Chapter 12
Continuum Mechanics and Electromagnetism

Abstract The combination of pure continuum mechanics and electromagnetism cannot be a simple linear superimposition. That explains why it took some time to arrive at a rational formulation of this exemplary coupled-field theory. In spite of the experimental discovery of simple coupled effects in the Nineteenth century (e.g., magnetostriction, piezoelectricity), one practically had to await the second half of the Twentieth century to find a rational theory of deformable magnetized, electrically polarisable and electricity conducting continua. This is due to a small group of mechanicians who possessed a good apprehending of electromagnetic theory. The role of scientists such as R.A. Toupin, R.D. Mindlin, A.C. Eringen, W.F. Brown, H.F. Tiersten, M. Lax, D.F. Nelson, K. Hutter and the author of this book was instrumental in this intellectual construct. This is reported in a vivid manner, without neglecting the constructive works of electrical engineers and some mathematical physicists. After a brief survey of Nineteenth-century developments in electromagnetism the emphasis is placed on the seminal role played by Toupin in the 1950s and 1960s and on the author's own contributions in the period 1970–1990 concerning the fundamentals and the formulation of nonlinear electro- and magneto-elasticity often in the footsteps of H.F. Tiersten. A particular attention is paid to the evolution of the notions of electromagnetic force, momentum and stress tensor, and electro-magneto-mechanical couplings at the energy level.

12.1 Introduction

For electromagnetism, the first half of the 19th century is the time of the land-clearers such as Ampère, Faraday, Gauss, Poisson, and Oersted. The second half of the 19th is the time of unification in a grand scheme involving electricity and magnetism on equal footing, and culminating in the works of Kelvin, Weber, Helmholtz, and above all, Maxwell (1873) and Heaviside (1892) (to whom we owe the presently used form of Maxwell's equations). In parallel, coupled effects

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of the electro-mechanical, magneto-mechanical and galvano-magnetic types were discovered. Among them electric conduction, piezoelectricity (Curie brothers) and magnetostriction (Joule) are still those that steer attention because of the many received applications. Then there followed a long period, early and first part of the 20th century, during which many relevant discussions were devoted to the relativistic framework, while electrical engineering took the front with applications to energy productive or transforming machines and to macroscopic electromechanical devices. It is only in the second part of the 20th century that we witness an in-depth thinking about the continuum representation of multi-physical couplings with works of R.A. Toupin, R.D. Mindlin, A.C. Eringen, W.F. Brown, H.F. Tiersten, M. Lax, and D.F. Nelson, to whom we associate ourselves as we clearly agree with many of these developments, in particular with due consideration of interaction forces, and this in a pre fast-computer age. In parallel one must account for the constructive works of electrical engineers such as Penfield, Haus and Livens, and physicists such as Lorentz and de Groot and Suttorp.

12.2 Prerequisite: 19th Century: Physics Versus Electrical Engineering

The thermomechanics of solely deformable material continua and the electromagnetism of vacuum are two well established bodies of knowledge. The main question arises when material continua and electromagnetic fields co-exist spatially. It is then agreed upon that the relevant Maxwellian fields in matter, magnetic field **H** and electric displacement **D**, differ from the characteristic electromagnetic fields of vacuum, the magnetic induction **B** and the electric field **E**, in such a way that with appropriate electromagnetic units (so-called Lorentz-Heaviside units; neither factor 4π nor coefficients ε_0 and μ_0) we have the equations

$$\mathbf{H} = \mathbf{B} - \mathbf{M},\ \mathbf{D} = \mathbf{E} + \mathbf{P}, \tag{12.1}$$

where **M** and **P** are the magnetization and electric polarization per unit volume, fields that differ from zero only in magnetized and electrically polarized matter, respectively, i.e., when the celebrated set of Maxwell's equations in a fixed laboratory frame reads in full generality as

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},\ \nabla.\mathbf{B} = 0, \tag{12.2}$$

and

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{J},\ \nabla.\mathbf{D} = q_f, \tag{12.3}$$

where c is the velocity of light in vacuum, $\mathbf{J}$ is the electric current vector, and q_f is the density of free electric charges. The first set (12.2) is valid everywhere and yields the notion of electromagnetic potentials. In general, to close the system of field Eqs. (12.2) and (12.3), we are to be given *electromagnetic constitutive equations*, e.g., to give an idea to the reader, functional relations of the type

$$\mathbf{M} = \mathbf{M}(\mathbf{H}, .),\ \mathbf{P} = \mathbf{P}(\mathbf{E}, .),\ \mathbf{J} = \mathbf{J}(\mathbf{E}, .), \tag{12.4}$$

where the dots stand for some other variables such as temperature or a strain in a deformable solid.

Remark 12.1 The formulation (12.2)–(12.3) indeed is the one given by Oliver Heaviside who is supposed to have said that "Maxwell's theory is none other than Maxwell's equations" (accordingly, no need for explanation!). Maxwell himself proposed a formulation in terms of the potentials (twenty equations for twenty variables), and sometimes even used the formalism of quaternions, fashionable at the time. The first of (12.2) is none other than Faraday's equation; the first of (12.3) is Ampère's equation amended to account for Maxwell's displacement current without which Maxwell could not have forecasted the existence of electromagnetic waves—experimentally checked by Heinrich Rudolph Hertz (1857–1894) in 1888. The second of (12.3) is the Gauss-Poisson's equation. As to the second of (12.2) it means that there are no sources of magnetic induction $\mathbf{B}$ or, in other words, magnetic monopoles do not exist, an assumption valid unless contradicted by some new experimental evidence. In all we may consider that Eqs. (12.2) and (12.3) are the results of collective—but not necessarily coordinated—efforts by scientists, some experimentalists, others more theoreticians or mathematicians, such as Oersted, Ampère, Faraday, Gauss, Poisson, Kelvin, Weber, Helmholtz, and above all, **James Clerk Maxwell** (1831–1879) and **Oliver Heaviside** (1850–1925) over a long stretch of time. Among the crucial steps in that lengthy story we like to single out (1) the discovery by the Danish physicist Hans Christian Oersted (1777–1851) of a link between electricity and magnetism (flowing electricity in a wire could cause the needle of a nearby magnetic compass to be deflected), (2) its mathematical formulation by André-Marie Ampère (1775–1836), (3) the discovery of the solenoid by Arago, (4) that of electromagnets by Sturgeon, and (5) the discovery of electromagnetic induction by Michael Faraday (1791–1867). Without these it would not have been possible to conceive electromagnetic machines to generate electricity (the electric dynamo; alternating current) and vice versa, those to produce motion (the electric motor). The invention of the battery (pile) by Volta was also crucial to have handy a source of electricity (direct current).

While other possibilities exist, the selection (12.4) of dependent variables is not gratuitous. It pertains to the *characteristic* electromagnetic fields of *matter*. Several remarks are in order. First, by taking the divergence of $(12.3)_1$ and accounting for $(12.3)_2$, we obtain the law of *conservation of electric charge*:

$$\frac{\partial q_f}{\partial t} + \nabla.\mathbf{J} = 0, \tag{12.5}$$

a *strict* conservation law. Second, by a usual manipulation, one also deduces from (12.2)–(12.3) an *energy identity* called the "Poynting-Umov theorem", such that

$$\mathbf{H}.\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J}.\mathbf{E} - \nabla.\mathbf{S},\ \mathbf{S} \equiv c\mathbf{E} \times \mathbf{H}, \tag{12.6}$$

without any hypothesis concerning the electromagnetic constitutive equations.

