Chapter 16
Epilogue

Abstract This concluding chapter first summarizes the historical developments exposed in a critical manner in all preceding chapters. It emphasizes the various nonlinear generalizations proposed in the Twentieth century as also the role played by remarkable schools and individuals in the fantastic progress reached in this period. This concerns more realistic material behaviors (accounting for micro-structures, involving coupled fields), a moreaxiomatic and thermodynamically justified approach, and a clear internationalization of engineering science. Simultaneously, progress in other collateral branches of sciences, both theoretical and experimental, has fostered a rapid, sometimes unexpected, progress in the-science of continuum mechanics. The latter has become more a mechanics of materials while developing tremendously its applicable side with performing numerical schemes andrequiring new developments in applied analysis and the interpretation in terms of advanced geometrical concepts. Final remarks points at the new marked interest of continuummechanics for living matter and the unavoidable relationship, both intellectually and numerically, between different scales of description, a trademark at the dawn of the Twenty first century.

16.1 What was Achieved

If we compare the main ideas and queries formulated at the dawn of the twentieth century—as recalled in Chap. 2—with the developments exposed in Chaps. 3–15, we witness a rather good fulfilment of the proposed programme.

First of all, the existing linear theories of elasticity and viscous fluids have been extended to a true and applicable theory of nonlinear elasticity—essentially for incompressible materials of the rubber-like type and more recently for bio-materials such as soft tissues—and non-Newtonian fluids (Chap. 3). The last case, because of its involvement of time, has necessitated a reflection that led to seriously accounting for the notion of objectivity in order to define properly invariant time derivatives.

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The perspicacious views and works of luminaries such as Rivlin, Oldroyd and others have been instrumental in this intellectual construct corroborated by appropriate experiments and fostered by socio-economical needs (industry of rubber and artificial textiles, paints, food industry, and all strongly viscous products).

In the linear theory of elasticity, two main ingredients have been introduced (Chap. 6). First anisotropy has established a better contact between classical continuum mechanics and the physics of materials, which is the realm of anisotropic crystals. But new materials are also anisotropic (e.g., fibre-reinforced materials). The second ingredient is the necessary consideration of the possible singularities of the elastic field. The way was paved by scientists such as A.E.H. Love. But there was a long way between the theoretical—kind of thought-experimental—notion of dislocation introduced by Vito Volterra and the real physical considerations on dislocations by G.I. Taylor and others. Similarly, the now obvious need to envisage the occurrence of cracks and their catastrophic expansion (in particular in aeronautics and nuclear-power industry) was dutifully answered by the formidable work achieved by the British school (Sneddon, Eshelby, Stroh, etc.) and then by teams both in the west (e.g., USA, France) and the east (Russia). The importance of some mathematical methods such as the application of the technique of complex variables by Kolosov and Muskhelishvili cannot be ignored in this context. This trend of research culminated in the theory of configurational forces (Chap. 14) with the seminal contributions of Eshelby, Cherepanov, Rice and others. Works of a more experimental nature and engineering-type such as those of Griffith and Irwin were of utmost importance in these developments.

Simultaneously, a necessary examination of the mathematical properties of the systems principally deduced from elasticity has led to a definite progress in the proof of theorems of uniqueness and existence. This is not gratuitous as there is no need to look for a classical solution (analytical or numerical) if we know in advance that the considered problem is ill-posed and a standard solution cannot exist. This progress was mostly based on the creation of a true applied functional analysis in the expert hands of mathematicians such as Sobolev, Leray, Schwartz, Lions, Magenes, etc., and more recently on its application in the UK, by Knops, and then John Ball in nonlinear elasticity (Chap. 6).

Of the three "real" mechanical behaviours mentioned in Sect. 2.2, friction and plasticity are certainly those which have commanded the largest number of works in the twentieth century. Plasticity and its application to the mechanics of structures made immense progress among those cultivating the ASME spirit, especially in Stanford and Brown (Chap. 4), but also in the UK with Rodney Hill and his disciples (Chap. 6), Poland (Chap. 8), and Russia with Ilyushin, Rabotnov and others (Chap. 11). We can say that, to the posthumous satisfaction of Pierre Duhem, elastoplasticity, but also allied theories of creep and damage (Odqvist, Hull, Kachanov, Rabotnov, Lemaitre) were given a good thermodynamic foundation thanks to the works of Hill, Mandel, Ziegler and the French school of continuum thermomechanics (Chap. 7). This definitely included one of Duhem's "nonsensical" branches of mechanics into a rational framework.

