Chapter 16
Basic Equations of Thermoelasticity

In this chapter the basic governing equations of thermoelasticity for three-dimensional bodies are recalled. The equilibrium equations of stresses, Cauchy's relations between the tractions and stresses, and the compatibility equations of strains in Cartesian coordinates are presented. The formulae for coordinate transformation of stress, strain and displacement components are included. A solution of Navier's equations is carried out wherein Goodier's thermoelastic potential is used in conjunction with harmonic functions of various types. The equilibrium equations, stress, strain, the compatibility equations, Navier's equations in cylindrical and spherical coordinates are also presented. [see also Chaps. 2, 3, 6, and 7.]

16.1 Governing Equations of Thermoelasticity

16.1.1 Stress and Strain in a Cartesian Coordinate System

Stress

The equilibrium equations of the elastic body from Eq. (2.21) are

$$\sigma_{ji,j} + F_i = 0 \quad (i, j = 1, 2, 3) \tag{16.1}$$

where σ_{ji} denote the components of stress, F_i mean the components of body force per unit volume. The components of stress satisfy symmetry relations

$$\sigma_{ij} = \sigma_{ji} \quad (i, j = 1, 2, 3) \tag{16.2}$$

Cauchy's fundamental relations are

$$\sigma_{ji} n_j = p_{ni} \quad (i, j = 1, 2, 3) \tag{16.3}$$

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where n_j denote the direction cosines between the external normal of the surface and each axis.

The formulae for coordinate transformation of stress components between the components of stress $(\sigma_{xx}, \sigma_{xy}, \ldots)$ referred to an old Cartesian coordinate system (x, y, z) and the components of stress $(\sigma_{x'x'}, \sigma_{x'y'}, \ldots)$ referred to a new Cartesian coordinate system (x', y', z') are

$$\sigma_{i'j'} = l_{i'k} l_{j'l} \sigma_{kl} \qquad (i', j' = 1, 2, 3) \tag{16.4}$$

where $l_{i'k}$ denote the direction cosines between the axis $x_{i'}$ of the new Cartesian coordinate system (x'_1, x'_2, x'_3) and the axis x_k of the old (x_1, x_2, x_3).

Strain

The strains are from Eq. (2.5)

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (i, j = 1, 2, 3) \tag{16.5}$$

where u_i are the components of displacement. The components of strain are symmetric

$$\epsilon_{ij} = \epsilon_{ji} \qquad (i, j = 1, 2, 3) \tag{16.6}$$

The transformation of coordinates between the the new Cartesian coordinate system $(x_{i'})$ and the old system (x_k) are

$$x_{i'} = l_{i'j} x_j, \quad x_i = l_{j'i} x_{j'} \qquad (i, i' = 1, 2, 3) \tag{16.7}$$

The relationship between the components of the displacement in each coordinate system are

$$u_{i'} = l_{i'j} u_j, \qquad u_i = l_{j'i} u_{j'} \qquad (i, i' = 1, 2, 3) \tag{16.8}$$

The coordinate transformation of strain components is

$$\epsilon_{i'j'} = l_{i'k} l_{j'l} \epsilon_{kl} \qquad (i', j' = 1, 2, 3) \tag{16.9}$$

16.1.2 Navier's Equations, Compatibility Equations and Boundary Conditions

Navier's equations

The constitutive equations for a homogeneous, isotropic body which are known as the generalized Hooke's law are

$$\epsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\delta_{ij}\right) + \alpha\tau\delta_{ij} \quad (i, j = 1, 2, 3) \tag{16.10}$$

an alternative form

$$\sigma_{ij} = 2\mu\epsilon_{ij} + (\lambda e - \beta\tau)\delta_{ij} \quad (i, j = 1, 2, 3) \tag{16.11}$$

where τ is the temperature change from the reference temperature T_0

$$\tau = T - T_0 \tag{16.12}$$

and G is the shear modulus, ν is Poisson's ratio, α is the coefficient of linear thermal expansion, λ and μ are the Lamé elastic constants, and β is the thermoelastic constant, and Θ denotes the sum of the normal stresses

$$\Theta = \sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{16.13}$$

e is the dilatation

$$e = \epsilon_{kk} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \tag{16.14}$$

and δ_{ij} is Kronecker's symbol

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{16.15}$$

The relationship between the elastic constants (E, G, ν, λ, μ) and the thermoelastic constant β is given by

$$\begin{aligned} 2G &= \frac{E}{1+\nu}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2\nu G}{1-2\nu} \\ \mu &= G, \quad \beta = \frac{\alpha E}{1-2\nu} = \alpha(3\lambda + 2\mu) \end{aligned} \tag{16.16}$$

Navier's equations of thermoelasticity, or the displacement equations of thermoelasticity expressed in terms of the components of displacement are from Eq. (3.22)

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} - \beta\tau_{,i} + F_i = 0 \qquad (i = 1, 2, 3) \tag{16.17}$$

an alternative form

$$(\lambda + 2\mu)u_{k,ki} - 2\mu\varepsilon_{ijk}\omega_{k,j} - \beta\tau_{,i} + F_i = 0 \qquad (i = 1, 2, 3) \tag{16.18}$$

where ∇^2 is the Laplacian operator defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{16.19}$$

and ω_k denote the rotations

$$\omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{16.20}$$

and ε_{ijk} is the alternating tensor, also known as permutation symbol, and is defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ represents an even permutation of } 123 \\ 0 & \text{if any two of the } ijk \text{ indices are equal} \\ -1 & \text{if } ijk \text{ represents an odd permutation of } 123 \end{cases} \tag{16.21}$$

Compatibility equations
The compatibility equations are from Eq. (2.18)

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0 \tag{16.22}$$

The compatibility equations (16.22) can be expressed in terms of the components of stress

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} + \alpha E\left(\frac{1}{1-\nu}\nabla^2\tau\delta_{ij} + \frac{1}{1+\nu}\tau_{,ij}\right) = -\left(\frac{\nu}{1-\nu}F_{k,k}\delta_{ij} + F_{i,j} + F_{j,i}\right) \tag{16.23}$$

These equations are called the Beltrami-Michell compatibility equations for thermoelasticity.

Boundary conditions
The boundary conditions have been explained in Chap. 3. The three kinds of boundary conditions are

(1) Traction boundary condition

$$\sigma_{ji}n_j = p_{ni} \quad (i = 1, 2, 3) \tag{16.24}$$

where p_{ni} denote the prescribed surface tractions, and n_j denote the direction cosines between the external normal and each axis.

(2) Displacement boundary condition

$$u_i = \overline{u}_i \quad (i = 1, 2, 3) \tag{16.25}$$

where the displacements with overbar denote the prescribed boundary displacements.

(3) Mixed boundary condition

$$\begin{aligned} \sigma_{ji}n_j &= p_{ni} \quad (i = 1, 2, 3) \text{ on the part of the boundary } B_1 \\ u_i &= \bar{u}_i \quad (i = 1, 2, 3) \text{ on the part of the boundary } B_2 \end{aligned} \tag{16.26}$$

16.1.3 General Solution of Navier's Equations

When the body forces are absent, the boundary-value problem for thermoelasticity may be written in the form

$$\mu\nabla^2 u_i + (\lambda + \mu)u_{k,ki} = \beta\tau_{,i} \quad \text{in the body } D \tag{16.27}$$

$$\sigma_{ji}n_j = p_{ni} \quad \text{on the part of the boundary } B_1 \tag{16.28}$$

$$u_i = \overline{u}_i \quad \text{on the rest of the boundary } B_2 \tag{16.29}$$

The general solution of Eq. (16.27) may be expressed as the sum of a complementary solution u_i^c and a particular solution u_i^p.

$$u_i = u_i^c + u_i^p \tag{16.30}$$

The particular solution u_i^p may be expressed in terms of a scalar potential function as follows

$$u_i^p = \Phi_{,i} \tag{16.31}$$

where Φ is called Goodier's thermoelastic potential, and Φ should satisfy the equation

$$\nabla^2\Phi = K\tau \tag{16.32}$$

where

$$K = \frac{\beta}{\lambda + 2\mu} = \frac{1+\nu}{1-\nu}\alpha \tag{16.33}$$

Goodier's thermoelastic potential for transient thermoelasticity can be calculated from

$$\Phi = \kappa K \int_{t_r}^{t} \tau\, dt + \Phi_r + (t - t_r)\Phi_0 \tag{16.34}$$

where t_r denotes the reference time, and Φ_r and Φ_0 denote solutions of the following Poisson's equation and Laplace's equation, respectively:

$$\nabla^2\Phi_r = K\tau_r \tag{16.35}$$

$$\nabla^2 \Phi_0 = 0 \tag{16.36}$$

where τ_r denotes the temperature change at the reference time.

The complementary solutions u_i^c for Navier's equations (16.27) are discussed in Chap. 6. The Boussinesq solution is

$$\boldsymbol{u}^c = \operatorname{grad} \varphi + 2 \operatorname{curl} [0, 0, \vartheta] + \operatorname{grad} (z\psi) - 4(1-\nu)[0, 0, \psi] \tag{16.37}$$

where φ, ϑ, and ψ are harmonic functions.

Hence, a typical example of the general solution of Navier's equations (16.27) is given as

$$\boldsymbol{u} = \operatorname{grad} \Phi + \operatorname{grad} \varphi + 2 \operatorname{curl} [0, 0, \vartheta] + \operatorname{grad} (z\psi) - 4(1-\nu)[0, 0, \psi] \tag{16.38}$$

where

$$\nabla^2 \Phi = K\tau, \quad \nabla^2 \varphi = 0, \quad \nabla^2 \vartheta = 0, \quad \nabla^2 \psi = 0 \tag{16.39}$$

16.1.4 Thermal Stresses in a Cylindrical Coordinate System

The equilibrium equations in a cylindrical coordinate system (r, θ, z) are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0 \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + F_\theta = 0 \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + F_z = 0
\end{aligned} \tag{16.40}$$

The components of stress in a cylindrical coordinate system expressed in terms of those of a Cartesian coordinate system are

$$\begin{aligned}
\sigma_{rr} &= \sigma_{xx} \cos^2 \theta + \sigma_{yy} \sin^2 \theta + \sigma_{xy} \sin 2\theta \\
\sigma_{\theta\theta} &= \sigma_{xx} \sin^2 \theta + \sigma_{yy} \cos^2 \theta - \sigma_{xy} \sin 2\theta \\
\sigma_{zz} &= \sigma_{zz} \\
\sigma_{r\theta} &= -\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\theta + \sigma_{xy} \cos 2\theta \\
\sigma_{\theta z} &= \sigma_{yz} \cos \theta - \sigma_{zx} \sin \theta \\
\sigma_{zr} &= \sigma_{zx} \cos \theta + \sigma_{yz} \sin \theta
\end{aligned} \tag{16.41}$$

The components of strain and dilatation e in a cylindrical coordinate system in terms of the components of displacement are

$$
\begin{aligned}
&\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z} \\
&\epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \epsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
&\epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
&e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}
\end{aligned}
\tag{16.42}
$$

where u_r, u_θ, u_z are the components of displacement in the r, θ, z directions, respectively.