Remark 12.2 Note that (12.6) is **not** the first law of thermodynamics (conservation of energy); it is just an identity relating to electromagnetic fields only, but these may be interacting with other fields as we shall see further down. In the West (12.6) is referred to as Poynting theorem after John Henry Poynting (1852–1914). But the Russian physicist at Moscow University, Nikolay A. Umov (1846–1915), was responsible for introducing the notion of *energy flux* (in liquid and elastic media) in 1874. Early disciples of Maxwell such as Poynting, Heaviside, and Larmor in the UK are called the "Maxwellians" (see Hunt 1991).

If we are in a *vacuum* (for which the three quantities in (12.4) vanish identically), long before the proof of her "invariance" theorem by Emmy Noether in 1918, Maxwell proved the existence of the following *vectorial* strict conservation law:

$$\frac{\partial \mathbf{p}^{\mathrm{em}.f}}{\partial t} - div\, \mathbf{t}^{\mathrm{em}.f} = \mathbf{0}, \tag{12.7}$$

wherein the electromagnetic momentum (in vacuum) and the so-called (symmetric) Maxwell stress tensor (stress tensor of *free* electromagnetic fields) are defined by

$$\mathbf{p}^{\mathrm{em}.f} = \frac{1}{c}\mathbf{E} \times \mathbf{B},\ \mathbf{t}^{\mathrm{em}.f} = \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B} - u^{\mathrm{em}.f}\mathbf{1},\ u^{\mathrm{em}.f} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \tag{12.8}$$

where the last quantity $u^{\mathrm{em}.f}$ is the electromagnetic energy of free fields per unit volume. As a matter of fact, in the same condition this verifies the conservation law (electromagnetic energy in vacuum)

$$\frac{\partial}{\partial t} u^{\mathrm{em}.f} + \nabla.\mathbf{S} = 0. \tag{12.9}$$

Equations (12.7) and (12.9) are peculiar expressions that hold here because of the inherent linearity of the electromagnetic constitutive equations ($\mathbf{H} = \mathbf{B}$, $\mathbf{D} = \mathbf{E}$, $\mathbf{J} = \mathbf{0}$) in vacuum. The latter serves as a (nonpolarized) *medium of comparison* for other electromagnetic media (an idea that will be successfully translated into mechanical behaviour by John R. Willis for studying effective properties of composites and deviations from a standard homogeneous elastic model in the 1970s; See Chap. 6).

Dealing with *energy* in a magnetized, electrically polarized, and conducting material in electromagnetism is a much more subtle matter as shown by the Eq. (12.6).

The latter can be integrated in a usual conservation form for a global volume only if the electromagnetic constitutive equations are *linear* and the body is *rigid*. Indeed, with simple constitutive equations $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \varepsilon\mathbf{E}$, for a rigid body occupying volume V bounded by regular boundary ∂V, of unit outward pointing normal $\mathbf{n}$, from (12.6) we would have the global balance of electromagnetic energy

$$\frac{d}{dt}\int_V u^{\mathrm{em}.m} dV = -\int_V \mathbf{J}.\mathbf{E} dV - \int_{\partial V} \mathbf{n}.\mathbf{S} dA, \tag{12.10}$$

with

$$u^{\mathrm{em}.m} = \frac{1}{2}(\varepsilon\mathbf{E}^2 + \mathbf{B}^2/\mu) = \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H}). \tag{12.11}$$

But this is **generally not true** as an expression for electromagnetic energy in an arbitrary deformable solid where (1) the electromagnetic constitutive equations may be strongly nonlinear and may even be dissipative (e.g., with relaxation, hysteresis), and (2) electromagnetic fields do **not** constitute an isolated thermodynamic system and they are in strong interaction with the deformation field. A consequence of this fact is that, if (12.6) is always true, it does not constitute a local statement of energy conservation for the *whole* mechanical-*plus*-electromagnetic system (sorry, the "plus" may be misleading with a connotation of simple "addition"). Similarly, Eq. (12.7) does not constitute an equation for conservation of so-called canonical momentum for the whole system. Much more work is required to reach this general result. What is remarkable is that, in spite of these words of caution, many authors have a natural tendency to think of an expression such as $(12.11)_2$ as a starting point in any electromagnetic continuum. This is particularly true in relativistically invariant theories where the a priori viewpoint of Minkowski (1908) concerning electromagnetic momentum and electromagnetic stress tensor (there the energy-momentum tensor) has been damaging. But Minkowski's reasoning is not based on a sophisticated physical model of field-matter interactions (Minkowski was a pure mathematician). The same remark also applies concerning another energetic quantity such as a Lagrangian density per unit volume. The density

$$l^{\mathrm{em}.f} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{12.12}$$

strictly applies only to electromagnetic fields in a vacuum although it was proposed by authors such as Voigt, following Thomson (Kelvin) and Maxwell, in analogy with a "mechanical" Lagrangian $L = K - W$ with kinetic and potential contributions. All this clearly means that part of the electromagnetic energy and of Lagrangian densities is stored also in the internal/free energy or "matter" Lagrangian for the combined mechanical-*plus*-electromagnetic medium that includes the missing interaction terms that should be expressed in terms of the essentially material fields (12.4).

One remark about the electric current: for all practical purposes, we note that the Joule term $\mathbf{J}.\mathbf{E}$ can be interpreted as a power expended by an electric force. Indeed, we can write as an example

$$\mathbf{J}.\mathbf{E} = (q\mathbf{v}).\mathbf{E} = (q\mathbf{E}).\mathbf{v} = \mathbf{f}.\mathbf{v}, \tag{12.13}$$

where $\mathbf{f} = q\mathbf{E}$ is seen in statics, according to Lorentz, as the elementary mechanical force acting on a point particle of electric charge q in an electric field $\mathbf{E}$. For a particle moving at velocity $\mathbf{v}$, we have the *Lorentz force*

$$\mathbf{f} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B} = q\tilde{\mathbf{E}},\ \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}, \tag{12.14}$$

where the electric field $\tilde{\mathbf{E}}$ is called the *electromotive intensity*.

To close this section of prerequisite we briefly recall the relationship of Maxwell's theory with electrical engineering.

First, Faraday's equation $(12.2)_1$ relates the circuitage voltage that appears when the flux linkage varies in time, as in electrical generators. Indeed, by use of Stokes' theorem applied to a surface element S leaning on a circuit C, one shows that the difference of potential is given by

$$e.m.f = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}, \tag{12.15}$$

or

$$e = \frac{d\lambda}{dt} \tag{12.16}$$

in terms of the flux linkage λ.

Second, Ampère's law $(12.3)_1$ relates the magnetic field that curls around a current flux, corrected for unsteady values of electric fields (this last correction is due to Maxwell; cf. the notion of displacement current). By applying Stokes' theorem to a surface element S leaning on a circuit C, one finds, for a coil of n turns of length l, the relation

$$\int_C \mathbf{H}.dl = nI \text{ or } H = nI/l, \tag{12.17}$$

where I is the current.

