Chapter 1
The Land Clearers and the "Classics"

Abstract This chapter has for object to remind the reader of the early developments of continuum mechanics-after the seminal works in mechanics by Descartes, Huygens, Newton and Leibniz-in the expert hands of the initiators of this science (the Bernoulli family, d'Alembert, Euler, Lagrange). This was rapidly followed by the foundational contributions of the first half of the Nineteenth century with Cauchy and Navier (in France), Piola (in Italy), Kirchhoff (in Germany), and those of various giants of science such as Green, Kelvin, Stokes, Maxwell, Boussinesq, Poiseuille, Clebsch, von Helmholtz, Voigt, Mohr, and Barré de Saint-Venant later in the century. The emphasis is placed on the role played by so-called "ingénieurs-savants", many of them educated at the French Ecole Polytechnique and the engineering schools inspired by this school all over Europe. Lamé, Navier and Duhamel in France and their Italian colleagues are examples of such people who harmoniously combined works in a much wanted contribution to civil engineering and a sure mathematical expertise in analysis. In contrast, the German and English contributors were more inclined towards an emerging true mechanical engineering and sometimes a burgeoning mathematical physics. This means that various national styles were being created despite the overall solution power of analysis and the birth of linear and tensor algebras.

In general a direct intrinsic notation is used for vectors and tensors, but a Cartesian index notation is introduced when a risk of confusion arises with the intrinsic one.

1.1 Analysis and Partial Differential Equations: 18th Century

We will be dealing with the mechanics of *continua*. Accordingly, the primary notion is that of *analysis* since the notion of continuity can only be defined within the mathematical specialty called analysis. We admit that with the works of,

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among others, René Descartes (1596–1650), Isaac Newton (1643–1727) and Gottfried W. Leibniz (1646–1716), we have at hand the standard formulation of analysis—also called differential and integral calculus—but for functions of one variable only. The necessary consideration of both time and space variations (in dynamics) and of multi-dimensional problems (in two or three space dimensions) requires the introduction of the notion of *partial derivative*. This we essentially owe to the Bernoulli’s—John (1667–1748) and Daniel (1700–1782), John’s son—and Jean Le Rond d’Alembert (1717–1783). In particular, the last author has formulated the first equation of wave motion—a second-order partial differential equation of the so-called hyperbolic type (finite velocity of propagation)—with its paradigmatic solution. Thus the path was paved for the fundamental works of Leonard Euler (1707–1783), Joseph Louis Lagrange (1736–1813) and Augustin Louis Cauchy (1789–1857).

1.2 Transition to the 19th Century

In possession of the appropriate tools, Euler, Lagrange and Cauchy were able to formulate the standard theory of *perfect* fluids and *perfectly* elastic solids, two cases in which ideal descriptions cope with what we now call *nondissipative* behaviours. It is this “perfection” that brings these modellings in a framework equivalent to that given by preceding and contemporary scientists to point and rigid-body mechanics, what was rapidly called “rational mechanics”. Only reason is at work in an intellectual construct that is entirely logical once the premises are assumed as postulates. This is reflected in the absence of figures in the book (1788) on “*Méchanique analitique*” (old French orthography) of Lagrange. These two cases are also the extreme cases—pure fluidity and pure elasticity—in the landscape so beautifully described in his “continuity of states” by Walter Noll in 1955. As we shall see, many of the developments in the 19th century and much more in the second half of the 20th century, deal with the formulation of “imperfect” cases now included in a thermo-mechanical theory of *thermodynamically irreversible* behaviours (fluid viscosity, visco-elasticity of solids, plasticity of solids, etc).

What is perhaps more to the point at this stage of our story are the following two elements. The first of these is the formulation of *variational principles* by Euler and Lagrange, culminating in the already cited “Méchanique” of Lagrange of 1788. This was to provide the essential tool in general *field theory* in the expert hands of William Rowan Hamilton (1805–1865) and others, but also to set forth the necessary basis of the modern formulation of the mechanics of continua both in its mathematical properties and the required numerical methods (e.g., finite-elements, optimization). The above mentioned “imperfect” cases cannot, in principle, be deduced from a variational formulation in the manner of Lagrange and Hamilton.

The second element is none other than the introduction of the notion of *stress tensor* (of course not called this when the notion of tensor did not exist yet) by Cauchy in his first theory of continua (1822, published in 1828). This is the object

σ that relates linearly the externally pointing unit normal $\mathbf{n}$ to a facet cut in a material body to the applied (in any direction) external traction $\mathbf{T}^d$ at this point of the facet, according to the now common formula

$$\mathbf{T}^d = \mathbf{n}.\sigma. \tag{1.1}$$

The object σ is often (but not always) a *symmetric* second-order tensor. It is also generally thought that the relation (1.1) does not involve any constitutive hypothesis—i.e., is independent of the considered material. We shall see when we consider generalized continua (cf. Chap. 13) that this vision is not exactly correct. In truth (1.1) is strictly valid only for so-called "simple" materials in Noll's classification (see Chap. 5). However, the formula (1.1), that is sufficiently general for many practical cases, is a decisive advance compared to the case of perfect fluids considered by Euler. Euler's case corresponds to an applied traction aligned with the unit normal $\mathbf{n}$, reducing thus the notion of stress to a unique scalar quantity, the pressure p, with (1.1) reduced to

$$\mathbf{T}^d = -p\mathbf{n}, \tag{1.2}$$

where the minus sign is conventional.