The compatibility equations of strain in a cylindrical coordinate system are

$$
\begin{aligned}
&2\frac{\partial^2 (r\epsilon_{r\theta})}{\partial r \partial \theta} = \frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} + r\frac{\partial^2 (r\epsilon_{\theta\theta})}{\partial r^2} - r\frac{\partial \epsilon_{rr}}{\partial r} \\
&2\frac{\partial^2 \epsilon_{zr}}{\partial r \partial z} = \frac{\partial^2 \epsilon_{zz}}{\partial r^2} + \frac{\partial^2 \epsilon_{rr}}{\partial z^2} \\
&2\frac{\partial^2 (r\epsilon_{\theta z})}{\partial \theta \partial z} = r^2\frac{\partial^2 \epsilon_{\theta\theta}}{\partial z^2} - 2r\frac{\partial \epsilon_{zr}}{\partial z} + r\frac{\partial \epsilon_{zz}}{\partial r} + \frac{\partial^2 \epsilon_{zz}}{\partial \theta^2} \\
&\frac{\partial}{\partial z}\left(-\frac{\partial \epsilon_{r\theta}}{\partial z} + \frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial \theta} + \frac{\partial \epsilon_{\theta z}}{\partial r}\right) = \frac{\partial^2}{\partial r \partial \theta}\left(\frac{\epsilon_{zz}}{r}\right) + \frac{1}{r}\frac{\partial \epsilon_{\theta z}}{\partial z} \\
&\frac{\partial}{\partial \theta}\left(\frac{\partial \epsilon_{r\theta}}{\partial z} - \frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial \theta} + \frac{\partial \epsilon_{\theta z}}{\partial r}\right) = \frac{\partial^2 (r\epsilon_{\theta\theta})}{\partial r \partial z} - \frac{\partial \epsilon_{rr}}{\partial z} - \frac{1}{r}\frac{\partial \epsilon_{\theta z}}{\partial \theta} \\
&\frac{\partial}{\partial r}\left(\frac{\partial \epsilon_{r\theta}}{\partial z} + \frac{1}{r}\frac{\partial \epsilon_{zr}}{\partial \theta} - \frac{\partial \epsilon_{\theta z}}{\partial r}\right) = \frac{1}{r}\frac{\partial^2 \epsilon_{rr}}{\partial \theta \partial z} - \frac{2}{r}\frac{\partial \epsilon_{r\theta}}{\partial z} + \frac{\partial}{\partial r}\left(\frac{\epsilon_{\theta z}}{r}\right)
\end{aligned}
\tag{16.43}
$$

The coordinate transformations of strain components between a cylindrical coordinate system and a Cartesian coordinate system are

$$
\begin{aligned}
&\epsilon_{rr} = \epsilon_{xx}\cos^2\theta + \epsilon_{yy}\sin^2\theta + \epsilon_{xy}\sin 2\theta \\
&\epsilon_{\theta\theta} = \epsilon_{xx}\sin^2\theta + \epsilon_{yy}\cos^2\theta - \epsilon_{xy}\sin 2\theta \\
&\epsilon_{zz} = \epsilon_{zz} \\
&\epsilon_{r\theta} = -\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\sin 2\theta + \epsilon_{xy}\cos 2\theta \\
&\epsilon_{\theta z} = \epsilon_{yz}\cos\theta - \epsilon_{zx}\sin\theta \\
&\epsilon_{zr} = \epsilon_{zx}\cos\theta + \epsilon_{yz}\sin\theta
\end{aligned}
\tag{16.44}
$$

The constitutive equations, or the generalized Hooke's law, for a homogeneous, isotropic body in a cylindrical coordinate system are

$$\begin{aligned}
\epsilon_{rr} &= \frac{1}{E}[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{rr} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\
\epsilon_{\theta\theta} &= \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{zz} + \sigma_{rr})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{\theta\theta} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\
\epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{zz} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\
\epsilon_{r\theta} &= \frac{\sigma_{r\theta}}{2G}, \quad \epsilon_{\theta z} = \frac{\sigma_{\theta z}}{2G}, \quad \epsilon_{zr} = \frac{\sigma_{zr}}{2G}
\end{aligned} \tag{16.45}$$

An alternative form

$$\begin{aligned}
\sigma_{rr} &= 2\mu\epsilon_{rr} + \lambda e - \beta\tau, \qquad \sigma_{r\theta} = 2\mu\epsilon_{r\theta} \\
\sigma_{\theta\theta} &= 2\mu\epsilon_{\theta\theta} + \lambda e - \beta\tau, \quad \sigma_{\theta z} = 2\mu\epsilon_{\theta z} \\
\sigma_{zz} &= 2\mu\epsilon_{zz} + \lambda e - \beta\tau, \quad \sigma_{zr} = 2\mu\epsilon_{zr}
\end{aligned} \tag{16.46}$$

where $\Theta = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz}$ and $e = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}$.

Navier's equations (16.17) for thermoelasticity can be expressed in a cylindrical coordinate system as

$$\begin{aligned}
&(\lambda + 2\mu)\frac{\partial e}{\partial r} - 2\mu\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) - \beta\frac{\partial \tau}{\partial r} + F_r = 0 \\
&(\lambda + 2\mu)\frac{1}{r}\frac{\partial e}{\partial \theta} - 2\mu\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) - \beta\frac{1}{r}\frac{\partial \tau}{\partial \theta} + F_\theta = 0 \\
&(\lambda + 2\mu)\frac{\partial e}{\partial z} - \frac{2\mu}{r}\left[\frac{\partial (r\omega_\theta)}{\partial r} - \frac{\partial \omega_r}{\partial \theta}\right] - \beta\frac{\partial \tau}{\partial z} + F_z = 0
\end{aligned} \tag{16.47}$$

where

$$\begin{aligned}
\omega_r &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right), \quad \omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) \\
\omega_z &= \frac{1}{2r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)
\end{aligned} \tag{16.48}$$

The solution of Navier's equations (16.47) without the body force can be expressed, for example, by the thermoelastic potential Φ and the Boussinesq harmonic functions as follows:

$$u_r = \frac{\partial \Phi}{\partial r} + \frac{\partial \varphi}{\partial r} + \frac{2}{r}\frac{\partial \vartheta}{\partial \theta} + z\frac{\partial \psi}{\partial r}$$

$$
\begin{aligned}
u_\theta &= \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta} - 2\frac{\partial \vartheta}{\partial r} + \frac{z}{r}\frac{\partial \psi}{\partial \theta} \\
u_z &= \frac{\partial \Phi}{\partial z} + \frac{\partial \varphi}{\partial z} + z\frac{\partial \psi}{\partial z} - (3-4\nu)\psi
\end{aligned} \tag{16.49}
$$

where the four functions Φ, φ, ϑ, and ψ must satisfy

$$
\nabla^2 \Phi = K\tau, \quad \nabla^2 \varphi = 0, \quad \nabla^2 \vartheta = 0, \quad \nabla^2 \psi = 0 \tag{16.50}
$$

and

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{16.51}
$$

16.1.5 Thermal Stresses in a Spherical Coordinate System

The equilibrium equations in a spherical coordinate system (r, θ, ϕ) are

$$
\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\theta r}\cot\theta) + F_r = 0 \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{r}\Big[(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}\Big] + F_\theta = 0 \\
&\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) + F_\phi = 0
\end{aligned} \tag{16.52}
$$

The components of stress in a spherical coordinate system expressed in terms of those of a Cartesian coordinate system are

$$
\begin{aligned}
\sigma_{rr} &= \sigma_{xx}\sin^2\theta\cos^2\phi + \sigma_{yy}\sin^2\theta\sin^2\phi + \sigma_{zz}\cos^2\theta \\
&\quad + \sigma_{xy}\sin^2\theta\sin 2\phi + \sigma_{yz}\sin 2\theta\sin\phi + \sigma_{zx}\sin 2\theta\cos\phi \\
\sigma_{\theta\theta} &= \sigma_{xx}\cos^2\theta\cos^2\phi + \sigma_{yy}\cos^2\theta\sin^2\phi + \sigma_{zz}\sin^2\theta \\
&\quad + \sigma_{xy}\cos^2\theta\sin 2\phi - \sigma_{yz}\sin 2\theta\sin\phi - \sigma_{zx}\sin 2\theta\cos\phi \\
\sigma_{\phi\phi} &= \sigma_{xx}\sin^2\phi + \sigma_{yy}\cos^2\phi - \sigma_{xy}\sin 2\phi \\
\sigma_{r\theta} &= \frac{1}{2}\sin 2\theta\,(\sigma_{xx}\cos^2\phi + \sigma_{yy}\sin^2\phi - \sigma_{zz}) \\
&\quad + \frac{1}{2}\sigma_{xy}\sin 2\theta\sin 2\phi + \sigma_{yz}\cos 2\theta\sin\phi + \sigma_{zx}\cos 2\theta\cos\phi \\
\sigma_{\theta\phi} &= -\frac{1}{2}\cos\theta\sin 2\phi\,(\sigma_{xx} - \sigma_{yy}) \\
&\quad + \sigma_{xy}\cos\theta\cos 2\phi - \sigma_{yz}\sin\theta\cos\phi + \sigma_{zx}\sin\theta\sin\phi \\
\sigma_{\phi r} &= -\frac{1}{2}\sin\theta\sin 2\phi\,(\sigma_{xx} - \sigma_{yy})
\end{aligned} \tag{16.53}
$$

$$+\ \sigma_{xy}\sin\theta\cos 2\phi + \sigma_{yz}\cos\theta\cos\phi - \sigma_{zx}\cos\theta\sin\phi$$

The components of strain and dilatation e in terms of the components of displacement are

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \\
\epsilon_{\phi\phi} &= \frac{u_r}{r} + \cot\theta\frac{u_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}, \quad \epsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\epsilon_{\theta\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \cot\theta\frac{u_\phi}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right) \\
\epsilon_{\phi r} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\
e &= \frac{\partial u_r}{\partial r} + 2\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \cot\theta\frac{u_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}
\end{aligned} \tag{16.54}$$

where u_r, u_θ, u_ϕ are the components of displacement in the r, θ, ϕ directions, respectively.

The compatibility equations of strain in a spherical coordinate system are

$$\begin{aligned}
&-\frac{\epsilon_{\theta\theta,\phi\phi}}{r^2\sin^2\theta} + \frac{\epsilon_{\theta\theta,\theta}}{r^2\tan\theta} - \frac{\epsilon_{\theta\theta,r}}{r} + \frac{2\epsilon_{r\theta,\theta}}{r^2} + \frac{2\epsilon_{r\theta}}{r^2\tan\theta} - \frac{2\epsilon_{\theta\theta}}{r^2} - \frac{\epsilon_{\phi\phi,\theta\theta}}{r^2} \\
&\quad - \frac{\epsilon_{\phi\phi,r}}{r} - \frac{2\epsilon_{\phi\phi,\theta}}{r^2\tan\theta} + \frac{\epsilon_{\phi r}}{r^2\sin\theta} + \frac{2\cos\theta\,\epsilon_{\theta\phi,\phi}}{r^2\sin^2\theta} + \frac{2\epsilon_{\theta\phi,\theta\phi}}{r^2\sin\theta} + \frac{2\epsilon_{rr}}{r^2} = 0 \\
&\frac{2\epsilon_{\phi r,\phi r}}{r\sin\theta} + \frac{\epsilon_{\phi r,\phi}}{r^2\sin\theta} - \frac{\epsilon_{rr,\phi\phi}}{r^2\sin^2\theta} - \frac{2\epsilon_{\phi\phi,r}}{r} - \epsilon_{\phi\phi,rr} \\
&\quad + \frac{2\epsilon_{r\theta,r}}{r\tan\theta} + \frac{2\epsilon_{r\theta}}{r^2\tan\theta} - \frac{\epsilon_{rr,\theta}}{r^2\tan\theta} + \frac{\epsilon_{rr,r}}{r} = 0 \\
&\frac{2\epsilon_{r\theta,r\theta}}{r} - \frac{2\epsilon_{\theta\theta,r}}{r} - \epsilon_{\theta\theta,rr} - \frac{\epsilon_{rr,r}}{r} - \frac{\epsilon_{rr,\theta\theta}}{r^2} + \frac{2\epsilon_{r\theta,\theta}}{r^2} = 0 \\
&\frac{\epsilon_{rr,\theta\phi}}{r^2\sin\theta} - \frac{\epsilon_{r\theta,\phi r}}{r\sin\theta} - \frac{\epsilon_{r\theta,\phi}}{r^2\sin\theta} + \frac{\epsilon_{\phi r}}{r^2\tan\theta} - \frac{\epsilon_{\phi r,\theta}}{r^2} - \frac{\epsilon_{\phi r,r\theta}}{r} \\
&\quad + \epsilon_{\theta\phi,rr} + \frac{2\epsilon_{\theta\phi,r}}{r} + \frac{\epsilon_{\phi r,r}}{r\tan\theta} - \frac{\cos\theta\,\epsilon_{rr,\phi}}{r^2\sin^2\theta} = 0 \\
&\frac{\epsilon_{\theta\theta,\phi r}}{r\sin\theta} - \frac{\epsilon_{\theta\phi,r\theta}}{r} - \frac{2\epsilon_{\theta\phi,r}}{r\tan\theta} + \frac{\cos\theta\epsilon_{r\theta,\phi}}{r^2\sin^2\theta} + \frac{\epsilon_{\phi r,\theta}}{r^2\tan\theta} \\
&\quad - \frac{\epsilon_{r\theta,\theta\phi}}{r^2\sin\theta} + \frac{\epsilon_{\phi r,\theta\theta}}{r^2} - \frac{\cos 2\theta\,\epsilon_{\phi r}}{r^2\sin^2\theta} - \frac{\epsilon_{rr,\phi}}{r^2\sin\theta} = 0 \\
&\frac{\epsilon_{\phi\phi,r\theta}}{r} + \frac{\epsilon_{\phi\phi,r}}{r\tan\theta} - \frac{\epsilon_{\phi r,\theta\phi}}{r^2\sin\theta} + \frac{2\epsilon_{r\theta}}{r^2} - \frac{\epsilon_{rr,\theta}}{r^2} \\
&\quad - \frac{\epsilon_{\theta\phi,\phi r}}{r\sin\theta} + \frac{\epsilon_{r\theta,\phi\phi}}{r^2\sin^2\theta} - \frac{\cos\theta\,\epsilon_{\phi r,\phi}}{r^2\sin^2\theta} - \frac{\epsilon_{\theta\theta,r}}{r\tan\theta} = 0
\end{aligned} \tag{16.55}$$

