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Gérard A. Maugin

Continuum Mechanics Through the Twentieth Century

A Concise Historical Perspective

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Preface

What is This Book About?

"You reasoned it out beautifully…

It is so long a chain, and yet every link rings true"

(Dr. Watson to Sherlock Holmes)

The proposed conspectus of the development of continuum mechanics throughout the twentieth century seems to be unique in its scope and ambition. Although reminding the reader of the early developments of the discipline with the magisterial works of our elders (from Newton to the late nineteenth century or more precisely the advent of the First World War), the book concentrates on the twentieth century, and more particularly on its second half, as witnessed by the author who has lived directly through these developments, and has humbly tried to contribute to them. But the main reason for this delineation is that the post World-War Two period saw an incredible burgeoning and progress in the enlargement of the field, its mathematization and its rational organization, i.e., both in its objectives and methods, to the benefit of fruitful applications to the mechanics of large classes of materials and in reciprocal interaction with other fields. This has been a fruitful period that saw a consolidation of analytical works, and a development of new aspects, both in modeling and mathematical approach.

This is placed in a context where the marked interest of the author for the history of Science, for Epistemology, and for People is obvious, having been much influenced by Pierre Duhem and Clifford Truesdell. Accordingly, this is also a study about individuals and scientific schools and institutions in an evolving social and historical context that experienced tragic events. We hope the book succeeds to present with objectivity a balanced appraisal of contributions from various parts of the world. The chosen approach emphasizes the importance of the role played by (i) organized professional groups, e.g., the A.S.M.E with its specific spirit in the

USA, (ii) traditional strongholds such as the University of Cambridge in the UK or the Technical University of Hannover in Germany, (iii) remarkable individuals such as Clifford Truesdell in the USA, or Paul Germain in France, or still Leonid Sedov in the Soviet Union, and others in Poland, Germany, and Japan, and (iv) a well-structured network of teaching institutions and research units whether in countries that inherited from the Austro-Hungarian Empire, or in the original system of "grandes écoles" in France and their copies all over the world, or the Academies of Sciences in formerly communist-led countries such as the Soviet Union. That is, the development of modern continuum mechanics; in spite of its technical subtleties (effects of nonlinearity, thermodynamic irreversibility, microstructure, and singularities) that are carefully scrutinized, is shown to take place within a true human background with its grandeurs and pettiness, and not as a purely abstract teleological evolution. This permeates an exposition which is, therefore, vivid and bears witness of an epoch making process, to which the author contributes both his technical expertize and his international experience.

In order to fulfil this ambitious project and to satisfy the various needs of potential readers, a three-way strategy has been implemented. After two preliminary chapters that take the reader to post World War I and underline the newly raised technical questions and the ongoing general reflections on the bases of continuum mechanics, three chapters are devoted to: (i) new progress in nonlinear aspects (in both elastic solids and the newly formulated rheology of non-Newtonian fluids), (ii) a specific spirit distilled to continuum mechanics by the influential organized group represented by the American Society of Mechanical Engineers (in particular with works in plasticity but also in coupled fields), and (iii) the aerial view of continuum mechanics introduced by the Truesdell School with its efforts at a true rationalization and axiomatization, as well as its construct of an efficient thermomechanics, and its positioning in a real historical perspective.

The second strategic line is implemented in the next six chapters where a more per-country or regional view has been chosen for reasons that should be clear enough. This is due to the existence—still true at the time of most of the second half of the twentieth century—of national styles, peculiar teaching and research institutions inherited from the past, and the role played by some remarkable individuals. This is the case in the UK, France, Poland, and Germany. The rest of western and southern European contributions are described in one lengthy chapter together with some indications on some Asian countries. It is to repair an unjust too frequent belittling of the role played by the Soviet Union and Russia that a long chapter is devoted to them. This allows for a more balanced view than usually given.

The third line consists in the deeper and more technical examination of four special avenues of developments which the author estimates to be most emblematic-and original-of the last 50 years and to which he can devote a more thoughtful approach having been much involved in these. They are: (i) the interaction between continuum mechanics and electromagnetism, (ii) the mechanics of generalized continua, (iii) the so-called configurational mechanics of continua, and (iv) relativistic continuum mechanics. These four avenues bring us closer to other fields of physics in conjunction with typical twentieth century developments (exploitation of coupled fields, physical acoustics, solid-state physics, the "mechanics of materials", and relativistic physics). An epilogue providing a general summary and pointing to recent and future developments (going to smaller scales, influence of powerful computational means, and a true internationalization of science) is given by way of conclusion. An appendix presenting about one hundred short biographies of the most fruitful contributing mechanicians in the considered period is given in encyclopedic form to the benefit of all readers who can satisfy a legitimate curiosity by finding there useful, albeit brief, information.

The bibliography given in different chapters is generous-in all, more than 900 entries. As it is conceived and written with an obvious generosity of information and a rather open mindedness to many styles and objects of works, the book, despite its inherent concision, should satisfy the curiosity and attract the interest of all those involved in the study and development of continuum mechanics as a general contemporary science.

In writing this book, I admit to have benefited from a powerful memory of names, dates, and research papers, on which is grafted a certain interest for foreign languages, at least in reading form. However, in the line of the teaching of one of my mentors, Paul Germain in Paris, I always pondered all contributions, trying to extract the best of each without prejudice, and being aware that scientific activity remains the product of human beings, with their qualities and deficiencies. But some enounced appraisals may be thought too severe, and others too lenient. In all cases, the author takes the full responsibility of his judgment. For some largely unknown reasons, but perhaps because of the exceptional quality and ego of many involved scientists in our field in the examined period of time, there have been vivid discussions on some privileged advances, not the least in the always debated thermomechanics of continua. This is sometimes reflected in the book where I do not hesitate to give my own viewpoint that might not be shared by all. In forming my views, I also had the chance to be scientifically formed in two countries (France and the USA) and to have entertained a professional position that allowed me to benefit from cooperative scientific stays in many countries. I was also lucky, and honored, to deliver series of graduate lectures to students and professors from all over the world at the *International Centre of Mechanical Sciences* (so-called *CISM*) in Udine, Italy, and this for a record number of eight times between 1977 and 2011. All these opportunities were dutifully exploited, first to establish enriching contacts and build enduring friendships, and next to get acquainted with other systems of higher education and research and their past history.

These words should be enough to explain to the reader the frame of mind in which this book was written, as a mixture of plain scientific facts and personal recollections. The level of required knowledge is that of graduate studies and of professional researchers in continuum mechanics. I have avoided too many equations, keeping only a few representative ones. As to the rather rich bibliography, either it serves to substantiate a specific information or it provides an idea of works published by some of the most fruitful scientists, often in the form of books.

Finally, very close to the spirit of autobiographical notes, from time to time I give within squared brackets [..] what I call a “personal touch”, what mostly consists in injecting some personal recollection to relax the reader from an obviously extremely serious reading of a dense text. I wish the reader an utmost pleasant and rewarding reading.

Heart full thanks are due to my worldwide friends who contributed valuable information, and to my colleague, Dr. Martine Rousseau (Paris), for her critical reading of most chapters.

Special thanks go to those who have been essential in the editing and production of this book, namely: Professor G.M.L. Gladwell, Editor-in-Chief, who unhesitatingly welcomed the book in his formidable series, Nathalie Jacobs and Cynthia Feenstra for their friendly editing at Springer in Dordrecht, and the production team at SPS in Chennai for their understanding and professional competence.

Paris, February 2013 The Author

Contents

Chapter 1
The Land Clearers and the "Classics"

Abstract This chapter has for object to remind the reader of the early developments of continuum mechanics-after the seminal works in mechanics by Descartes, Huygens, Newton and Leibniz-in the expert hands of the initiators of this science (the Bernoulli family, d'Alembert, Euler, Lagrange). This was rapidly followed by the foundational contributions of the first half of the Nineteenth century with Cauchy and Navier (in France), Piola (in Italy), Kirchhoff (in Germany), and those of various giants of science such as Green, Kelvin, Stokes, Maxwell, Boussinesq, Poiseuille, Clebsch, von Helmholtz, Voigt, Mohr, and Barré de Saint-Venant later in the century. The emphasis is placed on the role played by so-called "ingénieurs-savants", many of them educated at the French Ecole Polytechnique and the engineering schools inspired by this school all over Europe. Lamé, Navier and Duhamel in France and their Italian colleagues are examples of such people who harmoniously combined works in a much wanted contribution to civil engineering and a sure mathematical expertise in analysis. In contrast, the German and English contributors were more inclined towards an emerging true mechanical engineering and sometimes a burgeoning mathematical physics. This means that various national styles were being created despite the overall solution power of analysis and the birth of linear and tensor algebras.

In general a direct intrinsic notation is used for vectors and tensors, but a Cartesian index notation is introduced when a risk of confusion arises with the intrinsic one.

1.1 Analysis and Partial Differential Equations: 18th Century

We will be dealing with the mechanics of *continua*. Accordingly, the primary notion is that of *analysis* since the notion of continuity can only be defined within the mathematical specialty called analysis. We admit that with the works of,

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among others, René Descartes (1596–1650), Isaac Newton (1643–1727) and Gottfried W. Leibniz (1646–1716), we have at hand the standard formulation of analysis—also called differential and integral calculus—but for functions of one variable only. The necessary consideration of both time and space variations (in dynamics) and of multi-dimensional problems (in two or three space dimensions) requires the introduction of the notion of *partial derivative*. This we essentially owe to the Bernoulli's—John (1667–1748) and Daniel (1700–1782), John's son—and Jean Le Rond d'Alembert (1717–1783). In particular, the last author has formulated the first equation of wave motion—a second-order partial differential equation of the so-called hyperbolic type (finite velocity of propagation)—with its paradigmatic solution. Thus the path was paved for the fundamental works of Leonard Euler (1707–1783), Joseph Louis Lagrange (1736–1813) and Augustin Louis Cauchy (1789–1857).

1.2 Transition to the 19th Century

In possession of the appropriate tools, Euler, Lagrange and Cauchy were able to formulate the standard theory of *perfect* fluids and *perfectly* elastic solids, two cases in which ideal descriptions cope with what we now call *nondissipative* behaviours. It is this "perfection" that brings these modellings in a framework equivalent to that given by preceding and contemporary scientists to point and rigid-body mechanics, what was rapidly called "rational mechanics". Only reason is at work in an intellectual construct that is entirely logical once the premises are assumed as postulates. This is reflected in the absence of figures in the book (1788) on "*Méchanique analitique*" (old French orthography) of Lagrange. These two cases are also the extreme cases—pure fluidity and pure elasticity—in the landscape so beautifully described in his "continuity of states" by Walter Noll in 1955. As we shall see, many of the developments in the 19th century and much more in the second half of the 20th century, deal with the formulation of "imperfect" cases now included in a thermo-mechanical theory of *thermodynamically irreversible* behaviours (fluid viscosity, visco-elasticity of solids, plasticity of solids, etc).

What is perhaps more to the point at this stage of our story are the following two elements. The first of these is the formulation of *variational principles* by Euler and Lagrange, culminating in the already cited "Méchanique" of Lagrange of 1788. This was to provide the essential tool in general *field theory* in the expert hands of William Rowan Hamilton (1805–1865) and others, but also to set forth the necessary basis of the modern formulation of the mechanics of continua both in its mathematical properties and the required numerical methods (e.g., finite-elements, optimization). The above mentioned "imperfect" cases cannot, in principle, be deduced from a variational formulation in the manner of Lagrange and Hamilton.

The second element is none other than the introduction of the notion of *stress tensor* (of course not called this when the notion of tensor did not exist yet) by Cauchy in his first theory of continua (1822, published in 1828). This is the object

σ that relates linearly the externally pointing unit normal $\mathbf{n}$ to a facet cut in a material body to the applied (in any direction) external traction $\mathbf{T}^d$ at this point of the facet, according to the now common formula

$$\mathbf{T}^d = \mathbf{n}.\sigma. \tag{1.1}$$

The object σ is often (but not always) a *symmetric* second-order tensor. It is also generally thought that the relation (1.1) does not involve any constitutive hypothesis—i.e., is independent of the considered material. We shall see when we consider generalized continua (cf. Chap. 13) that this vision is not exactly correct. In truth (1.1) is strictly valid only for so-called "simple" materials in Noll's classification (see Chap. 5). However, the formula (1.1), that is sufficiently general for many practical cases, is a decisive advance compared to the case of perfect fluids considered by Euler. Euler's case corresponds to an applied traction aligned with the unit normal $\mathbf{n}$, reducing thus the notion of stress to a unique scalar quantity, the pressure p, with (1.1) reduced to

$$\mathbf{T}^d = -p\mathbf{n}, \tag{1.2}$$

where the minus sign is conventional.

We cannot simultaneously ignore that Cauchy was also instrumental in making much more precise the basic notions of analysis (convergence, limits, derivatives, integrals) all relevant to the mechanics of continua. We also owe to him a celebrated representation theorem for scalar-valued isotropic functions. This theorem provides a way for deducing the set of quantities—so-called *invariants*—on which such a function depends as a result of isotropy (equivalent response in any direction = invariance by the orthogonal group of transformations of material space). This important theorem for many mechanical behaviours of continua was recalled by Herrmann Weyl in his famous book on *classical groups* of 1946.

1.3 Finite Deformations: Piola, Kirchhoff, Boussinesq

Euler and Lagrange are usually considered as responsible for the kinematic descriptions of continua called, *Eulerian* and *Lagrangian*, respectively (although this may not be exactly true). In the first description, all dependent variables are expressed as function $f(\mathbf{x}, t)$ of the *actual* position $\mathbf{x}$—so-called placement in the modern jargon—of an infinitesimal element of matter at time t in Euclidean physical space and of the Newtonian time t itself. In the so-called Lagrangian vision the actual placement $\mathbf{x}$ is a function of time, but also of a previously occupied position, say $\mathbf{x}_0$, a so-called initial placement. That is,

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{x}_0, t). \tag{1.3}$$

The Italian scientist Gabrio Piola (1794–1850)—author of lengthy papers in the period 1825–1848 and honoured by a beautiful pedestal statue in his native

Milano—was a disciple of Lagrange. Accordingly, he prefers variational formulations. But more than that, he introduced the somewhat more abstract notion of "*material*" coordinates that we denote collectively by the symbol **X**. This "configuration", called the reference configuration K_R is chosen as a most convenient one for the problem under study. The resulting space-time parametrization of a general deformation mapping is therefore written as

$$\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t). \tag{1.4}$$

If this relation is sufficiently regular, i.e., with

$$\mathbf{F} = \nabla_R \tilde{\mathbf{x}} = \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}},\ J_F = \det \mathbf{F} > 0, \tag{1.5}$$

we can define the *inverse* motion

$$\mathbf{X} = \tilde{\mathbf{x}}^{-1}(\mathbf{x}, t). \tag{1.6}$$

Thus, on account of (1.4) and (1.3)

$$\mathbf{x} = \tilde{\mathbf{x}}\left(\tilde{\mathbf{x}}^{-1}(\mathbf{x}_0, t_0), t\right) = \hat{\mathbf{x}}(\mathbf{x}_0, t; t_0) = \bar{\mathbf{x}}(\mathbf{x}_0, t). \tag{1.7}$$

Although obviously not equipped with the notion of tensor transformations, Piola recognized that in the abstractly introduced configuration K_R described by the spatial parametrization **X** one could introduce a stress tensor (in fact not a standard second-order tensor), by the so-called *Piola transformation* (1836, 1848):

$$\mathbf{T} = J_F \mathbf{F}^{-1}.\sigma, \quad \sigma = J_F^{-1} \mathbf{F}.\mathbf{T}, \tag{1.8}$$

where $\mathbf{F}^{-1}$is the inverse of **F** such that

$$\mathbf{F}^{-1} = \frac{\partial \tilde{\mathbf{x}}^{-1}}{\partial \mathbf{x}},\ \mathbf{F}\mathbf{F}^{-1} = \mathbf{1}. \tag{1.9}$$

Conscious of the arbitrariness of the choice of his reference configuration K_R, Piola selects it as one of uniform density equal to one. Since we know that mass conservation is expressed by

$$\rho_R = \rho J_F, \tag{1.10}$$

Piola writes "his" transformation as

$$\rho \mathbf{T} = \mathbf{F}^{-1}.\sigma. \tag{1.11}$$

Although Piola could not write his transformation in this simple condensed intrinsic form, his writing of typical components reveals an understanding of a hidden algorithm that will later be interpreted within tensor algebra.

The concept of Piola stress was comforted by Gustav R. Kirchhoff (1824–1887), so that the object **T** in (1.7) is nowadays called the *first Piola-Kirchhoff stress*. A *second* Piola-Kirchhoff stress **S** can also be introduced by completing the *true* tensor transformation between stresses in the actual and reference configurations by the definition: (The symbol –*T* means the transpose of the inverse).

$$\mathbf{S} = \mathbf{T}.\mathbf{F}^{-T} = J_F \mathbf{F}^{-1}.\sigma.\mathbf{F}^{-T}, \tag{1.12}$$

where the superscript *T* denotes the operation of transposition. Both **T** and **S** have a deep thermodynamic significance.

Joseph V. Boussinesq (1842–1929), in his study of finite deformations introduces the stress object **B** such as [compare (1.11)]

$$\mathbf{B} = \mathbf{F}^{-1}.\,\sigma. \tag{1.13}$$

That is why we consider Piola, Kirchhoff and Boussinesq the founding fathers of the theory of finite transformations in spite of the pioneering works of Lagrange and Cauchy.

1.4 The French "Ingénieurs-savants"

In a further chapter (Chap. 7) we shall emphasize the role of the *Ecole Polytechnique* in the formation of a special scientific trend and "spirit" in the early 19th century: the appearance of what the British historian of sciences Ivor Grattan-Guinness (1993) calls "ingénieurs-savants". This is a group of alumni from that engineering school who received from their masters (Monge, Bossut, Lacroix, Lagrange, Cauchy, Fourier) a remarkable mathematical education although they were usually destined to work on engineering projects, essentially in civil engineering. They applied their mathematical technical skill and their physical ingenuity in fostering various facets of the "rational" mechanics of continua. Among these individuals, for our present purpose, we single out C.M.L. Navier (1785–1836), Gabriel Lamé (1795–1870), and J.M.C. Duhamel (1797–1872). Albeit a disciple of Laplace in his Newtonian particle-action-at-a-distance view, the first of these was instrumental in developing both continuum fluid mechanics and elasticity. In the case of fluids, he constructed what is now called the Navier-Stokes equation that involves shear motion and the allied viscosity. With this one enters the domain of *nonideal* fluids as compared to Euler's ideal fluid. In elasticity, he was responsible for the introduction of the so-called Navier equations for *isotropic* elasticity (although not on the basis of Cauchy's stress argument) in small strains. The difference in the technical approaches led to a thorough discussion about the number of existing elasticity coefficients (one or two in the case of linear isotropic elasticity?). As we know now, the correct answer is two, and these coefficients λ and μ are called after the second of our "ingénieurs-savants", Lamé.

In Cartesian indicial tensor notation and intrinsic notation, Hooke’s law for isotropic materials reads

$$\sigma_{ji} = \lambda e_{kk}\delta_{ji} + 2\mu e_{ij},\ \ \sigma = \lambda(tr\mathbf{e})\mathbf{1} + 2\mu\mathbf{e}. \tag{a}$$

In 1D in the x-direction this takes the simple form (all quantities are scalars)

$$\sigma = Ee,\ \ e = \partial u/\partial x. \tag{b}$$

This reflects mathematically the celebrated Hooke’s law according to which “elongation is proportional to the applied force”, i.e., (Robert Hooke 1635–1703)

$$\delta l/l_0 = kF, \tag{c}$$

where l_0 is the initial length, k is a coefficient of proportionality, and the force F should remain reasonably small. The coefficient E is called the Young modulus after Thomas Young (1773–1829), a polymath in competition with Jean-François Champollion (1790–1832) for the deciphering of Egyptian hieroglyphs, a competition that he lost. Hooke’s law (c) belongs to what we shall call “physical mechanics” as it is based on observation. Navier’s elasticity equations are the field equations obtained by substituting from (a) in the balance of linear momentum.

The third of the “ingénieurs-savants” is Duhamel, probably less known than the other two. But he was more a “savant” than an engineer as he never graduated from the Ecole Polytechnique, having been expelled from the school in 1816 with all his fellow classmates for political reasons. His originality stems from the fact that he was the first to study a problem of *coupled fields* in continuum mechanics, namely, *thermo-elasticity* (1838). Of course he could do that only after Carnot and Fourier had developed the necessary ingredients for treating heat conduction alone. But Duhamel had the right intuition in attacking this coupled-field problem, even though the best applications of that new field would be only in the 20th century. Furthermore, his was probably the first example of considering a *non-isotropic* material response since he was conscious that some directions may be more important than others and thus privileged contrary to the often assumed isotropy (no preferred direction). This matter was studied in detail from the viewpoint of epistemology by Gaston Bachelard, a French philosopher of sciences, in 1927.

In 1D the Hooke-Duhamel constitutive equation of thermo-elasticity reads

$$\sigma = Ee + m(\theta - \theta_0), \tag{d}$$

where θ is the thermodynamical temperature, θ_0 is a reference temperature, and m is the thermo-elasticity coupling coefficient. Thermal expansion is obtained by putting $\sigma = 0$ in (d) and solving for e, yielding thus

$$e^{\theta} = \alpha(\theta - \theta_0), \tag{e}$$

where $\alpha = -m/E$ is the coefficient of thermal expansion.

1.5 The British Giants: Green, Kelvin, Stokes, Maxwell

We are concerned with a group of British scientists whom we collectively call the "Cambridgians" (some may think that "Cantabridgians" would be better). They share a similar vision of the physical world and also a practically identical formation. They have been educated at Cambridge and some of them taught there also. They have also in common to have been influenced by the French school of mathematics of the late 18th century—early 19th century, a school that had chosen to exploit Leibniz' notation rather than Newton's one in analysis, e.g., from Bossut's and Lacroix's sources used at *Polytechnique* (cf. Bossut 1800). This was a happy choice of far reaching consequence because it contributed to a new blossom of British mathematical physics that brought British authors to the top of the field in the 19th century. After these "Cambridgians", there came the "Maxwellians"—who may also have been "Cambridgians"—among whom we must count Heaviside and Larmor.

(Abbé) Charles Bossut (1730–1814), a disciple of d'Alembert and a specialist of hydrodynamics, but also an underestimated historian of mathematics, was a remarkable pedagogue. His course of mathematics at the Military school of Mézières (cf. Sect. 7.1) was first published in 1781. Its last edition (1800) was in seven volumes, of which two were devoted to differential and integral calculus in the Leibniz notation. He was a colleague of Laplace and Lagrange at the Paris Academy of Sciences, but not in the same class as these two mathematicians-mechanicians from the point of view of creativity. He practically ended his career as an examiner in mathematics at the Ecole Polytechnique (1796–1808)—it seems to have been the oldest examiner ever at that school.

Among the "Cambridgians" George Green (1793–1841) is a very special case in the sense that he practically concluded his scientific life with his studies as an undergraduate at Cambridge. Indeed, a miller by profession and practically an autodidact, he wrote some of his most beautiful memoirs after having studied by himself the French pedagogues. It is as a consequence of these early successes that he was sent to Cambridge University where, among other things, he unfortunately learned gambling and drinking. For our purpose we obviously note the celebrated *Green theorem*, also called the divergence theorem (Green 1828). He also established the *Green reciprocity theorem* and introduced the notion of *Green function*, all extremely useful notions in problems both in electromagnetism and in continuum mechanics. In a nutshell pertaining to continuum mechanics, combined with Cauchy's lemma (1.1), Green's divergence theorem reads as follows if we consider a surface distribution of given traction $\mathbf{T}^d$on the regular boundary $\partial B$of a body B with ∂B equipped with unit outward normal $\mathbf{n}$:

$$\int_{\partial B} \mathbf{T}^d da = \int_{\partial B} \mathbf{n}.\sigma \; da = \int_B div\, \sigma \; dv, \tag{1.14}$$

hence the importance of this theorem—also attributed to Gauss—for the formulation of continuum mechanics in global form. In truth, the global balance of linear momentum written as

$$\frac{d}{dt}\int_B \rho \mathbf{v}\, dv = \int_B \rho \mathbf{f} dv + \int_{\partial B} \mathbf{T}^d da, \tag{1.15}$$

yields by localization on account of the assumed continuity of the present fields the standard local balance of linear momentum in the following form in the actual—Euler—frame of reference:

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + div\sigma, \tag{1.16}$$

where ρ is the matter density at time t, $\mathbf{v}$ is the velocity, and $\mathbf{f}$ is a density of bulk force per unit mass. Then (1.1) is none other than the *natural* boundary condition associated with (1.16) at the regular boundary ∂B. The left-hand side of (1.16) holds good because the elementary mass $dm = \rho dv$ is assumed constant in time.

Another notion introduced by George Green is that of *potential function* for elasticity. The introduction of such a potential means that any path of loading in the elastic regime of deformations and strains closes to zero energy expenditure. That is, let W be such a potential. According to Green, the Cauchy stress is derived from it by the derivative function

$$\sigma = \frac{\partial W}{\partial \mathbf{e}}\ or\ \sigma_{ji} = \frac{\partial W}{\partial e_{ij}}, \tag{1.17}$$

where the symmetric tensor $\mathbf{e}$ of components e_{ij} is defined by

$$\mathbf{e} = \left\{ e_{ij} = u_{(i,j)} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\right\}. \tag{1.18}$$

Here the vector $\mathbf{u}$ of Cartesian components u_i is the elastic displacement. In general, from (1.17) we have.

$$W|_{\mathbf{e}_1}^{\mathbf{e}_2} = \int_{\mathbf{e}_1}^{\mathbf{e}_2} \sigma : d\mathbf{e}, \tag{1.19}$$

which is none other than the elastic energy expended between the two limit strain states. For a closed circuit this yields that no energy was spent, a statement that is tautological with the definition of a potential. It is easy to imagine that this may have had a strong influence on the thoughts of Kelvin pondering the notion of energy conservation. This potential behaviour is translated into modern anthropomorphic language by saying that the elastic material "remembers" only one state, the initial one, usually a state of zero energy itself (virgin initial state) and providing a minimum of energy, hence the required *convexity* of the function. This convexity is trivially guaranteed in linear elasticity where W is quadratic in the strain. To obtain an explicit form of the elasticity constitutive equation, it is sufficient to know the expression of W. In the general isotropic case, one then applies the above mentioned Cauchy theorem for isotropic scalar-valued functions, and the true linear case results by considering only the contributions linear in the strain in the constitutive law, hence the Lamé-Navier expression with two coefficients λ and μ (cf. Eq. (a) above).

Those interested in the person of Green may visit his rebuilt windmill in Nottingham; this reconstruction and reviving of the mill was mostly due to the combined efforts of the local university faculty members such as Lawrence J. Challis and Antony J. M. Spencer, the latter himself a modern "Cambridgian" and renowned mechanician—see Chaps. 3 and 6.

In modern thermomechanics (e.g., in Maugin 2011), if W denotes the strain energy function (potential) per unit reference volume, we have the elasticity constitutive equations in the form

$$\mathbf{T} = \frac{\partial \bar{W}(\mathbf{F})}{\partial \mathbf{F}},\ \mathbf{S} = \frac{\partial \hat{W}(\mathbf{E})}{\partial \mathbf{E}},\ \mathbf{E}:=\frac{1}{2}\left(\mathbf{F}^T\mathbf{F}-\mathbf{1}\right), \tag{1.20}$$

so that $\mathbf{T}$ and $\mathbf{S}$ are also called the *nominal* stress and the *energetic* stress, respectively. If the second coinage is obvious, the first one comes from the fact that $\mathbf{T}$ represents the stress component per unit surface of the reference configuration. This comes from the property that oriented surface elements in the actual and reference configurations are related by

$$\mathbf{n}da = J_F \mathbf{N}.\mathbf{F}^{-1}dA,\ \mathbf{N}dA = J_F^{-1}\mathbf{n}.\mathbf{F}da, \tag{1.21}$$

so that

$$\mathbf{n}.\sigma da = \mathbf{N}.\mathbf{T}dA. \tag{1.22}$$

Equations (1.21)—relating actual and Lagrangian configurations—were established in hydrodynamics in 1874 by another "Cambridgian", Edward J. Nanson (1850–1936) who made an academic career in Australia (see Nanson 1874).

William Thomson—later called Lord Kelvin—(1824–1907) is the second of our "Cambridgians". He was a great admirer of Green's original memoir of 1828, and he had it re-published in 1846, after which Green's memoir became popular. During a scientific visit in Paris Thomson discovered the original work of Sadi Carnot (1796–1832) on the "motive power of heat" and also the work of B.P.E. Clapeyron (1799–1864), another "ingénieur-savant" (so was the case of Sadi Carnot—we shall return to all these French scientists in Chap. 7). It is by combining these influences and that of James Prescott Joule (1818–1889) that Thomson was led to a formulation of a principle now called the *conservation of energy* or "first law of thermodynamics". He was in fact but one of three co-discoverers of this "law", the other two being Julius R. Mayer (1814–1878) and Hermann von Helmholtz (1821–1894), both from Germany. But Thomson *aka* Kelvin, just as von Helmholtz, was an immense scientist with multiple scientific interests such as in electromagnetism, electrotechnics, and continuum mechanics. In the last field he was interested in both fluids and elastic solids. For further consideration (cf. Chap. 13), we note that like other scientists of this pre-Maxwellian period (even Cauchy!) he was trying to construct a model of continuum that could afford the propagation of light in the form of pure transverse waves, because this is what was observed according to Augustin Fresnel (1788–1827). This led to the notion of continuum capable of responding to a density of couple, a medium with internal rotation now called "*Kelvin medium*", a precursor of the generalized continua of which theories would be expended in the 20th century starting with the work of the Cosserat brothers.

The third "Cambridgian" of interest in the present context is George Gabriel Stokes (1819–1903) whose name is for ever associated with that of Navier for the Navier-Stokes equation that governs the flow of linear viscous fluids.

Newton's viscous constitutive law

$$\sigma = \eta \frac{\partial v}{\partial x} \tag{f}$$

in 1D, - where η is a viscosity coefficient—was experimentally checked by J.L.M. Poiseuille (1797–1869, an alumnus from *Ecole Polytechnique* who became a medical doctor and famed physiologist) in his study of blood flow, a first in " bio-mechanics".

For a *solid* in small deformation we can write

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial e}{\partial t} \equiv \dot{e}, \tag{g}$$

so that (f) yields

$$\sigma = \eta \dot{e}. \tag{h}$$

Stokes' name is also attached to a well known theorem of vector integral calculus in several dimensions (passing from the circulation along a closed line C to the flux of the *curl* across the surface S leaning on that line, i.e.,

$$\oint_C \mathbf{A}.dl = \int_S \mathbf{n}.(curl\,\mathbf{A})da.$$

This theorem is of the same nature as the divergence theorem evoked in (1.14)—it expresses the passing of an integral over a manifold of dimension n-1 to one of dimension n in the calculus of "exterior forms" (a generalized Stokes theorem); it is of obvious importance in electricity (for currents) and hydrodynamics (for vortices).

Our fourth giant "Cambridgian" is none other than James Clerk Maxwell (1831–1873) of electromagnetic fame. But few "electricians" (as Oliver Heaviside would have called them) know that Maxwell was also the author of a fundamental work on the mechanics of trusses (exploited in so-called graphic statics in pre-computer times) and that in his studies of viscous media he introduced a realistic model of rheological behaviour now called *Maxwell model of visco-elasticity*. In terms of rheological models using the vivid image of springs and dashpots this corresponds to a (Newtonian) viscous element—cf. (f) above—and a (Hookean) spring element put in series. This, like the so-called *Maxwell*-Cattaneo conduction law—that contains a relaxation time—was much influenced by Maxwell's work in the kinetic theory of gases.

In 1D Maxwell's visco-elastic constitutive equation can be expressed by

$$\frac{1}{\tau_M}\sigma + \frac{\partial \sigma}{\partial t} = E\frac{\partial e}{\partial t}. \tag{i}$$

This can be rewritten as the relaxation equation

$$\frac{\partial}{\partial t}(\sigma - \sigma_H) + \frac{1}{\tau_M}\sigma = 0, \quad \sigma_H = Ee. \tag{j}$$

In turn this can be compared to the visco-elasticity law proposed by Thomson (Kelvin) and Voigt as

$$\sigma = Ee + E\tau_{KV}\dot{e} = E(e + \tau_{KV}\dot{e}), \tag{k}$$

which clearly is a linear combination of Hooke's law (b) and Newton's law (h). This can also be written as the relaxation equation

$$\frac{\partial e}{\partial t} + \frac{1}{\tau_{KV}}(e - e_H) = 0, \; e_H = E^{-1}\sigma. \tag{l}$$

In terms of rheological models using the image of springs and dashpots this corresponds to a (Newtonian) viscous element and a spring element put in parallel.

For the sake of completeness, comparison, and further reference, we mention the Maxwell-Cattaneo law of heat conduction in the "relaxation" form [compare to (l)]

$$\frac{\partial q}{\partial t} + \frac{1}{\tau_q}(q - q_F) = 0, \; q_F = -\kappa\frac{\partial \theta}{\partial x}, \tag{m}$$

where q_F is the classical Fourier heat-conduction law with conduction coefficient κ in isotropic bodies.