Third, Gauss-Poisson's equation $(12.3)_2$ tallies the field lines emanating (hence the divergence) from a distribution of charges.

Finally, the last of Maxwell's equation $(12.2)_2$ reflects the circumstance that isolated magnetic poles do not exist. As a consequence a line of magnetic induction closes on itself. It does not "emanate" from a magnetic charge distribution as the latter does not exist.

Ampère's and Gauss-Poisson's equations are not used as such in circuitry, but the law of conservation of charges (12.5) yields, by integration, the circuitry equation

$$I = \frac{dq}{dt}, \tag{12.18}$$

where q is the electric charge. Then the system (12.15)–(12.18) is closed by the well known *constitutive equations* of passive circuit elements:

$$\begin{aligned} \lambda &= L\, I\, (inductor), \\ q &= C\, e\, (capacitor), \\ e &= R\, I\, (resistor), \end{aligned} \tag{12.19}$$

where L, C and R are an *inductance*, a *capacitance*, and a *resistance*, respectively. The last of these represents the celebrated *Ohm law*. There exist nonlinear generalizations of the constitutive Eq. (12.19). Added to Kirchhoff's laws of currents at nodes, the above set Eqs. (12.15) through (12.19) are all what one needs at the macro-scale of *electrical engineering*.

12.3 Passing to a Charged, Magnetized, Electrically Polarized, Deformable Continuum

12.3.1 A True 20th Century Adventure

For a true physicist the generalization of above given equations such as (12.7) and (12.9) is a difficult task that is identified with the evaluation of the forces, couples and energy sources arising from the interaction between a large number of electric charges in moving matter at a microscopic scale. This is in order to avoid any arbitrary or a priori macroscopic expressions that are hard to posit save by divination. Such an approach that we favour over any other methods accounts for the rich information about the interactions between the mechanical system and electromagnetic fields in matter that are gained from a particle model due initially to **Hendrik Anton Lorentz** (1853–1928). It was taken over by Dixon and Eringen (1964), Nelson (1979), and Maugin and Eringen (1977) to whom we owe the present formulation. This analysis belongs in the most rewarding improvements brought to the *electrodynamics of moving media* in the 20th century. It consists in evaluating the total force, couple and power acting on, or developed by, electromagnetic fields on the elementary electric charges contained in a stable cloud or representative volume element of volume ΔV, and introducing the approximations of multipoles, a truncation of these at a certain order, and a volume or phase-space average. Lorentz's vision is essentially that of a free space containing point charged particles. The starting point is the celebrated Lorentz force (Lorentz 1909) that is written as [cf. (12.14)]

$$\delta \mathbf{f}_\alpha = \delta q_\alpha \left(\mathbf{e}\,(\mathbf{r}_\alpha) + \frac{1}{c}\,\dot{\mathbf{x}} \times \mathbf{b}_\alpha(\mathbf{r}_\alpha) \right), \tag{12.20}$$

where $\mathbf{e}$ and $\mathbf{b}$ are the electric field and magnetic induction at the current placement $\mathbf{r}_\alpha$ of the elementary electric charge δq_α contained in ΔV. The computation then consists in evaluating the quantities (force, couple, power of forces)

$$\sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha,\ \sum_{\alpha \in \Delta V} (\mathbf{r}_\alpha \times \delta \mathbf{f}_\alpha),\ \sum_{\alpha \in \Delta V} \delta \mathbf{f}_\alpha . \dot{\mathbf{x}}_\alpha, \tag{12.21}$$

and then dividing by ΔV. This may be considered a naive volume average technique. A more advanced one would envisage a statistical average in phase space, and perhaps a formulation in a relativistic framework. Anyway this is the technique followed first by Lorentz and then by Dixon and Eringen (1964), Nelson (1979), and Maugin and Eringen (1977) in a Galilean approximation and by de Groot and Suttorp (1972) in a relativistic framework.

Remark 12.3: On the notation of fields. By way of example, let $\mathbf{M}$ denote the magnetization per unit volume in a fixed laboratory frame. Then derived fields are noted with a superimposed ornament. Thus $\tilde{\mathbf{M}}$ designates the volume magnetization in a frame co-moving with the element of matter; $\bar{\mathbf{M}}$ is the same but reported (i.e., convected back) to the material framework and $\mu = \tilde{\mathbf{M}}/\rho$ is the magnetization per unit mass. Similarly for the electric polarization $\mathbf{P}$, with $\tilde{\mathbf{P}}$, $\bar{\mathbf{P}}$ and $\pi = \tilde{\mathbf{P}}/\rho$. Note that both magnetization and electric polarization relate to matter and are *extensive* quantities, i.e., proportional to the volume of matter. This is most relevant in continuum thermo-mechanics.

12.3.2 Results from the Microscopic Model

From (12.20) and (12.21), expanding the expressions in terms of the internal coordinates $\xi_\alpha = \mathbf{x}_\alpha(t) - \mathbf{x}$, neglecting quadrupole contributions and higher-order multipoles, lengthy calculations lead to the following electromagnetic source terms of force, couple and energy per unit continuous volume:

$$\mathbf{f}^{\mathrm{em}} = q_f \tilde{\mathbf{E}} + \frac{1}{c}(\tilde{\mathbf{J}} + \mathbf{P}^*) \times \mathbf{B} + (\mathbf{P}.\nabla)\mathbf{E} + (\nabla \mathbf{B}).\tilde{\mathbf{M}}, \tag{12.22}$$

$$\mathbf{c}^{\mathrm{em}} = \mathbf{r} \times \mathbf{f}^{\mathrm{em}} + \tilde{\mathbf{c}}^{\mathrm{em}}, \tag{12.23}$$

$$w^{\mathrm{em}} = \mathbf{f}^{\mathrm{em}}.\mathbf{v} + \tilde{\mathbf{c}}^{\mathrm{em}}.\Omega + \rho h^{\mathrm{em}}, \tag{12.24}$$

where $\mathbf{r}$ refers to the centre of charges of the volume element, ρ is the matter density, and $\mathbf{v}$ is the physical velocity, Ω is the vorticity $\Omega = (\nabla \times \mathbf{v})/2$, and we have set

$$q_f(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha, \tag{12.25}$$

$$\mathbf{P}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha \xi_\alpha(\mathbf{x}, t), \tag{12.26}$$

$$\mathbf{M}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \frac{1}{2c} \delta q_\alpha \xi_\alpha \times \dot{\xi}_\alpha, \tag{12.27}$$

