y) = 0 \tag{17.72}$$

where a is a constant. The linearly independent solutions of Eq. (17.71) are

$$\begin{aligned} f(x) &= \begin{pmatrix} 1 \\ x \end{pmatrix} \text{ for } a = 0, \quad f(x) = \begin{pmatrix} \cos ax \\ \sin ax \end{pmatrix} \text{ for } a \neq 0 \\ g(y) &= \begin{pmatrix} 1 \\ y \end{pmatrix} \text{ for } a = 0, \quad g(y) = \begin{pmatrix} \cosh ay \\ \sinh ay \end{pmatrix} \text{ for } a \neq 0 \end{aligned} \tag{17.73}$$

and the linearly independent solutions of Eq. (17.72) are

$$f(x)=\begin{pmatrix}1\\x\end{pmatrix}\ \text{for}\ a=0,\quad f(x)=\begin{pmatrix}\cosh ax\\\sinh ax\end{pmatrix}\ \text{for}\ a\neq 0$$
$$g(y)=\begin{pmatrix}1\\y\end{pmatrix}\ \text{for}\ a=0,\quad g(y)=\begin{pmatrix}\cos ay\\\sin ay\end{pmatrix}\ \text{for}\ a\neq 0 \tag{17.74}$$

Now, we show that a function

$$p(x,y)=[Ax+By+C(x^2+y^2)]h(x,y) \tag{17.75}$$

is a biharmonic function, where a function $h(x,y)$ is harmonic, and A, B, C are arbitrary constants. Differentiation of Eq. (17.75) gives

$$\begin{aligned}\frac{\partial^2 p(x,y)}{\partial x^2}&=2Ch(x,y)+2(A+2Cx)\frac{\partial h(x,y)}{\partial x}\\&\quad+[Ax+By+C(x^2+y^2)]\frac{\partial^2 h(x,y)}{\partial x^2}\\\frac{\partial^2 p(x,y)}{\partial y^2}&=2Ch(x,y)+2(B+2Cy)\frac{\partial h(x,y)}{\partial y}\\&\quad+[Ax+By+C(x^2+y^2)]\frac{\partial^2 h(x,y)}{\partial y^2}\end{aligned} \tag{17.76}$$

As the function $h(x,y)$ is harmonic, we get

$$\nabla^2 p(x,y)=4Ch(x,y)+2(A+2Cx)\frac{\partial h(x,y)}{\partial x}+2(B+2Cy)\frac{\partial h(x,y)}{\partial y} \tag{17.77}$$

Differentiation of Eq. (17.77) gives

$$\begin{aligned}\frac{\partial^2\nabla^2 p(x,y)}{\partial x^2}&=12C\frac{\partial^2 h(x,y)}{\partial x^2}+2(A+2Cx)\frac{\partial^3 h(x,y)}{\partial x^3}\\&\quad+2(B+2Cy)\frac{\partial^3 h(x,y)}{\partial x^2\partial y}\\\frac{\partial^2\nabla^2 p(x,y)}{\partial y^2}&=12C\frac{\partial^2 h(x,y)}{\partial y^2}+2(B+2Cy)\frac{\partial^3 h(x,y)}{\partial y^3}\\&\quad+2(A+2Cx)\frac{\partial^3 h(x,y)}{\partial y^2\partial x}\end{aligned} \tag{17.78}$$

Therefore, we obtain

$$\begin{aligned}\nabla^4 p(x,y)&=12C\nabla^2 h(x,y)+2(A+2Cx)\frac{\partial}{\partial x}\nabla^2 h(x,y)\\&\quad+2(B+2Cy)\frac{\partial}{\partial y}\nabla^2 h(x,y)=0\end{aligned} \tag{17.79}$$

From Eqs. (17.73), (17.74) and (17.75), the particular solutions of a biharmonic equation (17.20) in a Cartesian coordinate system may be expressed as follows:

$$\begin{pmatrix} 1 \\ x \\ y \\ x^2+y^2 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ x \\ y \\ x^2+y^2 \end{pmatrix} \begin{pmatrix} \sin ax \\ \cos ax \end{pmatrix} \begin{pmatrix} \sinh ay \\ \cosh ay \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ x \\ y \\ x^2+y^2 \end{pmatrix} \begin{pmatrix} \sinh ax \\ \cosh ax \end{pmatrix} \begin{pmatrix} \sin ay \\ \cos ay \end{pmatrix} \qquad \text{(Answer)} \quad (17.80)$$

The notation for the product of three one-column matrices is explained by Eqs. (16.97) and (16.98).