The coordinate transformations of the displacement between a spherical coordinate system and a Cartesian coordinate system are

$$\begin{aligned} u_x &= u_r \sin\theta\cos\phi + u_\theta\cos\theta\cos\phi - u_\phi\sin\phi \\ u_y &= u_r \sin\theta\sin\phi + u_\theta\cos\theta\sin\phi + u_\phi\cos\phi \\ u_z &= u_r\cos\theta - u_\theta\sin\theta \end{aligned} \tag{16.56}$$

The coordinate transformations of the strain components between a spherical coordinate system and a Cartesian coordinate system are

$$\begin{aligned} \epsilon_{rr} &= \epsilon_{xx}\sin^2\theta\cos^2\phi + \epsilon_{yy}\sin^2\theta\sin^2\phi + \epsilon_{zz}\cos^2\theta \\ &\quad + \epsilon_{xy}\sin^2\theta\sin 2\phi + \epsilon_{yz}\sin 2\theta\sin\phi + \epsilon_{zx}\sin 2\theta\cos\phi \\ \epsilon_{\theta\theta} &= \epsilon_{xx}\cos^2\theta\cos^2\phi + \epsilon_{yy}\cos^2\theta\sin^2\phi + \epsilon_{zz}\sin^2\theta \\ &\quad + \epsilon_{xy}\cos^2\theta\sin 2\phi - \epsilon_{yz}\sin 2\theta\sin\phi - \epsilon_{zx}\sin 2\theta\cos\phi \\ \epsilon_{\phi\phi} &= \epsilon_{xx}\sin^2\phi + \epsilon_{yy}\cos^2\phi - \epsilon_{xy}\sin 2\phi \\ \epsilon_{r\theta} &= \frac{1}{2}\sin 2\theta\,(\epsilon_{xx}\cos^2\phi + \epsilon_{yy}\sin^2\phi - \epsilon_{zz}) \\ &\quad + \frac{1}{2}\epsilon_{xy}\sin 2\theta\sin 2\phi + \epsilon_{yz}\cos 2\theta\sin\phi + \epsilon_{zx}\cos 2\theta\cos\phi \\ \epsilon_{\theta\phi} &= -\frac{1}{2}\cos\theta\sin 2\phi\,(\epsilon_{xx} - \epsilon_{yy}) \\ &\quad + \epsilon_{xy}\cos\theta\sin 2\phi - \epsilon_{yz}\sin\theta\cos\phi + \epsilon_{zx}\sin\theta\sin\phi \\ \epsilon_{\phi r} &= -\frac{1}{2}\sin\theta\sin 2\phi\,(\epsilon_{xx} - \epsilon_{yy}) \\ &\quad + \epsilon_{xy}\sin\theta\cos 2\phi + \epsilon_{yz}\cos\theta\cos\phi - \epsilon_{zx}\cos\theta\sin\phi \end{aligned} \tag{16.57}$$

The constitutive equations for a homogeneous, isotropic body in a spherical coordinate system are

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{E}[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{\phi\phi})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{rr} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\ \epsilon_{\theta\theta} &= \frac{1}{E}[\sigma_{\theta\theta} - \nu(\sigma_{\phi\phi} + \sigma_{rr})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{\theta\theta} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\ \epsilon_{\phi\phi} &= \frac{1}{E}[\sigma_{\phi\phi} - \nu(\sigma_{rr} + \sigma_{\theta\theta})] + \alpha\tau = \frac{1}{2G}\left(\sigma_{\phi\phi} - \frac{\nu}{1+\nu}\Theta\right) + \alpha\tau \\ \epsilon_{r\theta} &= \frac{\sigma_{r\theta}}{2G}, \quad \epsilon_{\theta\phi} = \frac{\sigma_{\theta\phi}}{2G}, \quad \epsilon_{\phi r} = \frac{\sigma_{\phi r}}{2G} \end{aligned} \tag{16.58}$$

An alternative form

$$\begin{aligned}\sigma_{rr} &= 2\mu\epsilon_{rr} + \lambda e - \beta\tau, \quad \sigma_{r\theta} = 2\mu\epsilon_{r\theta}\\ \sigma_{\theta\theta} &= 2\mu\epsilon_{\theta\theta} + \lambda e - \beta\tau, \quad \sigma_{\theta\phi} = 2\mu\epsilon_{\theta\phi}\\ \sigma_{\phi\phi} &= 2\mu\epsilon_{\phi\phi} + \lambda e - \beta\tau, \quad \sigma_{\phi r} = 2\mu\epsilon_{\phi r}\end{aligned} \tag{16.59}$$

where $\Theta = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{\phi\phi}$ and $e = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}$.

Navier's equations (16.17) of thermoelasticity can be expressed in a spherical coordinate system as

$$\begin{aligned}&(\lambda+2\mu)\frac{\partial e}{\partial r} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial(\omega_\phi \sin\theta)}{\partial\theta} - \frac{\partial\omega_\theta}{\partial\phi}\right] - \beta\frac{\partial\tau}{\partial r} + F_r = 0\\ &(\lambda+2\mu)\frac{1}{r}\frac{\partial e}{\partial\theta} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial\omega_r}{\partial\phi} - \sin\theta\frac{\partial(r\omega_\phi)}{\partial r}\right] - \beta\frac{1}{r}\frac{\partial\tau}{\partial\theta} + F_\theta = 0\\ &(\lambda+2\mu)\frac{1}{r\sin\theta}\frac{\partial e}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial(r\omega_\theta)}{\partial r} - \frac{\partial\omega_r}{\partial\theta}\right] - \beta\frac{1}{r\sin\theta}\frac{\partial\tau}{\partial\phi} + F_\phi = 0\end{aligned} \tag{16.60}$$

where

$$\begin{aligned}\omega_r &= \frac{1}{2r\sin\theta}\left[\frac{\partial(u_\phi\sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\phi}\right], \quad \omega_\theta = \frac{1}{2r\sin\theta}\left[\frac{\partial u_r}{\partial\phi} - \sin\theta\frac{\partial(ru_\phi)}{\partial r}\right]\\ \omega_\phi &= \frac{1}{2r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right]\end{aligned} \tag{16.61}$$

The solution of Navier's equations (16.60) without the body force in a spherical coordinate system can be expressed, for example, by the thermoelastic potential Φ and the Boussinesq harmonic functions ϕ, ϑ, ψ:

$$\begin{aligned}u_r &= \frac{\partial\Phi}{\partial r} + \frac{\partial\varphi}{\partial r} + \frac{2}{r}\frac{\partial\vartheta}{\partial\phi} + r\cos\theta\frac{\partial\psi}{\partial r} - (3-4\nu)\psi\cos\theta\\ u_\theta &= \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \frac{2}{r\tan\theta}\frac{\partial\vartheta}{\partial\phi} + \cos\theta\frac{\partial\psi}{\partial\theta} + (3-4\nu)\psi\sin\theta\\ u_\phi &= \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} - 2\sin\theta\frac{\partial\vartheta}{\partial r} - 2\frac{\cos\theta}{r}\frac{\partial\vartheta}{\partial\theta}\\ &\quad + \frac{1}{\tan\theta}\frac{\partial\psi}{\partial\phi}\end{aligned} \tag{16.62}$$

where the four functions must satisfy

$$\nabla^2\Phi = K\tau, \quad \nabla^2\varphi = 0, \quad \nabla^2\vartheta = 0, \quad \nabla^2\psi = 0 \tag{16.63}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \tan\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{16.64}$$

16.2 Problems and Solutions Related to Basic Equations of Thermoelasticity

Problem 16.1. When the elastic body moves under the mechanical and thermal loads, find the equations of motion.

Solution. The motion of the element in the x direction is

$$\begin{aligned}&\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dydz - \sigma_{xx}dydz + \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy\right)dzdx - \sigma_{yx}dzdx \\ &+ \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}dz\right)dxdy - \sigma_{zx}dxdy + F_x dxdydz = \rho\frac{\partial^2 u_x}{\partial t^2}dxdydz\end{aligned} \tag{16.65}$$

where ρ means the density. Simplification of Eq. (16.65) gives

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + F_x = \rho\frac{\partial^2 u_x}{\partial t^2} \tag{16.66}$$

The two other equations of motion in the y and z directions can be obtained in the same way. Then, we get the equations of motion

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \qquad (i = 1, 2, 3) \qquad \text{(Answer)} \tag{16.67}$$

where u_i denote the components of displacement and the dot denotes partial differentiation with respect to the time.

Problem 16.2. Derive the compatibility equations (16.22).