Note the lack of symmetry between polarization and magnetization effects. We have also defined the intrinsic electromagnetic sources of couple, energy and stress by (here tr = trace; subscript s stands for the operation of symmetrisation)

$$\tilde{\mathbf{c}}^{\mathrm{em}} = \mathbf{P} \times \tilde{\mathbf{E}} + \tilde{\mathbf{M}} \times \mathbf{B}, \tag{12.28}$$

$$\rho h^{\mathrm{em}} = \widetilde{\mathbf{J}}\widetilde{\mathbf{E}} + \widetilde{\mathbf{E}}\mathbf{P}^* - \widetilde{\mathbf{M}}.\mathbf{B}^* + tr\left(\widetilde{\mathbf{t}}^{\,\mathrm{em}}(\nabla\mathbf{v})_s\right), \tag{12.29}$$

and

$$\widetilde{\mathbf{t}}^{\,\mathrm{em}} = \mathbf{P} \otimes \widetilde{\mathbf{E}} - \mathbf{B} \otimes \widetilde{\mathbf{M}} + (\widetilde{\mathbf{M}}\mathbf{B})\mathbf{1}, \tag{12.30}$$

where the following fields are those in a co-moving frame (*Galilean approximation*; first of these is the conduction current *per se*):

$$\tilde{\mathbf{J}} = \mathbf{J} - q_f \mathbf{v},\ \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B},\ \tilde{\mathbf{M}} = \mathbf{M} + \frac{1}{c}\mathbf{v} \times \mathbf{P} \tag{12.31}$$

and $\mathbf{E}$ and $\mathbf{B}$ are simple volume averages of $\mathbf{e}$ and $\mathbf{b}$. The first contribution in the *r-h-s* of (12.22) is none other than a "Lorentz force" [cf. (12.14)] since

$$\mathbf{f}_L = q_f \mathbf{E} + \frac{1}{c}(q_f \mathbf{v}) \times \mathbf{B} = q_f \tilde{\mathbf{E}}. \tag{12.32}$$

Finally, a left asterisk denotes a so-called convected (Oldroyd) time derivative such that (see Sect. 3.4)

$$\mathbf{P}^* = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + \mathbf{v}(\nabla.\mathbf{P}) = \frac{d\mathbf{P}}{dt} - (\mathbf{P}.\nabla)\mathbf{v} + \mathbf{P}(\nabla.\mathbf{v}). \tag{12.33}$$

The above given expressions where fields in the laboratory frame and those in a co-moving frame co-exist is only *Galilean invariant*, but this is sufficient for engineering purposes. Note that no $\tilde{\mathbf{P}}$ intervenes here for the good reason that $\tilde{\mathbf{P}} = \mathbf{P}$ in this approximation (Galilean approximation materialized by a lack of symmetry between electric polarization and magnetization).

12.3.3 Contributions in the Macroscopic Balance Laws

In principle, the above obtained *source terms,* once their origin forgotten, have to be carried into the classical balance laws of a continuum (with a possible *non symmetric* Cauchy stress), leaving however the internal/free energy of the medium to depend on the electromagnetic fields. A remarkable fact is that in spite of their farfetched outlook, some may be given a form that reminds us of some standard expression [such as in (12.6)]. For instance, Maugin and Eringen (1977) have shown that (12.24) can also be written as

$$w^{\mathrm{em}} = \mathbf{J}.\mathbf{E} + \mathbf{E}.\frac{\partial \mathbf{P}}{\partial t} - \mathbf{M}.\frac{\partial \mathbf{B}}{\partial t} + \nabla(\mathbf{v}(\mathbf{E}.\mathbf{P})) = -\frac{\partial u^{\mathrm{em}.f}}{\partial t} - \nabla.(\mathbf{S} - \mathbf{v}(\mathbf{E}.\mathbf{P})), \tag{12.34}$$

in which we identify the terms already present in (12.9).

The electromagnetic volume force defined in (12.22) is sometimes called the electromagnetic *ponderomotive force*, $\tilde{\mathbf{c}}^{\mathrm{em}}$ being then the *ponderomotive couple*. In 1974, Collet and Maugin proved the following remarkable identity at any regular material point:

$$\frac{\partial \mathbf{p}^{\mathrm{em}}}{\partial t} - div\, \mathbf{t}^{\mathrm{em}} = -\mathbf{f}^{\mathrm{em}}, \tag{12.35}$$

where

$$\mathbf{p}^{\mathrm{em}} = \mathbf{p}^{\mathrm{em}.f} = \frac{1}{c}\mathbf{E} \times \mathbf{B}, \tag{12.36}$$

$$\mathbf{t}^{\mathrm{em}} = \mathbf{t}^{\mathrm{em}.f} + \tilde{\mathbf{t}}^{\mathrm{em}}. \tag{12.37}$$

Since we are dealing with nonsymmetric second-order tensors, we must specify that their divergence is taken on the first index. Simultaneously, the ponderomotive couple is the axial vector associated with the skew part of $\tilde{\mathbf{t}}^{\mathrm{em}}$. The latter vanishes together with the source terms in (12.28) outside matter, and (12.35) reverts to (12.7) in a vacuum. Because of the source term in its *r-h-s* equation (12.35) is *not* a conservation law for the whole physical system. But it allows one to rewrite the balance law of linear momentum for the whole continuum in a specific form (see Maugin 1988, for these developments). We can also rewrite (12.22) emphasizing the occurrence of an *effective* Lorentz force $\mathbf{f}_L^{\mathrm{eff}}$ in the form

$$\mathbf{f}^{\mathrm{em}} = \mathbf{f}_L^{\mathrm{eff}} + div\, \tilde{\mathbf{t}}^{\mathrm{em}}, \tag{12.38}$$

with [compare to (12.20)]

$$\mathbf{f}_L^{\mathrm{eff}} = q^{\mathrm{eff}}\tilde{\mathbf{E}} + \frac{1}{c}\tilde{\mathbf{J}}^{\mathrm{eff}} \times \mathbf{B}, \tag{12.39}$$

where

$$q^{\text{eff}} = q_f - \nabla.\mathbf{P},\ \tilde{\mathbf{J}}^{\text{eff}} = \tilde{\mathbf{J}} + \mathbf{P}^* + c\nabla \times \tilde{\mathbf{M}}. \tag{12.40}$$

We easily check that there holds the identity

$$\frac{\partial \mathbf{p}^{\text{em}}}{\partial t} - div\, \mathbf{t}^{\text{em}.f} = -\mathbf{f}_L^{\text{eff}}. \tag{12.41}$$

Equations (12.35) and (12.41) are compatible, but they may suggest different ways to combine mechanics and electromagnetism in the balance of linear momentum as it may be tempting to many researchers to consider $\mathbf{f}_L^{\text{eff}}$ as the primitive interaction force because effective charge and electric current appear also in Maxwell's equations (cf. Eringen and Maugin 1990, p. 54) as natural perturbations of the vacuum equations, e.g., (12.3) also read

$$\nabla.\mathbf{E} = q_f - \nabla.\mathbf{P},\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M}\right). \tag{12.42}$$

These can be recast using convected fields and time derivatives yielding source expressions such as in (12.33). Finally, we remark that Eqs. (12.34) and (12.35) reduce to (12.9) and (12.7) in a vacuum, respectively.

Note While the above-given results are obtained, a similar treatment of Maxwell's equations in vacuum with source terms due to the individual electric charges, yields, after space average, the macroscopic equations (12.2) and (12.3)—this was the basic idea of Lorentz [on Maxwell's equations proper, see de Groot (1969) and Tiertsen (1990)].