Next, we show another type of the solutions of the biharmonic equation. A complex function $\varphi(z)$ is introduced.

$$z = x + iy, \quad \varphi(z) = p + iq \tag{17.81}$$

where $i^2 = -1$, and p and q are harmonic functions. Therefore

$$2p = \varphi(z) + \overline{\varphi(z)}, \quad 2iq = \varphi(z) - \overline{\varphi(z)} \tag{17.82}$$

We assume that $\varphi(z)$ is expressed by

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} (A_n z^n + B_n z^{-n}) \tag{17.83}$$

where A_n, B_n are real constants. Then, the harmonic function p of the real part of $\varphi(z)$ is written as

$$\begin{aligned} p &= \frac{1}{2}[\varphi(z) + \overline{\varphi(z)}] \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty}[A_n(z^n + \bar{z}^n) + B_n(z^{-n} + \bar{z}^{-n}) \\ &= 1 + \frac{1}{2}\sum_{n=1}^{\infty}(z^n + \bar{z}^n)\Big[A_n + B_n \frac{1}{(x^2+y^2)^n}\Big] \end{aligned} \tag{17.84}$$

We obtain the following harmonic functions from Eq. (17.84)

$$
\begin{aligned}
z + \bar{z} &= (x + iy) + (x - iy) = 2x \\
z^2 + \bar{z}^2 &= (x + iy)^2 + (x - iy)^2 = 2(x^2 - y^2) \\
z^3 + \bar{z}^3 &= (x + iy)^3 + (x - iy)^3 = 2x(x^2 - 3y^2) \\
z^4 + \bar{z}^4 &= (x + iy)^4 + (x - iy)^4 = 2(x^4 - 6x^2y^2 + y^4)
\end{aligned} \tag{17.85}
$$

In the similar way, we obtain the harmonic function q of the imaginary part of $\varphi(z)$

$$
\begin{aligned}
q &= \frac{1}{2i}[\varphi(z) - \overline{\varphi(z)}] \\
&= \frac{1}{2i}\sum_{n=1}^{\infty}[A_n(z^n - \bar{z}^n) + B_n(z^{-n} - \bar{z}^{-n})] \\
&= \frac{1}{2i}\sum_{n=1}^{\infty}(z^n - \bar{z}^n)\left[A_n - B_n\frac{1}{(x^2 + y^2)^n}\right]
\end{aligned} \tag{17.86}
$$

From Eq. (17.86) we obtain the following imaginary parts

$$
\begin{aligned}
z - \bar{z} &= (x + iy) - (x - iy) = 2iy \\
z^2 - \bar{z}^2 &= (x + iy)^2 - (x - iy)^2 = 4ixy \\
z^3 - \bar{z}^3 &= (x + iy)^3 - (x - iy)^3 = -2iy(y^2 - 3x^2) \\
z^4 - \bar{z}^4 &= (x + iy)^4 - (x - iy)^4 = 8izy(x^2 - y^2)
\end{aligned} \tag{17.87}
$$

Therefore, taking into the consideration of Eqs. (17.84)–(17.87), we obtain the alternative forms of the particular solutions of the biharmonic equation

$$
\begin{aligned}
&\begin{pmatrix} 1 \\ x \\ y \\ x^2 + y^2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ x \\ y \\ x^2 + y^2 \end{pmatrix}\begin{pmatrix} 1 \\ r^{-2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&\begin{pmatrix} 1 \\ x \\ y \\ x^2 + y^2 \end{pmatrix}\begin{pmatrix} 1 \\ r^{-4} \end{pmatrix}\begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix} \\
&\begin{pmatrix} 1 \\ x \\ y \\ x^2 + y^2 \end{pmatrix}\begin{pmatrix} 1 \\ r^{-6} \end{pmatrix}\begin{pmatrix} x(x^2 - 3y^2) \\ y(y^2 - 3x^2) \end{pmatrix} \\
&\begin{pmatrix} 1 \\ x \\ y \\ x^2 + y^2 \end{pmatrix}\begin{pmatrix} 1 \\ r^{-8} \end{pmatrix}\begin{pmatrix} x^4 - 6x^2y^2 + y^4 \\ xy(x^2 - y^2) \end{pmatrix}, \ldots
\end{aligned} \tag{Answer}
$$

in which $r = \sqrt{x^2 + y^2}$. The notation for the product of three one-column matrices is explained by Eqs. (16.97) and (16.98).