Solution. Using the definition of strain

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i} \tag{16.68}$$

we get

$$\begin{aligned}2\varepsilon_{ij,kl} &= u_{i,jkl} + u_{j,ikl}, \quad 2\varepsilon_{kl,ij} = u_{k,lij} + u_{l,kij} \\ 2\varepsilon_{ik,jl} &= u_{i,kjl} + u_{k,ijl}, \quad 2\varepsilon_{jl,ik} = u_{j,lik} + u_{l,jik}\end{aligned} \tag{16.69}$$

Therefore,

$$\begin{aligned}
&\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} \\
&\quad = \frac{1}{2}(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij} \\
&\quad - u_{i,kjl} - u_{k,ijl} - u_{j,lik} - u_{l,jik}) = 0
\end{aligned} \qquad \text{(Answer)}$$

Problem 16.3. Show that the strain components can be transformed by the following equations

$$\begin{aligned}
\epsilon_{x'x'} &= \epsilon_{xx} l_1^2 + \epsilon_{yy} m_1^2 + \epsilon_{zz} n_1^2 + 2(\epsilon_{xy} l_1 m_1 + \epsilon_{yz} m_1 n_1 + \epsilon_{zx} n_1 l_1) \\
\epsilon_{y'y'} &= \epsilon_{xx} l_2^2 + \epsilon_{yy} m_2^2 + \epsilon_{zz} n_2^2 + 2(\epsilon_{xy} l_2 m_2 + \epsilon_{yz} m_2 n_2 + \epsilon_{zx} n_2 l_2) \\
\epsilon_{z'z'} &= \epsilon_{xx} l_3^2 + \epsilon_{yy} m_3^2 + \epsilon_{zz} n_3^2 + 2(\epsilon_{xy} l_3 m_3 + \epsilon_{yz} m_3 n_3 + \epsilon_{zx} n_3 l_3) \\
\epsilon_{x'y'} &= \epsilon_{xx} l_1 l_2 + \epsilon_{yy} m_1 m_2 + \epsilon_{zz} n_1 n_2 + \epsilon_{xy}(l_1 m_2 + l_2 m_1) \\
&\quad + \epsilon_{yz}(m_1 n_2 + m_2 n_1) + \epsilon_{zx}(n_1 l_2 + n_2 l_1) \\
\epsilon_{y'z'} &= \epsilon_{xx} l_2 l_3 + \epsilon_{yy} m_2 m_3 + \epsilon_{zz} n_2 n_3 + \epsilon_{xy}(l_2 m_3 + l_3 m_2) \\
&\quad + \epsilon_{yz}(m_2 n_3 + m_3 n_2) + \epsilon_{zx}(n_2 l_3 + n_3 l_2) \\
\epsilon_{z'x'} &= \epsilon_{xx} l_3 l_1 + \epsilon_{yy} m_3 m_1 + \epsilon_{zz} n_3 n_1 + \epsilon_{xy}(l_3 m_1 + l_1 m_3) \\
&\quad + \epsilon_{yz}(m_3 n_1 + m_1 n_3) + \epsilon_{zx}(n_3 l_1 + n_1 l_3)
\end{aligned} \qquad (16.70)$$

Solution. Equations (16.9) are written as

$$\begin{aligned}
\varepsilon_{i'j'} &= l_{i'k} l_{j'l} \varepsilon_{kl} \\
&= l_{i'1} l_{j'1} \varepsilon_{11} + l_{i'1} l_{j'2} \varepsilon_{12} + l_{i'1} l_{j'3} \varepsilon_{13} + l_{i'2} l_{j'1} \varepsilon_{21} + l_{i'2} l_{j'2} \varepsilon_{22} \\
&\quad + l_{i'2} l_{j'3} \varepsilon_{23} + l_{i'3} l_{j'1} \varepsilon_{31} + l_{i'3} l_{j'2} \varepsilon_{32} + l_{i'3} l_{j'3} \varepsilon_{33}
\end{aligned} \qquad (16.71)$$

Using the notation

$$\begin{aligned}
l_{1'1} &= l_1 \quad l_{1'2} = m_1 \quad l_{1'3} = n_1 \\
l_{2'1} &= l_2 \quad l_{2'2} = m_2 \quad l_{2'3} = n_2 \\
l_{3'1} &= l_3 \quad l_{3'2} = m_3 \quad l_{3'3} = n_3
\end{aligned} \qquad (16.72)$$

and rewriting subscript $(1, 2, 3)$ as (x, y, z), we get

$$\begin{aligned}
\varepsilon_{x'x'} &= \varepsilon_{xx} l_1^2 + \varepsilon_{yy} m_1^2 + \varepsilon_{zz} n_1^2 + 2(\varepsilon_{xy} l_1 m_1 + \varepsilon_{yz} m_1 n_1 + \varepsilon_{zx} n_1 l_1) \\
\varepsilon_{y'y'} &= \varepsilon_{xx} l_2^2 + \varepsilon_{yy} m_2^2 + \varepsilon_{zz} n_2^2 + 2(\varepsilon_{xy} l_2 m_2 + \varepsilon_{yz} m_2 n_2 + \varepsilon_{zx} n_2 l_2) \\
\varepsilon_{z'z'} &= \varepsilon_{xx} l_3^2 + \varepsilon_{yy} m_3^2 + \varepsilon_{zz} n_3^2 + 2(\varepsilon_{xy} l_3 m_3 + \varepsilon_{yz} m_3 n_3 + \varepsilon_{zx} n_3 l_3) \\
\varepsilon_{x'y'} &= \varepsilon_{xx} l_1 l_2 + \varepsilon_{yy} m_1 m_2 + \varepsilon_{zz} n_1 n_2 \\
&\quad + \varepsilon_{xy}(l_1 m_2 + l_2 m_1) + \varepsilon_{yz}(m_1 n_2 + m_2 n_1) + \varepsilon_{zx}(n_1 l_2 + n_2 l_1)
\end{aligned}$$

$$\begin{aligned}\varepsilon_{y'z'} &= \varepsilon_{xx} l_2 l_3 + \varepsilon_{yy} m_2 m_3 + \varepsilon_{zz} n_2 n_3 \\ &\quad + \varepsilon_{xy}(l_2 m_3 + l_3 m_2) + \varepsilon_{yz}(m_2 n_3 + m_3 n_2) + \varepsilon_{zx}(n_2 l_3 + n_3 l_2) \\ \varepsilon_{z'x'} &= \varepsilon_{xx} l_3 l_1 + \varepsilon_{yy} m_3 m_1 + \varepsilon_{zz} n_3 n_1 \\ &\quad + \varepsilon_{xy}(l_3 m_1 + l_1 m_3) + \varepsilon_{yz}(m_3 n_1 + m_1 n_3) + \varepsilon_{zx}(n_3 l_1 + n_1 l_3) \qquad \text{(Answer)}\end{aligned}$$

Problem 16.4. Derive Navier's equations of thermoelasticity with motion of the body.

Solution. The equations of motion are given from Eq. (16.67)

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \quad (i = 1, 2, 3) \tag{16.73}$$

Substitution of Eq. (16.11) into Eq. (16.73) gives

$$\begin{aligned}&\sigma_{ji,j} + F_i - \rho \ddot{u}_i \\ &\quad = \mu(u_{i,jj} + u_{j,ji}) + \lambda u_{j,ji} - \beta \tau_{,i} + F_i - \rho \ddot{u}_i \\ &\quad = \mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} - \beta \tau_{,i} + F_i - \rho \ddot{u}_i = 0\end{aligned} \tag{16.74}$$

Then, we get

$$\mu \nabla^2 u_i + (\lambda + \mu) u_{k,ki} - \beta \tau_{,i} + F_i = \rho \ddot{u}_i \quad (i = 1, 2, 3) \qquad \text{(Answer)}$$

Problem 16.5. Derive the Beltrami-Michell compatibility equations (16.23).

Solution. The compatibility equations are from Eq. (16.22)

$$\begin{aligned}\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} &= 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\ \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} &= 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \\ \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} &= 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \\ \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \\ \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \\ \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)\end{aligned} \tag{16.75}$$

The constitutive equations are from Eq. (16.10)

$$\varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\Theta\delta_{ij}\right) + \alpha\tau\delta_{ij} \tag{16.76}$$

Substitution of Eq. (16.76) into the first three equations in Eq. (16.75) gives

$$\begin{aligned}
&\frac{\partial^2\sigma_{xx}}{\partial y^2} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial x^2}\right) + 2G\alpha\left(\frac{\partial^2\tau}{\partial y^2} + \frac{\partial^2\tau}{\partial x^2}\right) = 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}\\
&\frac{\partial^2\sigma_{yy}}{\partial z^2} + \frac{\partial^2\sigma_{zz}}{\partial y^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\Theta}{\partial z^2} + \frac{\partial^2\Theta}{\partial y^2}\right) + 2G\alpha\left(\frac{\partial^2\tau}{\partial z^2} + \frac{\partial^2\tau}{\partial y^2}\right) = 2\frac{\partial^2\sigma_{yz}}{\partial y\partial z}\\
&\frac{\partial^2\sigma_{zz}}{\partial x^2} + \frac{\partial^2\sigma_{xx}}{\partial z^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial z^2}\right) + 2G\alpha\left(\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial z^2}\right) = 2\frac{\partial^2\sigma_{zx}}{\partial z\partial x}
\end{aligned} \tag{16.77}$$

The summation of Eq. (16.77) gives

$$\begin{aligned}
&\nabla^2\Theta - \left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial^2\sigma_{zz}}{\partial z^2}\right) - \frac{2\nu}{1+\nu}\nabla^2\Theta + 4G\alpha\nabla^2\tau\\
&\quad = 2\left(\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yz}}{\partial y\partial z} + \frac{\partial^2\sigma_{zx}}{\partial z\partial x}\right)
\end{aligned} \tag{16.78}$$

Therefore,

$$\begin{aligned}
&\frac{1-\nu}{1+\nu}\nabla^2\Theta + 4G\alpha\nabla^2\tau = \left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{zx}}{\partial z\partial x}\right)\\
&\quad + \left(\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial^2\sigma_{yz}}{\partial y\partial z}\right) + \left(\frac{\partial^2\sigma_{zx}}{\partial z\partial x} + \frac{\partial^2\sigma_{yz}}{\partial y\partial z} + \frac{\partial^2\sigma_{zz}}{\partial z^2}\right)
\end{aligned} \tag{16.79}$$

Taking Eq. (16.1) into consideration, Eq. (16.79) becomes

$$\nabla^2\Theta = -\frac{2E}{1-\nu}\alpha\nabla^2\tau - \frac{1+\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) \tag{16.80}$$

The summation of the second and the third equation in Eq. (16.77) gives

$$\begin{aligned}
&\frac{\partial^2\sigma_{zz}}{\partial y^2} + \frac{\partial^2\sigma_{zz}}{\partial x^2} + \frac{\partial^2(\sigma_{xx}+\sigma_{yy})}{\partial z^2} - \frac{\nu}{1+\nu}\left(\nabla^2\Theta + \frac{\partial^2\Theta}{\partial z^2}\right)\\
&\quad + 2G\alpha\left(\nabla^2\tau + \frac{\partial^2\tau}{\partial z^2}\right) - 2\frac{\partial}{\partial z}\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y}\right)\\
&= \nabla^2\sigma_{zz} + \frac{\partial^2\Theta}{\partial z^2} - 2\frac{\partial}{\partial z}\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right) + 2G\alpha\left(\nabla^2\tau + \frac{\partial^2\tau}{\partial z^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\nu}{1+\nu}\frac{\partial^2 \Theta}{\partial z^2} + \frac{\nu}{1+\nu}\left[\frac{2E}{1-\nu}\alpha\nabla^2\tau + \frac{1+\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\right] \\
&= \nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} + \frac{\alpha E}{1+\nu}\frac{\partial^2\tau}{\partial z^2} + \frac{\alpha E}{1-\nu}\nabla^2\tau \\
&+ \frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + 2\frac{\partial F_z}{\partial z} = 0
\end{aligned} \tag{16.81}
$$

From Eq. (16.81), we get

$$
\begin{aligned}
&\nabla^2\sigma_{zz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} + \alpha E\left(\frac{1}{1+\nu}\frac{\partial^2\tau}{\partial z^2} + \frac{1}{1-\nu}\nabla^2\tau\right) \\
&= -\frac{\nu}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) - 2\frac{\partial F_z}{\partial z}
\end{aligned} \quad \text{(Answer)}
$$

The two other equations can be obtained in the same way. Then, we get the first, second and third equation in Eq. (16.23).