The attempts of Duhem to organize the “energetics” of many processes in a common frame were completed by the school of linear irreversible thermodynamics in Belgium and the Netherlands. But it is with Truesdell and his partners, Noll, Coleman, Toupin, that the whole field was re-organized in a more mathematical and axiomatized manner (Chap. 5). This fulfils one point in the prospective programme proposed by Hilbert. The offered thermodynamic formulation was audacious but with a certain efficacy. Amendments or generalizations (extended thermodynamics, introduction of internal variables of state, satisfaction of the causality of solutions) were advanced that much improved on the too much corseting original proposal. By the same token stringent conditions of invariance (e.g., objectivity) were duly enforced in continuum mechanics, probably under the influence of the flourishing of such principles in mathematical physics.

Another of Duhem’s “nonsensical” branches of mechanics was electromagnetism. With the pioneering works of Toupin, Mindlin, Eringen and others, this was successfully incorporated in modern continuum mechanics (Chap. 12). This provided a possibility to couple deformation and all types of electric and magnetic behaviours and to treat a large number of applications at the crossing point of mechanics, materials science, and electrical engineering with the same rationality as pure problems of continuum mechanics.

Poro-elasticity (in particular by Biot) and a theory of consolidation of soils allowed a fruitful co-operation between continuum mechanics and an emerging science of geo-materials to the benefit of civil-engineering applications. At the same time, thermo-elasticity, one of the first theories of coupled fields thanks to the pioneering work of Duhamel, developed tremendously to include couplings with electric fields such as in thermo-piezo-electricity, with many contributions from Japan and China. This will particularly apply to new microscopic electro-magneto-mechanical components known as MEMS.

Other thermo-mechanical couplings are those that necessarily play a fundamental role in phase transformations of deformable solids. The invaluable contributions of mechanicians of the continuum (from all over the world, but particularly from the USA, Russia, France, Germany, and Japan) to the mathematical description of such phenomena have brought this community in useful co-operational contact with the community of metallurgists and condensed-matter physics. No doubt that the way of approach and tools favoured by mechanicians—exploitation of balance laws, jump equations at moving discontinuities, considerations of configurational mechanics and variational formulations, mathematical refinements with special classes of functions—have permitted a rational but physically justified description that would have largely escaped the traditional tools of metallurgists. Works by applied mathematicians such as Jerald Ericksen, D.S. Kinderlehrer, John Ball and Richard D. James have been essential in such developments. Here the role played by the University of Minnesota should be underlined.

If we now recall the original works of Duhem and the Cosserats on elastic materials with a microstructure, then after a rest period of some 56 years, their original ideas developed into a real “industry” materializing in various paths to a truly generalized continuum mechanics as exposed in Chap. 13. Three essential lines have expanded, being represented by polar and micromorphic materials, materials described by higher order gradients of deformation, and the so-called non-local theory of materials. In all these we must acknowledge the leading role played by mechanicians such as Toupin, Mindlin, Eringen, Kröner, Kunin, Edelen, not to forget the German pioneers and their followers. What was again instrumental in the most recent developments of these research paths was a now obvious relation of such schemes of deformable matter with real materials, whether of natural origin (crystals of various types) or man-made new materials (composites, cellular materials, etc.). Of necessity this has led to considering representative length scales, and scale effects in general.