We cannot simultaneously ignore that Cauchy was also instrumental in making much more precise the basic notions of analysis (convergence, limits, derivatives, integrals) all relevant to the mechanics of continua. We also owe to him a celebrated representation theorem for scalar-valued isotropic functions. This theorem provides a way for deducing the set of quantities—so-called *invariants*—on which such a function depends as a result of isotropy (equivalent response in any direction = invariance by the orthogonal group of transformations of material space). This important theorem for many mechanical behaviours of continua was recalled by Herrmann Weyl in his famous book on *classical groups* of 1946.

1.3 Finite Deformations: Piola, Kirchhoff, Boussinesq

Euler and Lagrange are usually considered as responsible for the kinematic descriptions of continua called, *Eulerian* and *Lagrangian*, respectively (although this may not be exactly true). In the first description, all dependent variables are expressed as function $f(\mathbf{x}, t)$ of the *actual* position $\mathbf{x}$—so-called placement in the modern jargon—of an infinitesimal element of matter at time t in Euclidean physical space and of the Newtonian time t itself. In the so-called Lagrangian vision the actual placement $\mathbf{x}$ is a function of time, but also of a previously occupied position, say $\mathbf{x}_0$, a so-called initial placement. That is,

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{x}_0, t). \tag{1.3}$$

The Italian scientist Gabrio Piola (1794–1850)—author of lengthy papers in the period 1825–1848 and honoured by a beautiful pedestal statue in his native

Milano—was a disciple of Lagrange. Accordingly, he prefers variational formulations. But more than that, he introduced the somewhat more abstract notion of "*material*" coordinates that we denote collectively by the symbol **X**. This "configuration", called the reference configuration K_R is chosen as a most convenient one for the problem under study. The resulting space-time parametrization of a general deformation mapping is therefore written as

$$\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t). \tag{1.4}$$

If this relation is sufficiently regular, i.e., with

$$\mathbf{F} = \nabla_R \tilde{\mathbf{x}} = \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}},\ J_F = \det \mathbf{F} > 0, \tag{1.5}$$

we can define the *inverse* motion

$$\mathbf{X} = \tilde{\mathbf{x}}^{-1}(\mathbf{x}, t). \tag{1.6}$$

Thus, on account of (1.4) and (1.3)

$$\mathbf{x} = \tilde{\mathbf{x}}\left(\tilde{\mathbf{x}}^{-1}(\mathbf{x}_0, t_0), t\right) = \hat{\mathbf{x}}(\mathbf{x}_0, t; t_0) = \bar{\mathbf{x}}(\mathbf{x}_0, t). \tag{1.7}$$

Although obviously not equipped with the notion of tensor transformations, Piola recognized that in the abstractly introduced configuration K_R described by the spatial parametrization **X** one could introduce a stress tensor (in fact not a standard second-order tensor), by the so-called *Piola transformation* (1836, 1848):

$$\mathbf{T} = J_F \mathbf{F}^{-1}.\sigma, \quad \sigma = J_F^{-1} \mathbf{F}.\mathbf{T}, \tag{1.8}$$

where $\mathbf{F}^{-1}$is the inverse of **F** such that

$$\mathbf{F}^{-1} = \frac{\partial \tilde{\mathbf{x}}^{-1}}{\partial \mathbf{x}},\ \mathbf{F}\mathbf{F}^{-1} = \mathbf{1}. \tag{1.9}$$

Conscious of the arbitrariness of the choice of his reference configuration K_R, Piola selects it as one of uniform density equal to one. Since we know that mass conservation is expressed by

$$\rho_R = \rho J_F, \tag{1.10}$$

Piola writes "his" transformation as

$$\rho \mathbf{T} = \mathbf{F}^{-1}.\sigma. \tag{1.11}$$

Although Piola could not write his transformation in this simple condensed intrinsic form, his writing of typical components reveals an understanding of a hidden algorithm that will later be interpreted within tensor algebra.

The concept of Piola stress was comforted by Gustav R. Kirchhoff (1824–1887), so that the object **T** in (1.7) is nowadays called the *first Piola-Kirchhoff stress*. A *second* Piola-Kirchhoff stress **S** can also be introduced by completing the *true* tensor transformation between stresses in the actual and reference configurations by the definition: (The symbol $-T$ means the transpose of the inverse).

$$\mathbf{S} = \mathbf{T}.\mathbf{F}^{-T} = J_F \mathbf{F}^{-1}.\sigma.\mathbf{F}^{-T}, \tag{1.12}$$

where the superscript T denotes the operation of transposition. Both $\mathbf{T}$ and $\mathbf{S}$ have a deep thermodynamic significance.