Substitution of Eq. (16.76) into the fourth equation in Eq. (16.75) gives

$$
\begin{aligned}
&2G\left[\frac{\partial^2\varepsilon_{xx}}{\partial y\partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial\varepsilon_{yz}}{\partial x} + \frac{\partial\varepsilon_{zx}}{\partial y} + \frac{\partial\varepsilon_{xy}}{\partial z}\right)\right] \\
&= \frac{\partial^2}{\partial y\partial z}\left(\sigma_{xx} - \frac{\nu}{1+\nu}\Theta + 2G\alpha\tau\right) - \frac{\partial}{\partial x}\left(-\frac{\partial\sigma_{yz}}{\partial x} + \frac{\partial\sigma_{zx}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial z}\right) \\
&= \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} + 2G\alpha\frac{\partial^2\tau}{\partial y\partial z} \\
&\quad - \left[\frac{\partial}{\partial z}\left(\frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{xy}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial\sigma_{zx}}{\partial x}\right) - \frac{\partial^2\sigma_{yz}}{\partial x^2}\right] \\
&= \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} + 2G\alpha\frac{\partial^2\tau}{\partial y\partial z} \\
&\quad + \left[\frac{\partial}{\partial z}\left(\frac{\partial\sigma_{yz}}{\partial z} + F_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial\sigma_{yz}}{\partial y} + F_z\right) + \frac{\partial^2\sigma_{yz}}{\partial x^2}\right] \\
&= \nabla^2\sigma_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} + 2G\alpha\frac{\partial^2\tau}{\partial y\partial z} + \left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right) = 0
\end{aligned} \tag{16.82}
$$

From Eq. (16.82), we get the fifth equation in Eq. (16.23)

$$
\nabla^2\sigma_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} + \frac{\alpha E}{1+\nu}\frac{\partial^2\tau}{\partial y\partial z} = -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right) \quad \text{(Answer)}
$$

The fourth and sixth equations can be obtained in the same way.

Problem 16.6. Derive the equilibrium equations (16.40) in a cylindrical coordinate system.

Solution. The equilibrium equation of the forces in the r direction acting on the element is

$$\left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} dr\right)(r + dr)\, d\theta dz - \sigma_{rr} r\, d\theta dz$$
$$+ \left(\sigma_{zr} + \frac{\partial \sigma_{zr}}{\partial z} dz\right)\left[\pi(r + dr)^2 - \pi r^2\right]\frac{d\theta}{2\pi} - \sigma_{zr}\left[\pi(r + dr)^2 - \pi r^2\right]\frac{d\theta}{2\pi}$$
$$+ \left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial \theta} d\theta\right) drdz \cos\frac{d\theta}{2} - \sigma_{\theta r} drdz \cos\frac{d\theta}{2}$$
$$- \left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} d\theta\right) drdz \sin\frac{d\theta}{2} - \sigma_{\theta\theta} drdz \sin\frac{d\theta}{2}$$
$$+ F_r\left[\pi(r + dr)^2 - \pi r^2\right]\frac{d\theta}{2\pi} dz = 0 \tag{16.83}$$

After dividing Eq. (16.83) by $rdrd\theta dz$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + F_r = 0 \quad \text{(Answer)}$$

The equilibrium equation of the forces in the $\theta + d\theta/2$ direction acting on the element is

$$\left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} d\theta\right) drdz \cos\frac{d\theta}{2} + \left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial \theta} d\theta\right) drdz \sin\frac{d\theta}{2}$$
$$- \sigma_{\theta\theta} drdz \cos\frac{d\theta}{2} + \sigma_{\theta r} drdz \sin\frac{d\theta}{2}$$
$$+ \left(\sigma_{z\theta} + \frac{\partial \sigma_{z\theta}}{\partial z} dz\right)[\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi} - \sigma_{z\theta}[\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi}$$
$$+ \left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial r} dr\right)(r + dr) d\theta dz - \sigma_{r\theta} r d\theta dz$$
$$+ F_\theta\, [\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi} dz = 0 \tag{16.84}$$

Letting $\cos\frac{d\theta}{2} \to 1,\ \sin\frac{d\theta}{2} \to \frac{d\theta}{2}$, we get

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} dr d\theta dz + \sigma_{\theta r} dr d\theta dz + \frac{\partial \sigma_{\theta r}}{\partial \theta} dr d\theta dz \frac{d\theta}{2} + \frac{\partial \sigma_{z\theta}}{\partial z}\left(r + \frac{dr}{2}\right) dr d\theta dz$$
$$+ \sigma_{r\theta} dr d\theta dz + \frac{\partial \sigma_{r\theta}}{\partial r}(r + dr) dr d\theta dz + F_\theta \left(r + \frac{dr}{2}\right) dr d\theta dz$$
$$= \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + F_\theta\right) r dr d\theta dz$$
$$+ \left(\frac{\partial \sigma_{\theta r}}{\partial \theta}\frac{d\theta}{2} + \frac{\partial \sigma_{z\theta}}{\partial z}\frac{dr}{2} + \frac{\partial \sigma_{r\theta}}{\partial r} dr + F_\theta \frac{dr}{2}\right) dr d\theta dz = 0 \qquad (16.85)$$

After dividing Eq. (16.85) by $rdrd\theta dz$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + F_\theta = 0 \qquad \text{(Answer)}$$

The equilibrium equation of the forces in the z direction acting on the element is

$$\left(\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} dz\right)[\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi} - \sigma_{zz}[\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi}$$
$$+ \left(\sigma_{\theta z} + \frac{\partial \sigma_{z\theta}}{\partial \theta} d\theta\right) dr dz - \sigma_{\theta z} dr dz + \left(\sigma_{rz} + \frac{\partial \sigma_{rz}}{\partial r} dr\right)(r + dr) d\theta dz$$
$$- \sigma_{rz} r d\theta dz + F_z\,[\pi(r + dr)^2 - \pi r^2]\frac{d\theta}{2\pi} dz$$
$$= \frac{\partial \sigma_{zz}}{\partial z}\left(r + \frac{dr}{2}\right) dr d\theta dz + \frac{\partial \sigma_{z\theta}}{\partial \theta} d\theta dr dz + \sigma_{rz} dr d\theta dz$$
$$+ \frac{\partial \sigma_{rz}}{\partial r}(r + dr) dr d\theta dz + F_z\left(r + \frac{dr}{2}\right) dr d\theta dz = 0 \qquad (16.86)$$

After dividing Eq. (16.86) by $rdrd\theta dz$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + F_z = 0 \qquad \text{(Answer)}$$

Problem 16.7. Derive Eq. (16.47) in a cylindrical coordinate system.

Solution. Substitution of Eq. (16.46) into the first equation of Eq. (16.40) gives

$$\frac{\partial}{\partial r}(2\mu\epsilon_{rr} + \lambda e - \beta\tau) + \frac{2\mu}{r}\frac{\partial \epsilon_{\theta r}}{\partial \theta} + 2\mu\frac{\partial \epsilon_{zr}}{\partial z} + 2\mu\frac{\epsilon_{rr} - \epsilon_{\theta\theta}}{r} + F_r = 0 \qquad (16.87)$$

Using the strain-displacement relation (16.42), Eq. (16.87) reduces to

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} + 2\mu\left\{\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) - \frac{1}{r}\frac{1}{2r}\frac{\partial}{\partial \theta}\left[\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right]\right\}$$
$$- \beta\frac{\partial \tau}{\partial r} + F_r = 0 \tag{16.88}$$

Then, we can get

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - 2\mu\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) - \beta\frac{\partial \tau}{\partial r} + F_r = 0 \qquad \text{(Answer)}$$

where

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right), \quad \omega_z = \frac{1}{2r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)$$

We can obtain the second and third equations of Navier's equations (16.47) by the same technique.

Problem 16.8. Derive the solutions of Laplace's equation in a cylindrical coordinate system.

Solution. We consider the solutions of Laplace's equation in a cylindrical coordinate system by use of the method of separation of variables. Laplace's equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0 \tag{16.89}$$

We assume that the harmonic function can be expressed by the product of three unknown functions, each of only one variable

$$\varphi(r, \theta, z) = f(r)g(\theta)h(z) \tag{16.90}$$

Substitution of Eq. (16.90) into Eq. (16.89) gives

$$\frac{1}{f(r)}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)f(r) + \frac{1}{r^2}\frac{1}{g(\theta)}\frac{d^2 g(\theta)}{d\theta^2} + \frac{1}{h(z)}\frac{d^2 h(z)}{dz^2} = 0 \tag{16.91}$$

Equation (16.91) will be satisfied if the functions are selected as

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + \left(a^2 - \frac{b^2}{r^2}\right)f(r) = 0$$
$$\frac{d^2 g(\theta)}{d\theta^2} + b^2 g(\theta) = 0$$
$$\frac{d^2 h(z)}{dz^2} - a^2 h(z) = 0 \tag{16.92}$$

where a and b are constants. The first equation of Eq. (16.92) is called the Bessel's differential equation, and has two independent solutions $f(r) = J_b(ar)$ and $Y_b(ar)$ for $a \neq 0,\ |b| < \infty$. $J_b(ar)$ and $Y_b(ar)$ are the Bessel function of first kind of order b and of second kind of order b, respectively. Similarly, the first equation of Eq. (16.92) has two independent solutions $f(r) = 1$ and ln r for $a = b = 0$, and $f(r) = r^b$ and r^{-b} for $a = 0, b \neq 0$.

The linearly independent solutions of Eq. (16.92) are

$$
\begin{aligned}
&f(r) = \begin{pmatrix} 1 \\ \ln r \end{pmatrix} \quad \text{for} \quad a = b = 0, \quad f(r) = \begin{pmatrix} r^b \\ r^{-b} \end{pmatrix} \quad \text{for} \quad a = 0, \quad b \neq 0 \\
&f(r) = \begin{pmatrix} J_b(ar) \\ Y_b(ar) \end{pmatrix} \quad \text{for} \quad a \neq 0, \quad |b| < \infty \\
&g(\theta) = \begin{pmatrix} 1 \\ \theta \end{pmatrix} \quad \text{for} \quad b = 0, \quad g(\theta) = \begin{pmatrix} \sin b\theta \\ \cos b\theta \end{pmatrix} \quad \text{for} \quad b \neq 0 \\
&h(z) = \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \text{for} \quad a = 0, \quad h(z) = \begin{pmatrix} e^{az} \\ e^{-az} \end{pmatrix} \quad \text{for} \quad a \neq 0
\end{aligned} \tag{16.93}
$$

In another case, Eq. (16.91) will be satisfied if the functions are selected as

$$
\begin{aligned}
&\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} - \left(a^2 + \frac{b^2}{r^2}\right) f(r) = 0 \\
&\frac{d^2 g(\theta)}{d\theta^2} + b^2 g(\theta) = 0 \\
&\frac{d^2 h(z)}{dz^2} + a^2 h(z) = 0
\end{aligned} \tag{16.94}
$$

The first equation of Eq. (16.94) is called the modified Bessel's differential equation, and has two independent solutions $f(r) = I_b(ar)$ and $K_b(ar)$ for $a \neq 0,\ |b| < \infty$. $I_b(ar)$ and $K_b(ar)$ are the Bessel function of first kind of order b and of second kind of order b, respectively.

