Chapter 13
Thermal Stresses in Bars

In this chapter the concept of thermal stresses in bars is introduced for the simple case of a perfectly clamped bar subjected to arbitrary temperature change. The problems and solutions related to thermal stresses in bars are: a perfectly clamped bar, a clamped bar with a small gap, a clamped circular frustum, a bar with variable cross-sectional area, two bars attached to each other, three bars fastened to each other, truss of three bars, and three bars hanging from a rigid plate.

13.1 Thermal Stresses in Bars

When the temperature of a circular bar of length l changes from an initial temperature T_0 to its final temperature T_1, the free thermal elongation λ_T of the bar is defined by

$$\lambda_T = \alpha(T_1 - T_0)l = \alpha\tau l \tag{13.1}$$

where α is the coefficient of linear thermal expansion which is measured in one per one degree of the temperature 1/K, and τ denotes the temperature change given by

$$\tau = T_1 - T_0 \tag{13.2}$$

The free thermal strain is given by

$$\epsilon_T = \frac{\lambda_T}{l} = \alpha\tau \tag{13.3}$$

When an internal force and the temperature change act simultaneously in the bar, the normal strain is given by

$$\epsilon = \epsilon_s + \epsilon_T \tag{13.4}$$

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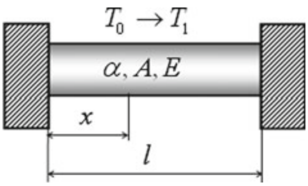


Fig. 13.1 A perfectly clampled bar

where ϵ_s denotes the strain produced by the internal force. The strain ϵ_s produced by the internal force is proportional to the normal stress σ

$$\epsilon_s = \frac{\sigma}{E} \tag{13.5}$$

where E denotes Young's modulus.

Hooke's law with the temperature change is

$$\epsilon = \frac{\sigma}{E} + \alpha\tau \tag{13.6}$$

When a perfectly clamped bar with length l and cross-sectional area A, shown in Fig. 13.1, is subjected to the uniform temperature change τ, the thermal stress is

$$\sigma = -\alpha E\tau \tag{13.7}$$

If the temperature change $\tau(x)$ is a function of the position x, the free thermal elongation λ_T of the bar of length l is

$$\lambda_T = \int d\lambda_T = \int_0^l \alpha\tau(x)\,dx = \alpha \int_0^l \tau(x)\,dx \tag{13.8}$$

The thermal strain ϵ_T is

$$\epsilon_T = \frac{\lambda_T}{l} = \frac{\alpha}{l}\int_0^l \tau(x)\,dx \tag{13.9}$$

The thermal stress in the perfectly clamped bar is

$$\sigma = -\frac{\alpha E}{l}\int_0^l \tau(x)\,dx \tag{13.10}$$

13.2 Problems and Solutions Related to Thermal Stresses in Bars

Problem 13.1. If the temperature in a mild steel rail with length 25 m is raised to 50 K, and the coefficient of linear thermal expansion for mild steel is 11.2×10^{-6} 1/K, what elongation is produced in the rail?

Solution. The elongation λ_T is from Eq. (13.1)

$$\lambda_T = \alpha\tau l = 11.2\times10^{-6}\times50\times25 = 14\times10^{-3}\,\text{m} = 14\,\text{mm} \qquad \text{(Answer)}$$

Problem 13.2. The temperature of a bar of length 1 m of mild steel is kept at 300 K. If the temperature at one end of the bar is raised to 380 K and at the other end to 480 K, and the temperature distribution is linear along the bar, what elongation is produced in the bar? The coefficient of linear thermal expansion for mild steel is 11.2×10^{-6} 1/K.

Solution. The temperature rise $\tau(x) = T_1(x) - T_0$ is

$$\tau(x) = T_1(x) - T_0 = \left[380 + (480-380)\frac{x}{1}\right] - 300 = 80 + 100x \qquad (13.11)$$

The free thermal elongation λ_T is

$$\begin{aligned}\lambda_T &= \int_0^1 \alpha\tau(x)dx = \alpha\int_0^1(80+100x)dx\\ &= 11.2\times10^{-6}\times\left[80x+50x^2\right]_0^1 = 1.456\times10^{-3}\,\text{m} = 1.46\,\text{mm} \qquad \text{(Answer)}\end{aligned} \qquad (13.12)$$

Problem 13.3. A bar of mild steel at 300 K is clamped between two walls. Calculate the thermal stress produced in the bar when the bar is heated to 360 K. The coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2\times10^{6}$ 1/K and $E = 206$ GPa, respectively.