Joseph V. Boussinesq (1842–1929), in his study of finite deformations introduces the stress object $\mathbf{B}$ such as [compare (1.11)]

$$\mathbf{B} = \mathbf{F}^{-1}.\,\sigma. \tag{1.13}$$

That is why we consider Piola, Kirchhoff and Boussinesq the founding fathers of the theory of finite transformations in spite of the pioneering works of Lagrange and Cauchy.

1.4 The French "Ingénieurs-savants"

In a further chapter (Chap. 7) we shall emphasize the role of the *Ecole Polytechnique* in the formation of a special scientific trend and "spirit" in the early 19th century: the appearance of what the British historian of sciences Ivor Grattan-Guinness (1993) calls "ingénieurs-savants". This is a group of alumni from that engineering school who received from their masters (Monge, Bossut, Lacroix, Lagrange, Cauchy, Fourier) a remarkable mathematical education although they were usually destined to work on engineering projects, essentially in civil engineering. They applied their mathematical technical skill and their physical ingenuity in fostering various facets of the "rational" mechanics of continua. Among these individuals, for our present purpose, we single out C.M.L. Navier (1785–1836), Gabriel Lamé (1795–1870), and J.M.C. Duhamel (1797–1872). Albeit a disciple of Laplace in his Newtonian particle-action-at-a-distance view, the first of these was instrumental in developing both continuum fluid mechanics and elasticity. In the case of fluids, he constructed what is now called the Navier-Stokes equation that involves shear motion and the allied viscosity. With this one enters the domain of *nonideal* fluids as compared to Euler's ideal fluid. In elasticity, he was responsible for the introduction of the so-called Navier equations for *isotropic* elasticity (although not on the basis of Cauchy's stress argument) in small strains. The difference in the technical approaches led to a thorough discussion about the number of existing elasticity coefficients (one or two in the case of linear isotropic elasticity?). As we know now, the correct answer is two, and these coefficients λ and μ are called after the second of our "ingénieurs-savants", Lamé.

In Cartesian indicial tensor notation and intrinsic notation, Hooke's law for isotropic materials reads

$$\sigma_{ji} = \lambda e_{kk}\delta_{ji} + 2\mu e_{ij},\ \ \boldsymbol{\sigma} = \lambda(tr\mathbf{e})\mathbf{1} + 2\mu\mathbf{e}. \tag{a}$$

In 1D in the x-direction this takes the simple form (all quantities are scalars)

$$\sigma = Ee,\ \ e = \partial u/\partial x. \tag{b}$$

This reflects mathematically the celebrated Hooke's law according to which "elongation is proportional to the applied force", i.e., (Robert Hooke 1635–1703)

$$\delta l/l_0 = kF, \tag{c}$$

where l_0 is the initial length, k is a coefficient of proportionality, and the force F should remain reasonably small. The coefficient E is called the Young modulus after Thomas Young (1773–1829), a polymath in competition with Jean-François Champollion (1790–1832) for the deciphering of Egyptian hieroglyphs, a competition that he lost. Hooke's law (c) belongs to what we shall call "physical mechanics" as it is based on observation. Navier's elasticity equations are the field equations obtained by substituting from (a) in the balance of linear momentum.

The third of the "ingénieurs-savants" is Duhamel, probably less known than the other two. But he was more a "savant" than an engineer as he never graduated from the Ecole Polytechnique, having been expelled from the school in 1816 with all his fellow classmates for political reasons. His originality stems from the fact that he was the first to study a problem of *coupled fields* in continuum mechanics, namely, *thermo-elasticity* (1838). Of course he could do that only after Carnot and Fourier had developed the necessary ingredients for treating heat conduction alone. But Duhamel had the right intuition in attacking this coupled-field problem, even though the best applications of that new field would be only in the 20th century. Furthermore, his was probably the first example of considering a *non-isotropic* material response since he was conscious that some directions may be more important than others and thus privileged contrary to the often assumed isotropy (no preferred direction). This matter was studied in detail from the viewpoint of epistemology by Gaston Bachelard, a French philosopher of sciences, in 1927.

In 1D the Hooke-Duhamel constitutive equation of thermo-elasticity reads

$$\sigma = Ee + m(\theta - \theta_0), \tag{d}$$

where θ is the thermodynamical temperature, θ_0 is a reference temperature, and m is the thermo-elasticity coupling coefficient. Thermal expansion is obtained by putting $\sigma = 0$ in (d) and solving for e, yielding thus

$$e^\theta = \alpha(\theta - \theta_0), \tag{e}$$

where $\alpha = -m/E$ is the coefficient of thermal expansion.