The linearly independent solutions of Eq. (16.94) are

$$
\begin{aligned}
&f(r) = \begin{pmatrix} 1 \\ \ln r \end{pmatrix} \quad \text{for} \quad a = b = 0, \quad f(r) = \begin{pmatrix} r^b \\ r^{-b} \end{pmatrix} \quad \text{for} \quad a = 0, \quad b \neq 0 \\
&f(r) = \begin{pmatrix} I_b(ar) \\ K_b(ar) \end{pmatrix} \quad \text{for} \quad a \neq 0, \quad |b| < \infty \\
&g(\theta) = \begin{pmatrix} 1 \\ \theta \end{pmatrix} \quad \text{for} \quad b = 0, \quad g(\theta) = \begin{pmatrix} \sin b\theta \\ \cos b\theta \end{pmatrix} \quad \text{for} \quad b \neq 0 \\
&h(z) = \begin{pmatrix} 1 \\ z \end{pmatrix} \quad \text{for} \quad a = 0, \quad h(z) = \begin{pmatrix} \sin az \\ \cos az \end{pmatrix} \quad \text{for} \quad a \neq 0
\end{aligned} \tag{16.95}
$$

Therefore, the particular solutions of Laplace's equation in a cylindrical coordinate system may be expressed as follows:

$$\begin{aligned}
&\begin{pmatrix}1\\ \ln r\end{pmatrix}\begin{pmatrix}1\\ \theta\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}, \quad \begin{pmatrix}r^b\\ r^{-b}\end{pmatrix}\begin{pmatrix}\sin b\theta\\ \cos b\theta\end{pmatrix}\begin{pmatrix}1\\ z\end{pmatrix}\\
&\begin{pmatrix}J_0(ar)\\ Y_0(ar)\end{pmatrix}\begin{pmatrix}1\\ \theta\end{pmatrix}\begin{pmatrix}e^{az}\\ e^{-az}\end{pmatrix}, \quad \begin{pmatrix}I_0(ar)\\ K_0(ar)\end{pmatrix}\begin{pmatrix}1\\ \theta\end{pmatrix}\begin{pmatrix}\sin az\\ \cos az\end{pmatrix}\\
&\begin{pmatrix}J_b(ar)\\ Y_b(ar)\end{pmatrix}\begin{pmatrix}\sin b\theta\\ \cos b\theta\end{pmatrix}\begin{pmatrix}e^{az}\\ e^{-az}\end{pmatrix}\\
&\begin{pmatrix}I_b(ar)\\ K_b(ar)\end{pmatrix}\begin{pmatrix}\sin b\theta\\ \cos b\theta\end{pmatrix}\begin{pmatrix}\sin az\\ \cos az\end{pmatrix} \qquad \text{(Answer)}
\end{aligned} \tag{16.96}$$

where $\begin{pmatrix}\sinh az\\ \cosh az\end{pmatrix}$ can be used instead of $\begin{pmatrix}e^{az}\\ e^{-az}\end{pmatrix}$.

In Eq. (16.96), we used the following notation for the product of three one-column matrices

$$\begin{pmatrix}f_1\\ f_2\end{pmatrix}\begin{pmatrix}g_1\\ g_2\end{pmatrix}\begin{pmatrix}h_1\\ h_2\end{pmatrix} = \begin{pmatrix}f_1g_1h_1\\ f_2g_1h_1\\ f_1g_2h_1\\ f_2g_2h_1\\ f_1g_1h_2\\ f_2g_1h_2\\ f_1g_2h_2\\ f_2g_2h_2\end{pmatrix} \tag{16.97}$$

Then, $\begin{pmatrix}I_b(ar)\\ K_b(ar)\end{pmatrix}\begin{pmatrix}\sin b\theta\\ \cos b\theta\end{pmatrix}\begin{pmatrix}\sin az\\ \cos az\end{pmatrix}$, for example means

$$\begin{pmatrix}I_b(ar)\\ K_b(ar)\end{pmatrix}\begin{pmatrix}\sin b\theta\\ \cos b\theta\end{pmatrix}\begin{pmatrix}\sin az\\ \cos az\end{pmatrix} = \begin{pmatrix}I_b(ar)\sin b\theta\sin az\\ K_b(ar)\sin b\theta\sin az\\ I_b(ar)\cos b\theta\sin az\\ K_b(ar)\cos b\theta\sin az\\ I_b(ar)\sin b\theta\cos az\\ K_b(ar)\sin b\theta\cos az\\ I_b(ar)\cos b\theta\cos az\\ K_b(ar)\cos b\theta\cos az\end{pmatrix} \tag{16.98}$$

Therefore, a product of the three one-column matrices in Eq. (16.96) produces 8 particular solutions of Laplace's equation, and Eq. (16.96) represent an ordered array of 8 × 6 particular harmonic solutions in cylindrical coordinates.

Problem 16.9. Derive the equilibrium equations (16.52) in a spherical coordinate system.

Solution. We consider the infinitesimal element in a spherical coordinate system.
The area of the infinitesimal element of ϕ plane is $rdrd\theta$
The area of the infinitesimal element of $(\phi + d\phi)$ plane is $rdrd\theta$
The area of the infinitesimal element of θ plane is $drr \sin\theta d\phi$
The area of the infinitesimal element of $(\theta + d\theta)$ plane is

$$\begin{aligned} drr \sin(\theta + d\theta)d\phi &= drr \sin\theta \cos d\theta d\phi + drr \cos\theta \sin d\theta d\phi \\ &\cong drr \sin\theta d\phi + drr \cos\theta d\theta d\phi \end{aligned}$$

The area of the infinitesimal element of r plane is $r^2 \sin\theta d\theta d\phi$
The area of the infinitesimal element of $(r + dr)$ plane is

$$(r + dr)^2 d\theta \sin\theta d\phi \cong (r^2 + 2rdr)d\theta \sin\theta d\phi$$

The volume of the infinitesimal element is $r^2 \sin\theta drd\theta d\phi$

The equilibrium equation of the forces in the r direction acting on the element is

$$\begin{aligned} &\left(\sigma_{rr} + \frac{\partial\sigma_{rr}}{\partial r}dr\right)(r^2 + 2rdr)\sin\theta d\theta d\phi - \sigma_{rr}r^2 \sin\theta d\theta d\phi \\ &\quad + \left(\sigma_{\theta r} + \frac{\partial\sigma_{\theta r}}{\partial\theta}d\theta\right)(\sin\theta + \cos\theta d\theta)rdrd\phi \cos\frac{d\theta}{2} - \sigma_{\theta r}\sin\theta rdrd\phi\cos\frac{d\theta}{2} \\ &\quad - \left(\sigma_{\theta\theta} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}d\theta\right)(\sin\theta + \cos\theta d\theta)rdrd\phi \sin\frac{d\theta}{2} - \sigma_{\theta\theta}\sin\theta rdrd\phi\sin\frac{d\theta}{2} \\ &\quad + \left(\sigma_{\phi r} + \frac{\partial\sigma_{\phi r}}{\partial\phi}d\phi\right)rdrd\theta\cos\frac{\sin\theta d\phi}{2} - \sigma_{\phi r}rdrd\theta\cos\frac{\sin\theta d\phi}{2} \\ &\quad - \left(\sigma_{\phi\phi} + \frac{\partial\sigma_{\phi\phi}}{\partial\phi}d\phi\right)rdrd\theta\sin\frac{\sin\theta d\phi}{2} - \sigma_{\phi\phi}rdrd\theta\sin\frac{\sin\theta d\phi}{2} \\ &\quad + F_r r^2 \sin\theta drd\theta d\phi = 0 \end{aligned} \tag{16.99}$$

Letting $\cos\frac{d\theta}{2} \to 1$, $\sin\frac{d\theta}{2} \to \frac{d\theta}{2}$, $\cos\frac{\sin\theta d\phi}{2} \to 1$, $\sin\frac{\sin\theta d\phi}{2} \to \frac{\sin\theta d\phi}{2}$, we get

$$\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r}r^2\sin\theta drd\theta d\phi + 2\sigma_{rr}r\sin\theta drd\theta d\phi + 2\frac{\partial\sigma_{rr}}{\partial r}r\sin\theta drd\theta d\phi dr \\ &\quad + \sigma_{\theta r}r\cos\theta drd\theta d\phi + \frac{\partial\sigma_{\theta r}}{\partial\theta}r\sin\theta drd\theta d\phi + \frac{\partial\sigma_{\theta r}}{\partial\theta}r\cos\theta drd\theta d\phi d\theta \\ &\quad - \sigma_{\theta\theta}r\sin\theta drd\theta d\phi \\ &\quad - \frac{1}{2}\left(\sigma_{\theta\theta}\cos\theta d\theta + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\sin\theta d\theta + \frac{\partial\sigma_{\theta\theta}}{\partial\theta}\cos\theta d\theta d\theta\right)rdrd\theta d\phi \end{aligned}$$

$$+ \frac{\partial \sigma_{\phi r}}{\partial \phi} r dr d\theta d\phi - \sigma_{\phi\phi} r \sin\theta dr d\theta d\phi - \frac{1}{2}\frac{\partial \sigma_{\phi r}}{\partial \phi} r \sin\theta dr d\theta d\phi d\phi$$
$$+ F_r r^2 \sin\theta dr d\theta d\phi = 0 \tag{16.100}$$

After dividing Eq. (16.100) by $r^2 \sin\theta dr d\theta d\phi$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\theta r}\cot\theta) + F_r = 0 \tag{Answer}$$

The equilibrium equation of the forces in the $(\theta + d\theta/2)$ direction acting on the element is

$$\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial r} dr\right)(r^2 + 2r dr)\sin\theta d\theta d\phi - \sigma_{r\theta} r^2 \sin\theta d\theta d\phi$$
$$+ \left(\sigma_{\theta r} + \frac{\partial \sigma_{\theta r}}{\partial \theta} d\theta\right)(\sin\theta + \cos\theta d\theta) r dr d\phi \sin\frac{d\theta}{2} + \sigma_{\theta r}\sin\theta r dr d\phi \sin\frac{d\theta}{2}$$
$$+ \left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} d\theta\right)(\sin\theta + \cos\theta d\theta) r dr d\phi \cos\frac{d\theta}{2} - \sigma_{\theta\theta}\sin\theta r dr d\phi \cos\frac{d\theta}{2}$$
$$+ \left(\sigma_{\phi\theta} + \frac{\partial \sigma_{\phi\theta}}{\partial \phi} d\phi\right) r dr d\theta \cos\frac{\cos\theta d\phi}{2} - \sigma_{\phi\theta} r dr d\theta \cos\frac{\cos\theta d\phi}{2}$$
$$- \left(\sigma_{\phi\phi} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi} d\phi\right) r dr d\theta \sin\frac{\cos\theta d\phi}{2} - \sigma_{\phi\phi} r dr d\theta \sin\frac{\cos\theta d\phi}{2}$$
$$+ F_\theta r^2 \sin\theta dr d\theta d\phi = 0 \tag{16.101}$$

Letting $\cos\frac{d\theta}{2} \to 1$, $\sin\frac{d\theta}{2} \to \frac{d\theta}{2}$, $\cos\frac{\cos\theta d\phi}{2} \to 1$, $\sin\frac{\cos\theta d\phi}{2} \to \frac{\cos\theta d\phi}{2}$, we get

$$2\sigma_{r\theta} r \sin\theta dr d\theta d\phi + \frac{\partial \sigma_{r\theta}}{\partial r} r^2 \sin\theta dr d\theta d\phi + 2\frac{\partial \sigma_{r\theta}}{\partial r} r \sin\theta dr d\theta d\phi dr$$
$$+ \sigma_{\theta r} r \sin\theta dr d\theta d\phi$$
$$+ \frac{1}{2}\left(\sigma_{\theta r}\cot\theta + \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{\theta r}}{\partial \theta}\cot\theta d\theta\right) r \sin\theta dr d\theta d\phi d\theta$$
$$+ \left(\sigma_{\theta\theta}\cot\theta + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\cot\theta d\theta\right) r \sin\theta dr d\theta d\phi$$
$$+ \frac{1}{\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} r \sin\theta dr d\theta d\phi - \left(\sigma_{\phi\phi} + \frac{1}{2}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} d\phi\right)\cot\theta r \sin\theta dr d\theta d\phi$$
$$+ F_\theta r^2 \sin\theta dr d\theta d\phi = 0 \tag{16.102}$$

After dividing Eq. (16.102) by $r^2 \sin\theta dr d\theta d\phi$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\theta} + \sigma_{\theta\theta}\cot\theta - \sigma_{\phi\phi}\cot\theta) + F_\theta = 0$$