Solution. The thermal stress σ is from Eq. (13.7)

$$\sigma = -\alpha E\tau = -138.4\times10^6\,\text{Pa} = -138\,\text{MPa} \qquad \text{(Answer)}$$

Problem 13.4. In Problem 13.2, calculate the thermal stress produced in the bar if it is clamped between two walls. The coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2\times10^{6}$ 1/K and $E = 206$ GPa, respectively.

Solution. As the summation of the free thermal elongation λ_T and the elongation λ_s due to the stress is zero, we get

$$\sigma = \frac{E\lambda_s}{l} = -\frac{E\lambda_T}{l} = -\frac{206 \times 10^9 \times 1.456 \times 10^{-3}}{1} = -300\,\text{MPa} \quad \text{(Answer)}$$

Problem 13.5. A bar of mild steel at 300 K is clamped between two walls in such a way that the initial stress is zero. Calculate the temperature when the thermal stress in the bar reaches the compressive strength ($\sigma_{BC} = 400$ MPa). The coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2 \times 10^6$ 1/K and $E = 206$ GPa, respectively.

Solution. The compressive thermal stress σ is given by Eq. (13.7). Therefore, the temperature rise τ is

$$\tau = \frac{\sigma_{BC}}{\alpha E} = \frac{400 \times 10^6}{11.2 \times 10^{-6} \times 206 \times 10^9} = 173.37 \tag{13.13}$$

Then

$$T_1 = T_0 + \tau = 300 + 173.37 = 473.37\,\text{K} = 473\,\text{K} \quad \text{(Answer)}$$

Problem 13.6. The temperature of a bar with a small gap $e = 1$ mm, shown in Fig. 13.2 is kept at 300 K. If the temperature at one end of the bar is raised to 380 K and at the other end to 480 K, and the temperature distribution is linear along the bar, calculate the thermal stress. Where length of the bar is 1 m, and the coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2 \times 10^6$ 1/K and $E = 206$ GPa, respectively.

Solution. The free thermal elongation is assumed to be longer than the gap. The summation of elongations due to the free thermal elongation and the elongation due to the stress is equal to the small gap e

$$\int_0^l \alpha\tau(x)dx + \frac{\sigma l}{E} = e \tag{13.14}$$

Then, we get

$$\sigma = -\frac{E}{l}\Big[\alpha \int_0^l \tau(x)dx - e\Big] \tag{13.15}$$

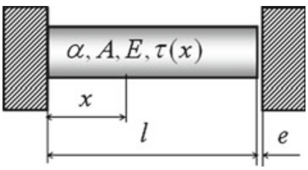


Fig. 13.2 A bar with a small gap

The free thermal expansion is given by Eq. (13.12). Therefore,

$$\sigma = -\frac{206 \times 10^9}{1}\left(1.46 \times 10^{-3} - 1 \times 10^{-3}\right) = -94.8 \times 10^6 = -94.8\,\text{MPa}$$

(Answer)

Problem 13.7. If a clamped circular frustum of mild steel with $d_0 = 1\,\text{cm}$, $d_1 = 2\,\text{cm}$, and $l = 2\,\text{m}$ is subjected to the temperature change $-50\,\text{K}$, calculate the resulting thermal stress. The coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2 \times 10^6\,1/\text{K}$ and $E = 206\,\text{GPa}$, respectively.

Solution. The free thermal elongation λ_T is

$$\lambda_T = \alpha \tau l \tag{13.16}$$

The cross-sectional area A_x at the position x is given by

$$A_x = \frac{\pi}{4}d_x^2 = \frac{\pi}{4}\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2 \tag{13.17}$$

Thus, the strain ϵ_x of the frustum at x due to an internal force Q becomes

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{Q}{EA_x} = \frac{4Q}{E\pi\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2} \tag{13.18}$$

and the elongation λ_s of the frustum due to the internal force Q equals

$$\begin{aligned}\lambda_s &= \int d\lambda_s = \int_0^l \epsilon_x\,dx = \int_0^l \frac{4Q}{E\pi\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2}\,dx \\ &= -\frac{4Ql}{E\pi(d_1 - d_0)}\left[\frac{1}{d_0 + (d_1 - d_0)\frac{x}{l}}\right]_0^l = \frac{4Ql}{E\pi d_1 d_0}\end{aligned} \tag{13.19}$$