Problem 17.7. Derive the solutions of Eq. (17.20) in the polar coordinate system.

Solution. First, we consider the solutions of Laplace's equation in the polar coordinate system by use of the method of separation of variables. Laplace's equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)h(r,\theta) = 0 \tag{17.88}$$

We assume that the harmonic function can be expressed by the product of two unknown functions, each of which has only one variable

$$h(r,\theta) = f(r)g(\theta) \tag{17.89}$$

Substitution of Eq. (17.89) into Eq. (17.88) gives

$$\begin{aligned} &\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} - \frac{n^2}{r^2}f(r) = 0 \\ &\frac{d^2 g(\theta)}{d\theta^2} + n^2 g(\theta) = 0 \end{aligned} \tag{17.90}$$

where n is the integer. The linearly independent solutions of Eq. (17.90) are

$$\begin{aligned} &f(r) = \begin{pmatrix} 1 \\ \ln r \end{pmatrix} \quad \text{for } n = 0, \quad f(r) = \begin{pmatrix} r^n \\ r^{-n} \end{pmatrix} \quad \text{for } n \neq 0 \\ &g(\theta) = \begin{pmatrix} 1 \\ \theta \end{pmatrix} \quad \text{for } n = 0, \quad g(\theta) = \begin{pmatrix} \sin n\theta \\ \cos n\theta \end{pmatrix} \quad \text{for } n \neq 0 \end{aligned} \tag{17.91}$$

Next, we consider the particular solution $p(r,\theta)$ which satisfies the equation

$$\nabla^2 p(r,\theta) = f(r)g(\theta) \tag{17.92}$$

The particular solution p is assumed to be expressed by the product of two functions, each of which has only one variable

$$p(r,\theta) = F(r)g(\theta) \tag{17.93}$$

Substitution of Eq. (17.93) into Eq. (17.92) gives

$$\frac{d^2 F(r)}{dr^2} + \frac{1}{r}\frac{dF(r)}{dr} - \frac{n^2}{r^2}F(r) = f(r) \tag{17.94}$$

and a particular solution $F(r)$ of Eq. (17.94) takes the form

$$F(r) = \begin{pmatrix} r^2/4 \\ r^2(\ln r - 1)/4 \end{pmatrix} \quad \text{when}\, f(r) = \begin{pmatrix} 1 \\ \ln r \end{pmatrix}$$
$$F(r) = \begin{pmatrix} r^3/8 \\ r\ln r/2 \end{pmatrix} \quad \text{when}\, f(r) = \begin{pmatrix} r \\ r^{-1} \end{pmatrix}$$
$$F(r) = \begin{pmatrix} r^{n+2}/(4n+4) \\ -r^{-n+2}/(4n-4) \end{pmatrix} \quad \text{when}\, f(r) = \begin{pmatrix} r^n \\ r^{-n} \end{pmatrix} \tag{17.95}$$

From Eqs. (17.91) and (17.95), the particular solutions of Eq. (17.20) in the polar coordinate system are

$$\begin{pmatrix} 1 \\ r^2 \\ \ln r \\ r^2 \ln r \end{pmatrix} \begin{pmatrix} 1 \\ \theta \end{pmatrix}, \quad \begin{pmatrix} r \\ r^{-1} \\ r^3 \\ r\ln r \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$
$$\begin{pmatrix} r^n \\ r^{-n} \\ r^{n+2} \\ r^{-n+2} \end{pmatrix} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \qquad \text{(Answer)} \tag{17.96}$$

In Eq. (17.96), we used the following notation for the product of two one-column matrices

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 g_1 \\ f_2 g_1 \\ f_3 g_1 \\ f_4 g_1 \\ f_1 g_2 \\ f_2 g_2 \\ f_3 g_2 \\ f_4 g_2 \end{pmatrix} \tag{17.97}$$

Then, for example

$$\begin{pmatrix} r^n \\ r^{-n} \\ r^{n+2} \\ r^{-n+2} \end{pmatrix} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} = \begin{pmatrix} r^n \cos n\theta \\ r^{-n} \cos n\theta \\ r^{n+2} \cos n\theta \\ r^{-n+2} \cos n\theta \\ r^n \sin n\theta \\ r^{-n} \sin n\theta \\ r^{n+2} \sin n\theta \\ r^{-n+2} \sin n\theta \end{pmatrix} \tag{17.98}$$

Problem 17.8. Derive Eq. (17.34).