1.6 The German School and its Giants: Kirchhoff, Clebsch, Voigt, Mohr, et al.

Many of the great German contributors to our subject matter could also be qualified of 'ingénieurs-savants" for they often were educated in Polytechnic schools ("Polytechnicum")—or *Technischen Hochschulen* in a more recent jargon—all more or less founded as imitations of the French *Ecole Polytechnique*. One of the first characters in that play is Karl Cuhlman (1821–1861). He received his engineering education at the Polytechnicum in Karlsruhe. He was himself active in the study of railways structures and bridges. He did mostly works related to the strength of materials and graphic statics. More important than him for our purpose is Franz E. Neumann (1798–1895) who graduated from the University of Berlin (Doctoral degree in mineralogy and crystallography). His work in elasticity was conducted in parallel with those of Navier, Cauchy and Poisson, establishing the number of elasticity constants for anisotropic materials. For isotropic elasticity he established without doubt that two coefficients—the Lamé coefficients—were necessary (and not only one as had been assumed by some French elasticians). He may also be considered one of the founding fathers of photo-elasticity after his study of double refraction in stressed transparent bodies. This he applied to thermal stresses (Cf. Duhamel). He was a reputed lecturer and author of highly appreciated books who mentored some of our relevant characters: Gustav R. Kirchhoff (1824–1887), Alfred Clebsch (1833–1872), and Woldemar Voigt (1850–1919). His influence on these scientists is mostly felt in the domain of elasticity.

The first of Neumann's disciples, Kirchhoff, was to become one of the German giants in continuum mechanics for the 19th century, although his reputation in electricity, spectroscopy, black-body radiation, and thermo-chemistry is at the same if not higher prestigious level. It is in Königsberg that Kirchhoff took lectures with Neumann. He later became a professor of physics in Breslau, Heidelberg and finally Berlin. We already cited Kirchhoff in relation to the Piola-Kirchhoff stress defined in (1.8). From this we can construct the *Piola-Kirchhoff format* of continuum mechanics. For instance, if we note the demonstrable identities

$$\nabla_R . \left(J_F \mathbf{F}^{-1} \right) = \mathbf{0}, \quad \nabla . \left(J_F^{-1} \mathbf{F} \right) = 0, \tag{1.23}$$

by applying $J_F \mathbf{F}^{-1}$ to the left of (1.16) and accounting for the continuity equation $\rho_0 = \rho J_F$ between actual and reference configurations, we obtain the balance of linear momentum in the form

$$\frac{\partial}{\partial t} \mathbf{p}_R - div_R \mathbf{T} = \rho_0 \mathbf{f} \quad or \ \frac{\partial}{\partial t} (\rho_0 v_i) - \frac{\partial}{\partial X^K} T_i^K = \rho_0 f_i. \tag{1.24}$$

That is, while now this equation makes use of independent time and space partial derivatives (since t and $X^K - K = 1, 2, 3$—form a set of time and space independent variables in this parametrization), the equation still has components in the actual configuration. Writing the associated natural boundary condition requires using the Nanson formulas (1.21) and (1.22). Note that (1.24) holds true because in the absence of growth or resorption of matter (see Chap. 14 for this case), the continuity equation in the Piola-Kirchhoff format can simply be written as

$$\left. \frac{\partial \rho_0}{\partial t} \right|_{\mathbf{X}} = 0. \tag{1.25}$$

Kirchhoff made another important contribution to continuum mechanics and the mechanics of structures by constructing a model theory for the bending of plates. The two-dimensional equation deduced from a variational principle (principle of virtual work) that governs the deflection w at the mid-surface of the plate reads

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = q, \tag{1.26}$$

where D is the flexural rigidity of the plate. To arrive at this equation, Kirchhoff had to formulate a reduced potential energy that accounts for a set of basic kinematic hypotheses concerning the section of the plate normal to the middle surface and the neglect of any stretching of the elements of the middle plane for small deflections. This much improved the tentative theory proposed earlier by Sophie Germain (1776–1831) even after correction of S. Germain's mistakes by Lagrange. In a modern vision, establishing the crucial Eq. (1.26) is one example of the reduction of a three-dimensional elasticity problem to one in two dimensions by an asymptotic procedure (cf. works by Ambartsumian, Gold'enveizer, Ciarlet and others in the period 1950–1980). Kirchhoff's theory is now referred to as the *Love-Kirchhoff theory of plates* after A.E.H. Love (1863–1940), another "Cambridgian" who nonetheless had his whole scientific career at Oxford. Love extended Kirchhoff's approach to the case of thin shells. But Kirchhoff also studied theoretically and experimentally the vibrations of plates on the basis of his model. He also subsequently extended his theory of plates to include the case of not too small deflections. All of Kirchhoff work on plates has provided the most important basis for the computation of thin-walled structures. Thomson (Kelvin), already cited, improved on Kirchhoff's theory of plates by specifying the boundary conditions concerning shearing forces and bending moments at an edge.

Another student of Neumann in Königsberg was Clebsch. He wrote a thesis in fluid mechanics. He became a professor at the Polytechnicum in Karlsruhe when he was only twenty five after spending a short time at the University in Berlin. It is while at Karlsruhe that he wrote a famous book on elasticity—*Theorie der Elastizität fester Körper*—when there existed only one such book, by Lamé, available. He wrote it for engineers, but with special emphasis on mathematical methods of solutions, often loosing the physical aspect. This gave the opportunity to Barré de Saint-Venant in his French translation of this book to expand the matter in such a way that the bulk of the book tripled in translation, resulting in a book that was more his than Clebsch's. The mathematical inclination of Clebsch and his remarkable gift for it resulted in Clebsch becoming a professor of pure mathematics and ending his brief career as one of the best German mathematicians of his period (works on variational problems, Abelian functions, invariant theory, algebraic geometry) in Göttingen after teaching in Giessen. Among his famous students we find Max Noether (the father of Fritz and Emmy Noether—see Chap. 14) and Felix Klein (1849–1925) who was to play an instrumental role in German mathematics. He was a co-founder of one of the best journals in mathematics, *Mathematische Annalen*. He died untimely of diphteria. His contribution to the mechanics of continua, achieved during his youngest research period, remains a fundamental one and was considerably enriched by Barré de Saint-Venant.

Woldemar Voigt was also one of the successful students and disciples of Neumann. But he more closely than others followed his master in devoting much work to the elasticity of crystals that culminated in his book "*Lehrbuch der Kristallphysik*" (First German edition, Teubner, Leipzig, 1910). It is during this work that he was led to introducing the recently created notions of *tensor* and tensor-triad in the theory of continua, so much that tensor algebra and analysis practically became synonymous with that field in the eyes of many physicists. Of course, the word "tensor" smells of its mechanical origin. It is less known that

Voigt anticipated the Lorentz-Poincaré transformation formulas in special relativity and that he was the first to propose a correct Lagrangian density in electrodynamics.

Another line or remarkable chain of German contributors to continuum mechanics in the large starts with Otto Mohr (1835–1918). This line extends to the middle of the 20th century with August Föppl (1854–1924), Ludwig Prandtl (1875–1953), and Theodor von Kármán (1881–1963). Mohr was a railway-structural engineer who graduated from the Polytechnicum in Hannover. He taught engineering mechanics first at the Polytechnicum in Stuttgart and then in Dresden. Following along the path of Karl Cuhlman (see above), he was very much interested in graphical methods (*Graphische Statik*). He is universally known for his two-dimensional representation of the stress state by means of so-called *Mohr circles*.

After starting his engineering studies at the Polytechnicum, August Föppl transferred to Stuttgart where he took courses with Mohr, but he finally graduated from the Polytechnicum in Karlsruhe. He was basically a structural engineer with a strong side interest in electricity. Regarding the later field, he popularized Maxwell's theory of electromagnetism in a book published in 1894—the first of its kind in Germany. This book is supposed to have left a definite print on Einstein as a young man. A talented teacher in Munich, he also published the most popular book on engineering mechanics in German-speaking countries. He counts among his students Ludwig Prandtl who worked with him on solid mechanics. Prandtl taught first at the Polytechnicum in Hannover and then at the university of Göttingen. He is considered to be the father of modern aerodynamics. His works in this field are marked by mathematical subtleties such as in his theory of the boundary layer. Together with Richard von Mises (1883–1953) he founded the (German) *Society of Applied Mathematics and Mechanics* (*G.A.M.M*). One of his co-workers in Göttingen was von Kármán who came from Hungary and would later become the founder of the *Jet Propulsion Laboratory* at Caltech and a prominent figure in aeronautical government agencies in the USA. A theory for large deflections of plates is named after Föppl and Kármán. Kármán was also responsible for the basic dynamic theory of elastic crystal lattices together with Max Born (of quantum-mechanical fame).

This overview of German contributions would not be complete without the repeated mention of Hermann von Helmholtz. A medical doctor and physiologist by formation, Helmholtz is one of the most brilliant and versatile mind of the 19th century. His formidable scientific production covers sensing physiology, ophthalmic optics, nerve physiology, acoustics, electromagnetism and mechanics. Of course, in the present context he is most well known for his co-discovery of the *first law of thermodynamics*, a law of conservation that includes all forms of energy, whether of mechanical, electrical, etc, origin. From the point of view of elasticity and acoustics he introduced the Helmholtz decomposition of a vector field—that is essential in many problems of elasticity and elastodynamics—as also the well known Helmholtz equation. He mentored many famous physicists, among them Max Planck, Wilhelm Wien, Henry Rowland, A.A. Michelson and Michael Pupin. He has successively taught in Königsberg, Bonn, Heidelberg, and Berlin.

Note that Heinrich Rudolph Hertz (1857–1894) was a student of Helmholtz and Kirchhoff in Berlin. He in fact became Helmholtz' assistant in 1880 for a period of three years. Hertz is known among mechanical circles for his works on the compression of elastic bodies and his theory of contact that he worked out while in Berlin. After Berlin, Hertz was a professor first in Kiel and then at the Polytechnicum in Karlsruhe, where he conducted his famous experimental work on electromagnetic waves, thus proving the correctness of Maxwell's equations. He also wrote a highly praised book on the principles of mechanics.

1.7 Concluding Remark

In this chapter which rapidly spanned the 18th and 19th centuries, we have identified the main landmarks in a concise historical view of the early developments of the science of continuum mechanics. We have explored its strengthening and consolidation in a true field of applied mathematics in the form of a mix of "rational mechanics" and engineering. Leaning on the firm bases of Newtonian axioms and necessarily starting with duly abstracted models exploiting essentially Cauchy's construct, we have witnessed a growing "mathematization" of the field. Starting with idealizations and abstractions that avoid the true complexity of the mechanical behaviour of existing materials, this development had mostly been the result of the hard work of many civil engineers although these individuals were equipped with a sound mathematical formation and a great ingenuity. Most of the breakthrough results were obtained in three countries, France, the United Kingdom, and Germany, in reason of the advance of these countries in civil engineering, their growing industrial needs, and the existence of appropriate schools often providing the needed "ingénieurs-savants". A marked tendency in the observed 19th century developments was, apart from necessary experiments, the will to solve problems with sophisticated mathematical tools, which tools were practically created purposefully for these solutions. The near future would be to better describe the real mechanical behaviour of materials at a macroscopic scale, incorporate more deeply the thermodynamic background, and also to take some time to ponder the general philosophy—its structure and principles - behind this science. This is the main nature of the progress achieved in the next period that we circumscribe to the time interval 1880–1914. This we consider to be a transition to the true, unfortunately agitated but simultaneously rich, 20th century, the object of this book.

1.8 Further Reading

Selected historical landmark contributions are to be found in Cauchy (1828), Duhamel (1837), Green (1828, 1839), Kirchhoff (1876), Piola (1836, 1848), and Weyl (1946). Epistemological study of Duhem's works is to be found in Bachelard

(1927), and of Cauchy's deep contributions in Belhoste (1991) and Dahan-Delmonico (1984–1985). An interesting overview of Green's life is given in Cannell (1993). Pertinent historical reviews in continuum and solid mechanics are given by Barré de Saint-Venant (1864), Todhunter (1886), Timoshenko (1953), Truesdell (1968, 1976, 1984) and Soutas-Little (2011). More broadly, Dugas (1950) and Szabò (1977) look at general mechanics, and Whittaker (1951) and Schreier(1991) to physics. Warwick (2003) focuses on mathematical physics at Cambridge. Gillispie (1974) remains a real mine concerning scientific biographies written by specialists.

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Chapter 2
Transition to the 20th Century

Abstract Chapter Two deals with the transition period between circa 1880 and 1914, which prepares the way for the Twentieth century. It also advocates an attitude towards a development that is characteristic of a period when many engineering scientists believe in a then fixed paradigm and no further evolution is thought possible in spite of a contemporary revolution in theoretical and mathematical physics. Of course this corresponds to a period of natural consolidation with the general creation of efficient engineering schools all over Europe and the appearance of newborn ones in the USA. Of particular interest in this rather quiet landscape are queries concerning going beyond the most traditional behaviours (linear elasticity and Newtonian viscous fluids). Here are distinguished the emerging attempts at the description of more involved behaviours such as visco-elasticity (Voigt, Boltzmann, Volterra), and friction and plasticity (Tresca, Barré de Saint-Venant, Lévy, Huber, Mises). In spite of the relative quietness of the period, new interests of investigation are considered, mainly in the dynamic frame, the consideration of continua with internal degrees of freedom (Duhem, the Cosserat brothers), and elements of homogenization theory. Perhaps more attractive at the time were the discussions about the general principles of mechanics by people like Hertz, Mach, Duhem (with his general energetics), Poincaré, Hamel and Hellinger. This pondering will prove extremely useful in the second half of the Twentieth century.

2.1 Setting the Stage

We consider the period extending between circa 1880 and 1914. The French call it the “Belle époque”. For the Germans and the Austrians it was “der guten alten Zeit”. Parodying a well known English writer, we could also say that “it was the best of times”, but the “worst of times” was to come soon. Indeed, Queen Victoria was ruling over an Empire that never saw the sun going down; Britannia ruled the

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world with the largest fleet ever. Victoria's son Edward was going to succeed her after enjoying life in Paris where the "French cancan" was being illustrated by the most fashionable painters. Oxford and Cambridge still were the best universities in the United Kingdom, if not in the world. Victoria's family was reigning in many countries in Europe. The Kaiser was taking care of a powerful industrial country where *Technische Hochschulen* had replaced the Polytechnic schools (with a much too French sounding name, according to Prussian philology). St Petersburg, also ruled by one of Victoria's family relation, had a strong Polytechnic Institute. The Austro-Hungarian Empire was ruled by a benevolent ageing emperor. Technical universities existed in all parts of this empire, whether in Austria (Vienna) itself but also in Hungary (Budapest) and Galicia (Krakow, Lvov).

Italy was a rather young kingdom but with old universities—among the oldest in the world (Ferrara, Bologna) -, but also with two Polytechnic schools in Torino and Milano and a *Scuola Normale Superiore* in Pisa, a remnant of French (Napoleonic) influence. Switzerland had his two polytechnic institutions in Zürich (German speaking) and Lausanne (French speaking), both created by alumni form French "grandes écoles". As to France, living in its third republic and thinking about a possible revenge against the Prussians who had taken over Alsace and Lorraine in a brief war in 1870–1871, it was extending its colonial empire by imitation of the British while having instituted a charge free education at almost all levels. But it kept the formation of its elite in the "*grandes écoles*" such as the *Ecole Normale Supérieure* and the *Ecole Polytechnique* and its engineering schools of application (example of so formed scientist: Henri Poincaré; See Chap. 7). The country was working towards an exemplary full laicity, separating the Church and the State. Marcelin Berthelot—a thermo-chemist—epitomized the hero of republican science opposite to the clerical "reaction" identified with Pierre Duhem, while Georges Clemenceau—the future French Prime Minister victor of WWI—was representing the radical atheist left on the political spectrum.

The Chinese Empire was soon to suffer mortal attacks from its own socialists. European countries had succeeded to unite against some of the Chinese in Peking. Japan was accommodating European and American culture and creating universities and schools as imitations of those in these two parts of the world. But its industry was growing, having transformed its shoguns into industrialists, and creating a network of universities to replace the former scarce teaching of "Hollandish studies" (as European science was denominated). Moreover, Japan could now defeat countries like Russia in a military confrontation.

Finally, the United States, recovering from a painful civil war but the beneficiaries of an important immigration from European countries (Ireland, Germany, Russia, Italy, Sweden, etc.), were building an enormous industrial potential where the automobile would soon become one of the main output. With Edison, Tesla, Pupin, Bell and others, electricity and telephone had left the laboratory to become essential elements in everyday life. Simultaneously, John D. Rockefeller was building his immense fortune with the exploitation of oil fields, while moguls from steel industry (Andrew Carnegie) and transcontinental railroads (Leland Stanford) made huge donations that contributed to the creation of new teaching institutions

(Carnegie Technical Schools, CALTECH, Stanford University)—that were going to play an important role in the development of mechanical engineering in the 20th century. Still the USA were not exactly part of the "concert" of the nations, although they were also involved in wars and occupations that now look quite colonial in character (Cuba, Mexico, Puerto Rico, Hawaii, the Philippines). Concerning higher education, while the Ivy League colleges (Harvard, Yale, Princeton, etc.) remained the most powerful institutions for the formation of the elite, the Massachusetts Institute of Technology, created in 1866, really opened engineering sections in the 1880s, and "Agricultural and Mechanical (A&M)" colleges were being founded in various states (Texas, Virginia, Ohio, etc.) for the training of technicians.

This sets the stage for new developments that are more international in nature, refer more than before to a real behaviour of materials in their mechanical response, and require a deeper thinking about the bases of the theory of continua, and mechanics in general. The following three sections are devoted to these aspects. Moreover, following the French physicist Léon Lecornu who writes in 1918, we can distinguish between "rational mechanics" (a pure construct of the mind), "physical mechanics" (based on observation and experiments) and "applied mechanics", the later in fact meaning "engineering mechanics". We shall use this denomination.

2.2 Describing More Real Mechanical Behaviours

According to Lecornu (1918), more realistic mechanical behaviours primarily come into the picture via observation and experiments. In agreement with this remark we single out the behaviours of friction, plasticity and visco-elasticity. All these have for main property to be related to dissipation.

2.2.1 Friction

It is Charles Augustin de Coulomb (1736–1806), a former student of the (military) "*Ecole du Génie*" of Mézières in France, a pioneer in geotechnical engineering, who created *the science of friction* in the 18th century. He did that at a time when the notion of vector did not exist so that we let the reader imagine the difficulties (still present with our students) met with questions of signs. This does not relate to continua, but still it may provide some constructive idea about the behaviour known as (perfect) plasticity. Furthermore, everyone experiences the production of heat wherever friction is in action. But Coulomb and his contemporaries did not have any knowledge about thermo-dynamics. As a matter of fact, we believe that a correct inclusion of the phenomenon of friction in irreversible thermodynamics had to await the second part of the 20th century to find a satisfactory formulation.

2.2.2 Plasticity

Here also it was soon realized that in testing elastic materials to a higher mechanical load a kind of limit—the elastic limit—appeared after which one could hardly control the deformation. In 1D one simply reaches a level, say σ_0, of stress (force per unit area of the section of the sample), at which one looses the control of the elongation. We now say that we observe *plastic flow*, while mathematically we formalize this by saying that we loose the uniqueness in the response in deformation. But real materials are three-dimensional, the stress is a more complex object (tensor) than a scalar, and the datum of one single scalar to characterize the entry into the *plastic regime* is not always sufficient. One must think in terms of a convenient representation of a tensorial state of stress and deformation. Thanks to Cauchy who related this to the representation in terms of ellipsoids, we also know that the length of *principal* axes of the ellipsoids representing stresses and strains are convenient representations of the actual state.

Henri E. Tresca (1814–1885), a professor at a Paris institution known as the *Conservatoire National des Arts et Métiers* (for short, *CNAM*) conducted in the early 1870s a series of fine experiments on metals whereby he constructed in an appropriate representation of the principal stresses the elastic limit of the said metals (cf. Tresca 1872). Practically simultaneously, Adhémar J.C. Barré de Saint–Venant (1797–1886) gave the mathematical formulation of these results (1871). Three important remarks are in order: first, it is noticed that no change in volume (so called *isochoric* deformation in the modern jargon) is observed during plastic deformation; second, the directions of the principal stresses coincide with those of the principal stresses (this assumes an *isotropic* response); third, the maximum shearing (or tangential) stress at a point is equal to a specific constant. This can be written as $\tau_M = k$. In mathematical terms, we have

$$Sup_{\alpha,\beta}\left|\sigma_\alpha - \sigma_\beta\right| = 2k, \quad \alpha, \beta = 1, 2, 3, \tag{2.1}$$

where the Greek indices label the principal stresses. Introducing the tangential stresses, this can also be expressed by the following set of three inequalities:

$$2|\tau_1| \equiv |\sigma_2 - \sigma_3| \le k, \quad etc, \tag{2.2}$$

by circular permutation. In an astute plane representation this is represented by a hexagon (see Maugin 1992, Fig. 1.18). The interior domain (a convex domain with angular corners) is the domain of elasticity. Although the criterion provided by (2.2) gives good results in the case of metals, this definition of the elastic limit by pieces of intersecting straight lines offers some difficulties in analytic treatment of problems. Nonetheless, Barré de Saint–Venant was able to give the solution of exemplary problems such as: the torsion of a circular shaft, the plane deformation of a hollow circular cylinder under the action of an internal pressure, etc. These are problems that we still give students to solve without the help of a computer (see, e.g., Maugin 1992, Appendix). It was a simple but formidable idea of Huber

(1903–1904, in Poland; see Chap. 8) to replace the hexagon of Tresca by a circumscribed circle (obviously a convex domain; see Maugin 1992, Fig. 1.18) of radius k and equation

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2k. \tag{2.3}$$

This elastic limit is said to provide a *maximum-distortion-energy theory* of yielding. This shows that a plasticity criterion must involve the notion of *energy*. But it will take almost a 100 years to include such criteria within a good thermo-mechanical description of plasticity (cf. Maugin 1992). We note that other criteria were proposed before to measure the energy of deformation, including by the famed William J.M. Rankine (1820–1872; a precursor of Pierre Duhem with his "science of energetics"; he also worked on the fatigue of metals)—maximum-stress theory—in Scotland and Beltrami (maximum-strain-energy theory) in Italy. Richard von Mises (1913) proposed the same criterion as Huber in 1913 on a pure mathematical basis in order to facilitate calculations. In the future other criteria would be proposed for anisotropic media, the plasticity of soils, the case of porous media, etc.

Maurice Lévy (1838–1910), proposed to discard the elastic behaviour—as negligible for some materials—and to consider only the plastic one, thus in so-called *rigid-plastic* bodies (cf. Lévy 1871). This is a rather highly singular behaviour since nothing happens to the strain, not even an elastic one, in so far as the plasticity threshold is not reached and then we have an uncontrolled plastic flow occurring along a plateau in stress.

Prandtl and von Kármán solved other problems of elasto-plasticity in the early 20th century. Other well known scientists who worked in plasticity in that period were Bauschinger (1886), and Mohr (1900).

> We do not know if the French are that much conservative but we can say here that they do not throw things away: according to my friend James Casey from Berkeley, you can still find the specimens used by Tresca in his (1870) experiments kept in a box in the basement of the actual *CNAM*. In early times *CNAM*, created during the French revolution, was an institution somewhat similar to the *Royal Institution* in London where both Humphry Davy and Michael Faraday gave public lectures and conducted experiments. It is complemented by a rich museum of science and technology.

Although we cannot claim that this belongs to "physical mechanics" (on the contrary) we mention here the remarkable mathematical model of dislocation obtained in a pure ideal construct—or a thought experiment—by the Italian mathematician Vito Volterra (1860–1940) in 1907, at a time when no dislocation had really been observed (this had to await 1956 with the use of electronic microscopy). This concerns lines along which discontinuities in the elastic displacement field occur. This is mentioned here because of its timeliness and the fact that it would later on play a role in the study of ductile materials (i.e., essentially plastic materials). [Volterra's astute thought-experiment consists in cutting a cylinder (devoid of its central axial part—because this "core" corresponds to a singularity-) displacing the two faces of the cut in a certain way and welding them back so as to create a special type of discontinuous material displacement (cf. Volterra 1907, 1909, 1939)].

2.2.3 *Visco-Elasticity*

Two models of visco-elasticity were mentioned in Chap. 1: the Maxwell and Kelvin-Voigt models that both involve a relaxation time, but with relaxation in stress and strain, respectively. Ludwig Boltzmann (1844–1906), renowned for his seminal work in the kinetic theory of gases and for his statistical definition of entropy, also proposed a model of visco-elasticity for solids, but in an original form. The idea is to take account in a good mathematical way of what happened in the past to the material: the past history of the strain should be involved, with an obvious more important influence of the recent past. In modern terms, this can be exemplified by a 1D stress–strain *functional* relation over time of the type (cf. Boltzmann 1874).

$$\sigma(x,t) = E_{relax} e(x,t) + \int_{-\infty}^{t} K(t-t')\dot{e}(x,t')dt', \tag{2.4}$$

where E_{relax} is the instantaneous modulus for relaxation, and K is a relaxation function. The later must be such as to favour the influence of the recent past, thus decreasing sufficiently fast in its argument. Using a modern illustrative jargon, we can say that the material so described possesses a *fading memory* of the past. In substituting (2.4) in the dynamical equation of linear momentum, we would be led to a new kind of equation, an *integro-differential equation*. It happens that functionals over time such as (2.4) and integro-differential equations were one of the fields to which Vito Volterra (already cited but also Volterra and Pérès 1936) contributed much with applications, not to mechanics, but to a kind of population dynamics [competition between species yielding a celebrated equation obtained independently by Alfred Lotka (1880–1949)]. The generalization of equations such as (2.4) will have a blossoming heritage in the 1960s (see Chaps. 5 and 11).

All three behaviours highlighted in the present section were still missing a good, if any, thermodynamic basis although some authors, e.g., Pierre Duhem, were pondering this matter, but as mere wishful thinking at the time.

2.3 New Interests of Investigation

2.3.1 *Dynamics*

With the pioneering work of Georg Bernhard Riemann (1826–1866) and those of William J.M. Rankine (1820–1872), Pierre H. Hugoniot (1815–1887), and Jacques C. E. Jouguet—known as Emile Jouguet (1871–1943)—one attacks the field of the *nonlinear dynamics* in continua. Having established the required equations governing the discontinuities of fields, these scientists could prove the existence and propagation of shock waves and also detonation waves (Jouguet 1906). Duhem also studied such waves in nonlinear elasticity and his friend Jacques Hadamard (1865–1963) provided a useful classification of propagating discontinuities depending on what is the order of the derivative of the basic field that is discontinuous (cf. Hadamard 1903). Jouguet was much influenced by Duhem in adopting a thermo-mechanical viewpoint. These studies in various schemes of deformable-solid mechanics will be taken over in the 1950–1970s, in particular by Truesdell, Ericksen and others (e.g., Peter J. Chen for a long time at Sandia National Laboratories; see his book, Chen 1976). Ernst Mach, in his experimental study of shock waves in fluids, was led to introducing the "Mach" number (as a measure of relative velocity compared to the sound speed) and the "Mach" angle (in the reflection of such shock waves). No need to mention the role of these studies in the future developments of aerodynamics. In that field, a seminal work was that of Prandtl with the notion of *boundary layer* and its elegant mathematical formulation using asymptotics for a mathematically singular problem.

2.3.2 Internal Degrees of Freedom

The period 1880–1910 saw the introduction of the idea that, perhaps, a material point in a continuum would be characterized by more than a simple translation (displacement) in space. It seems that Duhem (1893) was responsible for the idea to consider a triad of rigid vectors (so-called "directors") at each material point in order to describe the orientational changes in some kind of internal rotation. But this was more an idea than a true complete development. The Cosserat brothers were also led to consider the possible existence of internal couples (1909). They more or less were forced to do that by imposing an invariance (so-called *Euclidean invariance*) in a Lagrangian-Hamiltonian formulation, which invariance treats on an equal footing translations and rotations. This gave rise to the possible existence of a new type of internal force, the *couple stress* along with that of stress, and the possibility to have *non-symmetric* stresses. This was a first application of an argument of elementary group theory in continuum mechanics. As such, it was applauded by Elie Cartan (1869–1951), the famous geometer and specialist of the theory of Lie groups. Hellinger (1914) acknowledged the possible enrichments provided by Duhem and the Cosserats but without further elaboration. The argument of Euclidean invariance was exploited in papers by Sudria (1926, 1935). More recently it was applied by Toupin (1964) and Maugin (1970) for Cosserat continua and micromorphic ones, respectively.

Following the Cosserats, it is tempting to use a Lagrangian-Hamiltonian formulation to generalize the theory of classical elasticity, still in the absence of any dissipative process. Since small-strain elasticity corresponds to a theory that involves only the first gradient of the displacement in the energy, the next step would be to consider a better approximation of the displacement function at each material point, hence to envisage an energy density that depends also on the second-gradient of this displacement (and higher-order gradients if needed). This step was taken by J. Le Roux in 1911 and 1913 in his doctoral thesis published as two memoirs in the Scientific Annals of the *Ecole Normale Supérieure* in Paris. The second gradient of the displacement will be felt only in problems where a sufficiently spatially non-uniform state of strains exists. This is the case of torsion, a case duly examined by Le Roux in his pioneering work. But this type of considerations was left dormant for practically 50 years. We shall return to this matter and the Cosserats' media in detail with the developments of the 1960–1970s (Chap. 13).

2.3.3 Elements of Homogenisation

Just for the sake of completeness we mention the first, naïve, technique of homogenization for inhomogeneous bodies made of grains, e.g., for dielectrics by Maxwell. This technique is essentially one that is called the *rule of mixtures*, according to which effective properties are defined by an average accounting for the relative proportion of different components in the material. Precise mathematical techniques of homogenization will be proposed in the 1970–1980s only.

2.4 Pondering the Principles

When we carefully scrutinize the scientific atmosphere of the finishing 19th century and the dawn of the 20th century, we get the feeling that most scientists have agreed on a view that concludes that *everything has settled*. There is nothing in view such as an *epistemological rupture* (in the words of Gaston Bachelard) or a radical *change of paradigm* (in the words of Thomas Kuhn), although Albert Einstein is formulating his theory of special relativity (1905), and Max Planck is introducing his quantum (taken over by Einstein in his theory of the photo-electric effect; also 1905). This kind of attitude that marks an accomplishment, favours the reflection on the bases of the theory (Newton's one for mechanics) and the formulation of an axiomatic approach.

Concerning mechanics *per se*, even Heinrich Hertz (1857–1894) published a successful book on the principles of mechanics (Hertz 1899). Of course, Henri Poincaré (1854–1912), with his truly aerial view of science in its totality wrote beautiful books (e.g., *Science and Method*) that are still spot on, even though we may not share his epistemological views. Being himself an active participant in the field, he knows well the problem posed by the relativity of motion. He is the one who identified the group structure of Lorentz transformations between frames. Ernst Mach (1838–1916), an Austrian physicist and philosopher whose name is definitively attached to aerodynamics with the *Mach number*, wrote an articulated criticism of Newton's views in his book on the *Science of Mechanics* (cf. Mach 1911). The reading of this book is said to have deeply influenced the young Einstein. Mach pays special attention to the notion of inertia and its physical origin. Lecornu, writing in 1918, is more conscious of the forthcoming developments as he already knows elements of Einstein's gravitation theory.

In a different class we note the axiomatization of classical mechanics by Georg Hamel (1877–1954) in Hamel (1908). This was to have a long standing influence especially in Germany. The mathematician Ernst Hellinger (1883–1950), in an encyclopaedia article for mathematicians, provided in a nut shell a remarkable synthesis of the bases of continuum mechanics as of the 1910s. He captured all essential recent developments such as those due to Boltzmann (visco-elasticity), Duhem, and the Cosserat brothers. Finally, we shall consider the view of Duhem in greater detail.

Pierre Duhem (1861–1916) is a remarkable character who combines in one person a brilliant and sharp mind, a prolific writer and contributor to phenomenological physics, the champion of energetics, a philosopher of science, and the true creator of the history of medieval science. He made a big "mistake" early in his career. Aged only 24, as a student at the *Ecole Normale Supérieure* in Paris, he definitely criticized the work of Marcelin Berthelot in thermo-chemistry (he blatantly asserted that some principle regarding thermodynamic potentials and proposed by that hero of French republican science, was wrong—but Duhem was later proved to be absolutely right). This hindered the whole university career of Duhem who nonetheless produced a lot of good science, philosophy and epistemology, but in Bordeaux and not in Paris. In centralized France this was as bad as the original sin.

In the philosophy and methodology of science Duhem wrote two remarkable books, one on the *Aim and Structure of Physical Theory* (original French in 1906) and the other with a title repeating Plato's moto "*To save the phenomena*" (original French in 1908 with Greek title). In the first of these he exposes at length the under determination of theory by fact, the rejection of metaphysics and models (as used by, e.g., Kelvin and Maxwell in the UK and Boussinesq in France), and natural classification, rather than explanation, as the very object of physical theory (this can be discussed). The contents of the second book are clearly explained by its title. His view on the unifying role of thermodynamics in all of physical sciences (mechanics, electricity and magnetism, heat, etc.) is masterly but quite lengthily expended in his treatise on "energetics" or "general thermodynamics" (Duhem 1911). This has a flavour of axiomatic nature that will influence C.A. Truesdell in the 1950–60s. Duhem was a good friend of mathematicians Henri Poincaré (1854–1912) and Jacques Hadamard (1865–1963).

Personal touch. Both Duhem and Poincaré died untimely while Hadamard reached almost a hundred, using the facility of the library at *Institut Poincaré* in Paris until the end. According to the librarian—a certain Paul Belgodère—of this institute that the author knew as a student, Hadamard used to come to the library every afternoon in the late 1950s, asked to consult one of Poincaré's or Duhem's works of bygone days (say, from 1890 to 1905), and systematically fell asleep, being wakened up by the librarian at the closing time. The same scenario was repeated from day to day.