As the frustum is perfectly constrained in the x direction, the combined elongation of the free thermal elongation λ_T and the elongation λ_s due to the internal force Q must be zero

$$\lambda = \lambda_T + \lambda_s = 0 \tag{13.20}$$

From Eqs. (13.16), (13.19), and (13.20) the internal force Q is

$$Q = -\alpha E \tau \frac{\pi}{4} d_1 d_0 \tag{13.21}$$

Then, the thermal stress is

$$\sigma_x = \frac{Q}{A_x} = -\alpha E \tau \frac{d_1 d_0}{\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2} \tag{13.22}$$

If $d_1 > d_0$, the maximum thermal stress $(\sigma_x)_{\max}$ occurs at the minimum cross-sectional area and the minimum thermal stress $(\sigma_x)_{\min}$ occurs at the maximum cross-sectional area

$$(\sigma_x)_{\max} = -\alpha E \tau \frac{d_1}{d_0}, \quad (\sigma_x)_{\min} = -\alpha E \tau \frac{d_0}{d_1} \tag{13.23}$$

The thermal stress σ_x is calculated from Eq. (13.22)

$$\begin{aligned}\sigma_x &= -11.2 \times 10^{-6} \times 206 \times 10^9 \times (-50)\\ &\quad \times \frac{1 \times 10^{-2} \times 2 \times 10^{-2}}{\left[1 \times 10^{-2} + (2 \times 10^{-2} - 1 \times 10^{-2})\frac{x}{2}\right]^2}\\ &= \frac{230.7 \times 10^6}{\left(1 + \frac{x}{2}\right)^2}\,\text{Pa} = \frac{231}{\left(1 + \frac{x}{2}\right)^2}\,\text{MPa} \qquad \text{(Answer)}\end{aligned} \tag{13.24}$$

The maximum and minimum thermal stresses are from Eq. (13.24)

$$(\sigma_x)_{\max} = 231\,\text{MPa}, \quad (\sigma_x)_{\min} = \frac{231}{\left(1 + \frac{2}{2}\right)^2}\,\text{MPa} = 57.8\,\text{MPa} \qquad \text{(Answer)}$$

Problem 13.8. If the temperature of a clamped circular frustum of mild steel with $d_0 = 1$ cm, $d_1 = 2$ cm, and $l = 2$ m changes linearly from 0 K at one end to −50 K at the other end, calculate the resulting thermal stress. The coefficient of linear thermal expansion and Young's modulus are $\alpha = 11.2 \times 10^6$ 1/K and $E = 206$ GPa, respectively.

Solution. The distribution of the temperature change $\tau(x)$ is

$$\tau(x) = -50\frac{x}{l} \tag{13.25}$$

The free thermal elongation λ_T is

$$\lambda_T = \int_0^l \alpha \tau(x) dx = \alpha \int_0^l \frac{-50x}{l} dx = -\left[\frac{25\alpha x^2}{l}\right]_0^l = -0.56 \times 10^{-3}\ \text{m} \tag{13.26}$$

The cross-sectional area A_x at the position x is given by

$$A_x = \frac{\pi}{4} d_x^2 = \frac{\pi}{4}\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2 \tag{13.27}$$

Thus, the strain ϵ_x of the frustum at x due to an internal force Q becomes

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{Q}{EA_x} = \frac{4Q}{E\pi\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2} \tag{13.28}$$

and the elongation λ_s of the frustum due to the internal force Q equals

$$\begin{aligned}\lambda_s &= \int d\lambda_s = \int_0^l \epsilon_x \, dx = \int_0^l \frac{4Q}{E\pi\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2} dx \\ &= -\frac{4Ql}{E\pi(d_1 - d_0)}\left[\frac{1}{d_0 + (d_1 - d_0)\frac{x}{l}}\right]_0^l = \frac{4Ql}{E\pi d_1 d_0}\end{aligned} \tag{13.29}$$

As the frustum is perfectly constrained in the x direction, the summation of elongation of the free thermal elongation λ_T and the elongation λ_s due to the internal force Q must be zero

$$\lambda = \lambda_T + \lambda_s = 0 \tag{13.30}$$

From Eqs. (13.29) and (13.30) the internal force Q is

$$Q = -\frac{E\pi d_1 d_0 \lambda_T}{4l} \tag{13.31}$$

and the thermal stress is calculated to be

$$\begin{aligned}\sigma_x &= \frac{Q}{A_x} = -\frac{E\pi d_1 d_0 \lambda_T}{4l\frac{\pi}{4}\left[d_0 + (d_1 - d_0)\frac{x}{l}\right]^2} \\ &= \frac{(0.56 \times 10^{-3}) \times (206 \times 10^9) \times (1 \times 10^{-2}) \times (2 \times 10^{-2})}{2 \times \left[1 \times 10^{-2} + (2 \times 10^{-2} - 1 \times 10^{-2})\frac{x}{2}\right]^2} \\ &= \frac{115.36 \times 10^6}{\left(1 + \frac{x}{2}\right)^2} \text{ Pa} = \frac{115}{\left(1 + \frac{x}{2}\right)^2} \text{ MPa}\end{aligned} \qquad \text{(Answer)}$$

