Chapter 15
Heat Conduction

In this chapter the Fourier heat conduction equation along with the boundary conditions and the initial conditions for various coordinate systems are recalled. One-dimensional heat conduction problems in Cartesian coordinates, cylindrical coordinates and spherical coordinates are treated for both the steady and the transient temperature fields. The particular problems and solutions for heat conduction in a strip, a solid cylinder, a hollow circular cylinder and a hollow sphere are presented for various boundary conditions. [See also Chap. 22.]

15.1 Heat Conduction Equation

Heat conduction equation
The Fourier law of heat conduction is

$$q = -\lambda \frac{\partial T}{\partial n} \tag{15.1}$$

where q denotes the heat flux with dimension [W/m^2] and λ is the thermal conductivity of the solid with dimension [W/(m $\cdot$ K)]. Here, $\partial/\partial n$ denotes differentiation along out-drawn normal n to the isothermal surface.

The Fourier heat conduction equation for the homogeneous isotropic solid based on the Fourier law of heat conduction (15.1) is

$$c\rho \frac{\partial T}{\partial t} = \lambda \nabla^2 T + Q \tag{15.2}$$

An alternative form is

$$\frac{1}{\kappa} \frac{\partial T}{\partial t} = \nabla^2 T + \frac{Q}{\lambda} \tag{15.2$'$}$$

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where

$$\kappa = \frac{\lambda}{c\rho} \tag{15.3}$$

in which Q is the internal heat generation per unit volume per unit time, c is the specific heat with dimension [J/(kg · K)], ρ is the density with dimension [kg/m^3] of the solid, and κ means the thermal diffusivity with dimension [m^2/s], and the expression for the Laplacian operator ∇^2 is different for each coordinate system:

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} && : \text{for Cartesian coordinates} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} && : \text{for cylindrical coordinates} \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \\ & && : \text{for spherical coordinates}\end{aligned} \tag{15.4}$$

The heat conduction equation for a nonhomogeneous anisotropic solid is

$$c\rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_y \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial T}{\partial z}\right) + Q \tag{15.5}$$

where λ_x, λ_y, and λ_z denote the thermal conductivities along the x, y, and z directions, respectively, and depend on the position.

The heat conduction equation for a nonhomogeneous isotropic solid is

$$c\rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + Q \tag{15.6}$$

The heat conduction equation for homogeneous anisotropic solid is

$$c\rho\frac{\partial T}{\partial t} = \lambda_x \frac{\partial^2 T}{\partial x^2} + \lambda_y \frac{\partial^2 T}{\partial y^2} + \lambda_z \frac{\partial^2 T}{\partial z^2} + Q \tag{15.7}$$

The heat conduction equation for a homogeneous isotropic solid without internal heat generation is

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \nabla^2 T \tag{15.8}$$

The steady state heat conduction equation for the homogeneous isotropic solid with the internal heat generation Q is

$$\nabla^2 T + \frac{Q}{\lambda} = 0 \tag{15.9}$$

The steady state heat conduction equation for the homogeneous isotropic solid without internal heat generation is

$$\nabla^2 T = 0 \tag{15.10}$$

Boundary conditions

When heat transfer between the boundary surface of the solid and the surrounding medium occurs by convection, the boundary condition is

$$-\lambda \frac{\partial T}{\partial n} + q_b = h(T - \Theta) \tag{15.11}$$

where h denotes the heat transfer coefficient with dimension $[\mathrm{W}/(\mathrm{m}^2 \cdot \mathrm{K})]$, q_b means heat generation per unit area per unit time on the boundary surface and Θ is the temperature of the surrounding medium which is a given function of position and time.

When the surfaces of two solids are in perfect thermal contact, the temperature on the contact surface and the heat flow through the contact surface are the same for both solids

$$T_1 = T_2, \quad \lambda_1 \frac{\partial T_1}{\partial n} = \lambda_2 \frac{\partial T_2}{\partial n} \tag{15.12}$$

where subscripts 1 and 2 refer to the solid 1 and 2, respectively, and n is the common normal direction on the contact surface.

Initial condition

When the transient heat conduction Eq. (15.2) is considered, an initial condition which expresses the temperature distribution in the solid at initial time must be specified

$$T = \Phi(P) \tag{15.13}$$

where $\Phi(P)$ is the initial temperature distribution and P is a position in the solid.

15.2 One-Dimensional Heat Conduction Problems

Temperature in a strip

The heat conduction Eq. (15.9) simplifies to the form for one-dimensional steady state heat conduction problems of a homogeneous isotropic solid with the internal heat generation Q

$$\frac{d^2 T}{dx^2} = -\frac{Q}{\lambda} \tag{15.14}$$

If there is no internal heat generation Q, Eq. (15.14) reduces to

$$\frac{d^2T}{dx^2} = 0 \tag{15.15}$$

The steady temperature in a strip of width l with constant internal heat generation Q is given for the heat transfer boundary conditions

$$\begin{aligned} T &= T_a + (T_b - T_a)\frac{h_b(h_a x + \lambda)}{\lambda(h_a + h_b) + h_a h_b l} \\ &\quad + \frac{Ql^2}{2\lambda}\left[\frac{(2\lambda + h_b l)(h_b x + \lambda)}{\lambda(h_a + h_b)l + h_a h_b l^2} - \frac{x^2}{l^2}\right] \end{aligned} \tag{15.16}$$

where T_a and T_b are the temperatures of the surrounding media, h_a and h_b are the heat transfer coefficients, and subscripts a and b denote boundaries at $x = 0$ and $x = l$, respectively.

The heat conduction Eq. (15.8) simplifies to the form for one-dimensional transient heat conduction problems of a homogeneous isotropic solid

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{15.17}$$

The transient temperature in a strip of width l with the initial temperature $T_i(x)$ is given for the heat transfer boundary conditions

$$\begin{aligned} T(x,t) &= T_a + (T_b - T_a)\frac{h_b(h_a x + \lambda)}{\lambda(h_a + h_b) + h_a h_b l} \\ &\quad + 2\sum_{n=1}^{\infty} \frac{(\lambda^2 s_n^2 + h_b^2)(h_a \sin s_n x + \lambda s_n \cos s_n x)e^{-\kappa s_n^2 t}}{l(\lambda^2 s_n^2 + h_a^2)(\lambda^2 s_n^2 + h_b^2) + \lambda(h_a + h_b)(\lambda^2 s_n^2 + h_a h_b)} \\ &\quad \times \int_0^l \left\{T_i(x) - \left[T_a + (T_b - T_a)\frac{h_b(h_a x + \lambda)}{\lambda(h_a + h_b) + h_a h_b l}\right]\right\} \\ &\quad \times (h_a \sin s_n x + \lambda s_n \cos s_n x)\, dx \end{aligned} \tag{15.18}$$

where s_n are eigenvalues of the transcendental equation

$$\tan s_n l = \frac{\lambda s_n (h_a + h_b)}{\lambda^2 s_n^2 - h_a h_b} \tag{15.19}$$

Temperature in a hollow cylinder

The heat conduction Eq. (15.9) simplifies to the form for one-dimensional steady state heat conduction problems of the homogeneous isotropic cylinder with an internal heat generation Q

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -\frac{Q}{\lambda} \tag{15.20}$$

Furthermore, if there is no internal heat generation Q, Eq. (15.20) reduces to

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \tag{15.21}$$

The steady temperature in a hollow cylinder of inner radius a and outer radius b with constant internal heat generation Q is given for the heat transfer boundary conditions

$$\begin{aligned} T &= T_a + (T_b - T_a)\frac{\ln\dfrac{r}{a} + \dfrac{\lambda}{h_a a}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}} \\ &\quad - \frac{Q}{4\lambda}r^2 + \frac{Q}{4\lambda}\frac{\begin{pmatrix} a^2\left(1 - 2\dfrac{\lambda}{ah_a}\right)\ln\dfrac{b}{r} + b^2\left(1 + 2\dfrac{\lambda}{bh_b}\right)\ln\dfrac{r}{a} \\ + \dfrac{\lambda}{ah_a}b^2 + \dfrac{\lambda}{bh_b}a^2 + 2(b^2 - a^2)\dfrac{\lambda}{ah_a}\dfrac{\lambda}{bh_b} \end{pmatrix}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}} \end{aligned} \tag{15.22}$$

where subscripts a and b denote the boundaries at $r = a$ and $r = b$, respectively.

The heat conduction Eq. (15.8) simplifies to the form for one-dimensional transient heat conduction problems of a homogeneous isotropic cylinder

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{15.23}$$

The transient temperature in a hollow cylinder of inner radius a and outer radius b with the initial temperature $T_i(r)$ is given for the heat transfer boundary conditions

$$\begin{aligned} T &= T_a + (T_b - T_a)\frac{\ln(r/a) + \lambda/(h_a a)}{\ln(b/a) + \lambda/(h_a a) + \lambda/(h_b b)} \\ &\quad - \pi\sum_{n=1}^{\infty}\frac{T_a h_a - T_b h_b G_n}{(h_a^2 + \lambda^2 s_n^2) - (h_b^2 + \lambda^2 s_n^2)G_n^2} f(s_n, r)e^{-\kappa s_n^2 t} \\ &\quad - \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 f(s_n, r)}{(h_a^2 + \lambda^2 s_n^2) - (h_b^2 + \lambda^2 s_n^2)G_n^2}\int_a^b T_i(\eta) f(s_n, \eta)\eta d\eta\, e^{-\kappa s_n^2 t} \end{aligned} \tag{15.24}$$

where s_n are eigenvalues of the transcendental equation

$$[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)][h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)] \\ - [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)][h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] = 0 \tag{15.25}$$

and

$$G_n = \frac{h_a J_0(s_n a) + \lambda s_n J_1(s_n a)}{h_b J_0(s_n b) - \lambda s_n J_1(s_n b)} = \frac{h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)}{h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)} \tag{15.26}$$

$$f(s_n, r) = J_0(s_n r)[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] \\ - Y_0(s_n r)[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] \tag{15.27}$$

in which $J_0(sr)$ is the Bessel function of the first kind of order zero, and $Y_0(sr)$ is the Bessel function of the second kind of order zero.