For our main concern in this book and forthcoming chapters, the most relevant writing of Duhem is the one on the “evolution of mechanics” (Duhem 1903). In one section of this opus, Duhem examined what, at the time, he called the “*nonsensical” branches of mechanics*. What he means by this somewhat eccentric expression are the fields of physics, mechanics and electromagnetism that do not fit yet in his general framework of thermodynamics. It is interesting to note the list of these fields: so-called false equilibria, hysteresis phenomena, and electro-magnetic theory in materials. These are precisely dissipative phenomena such as thermodynamically irreversible reactions, friction, plasticity, etc. Now looked upon with our present knowledge, this sounds like a tentative proposal of research programme for the next generation, something quite equivalent in its own field to the Erlangen program (1872) of Felix Klein in geometry and the list (1900–1902) of unsolved—at the time—problems proposed by David Hilbert in pure mathematics—that in fact included the axiomatization of the whole of physics as Problem no six.

The flame of Duhem’s approach to general thermodynamics was successfully carried over by Th. De Donder (1872–1957) and other physicists from the Netherlands and Belgium between 1930 and 1970, resulting in the now commonly admitted *theory of irreversible processes* (S. De Groot, P. Mazur, I. Prigogine). However, both Duhem and these scientists did not possess the mathematical tools—such as convex analysis and nonlinear optimization—to deal with some of the properties (plasticity, hysteresis), so that they could deal only with *linear* irreversible processes. The solution would come in the 1970–1980s for *nonlinear* irreversible processes.

2.5 Concluding Remark

The considered interval of time was the last period during which the same scientists worked in so many different fields, but still within phenomenological physics. This marks the end of an era that was typically that of the 19th century. Examples of exceptions in more recent times will be Lev D. Landau and P. G. de Gennes. From now on, some specialization will be necessary, and this will be the case in practically all forthcoming chapters.

2.6 Further Reading

On the principles of mechanics first-rank contributions are by Barré de Saint-Venant (1851), Hertz (1899), Duhem (1906, 1908, 1911), Hamel (1908, 1927), Mach (1911) and Hellinger (1914). On Pierre Duhem see Ariew (2007) and Manville (1927). On Boussinesq we recommend Bois (2007). On the general history of the strength of materials, Timoshenko (1953) remains an unavoidable reference.

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Chapter 3
Rheology and Nonlinear Elasticity

Abstract This first specialized chapter deals with the awaited generalization to mechanical behaviours that deviate from linear elasticity and standard Newtonian viscous fluids, that is, elasticity in large deformations and the rheology of complex fluids. These extensions were kindled by the mechanics of rubber elasticity and artificial fabrics and of fluids with high viscosity and visco-plastic response. It happens that the same scientists were involved in these two lines as a result of a required focus on the bases of continuum mechanics, in particular the theory of finite deformations in a rational geometric background, and the need to account for complex flow features in some fluids. Ronald Rivlin, with his incommensurable contributions, is the great hero in this adventure. Other scientists whose work was seminal are initially E. Bingham, M. Reiner, L.G.R. Treloar, P. J. Flory, M.A. Mooney, and F.D. Murnaghan, and more recently J.G. Oldroyd, A.E. Green, J.L. Ericksen, C.A. Truesdell, B.D. Coleman, and W. Noll. The survey includes the models of neo-Hookean materials, Mooney-Rivlin materials, Rivlin-Ericksen fluids, and unsuccessful attempts such as those of Reiner-Rivlin fluids and hypoelasticity. Appropriately introduced tools have been those of Rivlin-Ericksen tensors, Oldroyd and Jaumann time derivatives, and invariant representations of scalar and tensorvalued functions. Through Rivlin and his co-workers the whole carries a strong print of British applied mathematics although Italian and Russian contributions to nonlinear elasticity cannot be overlooked. The mechanics of soft living tissues has now become the best field of application of these developments.

3.1 Beyond Standard Linear Elasticity and Viscous Fluids

3.1.1 General Remarks

As mentioned in Chap. 2, at the dawn of the twentieth century, we perceive shy attempts to venture in the domain of the continuum mechanics of more complex mechanical behaviours with the introduction of dissipative behaviours and some

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nonlinearity (such as in plasticity but not to the point of being able to solve difficult problems of evolution). But the object of continuum mechanics remains the same as before: *to evaluate the deformation or flow that results from the application of a system of forces to a body, whether solid or fluid*. In the early twentieth century, this is achieved only for elastic materials which undergo infinitesimal deformations (i.e., within linear elasticity) or for Newtonian viscous fluids, two cases fully developed in the nineteenth century. The set of equations to be considered consists, in the body, of the field equations, here written as the *Euler-Cauchy* equations of motion [in direct and indicial (Cartesian tensor) notations],

$$\rho \frac{d\mathbf{v}}{dt} = div\sigma + \rho\mathbf{f} \; or \; \rho \frac{d}{dt} v_i = \frac{\partial}{\partial x_j}\sigma_{ji} + \rho f_i, \tag{3.1}$$

where $\sigma = \{\sigma_{ji}\}$ stands for the symmetric Cauchy stress, $\mathbf{v} = \{v_i\}$ denotes the velocity field, and $\mathbf{f} = \{f_i\}$ represents an external bulk force per unit mass. Equation (3.1) is complemented by appropriate boundary conditions, and initial conditions in the case of dynamics.

In *linear* (isotropic, homogeneous) *elasticity*, we have the *Hookean* constitutive equation:

$$\sigma = \lambda(tr\mathbf{e})\mathbf{1} + 2\mu\mathbf{e} \; or \; \sigma_{ji} = \lambda e_{kk}\delta_{ji} + 2\mu e_{ji}, \tag{3.2}$$

where the infinitesimal strain $\mathbf{e}$ is defined as:

$$\mathbf{e} = (\nabla\mathbf{u})_S \; or \; e_{ji} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right). \tag{3.3}$$

Here $\mathbf{u} = \{u_i\}$ is the displacement vector, while λ and μ are the Lamé coefficients.

For Newtonian fluids, we have the *Navier-Stokes* constitutive equation:

$$\sigma = \lambda_v(tr\mathbf{D})\mathbf{1} + 2\mu_v\mathbf{D} \; or \; \sigma_{ji} = \lambda_v D_{kk}\delta_{ji} + 2\mu_v D_{ji}, \tag{3.4}$$

where the rate of strain $\mathbf{D}$—or symmetric velocity gradient—is defined as:

$$\mathbf{D} = (\nabla\mathbf{v})_S \; or \; D_{ji} = \frac{1}{2}\left(v_{j,i} + v_{i,j}\right). \tag{3.5}$$

Here $\mathbf{v} = \{v_i\}$ is the velocity vector, while λ_v and μ_v are the two viscosity coefficients. If the condition $3\lambda_v + 2\mu_v = 0$ is fulfilled, then (3.4) reduces to the constitutive equation of a *Stokesian* fluid:

$$\sigma = 2\mu_v\mathbf{D}^d \;, \; \mathbf{D}^d := \mathbf{D} - \frac{1}{3}(tr\mathbf{D})\mathbf{1}, \tag{3.6}$$

where now both $\boldsymbol{\sigma}$ and $\mathbf{D}^d$ are trace-less (i.e., deviatoric) tensors.

The two Eqs. (3.2) and (3.4) can be seen as deriving from some potential (energy in the first case, dissipation in the second one) as:

$$\sigma = \frac{\partial W}{\partial \mathbf{e}}\ ,\ W = \bar{W}(\mathbf{e}) = \frac{1}{2}\left(\lambda I_1^2 + 2\mu I_2\right)\ ,\ I_1 = tr\mathbf{e},\ I_2 = tr\mathbf{e}^2 \tag{3.7}$$

and

$$\sigma = \frac{\partial \Phi}{\partial \mathbf{D}},\ \Phi = \bar{\Phi}(\mathbf{D}) = \frac{1}{2}\left(\lambda_v I_1^2 + 2\mu_v I_2\right),\ I_1 = tr\mathbf{D},\ I_2 = tr\mathbf{D}^2, \tag{3.8}$$

following the views of Green and Rayleigh, respectively. The semi-positive definiteness of W requires that

$$\lambda + 2\mu \geq 0,\ \ \mu \geq 0. \tag{3.9}$$

A similar set of inequalities holds for the viscosity coefficients.

3.1.2 Non-Newtonian Fluids

Let us first consider the case of fluids. The Navier-Stokes equations apply to fluids of which we can say that they flow rather easily. First, they flow as soon as a small force is applied (no threshold) and they correspond to a simple model of proportionality in 1D, that of Newton given by equation (*f*) in Chap. 1, that we can rewrite as:

$$\tau = \eta\, \dot{\gamma},\ \eta = const.\ (Newton), \tag{3.10}$$

for a shear rate $\dot{\gamma}$. Here τ denotes the tangential stress and η depends at most on temperature. This was beautifully confirmed by Poiseuille's experiment concerning blood flow (1844, laminar flow in a cylindrical tube). The unit of viscosity, the "Poiseuille", was given to honour this scientist, but it is not an SI unit.

But early in the twentieth century the question was raised of the possible mathematical description of liquids that are manifestly viscous but they can flow only slowly and sometimes presenting a threshold in force for the activation of a real flow. In the latter case it seems that the behaviour is somewhat mixed between viscosity and plasticity. These fluids may be food stuff, fuels and biofluids, and in more recent times personal-care products (various gels and pastes), electronic and optical materials, and various polymers. This general problem was first identified by chemical engineers and chemists. Among them, Eugene Bingham (1878–1945), a professor and head of the Department of Chemistry at Lafayette College (not a research institution) in Pennsylvania, coined the appropriate term "rheology"—together with his friend Markus Reiner (1886–1976) from Palestine (the state of Israel did not exist yet) at the Technion—to denote the general study of such flows. They also founded a corresponding scientific society under the name *Society of Rheology* in 1929. The spot on motto of the Society is *Πάντα ρεῖ* or "panta rhei" [Greek for "everything flows", attributed to Heraclitus of Ephesus (c. 535–c. 475 BCE)]. Bingham revisited some ideas of Maxwell on what he calls "semi-fluids" and paid special attention to the case of so-called visco-plastic fluids (or *Bingham fluids* in modern terminology). This may be vividly illustrated by the tooth paste that comes out of the tube as a flow of product with a rigid core when one presses on the tube. Such a phenomenon may be observed in the extrusion production of some metallic bars at high temperature. Other strange phenomena were observed such as the so-called *Poynting effect* discovered in 1909—this is the same Poynting as in the Poynting theorem in electromagnetism (See Chap. 12)—and related to the existence of a difference in normal stresses or strains due to an impressed shear stress. To tell the truth, according to Rivlin (1984), Poynting discovered the effect not in a standard medium but in relation to light propagation in a kind of elastic medium (the ill-fated aether). More recently, K. Weissenberg (1949) discovered the "rod-climbing effect" according to which some "non-Newtonian" fluids have a tendency to climb along the rod that is vertically rotated (torsional flow) in a cylindrical container filled with such a fluid. Other non-linear effects proper to non-Newtonian fluids are swelling upon emergence from a tube and the bulging if allowed to flow downward in a through.

For further reference we can note the following nonlinear generalization of (3.10):

$$\tau = \eta(\dot{\gamma})\dot{\gamma},\ \eta(\dot{\gamma}) = m\dot{\gamma}^{n-1}, \tag{3.11}$$

where $m = \mu_v$, $n = 1$ corresponds to a Newtonian fluid, while *shear thickening* fluids are such that viscosity increases with the shear rate, i.e., $m \neq 0$, $n > 1$, and *shear thinning* fluids have a viscosity that decreases when driven to flow at a high shear rate. A *Bingham visco-plastic* fluid is such that

$$\eta(\dot{\gamma}) = \infty \ if \ \tau \leq \tau_g \ and \ \eta(\dot{\gamma}) = \mu_v + \dot{\gamma}^{-1}\tau \ if \ \tau \geq \tau_g, \tag{3.12}$$

where τ_g is a threshold in shear stress.

But a fluid body is not generally a one-dimensional object so that we need the proper formalism (in particular tensorial kinematic objects) for a true three-dimensional formulation of non-Newtonian fluids (see Sect. 3.2 below).

3.1.3 Nonlinear Elasticity

Simple generalizations of linear elasticity to a nonlinear framework were proposed for some metals at the end of the nineteenth century by considering a stress as a power expansion in the infinitesimal strain and then trying an identification of coefficients by testing the materials (see Bell's encyclopaedia article, 1973, Sect. 2.23), since (3.2) may appear as a first order approximation. The same kind of approximation may be constructed on the basis of a lattice model of continuum by considering nonlinear interactions between neighbouring "particles" in a discrete chain. In passing asymptotically to a continuum this applies to the case of very small deformation (of the order of 10^{-4}) expected, for instance, in electro-acoustic crystals.

But the obviously most interesting case, because offering a much more exciting challenge, is that of elastic materials likely to admit very large strains (say, of the order of 200 % or much more, e.g., 1,000 or 2,000 %) such as rubber-like materials, certain polymers, and some biological tissues. In these cases we are much better equipped than for non-Newtonian fluids, because the whole panoply of useful stress tensors and required finite-strain tensors has been developed in the nineteenth century and refined in the first half of the twentieth century. In particular, we mention the book of Murnaghan (1951) which was one of the first books to provide all the required mathematical tools in finite deformations.

Francis D. Murnaghan (1873–1976) is an interesting character. He was originally from Ireland and trained as a mathematician. He went to Johns Hopkins University in Baltimore where Harry Bateman (the applied mathematician) had just been appointed. Obtaining his PhD in 1916, he returned to Johns Hopkins in 1919 to become professor, and then head of the mathematics department in 1928, after a short stay at the Rice Institute in Houston, Texas. The later was a new institution founded thanks to a donation by Williams M. Rice; this became a University only in 1960. A man of many scientific interests, Murnaghan was in fact a rather pure mathematician with his main interest in the theory of group representations (classical groups, unitary and rotation (orthogonal) groups, symplectic groups). It is probably this specialty that brought him to write his famous book on finite deformations. This book still is a fundamental reference for all those interested in the application of finite strains and symmetries in continuum mechanics.

It is the proper relationship between the two sets of tensorial quantities (stresses and strains) that must be constructed as the relevant constitutive equations, for isotropic or anisotropic materials. In so far as possible, these should be thermodynamically admissible and derivable from a potential. We shall examine this matter in Sect. 3.2 where the seminal contributions of Ronald Rivlin (1915–2005) are emphasized. This happens to be also the case of non-Newtonian fluids where Rivlin left his name attached to various classes of fluids (Sect. 3.3).

3.2 Nonlinear Elasticity

3.2.1 Reminder

The following reminder is useful. The general deformation mapping is given by Eq. (1.4) between a reference configuration K_R and the actual configuration K_t. The direct and inverse deformation gradients are given by (1.5) and (1.9), respectively. The commonly used material tensor measures of finite-strains are the symmetric *Cauchy* and *Lagrange* material strain tensors such that (T stands for the operation of transposition):

$$\mathbf{C} := \mathbf{F}^T\mathbf{F},\ \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R). \tag{3.13}$$

For hyperelastic materials with an energy W per unit reference volume, the material stress–strain relation is given by [cf. Eq. (1.20)]:

$$\mathbf{S} = 2\,\frac{\partial\,\tilde{W}(\mathbf{C})}{\partial\,\mathbf{C}} = \frac{\partial\hat{W}(\mathbf{E})}{\partial\,\mathbf{E}}. \tag{3.14}$$

The Cauchy stress follows by inverting the last of (1.12), i.e.,

$$\sigma = J_F^{-1}\mathbf{F}.\mathbf{S}.\mathbf{F}^T = J_F^{-1}\mathbf{F}.\frac{\partial\hat{W}}{\partial\mathbf{E}}.\mathbf{F}^T. \tag{3.15}$$

The *Finger* strain tensor (after Joseph Finger, 1841–1925)—sometimes noted $\mathbf{c}^{-1}$—defined as:

$$\mathbf{c} = \mathbf{F}\mathbf{F}^T, \tag{3.16}$$

is also useful but in the actual configuration.

Many rubber elasticians (e.g., Treloar 2005) prefer to describe finite strains by means of the (principal) stretches along the three material orthogonal coordinates. These stretches are introduced thus. Consider a cube of edges of unit length in K_R. Under isothermal situations, after deformation this is transformed in a rectangular block with edge lengths λ_1, λ_2, λ_3. If the considered deformation is substantially incompressible—this is a reasonable working hypothesis for rubber like materials—we will have the constraint $\lambda_1\,\lambda_2\,\lambda_3 = 1$. Equivalently, we can think in terms of the principal axes of the strain ellipsoid introduced by Cauchy. In a general way the three basic invariants of tensor $\mathbf{C}$ are defined by:

$$I_1 = tr\mathbf{C},\, I_2 = \frac{1}{2}\left[(tr\mathbf{C})^2 - tr\left(\mathbf{C}^2\right)\right],\, I_3 = \det\,\mathbf{C}. \tag{3.17}$$

These are the three scalar invariants involved in the Cayley-Hamilton theorem applied to matrix $\mathbf{C}$. In terms of the stretches, this can be rewritten as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\, I_2 = \lambda_1^2\,\lambda_2^2 + \lambda_2^2\,\lambda_3^2 + \lambda_3^2\,\lambda_1^2,\, I_3 = \lambda_1^2\,\lambda_2^2\,\lambda_3^2. \tag{3.18}$$

If there exists a stored elastic energy W per unit reference volume, it must be such that $W = W(\lambda_1, \lambda_2, \lambda_3)$. Associated "material responses" (internal forces) are directly defined by the "Biot" constitutive relations:

$$f_\alpha = \frac{\partial W}{\partial\lambda_\alpha},\, \alpha = 1, 2, 3. \tag{3.19}$$

For an incompressible $(J_F = 1, I_3 = 1)$material, the corresponding forces per unit deformed area are then given by [cf. Eq. (3.15)]

$$\sigma_\alpha = \lambda_\alpha f_\alpha. \tag{3.20}$$

3.2.2 The Works of Treloar, Mooney and Rivlin

The main contributors to the development of rubber elasticity theory in the 1930–1950s certainly are Paul J. Flory (Nobel Prize 1974), Guth and James, Treloar, Mooney and Rivlin. They all worked closely with the related industry of rubber and early artificial fabrics. They all considered first an approach based on the physical description of polymers with long chains of molecules and thus necessarily exploiting arguments of statistical physics. But Ronald S. Rivlin (1915–2005)—although also a good experimentalist-, with his initial training as a mathematician in Cambridge, was the one who tried to use a pure mathematical standpoint to formulate the expression of relevant energies for rubber like materials. In doing so he contributed forcefully to the modern theory of continua in the large. He is probably the greatest of our heroes in the field (and also an original and difficult character; see below). In the theoretical and experimental developments of rubber elasticity, a critical and beneficial role was played in the UK by the British Rubber Producers Research Association (*BRPRA*) and then the British Rayon Research Association (*BRRA*).

Personal touch: Rivlin was educated at Cambridge (BA in Mathematics in 1937, MA in 1939). After a short stay at *General Electric Co* and two years as a Scientific Officer with the ministry of Aircraft Production during WWII, he spent nine years at the *BRPRA*, from 1944 to 1953, doing both seminal theoretical and experimental works with a one-year intermission/visit to the *National Bureau of Standards* in Washington (1946–1947) and a fruitful stay at the *Naval Research Laboratory* in Maryland (1952–1953). There his meetings with Jerald L. Ericksen and Richard A. Toupin were to produce also fundamental contributions. He did not return to the UK but joined Brown University (1953–1967) after which he settled at *the Centre for the Application of Mathematics* at Lehigh University (1967–1980). His works are marked by excellent applied mathematics, a clear overall vision of the field, and a sober style of writing (compared to Truesdell's grandiloquent style). In social contacts he had a sure sense of his own remarkable achievements, a specifically British sense of humour, a certain condescendence for the work of many people, and a devastating critical and sometimes unjust view of other great contemporary scientists. I witnessed his original behaviour on two specific occasions. One was at the Oberwolfach Center of Mathematics in the Black Forest—in the early 1980s—where he forgot that the other people present also needed some food at night! The other was at the IUTAM Colloquium celebrating his own former co-worker, Tony Spencer, where he spent the entirety of his lecture time to expand a sharp critic of the Royal Society (where he was not elected—although he definitely deserved this election) and to describe (his opinion) the ever worsening quality of papers published in the journals of the Society; see in Chap. 5 his critic of the Truesdellian school.

As a useful information we insist that most developments that follow concern **isotropic** bodies and most often **incompressible** ones.

Leslie R. G. Treolar (1906–1985), with a PhD from London (1938) was one of the most active concept builders of the mechanics of rubber as shown by his splendid book (first published in 1944, but with several editions—see the edition of 1975). He worked for the *BRPRA* before WWII and for the *BRRA* after WWII before joining the University of Manchester Institute of Science and Technology (period 1966–1974) as a Professor of Polymers and Fibre Science. He is most well known for his deduction from statistical physics of the following remarkably simple energy expression for an incompressible material such as made of long polymer chains (1946):

$$W = vNk_B\theta\,(I_1 - 3), \tag{3.21}$$

where N is the number of chain segments per unit volume, θ is the thermodynamic temperature, k_B is Boltzmann's constant, v is a coefficient that depends on the details of the assumed molecular model, and I_1 is the first invariant from (3.18). With p an arbitrary hydrostatic pressure accounting for incompressibility, Eqs. (3.19) and (3.20) yield

$$f_\alpha = \frac{\partial W}{\partial \lambda_\alpha} - \frac{p}{\lambda_\alpha},\ \sigma_\alpha = \lambda_\alpha \frac{\partial W}{\partial \lambda_\alpha} - p. \tag{3.22}$$

This checks well with a previous result of Guth and James, and Flory, for the tensile force f needed to extend a rod of unit cross section by a multiplicative factor λ, i.e.,

$$\sigma = 2vNk_B\theta\left(\lambda - \lambda^{-2}\right). \tag{3.23}$$

As to the shearing force necessary to maintain a simple shear of amount γ, it is given by:

$$\sigma = 2\,v\,N\,k_B\theta\,\gamma. \tag{3.24}$$

Experimental data are in fair agreement with the results (3.23) and (3.24). But note that all this applies to homogeneous deformations.

Now we turn to the works of Rivlin and co-workers. With a mathematical vision of the problem, the strain energy for an *isotropic* rubber-like material must be a function of the three basic invariants—reduced to two in the incompressible case—of the deformation [cf. Eqs. (3.17) or (3.18)]. This follows from a celebrated theorem due to Cauchy and reported in Murnaghan (1951). As a first approximation the following strain energy can be proposed:

$$W = C(I_1 - 3). \tag{3.25}$$

Rivlin (1948) called this a *neo-Hookean* form. The reason for this is that in small strains (3.25) reduces to the Hookean (pure shear) form

$$W = \frac{1}{2}\mu(\mathbf{e}_d)^2, \quad \mu = 2C. \tag{3.26}$$

Of course, in (3.25) the coefficient C must be determined experimentally although (3.25) strongly resembles (3.21). The tensorial equation replacing the second of (3.22) reads

$$\sigma = 2C\mathbf{c} - p\mathbf{1}. \tag{3.27}$$

By using such a constitutive relation, Rivlin was able to show that for simple shear of amount γ, not only a shear component of stress develops, but also unequal *normal components* (proportional to γ^2) that are acting in mutually perpendicular directions determined by the direction of shear and the normal to the plane of shear. The difference between two of these normal components is related to the *Poynting effect* mentioned in Sect. 3.1: the simple shear cannot be maintained by shearing surface loads alone (a hydrostatic pressure—*Kelvin effect*—keeping the volume constant is not enough)! Thus the theoretical proof of this effect is a crucial asset for the nonlinear theory based on (3.25) or its generalizations.

The continuum model (3.25) relies on the highly idealized molecular model yielding the energy density (3.21). One is therefore tempted—as was the case for Ronald Rivlin—to strictly apply Cauchy's representation theorem for the scalar valued function W for *isotropic* materials, consider from the start a function $W = W(I_1, I_2, I_3)$, and envisage an approximation accounting, say, for incompressibility. Thus, we have the reduction $W = W(I_1, I_2)$. For instance, for vulcanized rubbers, one can write

$$W = C(I_1 - 3) + F(I_2 - 3), \tag{3.28}$$

where function F is a monotonically decreasing function of its argument in the range of interest. It happens that Mooney (1940), on the basis of some experimental observations, has proposed an energy density of the form:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{3.29}$$

where both C_1 and C_2 are constants. This is a special case of (3.28). Thus model (3.29) is nowadays called a model of *Mooney-Rivlin material*. This has become the most popular model for incompressible rubber-like materials. Just for this Rivlin deserves to be honoured in the Hall of Fame of Elasticity.

3.2.2.1 Approximate Theories

These are theories that appear to be approximations of (3.29) by discarding some degrees in the deformation of line elements and the rotation of volume elements supposed to be small. A theory of this class had in fact been proposed by Murnaghan in a long paper of 1937.

3.2.3 Further Generalizations

3.2.3.1 Anisotropy

A technologically interesting case concerning the industry of tyres is that where rubber is reinforced by fibres. These fibres introduce locally a privileged direction of unit vector **d**. The problem of representation of the corresponding energy was initially considered by Ericksen and Rivlin (1954). The solution consists in applying the representation theorem for the full orthogonal group to a function of the finite strain *and* the vector **d.** This usually results in a function depending on six joined invariants, but one of them is none other than the unit length of **d**, and one has to account for the inextensibility of the fibres. This aspect was thoroughly discussed and applied in various problems by J.E. Adkins, A.C. Pipkin, T.G. Rogers, and A.J.M. Spencer. The last author, Anthony J.M. Spencer (1929–2008)—Tony for his many friends—, a very kind person and an alumnus from Cambridge, worked on his Doctoral degree with Frank Nabarro in Birmingham and then with Ian Sneddon in Keele. During a stay of 2 years at Brown, he established a fruitful co-operation with Rivlin and Albert Green, both original Britons. He rapidly became one of the most important contributors to the theory of invariants and its applications to the mechanics of continua (see his contribution of 1971). After serving 2 years at the Atomic Weapons Research Establishment in Aldermaston (UK), he joined the University of Nottingham. There he succeeded John Adkins as Professor of Theoretical Mechanics and Head of the Department in 1965 until retirement in 1994. His book of 1972 synthesizes his research results in the mechanics of fibre-reinforced materials. Spencer also is the author of a nicely readable little book on general continuum mechanics (1976).

Personal touch: During an extended stay (1985) at Nottingham the author had the occasion to befriend Tony and to witness his talent in the peaceful organization of his department, together with such bright scientists as David F. Parker, Arthur England, and the regretted Tryfan G. Rogers, among others.

3.2.3.2 Generalized Mooney-Rivlin Materials

In modern terms let us introduce a multiplicative decomposition of the deformation gradient **F** that singles out the dilatational contribution $J_F^{1/3}\mathbf{1}$ so that

$$\mathbf{F} = J_F^{1/3}\bar{\mathbf{F}} \textit{ and } \mathbf{C} = J_F^{2/3}\bar{\mathbf{C}}. \tag{3.30}$$

whence $\bar{\mathbf{F}}$ and $\bar{\mathbf{C}}$ refer to the distortional deformation alone. Using this formalism a compressible generalization of the Mooney-Rivlin material is described by an energy of the form:

$$W = \sum_{p,q=0}^{N} C_{pq}(\bar{I}_1 - 3)^p(\bar{I}_2 - 3)^q + \sum_{m=1}^{M} D^m (J_F - 1)^{2m}, \tag{3.31}$$

where the two summed terms represent the distortional and volumetric responses, respectively, and we have set

$$\bar{I}_1 = J_F^{-2/3} I_1, \; \bar{I}_2 = J_F^{-4/3} I_2. \tag{3.32}$$

A compressible Mooney-Rivlin material corresponds to the special case

$$W = C_{01}(\bar{I}_2 - 3) + C_{10}(\bar{I}_1 - 3) + D_1(J_F - 1)^2. \tag{3.33}$$

For $C_{01} = 0$ and $J_F = 1$, we recover the neo-Hookean material. Otherwise, in small strains, we recover the Hookean material with bulk modulus $\kappa = 3\lambda + 2\mu = 2D_1$ and $\mu = 2(C_{01} + C_{10})$. The general formula (3.31) includes a rich number of possibilities.

3.2.3.3 Odgen Model

In 1972, Ogden introduced another general model in the following form:

$$W = \sum_n \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3 \right), \tag{3.34}$$

where α_n may be either positive or negative. Such a model opened up a large variety of possibilities of modelling for rubberlike materials and biological tissues as well. This proved to be extremely useful as compared to previously proposed models especially in biomechanics, e.g., Fung's one given by:

$$W = \frac{1}{2}[C_{KLMN}E_{KL}E_{MN} + c(\exp(B_{KLMN}E_{KL}E_{MN}) - 1], \tag{3.35}$$

where E_{KL} is none other than the finite Lagrange strain of Eq. (3.13), c is a scalar, and C_{KLMN} and B_{KLMN} are tensors of material coefficients. Sometimes, this is referred to as *Fung's elastic material*.

With all these models the elasticity of large deformations is now understood as a rewarding and useful field of study. We may even say that "nonlinear elasticity" was rescued or saved from oblivion by the necessity to study the behaviour of many exploited rubberlike materials of polymeric type and the biomechanics of soft tissues. Examples of applications and other problems are to be found in Ogden (2003) and other contributions in the same volume.

3.2.3.4 Mullins Effect

This effect refers to the stress softening that is observed when a rubber specimen is subjected to cyclic loading. It was apparently first observed by the French scientists Bouasse and Carrière in 1903. But Mullins of the *BRPRA* described it in 1947. In this effect the resulting stress–strain response depends on the previously reached maximum load so that it can be said that the model is stress-history dependent. It resembles the *damage* of solid materials in which the elastic modulus is altered by the previous loading (in general decreases after cycles of loads). It is clear that a model involving an energy depending only on the standard strain invariants is not sufficient to describe this behaviour of the hysteretic type (or *pseudo*-elastic type).

At least one additional scalar parameter must be introduced to describe the effect. Ogden and Roxburg (1999) have proposed such a modelling that includes a damage variable $0 < \eta \leq 1$ with a typical energy density made of two terms:

$$W(\mathbf{F}, \eta) = \eta W_0(\mathbf{F}) + \phi(\eta). \tag{3.36}$$

We refer to these authors and further works by Dorfmann and Ogden for more on this type of approach that strongly resembles the thermodynamic formulation of modern theories of plasticity and damage (cf. Maugin 1999, and Chap. 5 hereinafter).

Note: Most non-French readers—and also most French students **and** professors—have probably never heard of Henri Bouasse (1866–1955). But Bouasse, a brilliant mind educated at the *ENS* in Paris (see Chap. 7 on the French masters) with doctoral degrees in both physics and mathematics, spent most of his career (1892–1937) at the University of Toulouse. He had a kind of *idée fixe*: he wrote a treatise in 45 (yes, forty-five) volumes—each 300–500 pages long-, about all fields of classical physics in reaction against what he estimated (his opinion) the bad quality of the teaching of physics in France, and against the "new physics" (relativity, quantum mechanics). This was called the "*Scientific Library for the Engineer and the Physicist*". His best volumes are those on acoustics and capillarity. This last one still is a useful reference. A few years ago, the present author bought a dozen of never read near mint hardbound volumes of this large opus at the flea market in Paris for less than fifty dollars the lot!.

3.3 Non-Newtonian Fluids

Now we turn to the possible generalizations of the fluid constitutive Eqs. (3.4) or (3.6). According to the historical sketch given by Coleman et al. (1966), the initial general idea of Stokes in 1845 about a viscous fluid constitutive equations would have been of the form (here, incompressible, and using modern notation)

$$\sigma = -p\mathbf{1} + \mathbf{f}(\mathbf{D}), \tag{3.37}$$

where the last term is a (symmetric deviatoric) tensor-valued function of the deviator of the rate of deformation tensor **D.** It is Reiner (1945) who, going beyond the linear Stokes fluid, proposed that **f** be a symmetric polynomial in **D.** He thus started with the assumption that

$$\sigma = \sum_{\alpha}^{n} \beta_\alpha \mathbf{D}^\alpha, \tag{3.38}$$

where the β_α's are functions of the density ρ. Upon using the Cayley-Hamilton theorem to **D** and invoking incompressibility, one finds that (3.38) reduces to

$$\sigma = -p\mathbf{1} + \beta_1 \mathbf{D} + \beta_2 \mathbf{D}^2, \tag{3.39}$$

where β_1 and β_2 are still functions of the remaining invariants $I_2 = trace\mathbf{D}^2$ and $I_3 = trace\mathbf{D}^3$. Apparently independently of Reiner, Rivlin proposed the constitutive Eq. (3.39) that is therefore referred to as that of a *Reiner-Rivlin fluid*. It was already remarked by Reiner that a model such as (3.39) yields not only non-linear viscosity but also normal stress effects. But if the two normal stresses are equal—as shown by applying (3.39) to specific flows—this does no explain the observed Poynting effect. Indeed, Oldroyd (1950) strongly criticized (3.39) as being unable to characterize so-called viscometric flows (See Coleman et al. 1966; Truesdell 1974, for this notion) if only a function of **D** alone is considered. The solution to this problem was given by Ericksen and Rivlin (1954) who showed that the stress had to depend on further time derivatives of the deformation gradient, e.g., the acceleration gradient tensor, etc. To that purpose they introduced what are now referred to as the *Rivlin-Ericksen tensors* noted $\mathbf{A}_n$. These are defined by a recurrence such as

$$\mathbf{A}_1 = \mathbf{D},\ \mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + (\nabla\mathbf{v}).\mathbf{A}_n + \mathbf{A}_n.(\nabla\mathbf{v})^T, \quad n \geq 1. \tag{3.40}$$

At first one may think that the new general constitutive equation should involve a dependence of the Cauchy stress $\boldsymbol{\sigma}$ on the Finger tensor $\boldsymbol{c}$ and the sequence of Rivlin-Ericksen tensors. For an incompressible isotropic *fluid* Ericksen and Rivlin assumed that the deformation itself is not involved and they proposed the general constitutive equation

$$\sigma = -p\mathbf{1} + \mathbf{f}(\mathbf{A}_1,\ \mathbf{A}_2, \ldots), \tag{3.41}$$

where **f** is an isotropic tensor-valued function of its arguments. Rivlin (1956) then showed that this can account for non-equal normal stresses in viscometric flows, thus definitely improving on the Reiner-Rivlin model.