The maximum and minimum thermal stresses are

$$(\sigma_x)_{max} = 115 \, \text{MPa}, \quad (\sigma_x)_{min} = 28.8 \, \text{MPa} \qquad \text{(Answer)}$$

Problem 13.9. If a bar with a small gap e between its free end and a rigid wall is subjected to the positive temperature change $\tau(x)$, and the cross-sectional area of the bar is given by $A(x)$, calculate the thermal stress produced in the bar.

Solution. The small elongation $d\lambda(x)$ of the small element dx is

$$d\lambda(x) = \epsilon(x)dx = \left[\frac{\sigma(x)}{E} + \alpha\tau(x)\right]dx \tag{13.32}$$

The elongation λ of the bar with length l is

$$\lambda = \int d\lambda = \int_0^l \left[\frac{\sigma(x)}{E} + \alpha\tau(x)\right]dx = \int_0^l \left[\frac{Q}{EA(x)} + \alpha\tau(x)\right]dx \tag{13.33}$$

in which Q is an internal force. The free thermal elongation is assumed to be longer than the gap. The summation of elongation due to the free thermal elongation and elongation due to the stress is equal to the small gap e

$$\int_0^l \alpha\tau(x)dx + \frac{1}{E}\int_0^l \frac{Q}{A(x)}dx = e \tag{13.34}$$

Then, we get

$$Q = -\frac{E}{\int_0^l \frac{1}{A(x)}dx}\left[\alpha\int_0^l \tau(x)dx - e\right] \tag{13.35}$$

Thermal stress is

$$\sigma = \frac{Q}{A(x)} = -\frac{E}{A(x)\int_0^l \frac{1}{A(x)}dx}\left[\alpha\int_0^l \tau(x)dx - e\right] \quad \text{(Answer)}$$

The maximum and minimum thermal stresses are

$$(\sigma)_{\max} = -\frac{E}{A(x)_{\min}\int_0^l \frac{1}{A(x)}dx}\left[\alpha\int_0^l \tau(x)dx - e\right]$$
$$(\sigma)_{\min} = -\frac{E}{A(x)_{\max}\int_0^l \frac{1}{A(x)}dx}\left[\alpha\int_0^l \tau(x)dx - e\right] \quad \text{(Answer)}$$

Problem 13.10. A hollow cylinder with a bar of the same length l and the same centerline, shown in Fig. 13.3 is subjected to different temperature changes τ_i, $(i = 1, 2)$. The hollow cylinder and the bar are connected to two rigid plates. Calculate the thermal stresses produced in both the hollow cylinder and the bar, and the elongations.

Solution. The elongations λ_i due to both the free thermal elongation and the thermal stress are

$$\lambda_1 = \alpha_1\tau_1 l + \frac{\sigma_1}{E_1}l, \quad \lambda_2 = \alpha_2\tau_2 l + \frac{\sigma_2}{E_2}l \tag{13.36}$$

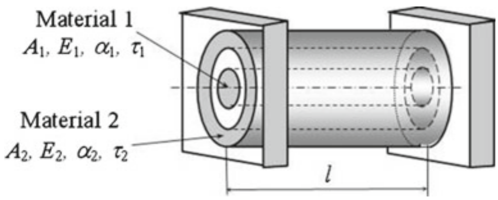


Fig. 13.3 A bar and a hollow cylinder with both ends clamped to rigid plates

where A_i, E_i, and α_i denote cross-sectional area, Young's modulus, and the coefficient of linear thermal expansion of the i-th material, respectively. Since the final length of both the cylinder and the bar after deformation is the same, the following relation holds

$$l + \alpha_1 \tau_1 l + \frac{\sigma_1}{E_1} l = l + \alpha_2 \tau_2 l + \frac{\sigma_2}{E_2} l \tag{13.37}$$