The equilibrium equation of the forces in $(\phi + d\phi/2)$ direction acting on the element is

$$\begin{aligned}
&\left(\sigma_{r\phi} + \frac{\partial \sigma_{r\phi}}{\partial r}dr\right)(r^2 + 2rdr)\sin\theta d\theta d\phi - \sigma_{r\phi}r^2\sin\theta d\theta d\phi \\
&\quad + \left(\sigma_{\theta\phi} + \frac{\partial \sigma_{\theta\phi}}{\partial \theta}d\theta\right)(\sin\theta + \cos\theta d\theta)rdrd\phi - \sigma_{\theta\phi}\sin\theta rdrd\phi \\
&\quad + \left(\sigma_{\phi r} + \frac{\partial \sigma_{\phi r}}{\partial \phi}d\phi\right)rdrd\theta\sin\frac{\sin\theta d\phi}{2} + \sigma_{\phi r}rdrd\theta\sin\frac{\sin\theta d\phi}{2} \\
&\quad + \left(\sigma_{\phi\phi} + \frac{\partial \sigma_{\phi\phi}}{\partial \phi}d\phi\right)rdrd\theta\cos\frac{\cos\theta d\phi}{2}\cos\frac{\sin\theta d\phi}{2} \\
&\quad - \sigma_{\phi\phi}rdrd\theta\cos\frac{\cos\theta d\phi}{2}\cos\frac{\sin\theta d\phi}{2} + \left(\sigma_{\phi\theta} + \frac{\partial \sigma_{\phi\theta}}{\partial \phi}d\phi\right)rdrd\theta\sin\frac{\cos\theta d\phi}{2} \\
&\quad + \sigma_{\phi\theta}rdrd\theta\sin\frac{\cos\theta d\phi}{2} + F_\phi r^2\sin\theta drd\theta d\phi = 0
\end{aligned} \tag{16.103}$$

Letting $\cos\frac{\sin\theta d\phi}{2} \to 1$, $\cos\frac{\cos\theta d\phi}{2} \to 1$, $\sin\frac{\sin\theta d\phi}{2} \to \frac{\sin\theta d\phi}{2}$, $\sin\frac{\cos\theta d\phi}{2} \to \frac{\cos\theta d\phi}{2}$, we get

$$\begin{aligned}
&2\sigma_{r\phi}r\sin\theta drd\theta d\phi + \frac{\partial \sigma_{r\phi}}{\partial r}r^2\sin\theta drd\theta d\phi + 2\frac{\partial \sigma_{r\phi}}{\partial r}rdr\sin\theta drd\theta d\phi \\
&\quad + \sigma_{\theta\phi}r\cos\theta drd\theta d\phi + \frac{\partial \sigma_{\theta\phi}}{\partial \theta}r\sin\theta drd\theta d\phi + \frac{\partial \sigma_{\theta\phi}}{\partial \theta}d\theta r\cos\theta drd\theta d\phi \\
&\quad + \left(\sigma_{\phi r} + \frac{1}{2}\frac{\partial \sigma_{\phi r}}{\partial \phi}d\phi\right)r\sin\theta drd\theta d\phi + \frac{\partial \sigma_{\phi\phi}}{\partial \phi}rdrd\theta d\phi \\
&\quad + \sigma_{\phi\theta}r\cos\theta drd\theta d\phi + \frac{1}{2}\frac{\partial \sigma_{\phi\theta}}{\partial \phi}d\phi r\cos\theta drd\theta d\phi \\
&\quad + F_\phi r^2\sin\theta drd\theta d\phi = 0
\end{aligned} \tag{16.104}$$

After dividing Eq. (16.104) by $r^2\sin\theta drd\theta d\phi$ and omitting higher infinitesimal terms, we obtain

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\cot\theta\sigma_{\theta\phi}) + F_\phi = 0 \quad \text{(Answer)}$$

Problem 16.10. Derive Navier's equations (16.60) in a spherical coordinate system.

Solution. Substitution of Hooke's law (16.59) into the first equation of the equilibrium equations (16.52) yields

$$\frac{\partial}{\partial r}(2\mu\epsilon_{rr}+\lambda e-\beta\tau)+\frac{2\mu}{r}\frac{\partial\epsilon_{r\theta}}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial\epsilon_{\phi r}}{\partial\phi}$$
$$+\frac{1}{r}[2(2\mu\epsilon_{rr}+\lambda e-\beta\tau)-(2\mu\epsilon_{\theta\theta}+\lambda e-\beta\tau)$$
$$-(2\mu\epsilon_{\phi\phi}+\lambda e-\beta\tau)+(2\mu\epsilon_{r\theta})\cot\theta]+F_r=0 \tag{16.105}$$

Using Eq. (16.54), Eq. (16.105) reduces to

$$(\lambda+2\mu)\frac{\partial e}{\partial r}-2\mu\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}\right)+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right.$$
$$\left.+\cot\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)-\frac{1}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^2 u_\phi}{\partial r\partial\phi}\right)\Big]$$
$$+\mu\left(\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)$$
$$+\frac{\mu}{r\sin\theta}\left(\frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial\phi^2}+\frac{\partial^2 u_\phi}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}\right)$$
$$+\frac{\mu}{r}\left[4\frac{\partial u_r}{\partial r}-4\frac{u_r}{r}-\frac{2}{r}\frac{\partial u_\theta}{\partial\theta}-3\cot\theta\frac{u_\theta}{r}-\frac{2}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\cot\theta\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right.$$
$$\left.+\cot\theta\frac{\partial u_\theta}{\partial r}\right]-\beta\frac{\partial\tau}{\partial r}+F_r=0 \tag{16.106}$$

From Eq. (16.106), we get

$$(\lambda+2\mu)\frac{\partial e}{\partial r}-\frac{\mu}{r\sin\theta}\left[\frac{\sin\theta}{r}\left(\frac{\partial u_\theta}{\partial\theta}+r\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{\partial^2 u_r}{\partial^2\theta}\right)\right.$$
$$\left.+\frac{\cos\theta}{r}\left(u_\theta+r\frac{\partial u_\theta}{\partial r}-\frac{\partial u_r}{\partial\theta}\right)-\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial^2 u_r}{\partial^2\phi}-\frac{\partial u_\phi}{\partial\phi}-r\frac{\partial^2 u_\phi}{\partial r\partial\phi}\right)\right]$$
$$-\beta\frac{\partial\tau}{\partial r}+F_r=0 \tag{16.107}$$

Taking into the consideration of

$$2\frac{\partial(\omega_\phi\sin\theta)}{\partial\theta}=2\sin\theta\frac{\partial\omega_\phi}{\partial\theta}+2\cos\theta\omega_\phi$$
$$=\frac{\sin\theta}{r}\left(\frac{\partial u_\theta}{\partial\theta}+r\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{\partial^2 u_r}{\partial^2\theta}\right)+\frac{\cos\theta}{r}\left(u_\theta+r\frac{\partial u_\theta}{\partial r}-\frac{\partial u_r}{\partial\theta}\right)$$
$$2\frac{\partial\omega_\theta}{\partial\phi}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial^2 u_r}{\partial^2\phi}-\frac{\partial u_\phi}{\partial\phi}-r\frac{\partial^2 u_\phi}{\partial r\partial\phi}\right) \tag{16.108}$$

equation (16.107) reduces to

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \frac{2\mu}{r\sin\theta}\left[\sin\theta\frac{\partial\omega_\phi}{\partial\theta} + \cos\theta\omega_\phi - \frac{\partial\omega_\theta}{\partial\phi}\right] - \beta\frac{\partial\tau}{\partial r} + F_r = 0 \quad (16.109)$$

We finally obtain

$$(\lambda + 2\mu)\frac{\partial e}{\partial r} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial(\omega_\phi\sin\theta)}{\partial\theta} - \frac{\partial\omega_\theta}{\partial\phi}\right] - \beta\frac{\partial\tau}{\partial r} + F_r = 0 \qquad \text{(Answer)}$$

We can obtain the second and third equations of Navier's equations (16.60) by same technique.

Problem 16.11. Derive the solutions of Laplace's equation in a spherical coordinate system.

Solution. We consider the solution of Laplace's equation in a spherical coordinate system by use of the method of separation of variables. Laplace's equation in a spherical coordinate system is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\varphi = 0 \qquad (16.110)$$

We assume that the harmonic function can be expressed by the product of three unknown functions, each of only one variable

$$\varphi(r,\theta,\phi) = f(r)g(\theta)h(\phi) \qquad (16.111)$$

Substitution of Eq. (16.111) into Eq. (16.110) gives

$$\frac{r^2}{f(r)}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)f(r) + \frac{1}{g(\theta)}\left(\frac{d^2}{d\theta^2} + \frac{1}{\tan\theta}\frac{d}{d\theta}\right)g(\theta)$$
$$+ \frac{1}{\sin^2\theta h(\phi)}\frac{d^2h(\phi)}{d\phi^2} = 0 \qquad (16.112)$$

Equation (16.112) will be satisfied if the functions are selected as follows:

$$\frac{d^2f(r)}{dr^2} + \frac{2}{r}\frac{df(r)}{dr} - \frac{\nu(\nu+1)}{r^2}f(r) = 0$$
$$\frac{d^2g(\theta)}{d\theta^2} + \frac{1}{\tan\theta}\frac{dg(\theta)}{d\theta} + \left[\nu(\nu+1) - \frac{\mu^2}{\sin^2\theta}\right]g(\theta) = 0$$
$$\frac{d^2h(\phi)}{d\phi^2} + \mu^2h(\phi) = 0 \qquad (16.113)$$

The linearly independent solutions of the first and the third equations of Eq. (16.113) are

$$f(r) = \begin{pmatrix} r^{\nu} \\ r^{-(\nu+1)} \end{pmatrix}$$
$$h(\phi) = \begin{pmatrix} 1 \\ \phi \end{pmatrix} \quad \text{for} \quad \mu = 0, \quad h(\phi) = \begin{pmatrix} \sin \mu\phi \\ \cos \mu\phi \end{pmatrix} \quad \text{for} \quad \mu \neq 0 \tag{16.114}$$

Application of the transformation of a variable $x = \cos\theta$ to the second equation in Eq. (16.113) gives

$$(1 - x^2)\frac{d^2 g(x)}{dx^2} - 2x\frac{dg(x)}{dx} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - x^2}\right]g(x) = 0 \tag{16.115}$$

The Eq. (16.115) is called the associated Legendre's differential equation, and the linearly independent solutions are given by

$$g(x) = \begin{pmatrix} P_{\nu}^{\mu}(x) \\ Q_{\nu}^{\mu}(x) \end{pmatrix} \tag{16.116}$$

Therefore, the particular solutions of the harmonic equation in a spherical coordinate system may be expressed as follows:

$$\begin{pmatrix} r^{\nu} \\ r^{-(\nu+1)} \end{pmatrix} \begin{pmatrix} P_{\nu}(\cos\theta) \\ Q_{\nu}(\cos\theta) \end{pmatrix} \begin{pmatrix} 1 \\ \phi \end{pmatrix}$$
$$\begin{pmatrix} r^{\nu} \\ r^{-(\nu+1)} \end{pmatrix} \begin{pmatrix} P_{\nu}^{\mu}(\cos\theta) \\ Q_{\nu}^{\mu}(\cos\theta) \end{pmatrix} \begin{pmatrix} \sin \mu\phi \\ \cos \mu\phi \end{pmatrix} \tag{Answer}$$

where μ and ν are constants, $P_{\nu}(\cos\theta)$ is the Legendre function of the first kind, $Q_{\nu}(\cos\theta)$ is the Legendre function of the second kind, $P_{\nu}^{\mu}(\cos\theta)$ is the associated Legendre function of the first kind, and $Q_{\nu}^{\mu}(\cos\theta)$ is the associated Legendre function of the second kind. The notation for the product of three one-column matrices is explained by Eqs.(16.97) and (16.98).

Problem 16.12. Express the displacements and strains in a spherical coordinate system by use of the thermoelastic potential Φ and the Boussinesq functions φ, ϑ, ψ.