16.2 The Influence of New Experimental Equipments and Computational Means

What could not be guessed at the dawn of the twentieth century were "things to come", future developments in both experimental and numerical means that would often be the consequence of progress in small-scale physics, especially wave and quantum mechanics that revolutionized solid-state physics. As a matter of fact, mechanics in the early twentieth century is still based on (1) standard observational means (e.g., the naked eye and optical microscopes) and testing machines in a most elementary—entirely mechanical—form, and (2) the search for analytical solutions, if not of graphical ones by hand. It is at this gauge that we must appreciate the extraordinary achievements of some people in analytical solutions, often based on astute Ansatze that reflect a gifted capture of the physics and symmetries of the looked for solutions. Still, practically only "simple" academic problems could be solved (e.g., in elasto-plasticity where problems are free-boundary ones).

With progress in atomic physics and the applications of modern physics to electronics, experimental means progressed at giant steps (think of electronic microscopy, atomic-force microscopy, image processing, etc.) with a tremendous decrease in scale of observation, while new means of computations were created (electronic computers, miniaturization) with easy access by the common user only in the nineteen seventies. These two facts created a new situation in which large computations of complicated real structures made of materials with complex constitutive equations (think of plastic-forming in large deformations with visco-elasto-plastic constitutive equations) could be performed in finite strains. Scientists like Juan Simo in the USA and people around Erwin Stein in Germany, who combined an excellent knowledge of continuum thermo-dynamics, performing computational methods, and mathematical results, had the most efficient background to realize such wonderful computations. This also applies to large computations in the bio-mechanics of soft tissues such as the practically complete mechanical simulation of the human heart structure with its multi-layered envelope made of variously oriented fibres (see Humphrey 2002).

Personal touch: In the same way as they cannot remember Bakelite telephones and the desperate look for a telephone booth with the requirement to carry dimes in your pocket to get in touch with the phone operator—see old black-and-white mystery US movies of the 1950s, (so-called "*films noirs*" in the jargon of aficionados); also remember the inenarrable sequence in "Dr Strangelove" when the British officer tries to enter in contact with the White House -, young readers may have difficulties to imagine a time (1950–1960s) at which only electro-mechanical desk computers existed. These were essentially used to make boring astronomical calculations, or to help tracing a curve starting from a painstakingly obtained analytical solution. Just to illustrate this, the author recalls that he first did some computations of fluid mechanics on an analogue computer in 1965. He had his first experience on a cumbersome—but extremely weak—digital electronic computer doing only simple algebraic operations in 1966, with programming in machine language. It is only going to the United States that he met more powerful large computers but with programming in Fortran language. One had to bring a thick batch of prepared punched cards to the computer centre and collect the results on large printed output sheets one or two days later. Only finite-difference schemes were available to treat problems of fluid mechanics. The finite-element method was invented only in 1965–1966 to make large computations on aeronautical structures; it took some time to become a commonly used tool.

Another consequence of this drastic development in both experimental and computational means was the rapid transformation of part of the mechanics of structures into a real *mechanics of materials*, that is, the due consideration of the intimate structure of the material with its inherent inhomogeneity, multi-component contributions, and transformations. It is only with modern fast computations that the mathematical theory of homogenization could be applied, delivering the effective coefficients of the replacement material by solving a set of exemplary problems on the relevant basic cell. Simultaneously, the new experimental means produced the appropriate images and measurements to confirm the numerical simulations. From these emerged this new science of the *mechanics of materials*, the last avatar of continuum mechanics. This gives a rather good visual perspective of developments to come in a near future.

A particular point to be emphasized is that while continuum mechanics was for a long time reserved to the study of *inert matter*, this new mechanics of materials now dares to attack the landscape of *living matter* in the framework of biomechanics and mechano-biology for the study of growth, resorption, ageing, remodelling, and morphogenesis. If we already mentioned that non-linear elasticity was in some sense saved from oblivion by its useful applications in biomechanics (Chap. 3; many works by Odgen and Holzapfel), most recent developments in biomechanics involve all new ingredients and ideas introduced in thermo-mechanics within the last 50 years: multiplicative decomposition of the finite deformation gradient, theory of mixtures, notion of internal variables of thermodynamic state, higher-order gradient theory, configurational forces, non-linear waves, and homogenisation techniques, all to a high degree of sophistication [see, for instance, in alphabetic order, Ciarletta and Maugin (2011), Ciarletta et al. (2012), Cowin (1996), Cowin and Hegedus (1976), Epstein (2012; Chap. 7), Epstein and Maugin (2000), Ganghoffer (2005), Humphrey (2002), Maugin (2011; Chap 10), Porubov and Maugin (2011), Rodriguez et al. (1994), Taber (1995)].