Temperature in a hollow sphere

The heat conduction Eq. (15.9) simplifies to the form for one-dimensional steady state heat conduction problems of the homogeneous isotropic sphere with the internal heat generation Q

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = -\frac{Q}{\lambda} \tag{15.28}$$

Furthermore, if there is no internal heat generation Q, Eq. (15.28) reduces to

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0 \tag{15.29}$$

The steady temperature in a hollow sphere of inner radius a and outer radius b with constant internal heat generation Q is given for the heat transfer boundary conditions

$$T = T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{h_a a} - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} - \frac{Q}{6\lambda}r^2 \\ - \frac{Q}{6\lambda}b^2 \frac{\left(\begin{array}{l} \left[1 - \left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{a h_a}\left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{b h_b}\right]\dfrac{a}{r} \\ -\left[\left(1 + 2\dfrac{\lambda}{b h_b}\right)\left(1 + \dfrac{\lambda}{a h_a}\right) - \left(\dfrac{a}{b}\right)^3\left(1 - 2\dfrac{\lambda}{a h_a}\right)\left(1 - \dfrac{\lambda}{b h_b}\right)\right] \end{array} \right)}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} \tag{15.30}$$

The heat conduction Eq. (15.8) simplifies to the form for one-dimensional transient heat conduction problems of a homogeneous isotropic sphere

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) \tag{15.31}$$

The transient temperature in a hollow sphere of inner radius a and outer radius b with the initial temperature $T_i(r)$ is given for the heat transfer boundary conditions

$$\begin{aligned} T(r,t) &= T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{h_a a} - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} \\ &+ \sum_{n=1}^{\infty} \frac{C_n}{r}\Big[(h_a a + \lambda)\sin s_n(r-a) + \lambda s_n a \cos s_n(r-a)\Big] e^{-\kappa s_n^2 t} \end{aligned} \tag{15.32}$$

where coefficients C_n are

$$C_n = \frac{\left(\begin{array}{c} 2[(h_b b - \lambda)^2 + \lambda^2 s_n^2 b^2] \displaystyle\int_a^b [T_i(\eta) - T_s(\eta)]\eta \\ \times [(h_a a + \lambda)\sin s_n(\eta - a) + \lambda s_n a \cos s_n(\eta - a)]\, d\eta \end{array}\right)}{\left(\begin{array}{c} (b-a)[(h_a a + \lambda)^2 + \lambda^2 s_n^2 a^2][(h_b b - \lambda)^2 + \lambda^2 s_n^2 b^2] \\ + \lambda[b(h_a a + \lambda) + a(h_b b - \lambda)][(h_a a + \lambda)(h_b b - \lambda) + \lambda^2 s_n^2 ab] \end{array}\right)} \tag{15.33}$$

and s_n are eigenvalues of the transcendental equation

$$\begin{aligned} &\Big[(h_a a + \lambda)\,(h_b b - \lambda) - \lambda^2 s_n^2 ab\Big] \sin s_n(b-a) \\ &+ \lambda s_n\,[b\,(h_a a + \lambda) + a\,(h_b b - \lambda)] \cos s_n(b-a) = 0 \end{aligned} \tag{15.34}$$

15.3 Problems and Solutions Related to Heat Conduction

Problem 15.1. When the boundary conditions of a strip are given by following three cases (1), (2) and (3), find the steady temperatures in the strip.

[1] Prescribed surface temperatures T_a and T_b at both surfaces $x = 0$ and $x = l$, respectively.
[2] Prescribed surface temperature T_a at the left surface $x = 0$ and constant heat flux q_b $(= -\lambda(dT/dx))$ at the right surface $x = l$.
[3] Constant heat flux q_a $(= \lambda(dT/dx))$ at the left surface $x = 0$ and prescribed surface temperature T_b at the right surface $x = l$.

Solution. The general solution of the governing Eq. (15.15) is

$$T = A + Bx \tag{15.35}$$

where unknown coefficients A and B are determined by each boundary condition.

[1] The boundary conditions are

$$T = T_a \quad \text{on} \quad x = 0, \qquad T = T_b \quad \text{on} \quad x = l \tag{15.36}$$

Substituting Eq. (15.35) into Eq. (15.36), unknown coefficients A and B can be determined as

$$A = T_a, \qquad B = (T_b - T_a)\frac{1}{l} \tag{15.37}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{x}{l} \tag{Answer}$$

[2] The boundary conditions are

$$T = T_a \quad \text{on} \quad x = 0, \qquad -\lambda\frac{dT}{dx} = q_b \quad \text{on} \quad x = l \tag{15.38}$$

Substituting Eq. (15.35) into Eq. (15.38), unknown coefficients A and B can be determined as

$$A = T_a, \quad B = -\frac{q_b}{\lambda} \tag{15.39}$$

The temperature becomes

$$T = T_a - \frac{q_b}{\lambda}x \tag{Answer}$$

[3] The boundary conditions are

$$\lambda\frac{dT}{dx} = q_a \quad \text{on} \quad x = 0, \quad T = T_b \quad \text{on } x = l \tag{15.40}$$

Substituting Eq. (15.35) into Eq. (15.40), unknown coefficients A and B can be determined as

$$A = T_b - \frac{q_a}{\lambda}l, \quad B = \frac{q_a}{\lambda} \tag{15.41}$$

The temperature becomes

$$T = T_b - \frac{q_a}{\lambda}(l - x) \tag{Answer}$$

Problem 15.2. When the boundary conditions of a hollow cylinder are given by following five cases (1)–(5), find the steady temperatures in the hollow cylinder.

[1] Prescribed surface temperatures T_a and T_b at both surfaces $r = a$ and $r = b$, respectively.

[2] Prescribed surface temperature T_a at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[3] Prescribed surface temperature T_b at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.
[4] Constant heat flux q_a $(= \lambda(dT/dr))$ at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[5] Constant heat flux q_b $(= -\lambda(dT/dr))$ at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.

Solution. The general solution of the governing Eq. (15.21) is

$$T = A + B \ln r \tag{15.42}$$

where unknown coefficients A and B are determined by each boundary condition.

[1] The boundary conditions are

$$T = T_a \quad \text{on} \quad r = a, \qquad T = T_b \quad \text{on} \quad r = b \tag{15.43}$$

Substituting Eq. (15.42) into Eq. (15.43), unknown coefficients A and B can be determined as

$$A = T_a - \frac{T_b - T_a}{\ln b - \ln a} \ln a, \quad B = \frac{T_b - T_a}{\ln b - \ln a} \tag{15.44}$$

The temperature becomes

$$T = T_a + (T_b - T_a) \frac{\ln \frac{r}{a}}{\ln \frac{b}{a}} \qquad \text{(Answer)}$$

[2] The boundary conditions are

$$T = T_a \quad \text{on} \quad r = a, \qquad -\lambda \frac{\partial T}{\partial r} = h_b (T - T_b) \quad \text{on} \quad r = b \tag{15.45}$$

Substitution of Eq. (15.42) into Eq. (15.45) gives

$$A = T_a - (T_b - T_a) \frac{\ln a}{\ln \frac{b}{a} + \frac{\lambda}{h_b b}}, \quad B = (T_b - T_a) \frac{1}{\ln \frac{b}{a} + \frac{\lambda}{h_b b}} \tag{15.46}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{\ln \dfrac{r}{a}}{\ln \dfrac{b}{a} + \dfrac{\lambda}{h_b b}} \qquad \text{(Answer)}$$

[3] The boundary conditions are

$$\lambda \frac{dT}{dr} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.47}$$

Substitution of Eq. (15.42) into Eq. (15.47) gives

$$A = T_a - (T_b - T_a)\frac{\ln a - \dfrac{\lambda}{h_a a}}{\ln \dfrac{b}{a} + \dfrac{\lambda}{h_a a}}, \quad B = (T_b - T_a)\frac{1}{\ln \dfrac{b}{a} + \dfrac{\lambda}{h_a a}} \tag{15.48}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{\ln \dfrac{r}{a} + \dfrac{\lambda}{h_a a}}{\ln \dfrac{b}{a} + \dfrac{\lambda}{h_a a}} \qquad \text{(Answer)}$$

[4] The boundary conditions are

$$\lambda \frac{dT}{dr} = q_a \quad \text{on} \quad r = a, \quad -\lambda \frac{dT}{dr} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.49}$$

Substitution of Eq. (15.42) into Eq. (15.49) gives

$$A = T_b - \frac{q_a a}{\lambda}(\ln b + \frac{\lambda}{h_b b}), \quad B = \frac{q_a a}{\lambda} \tag{15.50}$$

The temperature becomes

$$T = T_b + \frac{q_a a}{\lambda}\left(\ln \frac{r}{b} - \frac{\lambda}{h_b b}\right) \qquad \text{(Answer)}$$

[5] The boundary conditions are

$$\lambda \frac{dT}{dr} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad -\lambda \frac{dT}{dr} = q_b \quad \text{on} \quad r = b \tag{15.51}$$

Substitution of Eq. (15.42) into Eq. (15.51) gives

$$A = T_a + \frac{q_b b}{\lambda}(\ln a - \frac{\lambda}{h_a a}), \quad B = -\frac{q_b b}{\lambda} \tag{15.52}$$

The temperature becomes

$$T = T_a - \frac{q_b b}{\lambda}\left(\ln\frac{r}{a} + \frac{\lambda}{h_a a}\right) \tag{Answer}$$

Problem 15.3. When the boundary conditions of a hollow sphere are given by following five cases (1)–(5), find the steady temperatures in the hollow sphere.

[1] Prescribed surface temperatures T_a and T_b at both surfaces $r = a$ and $r = b$, respectively.
[2] Prescribed surface temperature T_a at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[3] Prescribed surface temperature T_b at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.
[4] Constant heat flux q_a $(= \lambda(dT/dr))$ at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[5] Constant heat flux q_b $(= -\lambda(dT/dr))$ at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.

Solution. The general solution of the governing Eq. (15.29) is

$$T = A + \frac{B}{r} \tag{15.53}$$

where unknown coefficients A and B are determined by each boundary condition.

[1] The boundary conditions are

$$T = T_a \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.54}$$

Substitution of Eq. (15.53) into Eq. (15.54) gives

$$A + \frac{B}{a} = T_a\,, \quad A + \frac{B}{b} = T_b \tag{15.55}$$

Equation (15.55) gives

$$A = T_a + \frac{T_b - T_a}{1 - \dfrac{a}{b}}, \quad B = -\frac{a(T_b - T_a)}{1 - \dfrac{a}{b}} \tag{15.56}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{1 - \dfrac{a}{r}}{1 - \dfrac{a}{b}} \quad \text{(Answer)}$$

[2] The boundary conditions are

$$T = T_a \quad \text{on} \quad r = a, \quad -\lambda\frac{dT}{dr} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.57}$$

Substitution of Eq. (15.53) into Eq. (15.57) gives

$$A = T_a + (T_b - T_a)\frac{1}{1 - \dfrac{a}{b} + \dfrac{a}{b}\dfrac{\lambda}{bh_b}}, \quad B = -\frac{a(T_b - T_a)}{1 - \dfrac{a}{b} + \dfrac{a}{b}\dfrac{\lambda}{bh_b}} \tag{15.58}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{1 - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{a}{b}\dfrac{\lambda}{bh_b}} \quad \text{(Answer)}$$

[3] The boundary conditions are

$$\lambda\frac{dT}{dr} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.59}$$

Substitution of Eq. (15.53) into Eq. (15.59) gives

$$A = T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{ah_a}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{ah_a}}, \quad B = -\frac{a(T_b - T_a)}{1 - \dfrac{a}{b} + \dfrac{\lambda}{ah_a}} \tag{15.60}$$

The temperature becomes

$$T = T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{ah_a} - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{ah_a}} \qquad \text{(Answer)}$$