Solution. The relationship between the Cartesian and the spherical coordinate systems is

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta$$
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \sqrt{x^2 + y^2}/z, \quad \tan\phi = \frac{y}{x} \tag{16.117}$$

The direction cosines in both coordinate systems are

$$
\begin{aligned}
&l_1 = \sin\theta\cos\phi, \quad m_1 = \sin\theta\sin\phi, \quad n_1 = \cos\theta \\
&l_2 = \cos\theta\cos\phi, \quad m_2 = \cos\theta\sin\phi, \quad n_2 = -\sin\theta \\
&l_3 = -\sin\phi, \quad m_3 = \cos\phi, \quad n_3 = 0
\end{aligned}
\tag{16.118}
$$

The partial derivatives are

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial y} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}
\end{aligned}
\tag{16.119}
$$

Displacements u_x, u_y, u_z in a Cartesian coordinate system are expressed by the thermoelastic potential Φ and the Boussinesq functions φ, ϑ, ψ from Eq. (16.38)

$$
\begin{aligned}
u_x &= \frac{\partial\Phi}{\partial x} + \frac{\partial\varphi}{\partial x} + 2\frac{\partial\vartheta}{\partial y} + z\frac{\partial\psi}{\partial x} \\
u_y &= \frac{\partial\Phi}{\partial y} + \frac{\partial\varphi}{\partial y} - 2\frac{\partial\vartheta}{\partial x} + z\frac{\partial\psi}{\partial y} \\
u_z &= \frac{\partial\Phi}{\partial z} + \frac{\partial\varphi}{\partial z} + z\frac{\partial\psi}{\partial z} - (3-4\nu)\psi
\end{aligned}
\tag{16.120}
$$

Displacements u_r, u_θ, u_ϕ in a spherical coordinate system are expressed by the displacements u_x, u_y, u_z in a Cartesian coordinate system

$$
\begin{aligned}
u_r &= u_x\sin\theta\cos\phi + u_y\sin\theta\sin\phi + u_z\cos\theta \\
u_\theta &= u_x\cos\theta\cos\phi + u_y\cos\theta\sin\phi - u_z\sin\theta \\
u_\phi &= -u_x\sin\phi + u_y\cos\phi
\end{aligned}
\tag{16.121}
$$

Substituting Eq. (16.120) into Eq. (16.121) and translating the partial differentials from the Cartesian to the spherical coordinate systems, we get

$$
\begin{aligned}
u_r &= u_x\sin\theta\cos\phi + u_y\sin\theta\sin\phi + u_z\cos\theta \\
&= \left[\left(\sin\theta\cos\phi\frac{\partial\Phi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\Phi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right. \\
&\quad + \left(\sin\theta\cos\phi\frac{\partial\varphi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\right) \\
&\quad + 2\left(\sin\theta\sin\phi\frac{\partial\vartheta}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial\vartheta}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\right) \\
&\quad \left. + r\cos\theta\left(\sin\theta\cos\phi\frac{\partial\psi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\psi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\sin\theta\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\left(\sin\theta\sin\phi\frac{\partial\Phi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\Phi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right.\\
&+\left(\sin\theta\sin\phi\frac{\partial\varphi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\varphi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\\
&-2\left(\sin\theta\cos\phi\frac{\partial\vartheta}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial\vartheta}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\right)\\
&\left.+r\cos\theta\left(\sin\theta\sin\phi\frac{\partial\psi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\psi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\sin\theta\sin\phi\\
&+\left[\left(\cos\theta\frac{\partial\Phi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\Phi}{\partial\theta}\right)+\left(\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\right.\\
&\left.+r\cos\theta\left(\cos\theta\frac{\partial\psi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)-(3-4\nu)\phi\right]\cos\theta\\
&=\frac{\partial\Phi}{\partial r}+\frac{\partial\varphi}{\partial r}+\frac{2}{r}\frac{\partial\vartheta}{\partial\theta}+r\cos\theta\frac{\partial\psi}{\partial r}-(3-4\nu)\psi\cos\theta
\end{aligned}
\tag{16.122}
$$

$$
\begin{aligned}
u_\theta &= u_x\cos\theta\cos\phi+u_y\cos\theta\sin\phi-u_z\sin\theta\\
&=\left[\left(\sin\theta\cos\phi\frac{\partial\Phi}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right.\\
&\quad+\left(\sin\theta\cos\phi\frac{\partial\varphi}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial\varphi}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\\
&\quad+2\left(\sin\theta\sin\phi\frac{\partial\vartheta}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\vartheta}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\right)\\
&\quad\left.+r\cos\theta\left(\sin\theta\cos\phi\frac{\partial\psi}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial\psi}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\cos\theta\cos\phi\\
&\quad+\left[\left(\sin\theta\sin\phi\frac{\partial\Phi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\Phi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right.\\
&\quad+\left(\sin\theta\sin\phi\frac{\partial\varphi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\varphi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\\
&\quad-2\left(\sin\theta\cos\phi\frac{\partial\vartheta}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial\vartheta}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\right)\\
&\quad\left.+r\cos\theta\left(\sin\theta\sin\phi\frac{\partial\psi}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial\psi}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]\cos\theta\sin\phi\\
&\quad-\left[\left(\cos\theta\frac{\partial\Phi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\Phi}{\partial\theta}\right)+\left(\cos\theta\frac{\partial\varphi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\varphi}{\partial\theta}\right)\right.\\
&\quad\left.+r\cos\theta\left(\cos\theta\frac{\partial\psi}{\partial r}-\frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)-(3-4\nu)\phi\right]\sin\theta\\
&=\frac{1}{r}\frac{\partial\Phi}{\partial\phi}+\frac{1}{r}\frac{\partial\varphi}{\partial\phi}+\frac{2}{r\tan\theta}\frac{\partial\vartheta}{\partial\phi}+\cos\theta\frac{\partial\psi}{\partial\phi}+(3-4\nu)\psi\sin\theta
\end{aligned}
\tag{16.123}
$$

$$
\begin{aligned}
u_\phi &= -u_x \sin\phi + u_y \cos\phi \\
&= -\Big[\Big(\sin\theta\cos\phi\frac{\partial\Phi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\Phi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\Big) \\
&\quad + \Big(\sin\theta\cos\phi\frac{\partial\varphi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\varphi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\Big) \\
&\quad + 2\Big(\sin\theta\sin\phi\frac{\partial\vartheta}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial\vartheta}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\Big) \\
&\quad + r\cos\theta\Big(\sin\theta\cos\phi\frac{\partial\psi}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\psi}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\Big)\Big]\sin\phi \\
&\quad + \Big[\Big(\sin\theta\sin\phi\frac{\partial\Phi}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\Big) \\
&\quad + \Big(\sin\theta\sin\phi\frac{\partial\varphi}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial\varphi}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial\varphi}{\partial\phi}\Big) \\
&\quad - 2\Big(\sin\theta\cos\phi\frac{\partial\vartheta}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial\vartheta}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial\vartheta}{\partial\phi}\Big) \\
&\quad + r\cos\theta\Big(\sin\theta\sin\phi\frac{\partial\psi}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial\psi}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\Big)\Big]\cos\phi \\
&= \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} - 2\sin\theta\frac{\partial\vartheta}{\partial r} - \frac{2\cos\theta}{r}\frac{\partial\psi}{\partial\theta} + \frac{1}{\tan\theta}\frac{\partial\psi}{\partial\phi}
\end{aligned} \tag{16.124}
$$

The displacement-strain relations (16.54) are

$$
\begin{aligned}
&\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \\
&\epsilon_{\phi\phi} = \frac{u_r}{r} + \cot\theta\frac{u_\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}, \quad \epsilon_{r\theta} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\Big) \\
&\epsilon_{\theta\phi} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \cot\theta\frac{u_\phi}{r} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\Big) \\
&\epsilon_{\phi r} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\Big)
\end{aligned} \tag{16.125}
$$

Substitution of Eqs. (16.122)–(16.124) into Eq. (16.125) gives

$$
\begin{aligned}
\epsilon_{rr} &= \frac{\partial^2\Phi}{\partial r^2} + \frac{\partial^2\varphi}{\partial r^2} - \frac{2}{r^2}\frac{\partial\vartheta}{\partial\phi} + \frac{2}{r}\frac{\partial^2\vartheta}{\partial r\partial\phi} + r\cos\theta\frac{\partial^2\psi}{\partial r^2} \\
&\quad - 2(1-2\nu)\cos\theta\frac{\partial\psi}{\partial r} \\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial\vartheta}{\partial\phi} - \frac{2}{r^2\sin^2\theta}\frac{\partial\vartheta}{\partial\phi}
\end{aligned}
$$

$$+\frac{2}{r^2\tan\theta}\frac{\partial^2\vartheta}{\partial\phi\partial\theta}+\cos\theta\frac{\partial\psi}{\partial r}+\frac{\cos\theta}{r}\frac{\partial^2\psi}{\partial\theta^2}+2(1-2\nu)\frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}$$

$$\epsilon_{\phi\phi}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\varphi}{\partial\theta}$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{2}{r^2\sin^2\theta}\frac{\partial\vartheta}{\partial\phi}-\frac{2}{r}\frac{\partial^2\vartheta}{\partial r\partial\phi}-\frac{2\cos\theta}{r^2\sin\theta}\frac{\partial^2\vartheta}{\partial\theta\partial\phi}$$
$$+\cos\theta\frac{\partial\psi}{\partial r}+\frac{\cos^2\theta}{r\sin\theta}\frac{\partial\psi}{\partial\theta}+\frac{\cos\theta}{r\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\epsilon_{r\theta}=\frac{1}{r}\frac{\partial^2\Phi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}$$
$$+\frac{1}{r^2}\frac{\partial^2\vartheta}{\partial\theta\partial\phi}-\frac{2}{r^2\tan\theta}\frac{\partial\vartheta}{\partial\phi}+\frac{1}{r\tan\theta}\frac{\partial^2\vartheta}{\partial r\partial\phi}$$
$$-2(1-\nu)\sin\theta\frac{\partial\psi}{\partial r}-2(1-\nu)\frac{\cos\theta}{r}\frac{\partial\psi}{\partial\theta}+\cos\theta\frac{\partial^2\psi}{\partial r\partial\theta}$$

$$\epsilon_{\theta\phi}=-\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial\Phi}{\partial\phi}+\frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\theta\partial\phi}-\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial\varphi}{\partial\phi}+\frac{1}{r^2\sin\theta}\frac{\partial^2\varphi}{\partial\theta\partial\phi}$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial\vartheta}{\partial\theta}-\frac{\sin\theta}{r}\frac{\partial^2\vartheta}{\partial r\partial\theta}-\frac{\cos\theta}{r^2}\frac{\partial^2\vartheta}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial^2\vartheta}{\partial\phi^2}$$
$$+\frac{1}{r\tan\theta}\frac{\partial^2\psi}{\partial\theta\partial\phi}+\left(1-2\nu-\frac{1}{\tan^2\theta}\right)\frac{1}{r}\frac{\partial\psi}{\partial\phi}$$

$$\epsilon_{r\phi}=\frac{1}{r\sin\theta}\frac{\partial^2\Phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial\Phi}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^2\varphi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial\varphi}{\partial\phi}$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial^2\vartheta}{\partial\phi^2}-\sin\theta\frac{\partial^2\vartheta}{\partial r^2}+\frac{\cos\theta}{r^2}\frac{\partial\vartheta}{\partial\theta}-\frac{\cos\theta}{r}\frac{\partial^2\vartheta}{\partial r\partial\theta}$$
$$+\frac{1}{\tan\theta}\frac{\partial^2\psi}{\partial r\partial\phi}-2(1-\nu)\frac{1}{r\tan\theta}\frac{\partial\psi}{\partial\phi} \qquad \text{(Answer)}$$