Example of a consistent constitutive equation after (3.41):

$$\sigma = -p\mathbf{1} + \alpha_1\mathbf{A}_1 + \alpha_2\mathbf{A}_2 + \alpha_3\mathbf{A}_1^2,$$

where the α_k's depend on *trace* $\mathbf{A}_1^2$ only.

Remark 3.1 A perhaps nicer definition than (3.40) can be given thus to the Rivlin-Ericksen tensors. Let $\mathbf{F}_t(\tau)$ denote the deformation gradient of the material point **X** at time $\tau \leq t$ relative to time t. Call $\mathbf{C}_t(\tau) = \mathbf{F}_t^T(\tau)\mathbf{F}_t(\tau)$ the corresponding Cauchy-Green strain. Then

$$\mathbf{A}_n := \frac{d^n}{d\tau^n}[\mathbf{C}_t(\tau)]|_{\tau=t}. \tag{3.42}$$

This definition is due to Noll after Green and Rivlin (1957).

Remark 3.2 The idea behind the representation (3.41) is that if, following Oldroyd's view (1950), the stress is to depend on the past history of the deformation gradient, then for sufficiently smooth deformations, Taylor's theorem enables one to express it in terms of the instantaneous values of the deformation gradient and its time derivatives. Equation (3.41) is what results when the instantaneous value of the strain is discarded (no standard elasticity) and incompressibility is imposed.

The last remark relates to the original model proposed by Oldroyd in 1950. There, this author proposed that the stress at time t be determined by an integro-differential tensor equation that relates the stress and the history of the convected metric (i.e., a strain) tensor. The approximations made on this modelling in order to be able to solve representative problems can be shown to provide solutions equivalent to those that would follow in the framework of a second-order approximation of the Rivlin-Ericksen constitutive equation. Although the idea of involving the history of the past deformation in a constitutive behaviour goes back to the hereditary model of Boltzmann (cf. Chap. 2 above), the first comprehensive properly invariant (tensorial) continuum theory involving these integral representations seems to be due to Green and Rivlin (1957). This resulted from a general *functional* relation of the type

$$\sigma(t) = \Phi[\mathbf{C}(\tau); \tau \leq t], \tag{3.43}$$

by means of an approximation by a series of multiple integrals (over past time), assuming the functional Φ to be Fréchet differentiable. This will not be expanded here because the present author thinks that this is not the most convenient mathematical form of constitutive equations in problem solving (see the notion of internal variable of state in Chap. 5). What we note, however, is that the functional relation (3.43) stands for Noll's (1958) definition of a so-called "*simple*" *material*, the "simple" being here rather euphemistic. The fundamental paper on the approximation of time functionals in the framework of so-called "vanishing memory" remains the paper by Coleman and Noll (1961). These functional models allow one to account for the modelling of *stress-relaxation*. This naturally brings us to the following item.

3.4 Rheological Models and Further Extensions

3.4.1 Zener's One-Dimensional Models

In Eqs. (i) and (k) of Chap. 1 we have recalled the Maxwell and Kelvin-Voigt models of linear visco-elasticity. These models account for a relaxation in stress and strain, respectively. They are easily represented by so-called rheological models, spring and dashpot in series, in the first case, spring and dashpot in parallel in the second case. These rheological models—see Zener (1948)—are practically hated and/or ridiculed by purists in continuum mechanics. Nonetheless they have a heuristic value and help the "simple minded" rheologist to build models that can become rather complex by a multitude of arrangements of simple elements—see Chap. 2 in Maugin (1992) and Vyalov (1986). But these models are one-dimensional. To go from them to true three-dimensional models one must have at hand the correct three-dimensional generalizations of the rates of strain and stress. If the former is easily constructed (see above the Rivlin-Ericksen tensors), the latter must be built so as to provide objective (i.e., independent of the observer) entities. The answer to this query was provided by Oldroyd in his landmark paper of 1950, with the notion of "Oldroyd time derivative", although we note that Jaumann (1911) had already proposed a solution of a different type.

3.4.2 Oldroyd's Time Derivative

This notion can be introduced thus. Consider the first of Eq. (3.15) and write the two-ways relations

$$\frac{\mathbf{S}}{J_F} = \mathbf{F}^{-1}.\sigma.\mathbf{F}^{-T}, \;\; \sigma = \mathbf{F}.\left(\frac{\mathbf{S}}{J_F}\right).\mathbf{F}^T. \tag{3.44a}$$

In modern vocabulary these two equations represent the "*pull back*" and "*push forward*" operations of *convection* between actual and reference configurations. They could be represented symbolically by the following obvious notation

$$(\mathbf{S}/J_F) = \overleftarrow{C}[\sigma], \; \sigma = \overrightarrow{C}[\mathbf{S}/J_F]. \tag{3.44b}$$

Then the *Oldroyd convected time derivative* $\hat{\sigma}$ of σ is defined as

$$\hat{\sigma} = \overrightarrow{C}\left[\frac{\partial}{\partial t}\left(\frac{\mathbf{S}}{J^F}\right)\right], \frac{\partial}{\partial t}\left(\frac{\mathbf{S}}{J_F}\right) = \overleftarrow{C}[\hat{\sigma}]. \tag{3.45}$$

The evaluation of $\hat{\sigma}$ in terms of the time derivative of σ and the gradient of the velocity requires only the knowledge of the expressions of the time derivatives of $\mathbf{F}$ and $\mathbf{F}^{-1}$. The result of this easy computation is

$$\hat{\sigma} = \dot{\sigma} - \sigma.(\nabla\mathbf{v}) - (\nabla\mathbf{v})^T.\sigma. \tag{3.46}$$

The acute observer from geometry will notice that this is none other than a *Lie derivative* in following the velocity field in an appropriate four-dimensional space–time. The so-called Truesdell time derivative $\tilde{\sigma}$ of σ is deduced by considering $\mathbf{S}$ rather than $(\mathbf{S}/J_F)$ in the above equations, resulting in the formula

$$\tilde{\sigma} = \dot{\sigma} - \sigma.(\nabla\mathbf{v}) - (\nabla\mathbf{v})^T.\sigma + \sigma(\nabla.\mathbf{v}). \tag{3.47}$$

Note that the exact expression of a Lie derivative depends on the variance of the tensorial object to which it is applied. This leads to a distinction between so-called "*upper*" and "*lower*" Oldroyd derivatives, but we do not need to enter this technical point here. Such derivatives have been used in constructing a lot of different three-dimensional models of non-linear visco-elasticity. It is not our purpose here to discuss this branch of rheological modelling. We prefer to refer the reader to an excellent book on the subject (e.g., Giesekus 1984).

In contrast the Jaumann time derivative (here noted D_J), while also objective, involves only the vorticity Ω—or rate of rotation tensor—rather than the whole velocity gradient. For instance, for a vector $\mathbf{V}$ and a second-order tensor σ we have the following expressions:

$$D_J\mathbf{V} = \dot{\mathbf{V}} - \Omega.\mathbf{V}, \quad D_J\sigma = \dot{\sigma} - \Omega.\sigma + \sigma.\Omega. \tag{3.48}$$

The last of these is equal to the Oldroyd derivative (3.46) up to (objective) terms linear in $\mathbf{D}$. The Jaumann derivative is a special case of *co-rotational time derivative* (Eringen and Maugin 1990, p. 17).

Oldroyd, Truesdell and Jaumann derivatives are of interest not only in rheology but also in the electrodynamics of continua (see Eringen and Maugin 1990, Volumes I and II). However, as a simple example from pure mechanics, we can cite the Truesdell isotropic (grade-zero) model of "hypo-elasticity"

$$\tilde{\sigma} = \lambda(\nabla.\mathbf{v})\mathbf{1} + 2\mu(\nabla\mathbf{v})_S. \tag{3.49}$$

With obvious time rates on both sides of this equation, this looks very much like the time derivative of Hooke's law. But it is "weaker" than the Hooke law since it relates the derivatives of functions rather than the functions themselves, hence the coinage of "*hypo*-elasticity". This kind of relationship between an objective time rate of a stress and the rate of strain was introduced by Truesdell with the hope to include the plastic behaviour in his "rational" scheme of continuum mechanics. This is now obsolete as plasticity now is most often presented within the framework of the thermodynamics of bodies with internal variables of state (cf. Maugin 1992). This is one more favorable argument for introducing objective time derivatives. Indeed, the idea that goes back to Coleman and Gurtin (1967)—and perhaps to Duhem—is to replace the time functional over the past history in Eq. (3.43) by a traditional function dependence on a set (as small as possible) of variables of which the time evolution is constrained by the second law of thermodynamics: they produce dissipation. If these variables are vector or tensor-valued, then we need to account for their dutifully defined objective time rates such as the above given ones. The case where the description of the complex fluid behaviour accounts for the time evolution of the internal microstructure is thus dealt with by Maugin and Drouot (1983)—see also Maugin (1999). We shall return to this type of approach in Chap. 5.

3.5 Concluding Remarks

In this chapter we have rapidly explored the innovative developments concerning the rheology of fluids and the birth of an applicable nonlinear theory of elasticity. The first of these bears traces of the influence of chemical engineers, while the second carries a strong print of British applied mathematics clearly dominated by the emblematic person of Ronald Rivlin. The great names are those of Bingham, Reiner, Treloar, Rivlin, Green, Oldroyd, Truesdell, Coleman and Noll. This adventure is mostly British and American, with the exception of Reiner. Successful developments have been achieved step after step. Some of these steps are now forgotten, e.g., the Reiner-Rivlin model and the hypoelasticity of Truesdell. Still these were useful in producing tools and a way of general thinking that provided sound bases for the whole of continuum mechanics. The most interesting period has been the one spanning between 1940 and 1970. This is not to say that no further progress was achieved afterwards and by scientists in other countries, in particular in Europe. New strongholds of rheological studies have appeared in the 1960–1980s as for instance in the USA with W. R. Schowalter at Princeton, A.S. Lodge in Wisconsin (formerly in Manchester, UK), L.G. Leal in Santa Barbara, D.D. Joseph in Minneapolis, L.G. Larson, in Canada with Pierre J. Carreau at Montreal Polytechnic, with O. Hassager in Denmark, H. Giesekus in Germany, and also in the UK with E.J. Hinch, K. Walters and R.I. Tanner, in Belgium with J.-M. Crochet, in Australia with R.R. Huilgol and N. Phan-Thien, in France with Angles d'Auriac and his Grenoble close co-workers (see Chap. 7 on the French masters), and J.-M. Piau first in Paris-Orsay and then in Grenoble. But most developments there are related to specifying special types of constitutive equations using the tools constructed by the great masters, accounting for a microstructure, solving particular problems and implementing numerical methods, where we acknowledge that the behaviour of non-Newtonian fluids poses difficulties. The European based journal *Acta Rheologica* played an important role in disseminating research, and this probably as much as the *Journal of Non-Newtonian fluid mechanics* or the *Journal of Rheology* in the USA.

On the non-linear elasticity front, we have emphasized again the contribution of Ronald Rivlin and his various co-workers in the USA (e.g., J.L. Ericksen), and in the UK (A.E. Green, A.J.M. Spencer,…). This again looks like a pure Anglo-American adventure. But this does not mean that no deep and fruitful studies were achieved in other places. In particular, we note the formidable achievements by the Italian school in the 1910s (e.g., E. Almansi) and in 1930s with a quantity of contributions in finite deformations, for instance by authors such as P. Burgatti, G. Armani, D. Bonvicini, U. Cisotti, B. Finzi, C. Tolotti, and A. Tonolo (see Truesdell 1952, for full citations) in the formal approach to finite-strain elasticity, and also around Antonio Signorini (1888–1963) and his "allievi", Carlo Cattaneo and Giuseppe Grioli, in the analytic solution of fundamental elasticity problems. Among these we must cite the Signorini's perturbation method in finite elasticity (1930) that allows one to solve a class of traction boundary-value problems, his work on finite-strain thermo-elasticity (1943 and on), and the celebrated "Signorini problem": find the elastic equilibrium configuration of an anisotropic non-homogeneous elastic body resting in a rigid frictionless surface and subjected only to its weight. Truesdell had an immense admiration for these works. Also, we cannot ignore any substantial contribution from the Soviet Union (e.g., by

Novozhilov 1953; Goldenblatt 1962; Lurie 1980, and their students; see Chap. 11 below).

But finite-strain elasticity mostly concerns rubber-like materials and the mechanics of soft biological tissues. No wonder, therefore, that most recent advances were accomplished within these two fields, with a remarkable creativity demonstrated in original developments of the *theory of growth* of biological tissues. These materials, first described as purely elastic, albeit not linear, are now governed by both elasticity *and* growth—both in volume and at the surface—due to the action of nutrients. In this line of fruitful research which goes much beyond the scope of this book, we find the works of S.C. Cowin, R. Skalak, L.A. Taber, E.K. Rodriguez, A. Hoger, M.E. Gurtin, M. Epstein, G.A. Maugin, S. Imatani, S. Guiligotti, A. Di Carlo, D. Ambrosi, L. Preziosi, J.-F. Ganghoffer, E. Kuhl, and P. Ciarletta.

Personal touch. I feel that the present chapter fully exhibits the dynamics of a field in expansion with its shy progresses, trials, errors, missed steps, and breakthroughs, and the fact that new developments rapidly made previous ones obsolete. Thus the Reiner-Rivlin modelling of fluids was rapidly superseded by a new model designed by Rivlin himself so that the former fell into oblivion almost instantaneously (on the historical scale). The "adventure" of "hypo-elasticity" is also exemplary: in Eringen's book of 1962—perhaps written while the field was not ripe enough-, some 50 pages are devoted to hypo-elasticity and its relation to plasticity. This has completely disappeared in the other Eringen text book of 1967. It is with some melancholy that I recollect a period at which I was trying to persuade some colleagues that we did not need hypo-elasticity at all in spite of the sacred words of pundits.

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Chapter 4
The American Society of Mechanical Engineers Spirit

Abstract Of great importance is the innovative and enduring influence that a well-organized professional society may have on the development of a science. This is the case of mechanical engineering with the American Society of Mechanical Engineers. As documented in this chapter, this society provided a specific forum to its members at a spot-on time. It brought a spirit that permeated many American works in continuum mechanics. This may be described as: good modelling (without too much abstraction and unnecessary formalism), good applied mathematics providing real applicable solutions with numbers and curves, and a specific interest in the relationship of these solutions with experimental facts. The prominent figure obviously is the founder of the Applied Mechanics Division of the ASME, Stephen P. Timoshenko. For easiness in presentation, a few most influential centres are highlighted in this chapter. These are Stanford (with Timoshenko himself), the M.I.T (with Eric Reissner), Brown (with William Prager) and Columbia (with Raymond Mindlin). Each of these is most representative of identified avenues of research: advanced strength of materials, mathematics applied to problems of engineering, tremendous and contagious developments in the theory of plasticity, and accurate dynamical theory of structural elements (e.g., plates and shells) and coupled fields (electroelasticity). This was to swarm all over the USA and then the whole world community of mechanics.

4.1 Introduction

One way to find out to which science group a scientist thinks he belongs is to ask him his own answer to that question (that is the most honest way). Another way is to find out in what scientific journals this person publishes most often since he is expecting this place to be the most suitable forum for his work and in turn to receive the best feedback possible. Accordingly, a professor who officially belongs to a department of engineering but publishes essentially in journals with the

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"applied mathematics" label may be called an "applied mathematician". The person herself should agree with that since there is no visible insult in this categorization. In this chapter we focus on a rather large group of contributors to continuum mechanics who are affiliated with the *American Society of Mechanical Engineers* (for short, the A.S.M.E) and who published essentially in the journals of this society, principally, the *Journal of Applied Mechanics* (Transactions of the A.S.M.E). We think this to be justified by the fact that both society and journals carry with them a specific spirit that can be delineated thus: good modelling (without too much abstraction and unnecessary formalism), good applied mathematics providing real applicable solutions with numbers and curves, and a specific interest in the relationship of these solutions with experimental facts. This departs very much from the other view that will be exposed in the next chapter.

Because of the clear-cut US professional context this concerns mostly American *mechanical engineers* or foreign scientists who made a career in the USA. Of course we are above all concerned by the *Applied Mechanics Division* (A.M.D) of the ASME. This division is involved in the fundamental and applied field of mechanics, including solids, fluids and systems. According to the Society's definition,

> it strives to foster the intelligent use of mechanics by engineers and to develop this science to serve the needs of the engineering community. Areas of activity cover all aspects of mechanics, irrespective of approach, including theoretical, experimental, and computational methodology. The field of mechanics, which is the study of how media responds to external stimuli, includes fundamental analytical and experimental studies in: Biomechanics, Composite materials, Computing methods, Dynamics, Elasticity, Experimental Methods, Fluid dynamics, Fracture, Geomechanics, Hydrodynamics, Lubrication, Mechanical properties of materials, Micromechanics, Plasticity and failure, Plates and shells, Wave propagation, other related fields.

Many of these sub-fields fall in the large subject matter of this book.

The possibilities of publishing papers originally fitting the ASME journals were much enlarged in the 1960s with the creation by Pergamon Press (Oxford, UK) of a series of international journals dealing with subjects of engineering sciences, but often at a more formal level (e.g. the *International Journal of Engineering Science*, the *International Journal of Solids and Structures*, the *International Journal of Non-linear Mechanics*, etc.). The *Journal of Elasticity*, founded in 1971 by Marvin Stippes (1922–1979), was also a good forum for the same papers, but in times it turned more to the style that we shall examine in Chap. 5. Also, many American mechanical engineers of British origin continued to publish in British journal of applied mathematics and mechanics. A short history of the AMD-ASME was given by Naghdi (1979).

4.2 The Stanford Connection and Timoshenko

The *ASME* was founded in 1880 explicitly "to provide a setting for engineers to discuss the concerns brought by the rise of industrialization and mechanization". Its *Applied Mechanics Division* is one of the oldest and largest divisions of the

ASME. This division was founded by Timoshenko and others, Timoshenko acting as its first chairman. The relevance of this division to our subject matter is made crystal-clear when we scan the list of recipients of the three famous medals bestowed by this division, the Timoshenko, Drucker and Koiter medals. Here we find, among others, the names of Goodier, Biot, Mindlin, Prager, Koiter, Lee, Eshelby, Ericksen, Naghdi, Argyris, Drucker, Irwin, Rivlin, Budiansky, Fung, Rice, Willis, Tvergaard, Maier, etc. between the years 1957 and 2000. All theses names appear in due place in the present book.

The prominent figure obviously is **Timoshenko** (1878–1972) of Ukrainian origin. Educated as a Railway engineer in St Petersburg, Timoshenko had a rather erratic career in the Russian Empire and Zagreb in Croatia before he moved to the USA in 1922 at the age of 44. At this time he had already contributed to several areas of engineering mechanics: complex structures, computation of eigen frequencies, simple approximate methods, stability of frames, etc. This was already an all round activity in the field. It is in Kiev in the years 1907–1911 that he developed an interest in studies of buckling and that he wrote the first version of his famous textbook on the *Strength of materials* (Timoshenko 1930). On his move to the USA he first worked for the Westinghouse Electric Corporation (1923–1927). He joined the University of Michigan in 1927 and transferred to Stanford in 1936 to stay there until he took a well deserved retirement in Germany in 1960 where his daughter was living. He published books on all aspects of engineering mechanics; these books were translated in more than thirty languages, a record achievement in the field. Among his many books (Timoshenko 1930, 1948, 1951, 1953, 1959, 1961) we like to single out his unique and well documented *History of the strength of materials* (1953). He may be considered the father of modern engineering mechanics in the USA. The Institute of Mechanics of the Ukrainian Academy of Sciences in Kiev is named after him.

Timoshenko was also a tremendous lecturer and supervisor of students' doctoral works (about 40 in the USA). Among those he mentored, we note in Michigan: Donnell (1930), Goodier (1931), Horger (1935), Hetényi (1936), and in Stanford: Lee (1940), Hoff (1942), and Popov (1946). His own most famous contributions are to the beam theory, the deflection of membranes, the bending of plates with Ritz method, and buckling in general. But the variety of his interests is also reflected in the title of the many books he wrote and the domains to which his doctoral students contributed, e.g. Hetényi (beams on elastic foundations, photoelasticity, general elasticity problems), Donnell (bending of beams, bucking of shells, thick plates, the Donnell-Vlasov equation), Goodier (thin-walled structures), Horger (fatigue, photo-elasticity), Hoff (aeronautical structures), Popov (strength of materials). Remarkably enough, four of Timoshenko's students (Goodier, Donnell, Hoff, Hetényi) became chairmen of the AMD of the ASME.

Herrmann (1921–2007), after stays at Columbia and Northwestern University, joined Stanford in 1970 to remain there until his retirement. He perpetuated the ASME spirit although he also created a new journal, the "International Journal of Solids and Structures". A Swiss/American polyglot born in Russia, he was for a long time editor of the English translation of the top Russian journal of mechanics and applied mathematics Prikladnaya Mekhanika i Matematika (P.M.M.). He worked in many fields including shell theory, stability of structures, vibrations of elastic bodies, wave propagation, fracture, and the theory of material forces (configurational mechanics). He was instrumental in bringing to Stanford Juan Simo (1952–1994), a foremost authority on computational mechanics in finite strains, and Alicia Golebiewska (1941–1983), a noted specialist of the theory of defects and configurational mechanics, both unfortunately for a rather short time.

4.3 The M.I.T Connection and Reissner

The MIT in Cambridge, Mass., with its announced banner, is most often seen as *the* temple of technology, what meant mechanical, civil, electrical and chemical engineering in the first half of the 20th century. In time, this has evolved by including new sectors of engineering as they were born, electronics, nuclear engineering, computer sciences, and then embracing all scientific fields to the highest degree. But in the period of interest here—say, when the writer was admitted to, but did not join, the MIT graduate school—"classical" engineering still was an obvious lighthouse. An excellent recruit for the MIT faculty in 1937 was Eric Reissner (1913–1996) who had previously been educated at the Technical University of Berlin. Rather typical of MIT policy, Reissner obtained a professorship in mathematics that he held from 1939 to 1969. This was the vision of mathematics at MIT at the time, by which should be understood "mathematics applied to problems of engineering". The mathematical dexterity and rigour of Reissner perfectly suited this definition. Reissner more than fulfilled the expectations of the faculty board by becoming one of the most productive, successful and internationally recognized engineer-scientist in the field of structural analysis with applications to both civil and aeronautical engineering. In particular, Reissner improved on solutions by Timoshenko in the elastic theory of beams, thin-walled structures, plates and shells. This is illustrated by his theory of shear deformation in plate theory. His works are marked by the exploitation of variational principles (e.g. the celebrated Hellinger–Reissner variational principle that accounts simultaneously for both displacement and stress conditions); cf. Reissner (1953) and an obvious easiness in dealing with complex analytic problems.

Personal touch: During a lecture by the writer in Blacksburg, Virginia, Prof. Reissner intervened publicly to tell that, even with special efforts on his part, he could not understand why the name of Hellinger was associated with his own name for the so-called two-field variational principle. This tells something on the personality of Eric Reissner with whom the author should have worked, had he joined the MIT for graduate studies.

The best known doctoral student of Reissner at MIT may have been James K. Knowles (1931–2009) who is the author of many seminal works in elasticity and phase-transition problems. This he achieved in close collaboration with Eli Sternberg (1917–1988) and younger colleagues, e.g. Horgan and Abeyaratne.

He was a professor at Caltech from 1965 until retirement. Knowles and Abeyaratne are the authors of a remarkable monograph on phase-transitions fronts (2006), a domain to which they contributed with energy and ingenuity in the period 1990–2000 with works on shape-memory alloys, the dynamics of propagating phase boundaries, and the kinetics of austenite–martensite phase transformations. Rohan Abeyaratne (PhD Caltech 1979) joined MIT in the late 1980s and became head of its Department of Mechanical engineering (2001–2008).

4.4 The Brown Connection and Prager

The disproportionately important contribution of Brown University to engineering and continuum mechanics is somewhat of a mystery. This rather small but old university resides in Providence, Rhode Island, in the smallest state of the USA with only about one million of inhabitants. It was founded in 1764, belongs to the Ivy League—along with Harvard, Yale, Princeton—and has the oldest undergraduate program in engineering in this class of colleges. A single division of engineering gathering small departments was created in 1926. Still the engineering faculty remains relatively small with—at the time of writing—about forty full-time members and a body of about one hundred and fifty graduate students. It is complemented by an active division of applied mathematics. But Brown succeeded to be almost the centre of the World for studies on elasticity and plasticity starting in the 1940–1960s. The following list of professors and PhD students at some time at Brown speaks for itself, looking like a real "dream team": Prager, Drucker, Rivlin, Symonds, Sternberg, Kestin, Rice, Weiner, Freund, Clifton, Needleman, Budiansky (PhD 1950), and in applied mathematics dealing with problems of continuum mechanics Gurtin (PhD 1962) and Dafermos.

Prager was the driving force behind all developments in plasticity theory at Brown. He is most well known for his proposal (1949) of the format of plasticity with kinematic hardening (plasticity surface moving with the evolving state of plastic strains; Prager (1955, 1961), his introduction of the notion of locking materials (i.e. materials exhibiting a saturation in strain; Prager (1957)) and his deeply thought books in the field of plasticity and general continuum mechanics (). Drucker (1951) gave his name to the *Drucker inequality* (non negative product of stress rate and plastic strain rate), i.e.

$$\dot{\sigma}_{ji}\dot{\varepsilon}^{p}_{ij} \geq 0, \tag{4.1}$$

and Drucker's stability postulate in the self explanatory form

$$W = \int_0^t \left(\sigma_{ji} - \sigma^0_{ji}\right)\dot{\varepsilon}^{p}_{ij} dt \geq 0, \tag{4.2}$$

where σ^0 is the original state of stresses and $[0,\ t]$ is a closed time-cycle of loading and unloading. Greenberg (1949), also at Brown, was one of those who proposed in 1949 a variational formulation of plasticity (minimum principle exploiting a convexity argument). This was improved by Budiansky and Pearson (1956/1957). Symonds (1951) introduced the ingenious notion of plastic hinges that allows one to treat the collapse of truss structures, and set forth the elements of shake down (or limit) analysis (Drucker et al. 1952). The seminal works of Rivlin were examined in Chap. 3. Gurtin and Sternberg (1962) worked in the linear theory of visco-elasticity (also Sternberg, 1964). Rice (1968) produced his famous works on path-independent integrals and the thermodynamics of plasticity, while Kestin became the most acute observer and critic of the thermo-mechanics of continua (see his celebrated treatise on thermodynamics 1966). Weiner expanded the statistical theory of elasticity (see his wonderful book of 1983). Freund produced his theoretical works on dynamic fracture, while Clifton performed landmark experiments in the same. Constantine Dafermos, a former PhD student of Ericksen, is an applied mathematician specialist of the dynamics of continua and hyperbolic systems. The world reputation—especially Prager's—of the Brown school, a true "Mecca" of plasticity, reached such a level that many foreign visitors came to Brown to get acquainted with the then most recent developments in plasticity (among them, Paul Germain from France in 1952–1953; See Chap. 7); also Hans Ziegler from Switzerland. Those formed in Brown then spread over the USA to continue the successful expansion of the Brown spirit.

4.5 The Columbia Connection and Mindlin

To be able to fully understand the University of Columbia in New York City, it might be requested be a born New-Yorker. Indeed so many people seem to have been born in New York, made their basic high-school training, college and university studies in the same city, and finally ended teaching also there. I even know some of these people who never travelled farther than New Jersey, spending in their youth some week-ends and later on some holidays in Atlantic City (otherwise famous for its Mafia connection and Frank Sinatra). One such character seems to have been Raymond D. Mindlin (1908–1978), although I met him abroad occasionally (CISME Lectures in Udine in 1970). Born in New York, Mindlin obtained all his university degrees (BA, BS, CE, and PhD) at Columbia where he taught from 1936 to 1975, with a few visits in Michigan to attend summer lectures from Timoshenko in the summers of 1933–1935, and a War scientific service at the Applied Physics Laboratory in Maryland in the period 1942–1945.

Mindlin's PhD work published in 1936 was already a masterpiece. He solved in it what is now called the "Mindlin problem": determine (analytically) the stresses in an elastic half-space subjected to a sub-surface point load. This is a generalization of results obtained by Kelvin and Boussinesq in the 19th centuries. It receives applications in geotechnical engineering. The roster of mechanical subjects treated, modelled and/or solved by Mindlin is extremely rich including such different items as: photoelasticity, classical elasticity problems, generalized elastic continua (strain-gradient theory, media with deforming microstructure and couple stresses—see Chap. 13), frictional contact and granular materials, waves and vibrations in isotropic and anisotropic plates (in the so-called Mindlin's theory of plates), wave propagation in rods and cylinders (cf. the Love-Mindlin lateral inertia), electro-elasticity and piezoelectric crystal resonators, crystal lattice theories. His work in vibrations of plates set forth standards in the theory of real and imaginary multiple coupled branches of dispersion. In the 1950s, he wrote on the subject a monograph for the US Army Signal Corps, which monograph stands out as a classic in the field (published in book form and edited recently by J. Yang 2007; Mindlin 2007). His theory of electro-elasticity with polarization gradients (1968–1972; cf. Chap. 12) opened up new horizons in the description of electromechanical couplings in materials that do not allow for the existence of standard linear piezoelectricity (for which no centre of symmetry is allowed). Mindlin has been a chairman of the *AMD* of the ASME. He collected many honours, among them a Presidential Medal of Merit (1946) for his scientific contribution to the War effort during WWII and the National Medal of Science in 1979 in recognition of his all round contributions to American engineering and applied physics. Collected works of Mindlin are given in Mindlin (1989).

One of Mindlin's doctoral students, Tiersten (1930–2006), seems to have inherited some traits of his mentor. Also born in New York, Tiersten also obtained all his degrees (BS, MS, and PhD) at Columbia. But he spent six years at Bell Telephone Laboratories (in nearby New Jersey), before joining the Rensselaer Polytechnic Institute in Troy (still in New York State) in 1967. But the writer succeeded to bring him to Paris for an international conference in 1983. He also had a continued and fruitful co-operation with the US Army Laboratory in Fort Monmouth in New Jersey, with Arthur Balluto, another New Yorker with mobility limited to New York City and the coast of New Jersey. Tiersten contributed to the elaboration of continuum theories exhibiting the role of a microstructure. His most powerful contributions, however, are in the field of piezoelectric couplings (cf. Tiersten 1969), vibrations, and the nonlinear theories of magnetized deformable bodies and electro-elasticity including thermal effects and the case of semi-conductors.

Raymond Parnes (born 1933, PhD 1962 with Mindlin), another New Yorker, who remained in New York and Columbia before moving to Israel became a noted specialist of problems in elasticity. Yih-Hsing Pao (born 1930, PhD with Mindlin in 1989) became in Cornell a well known specialist of physical acoustics and wave propagation in solids, with some excursion in electro-magneto-mechanical interactions. Lee carried the flame of piezo-electricity in the Department of Civil Engineering at Princeton with his PhD students Xanthippi Markenscoff and Jiashi Yang, themselves now professors of mechanics in California and Nebraska, respectively. George Herrmann spent some of his first years in North America in Columbia with Mindlin; he considered himself a disciple of both Prager (who advised him on his doctoral work in Zürich) and Mindlin, in the honour of whom he edited in 1974 a complimentary volume that provides a detailed technical description of Mindlin's scientific contributions by his main co-workers (Herrmann 1974).

Another colleague of Mindlin at Columbia was Boley (born 1924). Although born in Trieste (Italy), Boley may also be considered a New Yorker. Indeed, he obtained all his diplomas in New York City (College of the City of New York, Brooklyn Polytechnic—where Hoff was teaching) and left New York only for a short experience in industry, a short stay at Cornell (1968–1972), and a longer stay at Northwestern (1972–1986) where he served as Dean of Engineering. But he joined Columbia in 1952 until 1968 and back in 1986 until retirement, and then with emeritus status. A good friend but not a co-worker of Mindlin, he is most well known for his contribution to the theory of thermo-elasticity for which he wrote a classic in the field (1960) together with Columbia's colleague Weiner. He founded the journal titled "Mechanics Research Communications" of which the aim remains the rapid publication of short contributions (somewhat in the spirit of "letters").