The equilibrium of the internal forces is described by

$$\sigma_1 A_1 + \sigma_2 A_2 = 0 \tag{13.38}$$

Solving Eqs. (13.37) and (13.38) gives the stresses

$$\sigma_1 = -\frac{A_2 E_1 E_2 (\alpha_1 \tau_1 - \alpha_2 \tau_2)}{A_1 E_1 + A_2 E_2}$$
$$\sigma_2 = \frac{A_1 E_1 E_2 (\alpha_1 \tau_1 - \alpha_2 \tau_2)}{A_1 E_1 + A_2 E_2} \qquad \text{(Answer)}$$

Substitution of these stresses into Eq. (13.36) gives the elongations of the cylinder and the bar

$$\lambda_1 = \lambda_2 = \frac{(\alpha_1 \tau_1 E_1 A_1 + \alpha_2 \tau_2 E_2 A_2) l}{A_1 E_1 + A_2 E_2} \qquad \text{(Answer)}$$

Problem 13.11. Two circular bars, one is mild steel of length 50 cm and diameter 1 cm, and the other is aluminum of length 25 cm and diameter 2 cm, are attached to each other in series, placed between rigid walls, and subjected to the temperature change $\tau = T_1 - T_0$, as shown in Fig. 13.4. Calculate the temperature rise needed for the thermal stresses in the bars to reach the compressive strength. The coefficient of linear thermal expansion, Young's modulus and the compressive strength for mild steel are $\alpha_1 = 11.2 \times 10^6$ 1/K, $E_1 = 206$ GPa and 400 MPa, respectively. The coefficient of linear thermal expansion, Young's modulus and the compressive strength for aluminum are $\alpha_2 = 23.1 \times 10^6$ 1/K, $E_2 = 72$ GPa and 70 MPa, respectively.

Solution. The elongations of bar 1 and 2 are, respectively, given by

$$\alpha_1 \tau l_1 + \frac{\sigma_1}{E_1} l_1, \quad \alpha_2 \tau l_2 + \frac{\sigma_2}{E_2} l_2 \tag{13.39}$$

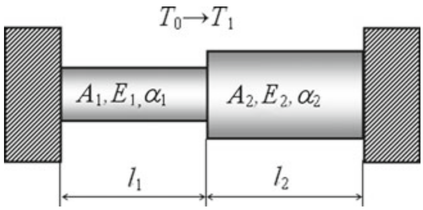


Fig. 13.4 Two bars attached to each other

As two bars are placed between rigid walls, the combined elongation of the bars is zero. Thus,

$$\alpha_1 \tau l_1 + \frac{\sigma_1}{E_1} l_1 + \alpha_2 \tau l_2 + \frac{\sigma_2}{E_2} l_2 = 0 \tag{13.40}$$

From the equilibrium condition of internal forces, the internal force in bar 1 is equal to the internal force in bar 2

$$\sigma_1 A_1 = \sigma_2 A_2 \tag{13.41}$$

From Eqs. (13.40) and (13.41), the thermal stresses σ_1 and σ_2 are given as

$$\sigma_1 = -\frac{\alpha_1 E_1 \tau \left(1 + \dfrac{\alpha_2 l_2}{\alpha_1 l_1}\right)}{1 + \dfrac{A_1 E_1 l_2}{A_2 E_2 l_1}}, \quad \sigma_2 = \sigma_1 \frac{A_1}{A_2} = -\alpha_1 E_1 \tau \frac{A_1}{A_2} \frac{1 + \dfrac{\alpha_2 l_2}{\alpha_1 l_1}}{1 + \dfrac{A_1 E_1 l_2}{A_2 E_2 l_1}} \tag{13.42}$$

Therefore, the necessary temperature rise for bar 1 is

$$\tau = -\frac{\sigma_1}{\alpha_1 E_1} \frac{1 + \dfrac{A_1 E_1 l_2}{A_2 E_2 l_1}}{1 + \dfrac{\alpha_2 l_2}{\alpha_1 l_1}} \tag{13.43}$$

Numerical calculation gives the temperature rise

$$\tau = 115.876\,\text{K} = 116\,\text{K} \tag{13.44}$$

On the other hand, the necessary temperature rise for bar 2 is given by

$$\tau = -\frac{\sigma_2}{\alpha_1 E_1} \frac{A_2}{A_1} \frac{1 + \dfrac{A_1 E_1 l_2}{A_2 E_2 l_1}}{1 + \dfrac{\alpha_2 l_2}{\alpha_1 l_1}} \tag{13.45}$$

Therefore, the necessary temperature rise is

$$\tau = 81.11\,\text{K} = 81\,\text{K} \tag{13.46}$$

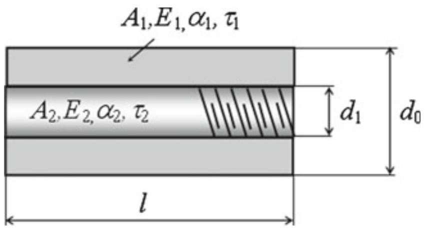


Fig. 13.5 A hollow cylinder with an inserted screw

Then, comparison between Eqs. (13.44) and (13.46) gives the necessary temperature rise 81 K.