Solution. The relationship for the displacement between a curvilinear coordinate system and a Cartesian coordinate system is given by Eq. (17.32)

$$u_\rho + iu_\theta = e^{-i\alpha}(u_x + iu_y) \tag{17.99}$$

When a small displacement dz is produced, a corresponding point ζ undergoes a small displacement $d\zeta$

$$dz = |dz|e^{i\alpha}, \quad d\zeta = |d\zeta|e^{i\theta} \tag{17.100}$$

From Eq. (17.100), we get

$$\begin{aligned}
e^{i\alpha} &= \frac{dz}{|dz|} = \frac{\omega'(\zeta)d\zeta}{|\omega'(\zeta)| \cdot |d\zeta|} = e^{i\theta}\frac{\omega'(\zeta)}{|\omega'(\zeta)|} = \frac{\zeta}{\rho}\frac{\omega'(\zeta)}{|\omega'(\zeta)|} \\
e^{-i\alpha} &= e^{-i\theta}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|} = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|} \\
e^{2i\alpha} &= \left[\frac{\zeta}{\rho}\frac{\omega'(\zeta)}{|\omega'(\zeta)|}\right]^2 = \frac{\zeta^2}{\rho^2}\frac{\omega'(\zeta)\omega'(\zeta)}{\omega'(\zeta)\overline{\omega'(\zeta)}} = \frac{\zeta^2}{\rho^2}\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}
\end{aligned} \tag{17.101}$$

Next, we introduce the new notation

$$\begin{aligned}
&\varphi(z) = \varphi(\omega(\zeta)) \equiv \phi(\zeta), \quad \psi(z) = \psi(\omega(\zeta)) \equiv \Psi(\zeta) \\
&\varphi'(z) = \frac{d\varphi(z)}{dz} = \frac{d\phi(\zeta)}{d\zeta}\frac{d\zeta}{dz} = \frac{1}{\omega'(\zeta)}\frac{d\phi(\zeta)}{d\zeta} = \frac{\phi'(\zeta)}{\omega'(\zeta)}
\end{aligned} \tag{17.102}$$

Substitution of Eqs. (17.28) with $c^* = 0$, (17.101) and (17.102) into Eq. (17.99) yields

$$\begin{aligned}
u_\rho + iu_\theta &= e^{-i\alpha}\frac{1}{2G}\left[\frac{3-\nu^*}{1+\nu^*}\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \left(\frac{\partial\chi_p}{\partial x} + i\frac{\partial\chi_p}{\partial y}\right)\right] \\
&= \frac{1}{2G}\frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\left[\frac{3-\nu^*}{1+\nu^*}\phi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\Psi(\zeta)}\right. \\
&\qquad \left. - \left(\frac{\partial\chi_p}{\partial x} + i\frac{\partial\chi_p}{\partial y}\right)\right]
\end{aligned} \tag{17.103}$$

Taking into the consideration the following relationship

$$\begin{aligned}
\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}\right) + i\left(\frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}\right) \\
&= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) + i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \\
&= 2\frac{\partial}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{\zeta}}\frac{d\bar{\zeta}}{d\bar{z}} = 2\frac{1}{\overline{\omega'(\zeta)}}\frac{\partial}{\partial \bar{\zeta}} = 2\frac{1}{\overline{\omega'(\zeta)}}\left(\frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial \bar{\zeta}} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial \bar{\zeta}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{1}{\overline{\omega'(\zeta)}}\left\{\frac{\partial}{\partial\rho}\frac{\partial\sqrt{\zeta\bar{\zeta}}}{\partial\bar{\zeta}} + \frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar{\zeta}}\Big[-\frac{i}{2}(\ln\zeta - \ln\bar{\zeta})\Big]\right\} \\
&= \frac{1}{\overline{\omega'(\zeta)}}\Big(\frac{\partial}{\partial\rho}\sqrt{\frac{\zeta}{\bar{\zeta}}} + i\frac{\partial}{\partial\theta}\frac{1}{\bar{\zeta}}\Big) = \frac{e^{i\theta}}{\overline{\omega'(\zeta)}}\Big(\frac{\partial}{\partial\rho} + i\frac{1}{\rho}\frac{\partial}{\partial\theta}\Big) \\
&= \frac{\zeta}{\rho}\frac{1}{\overline{\omega'(\zeta)}}\Big(\frac{\partial}{\partial\rho} + i\frac{1}{\rho}\frac{\partial}{\partial\theta}\Big)
\end{aligned}
\tag{17.104}
$$