4.6 Concluding Remarks

Here above we have selected a few places which, in our opinion, are representative of a style of some scientific/engineering developments in the 1940–1960s as they smell good the ASME spirit. This is not to say that these are the only such places. We cannot ignore other institutions where some luminaries contributed to definite advances in the same spirit. To the risk of missing some important places (but this is only due to our ignorance), such places are: Harvard (with Budiansky and Rice), Cornell (with Pao and Moon), Yale (with Onat), Purdue and Princeton (with Eringen), the University of Pennsylvania (with Hashin and others), Lehigh University (with Rivlin, Erdogan and others), the University of Michigan, the University of Chicago and Northwestern University in Evanston (with Achenbach and Bazant, and more recently Belytschko), the Illinois Institute of Technology, the University of Minnesota (with Ericksen), the University of Illinois at Urbana-Champaign, the University of Houston, the Texas A&M University, and, obviously, the University of California at Berkeley (with Naghdi), and Caltech. On perusing the short biographies of mechanicians given in the Appendix the reader should be able to form some good idea of the contribution of these institutions, taking however, account of the great mobility of many researchers (save for the above mentioned New Yorkers). It is the opinion of the present writer that some names already cited in this chapter (e.g. Gurtin) have also contributed to another "spirit", that of the axiomatization line launched in the 1940–1950s by Truesdell. This is examined in the next chapter.

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Chapter 5
Axiomatization and Thermo-Mechanics

Abstract In contradistinction with Chapter 4, the present chapter deals with a more voluntary tendency at axiomatization and abstraction, probably inherited from the early writings of Hamel in Germany and Duhem in France at the dawn of the Twentieth century. Such a program was essentially expanded under the magisterial leadership of Clifford Truesdell in the USA, after his careful historical perusal of mechanics from the origin to the 1940s. The pursued aim was a rational reconstruction of the whole of continuum mechanics in a somewhat Bourbakian style. Impressive encyclopedic contributions by Truesdell, Toupin and Noll were the lighthouses that "illuminated" the world community of mechanics. Simultaneously, a scientific journal (the A.R.M.A.) set forth standards and a definite style. A rather strict thermodynamic frame work was proposed by B.D. Coleman and W. Noll. The notions of fading memory and the required satisfaction of the Clausius-Duhem (thermodynamic) inequality are fundamental ingredients in this presentation. However, attractive as it was, some parts of this true credo imposed too much constraint on the thermomechanical modelling so that some freedom had to be granted and some generalization were necessary in a too much corseting frame work. As dutifully exposed in this chapter, this led to the conception of a rational extended thermodynamics (in particular by I. Müller) as also a less revolutionary but very efficient thermo-mechanics with well-chosen internal state variables.

5.1 Introduction

We have seen in Chap. 2 that the turn of the century around 1900 was pregnant of a mature insight in which the principles of mechanics in general and those of continuum mechanics in particular were discussed and somewhat formalized. Scientists such H. Hertz, E. Mach, H. Poincaré, G. Hamel, and perhaps above all P. Duhem were involved in this critical assessment trying to fix some definite doxa

G. A. Maugin, *Continuum Mechanics Through the Twentieth Century*,
Solid Mechanics and Its Applications 196, DOI: 10.1007/978-94-007-6353-1_5,

for this science in its classical framework. This may sound strange retrospectively because this period also corresponds to the birth of quantum mechanics and relativity, and the early development of the elementary-particle viewpoint, therefore requiring a harsh questioning of the bases of classical mechanics and of the notion of the "continuum" concept. Furthermore, there was an obvious effort to place everything in an historical context so that many authors—but this was already shown by Cauchy in his *Méchanique Analitique* and by Barré de Saint–Venant in his much enlarged translation of Clebsch's book—felt a need to give precise references from the past steps and achievements. **Clifford A Truesdell** (1915–2000), in the immediate post WWII period, engaged in a similar program of reconstruction and historical presentation of continuum mechanics. Duhem, with his complete devotion to the continuum view and his recourse to original sources, whatever the language used, probably is the model that Truesdell wanted to imitate and perhaps surpass definitely. Anyhow both Duhem and Truesdell are considered scientists of the continuum *and* historian of sciences, although Duhem specialized in the oldest texts with a keen interest in medieval science. Furthermore, Duhem was an advocated champion of the thermo-dynamic approach, something that Truesdell strongly encouraged with the works of B.D. Coleman and W. Noll, when he did not himself contribute directly to this trend.

5.2 Truesdell's Approach to Continuum Mechanics

C.A. Truesdell received a broad formation in mathematics and also a flavour if applied mathematics with Harry Bateman. His PhD thesis at Princeton during WWII was not along modern tracks. But he had also acquired a good knowledge of some foreign languages (Italian, French, German), what will help him in his future bibliographical search. In the introduction to a long critical synthesis that was published in 1952, he tells his own story of how he got involved in the field of nonlinear continuum mechanics. He began to study the foundations of continuum mechanics in 1946 and claims that within a few months "he had set the whole field in order, to his own satisfaction". But an editor told him that he had underestimated the work of earlier authors. This convinced him to return to the sources cited by authors of books, and then to the sources of these sources and so on. He also realized that he had overlooked the then recent works by Reiner, Rivlin and others—see Chap. 3. Anyway, the result of this historical search and a special effort at a synthesis was the long contribution published in the first issue (1952) of the *Journal of the Rational Mechanics and Analysis*, a kind of home journal for the Graduate Institute for Applied Mathematics at Indiana University in Bloomington. This opus of 175 pages written at the latest in 1949, was adorned by an incredible list of references—with the oldest references to the seventeenth century, a large number of footnotes, and an index of cited authors spreading over six pages on two columns. Truesdell (see the reprint of the preface in Truesdell 1984a, b) admitted later that he had made many mistakes and overlooked important authors. But this set forth the style and aim of further encyclopaedic articles, a rigour allied to an obsessive mania for citations and correcting other scientists, all this in a rich but sometimes pedantic language. Clearly, Truesdell was trying—and succeeded to a large degree—to supersede his great predecessors, e.g., Lagrange in his book on analytical mechanics with his historical notes, and of course Pierre Duhem.

While he was himself developing some aspects of continuum mechanics [theory of hypoelasticity (1955), study of wave motion (1961) in the line of Duhem and Hadamard, mixture theory or reacting media (1957)], he came to the idea to publish a general exposition of continuum mechanics. This was to be the two celebrated volumes in the *Handbuch der Physik* published by Springer-Verlag and edited by Siegfried Flügge. The agenda for these publications contained a full development of the field equations and their general properties, with an emphasis on principles of invariance for constitutive equations, and accounting for the latest works by Rivlin, Ericksen, Coleman, Toupin, and Noll. Rivlin was already a matured scientist with many seminal contributions (see Chap. 3). Ericksen and Toupin had done their doctoral studies with other people, but Walter Noll came from Berlin and wrote an original ambitious thesis in 1954 (Noll 1955). Volume III/1 of the *Handbuch* (also referred to for short as *CFT*) was written by Truesdell and Toupin (1960) and dealt with general principles with an appendix on tensors by Ericksen. Volume III/3 (for short, referred to as *NFTM*) dealt more closely with the formulation of nonlinear constitutive equations defining classes of ideal materials, and was authored by Truesdell and Noll (1965). The latter volume drew heavily on papers published by the same group of authors in the *Archives for Rational Mechanics and Analysis* that became the regular—but possibly not sufficiently self-critical—forum for the exposition of works by the "Truesdellian" school.

In retrospect the two volumes can be viewed as full of idiosyncrasies, prejudices, and harsh criticism (often expressed in footnotes) of the works by people from other lines of thought, if not condescendence for "lower level" scientists. Rivlin (1984, pp. 2799–2800), with his usual wit, commented on Truesdell's style and influence in the following words:

> In his writing Truesdell evidences a strong taste for the dramatic and so there has been created a fantasy world in which various *savants* produce stream of *principles*, *fundamental* theories, *capital* results, and work of *unusual depth*. No matter that, on examination and stripped of the, often irrelevant, mathematical verbiage with which they are surrounded, they frequently turn out to be known results in a disguise, or trivial, or physically unacceptable, or mathematically unsound, or some combination of these. Nonetheless, they have been widely and uncritically reproduced in the secondary literature and have provided the starting point for many, correspondingly flawed, theses and papers.

Of course we must leave aside the usual exaggerations of Rivlin in the mostly negative appraisal of works by other scientists. But there is some truth in the arguments vividly expressed in this true "war declaration".

Personal touch. I must be quite honest concerning this devastating critic of Truesdell's style in, and contribution to, continuum mechanics. As a young student in Paris, I felt a very strong attraction towards the "rational" presentation of continuum mechanics. It was

> not so much the pedantic and convoluted style that impressed me—because my knowledge of the language was insufficient to capture its exaggerations, subtleties or "defects" (depending on one's view). It was more the—perhaps misleading—feeling that, just like with Bourbaki's style much appreciated in Paris at the time, if you started at the beginning with very few basic definitions and followed rigorously the developments through the two volumes of the *Handbuch der Physik* written by Truesdell, Toupin and Noll, you could in principle apprehend the whole of continuum mechanics. Furthermore, with the Coleman-Noll rational thermodynamics—so much advertised by Truesdell (1969) as the *nec plus ultra* in the field—and the notion of fading memory as the solution of every evolution, I had found my "Holy book". It seemed that you did not need to read anything outside the *Archives for Rational Mechanics and Analysis* (and the last "Truesdellians" are still acting so). This, practically elevated to the status of a religion (but *sect* would be more appropriate), rapidly led its members to pronouncing excommunication and the like, a true church spirit. To be repelled by the sect it was sufficient that you be a graduate student with someone who did not belong to the sect! Furthermore, with more professional experience I realized, like people confronted to Bourbakism in mathematics, that the path of innovation and creation could be outside this credo that was corseting spirits. One had to take some liberties with the so-called "principles" enunciated by the Truesdellian School, in particular concerning thermodynamics, variational formulations, and the fruitful consideration of models. In summary, research is not truly done and written in the Truesdellian style. What remains, therefore, is a collection of useful expressions that have left a print on our mind by repetition and for lack of better expressions, so that we are all more a less a bit "Truesdellian", all the more that Truesdell succeeded in showing us the validity and fruitfulness of a general framework for a global science known as *continuum mechanics*.

In spite of these remarks, the Truesdellian style rapidly influenced the text-book market. This influence is illustrated by the books of Leigh (1968) and Eringen (1962, 1967), although the last author tried to take an original stand. Sometimes, the chapel spirit touches the ridiculous as in the subtitle of Smith's book of 1993 ("after Truesdell and Noll" that sounds like "the gospel according to…").

It is outside the scope of the present work to describe in detail the contents of *CFT*. Albeit not *the* Holy book, this impressive opus stands the passing of time as an unavoidable reference in the field. It indeed covers most of what was achieved in continuum mechanics and electromagnetism from the seventeenth century to circa 1958. It is perfectly Newtonian in the sense that the whole presentation is based on the statement of balance laws. Variational principles—efficient and fruitful as they have been in the overall development of mathematical physics and engineering approaches—are left aside or belittled, if not criticized in many footnotes. They are probably considered as too much "continental" compared to the British Newtonian tradition. This prejudice is not forgivable at a time when it is realized that it left out works that were going to have a glorious destiny (e.g., Eshelby's works and modern computational means). But apart from the introduced notation that became somewhat standard in nonlinear continuum mechanics, *CFT* also provides useful conducting threads in what Truesdell wants to be a "rational" construct, essentially along some basic principles and questions related to invariance. Concerning the first point, we mention the *principle of equipresence*. This is sometimes attributed to M. Brillouin (1900; the father of Léon Brillouin, the latter mentioned in Chap. 7 herein after). Hardly a principle at all (because always negated in the end), this is more like a precautionary measure advising the scientist to consider all possibilities of dependence of state functions and constitutive equations from the start. Concerning invariance, Truesdell (1984a, b) admits that he neglected too much this aspect in his 1952 contribution/review. We must see the influence of Noll behind this new emphasis, whence the *principle of material-frame indifference* (now often referred to as the *principle of objectivity*, and originally called "*the principle of isotropy of space*" according to Noll's comment on Zaremba's work of 1903). It is difficult to trace back the origin of this principle. There seems to be no early questioning about the formulation of elasticity where all fields are defined at the same actual Newtonian time and no time rates are involved. The question may have arisen from the formulation of mechanical behaviours that *do* involve time rates, typically flowing fluids. The present author remembers that he was once told (but by who?) that the great Boussinesq had pondered the question why the constitutive equation for linear viscous fluids involved only the symmetric part of the velocity gradient and not this whole gradient itself. As we know, the vorticity tensor (skewpart of the velocity gradient)—unless appropriately combined with another tensor having the same properties (e.g., in polar fluids)—cannot be there because it would yield a non-invariant constitutive equation under spatial rotations of the observer's frame. McCullagh in the nineteenth century indeed proposed a non-invariant theory—for a rotationally elastic aether—where the stress is directly proportional to the rotation matrix. The question was carefully examined in the early twentieth century by Zaremba (1903) and Jaumann (1911) while dealing with time fluxes of stresses or electromagnetic fields.

Stanisław Zaremba (1863–1942) was a Polish mathematician with a doctoral degree obtained in Paris under Darboux and Picard. His own most famous student was Wacław Sierpinski (set theory, topology, the "Sierpinski carpet" often illustrated in the theory of fractals). Gustav Jaumann (1863–1924) was an Austrian physicist, former assistant of Ernst Mach, with main interest in electromagnetism. They introduced so-called "objective" time derivatives (see Chap. 3). With the energy and justified insistence of Truesdell and co-workers, material-frame indifference (i.e., the requirement that constitutive behaviour should not depend on the observer) has become a tenet of the basic formulation of constitutive equations. Field equations, however, are not "objective" and satisfy only a certain relativity (Galileo's or Einstein's one) for they include inertial terms and possibly external forces. No matter Rivlin's critical comments, sometimes one (in particular the youth) needs clear statements of guiding principles. This is where the experience of the elder is most useful although it should not be abused. Furthermore, in full agreement with Duhem's view of the scientific method, the purpose of the work was not to explain, but to classify knowledge. This is even truer of the second volume, *NFTM*.

The volume *NFTM* is quite different from *CFT* in that it is more thought provoking, including the then last works by Noll and Coleman and their associates. Retrospectively, I think that it came a little too early as leaving not enough ripening time between its publication and the related research. The authors aim at generality with a simultaneous clarification of used terms in spite of the introduction of many neologisms, a trademark of the school. The general approach presented is based upon a few principles, namely, the principles of determinism, local action, and material frame-indifference. Noll (2002) has described the genesis of this volume. Practically, the whole book is based on the notion of *simple materials.* These are materials in which, according to Noll, the stress at a particle (material point) is determined by the cumulative history of the deformation gradient at the same particle. Typically, we will have a stress constitutive law of the *functional* form (compare (3.43))

$$\sigma(\mathbf{X},t) \ = \ \Phi[\mathbf{F}(\mathbf{X},t')], -\infty < t' \leq t. \tag{5.1}$$

Then all three fundamental principles can be expressed in a final and explicit mathematical form. Various materials are then characterized by invariant properties of their response functional. Such notions as "material uniformity", "homogeneity", "solid", "fluid", and "isotropy" are then made precise in mathematical terms. Furthermore, a sufficient smoothness in time of the relevant functional together with its approximation allow one to recover a visco-elastic behaviour of the Boltzmann type. Elasticity corresponds to the functional reduced to a standard function at the time of observation of the particle. "Simple fluids" are simple materials having the maximum possible isotropy group: they are necessarily *isotropic*. This general behaviour allows for the exhibition of stress-relaxation and long-range memory. But for viscometric flows mentioned in Chap. 3, the resulting response functional is shown to manifest itself through three experimentally useful *viscometric* functions only.

The writer thinks that the *NFTM* puts too much emphasis on the so-called *Cauchy elasticity* and *hypoelasticity* that have become obsolete. The former, contrary to Green's elasticity, does not require the notion of potential energy and is therefore constructed mathematically as an a priori invariant representation of the stress tensor in terms of a finite strain – a manner very similar to what was done for nonlinear viscous fluids by Reiner and Rivlin but in terms of a strain rate (Chap. 3). This technique of approach, introduced by Cauchy himself in his original work, suits well the isotropic case. We have already expressed a negative opinion about hypoelasticity. So much good mathematics for practically nothing! That is the destiny of scientific research.

With this the reader will easily understand the enthusiasm of young readers of the 1960s–1970s as reported by the writer in the above personal touch.

5.3 Rational Thermodynamics

For scientists like Duhem or Truesdell who envision phenomenological physics (continuum mechanics, electromagnetism, theory of heat) in its globality, accounting correctly for thermodynamic bases in the exposition of continuum mechanics is more than natural: it is a logical requirement. In this line of view, going further than the *NFTM* requires coupling mechanics and thermal effects to the same degree of generality. This was mostly achieved by Coleman and Noll in a series of papers which, although rigorous from the mathematical view point, may be discussed—or even thought heretic—from the point of view of physical thermodynamics.

There were efforts at combining mechanics and sound thermodynamics before WWII and until the early 1960s. The works of Eckart (1940, 1948) as well as the paper by Eringen (1960), and the general discussion by Bridgman (1945) must be singled out. But it is the school around de Donder in Belgium and the Netherlands that took the lead, finally expressing the so-called (linear) *theory of irreversible processes* (for short *TIP*) in its splendour in the celebrated book by de Groot and Mazur (1962). This was based on the thermostatic definition of temperature and entropy, and a supposedly infinitesimal deviation (hence the "linear") from equilibrium. An axiom of local (equilibrium) state is involved according to which "each part of a material system can be approximately considered at each time as being in thermal equilibrium". Accordingly, all thermodynamic evolutions should be "slow enough". This can be seriously discussed only by introducing and comparing time scales (see Maugin 1999, Sect. 3.2). This approach is supported by a microscopic analysis due to Onsager and Casimir.

The ambition of the school of *rational thermodynamics* is much above that of *TIP*. It takes as a model the rational mechanics of the mathematicians of the eighteenth and nineteenth centuries (especially the French mathematicians Lagrange and Cauchy) and the embryonic thermomechanics of Duhem (1911). It openly ignores, or bypasses, the experience acquired in thermostatics. It basic postulates seem to be that those notions that precisely could be defined only at equilibrium in thermostatics, exist a priori for any thermodynamical state whatever, even largely outside equilibrium. This attitude belongs to an axiomatic trend with a typical list of axioms giving the appearance of pure mathematics (this is described as a rigidly fixed doxa in Chap. 12 of Ignatieff 1996). The notions of temperature θ ($\theta > 0, \inf \theta = 0$) and entropy η (per unit mass) are a priori granted to any state, so that the formal bases of rational thermodynamics are the a priori statement of the second law and the usual first law, the former in the following global form (body B with regular boundary ∂B equipped with unit outward normal $\mathbf{n}$):

$$\frac{dS}{dt} \geq \int_B \rho s dB - \int_{\partial B} \mathbf{s}.\mathbf{n} da. \tag{5.2}$$

Here,

$$S = \int_B \rho \eta \, dB \tag{5.3}$$

while it is assumed that the entropy source s and the entropy (in)flux $\mathbf{s}$ are given by

$$s = h/\theta, \mathbf{s} = \mathbf{q}/\theta, \tag{5.4}$$

where h is a possible heat source per unit mass, and $\mathbf{q}$ is the heat (in)flux. Equation (5.2) is the differential form in time of the inequality of Clausius—who postulated it in 1854 for a vanishing right-hand side.

For a standard thermomechanical (simple) material the first law of thermodynamics combined to the balance of kinetic energy yields the *equation of internal energy* in the local form

$$\rho \dot{e} = \sigma : \mathbf{D} + \rho h - \nabla . \mathbf{q}, \tag{5.5}$$

while (5.2) provides the local inequality

$$\rho \dot{\eta} \geq \rho (h/\theta) - \nabla . (\mathbf{q}/\theta). \tag{5.6}$$

The combination of (5.5) and (5.6) leads to the *Clausius–Duhem inequality* as popularised by rational thermodynamics:

$$-\rho \left(\dot{\psi} + \eta \dot{\theta} \right) + \sigma : \mathbf{D} + \theta \mathbf{q} . \nabla \left(\theta^{-1} \right) \geq 0, \tag{5.7}$$

where $\psi = e - \eta \theta$ is the free (Helmholtz) energy per unit mass.

Then it is thought that the whole *past history*—i.e., the collection of all values taken by the fields of motion and temperature at a current point $\mathbf{x}$ for all past times and the present time—determines the thermo-mechanical behaviour at the present time, the only constraint being that the inequality (5.7) be respected. An obvious remark is that the recent past influences more the present state of a body than its distant past does. This is rigorously translated into the axiom of *fading memory* due to Coleman and Noll (1961; Dill 1975). The a priori considered constitutive equations are in the *functional* form (compare to (5.1))

$$\sigma(\mathbf{X}, t) = \Phi[\mathbf{F}(\mathbf{X}, t - s), \theta(\mathbf{X}, t - s)], s \in [0, +\infty) \tag{5.8}$$

for so-called *thermodynamically admissible processes*. This can also be generalized to deformable electromagnetic materials (see Eringen and Maugin 1990, Chap. 13; Fabrizio and Morro 2003). The application to thermo-elasticity in finite strains is reported in all modern books on continuum mechanics (e.g., Maugin 1999, Chap. 3).

This kind of approach due to Coleman and Noll was presented by Truesdell in a militant monograph (Truesdell 1969, 1984b) as also by other authors (e.g., Day 1972; Coleman 1964; Coleman and Owen 1974). It radiates a wonderful elegance to which it is difficult to resist intellectually. But it was seriously criticized by the tenants of *TIP* for its lack of touch with the experimental definition of concepts such as temperature. An international meeting held in Bussaco (Portugal) in 1973 to discuss the various trends in continuum thermodynamics was—I was told—snubbed by the invited members of the Trusdellian school (cf. Domingos et al. 1973).

The author's restrictions on the above view also relate to the special working hypotheses encapsulated in Eqs. (5.4–5.8). For instance:

1. The second of (5.4) is not generally true as is known in the theory of mixtures: the entropy flux is not always reducible to the ratio of the heat flux to the temperature;
2. Equations (5.1), (5.5) and (5.8) apply as such only to "simple materials". The reason for this is that there exists an intimate relationship (duality) between the existence of only one type of internal forces—the stresses—and the notion of simple materials (or a theory of first-order gradient as described by the author in more general framework; cf. Maugin 1980; see also Chap. 13 herein after). A strict application of the form (5.7) of the Clausius–Duhem inequality to other constitutive equations different from (5.8) led some authors to conclude to the thermodynamical inadmissibility of models that should necessarily exist. This is like negating physical evidence and logics on the basis of wrongly applied principles;
3. The notion of constitutive functional, although aesthetically pleasing, requires knowledge of the whole past state and is not the most manageable and realistic one in many applications save perhaps in some problems of biomechanics as emphasized by Epstein (2012, Chap. 3). In spite of this obvious difficulty, the Truesdellian School tried to include in this functional approach rather singular behaviours such as plasticity.
4. The notion of simple fluid with its reduction to a necessary isotropy may be misleading in fluids with an internal structure (e.g., liquid crystals).

As a consequence of the recognition of these shortcomings and intrinsic limitations, some "de-Truesdellisation" started to be manifested along several lines by more pragmatic scientists, having for main object to relax away from a strict orthodoxy—that would have frozen the field if not counteracted in time—while of course keeping the assets brought in by this approach, in particular an efficient methodology. This is what is examined in the few next sections.

Note: Bernard D. Coleman (born 1930) originally is a chemical engineer with a PhD from Yale obtained at the early of age of 24. Therefore, he should be a "Gibbsian" (remember Josiah W. Gibbs (1839–1903), one of the greatest and most original American scientists, was a graduate of Yale and one of the first PhD's in Engineering in the USA. He spent his whole scientific career at Yale after a three-year stay in Europe. He was responsible for the mathematical development of physical chemistry and for the invention of dyadic algebra, one of the paths to modern vector and tensor notations). Coleman's inclination towards more abstract formulations is probably due to Truesdell's and Noll's influence. Walter Noll (born 1925) is a German/American mathematician originally formed at T.U. Berlin and in Paris (where he was injected the "Bourbaki" virus). His thesis (Noll 1955) on the "continuity of states" in Bloomington, Indiana, was written during his stay there with C.A. Truesdell. It is clear that he was more familiar than Truesdell with modern mathematics (in particular topology and geometry). This he demonstrated with maestria in his mathematical theory of simple continua (Noll 1958, 1972) and in his approach to material uniformity and inhomogneities (Noll 1967). He is very much responsible for the extensive use of an intrinsic notation avoiding tensorial indices in contemporary continuum physics.

5.4 Deviation from the Standard Definition of the Entropy Flux

Further improvements of rational thermodynamics can be found in the works of Ingo Müller (1973, 1985), and in a remarkable paper by Liu (1972). First it was observed that the general notion of *coldness*—considered by several authors including Müller—as an integrating factor for entropy is essential. This reduces to the reciprocal of thermodynamic temperature with an appropriate scaling of temperature. Second, as the special choice $(5.4)_2$ is contradicted by kinetic theory (*dixit* Müller), some freedom must be left to the expression of the entropy flux **s** by letting it differ from the ratio of heat flux to temperature. That is, the second of (5.4) can be rewritten as

$$\mathbf{s} = \theta^{-1}\mathbf{q} + \mathbf{k}, \tag{5.9}$$

where the extra entropy flux **k**, in a phenomenological approach, has to be determined by a constitutive equation. Accordingly, it theoretically varies in form from one material to another. But it is possible that it proves to be nil after some tedious and cumbersome algebra. This is not the case for systems that exhibit a diffusion-like behaviour. As to Liu, he introduced the spot-on idea that in applying the condition of thermodynamic admissibility (satisfaction of the Clausius–Duhem inequality for all thermo-*dynamical* processes), it is generally thought necessary to account for the field equations (these are the conservation of mass, the balance of momentum, and the energy equation). This can be done by considering them as mathematical constraints and introducing them together with appropriate Lagrange multipliers in the Clausius–Duhem inequality. It is true, however, that in many cases this astute but cumbersome manipulation results in very little changes. Note that with (5.9) the C-D inequality (5.7) is replaced by

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + \sigma : \mathbf{D} + \nabla.(\theta\mathbf{k}) - \mathbf{s}.\nabla\theta \geq 0. \tag{5.10}$$

A possible expression for **k** follows from the above Liu procedure together with the other generalized constitutive equations.

5.5 Extended Thermodynamics

In order to go beyond standard rational thermodynamics, *extended thermodynamics*—essentially the child of Ingo Müller (cf. Müller and Ruggeri 1993) but also expanded by Jou et al. (1993)—envisages the consideration of the usual *dissipative fluxes* (e.g., viscous stresses, heat flux, electric conduction current) as additional independent variables. As a direct consequence the entropy itself becomes a function of these fluxes so that entropy will deviate from its thermostatic definition $\eta = \eta_s := -\partial\psi/\partial\theta$. This is in agreement with an early proposal made in 1953 by Machlup and Onsager (1953). Moreover, the dissipative fluxes will themselves satisfy *evolution-diffusion equations* inspired by higher-order kinetic-theory developments. This thoughtful interaction between two different levels of description of physical reality is original but would be rejected by tenants of pure phenomenology. However, this new thermodynamic approach is certainly comforted by the fact that it allows a satisfaction of causality, resulting in the end in hyperbolic systems of equations with a limited speed of propagation (see Müller and Ruggeri 1993). As an example, the modified heat-conduction law by Cattaneo in 1948—see Equation (m) in Chap. 1—that involves a relaxation time, enters this framework. But this extended thermodynamics bears such a strong print from the kinetic theory that it is difficult to apply it to complex solid-like behaviours exhibiting hysteresis such as plasticity. Another avenue must be opened to cope with such cases. The thermodynamics with internal variables of state seems to be the looked for framework.

5.6 Thermodynamics with Internal Variables of State

Rational thermodynamics with its functional constitutive equations of the type (5.8) and an a priori requirement for the knowledge of the whole past history is not a very convenient tool both in computations and in confrontation with experimental data. The *thermodynamics with internal variables of state*, while presenting the least deviation from the classical theory of irreversible processes, proposes to replace the functional dependence by that on a *finite* set of variables that satisfy evolution equations constrained by the second law. The idea of such variable can be traced back to Duhem (1911; according to Truesdell), but probably more to Bridgman (1945). A modern introduction is due to Coleman and Gurtin (1967). The best analyst of this thermodynamics has been Kestin (see, e.g., Bataille and Kestin 1979). Maugin and Muschik (1994) have specified and analysed many of its facets. A recent book on this thermodynamics is one by the author (Maugin 1999). Its general features can be presented as follows.

Internal variables of state are introduced in addition to the usual observable variables of state (e.g., deformation, temperature). They are supposed to account for the complex internal microscopic processes that occur in the material and manifest themselves at a macroscopic scale in the form of *dissipation*. They should not be mistaken for internal degrees of freedom that possess their own dynamical equations (see Chap. 13). Being of a pure dissipative nature, their time evolution is constrained by the second law of thermodynamics. Being internal and not observable—although certainly identifiable by a gifted physicist—they do not appear in the usual statement of the first law of thermodynamics as they are not directly acted upon by bulk or surface actions. But they do evolve under the action of external loads as a result of complex processes that follow from a re-distribution of internal forces (internal rearrangements of matter, etc.). For instance, the local density of dislocations (responsible for the macroscopic phenomenon of plasticity) evolves when a system of standard forces (tractions) is applied to a body.

For the sake of illustration but with a minimum of formalism, we note χ the set of observable variables and α that of internal variables. In all generality, the dependent variables (e.g., the stress σ) become simultaneously functions of the values of both χ and α. We can envisage the law of state (constitutive equation, here mechanical):

$$\sigma = \bar{\sigma}(\chi, \alpha), \tag{5.11}$$

evolution equation:

$$\dot{\alpha} = f(\chi, \alpha) + g(\chi, \alpha)\dot{\chi}; \tag{5.12}$$

We may suppose that an instantaneous time variation of χ does not cause an instantaneous variation of α (this is a question of time scales), so that we can set $g = 0$, reducing (5.12) to the equation

$$\dot{\alpha} = f(\chi, \alpha). \tag{5.13}$$

This must be compatible with the second law. Simultaneously, discarding thermal effects, the free energy density has the form

$$\psi = \bar{\psi}(\chi, \alpha). \tag{5.14}$$

The exploitation of the remaining standard Clausius–Duhem inequality (5.7) yields the constitutive equation (5.11) and the remaining dissipation as

$$\sigma = \frac{\partial \bar{\psi}}{\partial \chi}, \tag{5.15}$$

and

$$A\dot{\alpha} \geq 0, A = -\frac{\partial \bar{\psi}}{\partial \alpha}, \tag{5.16}$$

where A is the thermodynamic force associated with α. Now the constitutive theory is closed by proposing a relationship between the two members of the product in the left-hand side of the first of (5.16). If there exists a potential of dissipation Φ in the manner of Rayleigh (but expressed in terms of the force A), we will have the required evolution equation for α in the form

$$\dot{\alpha} = \frac{\partial \Phi}{\partial A}. \tag{5.17}$$

Contrary to Rayleigh's dissipation potential, we may select the degree of homogeneity of the function Φ at our convenience in so far as the inequality $(5.16)_1$ is satisfied. This is what allows one to construct a good theory of plasticity when the dissipation potential is homogeneous of degree one only. This provides a strain-rate independent plasticity, a valid approximation for many materials. In this case (5.17) is made more explicit as

$$\dot{\alpha} = \dot{\lambda} \frac{\partial F}{\partial A}, \tag{5.18}$$

where $\dot{\lambda}$ is a so-called plastic multiplier and $F(A) = 0$ is the equation of the elasticity surface delimiting the convex domain C in A space. The notation with superimposed dots in both sides of (5.18) emphasizes the absence of time scale in the process. Here we have

$$\dot{\lambda} \geq 0 \quad \textit{if } F = 0 \textit{ and } \dot{F} = 0, \tag{5.19a}$$

$$\dot{\lambda} = 0 \quad \textit{if } F < 0, \quad \textit{or } F = 0 \textit{ and } \dot{F} < 0. \tag{5.19b}$$

This can also be expressed in the formalism of *convex analysis* (cf. Rockafellar 1970)

$$\dot{\alpha} \in N_C(A) \Rightarrow \dot{\alpha}.(A - A^*) \geq 0, \quad \forall A^* \in \ C. \tag{5.20}$$

The first of these reads: "the time rate of α belongs to the cone of outward normals to the convex set C". This is a *law of normality* which dictates the direction of the time evolution once the elasticity limit is reached. With $\alpha = \varepsilon^p$ and $A = sigma$ we recognize the usual case of plasticity. This general formulation allows for the existence of angular or apex points along the surface (or limiting curve in a plane representation) of C. This is the case of Tresca's criterion of plasticity. The second of (5.20) is a *variational inequality* which means that plastic dissipation possibly occurs only once the elastic limit is reached.

This is documented at length with the accompanying mathematical framework in the author's book of 1992. There, is in fact applied the theory of so-called *generalized standard materials* due to Halphen and Nguyen Quoc Son (1975) where the existence of a convex dissipation potential is assumed together with the convexity of the energy density. In the framework delineated by these authors, some of the essential theorems of plasticity theory such as Drucker's inequality and Drucker's stability postulate—Eqs. (4.1, 4.2)—follow almost automatically. This is also the case of the Ilyushin postulate (see Chap. 11). The theory of fracture itself can be put in a similar general abstract framework (Sect. 7.7 in Maugin 1992).