[4] The boundary conditions are

$$\lambda\frac{dT}{dr} = q_a \quad \text{on} \quad r = a, \quad -\lambda\frac{dT}{dr} = h_b(T - T_b) \quad \text{on} \quad r = b \qquad (15.61)$$

Substitution of Eq. (15.53) into Eq. (15.61) gives

$$B = -\frac{a^2 q_a}{\lambda}, \quad A = T_b + \frac{aq_a}{\lambda}\frac{a}{b}\left(1 - \frac{\lambda}{bh_b}\right) \qquad (15.62)$$

The temperature becomes

$$T = T_b - \frac{aq_a}{\lambda}\frac{a}{b}\left(\frac{b}{r} - 1 + \frac{\lambda}{bh_b}\right) \qquad \text{(Answer)}$$

[5] The boundary conditions are

$$\lambda\frac{dT}{dr} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad -\lambda\frac{dT}{dr} = q_b \quad \text{on} \quad r = b \qquad (15.63)$$

Substitution of Eq. (15.53) into Eq. (15.63) gives

$$B = \frac{b^2 q_b}{\lambda}, \quad A = T_a - \frac{bq_b}{\lambda}\frac{b}{a}\left(1 + \frac{\lambda}{ah_a}\right) \qquad (15.64)$$

The temperature becomes

$$T = T_a - \frac{bq_b}{\lambda}\frac{b}{a}\left(1 + \frac{\lambda}{ah_a} - \frac{a}{r}\right) \qquad \text{(Answer)}$$

Problem 15.4. Find the steady temperature in a strip of width l with constant internal heat generation Q under heat transfer boundary conditions.

Solution. The steady state heat conduction equation is given by Eq. (15.14). The boundary conditions are

$$\lambda\frac{dT}{dx} = h_a(T - T_a) \quad \text{on} \quad x = 0, \quad -\lambda\frac{dT}{dx} = h_b(T - T_b) \quad \text{on} \quad x = l \qquad (15.65)$$

where T_a and T_b are the temperatures of the surrounding media, h_a and h_b are the heat transfer coefficients, and subscripts a and b denote boundaries at $x = 0$ and $x = l$, respectively. A general solution of Eq. (15.14) is

$$T = A + Bx - \frac{Q}{2\lambda}x^2 \tag{15.66}$$

The coefficients A and B can be determined from the boundary conditions (15.65)

$$\begin{aligned} A &= T_a + \frac{\lambda h_b(T_b - T_a)}{\lambda(h_a + h_b) + h_a h_b l} + \frac{Q}{2\lambda}\frac{\lambda l(2\lambda + h_b l)}{\lambda(h_a + h_b) + h_a h_b l} \\ B &= \frac{h_a h_b(T_b - T_a)}{\lambda(h_a + h_b) + h_a h_b l} + \frac{Q}{2\lambda}\frac{h_a l(2\lambda + h_b l)}{\lambda(h_a + h_b) + h_a h_b l} \end{aligned} \tag{15.67}$$

Substitution of Eq. (15.67) into Eq. (15.66) gives the temperature

$$\begin{aligned} T &= T_a + (T_b - T_a)\frac{h_b(h_a x + \lambda)}{\lambda(h_a + h_b) + h_a h_b l} \\ &\quad + \frac{Ql^2}{2\lambda}\left[\frac{(2\lambda + h_b l)(h_b x + \lambda)}{\lambda(h_a + h_b)l + h_a h_b l^2} - \frac{x^2}{l^2}\right] \end{aligned} \tag{Answer}$$

Problem 15.5. Find the steady temperature in a hollow cylinder of inner radius a and outer radius b with constant internal heat generation Q under heat transfer boundary conditions.

Solution. The steady heat conduction equation in the hollow cylinder is given by Eq. (15.20). The boundary conditions are

$$\begin{aligned} \lambda\frac{dT}{dr} &= h_a(T - T_a) \quad \text{on} \quad r = a \\ -\lambda\frac{dT}{dr} &= h_b(T - T_b) \quad \text{on} \quad r = b \end{aligned} \tag{15.68}$$

The general solution of Eq. (15.20) is

$$T = A + B\ln r - \frac{Q}{4\lambda}r^2 \tag{15.69}$$

The coefficients A and B can be determined from the boundary conditions (15.68)

$$A = T_a - (T_b - T_a)\frac{\ln a - \dfrac{\lambda}{h_a a}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}}$$

$$+\frac{Q}{4\lambda}\frac{\begin{pmatrix} a^2\ln b - b^2\ln a - 2\dfrac{\lambda}{bh_b}b^2\ln a - 2\dfrac{\lambda}{ah_a}a^2\ln b \\ +\dfrac{\lambda}{ah_a}b^2 + \dfrac{\lambda}{bh_b}a^2 + 2(b^2-a^2)\dfrac{\lambda}{ah_a}\dfrac{\lambda}{bh_b} \end{pmatrix}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}}$$

$$B = \frac{T_b - T_a}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}} + \frac{Q}{4\lambda}\frac{\left(1 + 2\dfrac{\lambda}{bh_b}\right)b^2 - \left(1 - 2\dfrac{\lambda}{ah_a}\right)a^2}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}} \tag{15.70}$$

Thus the temperature is

$$T = T_a + (T_b - T_a)\frac{\ln\dfrac{r}{a} + \dfrac{\lambda}{h_a a}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}}$$
$$-\frac{Q}{4\lambda}r^2 + \frac{Q}{4\lambda}\frac{\begin{pmatrix} a^2\left(1 - 2\dfrac{\lambda}{ah_a}\right)\ln\dfrac{b}{r} + b^2\left(1 + 2\dfrac{\lambda}{bh_b}\right)\ln\dfrac{r}{a} \\ +\dfrac{\lambda}{ah_a}b^2 + \dfrac{\lambda}{bh_b}a^2 + 2(b^2-a^2)\dfrac{\lambda}{ah_a}\dfrac{\lambda}{bh_b} \end{pmatrix}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_a a} + \dfrac{\lambda}{h_b b}} \quad \text{(Answer)}$$

Problem 15.6. Find the steady temperature in a hollow sphere of inner radius a and outer radius b with constant internal heat generation Q under heat transfer boundary conditions.

Solution. The steady heat conduction equation in the hollow sphere is given by Eq. (15.28). The boundary conditions are

$$\begin{aligned} \lambda\frac{dT}{dr} &= h_a(T - T_a) \quad \text{on} \quad r = a \\ -\lambda\frac{dT}{dr} &= h_b(T - T_b) \quad \text{on} \quad r = b \end{aligned} \tag{15.71}$$

The general solution of Eq. (15.28) is

$$T = A + \frac{B}{r} - \frac{Q}{6\lambda}r^2 \tag{15.72}$$

The coefficients A and B can be determined from the boundary conditions (15.71)

$$A = T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{h_a a}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}}$$
$$+ \frac{Q}{6\lambda}b^2 \frac{\left(1 + 2\dfrac{\lambda}{bh_b}\right)\left(1 + \dfrac{\lambda}{ah_a}\right) - \left(\dfrac{a}{b}\right)^3\left(1 - 2\dfrac{\lambda}{ah_a}\right)\left(1 - \dfrac{\lambda}{bh_b}\right)}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}}$$
$$B = -(T_b - T_a)\frac{a}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}}$$
$$- \frac{Q}{6\lambda}ab^2 \frac{1 - \left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{ah_a}\left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{bh_b}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} \tag{15.73}$$

Thus, the temperature is

$$T = T_a + (T_b - T_a)\frac{1 + \dfrac{\lambda}{h_a a} - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} - \frac{Q}{6\lambda}r^2$$
$$- \frac{Q}{6\lambda}b^2 \frac{\left(\begin{array}{l}\left[1 - \left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{ah_a}\left(\dfrac{a}{b}\right)^2 + 2\dfrac{\lambda}{bh_b}\right]\dfrac{a}{r} \\ -\left[\left(1 + 2\dfrac{\lambda}{bh_b}\right)\left(1 + \dfrac{\lambda}{ah_a}\right) - \left(\dfrac{a}{b}\right)^3\left(1 - 2\dfrac{\lambda}{ah_a}\right)\left(1 - \dfrac{\lambda}{bh_b}\right)\right]\end{array}\right)}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}}$$

(Answer)

Problem 15.7. Determine the one-dimensional steady temperature of a two-layered hollow cylinder for the heat transfer boundary conditions.

Solution. The temperature of each layer is

$$T_i = A_i + B_i \ln r \quad (i = 1, 2) \tag{15.74}$$

The boundary conditions at each boundary are

$$
\begin{aligned}
\lambda_1 \frac{dT_1}{dr} &= h_a(T_1 - T_a) \qquad \text{on} \quad r = a \\
T_1 = T_2\,, \quad \lambda_1 \frac{dT_1}{dr} &= \lambda_2 \frac{dT_2}{dr} \qquad \text{on} \quad r = c \\
-\lambda_2 \frac{dT_2}{dr} &= h_b(T_2 - T_b) \qquad \text{on} \quad r = b
\end{aligned}
\tag{15.75}
$$

We get the coefficients A_i and B_i in Eq. (15.74) from (15.74) and (15.75)

$$
\begin{aligned}
A_1 &= T_a - \frac{T_b - T_a}{D} b h_b \lambda_2 (a h_a \ln a - \lambda_1)\,, \quad B_1 = \frac{T_b - T_a}{D} a b h_a h_b \lambda_2 \\
A_2 &= T_a + \frac{T_b - T_a}{D} b h_b \left(a h_a \lambda_2 \ln \frac{c}{a} - a h_a \lambda_1 \ln c + \lambda_1 \lambda_2\right) \\
B_2 &= \frac{T_b - T_a}{D} a b h_a h_b \lambda_1
\end{aligned}
\tag{15.76}
$$

where

$$
D = \lambda_1 \lambda_2 (a h_a + b h_b) + a b h_a h_b \left(\lambda_1 \ln \frac{b}{c} + \lambda_2 \ln \frac{c}{a}\right) \tag{15.77}
$$

Then, the temperature of each layer is

$$
\begin{aligned}
T_1 &= T_a + \frac{T_b - T_a}{D} b h_b \lambda_2 \left(a h_a \ln \frac{r}{a} + \lambda_1\right) \\
T_2 &= T_a + \frac{T_b - T_a}{D} b h_b \lambda_2 \left\{a h_a \left(\ln \frac{c}{a} + \frac{\lambda_1}{\lambda_2} \ln \frac{r}{c}\right) + \lambda_1\right\}
\end{aligned}
\tag{Answer}
$$

Problem 15.8. Find the transient temperature in a strip, when the initial temperature is $T_i(r)$, and the boundary conditions of the strip are given by following two cases:

[1] Prescribed surface temperatures T_a and T_b at both surfaces $x = 0$ and $x = l$, respectively.
[2] Prescribed surface temperature T_a at $x = 0$ and constant heat flux $q_b (= -\lambda(\partial T/\partial x))$ at $x = l$.