12.3.4 Postulate of Equations Accounting for Informations from a Microscopic Model

This is the manner *à la* Newton-Cauchy dear to the Truesdellians. Global balance laws are written for linear and angular momenta along with the first and second laws of thermodynamics, in which electromagnetic source terms as recalled above are introduced. This is the viewpoint expanded in Eringen and Maugin (1990) and Maugin (1988), and other authors, in great detail. Of course the result depends on the microscopic model used to obtain the sources or else, on an a priori and somewhat arbitrary choice for these sources (**not** our viewpoint). The full application of the method shows its pertinence, albeit in spite of a complexity arising in the description of stresses. The latter are not symmetric a priori since there exists an applied couple (12.28), something that cannot be denied as otherwise there would not exist such an evident effect as the compass alignment with a magnetic field. But in the end the obtained thermo-mechanics proves to be satisfactory with an energy (internal or free-Helmholtz) containing part of the interactions, a part of constitutive origin. Among the results obtained we note the formula for the stresses $\mathbf{t}$ appearing in the local balance of linear momentum of a continuum (divergence of tensors taken on the first index; $\mathbf{f}$ = body force such as gravity, ρ = actual matter density; $\dot{\mathbf{v}}$ = acceleration)

$$\rho\dot{\mathbf{v}} = \mathbf{f} + \mathbf{f}^{\mathrm{em}} + div\,\mathbf{t}, \tag{12.43}$$

with a *nonsymmetric* Cauchy stress

$$\mathbf{t} = \mathbf{t}^{E} + \left(\mathbf{t}^{\mathrm{em}.f} - \mathbf{t}^{\mathrm{em}}\right) = \mathbf{t}^{E} - \tilde{\mathbf{t}}^{\mathrm{em}}, \tag{12.44}$$

or a total *symmetric (Cauchy)* stress τ such that

$$\tau = \mathbf{t} + \mathbf{t}^{\mathrm{em}} = \mathbf{t}^{E} + \mathbf{t}^{\mathrm{em}.f}, \tag{12.45}$$

where $\mathbf{t}^{E}$ is a symmetric "elastic" stress such that, in components (here symmetric and skewsymmetric parts)

$$t^{E}_{(ij)} = t_{(ji)} + \tilde{t}^{\mathrm{em}}_{(ji)},\ t^{E}_{[ji]} \equiv 0. \tag{12.46}$$

To the same degree of generality as (12.43), the local forms of the energy equation and inequality of entropy read (Eringen and Maugin 1990)

$$\rho\dot{e} = tr\left(\mathbf{t}(\nabla\mathbf{v})^{T}\right) - \mathbf{f}^{\mathrm{em}}.\mathbf{v} + w^{\mathrm{em}} - \nabla.\mathbf{q} + \rho h, \tag{12.47}$$

and

$$\rho\dot{\eta} \geq \rho h\theta^{-1} - \nabla.(\mathbf{q}\theta^{-1}), \tag{12.48}$$

where e, η, θ, $\mathbf{q}$ and h are the internal energy per unit actual mass, the entropy per unit actual mass, the thermodynamic temperature, the heat flux vector, and the external heat supply per unit actual mass, respectively. The electromagnetic energy "source" w^{em} is given by (12.24) with expressions (12.28) through (12.30) valid. Equivalent forms were given in (12.34). Another equivalent expression is given by

$$w^{\mathrm{em}} = \mathbf{f}^{\mathrm{em}}.\mathbf{v} + \rho\tilde{\mathbf{E}}.\dot{\pi} - \tilde{\mathbf{M}}.\dot{\mathbf{B}} + \tilde{\mathbf{J}}.\tilde{\mathbf{E}}, \tag{12.49}$$

where $\pi = \mathbf{P}/\rho$ is the electric polarization per unit mass. On introducing the Helmholtz free energy function per unit mass

$$\psi = e - \eta\,\theta, \tag{12.50}$$

and substituting from (12.47), (12.22) and (12.49) in (12.48), one arrives at the so-called *Clausius-Duhem inequality*

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + tr\left(\mathbf{t}(\nabla\mathbf{v})^T\right) + \tilde{\mathbf{J}}.\tilde{\mathbf{E}} + \rho\tilde{\mathbf{E}}.\dot{\pi} - \tilde{\mathbf{M}}.\dot{\mathbf{B}} - (\mathbf{q}/\theta).\nabla\theta \geq 0. \tag{12.51}$$

In a now well established tradition, this is conceived as a constraint on the formulation of constitutive equations for the fields $(\psi, \eta, \mathbf{t}, \tilde{\mathbf{J}}, \tilde{\mathbf{E}}, \tilde{\mathbf{M}}, \mathbf{q})$. The formulation (12.51) clearly emphasizes for electromagnetic processes the role of independent variables (causes) played by the pair $(\tilde{\mathbf{E}}, \nabla\theta)$, electric polarization and magnetic induction for galvanomagnetic effects.

What is important here is that, in deformable solids, one often prefers to reformulate the theory in terms of so-called *material* fields. To that effect we set the Piola transform or pull-back of electromagnetic fields (cf. Chap. 3)

$$\bar{\mathbf{B}} = J_F\mathbf{F}^{-1}.\mathbf{B},\ \bar{\mathbf{D}} = J_F\mathbf{F}^{-1}.\mathbf{D},\ \bar{\mathbf{P}} = \Pi = J_F\mathbf{F}^{-1}.\mathbf{P} = \rho_0\mathbf{F}^{-1}.\pi, \tag{12.52}$$

and

$$\bar{\mathbf{E}} = \mathbf{E}.\mathbf{F},\ \tilde{M}_K = J_F F^{-1}_{\mathrm{kp}}\tilde{M}_p,\ \bar{M}_K = \tilde{M}_i F_{iK}, \tag{12.53}$$

with

$$J_F = det\,\mathbf{F}, \rho_0 = \rho J_F,\ \mathbf{F} = \left\{F_{iK} = x_{i,K}\right\}. \tag{12.54}$$

We then check that the relations (12.1) translate in material components to

$$\bar{D}_K = J_F C^{-1}_{\mathrm{KL}}\bar{E}_L + \Pi_K\,,\ \bar{H}_K = J_F^{-1}C_{\mathrm{KL}}\bar{B}_L - \bar{M}_L, \tag{12.55}$$

with

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left\{C_{\mathrm{KL}} = x_{i,L}\,x_{i,L}\right\},\ \mathbf{C}^{-1} = (\mathbf{C})^{-1}. \tag{12.56}$$

First and second Piola-Kirchhoff stresses are defined by

$$\mathbf{T} = J_F\mathbf{F}^{-1}.\mathbf{t}\,,\ \mathbf{S} = \mathbf{T}.\mathbf{F}^{-T}. \tag{12.57}$$

Similar definitions hold for Piola-Kirchhoff stresses associated with the stresses $\mathbf{t}^E$ and $\mathbf{t}^{\mathrm{em}.f}$. Thus we write

$$\mathbf{T}^E = J_F\mathbf{F}^{-1}.\mathbf{t}^E,\ \mathbf{S}^E = \mathbf{T}^E.\mathbf{F}^{-T},\ \mathbf{T}^F = J_F\mathbf{F}^{-1}.\mathbf{t}^{\mathrm{em}.f}. \tag{12.58}$$