1.5 The British Giants: Green, Kelvin, Stokes, Maxwell

We are concerned with a group of British scientists whom we collectively call the "Cambridgians" (some may think that "Cantabridgians" would be better). They share a similar vision of the physical world and also a practically identical formation. They have been educated at Cambridge and some of them taught there also. They have also in common to have been influenced by the French school of mathematics of the late 18th century—early 19th century, a school that had chosen to exploit Leibniz' notation rather than Newton's one in analysis, e.g., from Bossut's and Lacroix's sources used at *Polytechnique* (cf. Bossut 1800). This was a happy choice of far reaching consequence because it contributed to a new blossom of British mathematical physics that brought British authors to the top of the field in the 19th century. After these "Cambridgians", there came the "Maxwellians"—who may also have been "Cambridgians"—among whom we must count Heaviside and Larmor.

(Abbé) Charles Bossut (1730–1814), a disciple of d'Alembert and a specialist of hydrodynamics, but also an underestimated historian of mathematics, was a remarkable pedagogue. His course of mathematics at the Military school of Mézières (cf. Sect. 7.1) was first published in 1781. Its last edition (1800) was in seven volumes, of which two were devoted to differential and integral calculus in the Leibniz notation. He was a colleague of Laplace and Lagrange at the Paris Academy of Sciences, but not in the same class as these two mathematicians-mechanicians from the point of view of creativity. He practically ended his career as an examiner in mathematics at the Ecole Polytechnique (1796–1808)—it seems to have been the oldest examiner ever at that school.

Among the "Cambridgians" George Green (1793–1841) is a very special case in the sense that he practically concluded his scientific life with his studies as an undergraduate at Cambridge. Indeed, a miller by profession and practically an autodidact, he wrote some of his most beautiful memoirs after having studied by himself the French pedagogues. It is as a consequence of these early successes that he was sent to Cambridge University where, among other things, he unfortunately learned gambling and drinking. For our purpose we obviously note the celebrated *Green theorem*, also called the divergence theorem (Green 1828). He also established the *Green reciprocity theorem* and introduced the notion of *Green function*, all extremely useful notions in problems both in electromagnetism and in continuum mechanics. In a nutshell pertaining to continuum mechanics, combined with Cauchy's lemma (1.1), Green's divergence theorem reads as follows if we consider a surface distribution of given traction $\mathbf{T}^d$ on the regular boundary ∂B of a body B with ∂B equipped with unit outward normal $\mathbf{n}$:

$$\int_{\partial B} \mathbf{T}^d da = \int_{\partial B} \mathbf{n}.\sigma \, da = \int_B div\, \sigma \, dv, \tag{1.14}$$

hence the importance of this theorem—also attributed to Gauss—for the formulation of continuum mechanics in global form. In truth, the global balance of linear momentum written as

$$\frac{d}{dt}\int_B \rho\mathbf{v}\,dv = \int_B \rho\mathbf{f}\,dv + \int_{\partial B} \mathbf{T}^d da, \tag{1.15}$$

yields by localization on account of the assumed continuity of the present fields the standard local balance of linear momentum in the following form in the actual—Euler—frame of reference:

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} + div\sigma, \tag{1.16}$$

where ρ is the matter density at time t, $\mathbf{v}$ is the velocity, and $\mathbf{f}$ is a density of bulk force per unit mass. Then (1.1) is none other than the *natural* boundary condition associated with (1.16) at the regular boundary ∂B. The left-hand side of (1.16) holds good because the elementary mass $dm = \rho dv$ is assumed constant in time.

Another notion introduced by George Green is that of *potential function* for elasticity. The introduction of such a potential means that any path of loading in the elastic regime of deformations and strains closes to zero energy expenditure. That is, let W be such a potential. According to Green, the Cauchy stress is derived from it by the derivative function

$$\sigma = \frac{\partial W}{\partial \mathbf{e}}\ or\ \sigma_{ji} = \frac{\partial W}{\partial e_{ij}}, \tag{1.17}$$

where the symmetric tensor $\mathbf{e}$ of components e_{ij} is defined by

$$\mathbf{e} = \left\{ e_{ij} = u_{(i,j)} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right\}. \tag{1.18}$$

Here the vector $\mathbf{u}$ of Cartesian components u_i is the elastic displacement. In general, from (1.17) we have.

$$W|_{\mathbf{e}_1}^{\mathbf{e}_2} = \int_{\mathbf{e}_1}^{\mathbf{e}_2} \sigma : d\mathbf{e}, \tag{1.19}$$

which is none other than the elastic energy expended between the two limit strain states. For a closed circuit this yields that no energy was spent, a statement that is tautological with the definition of a potential. It is easy to imagine that this may have had a strong influence on the thoughts of Kelvin pondering the notion of energy conservation. This potential behaviour is translated into modern anthropomorphic language by saying that the elastic material "remembers" only one state, the initial one, usually a state of zero energy itself (virgin initial state) and providing a minimum of energy, hence the required *convexity* of the function. This convexity is trivially guaranteed in linear elasticity where W is quadratic in the strain. To obtain an explicit form of the elasticity constitutive equation, it is sufficient to know the expression of W. In the general isotropic case, one then applies the above mentioned Cauchy theorem for isotropic scalar-valued functions, and the true linear case results by considering only the contributions linear in the strain in the constitutive law, hence the Lamé-Navier expression with two coefficients λ and μ (cf. Eq. (a) above).