Solution. When the heat transfer conditions at both surfaces are

$$
\lambda \frac{\partial T}{\partial x} = h_a(T - T_a) \quad \text{on} \quad x = 0, \qquad -\lambda \frac{\partial T}{\partial x} = h_b(T - T_b) \quad \text{on} \quad x = l \tag{15.78}
$$

the temperature is given by Eq. (15.18). Comparing between the heat transfer conditions (15.78) and each boundary condition, the temperature for each boundary condition can easily be obtained.

[1] The boundary conditions on $x = 0$ and $x = l$ for this problem are

$$
T = T_a \quad \text{on} \quad x = 0, \quad T = T_b \quad \text{on} \quad x = l \tag{15.79}
$$

Rewriting the boundary conditions (15.78) gives

$$T = T_a + \frac{\lambda}{h_a}\frac{\partial T}{\partial x} \quad \text{on} \quad x = 0, \quad T = T_b - \frac{\lambda}{h_b}\frac{\partial T}{\partial x} \quad \text{on} \quad x = l \tag{15.80}$$

Putting $h_a \to \infty$ and $h_b \to \infty$, Eq. (15.80) reduces to Eq. (15.79). Therefore, we can obtain the temperature from Eq. (15.18) after putting $h_a \to \infty$ and $h_b \to \infty$.

$$\begin{aligned} T(x,t) = T_a &+ (T_b - T_a)\frac{x}{l} \\ &+ 2\sum_{n=1}^{\infty}\left\{\int_0^l \left[T_i(x) - T_a - (T_b - T_a)\frac{x}{l}\right]\sin n\pi\frac{x}{l}dx\right\} \\ &\times \sin n\pi\frac{x}{l}e^{-\kappa(n\pi/l)^2 t} \end{aligned} \tag{Answer}$$

[2] The boundary conditions of this case are

$$T = T_a \quad \text{on} \quad x = 0, \quad -\lambda\frac{\partial T}{\partial x} = q_b \quad \text{on} \quad x = l \tag{15.81}$$

If we rewrite $h_b T_b = -q_b$, and put $h_a \to \infty$ and $h_b = 0$ in Eq. (15.78), Eq. (15.78) reduces to Eq. (15.81). Therefore, we can obtain the temperature from Eq. (15.18), after rewriting $h_b T_b = -q_b$ and putting $h_a \to \infty$ and $h_b = 0$.

$$\begin{aligned} T(x,t) = T_a &- \frac{q_b}{\lambda}x \\ &+ 2\sum_{n=1}^{\infty}\left\{\int_0^l \left[T_i(x) - T_a + \frac{q_b}{\lambda}x\right]\sin\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)dx\right\} \\ &\times \sin\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)e^{-\kappa[(2n-1)\pi/2]^2 t} \end{aligned} \tag{Answer}$$

Problem 15.9. When the boundary condition of the solid cylinder is heat transfer between the surface and the surrounding medium with the temperature T_a, and the initial temperature is $T_i(r)$, find the transient temperature in the solid cylinder.

Solution. When the boundary condition of the solid cylinder is heat transfer between the surface of the cylinder and the surrounding medium with the temperature T_a, and the initial temperature is $T_i(r)$, the equations to be solved are

(1) Governing equation

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{15.82}$$

(2) Boundary condition

$$-\lambda\frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a \tag{15.83}$$

(3) Initial condition

$$T = T_i(r) \quad \text{at} \quad t = 0 \tag{15.84}$$

The solution of Eq. (15.82) can be obtained by use of the method of separation of variables. We put the temperature to

$$T(r, t) = f(r)g(t) \tag{15.85}$$

Substitution of Eq. (15.85) into Eq. (15.82) leads to a pair of ordinary differential equation for $g(t)$ and $f(t)$

$$\frac{dg(t)}{dt} + \kappa s^2 g(t) = 0 \tag{15.86}$$

$$\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr} + s^2 f(r) = 0 \tag{15.87}$$

where s is arbitrary constant, and Eq. (15.87) is Bessel's differential equation of order zero.[1]

The linearly independent solutions of Eq. (15.86) are

$$g(t) = 1 \;\; \text{for} \;\; s = 0, \quad g(t) = \exp(-\kappa s^2 t) \;\; \text{for} \;\; s \neq 0 \tag{15.88}$$

and the linearly independent solutions of Eq. (15.87) are

$$f(r) = \begin{pmatrix} 1 \\ \ln r \end{pmatrix} \;\; \text{for} \;\; s = 0, \quad f(r) = \begin{pmatrix} J_0(sr) \\ Y_0(sr) \end{pmatrix} \;\; \text{for} \;\; s \neq 0 \tag{15.89}$$

where $J_0(sr)$ is the Bessel function of the first kind of order zero, and $Y_0(sr)$ is the Bessel function of the second kind of order zero. As these solutions exist for arbitrary values of s, the general solution of temperature $T(r, t)$ may be given by

$$T(r, t) = A_0 + B_0 r + \sum_{n=1}^{\infty}[A_n J_0(s_n r) + B_n Y_0(s_n r)]e^{-\kappa s_n^2 t} \tag{15.90}$$

As the Bessel function $Y_0(sr)$ and $\ln r$ are infinite at $r = 0$, B_0 and B_n must be zero for the solid cylinder. Therefore, the temperature for this problem reduces to

[1] G. N. Watson, Theory of Bessel Functions (2nd ed.), Cambridge University Press, Cambridge (1944).

$$T(r, t) = A_0 + \sum_{n=1}^{\infty} A_n J_0(s_n r) e^{-\kappa s_n^2 t} \tag{15.91}$$

where A_n are unknown coefficients.

Substitution of Eq. (15.91) into the boundary condition (15.83) gives

$$h_a(A_0 - T_a) + \sum_{n=1}^{\infty} A_n [h_a J_0(s_n a) - \lambda s_n J_1(s_n a)] e^{-\kappa s_n^2 t} = 0 \tag{15.92}$$

If $A_0 = T_a$ and s_n are eigenvalues of the eigenfunction

$$h_a J_0(s_n a) - \lambda s_n J_1(s_n a) = 0 \tag{15.93}$$

then, Eq. (15.92) is identically satisfied. Therefore, the temperature is

$$T = T_a + \sum_{n=1}^{\infty} A_n J_0(s_n r) e^{-\kappa s_n^2 t} \tag{15.94}$$

Using the initial condition (15.84), Eq. (15.94) yields

$$\sum_{n=1}^{\infty} A_n J_0(s_n r) = T_i(r) - T_a \tag{15.95}$$

Multiplying both sides of Eq. (15.95) by $r J_0(s_m r)$, and integrating from 0 to a, we find

$$A_m = \frac{2}{a^2[J_0^2(s_m a) + J_1^2(s_m a)]} \int_0^a [T_i(r) - T_a] J_0(s_m r) r dr \tag{15.96}$$

in which the following relations are used

$$\begin{aligned}
&\int_0^a J_0(s_m r) J_0(s_n r) r dr \\
&\quad = \frac{a}{s_m^2 - s_n^2} [s_m J_1(s_m a) J_0(s_n a) - s_n J_1(s_n a) J_0(s_m a)] \\
&\quad = \frac{a}{\lambda(s_m^2 - s_n^2)} [h_a J_0(s_m a) J_0(s_n a) - h_a J_0(s_n a) J_0(s_m a)] = 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad m \neq n \\
&\int_0^a J_0^2(s_m r) r dr = \frac{a^2}{2} [J_0^2(s_m a) + J_1^2(s_m a)]
\end{aligned} \tag{15.97}$$

Therefore, the temperature is given by

$$T(r,t) = T_a - \frac{2}{a^2}\sum_{n=1}^{\infty}\left\{\frac{1}{J_0^2(s_n a) + J_1^2(s_n a)}\int_0^a [T_a - T_i(r)] J_0(s_n r) r dr\right\} \times J_0(s_n r) e^{-\kappa s_n^2 t} \quad \text{(Answer)}$$

If the surface temperature on the surface $r = a$ is prescribed, the boundary condition is

$$T = T_a \quad \text{on} \quad r = a \tag{15.98}$$

For this case the temperature becomes

$$T(r,t) = T_a - \frac{2}{a^2}\sum_{n=1}^{\infty}\left\{\frac{1}{J_1^2(s_n a)}\int_0^a [T_a - T_i(r)] J_0(s_n r) r\, dr\right\} J_0(s_n r) e^{-\kappa s_n^2 t} \quad \text{(Answer)}$$

where s_n are eigenvalues of eigenfunction

$$J_0(s_n a) = 0 \tag{15.99}$$

Problem 15.10. Find the transient temperature in a hollow cylinder of inner radius a and outer radius b with the initial temperature $T_i(r)$ under heat transfer boundary conditions.

Solution. The problem to be solved consists of

(1) Governing equation:

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) \tag{15.100}$$

(2) Boundary conditions:

$$\begin{aligned} \lambda\frac{\partial T}{\partial r} &= h_a(T - T_a) \quad \text{on} \quad r = a \\ -\lambda\frac{\partial T}{\partial r} &= h_b(T - T_b) \quad \text{on} \quad r = b \end{aligned} \tag{15.101}$$

(3) Initial condition:

$$T = T_i(r) \quad \text{at} \quad t = 0 \tag{15.102}$$

The general solution of Eq. (15.100) is from (15.90)

$$T(r,t) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty}[A_n J_0(s_n r) + B_n Y_0(s_n r)] e^{-\kappa s_n^2 t} \tag{15.103}$$

Substitution of Eq. (15.103) into Eq. (15.101) gives

$$ah_a A_0 + B_0(ah_a \ln a - \lambda) - ah_a T_a + \sum_{n=1}^{\infty}\{A_n[ah_a J_0(s_n a) + \lambda s_n a J_1(s_n a)] + B_n[ah_a Y_0(s_n a) + \lambda s_n a Y_1(s_n a)]\}e^{-\kappa s_n^2 t} = 0$$
$$bh_b A_0 + B_0(bh_b \ln b + \lambda) - bh_b T_b + \sum_{n=1}^{\infty}\{A_n[bh_b J_0(s_n b) - \lambda s_n b J_1(s_n b)] + B_n[bh_b Y_0(s_n b) - \lambda s_n b Y_1(s_n b)]\}e^{-\kappa s_n^2 t} = 0 \quad (15.104)$$

Equation (15.104) gives

$$\begin{aligned} ah_a A_0 + B_0(ah_a \ln a - \lambda) &= ah_a T_a \\ bh_b A_0 + B_0(bh_b \ln b + \lambda) &= bh_b T_b \end{aligned} \quad (15.105)$$

and

$$\begin{aligned} A_n[ah_a J_0(s_n a) + \lambda s_n a J_1(s_n a)] + B_n[ah_a Y_0(s_n a) + \lambda s_n a Y_1(s_n a)] &= 0 \\ A_n[bh_b J_0(s_n b) - \lambda s_n b J_1(s_n b)] + B_n[bh_b Y_0(s_n b) - \lambda s_n b Y_1(s_n b)] &= 0 \end{aligned} \quad (15.106)$$