Then it is proved that Eqs. (12.43) and (12.51) can be rewritten as (here no body force)

$$\frac{\partial}{\partial t}\mathbf{p}^t_R\bigg|_X - div_R(\mathbf{T}^E + \mathbf{T}^F) = \mathbf{0},\ \mathbf{p}^t_R \equiv \rho_0\left(\mathbf{v} + \frac{1}{\rho c}\mathbf{E}\times\mathbf{B}\right), \tag{12.59}$$

and

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2}S^E_{\mathrm{KL}}\dot{C}_{\mathrm{KL}} + \bar{E}_K\dot{\Pi}_K - \bar{M}_K\dot{\bar{B}}_K + J_F(\tilde{\mathbf{J}}.\tilde{\mathbf{E}} - (\mathbf{q}/\theta).\nabla\theta) \geq 0, \tag{12.60}$$

where we have set

$$W = \rho_0 \psi, N = \rho_0\, \eta. \tag{12.61}$$

Once we have established constitutive equations for S^E_{KL}, $\bar{E}_K$ and $\bar{M}_K$, we can return to the original Eulerian fields, including the Cauchy stress $\mathbf{t}$ which is the stress present in the mechanical boundary condition. In order to complement Eq. (12.59) we also need Maxwell's equations expressed in the material framework. A first hint of this form of Maxwell equations was given by Walker et al. (1965), and McCarthy (1968). But the final form was definitely set by Lax and Nelson (1976).

Equation (12.60) is perfectly equipped to treat both reversible and irreversible coupled electro-magneto-deformable properties. Irreversible properties have been dealt with in particular by Maugin and co-workers (electric relaxation, hysteresis). However, both Eqs. (12.59) and (12.60), although fully dynamical in the Galilean approximation, are *not* equipped to treat the case of electromagnetic materials endowed with electromagnetic *internal degrees* of freedom (see Sects. 12.4 and 12.5 for these).

In this section we have presented the continuum dynamic theory—with the restrictions just mentioned—in its achieved form. Of course the path to this final form (reported in the author's formalism and its obvious shortcomings) was long and paved by many researchers, using various methods of approach. In particular, we must single out the enlightening book of Livens (1962)—George Henry Livens (1886–1950) was a Cambridgian whose main work was in effect electrical theory, the work of Tiersten and Tsai (1972) at the Rensselaer Polytechnic, the theory developed by Kolumban Hutter and Y.-H. Pao at Cornell in the early 1970s—see the book by Hutter and Van de Ven (1978), the variational approach by electrical engineers (Penfield and Haus 1967) at M.I.T, the treatise of Truesdell and Toupin (1960) with its appropriate sections, the papers by Alblas (1974) in the Netherlands, the remarkable works of physicists Lax and Nelson synthetized in Nelson's (1979) book, and the often unjustly ignored works by L.I. Sedov and his co-workers at Moscow State University in the 1960s–1970s (see Sedov's books on continuum mechanics; Chap. 11). We shall shortly deal with the fundamental role played by the pioneers such as Toupin, Brown, Eringen, Mindlin and Tiersten. In more recent works overlapping the early 21st century we find the variational formulations of Trimarco and Maugin (2001)—also in Maugin (1993a), Chap. 8, and Dorfmann and Ogden (see all these in the Udine course of 2009 published by Ogden and Steigmann 2011). These formulations require the addition of an interaction term between matter and fields to the Lagrangian density (12.12). These continuum-mechanics approaches supersede all presentations by well-known physicists, even the classic books on electrodynamics by Jackson and Landau and Lifshitz.

12.4 Theory of Elastic Dielectrics and Generalizations

12.4.1 Toupin's Theory

In the above reported developments a fundamental role was played by a beautiful work published by Toupin in 1956. A few words of gossip may be exceptionally introduced at this point. In his autobiographic notes (reprinted in his collected works edited by Barenblatt and Joseph (1997), Rivlin tells that he met Toupin at the National Research Laboratory in Maryland where both were visiting in 1953. Toupin was then working on a PhD with Melvin Lax at the University of Syracuse (Lax left Syracuse to join Bell Telephone Laboratories and ended his brilliant career of solid-state physicist at the City College of New York where we visited him). Rivlin advised Toupin to change his research subject to the theory of deformable dielectrics where he foresaw some promising developments in the finite-strain framework of continuum mechanics, what Toupin did with the success we know. Indeed, Toupin's publication of his "Elastic dielectric" paper in the *J.R.M.A.* in 1956 proved to be a true milestone. This was followed by another paper about dynamics in 1963. Of course, this is now contained in the general presentation given in the foregoing section.

It must be realized that before Toupin's landmark work, the only well formulated and very much applied theory of *electro-mechanical interactions* in continuum mechanics was the standard theory of *linear* piezoelectricity. This went back to the original discovery of the effect by the Pierre and Jacques Curie in Paris in 1881. It was recognized that this required the consideration of crystals having no centre of symmetry [in order to allow a *linear* relation between a second-order tensor (e.g., strain) and a vector (electric field or polarization)]. The relevant Cartesian tensor formulation was established and the effect became popular through its exploitation in "sonars" (cf. the pioneering work by Paul Langevin during WWI). In the 1940s and 1950s, the importance of piezoelectric couplings was duly recognized in devices of *signal processing* exploiting the short wavelength of piezoelectric waves compared to electromagnetic ones, and using the properties of piezoelectric vibrations of structures (e.g., plates). The collaboration between the US Army Signal Corps Laboratory in Fort-Monmouth, New Jersey, and Raymond Mindlin culminated in a beautiful lengthy technical Army report by Mindlin that was only recently published in book form (Mindlin 2006). Harry Tiersten, a former student of Mindlin, put some of these in Lagrangian-Hamiltonian variational form in a small monograph (Tiersten 1969). Now back to Toupin.

Toupin's theory can be extracted from the contents of Sect. 12.3 by discarding magnetic effects and all forms of dissipation. Thus we obtain the following reduction of (12.60) to an equality for *hyperelastic dielectric solids* ($q_f = 0$, $\tilde{\mathbf{M}} = \mathbf{0}$, $\tilde{\mathbf{J}} = \mathbf{0}$):

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2} S^{E}_{\mathrm{KL}} \dot{C}_{\mathrm{KL}} + \bar{E}_K \dot{\Pi}_K = 0, \tag{12.62}$$

from which there follows the constitutive equations

$$S_{\mathrm{KL}}^{E} = 2\frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}},\, \bar{E}_{K} = \frac{\partial \hat{W}}{\partial \Pi_{K}},\, N = -\frac{\partial \hat{W}}{\partial \theta}, \tag{12.63}$$

wherein the free energy per unit undeformed volume is given by

$$W = \hat{W}(C_{KL}, \Pi_{K}, \theta). \tag{12.64}$$

Accordingly, the following constitutive equations are obtained for the "elastic" stress and the material electric field

$$t_{ji}^{E} = 2J_{F}^{-1}F_{jK}F_{iL}\frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}},\, \bar{E}_{K} = \frac{\partial \hat{W}}{\partial \Pi_{K}}, \tag{12.65}$$

Then, after (12.14),

$$\mathbf{t} = \mathbf{t}^{E} - \mathbf{P} \otimes \tilde{\mathbf{E}} = \mathbf{t}^{E} - J_{F}^{-1}\mathbf{F}.\Pi \otimes \tilde{\mathbf{E}}, \tag{12.66}$$

hence in components for the Cauchy stress

$$t_{ji} = J_{F}^{-1}F_{jK}\left(2\frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}} - \Pi_{K}\frac{\partial \hat{W}}{\partial \Pi_{L}}\right)F_{iL}. \tag{12.67}$$