Those interested in the person of Green may visit his rebuilt windmill in Nottingham; this reconstruction and reviving of the mill was mostly due to the combined efforts of the local university faculty members such as Lawrence J. Challis and Antony J. M. Spencer, the latter himself a modern "Cambridgian" and renowned mechanician—see Chaps. 3 and 6.

In modern thermomechanics (e.g., in Maugin 2011), if W denotes the strain energy function (potential) per unit reference volume, we have the elasticity constitutive equations in the form

$$\mathbf{T} = \frac{\partial \bar{W}(\mathbf{F})}{\partial \mathbf{F}}, \; \mathbf{S} = \frac{\partial \hat{W}(\mathbf{E})}{\partial \mathbf{E}}, \; \mathbf{E} \colon = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right), \tag{1.20}$$

so that $\mathbf{T}$ and $\mathbf{S}$ are also called the *nominal* stress and the *energetic* stress, respectively. If the second coinage is obvious, the first one comes from the fact that $\mathbf{T}$ represents the stress component per unit surface of the reference configuration. This comes from the property that oriented surface elements in the actual and reference configurations are related by

$$\mathbf{n}da = J_F \mathbf{N}.\mathbf{F}^{-1}dA, \; \mathbf{N}dA = J_F^{-1}\mathbf{n}.\mathbf{F}da, \tag{1.21}$$

so that

$$\mathbf{n}.\sigma da = \mathbf{N}.\mathbf{T}dA. \tag{1.22}$$

Equations (1.21)—relating actual and Lagrangian configurations—were established in hydrodynamics in 1874 by another "Cambridgian", Edward J. Nanson (1850–1936) who made an academic career in Australia (see Nanson 1874).

William Thomson—later called Lord Kelvin—(1824–1907) is the second of our "Cambridgians". He was a great admirer of Green's original memoir of 1828, and he had it re-published in 1846, after which Green's memoir became popular. During a scientific visit in Paris Thomson discovered the original work of Sadi Carnot (1796–1832) on the "motive power of heat" and also the work of B.P.E. Clapeyron (1799–1864), another "ingénieur-savant" (so was the case of Sadi Carnot—we shall return to all these French scientists in Chap. 7). It is by combining these influences and that of James Prescott Joule (1818–1889) that Thomson was led to a formulation of a principle now called the *conservation of energy* or "first law of thermodynamics". He was in fact but one of three co-discoverers of this "law", the other two being Julius R. Mayer (1814–1878) and Hermann von Helmholtz (1821–1894), both from Germany. But Thomson *aka* Kelvin, just as von Helmholtz, was an immense scientist with multiple scientific interests such as in electromagnetism, electrotechnics, and continuum mechanics. In the last field he was interested in both fluids and elastic solids. For further consideration (cf. Chap. 13), we note that like other scientists of this pre-Maxwellian period (even Cauchy!) he was trying to construct a model of continuum that could afford the propagation of light in the form of pure transverse waves, because this is what was observed according to Augustin Fresnel (1788–1827). This led to the notion of continuum capable of responding to a density of couple, a medium with internal rotation now called "*Kelvin medium*", a precursor of the generalized continua of which theories would be expended in the 20th century starting with the work of the Cosserat brothers.

The third "Cambridgian" of interest in the present context is George Gabriel Stokes (1819–1903) whose name is for ever associated with that of Navier for the Navier-Stokes equation that governs the flow of linear viscous fluids.

Newton's viscous constitutive law

$$\sigma = \eta \frac{\partial v}{\partial x} \tag{f}$$

in 1D, - where η is a viscosity coefficient—was experimentally checked by J.L.M. Poiseuille (1797–1869, an alumnus from *Ecole Polytechnique* who became a medical doctor and famed physiologist) in his study of blood flow, a first in " bio-mechanics".

For a *solid* in small deformation we can write

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial e}{\partial t} \equiv \dot{e}, \tag{g}$$

so that (f) yields

$$\sigma = \eta \dot{e}. \tag{h}$$

Stokes' name is also attached to a well known theorem of vector integral calculus in several dimensions (passing from the circulation along a closed line C to the flux of the *curl* across the surface S leaning on that line, i.e.,

$$\oint_C \mathbf{A}.dl = \int_S \mathbf{n}.(curl\,\mathbf{A})da.$$

This theorem is of the same nature as the divergence theorem evoked in (1.14)—it expresses the passing of an integral over a manifold of dimension n-1 to one of dimension n in the calculus of "exterior forms" (a generalized Stokes theorem); it is of obvious importance in electricity (for currents) and hydrodynamics (for vortices).