Solving Eq. (15.105) for A_0 and B_0, we get

$$A_0 = T_a - (T_b - T_a)\frac{\ln a - \dfrac{\lambda}{ah_a}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{ah_a} + \dfrac{\lambda}{bh_b}}, \quad B_0 = \frac{T_b - T_a}{\ln\dfrac{b}{a} + \dfrac{\lambda}{ah_a} + \dfrac{\lambda}{bh_b}} \quad (15.107)$$

Equation (15.106) is satisfied, if s_n are eigenvalues of the transcendental equation

$$[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)][h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)] - [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)][h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] = 0 \quad (15.108)$$

Referring to Eq. (15.108) we may put

$$G_n = \frac{h_a J_0(s_n a) + \lambda s_n J_1(s_n a)}{h_b J_0(s_n b) - \lambda s_n J_1(s_n b)} = \frac{h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)}{h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)} \quad (15.109)$$

Equation (15.108) can be written as

$$h_b f_0(s_n, b) - \lambda s_n f_1(s_n, b) = 0 \quad (15.110)$$

where

$$f_i(s_n, r) = J_i(s_n r)[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] - Y_i(s_n r)[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] \quad (i = 0, 1) \tag{15.111}$$

The temperature (15.103) reduces to

$$T = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} A_n f_0(s_n, r) e^{-\kappa s_n^2 t} \tag{15.112}$$

where $A_n/[a h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)]$ is rewritten as A_n. From the initial condition (15.102), Eq. (15.112) reduces to

$$\sum_{n=1}^{\infty} A_n f_0(s_n, r) = T_i(r) - (A_0 + B_0 \ln r) \tag{15.113}$$

Multiplying $r f_0(s_m r)$ to both sides of Eq. (15.113) and integrating form a to b, we get[2]

$$A_m = \frac{\pi^2 s_m^2}{2[(h_b^2 + \lambda^2 s_m^2) G_m^2 - (h_a^2 + \lambda^2 s_m^2)]} \times \int_a^b [T_i(r) - (A_0 + B_0 \ln r)] f_0(s_m, r) r dr \tag{15.114}$$

Substitution of Eq. (15.107) into Eq. (15.114), we get

$$A_m = \frac{\pi^2 s_m^2}{2[(h_b^2 + \lambda^2 s_m^2) G_m^2 - (h_a^2 + \lambda^2 s_m^2)]} \int_a^b T_i(r) f_0(s_m, r) r dr - \frac{\pi (T_b h_b G_m - T_a h_a)}{(h_b^2 + \lambda^2 s_m^2) G_m^2 - (h_a^2 + \lambda^2 s_m^2)} \tag{15.115}$$

Rewriting $f_0(s_m, r)$ as $f(s_m, r)$, the temperature can be expressed as

$$T = T_a + (T_b - T_a) \frac{\ln(r/a) + \lambda/(h_a a)}{\ln(b/a) + \lambda/(h_a a) + \lambda/(h_b b)} - \pi \sum_{n=1}^{\infty} \frac{T_a h_a - T_b h_b G_n}{(h_a^2 + \lambda^2 s_n^2) - (h_b^2 + \lambda^2 s_n^2) G_n^2} f(s_n, r) e^{-\kappa s_n^2 t} - \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{s_n^2 f(s_n, r)}{(h_a^2 + \lambda^2 s_n^2) - (h_b^2 + \lambda^2 s_n^2) G_n^2} \int_a^b T_i(\eta) f(s_n, \eta) \eta d\eta \, e^{-\kappa s_n^2 t}$$

(Answer)

[2] see: Problem 15.11.

Problem 15.11. Derive Eqs. (15.114) and (15.115) from Eq. (15.113) in Problem 15.10.

Solution. From Eq. (15.113), we have

$$\sum_{n=1}^{\infty} A_n f_0(s_n, r) = T_i(r) - (A_0 + B_0 \ln r) \tag{15.116}$$

Multiplying $r f_0(s_m, r)$ to Eq. (15.116) and integrating form a to b, we get

$$\int_a^b f_0(s_m, r) \sum_{n=1}^{\infty} A_n f_0(s_n, r) r dr = \int_a^b f_0(s_m, r)[T_i(r) - (A_0 + B_0 \ln r)] r dr \tag{15.117}$$

where

$$\begin{aligned} f_i(s_n, r) &= J_i(s_n r)[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] \\ &\quad - Y_i(s_n r)[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] \qquad (i = 0, 1) \end{aligned} \tag{15.118}$$

Before performing integration of Eq. (15.117), we calculate the following functions:

$$\begin{aligned} f_0(s_n, a) &= -\lambda s_n \frac{2}{\pi a s_n} = -\frac{2\lambda}{\pi a} \\ f_1(s_n, a) &= \frac{2h_a}{\pi s_n a} \\ f_0(s_n, b) &= J_0(s_n b)[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] \\ &\quad - Y_0(s_n b)[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] \\ &= J_0(s_n b) G_n [h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)] \\ &\quad - Y_0(s_n b) G_n [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)] \\ &= G_n \lambda s_n [J_1(s_n b) Y_0(s_n b) - Y_1(s_n b) J_0(s_n b)] \\ &= G_n \lambda s_n \frac{2}{\pi s_n b} = G_n \frac{2\lambda}{\pi b} \\ f_1(s_n, b) &= J_1(s_n b)[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] \\ &\quad - Y_1(s_n b)[h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] \\ &= J_1(s_n b) G_n [h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)] \\ &\quad - Y_1(s_n b) G_n [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)] \\ &= G_n h_b [J_1(s_n b) Y_0(s_n b) - Y_1(s_n b) J_0(s_n b)] = G_n \frac{2h_b}{\pi s_n b} \end{aligned} \tag{15.119}$$

in which the following formula of Bessel functions is used:

$$J_{n+1}(z)Y_n(z) - J_n(z)Y_{n+1}(z) = \frac{2}{\pi z} \tag{15.120}$$

and

$$G_n = \frac{h_a J_0(s_n a) + \lambda s_n J_1(s_n a)}{h_b J_0(s_n b) - \lambda s_n J_1(s_n b)} = \frac{h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)}{h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)} \tag{15.121}$$

By use of Eq. (15.119), the integration of the left side in Eq. (15.117) becomes

$$\begin{aligned}
&\int_a^b f_0(s_m, r) f_0(s_n, r) r dr \\
&\quad = \frac{b}{s_m^2 - s_n^2}\Big[s_m f_1(s_m, b) f_0(s_n, b) - s_n f_1(s_n, b) f_0(s_m, b)\Big] \\
&\qquad - \frac{a}{s_m^2 - s_n^2}\Big[s_m f_1(s_m, a) f_0(s_n, a) - s_n f_1(s_n, a) f_0(s_m, a)\Big] \\
&= \frac{b}{s_m^2 - s_n^2}\Big[\frac{h_b}{\lambda s_m} s_m f_0(s_m, b) f_0(s_n, b) - \frac{h_b}{\lambda s_n} s_n f_0(s_n, b) f_0(s_m, b)\Big] \\
&\qquad - \frac{a}{s_m^2 - s_n^2}\Big[s_m \frac{2h_a}{\pi s_m a}\Big(-\frac{2\lambda}{\pi a}\Big) - s_n \frac{2h_a}{\pi s_n a}\Big(-\frac{2\lambda}{\pi a}\Big)\Big] = 0 \\
&\int_a^b f_0^2(s_m, r) r dr = \frac{b^2}{2}\Big[f_0^2(s_m b) + f_1^2(s_m b)\Big] - \frac{a^2}{2}\Big[f_0^2(s_m a) + f_1^2(s_m a)\Big] \\
&\quad = \frac{b^2}{2}\frac{h_b^2 + \lambda^2 s_m^2}{h_b^2} f_1^2(s_m b) - \frac{a^2}{2}[f_0^2(s_m a) + f_1^2(s_m a)] \\
&\quad = \frac{b^2}{2}\frac{h_b^2 + \lambda^2 s_m^2}{h_b^2}\Big(G_m \frac{2h_b}{\pi s_m b}\Big)^2 - \frac{a^2}{2}\Big[\Big(\frac{2h_a}{\pi s_m a}\Big)^2 + \Big(-\frac{2\lambda}{\pi a}\Big)^2\Big] \\
&\quad = \frac{2}{\pi^2 s_m^2}\Big[(h_b^2 + \lambda^2 s_m^2)G_m^2 - (h_a^2 + \lambda^2 s_m^2)\Big]
\end{aligned} \tag{15.122}$$

From Eqs. (15.117) and (15.122), A_m is determined as

$$\begin{aligned}
A_m = &\frac{\pi^2 s_m^2}{2[(h_b^2 + \lambda^2 s_m^2)G_m^2 - (h_a^2 + \lambda^2 s_m^2)]} \\
&\times \int_a^b [T_i(r) - (A_0 + B_0 \ln r)] f_0(s_m, r) r dr
\end{aligned} \tag{Answer}$$

Calculating the following integral:

$$\begin{aligned}
&\int_a^b (A_0 + B_0 \ln r) f_0(s_m, r) r dr \\
&\quad = \Big[(A_0 + B_0 \ln r)\frac{r}{s_m} f_1(s_m, r) - \int \frac{B_0}{s_m} f_1(s_m, r) dr\Big]_a^b
\end{aligned}$$

$$
\begin{aligned}
&= \left[(A_0 + B_0 \ln r)\frac{r}{s_m} f_1(s_m, r) + \frac{B_0}{s_m^2} f_0(s_m, r)\right]_a^b \\
&= \frac{2}{\pi s_m^2}\left\{(h_b G_m - h_a)A_0 + B_0\left[\left(\ln b + \frac{\lambda}{b h_b}\right)h_b G_m \right.\right. \\
&\quad \left.\left. - \left(\ln a - \frac{\lambda}{a h_a}\right)h_a\right]\right\} \\
&= \frac{2}{\pi s_m^2}(h_b G_m - h_a)T_a + \frac{2}{\pi s_m^2}(T_b - T_a)h_b G_m \\
&= \frac{2}{\pi s_m^2}(T_b h_b G_m - T_a h_a) \qquad (15.123)
\end{aligned}
$$

we get

$$
\begin{aligned}
A_m = &\frac{\pi^2 s_m^2}{2[(h_b^2 + \lambda^2 s_m^2)G_m^2 - (h_a^2 + \lambda^2 s_m^2)]} \int_a^b T_i(r) f_0(s_m, r) r dr \\
&- \frac{\pi(T_b h_b G_m - T_a h_a)}{(h_b^2 + \lambda^2 s_m^2)G_m^2 - (h_a^2 + \lambda^2 s_m^2)} \qquad \text{(Answer)}
\end{aligned}
$$

Problem 15.12. Find the solution of the differential equation

$$
\frac{d^2 f(r)}{d^2 r} + \frac{1}{r}\frac{df(r)}{dr} - q^2 f(r) = g(r) \qquad (15.124)
$$

Equation (15.124) is appeared in the derivation of the transient temperature in the cylinder when Laplace transform technique is used.