Toupin's theory is not exactly this because temperature effects are not included and, astutely, Toupin makes a difference between the Maxwellian field $\mathbf{E}$ and a *local* electric field, noted $\mathbf{E}^{L}$ provided by a constitutive equation, so that we in fact have a kind of *local balance law for electric fields*:

$$\mathbf{E} + \mathbf{E}^{L} = \mathbf{0},\, \bar{E}_{K}^{L} = E_{i}^{L}F_{iK} = -\frac{\partial \hat{W}}{\partial \Pi_{K}}. \tag{12.68}$$

Contrary to the theory of linear piezoelectricity where all nonlinear terms in the fields are discarded, Toupin's theory still includes nonzero ponderomotive force and couple (hence a non symmetric Cauchy stress) given by (compare to the general expression in (12.22) and (12.28))

$$\mathbf{f}^{\mathrm{em}} = (\mathbf{P}.\nabla)\mathbf{E} = (\nabla\mathbf{E}).\mathbf{P},\, \tilde{\mathbf{c}}^{\mathrm{em}} = \mathbf{P} \times \mathbf{E}, \tag{12.69}$$

where the transformation of the first of these follows from the quasi-static electric equation $\nabla \times \mathbf{E} = \mathbf{0}$.

Toupin's theory is potentially rich of many effects and generalizations. First, it can include electro-elastic interactions at any order (piezoelectricity, electrostriction, and higher order effects in the electric field). Second, it is not limited to small deformations, and can therefore be applied in modern technology to finitely deformable polymeric dielectrics. Finally, the writing of the first of (12.68) that looks somewhat artificial and unnecessary, is gross of further generalizations that we are going to examine. Toupin's theory was presented in a variational form by Eringen (1963). For sure, it influenced all works after 1956.

12.4.2 Generalizations

It was discovered by Mindlin (1968; see also Herrmann 1974) and others that in the presence of a centre of symmetry—that forbids the existence of linear piezoelectricity, there still exist a possibility of a linear coupling between deformation and a gradient of electric polarization, for nonuniformly polarized materials. This is rare but possible for ionic crystals such as alkali halides (e.g., *NaCl*, *KCl*). In this theory the generalization of (12.68) reads

$$\mathbf{E} + \mathbf{E}^L + \rho^{-1} div\, \hat{\mathbf{E}}^L = \mathbf{0}, \tag{12.70}$$

where the new tensor $\hat{\mathbf{E}}^L$ is principally determined by the gradient $\nabla \mathbf{P}$, while the vector $\mathbf{E}^L$ remains determined by the electric polarization itself. In a nonlinear theory, both of these quantities will contribute to the skew symmetric part of the Cauchy stress (Collet and Maugin 1974). In this theory called "the theory of polarization gradients", Eq. (12.70) has the true status of a field equation on the same footing as the standard equation of equilibrium. This theory is entirely corroborated by the appropriate approach from lattice dynamics, as shown by Askar et al. (1970) [P.C.Y. Lee also was a PhD student of Mindlin]. The full formulation of this theory with applications to nontrivial physical effects is to be found in Mindlin's synthesis of 1972, but above all in Chap. 7 of our book (Maugin 1988) with a generous relevant bibliography.

In his original work of 1963, Toupin alludes to the possibility of having an inertial (polarization) term in the right-hand side of the above given Eq. $(12.68)_1$. In a successful attempt at a dynamical theory of ferroelectric crystals, Maugin and Pouget (1980) formulated a complete theory in the finite-strain framework of continuum thermo-mechanics in which a field equation governing the electric polarization is obtained in a form that looks like (12.70) but with a polarization inertia in its right-hand side, i.e.,

$$\mathbf{E} + \mathbf{E}^L + \rho^{-1} div\, \hat{\mathbf{E}}^L = d_E \ddot{\mathbf{P}} \tag{12.71}$$

where the tensor field $\hat{\mathbf{E}}^L$ is related to the interaction between neighbouring permanent electric dipoles. This theory is also justified by a lattice-dynamics approach as shown by Pouget et al. in 1986 for ferroelectrics of the molecular-group type (e.g., $NaNO_2$). More on this model and wave propagation (including the structure and motion—as solutions—of ferroelectric domain walls is to be found in the book of Maugin et al. (1992)—also Bassiouny et al. 1988. The theory was extended to the case of elastic antiferroelectrics by Soumahoro and Pouget (1994).

12.5 Theory of Magneto-Elastic Continua

The magneto-elastic coupling called *magnetostriction* was discovered by Joule in the 19th century. It results in a very small strain upon the application of a longitudinal magnetic field to a bar, and this independently of the direction of the field; Hence it intensity varies like the square of that field and it is, basically, a nonlinear effect. In spite of its smallness this effect is important because a corresponding linear effect—linear piezomagnetism—is much more rare than piezoelectricity in natural conditions. However it can appear as linearized magnetostriction about an intense magnetic field. Note also that with the discovery of "giant" magnetostriction in some compounds the effect is improved by two orders of magnitude so that magnetostriction may be envisaged in competition with some piezoelectric devices (for the physical viewpoint on magnetostriction see the book by du Trémolet 1993).

After many works in the field, William F. Brown Jr proposed a serious continuum theory of magneto-elastic interactions in his book of 1966, in a collection edited by Truesdell. Simultaneously, Tiersten (1964, 1965)—he had been a PhD student of Mindlin and worked at Bell Labotatories for sometimes before joining the Rensselaer Polytechnic—following works by the solid-state physicist Kittel (1958) dealing with the interaction between elastic and magnetic-spin waves, proposed in 1964 a theory of elastic hard ferromagnets in the finite-strain framework that accounts for the presence of a density of magnetic spin and the interaction between neighbouring spins (or magnetic dipoles). This he complemented with an astute variational formulation in Tiersten (1965). This modelling was taken over by Maugin (1971) in his Princeton PhD thesis and in papers by Maugin and Eringen (1972). It was also exposed by Akhiezer et al. (1968) in a famous book on spin waves. What is remarkable is that in this theory the equation governing the magnetic spin density has the following form:

$$\gamma^{-1}\dot{\mu} = \mu \times (\mathbf{B} + \mathbf{B}^L + \rho^{-1}\, div\, \hat{\mathbf{B}}^L), \qquad (12.72)$$

which guarantees that the magnetization per unit mass has a prescribed modulus (condition of saturation). Here γ is the so-called gyromagnetic ratio of the material, and the "local" fields $\mathbf{B}^L$ and $\hat{\mathbf{B}}^L$ are primarily determined by the magnetization and its gradient, respectively, reflecting in the continuum framework the effects of magnetic anisotropy (preferential directions of magnetization) and Heisenberg exchange forces between neighbouring spins. In principle, both may have dissipative contributions associated with them. It was shown by Maugin (1972, 1975) that the first yields a correct formulation of the effect of spin-lattice relaxation in deformable ferromagnets (using the notion of Jaumann co-rotational time derivative). Later on the theory was extended to the case of deformable ferrimagnets and antiferromagnets (Maugin 1976; Maugin and Sioké-Rainaldy 1983) by adopting the idea of the French physicist Louis Néel of the co-existence of multiple magnetic sub-lattices. For all these and a rather complete presentation of dynamical processes (coupled waves), we refer the reader to Chap. 7 in Maugin (1988) and Chap. 9 in Eringen and Maugin (1990). At the same time, Sabir and Maugin (1990) provided a phenomenological theory of irreversible magnetic hysteresis coupled to stresses by applying the thermomechanical framework using internal variables of state (compare Sect 5.6 and Maugin 1993b) by analogy with plasticity and visco-plasticity. This concurs with Néel's theory of the 1940s.