Problem 13.12. A hollow cylinder with an inserted screw, shown in Fig. 13.5 is subjected to different temperature changes τ_i, $(i = 1, 2)$. Calculate the thermal stresses produced in both the hollow cylinder and the screw.

Solution. The elongations λ_i due to both the free thermal elongation and the thermal stress are

$$\lambda_1 = \alpha_1 \tau_1 l + \frac{\sigma_1}{E_1} l, \quad \lambda_2 = \alpha_2 \tau_2 l + \frac{\sigma_2}{E_2} l \tag{13.47}$$

where A_i, E_i, and α_i denote cross-sectional area, Young's modulus, and the coefficient of linear thermal expansion of the i-th material, respectively. Since the final length of both the hollow cylinder and the screw after deformation is the same, the following relation holds

$$l + \alpha_1 \tau_1 l + \frac{\sigma_1}{E_1} l = l + \alpha_2 \tau_2 l + \frac{\sigma_2}{E_2} l \tag{13.48}$$

The equilibrium condition of the internal forces is described by

$$\sigma_1 A_1 + \sigma_2 A_2 = 0 \tag{13.49}$$

Solving Eqs. (13.48) and (13.49) gives the thermal stresses

$$\begin{aligned} \sigma_1 &= -\frac{E_1 E_2 A_2 (\alpha_1 \tau_1 - \alpha_2 \tau_2)}{A_1 E_1 + A_2 E_2} \\ \sigma_2 &= \frac{E_1 E_2 A_1 (\alpha_1 \tau_1 - \alpha_2 \tau_2)}{A_1 E_1 + A_2 E_2} \end{aligned} \qquad \text{(Answer)}$$

Problem 13.13. A copper tube is fastened by a mild steel bolt, as shown in Fig. 13.6. The length of the tube is 50 cm, and the cross-sectional areas of the bolt and the tube are $A_s = 1\,\text{cm}^2$ and $A_c = 2\,\text{cm}^2$, respectively. Calculate the thermal stresses produced if the system is subjected to the temperature change of 80 K. The coefficient of linear thermal expansion and Young's modulus for mild steel are $\alpha_s = 11.2 \times 10^6$ 1/K and $E_s = 206\,\text{GPa}$, respectively. The coefficient of linear thermal expansion

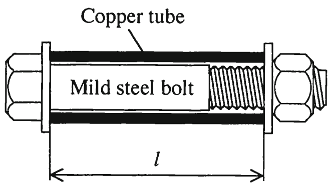


Fig. 13.6 A copper tube fastened by a mild steel bolt

and Young's modulus for copper are $\alpha_c = 16.5 \times 10^6$ 1/K and $E_c = 120$ GPa, respectively.

Solution. Since the final length of both the copper tube and the mild steel bolt after deformation is the same, the following relation holds

$$l + \alpha_s \tau l + \frac{\sigma_s}{E_s} l = l + \alpha_c \tau l + \frac{\sigma_c}{E_c} l \qquad (13.50)$$

The equilibrium condition of the internal forces is described by

$$\sigma_s A_s + \sigma_c A_c = 0 \qquad (13.51)$$

Solving Eqs. (13.50) and (13.51) gives the stresses

$$\sigma_s = -\frac{E_s E_c A_c (\alpha_s - \alpha_c)\tau}{A_s E_s + A_c E_c}, \quad \sigma_c = \frac{E_s E_c A_s (\alpha_s - \alpha_c)\tau}{A_s E_s + A_c E_c} \qquad (13.52)$$

The numerical results are

$$\sigma_s = 47.001 \times 10^6 \text{ Pa} = 47 \text{ MPa}, \quad \sigma_c = -23.5 \text{ MPa} \qquad \text{(Answer)}$$

Problem 13.14. In the foregoing problem, calculate the maximum tolerable temperature rise such that stresses in the system do not exceed the compressive or the tensile strength. The tensile strengths of the steel and the copper are $\sigma_{st} = 400$ MPa and $\sigma_{ct} = 300$ MPa, respectively. We assume the compressive strength has the same magnitude as the tensile strength. The safety factor (defined by the ratio of yield stress or the tensile strength and the tolerable stress) is $f = 3$.