we obtain the displacement

$$
\begin{aligned}
u_\rho + iu_\theta = \frac{1}{2G}\frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\Big[&\frac{3-\nu^*}{1+\nu^*}\phi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\phi'(\zeta)} - \overline{\Psi(\zeta)} \\
&- \frac{\zeta}{\rho}\frac{1}{\overline{\omega'(\zeta)}}\Big(\frac{\partial}{\partial\rho} + i\frac{1}{\rho}\frac{\partial}{\partial\theta}\Big)\chi_p\Big]
\end{aligned}
\tag{Answer}
$$

Problem 17.9. Derive Eq. (17.35).

Solution. The relationship for the stress between a curvilinear coordinate system and a Cartesian coordinate system is given by Eq. (17.33):

$$
\begin{aligned}
\sigma_{\rho\rho} + \sigma_{\theta\theta} &= \sigma_{xx} + \sigma_{yy} \\
\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\theta\rho} &= e^{2i\alpha}(\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})
\end{aligned}
\tag{17.105}
$$

Substitution of Eq. (17.27) into Eq. (17.105) yields

$$
\begin{aligned}
\sigma_{\rho\rho} + \sigma_{\theta\theta} &= 4Re[\varphi'(z)] - \alpha^* E^* \tau \\
\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\theta\rho} &= e^{2i\alpha}\Big\{2[\bar{z}\varphi''(z) + \psi'(z)] + \Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)^2\chi_p\Big\}
\end{aligned}
\tag{17.106}
$$

By transforming the variable from z to ζ, and using Eqs. (17.101) and (17.102) in Problem 17.8, Eq. (17.106) reduce to

$$
\begin{aligned}
\sigma_{\rho\rho} + \sigma_{\theta\theta} &= 4Re\Big[\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big] - \alpha^* E^* \tau \\
\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\theta\rho} &= \frac{\zeta^2}{\rho^2}\frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\Big\{2\Big[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Big(\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big)' + \frac{\Psi'(\zeta)}{\omega'(\zeta)}\Big] \\
&\qquad + \Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)^2\chi_p\Big\}
\end{aligned}
\tag{17.107}
$$

Taking into the consideration of the relationship

$$\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = \Big(\frac{\partial}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}\Big) - i\Big(\frac{\partial}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}\Big)$$
$$= \Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\Big) - i^2\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big) = 2\frac{\partial}{\partial z} = 2\frac{\partial}{\partial \zeta}\frac{d\zeta}{dz} = 2\frac{1}{\omega'(\zeta)}\frac{\partial}{\partial \zeta}$$
$$\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)^2 = 4\frac{1}{\omega'(\zeta)}\frac{\partial}{\partial \zeta}\Big[\frac{1}{\omega'(\zeta)}\frac{\partial}{\partial \zeta}\Big]$$
$$= 4\frac{1}{\omega'(\zeta)}\Big[\frac{1}{\omega'(\zeta)}\frac{\partial^2}{\partial \zeta^2} - \frac{\omega''(\zeta)}{[\omega'(\zeta)]^2}\frac{\partial}{\partial \zeta}\Big] \tag{17.108}$$

we obtain the stress

$$\sigma_{\rho\rho} + \sigma_{\theta\theta} = 4Re\Big[\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big] - \alpha^* E^* \tau$$
$$\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\theta\rho} = 2\frac{\zeta^2}{\rho^2}\frac{1}{\overline{\omega'(\zeta)}}\Big\{\overline{\omega(\zeta)}\Big[\frac{\phi'(\zeta)}{\omega'(\zeta)}\Big]' + \Psi'(\zeta)\Big\}$$
$$+ 4\frac{\zeta^2}{\rho^2}\frac{1}{\overline{\omega''(\zeta)}}\Big[\frac{1}{\omega'(\zeta)}\frac{\partial^2 \chi_p}{\partial \zeta^2} - \frac{\omega'(\zeta)}{[\omega'(\zeta)]^2}\frac{\partial \chi_p}{\partial \zeta}\Big] \quad \text{(Answer)}$$