We note the resemblance of the expression within parentheses in the right-hand side of (12.72) with the left-hand side of (12.70) and (12.71). As a matter of fact, whenever exchange interactions and gyromagnetic effects are discarded, Eq. (12.72) reduces to a balance equation for the Maxwellian and local magnetic inductions in the form

$$\mathbf{B} + \mathbf{B}^L = \mathbf{0}, \tag{12.73}$$

a form entirely analogous to that of Toupin's equation (12.68). The resulting theory for soft ferromagnets and paramagnets could be derived from the theory exposed in Sect. 12.3. But the more general theory contained in both Eqs. (12.71) and (12.72) was shown to be derivable from a modern formulation of the *principle of virtual power* (d'Alembert's principle) by Collet and Maugin (1974) by considering that electric polarization and magnetization provide additional *internal degrees of freedom*, on equal footing with the classical deformation motion [general theory in Maugin (1980)]. The coupling between these internal degrees of freedom and stresses then appear naturally in writing the power expended by internal forces upon the constraint of being objective. This safe and powerful method was further used in all models of electro-magneto-mechanical interactions, including in complex modellings such as that of deformable semi-conductors (Daher and Maugin 1986; Maugin and Daher 1986), after initial studies on piezoelectric semiconductors by Ancona and Tiersten (1983).

12.6 Concluding Remarks

In the above given survey we emphasized the evolution in the very bases of the theory of electro-magneto-elastic interactions, noting the seminal role played by Richard A. Toupin and Raymond D. Mindlin. This should be complemented by a description of the many applications treated during the period 1960–2010 and witnessed by the present writer in a very active position. This would prove to be a formidable task, perhaps not as instructive as imagined. What we can notice is that, apart from the general principles already scrutinized, many of the applicative developments have more or less followed the trends of the corresponding pure mechanical developments in the same period.

Concerning *wave propagation*, the very new ingredient in the linear theory of piezoelectricity was the discovery in 1968 of the so-called Bleustein-Gulyaev piezoelectric surface wave simultaneously by J.L. Bleustein (a co-worker of Tiersten) in the USA and Yu. V. Gulyaev in the USSR (cf. Chap. 11). This is a shear horizontal surface wave (like the celebrated Love surface mode) of which the propagation is allowed by the perturbation created at the surface by a piezoelectric coupling. Many other wave problems including bulk waves, surface waves, shock waves, and solitary waves have been treated in particular—for complex models of interactions—by the writer in Paris in collaboration with many researchers among whom we must single out Bernard Collet, Joel Pouget, Anaclet Fomethe, and Naoum Daher (see Maugin et al. 1992; Maugin 1988). Other centres of study of some of these waves were in Besançon (France) with J.-J. Gagnepain and M. Planat, but also in Tiersten's environment at the Rensselaer Polytechnic in Troy (USA), with David F. Parker in Nottingham (UK), and in the active team of A.N. Guz in Kiev (Ukraine). Special attention was paid to superconducting deformable solids by S.A. Zhou (1999).

The stability of magnetoelastic structures received much attention in relation to the fantastic strength of electromagnets used in magnetically-levitating trains and in some thermo-nuclear technologies. This culminated in the splendid book of Moon (1984). Other studies along the same line were conducted in Japan (with K. Miya in Tokyo and J. Tani in Sendai), and in Europe (G.A. Maugin and C. Goudjo in Paris, A.A.F Van de Ven in Eindhoven).

The mechanics of slender structures (plates, shells) coupled to electromagnetic properties was perfected to a high degree of analysis by Academician S.A. Ambartsumian and co-workers in Yerevan (Armenia) applying the asymptotic integration method introduced in pure mechanics by Golden'veizer and Ambartsumian himself—see Ambartsumian et al. (1977), while the zoom technique of P.G. Ciarlet and P. Destuynder was exploited by Attou and Maugin (1990) for piezoelectric plates.

Homogenization techniques first applied in pure continuum mechanics in the 1980s were rapidly applied to electro- and magneto-elasticity. Here we must cite the much original work in dynamics of Turbé and Maugin (1991) using a Bloch expansion of waves, and the application to nonlinear electroelasticity by Rodriguez-Ramos et al. (2004). Homogenization schemes have also been introduced in ferromagnetic bodies in order to account for the influence of their microstructure in industrial applications, for instance, by the group of René Billardon in Cachan (France) including Laurent Hirsinger, Nicolas Buiron, Olivier Hubert, and Laurent Daniel, and also in Metz and Besançon (France).

The theory of structural defects (e.g., dislocations) in elastic dielectrics has been carefully approached especially by V.I. Alshits (Moscow), A. Radowicz (Kielce, Poland) and J.P. Nowacki (Warsaw)—see Nowacki's book (2006). The corresponding theory of configurational forces—acting on defects, shock waves, phase-transition fronts—has also been extensively expanded in electro- ad magneto-elasticity (see Chap. 14 below). The works of R.M. McMeecking et al. should also be noted in conjunction with crack studies.

The relationship between complex models of electromagnetic deformable materials (e.g., ferroelectrics, ferromagnets) and *generalized continuum mechanics* will be briefly discussed in Chap. 13 [see also Chap. 1 in the book edited by Altenbach and Eremeyev (2012)].

Numerically convenient variational formulations and applications of the non-linear electroelasticity and magnetoelasticity have recently been given by several groups of authors, e.g., R.W. Ogden, A. Dorfmann and R. Bustamante on the one hand, D.J. Steigmann et al. on the other, and also Paul Steinmann and co-workers, and N. Triantafyllidis et al.

Finally, we mention that a specific scientific journal entitled the *International Journal of Applied Electromagnetics and Mechanics* was founded in 1991 by Kenzo Miya (Tokyo), Richard Hsieh (Stockholm) and G.A. Maugin (Paris), simultaneously with a successful series of technical volumes called "*Applied Electromagnetics and Mechanics*" published by I.O.S, in The Netherlands and Japan.

Personal touch: In addition to his doctoral students, co-workers in Paris, and visiting research associates, the author has had, or still entertain, friendly relations with many of the strongly involved actors: A.C. Eringen, R.D. Mindlin, H.F. Tiersten, M. Lax, D.F. Nelson, F.C. Moon, W. Nowacki, J.P. Nowacki, A. Askar, L.I. Sedov, S.A. Ambartsumian, S.R. de Groot, K. Hutter, D.F. Parker, R.W. Ogden, V.I. Alshits, A. Dorfmann, E. du Trémolet, R. Billardon, K. Miya and R.K.T. Hsieh. Unfortunately, he never met Richard Toupin.

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