Solution. The stresses due to the temperature change τ are given by Eq. (13.52), namely

$$\sigma_s = -\frac{E_s E_c A_c (\alpha_s - \alpha_c)\tau}{A_s E_s + A_c E_c}, \quad \sigma_c = \frac{E_s E_c A_s (\alpha_s - \alpha_c)\tau}{A_s E_s + A_c E_c} \qquad (13.53)$$

The tolerable stress of a mild steel bolt is

$$\sigma_{sa} = \frac{\sigma_{st}}{f} \tag{13.54}$$

The maximum tolerable temperature rise τ of the mild steel bolt is given by

$$\begin{aligned}
\tau &= -\frac{\sigma_{sa}}{\alpha_s E_s\left(1-\frac{\alpha_c}{\alpha_s}\right)\Big/\left(1+\frac{A_s E_s}{A_c E_c}\right)} = -\frac{\sigma_{st}}{f}\frac{1+\frac{A_s E_s}{A_c E_c}}{\alpha_s E_s\left(1-\frac{\alpha_c}{\alpha_s}\right)} \\
&= -\frac{400\times 10^6}{3} \\
&\quad\times \frac{1+1\times 10^{-4}\times 206\times 10^9/(2\times 10^{-4}\times 120\times 10^9)}{11.2\times 10^{-6}\times 206\times 10^9\times[1-16.5\times 10^{-6}/(11.2\times 10^{-6})]} \\
&= 226.944 = 227\,\mathrm{K}
\end{aligned} \tag{13.55}$$

Tolerable stress of the copper tube is

$$\sigma_{ca} = \frac{\sigma_{ct}}{f} \tag{13.56}$$

The maximum tolerable temperature rise τ of the copper tube is given by

$$\begin{aligned}
\tau &= \frac{\sigma_{ca}}{\alpha_s E_s \frac{A_s}{A_c}\left(1-\frac{\alpha_c}{\alpha_s}\right)\Big/\left(1+\frac{A_s E_s}{A_c E_c}\right)} = \frac{\sigma_{ct}}{f}\frac{A_c}{A_s}\frac{1+\frac{A_s E_s}{A_c E_c}}{\alpha_s E_s\left(1-\frac{\alpha_c}{\alpha_s}\right)} \\
&= -\frac{300\times 10^6}{3}\frac{2\times 10^{-4}}{1\times 10^{-4}} \\
&\quad\times \frac{1+1\times 10^{-4}\times 206\times 10^9/(2\times 10^{-4}\times 120\times 10^9)}{11.2\times 10^{-6}\times 206\times 10^9\times[1-16.5\times 10^{-6}/(11.2\times 10^{-6})]} \\
&= 340.416 = 340\,\mathrm{K}
\end{aligned} \tag{13.57}$$

Therefore, from Eqs. (13.55) and (13.57), the maximum tolerable temperature rise is 227 K.

Problem 13.15. A bar of mild steel of cross-sectional area A_s is placed between two parallel bars of copper of cross-sectional area A_c, shown in Fig. 13.7. When the three bars of same length l are bonded together and are subjected to a temperature change of τ_s in the bar of mild steel and τ_c in the bar of copper, calculate the thermal stresses produced in each bar.

Solution. The final lengths of middle steel and two copper bars are same

$$l + \alpha_s \tau_s l + \frac{\sigma_s l}{E_s} = l + \alpha_c \tau_c l + \frac{\sigma_c l}{E_c} \tag{13.58}$$

While the equilibrium condition of internal forces gives

$$\sigma_s A_s + 2\sigma_c A_c = 0 \tag{13.59}$$

From Eqs. (13.58) and (13.59) we get

$$\sigma_s = \frac{E_s(\alpha_c \tau_c - \alpha_s \tau_s)}{1 + \dfrac{A_s E_s}{2A_c E_c}}, \qquad \sigma_c = -\frac{\sigma_s A_s}{2A_c} \qquad \text{(Answer)}$$

Problem 13.16. Calculate the thermal stresses produced in the bars of the truss shown in Fig. 13.8, if the temperature changes of the bars are τ_i.