Problem 17.10. Airy's stress function F related to the components of stress

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{17.109}$$

is usually used in isothermal plane problems, where a governing equation of F is $\nabla^4 F = 0$. Prove that the thermal stress in plane problems can be expressed by

$$\sigma_{xx} = \frac{\partial^2}{\partial y^2}(F - 2\mu\Phi), \quad \sigma_{yy} = \frac{\partial^2}{\partial x^2}(F - 2\mu\Phi)$$
$$\sigma_{xy} = -\frac{\partial^2}{\partial x \partial y}(F - 2\mu\Phi) \tag{17.110}$$

where Φ is Goodier's thermoelastic potential and F is Airy's stress function.

Solution. Using Eqs. (17.37) and (17.38), the strains are expressed by

$$\varepsilon_{xx} = \varepsilon^c_{xx} + \Phi,_{xx}, \quad \varepsilon_{yy} = \varepsilon^c_{yy} + \Phi,_{yy}, \quad \varepsilon_{yx} = \varepsilon^c_{xy} + \Phi,_{xy} \tag{17.111}$$

From Eqs. (17.1′) and (17.111), we get

$$\sigma_{xx} = (\lambda^* + 2\mu)\varepsilon^c_{xx} + \lambda^*\varepsilon^c_{yy} - 2\mu\Phi,_{yy} + (\lambda^* + 2\mu)\nabla^2\Phi - \beta^*\tau$$
$$\sigma_{yy} = (\lambda^* + 2\mu)\varepsilon^c_{yy} + \lambda^*\varepsilon^c_{xx} - 2\mu\Phi,_{xx} + (\lambda^* + 2\mu)\nabla^2\Phi - \beta^*\tau$$
$$\sigma_{xy} = 2\mu\varepsilon^c_{xy} + 2\mu\Phi,_{xy} \tag{17.112}$$

Since the governing equation for Goodier's thermoelastic potential function Φ is given by Eq. (17.39), Eq. (17.112) reduces to

$$\begin{aligned}
\sigma_{xx} &= (\lambda^* + 2\mu)\varepsilon^c_{xx} + \lambda^*\varepsilon^c_{yy} - 2\mu\Phi,_{yy} = \sigma^c_{xx} - 2\mu\Phi,_{yy} = F,_{yy} - 2\mu\Phi,_{yy} \\
\sigma_{yy} &= (\lambda^* + 2\mu)\varepsilon^c_{yy} + \lambda^*\varepsilon^c_{xx} - 2\mu\Phi,_{xx} = \sigma^c_{yy} - 2\mu\Phi,_{xx} = F,_{xx} - 2\mu\Phi,_{xx} \\
\sigma_{xy} &= 2\mu\varepsilon^c_{xy} + 2\mu\Phi,_{xy} = \sigma^c_{xy} + 2\mu\Phi,_{xy} = -(F,_{xy} - 2\mu\Phi,_{xy}) \qquad \text{(Answer)}
\end{aligned}$$

Next, we derive the governing equation of Airy's stress function F. Substitution of Eq. (17.1) into Eq. (17.4) gives

$$\begin{aligned}
&\frac{1}{E^*}\frac{\partial^2\sigma_{xx}}{\partial y^2} - \frac{\nu^*}{E^*}\frac{\partial^2\sigma_{yy}}{\partial y^2} + \alpha^*\frac{\partial^2\tau}{\partial y^2} + \frac{1}{E^*}\frac{\partial^2\sigma_{yy}}{\partial x^2} - \frac{\nu^*}{E^*}\frac{\partial^2\sigma_{xx}}{\partial x^2} + \alpha^*\frac{\partial^2\tau}{\partial x^2} \\
&\quad = 2\frac{1+\nu^*}{E^*}\frac{\partial^2\sigma_{xy}}{\partial x\partial y}
\end{aligned} \tag{17.113}$$