In contrast to the functional theory embedded in the rational thermodynamics of Coleman and Noll, the theory with internal variables of state yields a mathematical problem in terms of evolution equations. This is much more adapted to numerical computations as it directly fits techniques and a good mathematical frame known in other fields such as nonlinear programming. This explains the remarkable efficacy and success met by this theory within the last 30 years. The applications are multiple in plasticity, visco-elasticity, visco-plasticity, damage, phase-transformation theory, electromagnetic hysteresis, etc. From the formal point of view of thermodynamics, as remarked before, this is the least deviation from the well understood theory of irreversible processes. What must be amended is the axiom of local (equilibrium) state. This was formulated in a persuasive manner by Bataille and Kestin (1979) by introducing the notions of *accompanying processes* and *local accompanying states* (for each instantaneous value of the internal variables) to replace the axiom of local state (see Chap. 4 in Maugin 1999). This fruitful idea may be originally due to Meixner (1972). The last question concerns the number of internal variables to be considered. For obvious convenience, it should be kept as small as possible and, although not directly controllable, these variables should be easily identifiable and measurable by a gifted experimentalist. A great asset of the approach is that in general a few internal variables or a well chosen tensor one is sufficient. Other views on the theory of internal variables of state have been given by Day (1976), Grmela and Öttinger (1977), Maugin and Drouot (1983) and Maugin (1990)—the latter with a possibility of diffusion yielding evolution-diffusion equations for the said variables and a consistent re-definition of the entropy flux vector.

At the time of writing of this book, the ultimate progress in a rational approach to continuum thermodynamics in the presence of dissipative processes seems to be along a line expanded by authors such as A. N. Beris and B. J. Edwards (see their book of 1994), Grmela (1984), and Hans C. Öttinger (Zurich; see his book of 2005). These authors succeeded in introducing a kind of bracket formulation of kinetic equations (for both observable and internal variables) with many applications to complex non-Newtonian fluids. The last version of this approach is called *GENERIC* (short for "**G**eneral **E**quation for the **N**on-**E**quilibrium **R**eversible-**I**rreversible **C**oupling"). In a nut-shell this is represented by a general evolution equation of the type

$$\frac{dx}{dt} = L(x).\frac{\delta E(x)}{\delta x} + M(x).\frac{\delta S(x)}{\delta x}, \tag{5.21}$$

where (Öttinger 2005, p. 11) "x represents a set of independent variables that are required for a complete description of the system outside equilibrium, the real *functionals* $E(x)$ and $S(x)$ are the total energy and entropy—generating functions—expressed in terms of the state variables x, and $L(x)$ and $M(x)$ are the Poisson and friction matrices representing geometric structures and dissipative material properties in terms of linear operators". The fundamental point here remains the choice of the most representative variables x, because, whereas the formalism (5.21) is canonical, there is no universal set of non-equilibrium variables. A statement equivalent to (5.21) is one in terms of appropriately defined brackets where, with

$$\frac{dA}{dt} = \frac{\delta A(x)}{\delta x}.\frac{dx}{dt}, \tag{5.22}$$

$$\frac{dA}{dt} = \{A,\, E\} + [A, S]. \tag{5.23}$$

Here the first bracket is an antisymmetric Poisson bracket, while the second is a symmetric "dissipative" bracket. Accordingly, the first and second laws of thermodynamics read

$$\frac{dE}{dt} = \{E, E\} = 0, \frac{dS}{dt} = [S, S] \geq 0. \quad (5.24)$$

One difficulty is in the definition of the second bracket. We do not pursue the interesting details of this elegant and powerful approach that brings irreversible thermodynamics in the domain of symplectic geometry (see Öttinger's book 2005).

5.7 Conclusions

The contents of this chapter, somewhat an intermezzo between the elasticity of Chap. 3 and the return to elasticity and applied plasticity in Chap. 6, probably helped the author, as also the reader, to ponder the question of the axiomatization and modern mathematization of a field of physics. Such an endeavour was among the list of fundamental problems proposed by David Hilbert in 1900 (this was the sixth problem posed at the second international congress of mathematics) and obviously a wish of Pierre Duhem. Truesdell and his co-workers claim to have fulfilled such a programme for continuum physics in the Newtonian-Galilean framework. Some of them (Toupin, Noll) also tried their hands at the relativistic framework, but without much response and enthusiasm from relativists. What the author gathers from the above is the usefulness of such an axiomatization when a field has quietly reached a stable state of progress and no new paradigm seems to appear. This is more difficult and perhaps not justified when the field still is in expansion and presents great expectations of further developments. Otherwise, fixing the framework at some definite time and deciding that everything that does not fit in is pure heresy, may be the source of a true intellectual terrorism. This happened with some "Truesdellians". As a consequence, on the one hand Truesdell and others succeeded in framing the whole of continuum thermodynamics in an attractive logically built body of knowledge. On the other hand, a reaction from many engineers who felt insulted and belittled, has been to fully separate again fluid and solid mechanics in two different subject matters. This is also an intellectual faux pas. Other general views on the axiomatics of continuum thermodynamics are given by Gurtin and Williams (1971), Hutter (1977), Muschik (1986) and Ericksen (1991).

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Chapter 6
The British School of Elasticity, Plasticity and Defects: Applied Mathematics

Abstract Although pertaining to specific aspects of the development of continuum mechanics in the period of interest, it happens that this coincides with a technical expertise in applied mathematics particularly well cultivated in the United Kingdom, hence, an unavoidable regional bias in spite of the international nature of science. The prevailing influence of some institutions such as the University of Cambridge is obvious, while, unexpectedly, research fostered by technical problems met during the Second World War, also had a strong influence on the selection of projects. A recurring theme is a specific interest in mathematical problems posed by the theory of elasticity, no doubt a consequence of the enduring influence of past "elasticians" of great mathematical dexterity among whom A.E.H. Love must be singled out. A clear-cut emphasis was placed on problems dealing with the existence of field singularities such as happens with cracks, dislocations, and other material defects. Here great names are those of A.A. Griffith, Ian Sneddon, "Jock" Eshelby, and A.N. Stroh. Simultaneously, an "immoderate" but fruitful taste for problems of elastic wave propagation with applications in both mechanics and geophysics was demonstrated and still remains a subject of attraction. Furthermore, a geometrical approach to defect theory was proposed by a group around Bruce Bilby, while Rodney Hill produced among the most powerful results in plasticity theory and homogenisation procedure. Still it is the mathematical dexterity and elegance allied to a deep physical insight that best characterizes most of these works.

6.1 The General Landscape

This chapter is devoted to the aspects of the theory of elasticity cultivated in the United Kingdom during the twentieth century at the exclusion of those concerning nonlinear elasticity (e.g., the works of R. S. Rivlin, A. E. Green, A.J.M. Spencer, and R.W. Ogden) which have already been examined in Chap. 3. That means that we are here concerned by linear elasticity and its allied problems such as the

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existence of singularities and cracks, the question of defects such as dislocations, some problems of wave propagation, and the transition to a plastic regime of deformation. Practically no constitutive modelling and little thermodynamic bases are involved. This agrees well with a pragmatic and non-speculative attitude entertained by a large part of the British school of continuum mechanics. It fits well in the British tradition of applied mathematics which, in the past, essentially dealt with the solution of partial differential equations with its paraphernalia of special functions and the like, what requires a high dexterity for which the British have not been surpassed. Of course, the period concerned is pre-computer time. Some scientific centres emerge naturally in this landscape, as the traditional universities of Cambridge and Oxford, but also Bristol, Sheffield, Nottingham, Manchester, Keele, and Glasgow and Edinburgh in Scotland. As to the top individualities, they certainly are Lord Rayleigh, A. E. H. Love, G. I. Taylor, I. N. Sneddon, and Rodney Hill; and in more recent times, A. J. M. Spencer, P. Chadwick, J. D. Eshelby, A. N. Stroh, J. R. Willis, R. Knops, and J. Ball.

6.2 The Tradition of Applied Mathematics, Elasticity and Waves

Perhaps that the best characterization of the involved style of applied mathematics is illustrated by the book written by the Jeffreys—husband and wife—in 1946 on the methods of mathematical physics (cf. Jeffreys and Jeffreys 1946). It provides the essential elements that an applied mathematician working in continuum mechanics and geophysics must grasp. Sir Harold Jeffreys (1891–1989) was himself a famous geophysicist and astronomer. He was educated in Newcastle-upon-Tyne and Cambridge. He spent his entire scientific career in Cambridge, teaching there in succession mathematics, geophysics, and astronomy. This emphasis on geophysics may sound strange to the reader. But it happens that the other three contributors that we mention now are also related to geophysical studies to a greater or lesser extent.

One is Lord Rayleigh (J. W. Strutt 1842–1919), also a Cambridgian, Nobel Prize in Physics in 1904, author of the celebrated treatise on the "Theory of sound", who gave in 1887 the first *surface wave solution* in elasticity (propagation at the surface of a free semi-infinite, linear elastic isotropic half space). The author recommends the reading of this beautiful paper (Rayleigh 1887) to all readers as a perfect example of how a scientific paper should be written. Among his many works that cover all fields of the physics of the time, Rayleigh also contributed to the theory of vibrations of various structures (strings, bars, membranes, plates, shells). Bear his name the "Betti-Rayleigh" reciprocity theorem (exploiting generalized forces and generalized coordinates) and the "Rayleigh–Ritz" method for computing frequencies from energy considerations. The connection with geophysics is that elastic surface waves play a fundamental role in earthquakes.

This is comforted by the second character, A. E. H. Love (1863–1940), another Cambridgian, but who spent his whole academic career at Oxford. He may be considered both an elastician and a geophysicist, but above all an applied mathematician. He is the author of two landmark treatises. One is none other than the now classic "Treatise on the mathematical theory of elasticity" (First edition published in 1892–1893 in two volumes), and the other is the no less classic work on the structure of the Earth entitled "Some problems in geodynamics" published in 1911. The first of these, reaching four editions, set up standards for several decades in the theory of elasticity. The second (Love 1911), obviously related to geophysics, introduces the notion of "Love" surface waves: these are elastic surface waves with a so-called shear-horizontal polarization that can propagate at the surface of a body made of an elastic half-space on which a layer of "slower" elastic material is superimposed and perfectly glued. These waves are said to be the most destructive ones for structures (buildings, etc.) during an earthquake.

Along the same line, a third actor is also a Cambridgian and a geophysicist, Robert Stoneley (1894–1976) who became one of the most famous British seismologists. In one of his first papers (Stoneley 1924) he considered the possible propagation of waves guided by the plane interface between two welded elastic solids. He proved the existence of this possibility for certain ranges of elastic coefficients of the two media. These waves are rightly called "Stoneley waves". In his successful career he wrote, either alone or in collaboration, many papers on surface waves and some on micro seisms and tsunamis (sea waves produced by earthquakes).

Two remarks are in order concerning surface waves in elasticity. First, at the time of discovery of the exemplary surface waves of the Rayleigh, Love and Stoneley types, only applications to geophysics could be conceived. The last type, Stoneley's, was in fact first thought to be only a mathematical curiosity. But now we have a whole group of applications in nondestructive testing devices in physical acoustics (detection of material defects and obstacles on the path of the waves) and surface-wave devices in the treatment of electro-mechanical signals (e.g., in convolver and correlator "machines" as used in the treatment of RADAR signals, cf. Maugin 1985). Second, the dynamical properties of propagating surface elastic waves—e.g., the Rayleigh wave velocity—are also characteristic properties in dynamical fracture, i.e., the propagation of cracks, our next object of attention.

6.3 Cracks, Always Cracks

6.3.1 Griffith's Theory

The English engineer **Alan Arnold Griffith** (1893–1963) can be called the father of the modern theory of fracture. He was educated (B.Eng., M.Eng, D.Eng.) at the school of Mechanical Engineering of the University of Liverpool, and then had a very fruitful career in aeronautics. Before he got involved in the aerodynamic theory of turbine design—that was to lead to the successful technical development of the jet engine—he achieved two works of remarkable consequences. One was the conception in 1917—together with G. I. Taylor (the future Sir Geoffrey; see Griffith and Taylor 1917)—of the use of soap films in solving torsion problems, providing a membrane analogy as a method of solution of complex elasticity problems. The other paper published in 1920 single-handedly opened the way for a true mechanical and thermodynamic theory of fracture in the elastic regime (so-called brittle fracture). It has become a "Metallurgical classic" (Griffith 1920).

According to G. I. Taylor's words (cited by Gilman 1998) that we paraphrase: Griffith

> realized that the weakening of a material by a crack could be treated as an equilibrium problem in which the reduction on strain elastic energy of the material, when the crack extends, could be equated to the increase in surface energy due to the increase in surface area at the crack.

The beauty of Griffith's theory stems from the fact that "it uses the elegance of a thermodynamic argument to deal with the singularity that appears at the tip of a crack in linear elasticity" (Taylor's words). But Griffith's theory, in its original form, only applies directly to truly brittle substances such as hard non-metallic glasses. Further improvements had to account for a possible anisotropy in elastic behaviour, an application to inelastic materials, dependence on time, stress state, and other factors. Anyway, with the efforts of George R. Irwin (1907–1998) —at the US Naval Research Laboratory in Washington—and others, it became the basis of a new branch of engineering mechanics called "fracture mechanics". A thermo-mechanical and mathematical approach is given in the author's textbook of 1992. Nowadays, Griffith's approach, together with the notions of stress-intensity factor (developed by Irwin), energy-release rate and path-independent integrals of fracture due to Eshelby, Cherepanov and Rice, belongs in the theory of "configurational forces" (cf. Chap. 14 below and the author's book of 2011).

6.3.2 Sneddon's Mathematical Works

The singularity of the stress field at the tip of a crack was masterly dealt with in a celebrated paper of Westergaard (1939). Using a complex-variable representation of plane elastic problems, this author established the $1/\sqrt{r}$ singularity for stresses and the accompanying formulas for the so-called stress-intensity factor. This was done in the USA. But we can return to the UK with a touching personality and gifted analyst, **Ian N. Sneddon** (1919–2000).

A Scotsman educated in Glasgow and Cambridge, Sneddon's early career was necessarily marked by the experience of WWII. He served as a Junior Scientific Officer at the British Ministry of Supply during the period 1942–1945. In this position he spent some time at the Cavendish Laboratory in the solid mechanics group and started to work on problems involving the fracture of metals. He met and worked with the famed physicist (Sir) Nevill F. Mott (1905–1996; Nobel Prize in Physics, 1977). This co-operation resulted in two important facts. One was the publication in 1948 of a pioneering book (Mott and Sneddon 1948) on wave mechanics (something surprising for a scientist who devoted all his life to continuum mechanics). The second fact was to decide on the future of Sneddon as an applied mathematician.

Indeed, Mott suggested to Sneddon to examine the determination of the stress field in an infinite elastic body containing a disc-shaped crack. This was related to the mechanical resistance of steel tank armour plates where bubbles of gas flattened in the roll process transformed into disc-shaped cracks. This led Sneddon to his landmark paper of 1946 which provided the first three-dimensional solution of a problem in elasticity involving a crack. This featured the first use of integral transforms in crack theory (Sneddon 1946). The paper was instrumental in stimulating research on mixed boundary-value problems. It exploited a novel technique already introduced by Sneddon in 1945 to deal with the so-called axisymmetric Boussinesq problem (determination of the displacement and stress fields produced in a semi-infinite isotropic elastic solid by pressing a rigid punch normally to its surface). This involved solving a system of *dual integral equations*, a recurring matter in problems of waves impinging an interface crack between two elastic solids. From then on Sneddon became the acknowledged specialist of the mathematics of such problems. This led him to write pioneering books on Fourier transforms (Sneddon 1951), integral transforms in general, the mathematics of elasticity and cracks (Sneddon and Berry 1958; Sneddon and Lovengrub 1969), and mixed boundary-value problems in potential theory (see bibliography below). A truly never tired writer of books and papers, Sneddon was also a pleasant companion always enjoying to tell stories on science, the Second World War, and people.

Personal touch: On a social gathering in a café in Oxford, I heard Mary Sneddon kindly tell her husband (who had found a benevolent listener in the present writer): "you already told that story several times to Gerard; perhaps he would like to hear something else".

Brilliant as he was, Sneddon became in Keele the youngest professor of mathematics in the UK at the age of 30. He was later on to move to his dear Glasgow (1956–1984). He was succeeded by Ray Ogden on his chair of Mathematics. It is there that Sneddon co-authored with J. G. Defares an original book on an "Introduction to the mathematics of biology and medicine" (1961). It was while still at Keele that he supervised the PhD. Thesis (1955) of Tony Spencer who had started to work with Frank Nabarro (1915–2006) on the brittle fracture of elastic–plastic materials in Birmingham. Sneddon also developed a keen interest in thermo-elasticity (cf. Chadwick and Sneddon 1958; Chadwick 1960). Peter Chadwick (born 1931) had a long career at the University of East Anglia in Norwich (1965–1991). He had been a colleague of Tony Spencer at Aldermaston (Atomic Weapons Establishment in the UK). He was among the first authors to produce a paper on waves in electricity conducting deformable solids (Chadwick 1956). Later on he became interested in the thermo-mechanics of rubberlike materials, but also in general elastic-wave problems and the Stroh formalism (see below Chadwick and Smith 1977; Chadwick 1997). He is the author (Chadwick 1976) of a short but very efficient textbook on continuum mechanics.

Crack studies continued non-stop in the UK representing some kind of endemic/totem field for British applied mathematics. The field was of interest to both mechanicians and materials scientists. Here we should mention the seminal works of Eshelby, Stroh and Willis. Of necessity, the studies of cracks (a macroscopic easily observed process) and dislocations (a microscopic phenomenon not visible at the naked eye) are intermixed. The vivid view of a dislocation associated with a missing half plane of atoms in an otherwise ordered regular arrangement of atoms and the vision of a through crack as a semi-infinite plane cut in an elastic body favour a common view within a general theory, that of configurational forces (Cf. Chap. 14). The "sucking force" acting on the tip of a crack and making the crack extend in an elastic body and the force acting on a dislocation line—the border of the missing plane of atoms—established by Peach and Koehler in 1950 therefore find a common framework in the theory of configurational forces. Furthermore, the evolution of dislocation patterns seen as the microscopic mechanism behind the macroscopically observed plasticity property of ductile materials provides the leading thread of the next sections. No wonder also that we find the same names and groups of authors involved in these various but intellectually close interests.

6.3.3 Eshelby, Stroh and Co-Workers

According to Alfred Seeger from Stuttgart, in the late 1940s-early 1950s the "British schools (of physical metallurgy and structural defects) under (Sir) Nevill Mott in Bristol and (Sir) Alan H. Cottrell (1919–2012) in Birmingham were most prominent and influential in the world at that time". For instance both Alfred Seeger (born 1927) from Stuttgart and Jacques Friedel (born 1921) from Paris—who both became eminent contributors to the theory of crystal defects—were visiting scientists at Bristol when **John** ("Jock") **D. Eshelby** (1916–1981) entered the field of dislocations through his PhD Thesis ("Stationary and moving dislocations", University of Bristol, 1949) after working on defence related projects (1940–1946). From here on all his published works permeate a striking elegance, analytic dexterity, imagination, physical acumen, and a deep knowledge of fundamental mathematical physics. His work on dislocation theory will be examined in the next section. In this early period at Bristol in parallel with his dislocation studies, Eshelby wrote his fundamental paper on "The force on an elastic singularity" (Eshelby 1951). This paper proposed a proper intellectual construct to apprehend the notion of singularity-driving forces (now included in the theory of configurational forces; see Chap. 14). However, it was not until 1968, when Genady P. Cherepanov and Jim R. Rice published their works on path-independent integrals in fracture, that Eshelby's innovative work of 1951 was fully understood and recognized as a basic theoretical and practical instrument in fracture theory. From that year on Eshelby devoted most of his research and diffusion of knowledge to fracture theory, being invited to write several overviews on the subject matter (e.g., Eshelby 1970, 1971, 1975a, b, 1982; Eshelby and Bilby 1968). C. Atkinson—who went to teach at Imperial College in London—was one of his co-workers in this theme of "configurational forces" (see, e.g., Atkinson and Eshelby 1968). For landmark papers in the field see Cherepanov (1998).

Eshelby (1957, 1961) also entered the mechanics of some complex elastic materials (e.g., with inclusions of a foreign constituent) by solving the problem of ellipsoidal inclusions of which elastic coefficients differ from those of the matrix. The method introduced by Eshelby in elasticity in this memorable paper was inspired by problems of electrostatics (the ellipsoidal form allows for a homogeneous strain inside the inclusion, in the same way as an electrically polarizable ellipsoid—or a degenerate form of this shape—admits a uniform electric polarization—see the notion of demagnetising factor in electromagnetism). It itself inspires those who formalized it in more complex cases (e.g., elasto-plasticity; cf. works by E. Kröner, A. Zaoui, M. Berveiller). After leaving Bristol, Eshelby worked in Birmingham (1953–1964), where Nabarro had moved. He spent two years in Cambridge in the Metal Physics Group at the Cavendish Laboratory (1964–1966), after which he joined the Department of the Theory of Materials at the University of Sheffield where he stayed until the end of his life. He was of course elected a Fellow of the Royal Society of London.

It is while still at Bristol that Eshelby supervised the initial research work of another remarkable scientist, **Alan N. Stroh** (1926–1962), originally from South Africa. Nevill Mott replaced Eshelby in this tutorial role when the latter left Bristol in 1952. Stroh obtained his PhD in Bristol in 1952. Then he moved for a couple of years to the Cavendish Laboratory in Cambridge, and thereafter joined the Department of Physics in Sheffield. He went to the USA at MIT in 1958, but he was killed in a car accident in 1962 while he was on the move to a new post at the Boeing Scientific Research Laboratories in Seattle. He published papers with J. D. Eshelby, F. C. Frank, B. A. Bilby and L. R. T. Gardner, all on dislocation theory and kinks. He also published a few papers on cracks and brittle fracture (Stroh 1954, 1955a, b, 1957, 1960, 1962a, b). However, it is with two of his papers (Stroh 1958, 1962a) that Stroh laid down the foundations of a new formalism for treating the two-dimensional deformations of *anisotropic* elastic media, and deserves to have his name engraved at the pantheon of elasticity and wave propagation (if such a thing exists at all). What he cleverly did was to introduce a six-dimensional vector of unknowns comprised of the elastic displacement and the traction (not the stress) in a direction, and the appropriate matrix formalism in R^6. This reduces the mathematical problem to finding eigenvalues and two associated eigenvectors in the proper space. Obviously, this "sextic" formalism, now referred to as "Stroh formalism", is most convenient in quasi-automatically accounting for boundary and transition conditions in elastic structures such as multi-layers and in treating the wave propagation in such structures. This mathematically elegant and technically powerful formalism was discussed, perfected and applied to many cases by, among others, D. M. Barnett (Stanford), J. Lothe (Norway), P. Chadwick (UK; already cited), V. I. Alshits (Moscow), and T. C. T. Ting (1996) who has clearly demonstrated the general superiority of Stroh's formalism over the technique proposed by the Russian scientist S. G. Lekhnitskii (1963) for treating elastostatics and steady-wave motion in anisotropic elastic bodies. Furthermore, the sextic formalism can be given a Hamiltonian interpretation (Y. B. Fu 2007). As we know, Stroh's promising career was suddenly interrupted on September 21, 1962.

6.3.4 Willis' Works

John R. Willis (born 1940) was educated in London (PhD 1964 at the Imperial College). He was a research associate at the Courant Institute of Mathematical Sciences in New York (1964–1965), and then went on to Cambridge (1965–1972), the University of Bath (1972–1994; 2000–2001), and became a professor of Theoretical Solid Mechanics at the University of Cambridge (1994–2000, 2001–2007). His long fruitful roster of works illustrates perfectly the continued British interest in the mathematical investigation of problems that arise in the mechanics of solids: the inclusion problem (cf. Eshelby), the theory of dislocations, fracture mechanics, the elastodynamics of crack propagation (cf. Eshelby, Stroh), statics and dynamics of composite materials, and the homogenisation of composites. Among the names of his main co-workers we note those of R. Bullough, R. Burridge, F. J. Sabina (from Mexico), D. R. S. Talbot, P. Ponte Castaneda (from Pennsylvania, USA), A. B. Movchan, V. P. Smyshltyaev, N. A. Fleck, and G. W. Milton (from Utah, USA).

In Willis' formidable list of publications, we identify his early works concerned with anisotropic elasticity (inclusion problem, Willis 1964, 1965; second-order effects of dislocations, Willis 1967, 1971), the fracture mechanics of interfacial cracks (Willis 1971, 1972), and the equation of motion for propagating cracks. In elastic-wave studies and the mechanics of composites, he introduced (Willis 1980) the seminal idea of "polarization" (inspired by the vacuum as a model of comparison in electromagnetism), variational principles for dynamic problems in inhomogeneous media (Willis 1981a, b), variational estimates of effective dynamical properties in random composites (with Talbot and Willis 1982), the homogenisation of various nonlinear composites and the bounds on overall properties of the same (e.g., 1991). These variational formulations are creative landmarks. In crack propagation, he introduced dynamic weight functions and successfully dealt with perturbation problems (with A. B. Movchan). In this framework he has proved the existence of a new type of waves, so-called "crack-front waves". Recent works with Fleck (2009) provide a mathematical basis for the strain-gradient theory of plasticity (cf. Chap. 13 herein after). All these contributions are of utmost importance in solid mechanics and place Willis among the outstanding specialists of advanced mathematical techniques in this field of application. Still his style of writing remains remarkably sober.

6.4 Dislocations and Plasticity

6.4.1 The Pioneers

After plastic deformation slip bands are usually observed at the previously polished surfaces of single crystals. It is **Geoffrey I. Taylor** (1886–1975) who in 1934 first proposed an explanation of this observed effect by the sliding mechanism by *crystal defects*, which he identified with the mathematically conceived *dislocations* of Vito Volterra in 1905 (see Chap. 2). This idea was practically advanced simultaneously by M. Polanyi and E. Orowan. This insight is all the more remarkable that the experimental proof of the existence of dislocations as individual objects had to await the 1950s after the invention of electron microscopy. A word is necessary about these three scientists.

Both Polanyi and Orowan were originally from Hungary. Michael Polanyi (1891–1976) took a chair of physical chemistry at the University of Manchester after he left Germany in 1933. He made essential contributions to crystallography including dislocation theory. Egon Orowan (1902–1989) was educated at TH Berlin. After working for sometime in Germany and Hungary, he moved to the University of Birmingham to work with Rudolph Peierls and then to Cambridge with W. L. Bragg. He finally moved to MIT in the USA in 1950. Of course, G. I. Taylor is a scientist of another calibre (see his biography by G. K. Batchelor 1994—probably the most well known student of Taylor). A Cambridgian like many others, he is known above all for his outstanding contributions to dynamical meteorology and fluid mechanics, especially in turbulent flows. His name remains for ever attached to such phenomena as the Rayleigh–Taylor instability and Taylor vortices. During WWII he became famous for his theoretical prediction of the yield (16.8 kilotons of TNT) of the first atomic explosion by working out a similarity solution for a blast wave and examining the (then unclassified) relevant pictures of the explosion that were released. But Taylor was a man of many scientific interests. Our colleagues from fluid mechanics are usually quite surprised to learn that he played a seminal role in plasticity theory (Taylor and Quinney 1931) and dislocation theory via his breakthrough paper of 1934 (Taylor 1934). It may be anecdotic to mention that one his grandfathers was none other than George Boole of Boolean algebra fame.

6.4.2 Eshelby's Contributions

It is true that Taylor's insight was critical in developing several aspects of the modern sciences of solid mechanics and materials science (metallurgy). This was happily completed by J. M. Burgers' introduction of the notion of "Burgers" vector in 1939 (See Chap. 10). The theory of dislocations was taken over by metallurgists and solid-state scientists such as F. C. Frank and F. R. N. Nabarro.

We recall that J. D. Eshelby had entered the field of dislocations through his PhD Thesis of 1949. Many of his publications in the period 1949–1968 are devoted to the field of dislocation theory although we cannot ignore the already mentioned epoch-making contribution to the theory of configurational forces and his no less important contribution to the problem of elastic inclusions. Among his co-authors on the subject of dislocations and defects we note a group of famed scientists: B. A. Bilby, F. R. N. Nabarro, A. N. Stroh, F. C. Frank, W. T. Read (the latter two with their names associated for ever in the "Frank-Read" source of dislocations), and W. B. Shockley from Bell Telephone Laboratories (Nobel Prize in Physics for the invention of the transistor). The quality of a researcher is also to be judged from that of his co-authors. One of the masterpieces of Eshelby was the paper he co-authored with Frank and Nabarro on the equilibrium of a linear array of dislocations (Eshelby et al. 1951).

Of noticeable value and great ingenuity in the mathematical solution, we like to emphasize the recurring theme of the motion of a dislocation in Eshelby's works. He pondered this matter for several years (Eshelby 1949, 1953, 1956, 1982). What here is typically a mark of Eshelby's is his deep understanding of mathematical physics and the evident analogies he draws from electro-magnetism (relativistic factor, Lennard potentials).

6.4.3 Geometry, Dislocations, and Plasticity

Bruce A. Bilby (FRS 1977) has been a fruitful contributor to the theory of dislocations for more than fifty years starting with the publication of a breakthrough paper with Cottrell (Cottrell and Bilby 1949). Another famous paper co-authored with Cottrell and Swinden is dated 1964 (cf. Bilby et al. 1964). These works, although innovative, may be said to be in the standard framework of dislocation theory, like most of the works by this author. But in 1955 Bilby and some of his colleagues in the Department of Metallurgy in Sheffield started to develop a geometric theory of the continuous distribution of dislocations. This must have been inspired by the early work of K. Kondo in Japan (See Sect. 10.7 herein after). Kondo himself had noticed that some geometric ideas at work in Einstein's theory of gravitation could be useful in order to represent the special deformation field of defective elastic solids. This would be taken over by E. Kröner in Germany in his theory of strain incompatibility. What Bilby and co-workers did was much more ambitious. They remarked that in the presence of an assumed continuous distribution of dislocations—this means a very high density of dislocation lines—the notion of "good crystal" no longer exists. So there appear difficulties to define Burgers'circuits and a dislocation density tensor. This dislocation density can be defined only by defining the dislocated state by relating a local basis at each point to that of a reference lattice. Then (in words adapted from the authors' abstract with little alteration)

> the geometry of the continuously dislocated crystal is most conveniently analyzed by treating the manifold of lattice points in the final state as a non-Riemannian one with a single asymmetric connexion. The coefficients of that connexion are then expressed in terms of the generating deformations that relate the dislocated crystal to the reference lattice.

In these conditions the local dislocation density can be identified with the *torsion tensor* associated with the connexion [see Sect. 14.3 below, especially the last of Eq. (14.29)]. The introduced connexion possesses the property of so-called *distant parallelism* [a notion introduced in non-Riemannian geometry by the French geometer Elie Cartan (1869–1951); See Maugin 1993, p. 128]. Bilby and his co-workers developed their theory in a series of six long papers between 1955 and 1966. It is aesthetically rewarding but probably of limited use (In particular, Bilby et al. 1955).

Furthermore Bilby et al. (1957, presented in 1956) were the first to introduce a *multiplicative decomposition* of the finite deformation gradient in the presence of defects and plasticity. This was to have a glorious future in the description of the deformation field where local rearrangements of matter take place (e.g., in plasticity, visco-plasticity, damage, heat conduction, phase transformations). This decomposition is sometimes attributed to E. H. Lee in 1969—see Maugin 2011, Chap. 6. Note that Stroh was one of the co-authors of this 1957 contribution.

In his most recent works (early 2000s) Bilby also worked on the continuum theory of damage.

6.5 Rodney Hill: Mathematical Plasticity

With the fundamental works of **Rodney M. Hill** (1921–2011), we return to the (perhaps dry) mathematics of phenomenological plasticity. It seems that Rodney Hill was a remarkable person in many ways. Educated in mathematics at Cambridge, he worked on defence projects under Nevill Mott during WWII, and joined Egon Orowan (already cited) in 1946 at the Cavendish Laboratory in Cambridge where he became a specialist of plasticity. His doctoral thesis (1949) was devoted to "Theoretical studies of the plastic deformation of metals". He was in Sheffield for a short time and then for three years in Bristol. This was the time at which he founded the *Journal of the Mechanics and Physics of solids* (1952) that was to become one of the most influential journals in the field. In Nottingham from 1953 to 1963, he created there the Department of Theoretical Mechanics in 1960. He finally joined the University of Cambridge in 1963 first as a research fellow and then on a personal professorship (1972). His many scientific works are characterized by fresh thinking, concision, and an unsurpassed scholarship.

Hill's outstanding works in mathematical plasticity were performed at a burgeoning time for this theory. This was reported in Chap. 4 when dealing with the case of William Prager and Brown University, although the Soviet Union had also a share in this movement (cf. Kachanov and Ilyushin in Chap. 11). But Hill brought a fresh view by introducing seminal ideas while providing a remarkable synthetic approach for the period. The most powerful idea probably is that of *maximum plastic work* (Hill 1948) now classically referred to as the principle of *maximal dissipation* (the French call it the Hill-Mandel maximal-dissipation principle). In modern form this principle can be expressed by the following *variational inequality* [cf. Eq. (5.20); see also Maugin 1992]:

$$(\sigma - \sigma^*) : \dot{\varepsilon}^p \geq 0 , \quad \forall \, \sigma* \in C, \tag{6.1}$$

where σ represents the stress tensor whose representative point in the appropriate space should remain in a convex set C, and $\dot{\varepsilon}^p$is the rate of plastic strain. Equation (5.20) is none other than a modern representation of (6.1) within the framework of the convex analysis with internal variables of state, plastic strain being possibly one such variable. The important feature of Eq. (6.1) is that together with the convexity of the strain energy, it guarantees the validity of Drucker's inequality and Drucker's postulate—cf. Eqs. (4.1) and (4.2). In other words, it provides a solid foundation for the plasticity of materials that accept the normality law contained in (6.1). These works eventually led to general studies of uniqueness and stability in nonlinear continuum mechanics. This makes Rodney Hill the father of modern plasticity theory in a thermodynamic context. This was expanded by Hill in his book of 1950, published while he was just reaching age twenty nine. This book remains *the* classic opus and indispensable reference in plasticity theory. As mentioned before concerning Hill's style, the book was written with a typical economy of thought and words. The book also generously presented the treatment of typical plasticity problems by means of the theory of slip lines, a then recent introduction for solutions of plasticity problems before the advent of computers. This applies in quasi-static loadings to plane strain problems in rigid-plastic bodies. Hill gave such a solution for the exemplary problem of a rigid punch indenting a rigid-plastic half-space. With the publication of his book at the early age of 29, Hill was recognized as a leading authority in the field.