Solution. The solution of homogeneous differential equation of Eq. (15.124) is

$$
f(r) = A I_0(qr) + B K_0(qr) \qquad (15.125)
$$

where $I_0(qr)$ and $K_0(qr)$ are modified Bessel functions.

We introduce the method of variation of parameters to obtain the particular solution of Eq. (15.124). We put $f(r)$ instead of Eq. (15.125)

$$
f(r) = A(r) I_0(qr) + B(r) K_0(qr) \qquad (15.126)
$$

Differentiation of Eq. (15.126) with respect to r gives

$$
\begin{aligned}
\frac{df(r)}{dr} &= \frac{dA(r)}{dr} I_0(qr) + \frac{dB(r)}{dr} K_0(qr) + A(r)\frac{dI_0(qr)}{dr} + B(r)\frac{dK_0(qr)}{dr} \\
\frac{d^2 f(r)}{dr^2} &= \frac{d}{dr}\left[\frac{dA(r)}{dr} I_0(qr) + \frac{dB(r)}{dr} K_0(qr)\right] + \frac{dA(r)}{dr}\frac{dI_0(qr)}{dr} \\
&\quad + \frac{dB(r)}{dr}\frac{dK_0(qr)}{dr} + A(r)\frac{d^2 I_0(qr)}{dr} + B(r)\frac{d^2 K_0(qr)}{dr} \qquad (15.127)
\end{aligned}
$$

Substitution of Eqs. (15.127) into Eq. (15.124) yields

$$\left(\frac{d}{dr}+\frac{1}{r}\right)\left[\frac{dA(r)}{dr}I_0(qr)+\frac{dB(r)}{dr}K_0(qr)\right] + \frac{dA(r)}{dr}\frac{dI_0(qr)}{dr}+\frac{dB(r)}{dr}\frac{dK_0(qr)}{dr}=g(r) \tag{15.128}$$

Equation (15.128) can be satisfied when we take

$$\begin{aligned}\frac{dA(r)}{dr}I_0(qr)+\frac{dB(r)}{dr}K_0(qr)&=0\\ \frac{dA(r)}{dr}\frac{dI_0(qr)}{dr}+\frac{dB(r)}{dr}\frac{dK_0(qr)}{dr}&=g(r)\end{aligned} \tag{15.129}$$

Solving Eq. (15.129), we get

$$\begin{aligned}\frac{dA(r)}{dr}&=g(r)\frac{K_0(qr)}{q[I_0(qr)K_1(qr)+I_1(qr)K_0(qr)]}\\ \frac{dB(r)}{dr}&=-g(r)\frac{I_0(qr)}{q[I_0(qr)K_1(qr)+I_1(qr)K_0(qr)]}\end{aligned} \tag{15.130}$$

where

$$\frac{dI_0(qr)}{dr}=qI_1(qr),\quad \frac{dK_0}{dr}=-qK_1(qr) \tag{15.131}$$

Using the following formula

$$I_0(qr)K_1(qr)+I_1(qr)K_0(qr)=\frac{1}{qr} \tag{15.132}$$

Equation (15.130) reduce to

$$\frac{dA(r)}{dr}=rg(r)K_0(qr),\quad \frac{dB(r)}{dr}=-rg(r)I_0(qr) \tag{15.133}$$

Then, we obtain

$$A(r)=\int_r rg(r)K_0(qr)dr,\quad B(r)=-\int_r rg(r)I_0(qr)dr \tag{15.134}$$

Therefore the general solution is given by

$$f(r)=AI_0(qr)+BK_0(qr)+\int_r \eta g(\eta)[I_0(qr)K_0(q\eta)-K_0(qr)I_0(q\eta)]d\eta \quad \text{(Answer)} \tag{15.135}$$

Problem 15.13. Find the transient temperature in a hollow cylinder, when the initial condition is $T_i(r)$, and the boundary conditions of the hollow cylinder are given by following five cases (1)–(5):

[1] Prescribed surface temperatures T_a and T_b at both surfaces $r = a$ and $r = b$, respectively.
[2] Prescribed surface temperature T_a at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[3] Prescribed surface temperature T_b at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.
[4] Constant heat flux $q_a (= \lambda(\partial T/\partial r))$ at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[5] Constant heat flux $q_b (= -\lambda(\partial T/\partial r))$ at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.

Solution. When the both surfaces are heat transfer conditions

$$\lambda \frac{\partial T}{\partial r} = h_a (T - T_a) \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = h_b (T - T_b) \quad \text{on} \quad r = b \tag{15.136}$$

the temperature is given by Eq. (15.24). Comparing between the heat transfer conditions (15.136) and each boundary condition, the temperature for each boundary condition can be obtained.

[1] The boundary conditions on $r = a$ and $r = b$ for this problem are

$$T = T_a \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.137}$$

Rewriting the boundary conditions (15.136) gives

$$T = T_a + \frac{\lambda}{h_a} \frac{\partial T}{\partial r} \quad \text{on} \quad r = a, \quad T = T_b - \frac{\lambda}{h_b} \frac{\partial T}{\partial r} \quad \text{on} \quad r = b \tag{15.138}$$

Putting $h_a \to \infty$ and $h_b \to \infty$ in Eq. (15.138), Eq. (15.138) reduces to Eq. (15.137). Therefore, we can obtain the temperature from Eq. (15.24) after putting $h_a \to \infty$ and $h_b \to \infty$.

$$T = T_a + (T_b - T_a)\frac{\ln\dfrac{r}{a}}{\ln\dfrac{b}{a}} - \pi\sum_{n=1}^{\infty}\frac{T_a J_0(s_n b) - T_b J_0(s_n a)}{J_0^2(s_n b) - J_0^2(s_n a)} J_0(s_n b) f(s_n, r)\, e^{-\kappa s_n^2 t} - \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 J_0^2(s_n b) f(s_n, r)}{J_0^2(s_n b) - J_0^2(s_n a)}\int_a^b T_i(\eta) f(s_n, \eta)\eta\, d\eta\, e^{-\kappa s_n^2 t} \quad \text{(Answer)}$$

where

$$f(s_n, r) = Y_0(s_n a) J_0(s_n r) - J_0(s_n a) Y_0(s_n r) \tag{15.139}$$

and s_n are eigenvalues of the eigenfunction $f(s_n, b) = 0$.

[2] The boundary conditions of this case are

$$T = T_a \quad \text{on} \quad r = a, \quad -\lambda\frac{\partial T}{\partial r} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.140}$$

If we put $h_a \to \infty$ in Eq. (15.138), Eq. (15.138) reduces to Eq. (15.140). Therefore, we can obtain the temperature from Eq. (15.24) after putting $h_a \to \infty$.

$$T = T_a + (T_b - T_a)\frac{\ln\dfrac{r}{a}}{\ln\dfrac{b}{a} + \dfrac{\lambda}{h_b b}} - \pi\sum_{n=1}^{\infty}\frac{(T_a - T_b G_n h_b) f(s_n, r)}{1 - G_n^2(h_b^2 + \lambda^2 s_n^2)} e^{-\kappa s_n^2 t} - \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 f(s_n, r) e^{-\kappa s_n^2 t}}{1 - G_n^2(h_b^2 + \lambda^2 s_n^2)}\int_a^b T_i(\eta) f(s_n, \eta)\eta\, d\eta \quad \text{(Answer)}$$

where $f(s_n, r)$ and G_n are given by

$$f(s_n, r) = Y_0(s_n a) J_0(s_n r) - J_0(s_n a) Y_0(s_n r)$$
$$G_n = \frac{J_0(s_n a)}{h_b J_0(s_n b) - \lambda s_n J_1(s_n b)} = \frac{Y_0(s_n a)}{h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)} \tag{15.141}$$

and s_n are eigenvalues of the eigenfunction

$$[h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)] J_0(s_n a) - [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)] Y_0(s_n a) = 0 \tag{15.142}$$

[3] The boundary conditions of this case are

$$\lambda \frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.143}$$

By comparison between Eqs. (15.138) and (15.143), Eq. (15.138) reduces to Eq. (15.143) if we put $h_b \to \infty$. Therefore, we can obtain the temperature from Eq. (15.24) after putting $h_b \to \infty$.

$$\begin{aligned} T = T_a &+ (T_b - T_a) \frac{\ln \dfrac{r}{a} + \dfrac{\lambda}{h_a a}}{\ln \dfrac{b}{a} + \dfrac{\lambda}{h_a a}} \\ &- \pi \sum_{n=1}^{\infty} \frac{(T_a h_a - T_b G_n) f(s_n, r)}{h_a^2 + \lambda^2 s_n^2 - G_n^2} e^{-\kappa s_n^2 t} \\ &- \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{s_n^2 f(s_n, r) e^{-\kappa s_n^2 t}}{h_a^2 + \lambda^2 s_n^2 - G_n^2} \int_a^b T_i(\eta) f(s_n, \eta) \eta \, d\eta \end{aligned} \quad \text{(Answer)}$$

where $f(s_n, r)$ and G_n are given by

$$\begin{aligned} f(s_n, r) &= [h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)] J_0(s_n r) \\ &\quad - [h_a J_0(s_n a) + \lambda s_n J_1(s_n a)] Y_0(s_n r) \\ G_n &= \frac{h_a J_0(s_n a) + \lambda s_n J_1(s_n a)}{J_0(s_n b)} = \frac{h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)}{Y_0(s_n b)} \end{aligned} \quad \text{(Answer)}$$

and s_n are eigenvalues of the eigenfunction $f(s_n, b) = 0$.