Simplification of Eq. (17.113) reduces to

$$\begin{aligned}
&\frac{\partial^2\sigma_{xx}}{\partial y^2} - 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu^*\left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2}\right) \\
&\quad = -\alpha^* E^*\nabla^2\tau
\end{aligned} \tag{17.114}$$

Substitution of Eq. (17.110) into Eq. (17.114) gives

$$\begin{aligned}
&\frac{\partial^2(F,_{yy} - 2\mu\Phi,_{yy})}{\partial y^2} + 2\frac{\partial^2(F,_{xy} - 2\mu\Phi,_{xy})}{\partial x\partial y} + \frac{\partial^2(F,_{xx} - 2\mu\Phi,_{xx})}{\partial x^2} \\
&\quad - \nu^*\Big[\frac{\partial^2(F,_{yy} - 2\mu\Phi,_{yy})}{\partial x^2} - 2\frac{\partial^2(F,_{xy} - 2\mu\Phi,_{xy})}{\partial x\partial y} + \frac{\partial^2(F,_{xx} - 2\mu\Phi,_{xx})}{\partial y^2}\Big] \\
&= -\alpha^* E^*\nabla^2\tau
\end{aligned} \tag{17.115}$$

Simplification of above equation reduces to

$$\nabla^4 F - 2\mu\nabla^2\left(\nabla^2\Phi - \frac{\alpha^* E^*}{2\mu}\tau\right) = 0 \tag{17.116}$$

By the use of Eq. (17.39), Eq. (17.116) reduces to

$$\nabla^4 F = 0 \tag{17.117}$$

Problem 17.11. Prove that the components of thermal stress in plane problems can be expressed by

$$\sigma_{xx} = 2\mu\Big[-\Phi_{,yy} + x\phi_{1,yy} + y\phi_{2,yy} + \frac{2}{1+\nu^*}(\phi_{1,x} + \nu^*\phi_{2,y})\Big]$$

$$\sigma_{yy} = 2\mu\Big[-\Phi_{,xx} + x\phi_{1,xx} + y\phi_{2,xx} + \frac{2}{1+\nu^*}(\phi_{2,y} + \nu^*\phi_{1,x})\Big]$$
$$\sigma_{xy} = 2\mu\Big[\Phi_{,xy} - (x\phi_1 + y\phi_2)_{,xy} + \frac{1-\nu^*}{1+\nu^*}(\phi_{1,y} + \phi_{2,x})\Big] \tag{17.118}$$

where Φ denotes Goodier's thermoelastic potential given by Eq. (17.38) and two harmonic functions ϕ_1, ϕ_2 are given by Eq. (17.41).

Solution. The displacement may be expressed from Eqs. (17.38) and (17.41)

$$u_x = \Phi_{,x} + \frac{3-\nu^*}{1+\nu^*}\phi_1 - x\phi_{1,x} - y\phi_{2,x}$$
$$u_y = \Phi_{,y} + \frac{3-\nu^*}{1+\nu^*}\phi_2 - x\phi_{1,y} - y\phi_{2,y} \tag{17.119}$$

Equation (17.119) give the strains

$$\varepsilon_{xx} = \Phi_{,xx} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{1,x} - x\phi_{1,xx} - y\phi_{2,xx}$$
$$\varepsilon_{yy} = \Phi_{,yy} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{2,y} - x\phi_{1,yy} - y\phi_{2,yy}$$
$$\varepsilon_{xy} = \Phi_{,xy} + \frac{1-\nu^*}{1+\nu^*}(\phi_{1,y} + \phi_{2,x}) - x\phi_{1,xy} - y\phi_{2,xy} \tag{17.120}$$

From Eqs. (17.1′) and (17.120), we get

$$\sigma_{xx} = (\lambda^* + 2\mu)(\Phi_{,xx} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{1,x} - x\phi_{1,xx} - y\phi_{2,xx})$$
$$+ \lambda^*(\Phi_{,yy} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{2,y} - x\phi_{1,yy} - y\phi_{2,yy}) - \beta^*\tau$$
$$\sigma_{yy} = (\lambda^* + 2\mu)(\Phi_{,yy} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{2,y} - x\phi_{1,yy} - y\phi_{2,yy})$$
$$+ \lambda^*(\Phi_{,xx} + 2\frac{1-\nu^*}{1+\nu^*}\phi_{1,x} - x\phi_{1,xx} - y\phi_{2,xx}) - \beta^*\tau$$
$$\sigma_{xy} = 2\mu[\Phi_{,xy} + \frac{1-\nu^*}{1+\nu^*}(\phi_{1,y} + \phi_{2,x}) - x\phi_{1,xy} - y\phi_{2,xy}] \tag{17.121}$$