Hill made many other memorable contributions to the mechanics of solids in his more than 150 published articles. What I like to single out in this long roster, are the foundational papers that he produced in the mechanics of composite materials, i.e., his essential contributions the theory of *homogenization* (Hill 1965, 1987). In this approach one first selects an Representative Volume Element (REV) V that characterizes the size of variations (in coordinates y) of properties at a microscopic scale. Let σ and ε the stress and strain tensors at that scale. At the macroscopic scale of observation, let Σ and E denote the volume averages of the stress and strain defined by (x denotes the macroscale coordinates)

$$\Sigma(x) = <\sigma> \equiv |V|^{-1} \int_V \sigma(x, y) \, dy, \tag{6.2}$$

$$E(x) = <\varepsilon> \equiv |V|^{-1} \int_V \varepsilon(x, y) \, dy. \tag{6.3}$$

The crucial Hill *principle of macrohomogeneity* is expressed as follows:

$$< \bar{\sigma} : \varepsilon(\bar{\mathbf{u}}) > \; = \; \bar{\Sigma} : \bar{E} \qquad (6.4)$$

where $\bar{\sigma}$ and $\bar{\mathbf{u}}$ are, respectively, a statically admissible stress field and a kinematically admissible displacement field, i.e., fields respecting the data in forces and displacements, respectively. The energy equivalence contained in Eq. (6.4) is proved for the three basic types of boundary conditions that may take place at the boundary of V (what is called the *local* problem of homogenization). The French also refer to (6.4) as the Hill-Mandel principle following Mandel (1971). But in statistical theories of composite materials (e.g., in Kröner 1972), (6.4) is but an *ergodic hypothesis*. Anyway, Hill's condition (6.4) provided the basic hypothesis of the energy type in all future approaches to homogenization (i.e., replacing a strongly inhomogeneous medium with rapid spatial variations in its mechanical properties by an equivalent "homogenized" medium).

Hill was a man of few words, personally quite reserved and eminently quiet. This may explain why he did not mentor so many students although he did not hesitate to provide advices. One of his successful PhD students was Ray W. Ogden at Cambridge. Robin Knops (see next section) also wrote a doctoral thesis under the supervision of Hill, but in Nottingham.

Hill's works are incorporated in the author's book of 1992 on plasticity in homogeneous and inhomogeneous solids. Hopkins and Sewell (1982) have edited a very interesting special volume of relevant contributions in the honour of Hill.

6.6 Mathematical Problems

Now we go one step further towards applied functional analysis and pure analysis. The first scientist in this class is **Robin J. Knops** (born 1932). In 1952 Knops won an Open Entrance Scholarship to Nottingham University. As mentioned above he wrote a PhD thesis under Rodney Hill when the latter was in Nottingham. This memoir was devoted to inequalities bounding elastostatic solutions and properties produced by a variation in elastic moduli (Knops 1958). This gave rise to his first published paper in 1958. Then Knops' career unwound as follows. He stayed at Nottingham until 1962, taking a leave of absence at Brown in 1961–1962. On his return to the UK he accepted a call from A. E. Green to join Newcastle as a lecturer in continuum mechanics. He was promoted Lecturer in this field in 1968. In 1966, he wrote one of the first papers on the continuum mechanics of electrostriction for two-dimensional problems (Knops 1966). In 1971 he was appointed Professor of Mathematics and Head of Department at Heriot-Watt University (founded 1821) in Edinburgh. This institution had achieved university status in 1966 only. Knops was very active not only with his own research but also as a successful organizer in an expanding department that created a favourable environment for research in non-linear analysis and mechanics, and reached international reputation through his tireless efforts. In particular, he was successful in attracting creative and innovative scientists, a fact well illustrated by the doctoral work of Stefan Müller from Germany, and the coming of John Ball to Heriot-Watt.

Most of Knops' mathematical researches are in the framework of applied functional analysis. This means a focused interest for problems of uniqueness, existence and non-existence, ill-posed problems, stability, continuous dependence of solutions on data, decay and growth of solutions, and spatial and asymptotic behaviour (e.g., Saint–Venant principle), This is supported by a long roster of publications, often in collaboration with other noted analysts, e.g., Michael Hayes and J. N. Flavin from Ireland, L. E. Payne from Cornell University, R. N. Hills, Piero Villaggio from Pisa, and Brian Straughan from Edinburgh who himself became a professor at Heriot-Watt and contributed (Straughan 1991, 1998) to the solution of problems of stability and convection. Robin Knops has contributed superbly written useful syntheses on uniqueness problems in linear elasticity (Knops and Payne 1971) and elastic stability (Knops and Wilkes 1973; Knops 2001), all real mines of information and frequently referenced authoritative works.

(Sir) **John Macleod Ball** (born 1948) is a true analyst whose main field of research includes basic mathematical properties in finite-strain elasticity, the calculus of variation in the large, and infinite-dimensional systems. He was educated in Cambridge and obtained his doctoral degree at the University of Sussex under the supervision of David E. Edmunds. He was on a post-doctoral fellowship at Brown in 1972–1974. He then joined the Department of Mathematics at Heriot-Watt University where he remained until 1996. He finally became the Sedleian Professor of Natural Philosophy at Oxford, where he was most instrumental in organizing the Oxford Centre for Non-linear Partial Differential Equations. He visited for extended periods Berkeley, Paris (UPMC) and the Institute for Advanced Studies in Princeton.

Ball's most famous paper (Ball 1977) deals with convexity conditions and existence theorems in nonlinear elasticity. In this paper and others he proved for the first time in the history of nonlinear elasticity the existence of configurations that achieve minimal energy under realistic conditions. He introduced the notions of quasi-convexity and poly-convexity in elasticity. Quasi-convexity was originally introduced by Morrey (1952) in pure mathematics. Poly-convexity is a weaker form of convexity introduced to deal with functions defined on a space of matrices (case of deformations). Another landmark paper (Ball and James 1987) was co-authored with Richard D. James (from Minneapolis) and showed the possibility of having cases without energy-minimizing configurations with applications to the theory of martensites, materials with a fine structure resulting from solid-to-solid phase transformations. Indeed, for martensites there exists neither true minimizer nor true infimum, but the minimizer-infimum can be approached indefinitely closely by a sequential development of finer and finer structures, what is experimentally verified. It is no question here to discuss the immense contributions of John Ball to applied mathematics, as they far exceed the understanding of the author. It was more than justified that Ball, a remarkable mathematician of international standing, received many honours including prizes, honorary doctorates, and memberships in academies. Among his co-workers is François Murat from Paris (cf. Ball and Murat 1984).

Note. There are many "John Ball" in the history of Great Britain. But our mathematician should not be mistaken for John Ball (died 1381; a kind of mythical figure, leader of a rebellion of peasants and craftsmen in the fourteenth century), John Ball (1818–1888; honourable member of the Royal Geographical Society) or John Ball (1861–1940; a most famous golfer).

6.7 Conclusion: From and Back to Materials Science via Mathematics

The perspicacious reader will have uncovered an Ariadne's thread in the above described developments. One fact was essential although not very pleasant. It was the tragic occurrence of the Second World War. Two individuals were the initiators of some of these developments, Sir Nevill Mott and Sir Alan Cottrell (and also E. Orowan). Albeit not themselves mechanicians, they cleverly invited young applied mathematicians drafted in their laboratories to ponder problems concerning materials and their defects in terms of what these young men knew best, applied mathematics. This fitted precisely in a favourable pre-existing spirit that expanded previously with such individuals as Maxwell, Rayleigh and Love, while creating a real network of interrelations and co-operations.

Thus was born a *mechanics of materials* (not the traditional mechanics of structures dear to the nineteenth century engineers and also to most continental mechanicians of the first part of the twentieth century). Of course with an acquired momentum of its own, this framework developed its own autonomy. But it was noticed in the 1970s–1980s that new modellings were required following the demands of metallurgists (now often called "materials scientists"). Thus a return to an intimate co-operation between applied mathematicians, mechanicians of the continuum, and materials scientists has become a necessity. This is true in the study of the fine structure of martensites, but also with the inception of new man-made materials for which one must know and model the mechanical properties, all the more that some of these properties may be exceptional (think of honeycomb structures in aeronautics). This is well illustrated by the works of Michael F. Ashby (born 1935) and Norman A. Fleck (born 1958), both in Engineering at Cambridge, although similar trends are observed all over the world. The former is a true materials engineer entirely educated and formed at Cambridge (PhD 1961). After long term stays in Göttingen (1962–1965) and at Harvard (1966–1973), he became a Research Professor at Cambridge. Suffice it to mention his many works on such new materials as *cellular materials* (see Ashby and Gibson 1997), and his general views on engineering materials (book by Ashby and Jones 1996). As to Fleck, with a PhD obtained in Cambridge in 1984, he turned his attention to a variety of interests including the mechanics of metallic foams, powder compaction, and strain-gradient plasticity, with applications of micromechanical modelling.

At the same time, Guy T. Houlsby, also with a PhD (1981) from Cambridge (on the plasticity of soils), developed in Civil Engineering at Oxford a marked interest for the thermomechanics of geotechnical materials such as clay and granular materials (see Collins and Houlsby 1997; Houlsby and Puzin 2002, 2006). All these works are clear deviations from the previous "applied mathematics" trend. They point to the future evolution at the beginning of the twenty first century. This justifies the title of this concluding section.

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Chapter 7
The French Masters

Abstract The French case is peculiar because of the well-known "French exception". In the present case, this exception is provided—in spite of attempts at changes—by the enduring distinction between university education and the celebrated engineering schools familiarly known in French as "Grandes Ecoles" and of which the Ecole polytechnique remains the world acknowledged paragon. After an attempt at explaining this duality in higher French education as well as the ever present centralization of all things in France, of necessity the development of mechanical engineering sciences in this country through the Twentieth century is examined by schools and centres of influence with mention of the most remarkable individuals who have allowed a renascence of continuum mechanics in the country: namely, the University of Paris (Paris 6 also called UPMC to be more precise) with Paul Germain, the Ecole Polytechnique with Jean Mandel, the University of Grenoble with its polytechnic institute, and other centres which have developed in the period 1950–2000 in spite of the Parisian Jacobinism. Each centre has succeeded to develop special trends in continuum mechanics at the international level of competition, often in the fields of continuum thermomechanics, nonlinear deformations, plasticity and visco-plasticity, rheology, fracture mechanics, coupled fields, homogenization techniques, and other mathematical methods. This is presented in great detail with as much neutrality as possible from the part of a long-time Parisian.

7.1 The Originality of the French System of Higher Education: A Necessary Introductory Explanation

One cannot deal with the 20th century French system of education and research without trying to explain the historical reasons for its originality and singularity, mostly in its strange dichotomy between universities and so-called "grandes écoles". By these we here mean essentially "engineering schools", a system that is

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now foreign to most people around the world although it provided a leading example in many countries in the 19th century.

Before the great revolution of 1789 there existed universities in France in various places, Paris (the "Sorbonne" whose name derives from that of a *Sieur Sorbon*) and Montpellier being the most prestigious and oldest ones. The common curricula were law and medicine; many students graduated in both simultaneously. A few exceptions to this rule were schools created with a special technical purpose in the middle of the 18th century. These were the Royal school of bridges and Highways (*E.R.P.C = Ecole Royale des Ponts and Chaussées*, founded in 1747) in Paris and the Military school of Mézières (in the Champagne-Ardennes region, North-East of France). This opposed "civil engineers" and "military engineers", although both were instructed to serve the kingdom, in—what is now called—civil engineering and artillery, respectively. The second school is not well known outside France, but it was the ancestor of the present *Ecole Polytechnique*. Napoleon Buonaparte (not yet "Bonaparte"), sometimes presented by his enemies as ignorant and uncultured, was a student there acquiring an excellent formation in mathematics and mechanics. This allowed him to participate actively in the sessions of the French Academy of Sciences—in the Section "Mechanics"—where he nominated himself in the best dictatorial tradition, and this when he was not trying to conquer Europe during his multiple wars.

Scientific research by "savants" before the French revolution was conducted either on a private basis by members of the clergy with some leisure time, gifted rich amateurs or by members of the Academy of Sciences in Paris (founded by King Louis XIV in the 17th century as a reaction to the founding of the Royal Society of London). These members—who could be rather young—were paid by the state so that this academy resembled one of the modern state research institutions (e.g., the Soviet academy of sciences or the French *C.N.R.S* in the 20th century).

The 1789 revolution changed everything, starting by abolishing universities. The main idea was to suppress these supposedly reactionary temples while replacing the self proclaimed elite by an elite based on pure scholarly merits. Remember that one had to be a noble to enter the Mézières school (Buonaparte's family was assimilated to small nobility from Corsica). The selection of this new elite had to be through entrance competition (like for the selection of the mandarins in imperial China) to specially devised schools. From that viewpoint the *E.R.P.C* was readily transformed into the *E.N.P.C* (*Ecole* Nationale *des Ponts et Chaussées*) while former professors from Mézières, most actively the mathematician Gaspard Monge, worked out the creation of the *Ecole Polytechnique* in Paris. This was to become a model in many countries, e.g. in Switzerland (the Polytechnicum in Zürich), Germany (various polytechnic schools in Darmstadt, Munich, Aachen, etc.; later renamed "Technische Hochschulen", and then "Technische Universitäten"), Austria (Vienna), Russia (St Petersburg), Sweden (Stockholm), Italy (Torino, Milano), Poland, and much later even in the USA (the *M.I.T.* in Cambridge, Mass.). In principle the *Ecole Polytechnique*—simply referred to as "*X*", the unknown variable in mathematics—offered a rather general education with an emphasis on mathematics, so that students had to further attend another school—said of application—to become purposeful servants of the state. The *E.N.P.C* was one of these schools. Those who completed this curriculum became members of the so-called "corps" ("bodies"). To be a member of these "corps" was—and still is—the highest ambition in the French state service. Many of the great French scientists (specially in mechanics) of the early 19th century belong to this elite, to name a few: Cauchy, Navier, Lamé, Clapeyron, Poncelet, Liouville, Coriolis, Arago and Barré de Saint–Venant, while Monge, Lagrange, Laplace, Ampère and Fourier taught at the Polytechnique school. Schools of application for "polytechnicians" were created according to technical needs, e.g. the school of Mines in Paris (now called *E.N.S.M.P*) with branches in Nancy, St Etienne and Alès (i.e. where there were mines), and then in the 20th century the school of Aeronautics (*E.N.S.A*. nicknamed "Sup Aéro", the first of its kind in the world in 1909), and the school of Telecommunications (*E.N.S.T*). Thus Henri Poincaré belonged to the "corps des Mines". Each specialization school depended—and still depends—on the corresponding Ministry (Equipment, Defence, Post and Telecommunications, etc.). Many of the already cited people were to become what Ivor Grattan-Guinness (1993) calls "ingénieurs-savants", often sharing their time between technical works (e.g. design of bridges or new harbours) and true scientific research.

In parallel with *grandes écoles*, Napoleon's Empire (Buonaparte had become self-crown emperor in 1804) reorganised teaching at the primary ("Ecoles primaires" from age 6 to age 12) and secondary ("lycées" from age 12 to age 18) levels. The formation of teachers for these two levels was to be delivered in the "Ecoles normales (of teachers)"—one per department—and the (unique) *Ecole Normale Supérieure* (*ENS*) in Paris, respectively. Entrance in the latter had to be by competition. As the level of this examination was similar to that to the admission to *Polytechnique*, these two schools became rivals, although not intended to fulfil the same purpose. It was soon discovered that alumni from the *ENS* could also become creative scientists in pure science. Fourier was among the first alumni from the *ENS*. Under the influence of Louis Pasteur, the *ENS* became a true centre of formation of high-level scientists late in the 19th century. Another originality is that students in the cited "grandes écoles" were paid by the state, so that they owed some years of service to the state as a partial re-payment for this facility.

But what about universities, which continued to form lawyers, medical doctors, humanists, and a few scientists, when they were re-instituted in the 19th century? To speak of universities in the plural form is a mistake. There was only *one* university, called imperial or national, provinces welcoming only *faculties*; thus the Faculty of sciences in Bordeaux where Pierre Duhem spent most of his career. All of these were under the directorship of the Ministry of Education that made all nominations. This system, far from the present democratic system, could not avoid political biases (of which Duhem was a victim, although himself a brilliant alumnus from the *ENS*). This centralized system with an ultimate recognition reached only once one was nominated in Paris, favoured a kind of careerism with a typical path from education in Paris (preferably the *ENS* or *Polytechnique*), a succession to a few positions in provinces such as in Rennes, Lille, or Poitiers, and then back to the Sorbonne in Paris at the apogee. This system lasted practically until the 1960s. Except for hard-tempered and very gifted individuals such as Boussinesq—originally student at the Faculty in Montpellier—it was hard to follow a similar conventional path for students from provincial faculties.

The noted dichotomy was not much in favour for research in the "grandes écoles". It is not until the 1960s that research laboratories were established in many of the "grandes écoles", in mechanics principally through the influence of Paul Germain, and the encouragement and financial support of the *Centre National de la Recherche Scientifique* (*CNRS*). The later was created in 1939 by the nuclear physicist and Nobel-Prize winner Frédéric Joliot-Curie, certainly inspired by the Soviet Academy of Sciences.

We have not mentioned two original institutions. One is the "Collège de France" in Paris, created by king François I early in the 16th century. This is considered to be the highest institution of education in France where a selected group of professors teach a few hours per year the last developments in their speciality, usually their own most recent research. This institution delivers no diploma and the lectures are open to all. The second institution is the *Ecole Supérieure de Physique et Chimie Industrielles* (*ESPCI*). This « *grande école* » depends on the city of Paris: It was created after the defeat of the French opposite to the Prussian in 1870, with a view to accommodate refugees from a technical school in Mulhouse in the then Prussian occupied Alsace. The curriculum of the *ESPCI* puts the emphasis on experimental observation and training.

7.2 The University of Paris and Paul Germain

The history of the development of modern continuum mechanics in France bears the print of the French centralized educational system: "No hope outside Paris". Furthermore, in the best French tradition of the 19th century, "mechanics" had to be understood as "rational mechanics", a field of applied analysis. Accordingly, university courses in the field were often delivered by professors expecting a future true chair of mathematics, analysis or geometry. This did not encourage much research in the field of continuum mechanics, fluid or solid mechanics. Boussinesq in Montpellier was an exception. In Paris, because of his interest in the rapid development of aeronautics in the 1930s, Joseph Pérès—another alumnus from the *ENS*, and co-worker of Vito Volterra on functional equations—felt that various lines of research had to be promoted, in particular, in the field of theoretical and experimental aerodynamics. He persuaded three brilliant individuals, Lucien Malavard (Aeronautical engineer from "Sup Aéro"), Germain and Roger Siestrunck—the last two, respectively mathematician and physicist from the *ENS*—to work on advanced problems of fluids mechanics. Malavard was to develop some of the first analogue numerical simulations, and then created with *CNRS* the first French centre of computational mechanics in Orsay (south of Paris). Siestrunck devoted himself first to aerodynamics and then to applied mechanics in the large. Germain, as we shall see, was the pivotal and crucial element in founding a true school of continuum thermo-mechanics. A biography of Pérès is given by Germain (1977).

Paul Germain (1920–2009) was destined to become a geometer. But his scientific career took a new turn at the end of WWII when Pérès sent him to the UK for a stay at the *National Physical Laboratory*. As a matter of fact, this kind of visit abroad became a mark of Germain who did not hesitate (this was not frequent at the time) to make extended stays in various foreign institutions, notably in the USA at Caltech, Brown and Stanford, and to get the best out of these foreign experiences. Note that the system of sabbatical leaves did not exist at the time in France. Simultaneously, Germain did not escape from the traditional tour of provincial universities (Poitiers, Lille) before obtaining a chair in Paris in 1956 and also becoming later on General Director of the *Office National d'Etudes et de Recherches Aéronautiques* (*ONERA*). It is in Paris that Germain accomplished a tremendous pedagogical, organizational and research work in continuum mechanics. This he did with a remarkable open-mindedness by gathering a group of young researchers from various horizons, although professorships were practically still reserved to alumni from the ENS. There was also an active campaign of inviting foreign visitors who all left a print on the locally expanded research themes. For some time the emphasis was still placed on mathematical problems of fluid dynamics and magneto-hydrodynamics. In particular, Germain worked out and encouraged the application of new methods of asymptotics. (e.g. matched asymptotic expansions). He also had to implement a curriculum in general continuum mechanics. He had started to write organized notes for undergraduate studies when still in Poitiers and Lille. These notes materialized in a book published in 1962. This opus had a formidable efficacy and became a standard text for the whole country. Although of a larger format than Chairman Mao Tse Toung's little red book, because of its red cover Germain's 1962 book was often called "Popol's" (affectuous diminutive for Paul) red book. Simultaneously, Germain started to give a selection of lectures at the graduate level where he introduced the most recent works of Truesdell, Toupin, Noll, Coleman, and Rivlin. Thus he lectured on the general structure of constitutive theory, on hereditary materials and viscoelasticity, on thermodynamic principles with the Clausius–Duhem inequality, etc. These lectures were directly abstracted from research papers essentially published in the *A.R.M.A*. before the publication of the celebrated volume III/3 of the *Handbuch der Physik* by Truesdell and Noll. This was material hard to swallow by young minds (according to the author's own experience).

Among his direct assistants and doctoral students, a variety of fields were explored and improved in works at an international level. The group included Duvaut, Lanchon, Hartman, Muller, and then Piau, Drouot, Sidoroff, Gérard, Maugin, and others, many of them alumni from "grandes écoles". This work force rapidly contributed to newly opened research trends. Thus Duvaut worked on nonlinear elasticity and waves and then was instrumental in implementing applied functional analysis and variational inequalities in mechanics (with a landmark pioneering book co-authored by Lions and published in 1972). Hartman worked in the field of polar materials, a field that had been practically left untouched in France since the Cosserats. Hélène Lanchon worked with Duvaut. Patrick Muller paid special attention to the strength of materials and the rewriting of Germain's book for undergraduates (See Germain and Muller 1997). Monique Piau (1975) was among the first French scientists to consider finite-strain elastoplasticity and wave propagation therein. Raymonde Drouot dealt with non-Newtonian fluids under the supervision of Ratip Berker (a Turkish mathematician who obtained his state doctoral degree in Lille before WWII under Kampé de Fériet) and Michel Lucius, himself a disciple of Noll. Sidoroff was instrumental in introducing the notion of internal variables of state and multiplicative decomposition of the deformation gradient in finite-strain viscoelasticity (Sidoroff 1975). Gérard worked on wave propagation in spherical structures (typically the Earth). As an original move for the period, Maugin continued his graduate studies at Princeton to come back later in 1972 after serving in the French Air Force.

In parallel but not necessarily under Germain's supervision, a true school of asymptotic studies and nonlinear waves developed under the leadership of Jean-Pierre Guiraud (himself a former co-worker of Germain at *ONERA*), Maurice Roseau and Henri Cabannes: This group included Thérèse Lévy, Philippe Gatignol, Bois, Renée Gatignol, Daniel Euvrard, Roger Peyret, Jacques Mauss, Alain Rigolot, and Enrique/Evariste Sanchez-Palencia. The latter—originally formed as an aeronautical engineer in Madrid—was among those who created the asymptotic method of homogenisation (Sanchez-Palencia 1980). Other mathematical problems were studied by Maurice Roseau, Claude Do, and Pierre Brousse. The resulting burgeoning was formidable. The more recent generation included Michel Potier-Ferry (stability) and Suquet (mathematical problems in plasticity, homogenisation; e.g. Suquet 1985, 1987). Furthermore, the formed doctoral students were to spread the "Gospel" in various provincial universities, including Rouen, Lille, Compiègne, Bordeaux, Toulouse, Nancy, Metz, Lyon, Nantes, Grenoble, Montpellier, and Marseille, this time with very little hope to return to Paris, but creating a real national web.

The influence of Paul Germain was also felt in studies encouraged at the *ONERA*, not only in fluid mechanics and aerodynamics (this was the time of development of the "Concorde" supersonic commercial plane and of famous military fighters), but also in solid mechanics with Jean Lemaître (born 1934) who, after studies on fatigue and viscoelasticity, became famous for his theory of damage in solids developed together with Jean-Louis Chaboche (cf. Lemaître and Chaboche 1985). He later on became a professor at the University of Paris (now Université Pierre et Marie Curie) while creating the "Laboratoire de Mécanique et Technologie" in Cachan (south suburb of Paris) with the help of Raymond Siestrunck.

The flow of original research by Germain was obviously slowed down by his many university and organizational activities, but also by the fact that he was practically in charge of the French Academy of Sciences of Paris, in the position of "perpetual" (albeit limited to age 75) secretary. Nonetheless, he contributed efficiently to a modern formulation of the principle of virtual power that proved to be the safest and best available method to devise theories of complex media and specific structural elements (plates, shells). This was implemented in complex theories of electromagnetic continua by Maugin on his return from the USA.

Paul Germain published in 1973 the first volume—with an emphasis on thermodynamic bases—of a general course on continuum mechanics but the other volumes were never published (this was to become known a “Popol’s green book” as compared to the previously published “red” one). In 1975 he became for ten years *the* professor of mechanics at the *Ecole Polytechnique*, while remaining attached to the research laboratory he had contributed to create at the university in Paris. The *Polytechnique* course itself gave birth to a two-volume book in French (Germain 1986) of which the English translation was never finalized (although the author together with his English speaking wife was ready to supervise this translation). Germain published his last scientific paper in 1998, and died early in 2009. No other scientist had such a wide and marked influence in both teaching and research in continuum mechanics in France in the last fifty years. The original research unit (first “group of continuum mechanics” and then *Laboratoire de Mécanique Théorique*) created by Paul Germain was later renamed *Laboratoire de Modélisation en Mécanique* (with successive directors: Renée Gatignol, Rigolot, and Maugin). This one was integrated in a much larger unit called the *Institut Jean Le Rond d’Alembert* (including physical and musical acoustics and energetics with heavy experimental facilities) in 2007 by the author. Biographical elements on Germain are to be found in Germain (1990), Maugin et al (2000) and Maugin (2010).

7.3 The Ecole Polytechnique and Jean Mandel

Jean Mandel (1907–1982) belonged to the “Corps de Mines” (formation: $X +$ Mines). This, in a sense, explains that. Before Germain became professor in 1975, only engineers belonging to one of the “corps” (*Mines* or *Ponts and Chaussées*) could possibly become professor of Mechanics at the *Ecole Polytechnique*. But Mandel who taught mechanics at *Polytechnique* from 1942 to 1973—occupying the chair of professor in the period 1951–1973, was also exceptional from other viewpoints. On the one hand (the “bad” side), he did not speak English. On the other hand (the “good” side) he was both a theoretician and an experimentalist. His first interest as “mining engineer” was necessarily in soil mechanics where he already achieved remarkable work. But more generally his scientific interest was in the anelastic behaviour of materials (plasticity, visco-elasticity, visco-plasticity). He is recognized as a pioneer in the application of plasticity to soil and rock mechanics. He created the *Laboratoire de Mécanique des Solides* at *Polytechnique* in 1961, a laboratory common to *Polytechnique* and the application schools of *Mines* and *Ponts and Chaussées*, hence its acknowledged strength. He also created the “French Group of Rheology” in 1964, of which he was the first president. But he was somewhat alien to the University system. He did not encourage so much his students to take a Doctoral degree. At the time "grandes écoles"—how "grandes" they may have been—could not deliver Doctoral degrees; This privilege was reserved to universities. As a consequence Mandel's students had to register at a university with a university supervisor (e.g. very often, Paul Germain or Raymond Siestrunck in Paris). Indeed some of his students were very successful in the French system (so much as becoming head of the French Atomic Energy Commission) without holding a Doctoral degree, no doubt the "Corps" helping in this strategy. Strangely enough, Mandel was never elected to the French Academy of Sciences, a position that he clearly deserved in view of his great scientific achievements that we briefly examine now.

Among Mandel's scientific achievements we must single out the following essential contributions because they had a direct influence on his students and French and world continuum mechanics in the large. Thus he solved the problem of soil deformation under a load accounting for what is called consolidation (expulsion of water from the soil in time), obviously a situation more realistic than the standard Boussinesq problem. He dealt experimentally with the creep of plastic materials and in particular plexiglas. He was also active in viscoelasticity of the Boltzmannian type. In problem solving of this theory he favoured the exploitation of the Laplace-Carson transform instead of the standard Laplace transform, and proposed, simultaneously with Lee in the USA, but independently, the "correspondence principle" between the linear dynamic viscoelastic problem and the corresponding elastic one. In dynamic plasticity he demonstrated the existence of additional wave fronts with velocity bounded by those of the pure elastic waves. In homogenisation, he formulated independently of Rodney Hill the now-called *Hill-mandel principle of macrohomogeneity*, that proposes an energy equivalent between micro and macro scales, a principle later shown as an ergodic hypothesis by E. Kroner. Finally, he was a critical but constructive contributor to the general thermo-mechanics of continua (see his contribution at the "thermodynamics of continua" meeting in Bussaco, Portugal, in 1972), especially in the theory of finite-strain elasto-plasticity where he solved the problem of the indetermination of the so-called intermediate or released configuration (yielding a multiplicative decomposition of the deformation gradient up to an orthogonal transformation) by introducing the idea of the *director frame*—that specifies the orientation of the matter element. In a crystal, this frame is none other than the lattice frame, from which it follows that the elastic deformation *per se* is nothing but the lattice deformation. In a polycrystal one would take as director frame any one of the frames attached to the constituent crystals (Mandel 1971).

Like Germain, Mandel rarely co-signed papers with his students or disciples. But his works had a deep influence on their works which he discussed thoroughly with them. He thus created a true school that produced a nice roster of original works. Among these we must cite the creative works of Bui on fracture (e.g. Bui 1978), those of Zarka on the plasticity of the monocrystal (Zarka 1972, 1973), those of André Zaoui on the case of polycrystal, the introduction of the notion of "generalized standard materials" (dealing with internal variables of state, convexity, and the existence of a dissipation potential) by Halphen and Nguyen QS

(1975), and the works on finite-strain anelasticity, numerical schemes and stability matters by Nguyen QS (2000) and Claude Stolz. All these works did not have first much influence outside of France because of the lack of enthusiasm showed by Mandel's students—who somewhat imitated their master—for the use of the English language. But now these works are internationally recognized. I gave a mathematical presentation of most of these works in my book on the "thermo-mechanics of plasticity and fracture" (Maugin 1992).

Somewhat separate but also influenced by Mandel's general ideas and teaching, we note the importance of the *Laboratoire Central des Ponts et Chausées*, in Paris and then in Marne-la-Vallée, east of Paris. Here we should note the remarkable works of Frémond (2002), a disciple of Moreau in the application of convex analysis to non-smooth problems of continuum mechanics, including shocks, and then in the framework of the *Institut Navier of Civil Engineering* founded in 2003 the creative works of its director, Olivier Coussy (1953–2010) who, in a rigorous almost Truesdellian style, much improved the thermo-mechanics of porous media by considering finite strains, thermal effects, unsaturated poro-elastic solids, and involving physico-chemical properties (cf. Coussy 1995, 2010).

As a final remark on the case of Mandel's works and influence, we must note that many of the works were achieved in close relation with the French National Electricity distribution company (*Electricité de France*), the *Ponts and Chaussées* administration, and the ongoing atomic-energy developments (construction of atomic-power plants for electricity production). No doubt again: the intimate connection between members of the "corps" here played a definite role.

Mandel's course at *Polytechnique* was published in two volumes in Paris in 1966 (no foreign editions). Although it contains many deep thoughts about the matter, it did not get the same reception as Germain's book of 1962 that was translated in many languages. Mandel (1974) also published more advanced lecture notes in French but in Warsaw. A biography of Mandel is given by Habib (1983).

7.4 The University of Grenoble and its Polytechnic Institute (INPG)

The city of Grenoble is situated in the Isère Department in the south-west of France, in a valley with direct access to the French Alps in a region called the "Dauphiné". It is thus very popular with French students who like skiing during the week end in the snowy season. It has always been a strong hold of local government (so-called provincial "parliament" before the French revolution of 1789). In Napoleonic times it had a famous prefect for the Isère Department in the person of Jean-Baptiste Joseph Fourier, the scientist of heat-conduction and series and integrals fame. The gentleman did his best in this capacity although it is not what he did best in his life! Apart from the deciphering of Egyptian hieroglyphs by Jean-François Champollion, the city of Grenoble did not play any great role in the national education landscape in France until the development of hydrological production of electricity from dams developed at some rapid pace. The aluminium industry followed with its large consumption of electricity. Simultaneously a school of engineering specialized in hydrology and then geophysical problems—initially the *EIH = Ecole des Ingénieurs Hydrauliciens* founded in 1929, and later on transformed into the *ENSHMG = Ecole Nationale Supérieure d'hydraulique et de Mécanique de Grenoble*—took a specific importance (very much with students fans of skiing—in my young years this school was nick-named "Sup-ski", an expression we need not explain if we remember that the prefix "sup" was used to qualify all specialized schools of engineering attended after a general curriculum—e.g. "Sup-Aéro" for the already named *ENSA* in Paris). This engineering school worked in conjunction with the appropriate departments of the University of Grenoble. Later on, with the successful efforts of Louis Néel (Nobel Prize winner in physics for his works on ferro- and ferri-magnetism), a research centre for atomic studies was also installed at Grenoble while a school specialized in applied (numerical) mathematics was also created. With these new structures and the creation of laboratories affiliated with *CNRS*, research bloomed in Grenoble that became one of the most active scientific centres in France.