[4] The boundary conditions of this case are

$$\lambda \frac{\partial T}{\partial r} = q_a \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.144}$$

By comparison between Eqs. (15.136) and (15.144), after rewriting $h_a T_a = -q_a$ and putting $h_a \to 0$, Eq. (15.136) reduces to Eq. (15.144). Therefore, we can obtain the temperature from Eq. (15.24), after rewriting $h_a T_a = -q_a$ and putting $h_a \to 0$.

$$\begin{aligned} T = T_b &+ \frac{q_a a}{\lambda} \left(\ln \frac{r}{b} - \frac{\lambda}{h_b b} \right) \\ &+ \pi \sum_{n=1}^{\infty} \frac{(q_a + T_b G_n h_b \lambda s_n) f(s_n, r)}{\lambda s_n [1 - G_n^2 (h_b^2 + \lambda^2 s_n^2)]} e^{-\kappa s_n^2 t} \end{aligned}$$

$$-\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 f(s_n,r)e^{-\kappa s_n^2 t}}{1-G_n^2(h_b^2+\lambda^2 s_n^2)}\int_a^b T_i(\eta)f(s_n,\eta)\eta\,d\eta \qquad \text{(Answer)}$$

where $f(s_n, r)$ and G_n are given by

$$f(s_n,r) = Y_1(s_n a)J_0(s_n r) - J_1(s_n a)Y_0(s_n r)$$
$$G_n = \frac{J_1(s_n a)}{h_b J_0(s_n b) - \lambda s_n J_1(s_n b)} = \frac{Y_1(s_n a)}{h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)} \qquad (15.145)$$

and s_n are eigenvalues of the eigenfunction

$$[h_b Y_0(s_n b) - \lambda s_n Y_1(s_n b)]J_1(s_n a) - [h_b J_0(s_n b) - \lambda s_n J_1(s_n b)]Y_1(s_n a) = 0 \qquad (15.146)$$

[5] The boundary conditions of this case are

$$\lambda\frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad -\lambda\frac{\partial T}{\partial r} = q_b \quad \text{on} \quad r = b \qquad (15.147)$$

By comparison between Eqs. (15.136) and (15.147), after rewriting $h_b T_b = -q_b$ and putting $h_b \to 0$, Eq. (15.136) reduces to Eq. (15.147). Therefore, we can obtain the temperature from the temperature Eq. (15.24), after rewriting $h_b T_b = -q_b$ and putting $h_b \to 0$.

$$T = T_a - \frac{q_b b}{\lambda}\left(\ln\frac{r}{a} + \frac{\lambda}{h_a a}\right)$$
$$-\pi\sum_{n=1}^{\infty}\frac{(T_a h_a \lambda s_n - q_b G_n)f(s_n,r)}{\lambda s_n(h_a^2+\lambda^2 s_n^2 - G_n^2)}e^{-\kappa s_n^2 t}$$
$$-\frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{s_n^2 f(s_n,r)e^{-\kappa s_n^2 t}}{h_a^2+\lambda^2 s_n^2 - G_n^2}\int_a^b T_i(\eta)f(s_n,\eta)\eta\,d\eta \qquad \text{(Answer)}$$

where $f(s_n, r)$ and G_n are given by

$$f(s_n,r) = [h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)]J_0(s_n r) - [h_a J_0(s_n a) + \lambda s_n J_1(s_n a)]Y_0(s_n r)$$
$$G_n = \frac{h_a J_0(s_n a) + \lambda s_n J_1(s_n a)}{J_1(s_n b)} = \frac{h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)}{Y_1(s_n b)} \qquad (15.148)$$

and s_n are eigenvalues of the eigenfunction

$$[h_a Y_0(s_n a) + \lambda s_n Y_1(s_n a)]J_1(s_n b) - [h_a J_0(s_n a) + \lambda s_n J_1(s_n a)]Y_1(s_n b) = 0 \qquad (15.149)$$

Problem 15.14. Find the transient temperature in the hollow sphere, when the initial condition is $T_i(r)$, and the boundary conditions of a hollow sphere are given by following five cases (1)–(5):

[1] Prescribed surface temperatures T_a and T_b at both surfaces $r = a$ and $r = b$, respectively.
[2] Prescribed surface temperature T_a at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[3] Prescribed surface temperature T_b at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.
[4] Constant heat flux $q_a (= \lambda(\partial T/\partial r))$ at the inner surface $r = a$, and heat transfer between the outer surface and the surrounding medium with temperature T_b at the outer surface $r = b$.
[5] Constant heat flux $q_b (= -\lambda(\partial T/\partial r))$ at the outer surface $r = b$, and heat transfer between the inner surface and the surrounding medium with temperature T_a at the inner surface $r = a$.

Solution. When at both surfaces the heat transfer conditions are

$$\lambda \frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.150}$$

the temperature is given by Eq. (15.32). By comparing between the heat transfer conditions (15.150) and each boundary condition, the temperature for each boundary condition can be obtained.

[1] The boundary conditions on $r = a$ and $r = b$ for this problem are

$$T = T_a \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.151}$$

Rewriting the boundary conditions (15.150) gives

$$T = T_a + \frac{\lambda}{h_a}\frac{\partial T}{\partial r} \quad \text{on} \quad r = a, \quad T = T_b - \frac{\lambda}{h_b}\frac{\partial T}{\partial r} \quad \text{on} \quad r = b \tag{15.152}$$

If we put $h_a \to \infty$ and $h_b \to \infty$ in Eq. (15.152), Eq. (15.152) reduces to Eq. (15.151). Therefore, we can obtain the temperature from Eq. (15.32) after putting $h_a \to \infty$ and $h_b \to \infty$

$$T = T_a + (T_b - T_a)\frac{1 - \dfrac{a}{r}}{1 - \dfrac{a}{b}} + \frac{2}{(b-a)r}\sum_{n=1}^{\infty} \sin s_n(r-a) e^{-\kappa s_n^2 t}$$
$$\times \int_a^b \left[T_i(\eta) - T_a - (T_b - T_a)\frac{1 - \dfrac{a}{\eta}}{1 - \dfrac{a}{b}} \right] \eta \sin s_n(\eta - a) d\eta \qquad \text{(Answer)}$$

where $s_n = n\pi/(b-a)$.

[2] The boundary conditions of this case are

$$T = T_a \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.153}$$

If we put $h_a \to \infty$ in Eq. (15.152), Eq. (15.152) reduces to Eq. (15.153). Therefore, we can obtain the temperature from Eq. (15.32), after putting $h_a \to \infty$.

$$\begin{aligned} T = T_a + (T_b - T_a) \frac{1 - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} + \frac{2}{r} \sum_{n=1}^{\infty} \sin s_n (r-a) e^{-\kappa s_n^2 t} \\ \times \frac{\lambda^2 s_n^2 b^2 + (h_b b - \lambda)^2}{(b-a)[\lambda^2 s_n^2 b^2 + (h_b b - \lambda)^2] + \lambda b (h_b b - \lambda)} \\ \times \int_a^b \left[T_i(\eta) - T_a - \frac{(T_b - T_a)\left(1 - \dfrac{a}{\eta}\right)}{1 - \dfrac{a}{b} + \dfrac{a}{b}\dfrac{\lambda}{h_b b}} \right] \eta \sin s_n (\eta - a)\, d\eta \end{aligned}$$

(Answer)

where s_n are eigenvalues of the eigenfunction

$$(h_b b - \lambda) \sin s_n (b-a) + \lambda s_n b \cos s_n (b-a) = 0 \tag{15.154}$$

[3] The boundary conditions of this case are

$$\lambda \frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad T = T_b \quad \text{on} \quad r = b \tag{15.155}$$

If we put $h_b \to \infty$ in Eq. (15.152), Eq. (15.152) reduces to Eq. (15.155). Therefore, we can obtain the temperature from Eq. (15.32) after putting $h_b \to \infty$.

$$\begin{aligned} T = T_a + (T_b - T_a) \frac{1 + \dfrac{\lambda}{h_a a} - \dfrac{a}{r}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a}} \\ + \frac{2}{r} \sum_{n=1}^{\infty} \frac{(h_a a + \lambda) \sin s_n (r-a) + \lambda s_n a \cos s_n (r-a)}{(b-a)[\lambda^2 s_n^2 a^2 + (h_a a + \lambda)^2] + \lambda a (h_a a + \lambda)} e^{-\kappa s_n^2 t} \end{aligned}$$

$$\times \int_a^b \left[T_i(\eta) - T_a - (T_b - T_a) \left(\frac{1 + \dfrac{\lambda}{h_a a} - \dfrac{a}{\eta}}{1 - \dfrac{a}{b} + \dfrac{\lambda}{h_a a}} \right) \right]$$
$$\times \eta[(h_a a + \lambda) \sin s_n(\eta - a) + \lambda s_n a \cos s_n(\eta - a)]\, d\eta \qquad \text{(Answer)}$$

where s_n are eigenvalues of the eigenfunction

$$(h_a a + \lambda) \sin s_n\, (b - a) + \lambda s_n a \cos s_n\, (b - a) = 0 \tag{15.156}$$

[4] The boundary conditions of this case are

$$\lambda \frac{\partial T}{\partial r} = q_a \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = h_b(T - T_b) \quad \text{on} \quad r = b \tag{15.157}$$

If we rewrite $h_a T_a = -q_a$ and put $h_a \to 0$ in Eq. (15.150), Eq. (15.150) reduces to Eq. (15.157). Therefore, we can obtain the temperature from Eq. (15.32) after rewriting $h_a T_a = -q_a$ and putting $h_a \to 0$.

$$T = T_b - \frac{q_a a}{\lambda} \frac{a}{b} \left(\frac{b}{r} - 1 + \frac{\lambda}{h_b b} \right)$$
$$+ \frac{2}{r} \sum_{n=1}^{\infty} \frac{[\lambda^2 s_n^2 b^2 + (h_b b - \lambda)^2][\sin s_n(r - a) + s_n a \cos s_n(r - a)]}{\begin{pmatrix} (b - a)(1 + s_n^2 a^2)[\lambda^2 s_n^2 b^2 + (h_b b - \lambda)^2] \\ + [b\lambda + a(h_b b - \lambda)][\lambda s_n^2 ab + (h_b b - \lambda)] \end{pmatrix}}$$
$$\times e^{-\kappa s_n^2 t} \int_a^b \left[T_i(\eta) - T_b + \frac{q_a a}{\lambda} \frac{a}{b} \left(\frac{b}{\eta} - 1 + \frac{\lambda}{h_b b} \right) \right]$$
$$\times \eta[\sin s_n(\eta - a) + s_n a \cos s_n(\eta - a)]\, d\eta \qquad \text{(Answer)}$$

where s_n are eigenvalues of the eigenfunction

$$\left(h_b b - \lambda - \lambda s_n^2 ab \right) \sin s_n\, (b - a) + s_n\, [a\, (h_b b - \lambda) + b\lambda] \cos s_n\, (b - a) = 0 \tag{15.158}$$

[5] The boundary conditions of this case are

$$\lambda \frac{\partial T}{\partial r} = h_a(T - T_a) \quad \text{on} \quad r = a, \quad -\lambda \frac{\partial T}{\partial r} = q_b \quad \text{on} \quad r = b \tag{15.159}$$

If we rewrite $h_b T_b = -q_b$ and put $h_b \to 0$ in Eq. (15.150), Eq. (15.150) reduces to Eq. (15.159). Therefore, we can obtain the temperature from Eq. (15.32) after rewriting $h_b T_b = -q_b$ and putting $h_b \to 0$.

$$T = T_a - \frac{q_b b}{\lambda} \frac{b}{a} \left(1 - \frac{a}{r} + \frac{\lambda}{h_a a} \right)$$

$$+\frac{2}{r}\sum_{n=1}^{\infty}\frac{(s_n^2b^2+1)[(h_aa+\lambda)\sin s_n(r-a)+\lambda s_na\cos s_n(r-a)]}{\begin{pmatrix}(b-a)(1+s_n^2b^2)[\lambda^2s_n^2a^2+(h_aa+\lambda)^2]\\+[b(h_aa+\lambda)-a\lambda][\lambda s_n^2ab-(h_aa+\lambda)]\end{pmatrix}}$$
$$\times e^{-\kappa s_n^2t}\int_a^b\left[T_i(\eta)-T_a+\frac{q_bb}{\lambda}\frac{b}{a}\left(1-\frac{a}{\eta}+\frac{\lambda}{h_aa}\right)\right]$$
$$\times\eta[(h_aa+\lambda)\sin s_n(\eta-a)+\lambda s_na\cos s_n(\eta-a)]\,d\eta \qquad \text{(Answer)}$$

where s_n are eigenvalues of the eigenfunction

$$\left(h_aa+\lambda+\lambda s_n^2ab\right)\sin s_n\,(b-a)-s_n\,[b\,(h_aa+\lambda)-a\lambda]\cos s_n\,(b-a)=0 \tag{15.160}$$

Problem 15.15. When a solid cylinder with the initial temperature $T_i(r)$ is exposed to heat transfer between the surface of radius a and the surrounding medium with time dependent temperature $T_a(t)$, find the transient temperature in the solid cylinder.