Our fourth giant "Cambridgian" is none other than James Clerk Maxwell (1831–1873) of electromagnetic fame. But few "electricians" (as Oliver Heaviside would have called them) know that Maxwell was also the author of a fundamental work on the mechanics of trusses (exploited in so-called graphic statics in pre-computer times) and that in his studies of viscous media he introduced a realistic model of rheological behaviour now called *Maxwell model of visco-elasticity*. In terms of rheological models using the vivid image of springs and dashpots this corresponds to a (Newtonian) viscous element—cf. (f) above—and a (Hookean) spring element put in series. This, like the so-called *Maxwell*-Cattaneo conduction law—that contains a relaxation time—was much influenced by Maxwell's work in the kinetic theory of gases.

In 1D Maxwell's visco-elastic constitutive equation can be expressed by

$$\frac{1}{\tau_M}\sigma + \frac{\partial \sigma}{\partial t} = E\frac{\partial e}{\partial t}. \quad \text{(i)}$$

This can be rewritten as the relaxation equation

$$\frac{\partial}{\partial t}(\sigma - \sigma_H) + \frac{1}{\tau_M}\sigma = 0, \quad \sigma_H = Ee. \quad \text{(j)}$$

In turn this can be compared to the visco-elasticity law proposed by Thomson (Kelvin) and Voigt as

$$\sigma = Ee + E\tau_{KV}\dot{e} = E(e + \tau_{KV}\dot{e}), \quad \text{(k)}$$

which clearly is a linear combination of Hooke's law (b) and Newton's law (h). This can also be written as the relaxation equation

$$\frac{\partial e}{\partial t} + \frac{1}{\tau_{KV}}(e - e_H) = 0, \; e_H = E^{-1}\sigma. \quad \text{(l)}$$

In terms of rheological models using the image of springs and dashpots this corresponds to a (Newtonian) viscous element and a spring element put in parallel.

For the sake of completeness, comparison, and further reference, we mention the Maxwell-Cattaneo law of heat conduction in the "relaxation" form [compare to (l)]

$$\frac{\partial q}{\partial t} + \frac{1}{\tau_q}(q - q_F) = 0, \; q_F = -\kappa\frac{\partial \theta}{\partial x}, \quad \text{(m)}$$

where q_F is the classical Fourier heat-conduction law with conduction coefficient κ in isotropic bodies.

1.6 The German School and its Giants: Kirchhoff, Clebsch, Voigt, Mohr, et al.

Many of the great German contributors to our subject matter could also be qualified of 'ingénieurs-savants" for they often were educated in Polytechnic schools ("Polytechnicum")—or *Technischen Hochschulen* in a more recent jargon—all more or less founded as imitations of the French *Ecole Polytechnique*. One of the first characters in that play is Karl Cuhlman (1821–1861). He received his engineering education at the Polytechnicum in Karlsruhe. He was himself active in the study of railways structures and bridges. He did mostly works related to the strength of materials and graphic statics. More important than him for our purpose is Franz E. Neumann (1798–1895) who graduated from the University of Berlin (Doctoral degree in mineralogy and crystallography). His work in elasticity was conducted in parallel with those of Navier, Cauchy and Poisson, establishing the number of elasticity constants for anisotropic materials. For isotropic elasticity he established without doubt that two coefficients—the Lamé coefficients—were necessary (and not only one as had been assumed by some French elasticians). He may also be considered one of the founding fathers of photo-elasticity after his study of double refraction in stressed transparent bodies. This he applied to thermal stresses (Cf. Duhamel). He was a reputed lecturer and author of highly appreciated books who mentored some of our relevant characters: Gustav R. Kirchhoff (1824–1887), Alfred Clebsch (1833–1872), and Woldemar Voigt (1850–1919). His influence on these scientists is mostly felt in the domain of elasticity.

The first of Neumann's disciples, Kirchhoff, was to become one of the German giants in continuum mechanics for the 19th century, although his reputation in electricity, spectroscopy, black-body radiation, and thermo-chemistry is at the same if not higher prestigious level. It is in Königsberg that Kirchhoff took lectures with Neumann. He later became a professor of physics in Breslau, Heidelberg and finally Berlin. We already cited Kirchhoff in relation to the Piola-Kirchhoff stress defined in (1.8). From this we can construct the *Piola-Kirchhoff format* of continuum mechanics. For instance, if we note the demonstrable identities

$$\nabla_R.\left(J_F\mathbf{F}^{-1}\right) = \mathbf{0}, \quad \nabla.\left(J_F^{-1}\mathbf{F}\right) = 0, \tag{1.23}$$

by applying $J_F\mathbf{F}^{-1}$ to the left of (1.16) and accounting for the continuity equation $\rho_0 = \rho J_F$ between actual and reference configurations, we obtain the balance of linear momentum in the form

$$\frac{\partial}{\partial t}\mathbf{p}_R - div_R\mathbf{T} = \rho_0\mathbf{f} \quad or\ \frac{\partial}{\partial t}(\rho_0 v_i) - \frac{\partial}{\partial X^K}T_i^K = \rho_0 f_i. \tag{1.24}$$