By the use of Eqs. (17.39) and (17.42), we can obtain

$$\sigma_{xx} = 2\mu(-\Phi_{,yy} - x\phi_{1,xx} - y\phi_{2,xx}) + 2\frac{1-\nu^*}{1+\nu^*}[(\lambda^* + 2\mu)\phi_{1,x} + \lambda^*\phi_{2,y}]$$
$$\sigma_{yy} = 2\mu(-\Phi_{,xx} - x\phi_{1,yy} - y\phi_{2,yy}) + 2\frac{1-\nu^*}{1+\nu^*}[(\lambda^* + 2\mu)\phi_{2,y} + \lambda^*\phi_{1,x}]$$
$$\sigma_{xy} = 2\mu[\Phi_{,xy} + \frac{1-\nu^*}{1+\nu^*}(\phi_{1,y} + \phi_{2,x}) - x\phi_{1,xy} - y\phi_{2,xy}] \tag{17.122}$$

Material constants are rewritten as for plane strain

$$
\begin{aligned}
&\nu^* = \frac{\nu}{1-\nu} \quad \leftrightarrow \ \nu = \frac{\nu^*}{1+\nu^*} \\
&\lambda^* = \lambda \ = \frac{2\nu\mu}{1-2\nu} = \frac{2\mu\nu^*/(1+\nu^*)}{1-2\nu^*/(1+\nu^*)} = \frac{2\mu\nu^*}{1-\nu^*} \\
&\frac{1-\nu^*}{1+\nu^*}(\lambda^* + 2\mu) = 2\mu\frac{1-\nu^*}{1+\nu^*}\Big(\frac{\nu^*}{1-\nu^*} + 1\Big) = 2\mu\frac{1}{1+\nu^*} \\
&\frac{1-\nu^*}{1+\nu^*}\lambda^* = 2\mu\frac{1-\nu^*}{1+\nu^*}\frac{\nu^*}{1-\nu^*} = 2\mu\frac{\nu^*}{1+\nu^*}
\end{aligned}
\tag{17.123}
$$

and for plane stress

$$
\begin{aligned}
&\nu^* = \nu, \quad \lambda^* = \frac{2\mu\lambda}{\lambda+2\mu} = \frac{2\mu 2\nu\mu/(1-2\nu)}{2\nu\mu/(1-2\nu)+2\mu} = \frac{2\mu\nu}{1-\nu} = \frac{2\mu\nu^*}{1-\nu^*} \\
&\frac{1-\nu^* *}{1+\nu^*}(\lambda^* + 2\mu) = 2\mu\frac{1-\nu^*}{1+\nu^*}\Big(\frac{\nu^*}{1-\nu^*} + 1\Big) = 2\mu\frac{1}{1+\nu^*} \\
&\frac{1-\nu^*}{1+\nu^*}\lambda^* = 2\mu\frac{1-\nu^*}{1+\nu^*}\frac{\nu^*}{1-\nu^*} = 2\mu\frac{\nu^*}{1+\nu^*}
\end{aligned}
\tag{17.124}
$$

By the use of Eqs. (17.123), (17.124) and (17.42), Eq. (17.122) reduce to

$$
\begin{aligned}
\sigma_{xx} &= 2\mu\Big[-\Phi,_{yy} + x\phi_1,_{yy} + y\phi_2,_{yy} + \frac{2}{1+\nu^*}(\phi_1,_x + \nu^*\phi_2,_y)\Big] \\
\sigma_{yy} &= 2\mu\Big[-\Phi,_{xx} + x\phi_1,_{xx} + y\phi_2,_{xx} + \frac{2}{1+\nu^*}(\phi_2,_y + \nu^*\phi_1,_x)\Big] \\
\sigma_{xy} &= 2\mu\Big[\Phi,_{xy} - (x\phi_1 + y\phi_2),_{xy} + \frac{1-\nu^*}{1+\nu^*}(\phi_1,_y + \phi_2,_x)\Big] \qquad \text{(Answer)}
\end{aligned}
$$