16.3 Towards Interactions Between Scales

For a long time continuum mechanics benefited in its simplest form—Hooke's law with two Lamé coefficients—to the evaluation of the strength of large structures. By this we mean human scale and above. The main trait that clearly emerges from the above reminder is a complexification of the modelling allied to a focus on smaller scales with a neat tendency towards the crystal size, microstructure, and an approach to the discrete description. Already mentioned examples relate to the fields of dislocations and phase transformations. Thanks to the power of present-day computational tools, it is now possible to simulate the movement of a large ensemble of interacting dislocations, as also to relate meso- and macro-scopic mechanical responses to it. It is this mutual enrichment between scales that is most characteristic of the developments in the beginning of the twenty first century. The new *multi-scale techniques* involving matching between continuum (finite-elements) and atomistic computations vividly illustrate this tendency (see, e.g., Tadmor and Miller 2011).

Along a somewhat different line one may wonder if the exploitation of direct simulation techniques such as *molecular dynamics*—with an appropriate choice of interaction potentials (see Rapaport 1995)—does not relegate the very concept of continuum to the "dark ages" of phenomenological physics. But if this technique yields spectacular results in some cases (e.g., propagation of cracks and other local structural rearrangements) there is no obvious proof that this may replace the continuum approach—appropriately discretized—in the computed response of structural elements at any scale.

Another question is whether the development of *nano-mechanics* brings a true revolution in the field (see Bhushan 2007; Liu et al. 2006). Of course one has to account for scale effects, noticeably for the natural enhancement of surface effects. Surprisingly enough, many mechanical engineers approach this mechanics with a rather simple adaptation of tools used in macroscopic physics. Progress will necessarily be done in this field.

Also, we cannot avoid a return to geometrical concepts. No doubt that geometry is the basis on which the kinematics and deformation theory of continuum mechanics rely. Until recently only geometry as made analytical by René Descartes and considering the three-dimensional Euclidean space as the normal arena of continuum mechanics was acknowledged as the standard background. Differential geometry as formulated by nineteenth century mathematicians (above all Gauss and Riemann) was the tool that introduced the notions of metric and eventually curvature (and therefore, by negation, a good definition of flatness; think of the Navier-Saint–Venant equations of compatibility). Two facts have complicated the picture. One was the influence of the consideration of non-Euclidean spaces in gravitational theory following Einstein and others. The second, in fact related to the first, was the recognition that taking account of the presence of many structural defects requires abandoning the peculiarity of the Euclidean nature of material space in favour of more general concepts introduced by modern differential geometers such as Elie Cartan: non-Riemannian spaces and the allied incursion of group theoretical concepts. A fundamental question is raised for the future of such developments that have already reached an incredible level of sophistication which unfortunately drastically reduces the potential readership to a happy few while of course becoming extremely strict from a mathematical viewpoint. A similar problem appears in the geometric approach to unified theories of physics that is apprehended by a very few. In mechanics, this will require from a selected group of scientists an education of equivalently high level in modern differential geometry, materials science, and continuum mechanics. Some published books go in that direction; see Epstein 2010; Frankel 2004.

Personal touch. It was the idea of the author to initiate in 1997 a series of International seminars on the subject of *Geometry, Continua and Microstructure* with a view to gather informally geometers, mechanicians of the continuum, and materials scientists. Eight such seminars were held in various European countries since 1997.

As a final but trivial remark, like in all scientific fields, the second part of the twentieth century has witnessed an internationalisation in notation and research themes. All involved personnel now read the same scientific journals that have gained a true international public, and they practically all have access to a stupendous flux of information by electronic means. Although local scientific traditions and masters are still active, the "provincial" print that we highlighted in some chapters is fading away, giving a chance to all, even in remote or more recently scientifically developed regions, to participate in the marvellous adventure of science of which, obviously, continuum mechanics is only a very small part, but often one at the meeting point of various scientific disciplines.

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