Solution. The equations to be solved are

(1) Governing equation

$$\frac{1}{\kappa}\frac{\partial T}{\partial t}=\frac{\partial^2T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r} \tag{15.161}$$

(2) Boundary condition

$$-\lambda\frac{\partial T}{\partial r}=h_a[T-T_a(t)] \quad \text{on} \quad r=a \tag{15.162}$$

(3) Initial condition

$$T=T_i(r) \quad \text{at} \quad t=0 \tag{15.163}$$

Applying the Laplace transform with respect to the time t and taking the initial condition into consideration, we obtain
Governing equation:

$$\frac{d^2\bar{T}}{dr^2}+\frac{1}{r}\frac{d\bar{T}}{dr}-q^2\bar{T}=-\frac{1}{\kappa}T_i(r) \tag{15.161$'$}$$

Boundary condition:

$$-\lambda\frac{d\bar{T}}{dr}=h_a(\bar{T}-\bar{T}_a) \quad \text{on} \quad r=a \tag{15.162$'$}$$

where $q^2=p/\kappa$.

The general solution of Eq. (15.161′) for a solid cylinder is obtained from Eq. (15.135) of Problem 15.12 by putting $B=0$ and $g(\eta)=-T_i(\eta)/\kappa$

$$\bar{T} = AI_0(qr) - \frac{1}{\kappa}\int_0^r T_i(\eta)\eta[I_0(qr)K_0(q\eta) - I_0(q\eta)K_0(qr)]d\eta$$
$$= AI_0(qr) + G(qr) \tag{15.164}$$

where

$$G(qr) = -\frac{1}{\kappa}\int_0^r T_i(\eta)\eta[I_0(qr)K_0(q\eta) - I_0(q\eta)K_0(qr)]d\eta \tag{15.165}$$

Differentiation of Eq. (15.164) with respect to r gives

$$\frac{d\bar{T}}{dr} = AqI_1(qr) + G_1(qr) \tag{15.166}$$

where

$$G_1(qr) \equiv \frac{dG(qr)}{dr} = -\frac{1}{\kappa}\int_0^r T_i(\eta)\eta q[I_1(qr)K_0(q\eta) + I_0(q\eta)K_1(qr)]d\eta \tag{15.167}$$

The boundary condition (15.162′) gives

$$A\lambda qI_1(qa) + \lambda G_1(qa) + Ah_aI_0(qa) + h_aG(qa) = h_a\bar{T}_a \tag{15.168}$$

Then, A is given by

$$A = \frac{h_a\bar{T}_a}{\lambda qI_1(qa) + h_aI_0(qa)} - \frac{\lambda G_1(qa) + h_aG(qa)}{\lambda qI_1(qa) + h_aI_0(qa)}$$
$$= \frac{h_a\bar{T}_a}{\lambda qI_1(qa) + h_aI_0(qa)} + \frac{\lambda qK_1(qa) - h_aK_0(qa)}{\lambda qI_1(qa) + h_aI_0(qa)}$$
$$\times \frac{1}{\kappa}\int_0^a T_i(\eta)\eta I_0(q\eta)d\eta + \frac{1}{\kappa}\int_0^a T_i(\eta)\eta K_0(q\eta)d\eta \tag{15.169}$$

Hence, the temperature in the Laplace transformed domain is

$$\bar{T} = \frac{h_a\bar{T}_aI_0(qr)}{\lambda qI_1(qa) + h_aI_0(qa)}$$
$$+ \frac{\lambda qK_1(qa) - h_aK_0(qa)}{\lambda qI_1(qa) + h_aI_0(qa)}\frac{I_0(qr)}{\kappa}\int_0^a T_i(\eta)\eta I_0(q\eta)d\eta$$
$$+ \frac{I_0(qr)}{\kappa}\int_0^a T_i(\eta)\eta K_0(q\eta)d\eta + G(qr) \tag{15.170}$$

or an alternative form

$$\bar{T} = \bar{T}_a\bar{T}_1 + \bar{T}_2 + \bar{T}_3 \tag{15.171}$$

where

$$\bar{T}_1 = \frac{h_a I_0(qr)}{\lambda q I_1(qa) + h_a I_0(qa)} \tag{15.171$'$}$$

$$\bar{T}_2 = \frac{\lambda q K_1(qa) - h_a K_0(qa)}{\lambda q I_1(qa) + h_a I_0(qa)} \frac{I_0(qr)}{\kappa} \int_0^a T_i(\eta)\eta I_0(q\eta)d\eta \tag{15.171$''$}$$

$$\bar{T}_3 = \frac{I_0(qr)}{\kappa} \int_0^a T_i(\eta)\eta K_0(q\eta)d\eta + G(qr) \tag{15.171$'''$}$$

The inverse Laplace transform of Eq. (15.170) reduces to calculation of the sum of the residues at the poles in the inner region with the contour. Since Eq. (15.171$'''$) has no pole, the inverse Laplace transform of Eq. (15.171$'''$) reduces to zero. Equations (15.171$'$) and (15.171$''$) have poles at $p = -\kappa s_n^2$, and s_n are eigenvalues of the eigenfunction

$$\lambda s_n J_1(s_n a) - h_a J_0(s_n a) = 0 \tag{15.172}$$

The residue of $\bar{T}_1$ is

$$\begin{aligned}
&\left.\frac{h_a I_0(qr)e^{pt}}{\frac{d}{dp}[\lambda q I_1(qa) + h_a I_0(qa)]}\right|_{p=-\kappa s_n^2} = \left.\frac{h_a I_0(qr)e^{-\kappa s_n^2 t}}{\frac{1}{2q\kappa}\frac{d}{dq}[\lambda q I_1(qa) + h_a I_0(qa)]}\right|_{q=is_n} \\
&= \frac{2is_n\kappa h_a I_0(is_n r)e^{-\kappa s_n^2 t}}{a[\lambda i s_n I_0(is_n a) + h_a I_1(is_n a)]} = \frac{2is_n\kappa h_a J_0(s_n r)e^{-\kappa s_n^2 t}}{a[\lambda i s_n J_0(s_n a) + i h_a J_1(s_n a)]} \\
&= \frac{2s_n\kappa h_a J_0(s_n r)e^{-\kappa s_n^2 t}}{a[\lambda s_n J_0(s_n a) + h_a J_1(s_n a)]} = \frac{2\kappa\lambda s_n^2 h_a J_0(s_n r)e^{-\kappa s_n^2 t}}{a(\lambda^2 s_n^2 + h_a^2)J_0(s_n a)}
\end{aligned} \tag{15.173}$$

where $i^2 = -1$. On the other hand,

$$\begin{aligned}
&\lambda q K_1(qa) - h_a K_0(qa)|_{q=is_n} \\
&= \lambda i s_n(-\frac{\pi}{2})[J_1(s_n a) - iY_1(s_n a)] - h_a(-\frac{\pi}{2}i)[J_0(s_n a) - iY_0(s_n a)] \\
&= \frac{\pi}{2}i[h_a J_0(s_n a) - \lambda s_n J_1(s_n a)] + \frac{\pi}{2}[h_a Y_0(s_n a) - \lambda s_n Y_1(s_n a)]
\end{aligned} \tag{15.174}$$

Using Eq. (15.172), Eq. (15.174) reduces to

$$\begin{aligned}
&\lambda q K_1(qa) - h_a K_0(qa)|_{q=is_n} \\
&= \frac{\pi}{2}\left\{h_a Y_0(s_n a) - \frac{\lambda s_n}{J_0(s_n a)}\left[J_1(s_n a)Y_0(s_n a) - \frac{2}{\pi s_n a}\right]\right\}
\end{aligned}$$

$$= \frac{\pi}{2}[h_a J_0(s_n a) - \lambda s_n J_1(s_n a)] \frac{Y_0(s_n a)}{J_0(s_n a)} + \frac{\lambda}{a J_0(s_n a)}$$
$$= \frac{\lambda}{a J_0(s_n a)} \tag{15.175}$$

Residue of $\bar{T}_2$ is

$$\frac{\lambda q K_1(qa) - h_a K_0(qa)}{\dfrac{d}{dp}[\lambda q I_1(qa) + h_a I_0(qa)]} \frac{I_0(qr)}{\kappa} \int_0^a T_i(\eta) \eta I_0(q\eta) d\eta e^{pt} \Big|_{p=-\kappa s_n^2}$$
$$= \frac{2\lambda^2 s_n^2 J_0(s_n r) e^{-\kappa s_n^2 t}}{a^2(\lambda^2 s_n^2 + h_a^2) J_0^2(s_n a)} \int_0^a T_i(\eta) \eta J_0(s_n \eta) d\eta \tag{15.176}$$

We get

$$L^{-1}[\bar{T}_a \bar{T}_1] = \int_0^t T_a(\tau) T_1(t-\tau) d\tau$$
$$= \frac{2\kappa\lambda h_a}{a} \int_0^t T_a(\tau) \sum_{n=1}^{\infty} \frac{s_n^2 J_0(s_n r) e^{-\kappa s_n^2 (t-\tau)}}{(\lambda^2 s_n^2 + h_a^2) J_0(s_n a)} d\tau \tag{15.177}$$

Therefore, the temperature is given by

$$T = \frac{2\kappa\lambda h_a}{a} \int_0^t T_a(\tau) \sum_{n=1}^{\infty} \frac{s_n^2 J_0(s_n r) e^{-\kappa s_n^2 (t-\tau)}}{(\lambda^2 s_n^2 + h_a^2) J_0(s_n a)} d\tau$$
$$+ \frac{2\lambda^2}{a^2} \sum_{n=1}^{\infty} \frac{s_n^2 J_0(s_n r) e^{-\kappa s_n^2 t}}{(\lambda^2 s_n^2 + h_a^2) J_0^2(s_n a)} \int_0^a T_i(\eta) \eta J_0(s_n \eta) d\eta \qquad \text{(Answer)}$$