That is, while now this equation makes use of independent time and space partial derivatives (since t and $X^K - K = 1, 2, 3$—form a set of time and space independent variables in this parametrization), the equation still has components in the actual configuration. Writing the associated natural boundary condition requires using the Nanson formulas (1.21) and (1.22). Note that (1.24) holds true because in the absence of growth or resorption of matter (see Chap. 14 for this case), the continuity equation in the Piola-Kirchhoff format can simply be written as

$$\left.\frac{\partial\rho_0}{\partial t}\right|_{\mathbf{X}} = 0. \tag{1.25}$$

Kirchhoff made another important contribution to continuum mechanics and the mechanics of structures by constructing a model theory for the bending of plates. The two-dimensional equation deduced from a variational principle (principle of virtual work) that governs the deflection w at the mid-surface of the plate reads

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q, \tag{1.26}$$

where D is the flexural rigidity of the plate. To arrive at this equation, Kirchhoff had to formulate a reduced potential energy that accounts for a set of basic kinematic hypotheses concerning the section of the plate normal to the middle surface and the neglect of any stretching of the elements of the middle plane for small deflections. This much improved the tentative theory proposed earlier by Sophie Germain (1776–1831) even after correction of S. Germain's mistakes by Lagrange. In a modern vision, establishing the crucial Eq. (1.26) is one example of the reduction of a three-dimensional elasticity problem to one in two dimensions by an asymptotic procedure (cf. works by Ambartsumian, Gold'enveizer, Ciarlet and others in the period 1950–1980). Kirchhoff's theory is now referred to as the *Love-Kirchhoff theory of plates* after A.E.H. Love (1863–1940), another "Cambridgian" who nonetheless had his whole scientific career at Oxford. Love extended Kirchhoff's approach to the case of thin shells. But Kirchhoff also studied theoretically and experimentally the vibrations of plates on the basis of his model. He also subsequently extended his theory of plates to include the case of not too small deflections. All of Kirchhoff work on plates has provided the most important basis for the computation of thin-walled structures. Thomson (Kelvin), already cited, improved on Kirchhoff's theory of plates by specifying the boundary conditions concerning shearing forces and bending moments at an edge.

Another student of Neumann in Königsberg was Clebsch. He wrote a thesis in fluid mechanics. He became a professor at the Polytechnicum in Karlsruhe when he was only twenty five after spending a short time at the University in Berlin. It is while at Karlsruhe that he wrote a famous book on elasticity—*Theorie der Elastizität fester Körper*—when there existed only one such book, by Lamé, available. He wrote it for engineers, but with special emphasis on mathematical methods of solutions, often loosing the physical aspect. This gave the opportunity to Barré de Saint-Venant in his French translation of this book to expand the matter in such a way that the bulk of the book tripled in translation, resulting in a book that was more his than Clebsch's. The mathematical inclination of Clebsch and his remarkable gift for it resulted in Clebsch becoming a professor of pure mathematics and ending his brief career as one of the best German mathematicians of his period (works on variational problems, Abelian functions, invariant theory, algebraic geometry) in Göttingen after teaching in Giessen. Among his famous students we find Max Noether (the father of Fritz and Emmy Noether—see Chap. 14) and Felix Klein (1849–1925) who was to play an instrumental role in German mathematics. He was a co-founder of one of the best journals in mathematics, *Mathematische Annalen*. He died untimely of diphteria. His contribution to the mechanics of continua, achieved during his youngest research period, remains a fundamental one and was considerably enriched by Barré de Saint-Venant.

Woldemar Voigt was also one of the successful students and disciples of Neumann. But he more closely than others followed his master in devoting much work to the elasticity of crystals that culminated in his book "*Lehrbuch der Kristallphysik*" (First German edition, Teubner, Leipzig, 1910). It is during this work that he was led to introducing the recently created notions of *tensor* and tensor-triad in the theory of continua, so much that tensor algebra and analysis practically became synonymous with that field in the eyes of many physicists. Of course, the word "tensor" smells of its mechanical origin. It is less known that

Voigt anticipated the Lorentz-Poincaré transformation formulas in special relativity and that he was the first to propose a correct Lagrangian density in electrodynamics.

Another line or remarkable chain of German contributors to continuum mechanics in the large starts with Otto Mohr (1835–1918). This line extends to the middle of the 20th century with August Föppl (1854–1924), Ludwig Prandtl (1875–1953), and Theodor von Kármán (1881–1963). Mohr was a railway-structural engineer who graduated from the Polytechnicum in Hannover. He taught engineering mechanics first at the Polytechnicum in Stuttgart and then in Dresden. Following along the path of Karl Cuhlman (see above), he was very much interested in graphical methods (*Graphische Statik*). He is universally known for his two-dimensional representation of the stress state by means of so-called *Mohr circles*.