Solution. The relation between the elongation of bar 1 and bar 2 is

$$\lambda_2 = \lambda_1 \cos\theta \tag{13.60}$$

Therefore,

$$\alpha_2 \tau_2 l_2 + \frac{\sigma_2 l_2}{E_2} = \left(\alpha_1 \tau_1 l_1 + \frac{\sigma_1 l_1}{E_1}\right)\cos\theta \tag{13.61}$$

The relation between the length of bar 1 and bar 2 gives

$$l_1 = l_2 \cos\theta \tag{13.62}$$

Substitution of Eq. (13.62) into (13.61) reduces to

$$\alpha_2 \tau_2 + \frac{\sigma_2}{E_2} = \left(\alpha_1 \tau_1 + \frac{\sigma_1}{E_1}\right)\cos^2\theta \tag{13.63}$$

Then

$$\frac{\sigma_1}{E_1}\cos^2\theta - \frac{\sigma_2}{E_2} = -\alpha_1 \tau_1 \cos^2\theta + \alpha_2 \tau_2 \tag{13.64}$$

The equilibrium of internal forces requires

$$\sigma_1 A_1 + 2\sigma_2 A_2 \cos\theta = 0 \tag{13.65}$$

Solution of Eqs. (13.64) and (13.65) gives

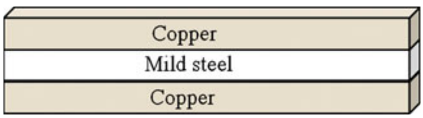
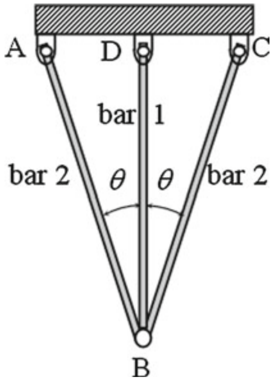


Fig. 13.7 Three bars with rectangular cross section fastened to each other

Fig. 13.8 Truss of three bars



$$\sigma_1 = -\alpha_1 E_1 \tau_1 \frac{\cos^2\theta - \dfrac{\alpha_2\tau_2}{\alpha_1\tau_1}}{\cos^2\theta + \dfrac{E_1}{E_2}\dfrac{A_1}{2A_2}\dfrac{1}{\cos\theta}}$$

$$= -\alpha_1 E_1 \tau_1 \frac{1 - \dfrac{\alpha_2\tau_2}{\alpha_1\tau_1\cos^2\theta}}{1 + \dfrac{A_1 E_1}{2A_2 E_2 \cos^3\theta}}$$

$$\sigma_2 = -\frac{A_1}{2A_2\cos\theta}\sigma_1 \qquad \text{(Answer)}$$

Problem 13.17. Calculate the thermal stresses produced in the bars which hang from a rigid plate shown in Fig. 13.9, if the temperature changes of the bars are τ_i. The weight of the rigid plate may be neglected.

Solution. The elongations of each bar are

$$\lambda_i = \frac{\sigma_i}{E_i} l + \alpha_i \tau_i l \quad (i = 1, 2, 3) \tag{13.66}$$

The equilibrium condition of the internal forces in each bar requires

$$\sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3 = 0 \tag{13.67}$$

The equilibrium of the moments at the point A is

$$\sigma_2 A_2 a + \sigma_3 A_3 (a + b) = 0 \tag{13.68}$$

The relation between the elongation of each bar is

$$(\lambda_3 - \lambda_1) : (\lambda_2 - \lambda_1) = (a + b) : a \tag{13.69}$$

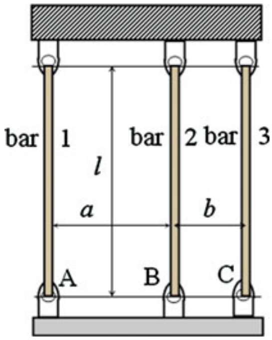


Fig. 13.9 Three bars on whih hangs a rigid plate

Solution of Eqs. (13.67), (13.68) and (13.69) gives

$$\sigma_1 = A_2 A_3 b \frac{C}{D}, \quad \sigma_2 = -A_1 A_3 (a+b) \frac{C}{D}, \quad \sigma_3 = A_1 A_2 a \frac{C}{D} \qquad \text{(Answer)}$$

in which

$$\begin{aligned} C &= -b\alpha_1 \tau_1 + (a+b)\alpha_2 \tau_2 - a\alpha_3 \tau_3 \\ D &= a^2 \frac{A_1 A_2}{E_3} + b^2 \frac{A_2 A_3}{E_1} + (a+b)^2 \frac{A_1 A_3}{E_2} \end{aligned} \qquad (13.70)$$