After starting his engineering studies at the Polytechnicum, August Föppl transferred to Stuttgart where he took courses with Mohr, but he finally graduated from the Polytechnicum in Karlsruhe. He was basically a structural engineer with a strong side interest in electricity. Regarding the later field, he popularized Maxwell's theory of electromagnetism in a book published in 1894—the first of its kind in Germany. This book is supposed to have left a definite print on Einstein as a young man. A talented teacher in Munich, he also published the most popular book on engineering mechanics in German-speaking countries. He counts among his students Ludwig Prandtl who worked with him on solid mechanics. Prandtl taught first at the Polytechnicum in Hannover and then at the university of Göttingen. He is considered to be the father of modern aerodynamics. His works in this field are marked by mathematical subtleties such as in his theory of the boundary layer. Together with Richard von Mises (1883–1953) he founded the (German) *Society of Applied Mathematics and Mechanics* (*G.A.M.M*). One of his co-workers in Göttingen was von Kármán who came from Hungary and would later become the founder of the *Jet Propulsion Laboratory* at Caltech and a prominent figure in aeronautical government agencies in the USA. A theory for large deflections of plates is named after Föppl and Kármán. Kármán was also responsible for the basic dynamic theory of elastic crystal lattices together with Max Born (of quantum-mechanical fame).

This overview of German contributions would not be complete without the repeated mention of Hermann von Helmholtz. A medical doctor and physiologist by formation, Helmholtz is one of the most brilliant and versatile mind of the 19th century. His formidable scientific production covers sensing physiology, ophthalmic optics, nerve physiology, acoustics, electromagnetism and mechanics. Of course, in the present context he is most well known for his co-discovery of the *first law of thermodynamics*, a law of conservation that includes all forms of energy, whether of mechanical, electrical, etc, origin. From the point of view of elasticity and acoustics he introduced the Helmholtz decomposition of a vector field—that is essential in many problems of elasticity and elastodynamics—as also the well known Helmholtz equation. He mentored many famous physicists, among them Max Planck, Wilhelm Wien, Henry Rowland, A.A. Michelson and Michael Pupin. He has successively taught in Königsberg, Bonn, Heidelberg, and Berlin.

Note that Heinrich Rudolph Hertz (1857–1894) was a student of Helmholtz and Kirchhoff in Berlin. He in fact became Helmholtz' assistant in 1880 for a period of three years. Hertz is known among mechanical circles for his works on the compression of elastic bodies and his theory of contact that he worked out while in Berlin. After Berlin, Hertz was a professor first in Kiel and then at the Polytechnicum in Karlsruhe, where he conducted his famous experimental work on electromagnetic waves, thus proving the correctness of Maxwell's equations. He also wrote a highly praised book on the principles of mechanics.

1.7 Concluding Remark

In this chapter which rapidly spanned the 18th and 19th centuries, we have identified the main landmarks in a concise historical view of the early developments of the science of continuum mechanics. We have explored its strengthening and consolidation in a true field of applied mathematics in the form of a mix of "rational mechanics" and engineering. Leaning on the firm bases of Newtonian axioms and necessarily starting with duly abstracted models exploiting essentially Cauchy's construct, we have witnessed a growing "mathematization" of the field. Starting with idealizations and abstractions that avoid the true complexity of the mechanical behaviour of existing materials, this development had mostly been the result of the hard work of many civil engineers although these individuals were equipped with a sound mathematical formation and a great ingenuity. Most of the breakthrough results were obtained in three countries, France, the United Kingdom, and Germany, in reason of the advance of these countries in civil engineering, their growing industrial needs, and the existence of appropriate schools often providing the needed "ingénieurs-savants". A marked tendency in the observed 19th century developments was, apart from necessary experiments, the will to solve problems with sophisticated mathematical tools, which tools were practically created purposefully for these solutions. The near future would be to better describe the real mechanical behaviour of materials at a macroscopic scale, incorporate more deeply the thermodynamic background, and also to take some time to ponder the general philosophy—its structure and principles - behind this science. This is the main nature of the progress achieved in the next period that we circumscribe to the time interval 1880–1914. This we consider to be a transition to the true, unfortunately agitated but simultaneously rich, 20th century, the object of this book.

1.8 Further Reading

Selected historical landmark contributions are to be found in Cauchy (1828), Duhamel (1837), Green (1828, 1839), Kirchhoff (1876), Piola (1836, 1848), and Weyl (1946). Epistemological study of Duhem's works is to be found in Bachelard

(1927), and of Cauchy's deep contributions in Belhoste (1991) and Dahan-Delmonico (1984–1985). An interesting overview of Green's life is given in Cannell (1993). Pertinent historical reviews in continuum and solid mechanics are given by Barré de Saint-Venant (1864), Todhunter (1886), Timoshenko (1953), Truesdell (1968, 1976, 1984) and Soutas-Little (2011). More broadly, Dugas (1950) and Szabò (1977) look at general mechanics, and Whittaker (1951) and Schreier(1991) to physics. Warwick (2003) focuses on mathematical physics at Cambridge. Gillispie (1974) remains a real mine concerning scientific biographies written by specialists.

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