No need to emphasize that the teaching and laboratories at the *EIH* and the *ENSHMG* were very much influenced by the proximity, and the engineers, of the *Société Grenobloise d'Etudes et d'Applications Hydrauliques* (*SOGREAH*), a company very much involved in the design and safety studies of hydraulic dams, especially after WWII. In 1957 a graduate curriculum was created jointly at the University and the engineering schools. Continuum mechanics was part of this curriculum. The main actors in this action were Julien Kravchenko (a mathematician from a family of Polish-Ukrainian origin, formed in Paris at the celebrated *ENS* and who did analytic works in fluid mechanics), Jean Biarez (an engineer to become later on professor of Soil Mechanics at the *Ecole Centrale* in Paris) and Paul Anglès d'Auriac. The latter, engineer formed at the *Ecole Polytechnique* in Paris, and at the time Scientific director of the *SOGREAH*, proved to be a remarkable pedagogue. He introduced in his teaching of continuum mechanics as a fundamental science the most modern elements at the time (including works by the American mechanicians such as Truesdell et al.). He published one of the first monographs on the subject in French (1955, with a preface by Kravchenko). He was also constantly using tensorial analysis, a formalism not so much entertained by French engineers at that time. Unfortunately, the chosen place of publication was ill-fated. Most French authors, except a few from Grenoble, completely ignored this monograph. After leaving the *SOGREAH* Anglès d'Auriac became Professor of Mechanics at the University of Grenoble but he does not seem to have been active in scientific publications in journals—with the exception of general views on continuum mechanics at some colloquia—e.g. his contribution of 1966 at the IUTAM Symposium on the "Rheology and soil mechanics". Nonetheless, Anglès d'Auriac had a strong influence on several doctoral students who established a true school of rheology in Grenoble (among them, Philippe Le Roy and Jean-Marie Pierrard). This has lasted until the present time with the venue at

Grenoble from Paris of Professor Monique Piau and her husband, Jean-Michel Piau. Also, with the many friendly connections of Kravchenko in Poland, a flux of Polish scientists was established between Warsaw and Grenoble, including in particular specialists of plasticity and viscoplasticity (Nowacki and Sawczuk).

7.5 Other Places: Marseille, Lille, Toulouse, Montpellier, Poitiers

In comparison with the already examined three centres, the activity and creativity of the other French centres seem to be more modest. Still, their creation is also related to the action of Joseph Pérès with the creation of a *Ministry of the Air Force* in 1929 and the inception of various Institutes of fluid mechanics (in French, *IMF*) in the 1930s and 1940s in Marseille, Lille and Toulouse. This was very much the result of the marked interest of Pérès for aeronautics and the related theoretical and experimental studies in aerodynamics and turbulence.

The first *Institut de Mécanique des Fluides* (*IMF*) was created by Pérès when this scientist was in post in **Marseille** in the 1930s. This was later complemented by an Institute specialized in the study of Turbulence. Furthermore, with studies on submarine armaments, a laboratory devoted to submarine acoustics was also developed starting in the 1940s. This was later integrated in a larger laboratory of physical research destined to become the actual *Laboratoire de Mécanique et Acoustique*. Under the influence of various scientists (e.g. Vogel and Nayroles) this laboratory became involved in studies on deformable solids, with a special emphasis on mathematical problems in general wave problems, viscoelasticity, numerical techniques, and homogenisation of composites (in particular with Suquet who had been educated in Paris). Musical acoustics was also expanded but this is another story.

The University of **Lille**, in the north of France, had a different experience. First of all, Joseph V. Boussinesq (1842–1929), the famous elastician who also worked in fluid mechanics—coming from Montpellier—was a professor there before joining the University of Paris when he was elected to the French Academy of Sciences. He left a deep print in Lille. Another *IMF* was created in Lille in the 1930s, of which Marie-Joseph Kampé de Fériet (1893–1982), basically a theoretician, took the leadership. He was joined by André Martinot-Lagarde (1903–1982), a graduate from the *ENS* in Paris. They both developed theoretical and experimental studies in aerodynamics, creating there a true school. The *IMF* still exists being attached to the now *Office National d'Etudes et de Recherches Aérospatiales* (*ONERA*) in contact with mathematicians of the University and the actual *Laboratoire de Mécanique de Lille*. Arthur Dyment was fruitfully active as one of its directors. Concerning continuum mechanics *per se*, we remember that Paul Germain had taught there. In the 1960s he was succeeded by Gérard Gontier who wrote a rather nice and complete course on continuum mechanics (1969).

This book included many of the then recent developments from the Truesdellian school, and was somewhat in competition with Germain's book of 1962 and Mandel's course of 1966. Apparently, this was not as successful as these two books: probably the disadvantage of not being in Paris! Furthermore, there does not seem to have been a true school pondering new developments. The big man there still was Kampé de Fériet. He has made remarkable contributions not only to theoretical and experimental mechanics but also to the theory of hypergeometric functions, stochastic functions and information theory. The teaching of good continuum mechanics was revisited with the coming of Pierre-Antoine Bois (2000) from Paris. The latter is a specialist of theoretical fluid mechanics with a specific interest in asymptotic methods and the Boussinesq approximation, a spot on subject for Lille.

The university of **Toulouse** still had another different experience. This time, while an *IMF* was also created in this beautiful historical city—a true provincial capital—the IMF benefited from the presence of a recently opened engineering school, the *Institut d'Electrotechnique et Mécanique Appliquée* de l'Université de Toulouse in 1907. This was to become the *Ecole Nationale Supérieure d'Electrotechnique et Hydraulique de Toulouse* (*ENSEHT*) under the directorship of Leopold Escande (1912–1980), a renowned specialist of computations of dams and problems in hydraulics. With the addition of electronics and applied mathematics—soon transformed into "informatics"—at the proper place in its name, the school was given the longest acronysm ever: "*ENSEEIHT*", while being integrated in 1948 in a national network of *Ecoles Nationales Supérieures d'Ingénieurs* (*ENSI*) that share a common entrance competition exam. The school is the largest "Grande Ecole" in the south west of France. Jean Nougaro (1922–1980) succeeded Escande. Both Escande and Nougaro were instrumental in building a true local school of mechanics dealing with all aspects of fluid mechanics and energetics, the more mathematically minded mechanicians remaining as well members of the department of mathematics at the university. Here also the local school was enriched with the recruitment of young professors coming from Paris in the early 1970s, e.g. Christian Hartman and Jacques Mauss. The *IMFT* remains a stronghold of original research in fluid mechanics in France. This became even more true when Toulouse became the French capital of Aeronautics and Space research and development with the local design and construction of very successful commercial planes. The *ENSA* ("Sup Aéro") moved from Paris to Toulouse in 1968, in fact joining another pre-existing school devoted to more technological matters. This whole group of institutions constitutes the strongest pole in aeronautical and space teaching and research in France.

Montpellier is a nice city in the Languedoc-Roussillon region, very close to the Mediterranean sea and also close to the North east of Spain with which it shares the Occitan-Catalan spirit. This city had one of the first French universities in the Middle Ages. Its medical school, the first of its type in the world, was founded in 1220. It was for a long time the best medical school forming the first French masters of this science in the Renaissance period. In the author's opinion, Montpellier has only one defect: it is too windy. The Faculty of sciences was re-opened in 1810 after its closure for about 20 years. But with the centralization typical of France, it was difficult for Montpellier to remain at the level of Paris. However, continuum mechanics and the relevant mathematics burgeoned there in the 1960s–1980s thanks to the action of some individuals; among them Jean-Jacques Moreau and Olivier Maisonneuve. The university is now called the *Université des Sciences et Techniques*, alias *Montpellier-2*.

Moreau, a gifted mathematician formed at the *ENS* and the University of Paris, first worked on theoretical fluid mechanics. But much more than that, he became—with Rockafellar—one of the most productive contributors to *convex analysis*. Moreau conducted a seminar in this field during many years in Montpellier, also with a landmark series of lectures at *Collège de France* in Paris (Moreau 1966). This proved to provide a breakthrough in continuum mechanics. As a matter of fact, this provided the looked for mathematical formalism to treat mathematical problems involving unilateral constraints, friction, and plastic behaviour (see Moreau 1971). Among the successful applicants of this formalism in continuum mechanics, we must single out Bernard Halphen and Nguyen QS (already cited as disciples of Mandel at *Polytechnique*) in their thermo-mechanical theory of so-called "generalized standard materials" (simultaneous existence of a potential energy and of a pseudo-potential of dissipation subjected to convexity conditions), Michel Frémond at the *Ponts and Chaussées*, Bernard Nayroles, first in Poitiers and then in Marseille, Suquet, successively in Paris, Montpellier and Marseille, Michel Jean in Marseille, and André Chrysochos (thermo-mechanics) in Montpellier. The enormous success of his work on convex analysis should have justified the election of Moreau at the Paris Academy of Sciences. Moreau and his close colleagues went on to apply these concepts in numerical applications resulting in spectacular simulations of the fall of structures and the flow of granular materials affected by frictional forces.

Olivier Maisonneuve, himself a noted specialist of the mathematics of structures formed in Poitiers, was instrumental in building a successful research unit at the university, which combined both mathematicians (including Moreau) and more engineering oriented researchers.

Finally, we mention the case of **Poitiers**. This city, beautiful with its Roman-Gothic monuments, situated in the central west part of France, is historically rich. There exists in Poitiers an engineering school of mechanics and aeronautics (*E.N.S.M.A* = *Ecole Nationale Supérieure de Mécanique et d'Aéronautique*), created in 1948 as the *Institut de Mécanique et d'Aérotechnique de Poitiers*. Paul Germain had taught there in the early 1950s. In the 1960s–1980s two individuals gave a special impulse to researches in mechanics; they are Thierry Alziary de Roquefort who worked in supersonic flows and mixing layers, and Alexis Lagarde who devised beautiful experimental optical techniques such as in photo-elasticity. The school has developed extensive researches in aerodynamics, aerothermics, detonics and the mechanics and physics of materials. Somewhat outside these main lines, we also note the works of Claude Vallée and co-workers mostly on the application of differential geometry to large deformations of solids and specific structures (shells).

7.6 Concluding Remarks

Not all teaching and research centres of interest have been scanned in the foregoing sections. In particular, we have left out the so-called "Ecoles Centrales", of which the Paris one (*ECP*) is the oldest and most celebrated one. It provided many of the famous railroad engineers in the 19th and 20th centuries as well as a large number of industry managers. Its sister school in Lyon—the *ECL*—developed extensive researches in both fluid and solid mechanics. François Sidoroff, a former student of Germain in Paris, taught there a beautiful course on continuum mechanics while pursing his nice work on the thermo-mechanical modelling of anelastic and damaged materials. Both *ECP* and *ECL* were, and are, very active in the developments of numerical approaches in solid and soil mechanics. The Technological University at **Compiègne** (*UTC*), north of Paris, was created as an imitation of American institutes of technology. The research emphasis was placed on numerical computations of structures, acoustics, and some parts of biomechanics.

The *Institut National des Sciences Appliquées* (*INSA*) in a suburb of **Lyon** was also created to provide a more democratic type of recruitment than the standard "grandes écoles" (i.e. direct entrance upon perusing the student's record from the high school and thus avoiding—or incorporating—these special two or three years of preparation that are followed by incredible entrance competition exams for the admission to traditional "grandes écoles"). Although considered with some scorn by the more traditionally formed engineers, this proved to be an excellent idea. Soon some very good research was done there especially in the physics and mechanics of materials. One of its alumni, Marcel Berveiller, developed under the supervision of André Zaoui at the Paris-North University the Kröner technique for evaluating the effective properties of elasto-plastic polycrystals (Berveiller and Zaoui 1978). Berveiller contributed then to the creation of a true school of physics and mechanics at the University of **Metz**. Other *INSAs* have been opened in other parts of France, especially in Rouen. In Nantes, on the Atlantic coast, an *Ecole Supérieure de Mécanique* was transformed in the *ECN*. A somewhat similar school dealing also with electricity had been established in **Nancy**, in Loraine. In that school, thermal sciences, theoretical fluid mechanics—including Non-Newtonian fluids—and some modern thermo-mechanics were the main objects of research (e.g. recently by Jean-François Ganghoffer (cf. Ganghoffer 2003) and Christian Cunat), sometimes with active contribution of former students from the Paris school (e.g. Hélène Lanchon-Ducauquis).

Besançon, in the heart of the watch and clock industry (this is close to Switzerland) opened a school specialized in forming technicians for this industry in 1902. This was transformed into the *Institute for chronometry and micromechanics* in 1928. With the technical evolution of the 1960s–1980s, this institute became the *Ecole Nationale Supérieure de Mécanique et Micro-mécanique* (*ENSMM*), incorporating much robotics and mechanics of materials (e.g. with

Christian Lexcellent, a specialist of the modern thermo-mechanical modelling of shape-memory alloys).

The above described landscape of continuum mechanics in France in the 20th century and more particularly in its second half, bears the print of remarkable individuals, above all Joseph Pérès, Paul Germain and Jean Mandel. They succeeded to influence both subjects of study and institutions of research in the field, in both universities and "grandes écoles", to the point of re-placing France at the international level of competition that had somewhat disappeared in the interval between the two world wars. In the general theme of study, we witness an evolution from the traditional "rational mechanics" to a more physical view, sometimes yielding a true mechanics of materials. From the standpoint of basic teaching it seems that the general approach coupling intimately continuum mechanics and thermodynamics has been definitely adopted. For instance, the thermodynamics with internal variables of state has been integrated in most modelling all over the country.

7.7 Remark on Isolated Cases

This chapter would not be complete without the mention of two isolated cases of interest. One is the solitary work in elasticity by the Cosserat brothers at the end of the 19th century and in the first years of the 20th century (Cosserat and Cosserat 1909). As we know, this led to the publication by these authors of their (now) celebrated book on the theory of deformable bodies (1909). This originally received few echoes. It seems that only Joachim Sudria (1875–1950) published along the same line in Sudria (1926, 1935) (variational formulation of non-linear elasticity including couple stresses and consideration of a so-called Euclidean invariance, probably the first application of group theory to continuum mechanics).

Another isolated case is the publication by Brillouin (1889–1969)—a physicist formed at the *ENS* and of the Nobel-prize calibre (quantum theory of solids, Brillouin scattering, Brillouin zones, WKB method, Brillouin-Wigner formula, notion of neg-entropy, etc.) who later taught at Columbia and Harvard and became a specialist of information theory at IBM in the USA—of a remarkable book on the exploitation of *Tensors in Mechanics and Elasticity* (original French edition in Paris, Brillouin 1938; Dover 1946; English translation, Academic Press, New York, 1963). This was the first book of its kind in France and in the world with the exception of books on general relativity that necessarily involved tensors. It was supposed to be the first volume of a two-volume introductory course to theoretical physics, but the advent of WWII interrupted this project. We remind the reader that it is Woldemar Voigt (Germany, 1850–1919) who had identified tensors as the appropriate mathematical notion in his studies of the physics of crystals. Brillouin's book includes his own results in wave mechanics, radiation stresses, and the quantum theory of the solid state. This makes it one of the most original books in its class.

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Chapter 8
The Polish Strength

Abstract Poland is a country that suffered much from it various neighbors for almost 200 years. Having finally reached a certain stability after World War II, but under the acute and "benevolent" control of its big eastern brother, the Polish mechanics community succeeded in developing a remarkable research activity. Such activity justifies this independent chapter, all the more that the author knows well the country, having started his friendly visits there in the 1970s. No doubt that this development, out of proportion with the size of the country, is due to the excellence and hard work of a selected group of engineers, physicists, and applied mathematicians, among them W. Olszak, W. Nowacki, S. Kaliski, A. Sawszuk, and H. Zorski, and their disciples. These people rebuilt Polish mechanics on a ground that was solidly established early in the Twentieth century (by scientists such as Huber, Zaremba, Natanson, Zorawski, and Banach) as recalled at the beginning of the chapter. As exposed next, the main subject matters entertained in the second half of the Twentieth century have been plasticity, thermoelasticity, coupled fields (electroelasticity), wave dynamics, and generalized continuum mechanics in its different avatars. This undoubtedly received world applause. The positive role played by the Polish Academy of Sciences with its research centres is emphasized.

8.1 Historical Remark

It is not possible to grasp and appreciate the 20th century developments of Polish continuum mechanics without a clarifying brief historical survey of Poland in the 19th and 20th centuries.

Of course the great, almost mythical, scientific figure of Poland is Nicolaus Copernicus (Mikołaj Kopernik 1473–1543). The second figure is the internationally known Marie Curie (née Maria Skłodowska). Although Polish, Copernicus was a Renaissance man, a quadrilingual polyglot, who belongs to the international world of education as he received a large part of his formation in Italy and the region where he lived was strongly influenced by the Germanic Lutheran world.

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Marie Curie was scientifically educated in France and spent her whole career in Paris, being French by marriage with Pierre Curie, himself a remarkable physicist (symmetry, piezoelectricity, Curie temperature, etc.) and probably a deeper thinker than his wife. But Marie Curie in some sense epitomizes the typical Polish scientist at the end of the 19th century because she had to emigrate from her native country then under Russian rule where she could not receive any university education. This characterizes one of the historical points that we want to stress.

Indeed, after the Vienna congress of 1815 that restructured a large part of post-Napoleonic Europe, the Grand Duchy of Warsaw was split and its various pieces were redistributed. One part fell under Prussian rule (the Poznan area), one part including Warsaw—was attributed to the Russian Empire, and the "Galicia" region (Krakow and Lwow—now called Lviv in Ukraine) rapidly became part of the multi-national Austro-Hungarian Empire. Higher and technical educations fell under the direction of various more or less generous systems. For instance, a "*School of Civil Engineering, Ways and Bridges*" copying the French *Ecole des Ponts & Chaussées* (see Chap. 10) was opened in 1823 in Warsaw. But this was closed by the Russian authorities after the 1830–1831 insurrection. A Warsaw polytechnic school was re-opened only in 1901 with Russian language of instruction. Many Polish engineers were therefore educated in Russia, essentially at the St Petersburg University and the St Petersburg *Institute for Engineers of Ways and Communications* (another imitation of the French *Ponts & Chaussées* school). In Galicia, a polytechnic school was created in Lwow while an Academy of Arts and Sciences opened in Krakow (1871–1919). Both admitted Polish scientists independently of their citizenship. But as mentioned above, many of the Polish engineers were educated in Russia. Also, in the period 1825–1875, many Polish engineers were educated in France, especially at the *Ecole Polytechnique* and the *Ecole des Ponts & Chaussées*. Many diplomed engineers emigrated from Poland to join a variety of countries in Europe, North and South Americas, and even to Turkey and Australia. This explains the practice of many foreign languages by Polish scientists, notably German and/or Russian, and also French as a more literary and elite tradition in Slavic countries.

While Poland regained its independence in 1919 it was to suffer its more drastic period during WWII when it lost three quarters of its intellectuals and people with academic diploma. The blow was terrible so that practically everything had to be started again from scratch in 1945–1946. The communist regime that shortly followed re-organized science and technological research along its own lines (see below) with a strong print of Soviet influence.

8.2 Polish Mechanicians in the Early 20th Century

The early 20th century in continuum mechanics in Poland is marked by a few scientists and engineers who had most of their career in occupied Poland (until 1919). Among them we must single out Marian Smoluchowski (1872–1917), not strictly an engineer—he contributed much to the theory of Brownian motion in friendly competition with A. Einstein. A professor first at the University of Lwow in theoretical physics and then in Krakow in experimental physics, he published in French, German and Polish. In addition to his physics specialty, he was also interested in various problems of mechanics, including temperature-dependent elasticity moduli, and problems dealing with aerodynamics and viscous fluids. Other known contributors to continuum mechanics in the first half of the 20th century were, among others, Władysław Natanson (1864–1937), Stanisław Zaremba (1863–1942), Kaziemierz Zorawski (1866–1953), Maksymilian Huber (1872–1950), and Stefan Banach (1895–1945). Natanson, much praised by C.A. Truesdell, had a marked interest in problems of irreversible thermodynamics and the hydrodynamics of viscous fluids, before he turned to less mundane physical subjects. Zaramba who studied in St Petersburg, Paris and Berlin, taught in Krakow. He is famous among our community for the introduction of the now called *Zaramba-Jaumann derivative* or co-rotational derivative. This provides a so-called *objective* time derivative (invariant by time-dependent rotation of a rigid frame). This object allows one to formulate in the best manner a truly invariant theory of elasto-plasticity in finite strains, as also complex theories of electro-magneto-deformable media (cf. works by G.A. Maugin). He was also interested in the theory of invariants and variational inequalities (he influenced Arthur Korn along this line), both to become later of great importance in continuum mechanics. Zorawski's name is attached in fluid mechanics to theorems dealing with rotation and flux conservation. By some coincidence, Maria Skłodowska—the future Madame Curie—was employed as a home teacher in Zorawski's home before her departure to Paris.

M. T. Huber deserves a special notice because of the importance of his work for plasticity. Especially gifted as a young student at the Imperial-Royal Polytechnic in Lwow, he published his first paper when he was only 18 years old. His initial works dealt with the contact problem, already approached but not entirely solved by Hertz. His most well known paper, however, was a paper in Polish published in four parts in 1903 and entitled (English translation) "Specific strain work as a measure of material effort—A contribution to the foundations of the strength theory". This paper was translated into English only in 2004 [Arch. Mech. (PL), 56/3, 173–190.] It in fact introduces the strain energy of distortion as a criterion for measuring the yield of elasto-plastic materials as proved later on by Hencky (1924). Richard von Mises obtained a similar criterion in 1913. Often this criterion is unjustly referred to only as Mises' criterion. This was a nice step forward in that the new criterion replaced the Tresca-Saint–Venant criterion of 1871–1872 that can only be represented by a *set* of inequalities. Indeed, in the appropriate stress space, the section of an hexagonal prism is replaced by a circumscribed circle, providing a mathematical facility of treatment of elasto-plastic problems. The two Tresca-Saint–Venant and Huber-Mises criteria give equal resistance to simple traction and to simple compression, but different resistances in pure shear. In the case of metals, the experimental results are contained between these two criteria, although they are generally closer to the Huber-Mises criterion as shown by well-known experiments conducted by G. I. Taylor and H. Quinney in 1931 (see Chap. 1 in Maugin's book on plasticity and fracture, 1992). M. Huber corresponded with S. P. Timoshenko and B.G. Galerkin. Back in Poland after being a prisoner in Russia during WWI he returned to his alma mater in Lwow where he became the rector of the Polytechnic school. In 1928 he joined the Warsaw Polytechnic. He devoted some of his work to the study of plates, in particular anisotropic ones. He had also many other scientific interests as for instance translating into Polish Einstein's book on *special and general relativity*, and Marie Curie's book on *radioactivity*. But his name for ever remains attached to elasto-plasticity where he had excellent disciples, including Wacław Olszak who was to play a fundamental role in Polish mechanics after WWII.

In this group of strong personalities we must also include the famous mathematician Stefan Banach who, in addition to devoting his research mostly to the foundations of functional analysis and introducing the notion of spaces that bear his name, taught theoretical mechanics, writing an influential and marvellous book on mechanics that was translated both in English and in French in Warsaw in the early 1950s.

> Personal touch: The author took his foreign-language requirement—German and French—for the Ph.D. at Princeton with Marian Smoluchowski's son, Roman (born in Zakopane in 1910—died in Texas, 1996), then a professor of solid-state physics at Princeton. R. Smoluchowski was known among students for his kindness and generosity in passing them at the language tests—he himself wrote in at least five different languages.

8.3 Reconstruction of Polish Mechanics After WWII

It seems that three individuals have been most active in the revival of Polish continuum mechanics after WWII. These are Wacław Olszak (1902–1980), Witold Nowacki (1911–1986), and Sylwester Kaliski (1925–1978).

Wacław Olszak was born in Silesia then part of the Austro-Hungarian Empire. He studied civil engineering at the Technische Hochschule in Vienna (1920–1925) and then in Paris at the Faculty of Mathematics (1925–1927). He also attended violin classes at the Vienna conservatory where he became a good performer. He wrote one doctoral thesis in Civil Engineering in Vienna (1933) and another one with M. Huber at the Polytechnic in Warsaw in 1934. He was at the time interested in mathematical problems in elasticity. Engaged in forced labour by the German, he spent WWII as a worker and driver. He returned to Poland in 1946 to join the Krakow Polytechnic. He became interested in the mechanics of pre-stressed concrete structures. He accepted the chair of the Strength of Materials at the Warsaw Polytechnic in 1952. The following year he participated actively in the creation of the celebrated *Institute of Fundamental Technological Problems* (abridged to I.P.P.T in Polish) of the Polish Academy of Sciences, where he was in charge of the Department of Continuum Mechanics. He became its general director in 1964 until 1969. He was really quite open-minded concerning the various developing trends in continuum mechanics. But he himself concentrated on the theory of plasticity (cf. Olszak and Sawczuk 1967). Under his leadership the I.P.P.T became a world renowned centre for this speciality. Among his many disciples were Antoni Sawczuk (1927–1984, cf. Sawczuk ed. 1973) , Zenon Mroz (cf. Mroz 1963, 1967), Wojciech K. Nowacki (1938–2009), and Piotr Perzyna. The first of these scientists established a strong co-operative link with French research centres in plasticity (in particular Grenoble, see Sect. 7.4) while W.K. Nowacki did the same with the *Ecole Polytechnique* in Paris, specializing in finite strains in elasto-plasticity and wave-front propagation (cf. W.K. Nowacki 1978). P. Perzyna developed an original theory of visco-plasticity (cf. Perzyna 1966), incorporating the latest developments in thermo-mechanics by American scientists (Coleman, Noll, Truesdell, Gurtin). This was a very original move at the time in countries east of the Iron Curtain. These scientists themselves had students who continued along the same line, constantly enriching the field at an international level. Professor Olszak was also very active at the European and international scales, becoming the resident Rector of the newly created *International Centre for Mechanical Sciences* (*C.I.S.M* in its abridged original French form) in Udine, Italy in 1969. He was also instrumental in creating the Polish journal *Archives of Mechanics*. A polyglot like many of his Polish peers educated before WWII, Olszak published not only in Polish but also in German, French, English and Hungarian. He also spoke other Roman languages.

Personal touch: Basing on the author’s experience W. Olszak was a very kind and soft speaking person.

Witold Nowacki was born in the north east of Poland (then under Prussian rule). First trained as a civil engineer this led him to participate in the construction of both secular and religious buildings. He was first Professor at the Gdansk University of Technology. Captured by the Germans and kept as a prisoner of war in Woldenberg in a camp for officers, he had plenty of forced free time to ponder the possible re-organization of science in future Poland in post-WWII. On his appointment at the Warsaw Polytechnic in 1952 after gaining his doctoral degree in 1945, he became involved in the re-organization of the Polish Academy of Sciences (for short in Polish: *P.A.N*) of which he finally became president in 1978. W. Nowacki was a shy and somewhat reserved person. Nonetheless, he was a strong and powerful organizer. Together with M. Huber, W. Olszak and W. Wierzbicki he created the “*Archives de Mécanique Appliquée*” (*Archiwum Mechaniki Stosowanej*) in 1949. This was to become *Archives of Mechanics.* He supervised the publication of the “*Bulletin de l’Académie Polonaise des Sciences*” (Technical sciences). As already mentioned he was also active in the development of the Polish Academy of Sciences and the growth of the I.P.P.T. Among his scientific interests that spanned all fields of applied mathematics and tremendous developments, we find the analytical study of generalized continuum mechanics (especially Cosserat continua; cf. Nowacki 1986b) and of multi-field problems (thermo-elasticity and magneto-electro-elasticity; cf. Nowacki 1986b), encouraging the study of dynamical problems. He was internationally applauded and received many honours all over the world, in particular in France, the UK and the USSR. He was among the founders of the *CISM* in Udine. His disciples and co-workers in Poland and outside are too numerous to be cited. As a final note, we emphasize that he served as a protector of many Polish scientific "rebels" against the political system at work during his Warsaw years.

Personal touch: The author took a course (in German) on polar elasticity with W. Nowacki in Udine, Italy, in the summer of 1970. That is when he established a close and enduring contact with Polish mechanics, to the point of being elected to the Polish Academy of Sciences in 1994. Witold Nowacki somewhat belonged to a vanishing class of scientists; he told the author in 1975 that someone does not comprehend a field in its subtleties and totality until he has written a book on it; I tried to follow his advice.

Sylwester Kaliski is a totally different kind of personality. Most of his scientific career is connected with the Military Technical Academy in Warsaw (in Polish *WAT* = *Wojskowej Akademi Technicznej*). Initially formed at the Gdansk Polytechnic (1951), he obtained his doctoral degree at *WAT*, where he finally reached the rank of general and which he directed from 1967 to 1974. He did many works in coupled field theory (thermo-elasticity, magneto-elasticity), often trying to combine theoretical physics and continuum mechanics. A smart and gifted scientist he was not an easy person and was quite ambitious, to the point of dreaming of becoming the Polish "Edward Teller", with projects on the "Polish bomb" and thermo-nuclear fusion where he claimed to have reached tremendous temperature levels with a laser apparatus. He died untimely in a car accident. The years at *WAT* in the early fifties were spent under the supervision of Russian army people. This may have been rather unpleasant to Polish scientists reputed for their nationalistic feelings. Still many young people found a job there where they acquired their basic formation simultaneously in mathematical physics and continuum mechanics. This should have been a copy of the French *Ecole Polytechnique* (still under the directorship of a general at the time of writing) with the aim of forming military engineers of high level. C. Z. Rymarz—who became a colonel—was one of them. Henryk Zorski (1927–2003) was also among these young people who later joined the I.P.P.T and often built a bridge between mathematical physics, solid state physics, and continuum mechanics. In turn he influenced younger people such as Dominik Rogula (1965), Kazimierz Sobczyk, J. Kapelewski, J. Petykiewicz and others. Newly expanded fields of mechanics were the interaction of crystal defects (dislocations and disclinations), the truly *nonlocal* theory of continua, and many problems of coupled fields and both surface and bulk waves. In 1991 *WAT* became the *Institute of Technical Physics* situated on the well named "Kaliski" street.

Along a different line, Krzysztof Wilmanski (1940–2012) is more difficult to categorize because of his great mobility. He was basically educated in his native city of Łodź (M.Sc. 1962, Ph.D. 1965), but he obtained his habilitation at the IPPT in Warsaw in 1970. He left Poland for Germany in the early 1980s although he was officially affiliated with the IPPT between 1966 and 1986. He was a close friend of Henryk Zorski (see Wilmanski 2004). He had also made a short stay at Johns Hopkins and taught in Baghdad (Irak) on an educational co-operative programme.

In Germany, he was successively in Berlin, Paderborn, Hamburg-Harburg, TU Berlin, Univ. Essen and at the Weierstrass Institute for applied mathematics and statistics (1996–2005) in Berlin before joining the University of Zielona-Gora—close to the German border—back in Poland (2005–2010). A polyglot who published very early (as soon as 1962), Wilmanski demonstrated a strong attraction towards axiomatics and abstraction in his first works (e.g., his first lengthy work—his habilitation—on phenomenological thermodynamics, 1974). He may be considered a disciple of the Truesdellian school, but also an advocate of Ingo Müller's views. He in fact co-operated with Müller in the mixed continuum–statistical-mechanical approach to pseudo-elastic bodies (i.e., elastic bodies presenting small hysteresis loops, very much like the electric response of ferroelectrics because of a non-convex potential). His other multiple scientific interests include the theory of mixtures, phase transformations, non-Newtonian fluids, acoustic waves, and crystal plasticity. But above all he developed a keen interest for the thermomechanical description (finite strains, thermal effects, wave properties; cf. Wilmanski 1996) of porous media when he was in Reint de Boer's group in Essen. This interest has remained active and productive at the Weierstrass Institute and on, with fruitful contacts established in the community of geophysical sciences, especially in Italy. He also contributed general texts on continuum thermomechanics (Wilmanski 1998, 2008). Probably because of his extreme mobility, he supervised very few Ph.D. theses, among these few, those of Marek Elzanowski (who became a professor in Portland, USA) at the IPPT and of Bettina Albers in Berlin.

Also, we cannot forget Jozef Joachim Telega (1943–2005)—a dear friend of the author—initially formed in Gliwice (Silesia) who was at the I.P.P.T from 1977 to his death. There he developed single-handedly an activity in applied functional analysis with a specific interest in variational methods and homogenisation techniques very much along the French line (he had spent some time in Paris). In his last years, strongly impeded by a degenerative illness of the bones, he developed a personal interest in orthopaedic biomechanics exercising in parallel a very intense editorial activity to the benefit of both Polish and international communities.

This vast landscape description would not be complete without mentioning some of the Polish applied mathematicians and mechanicians who left early enough and created research centres elsewhere. Here we must first name Olgierd C. Zienkiewicz (1921–2009)—one of the creators of the finite-element method (see the world renowned book by Zienkiewicz 1971)—who left Poland at the beginning of WWII, and developed a whole school of computational mechanics in the UK after completing his university education in England. Also, Richard B. Hetnarski (born 1928), educated in Gdansk and Warsaw, carried to the USA the spirit of Polish thermo-elasticity in 1969, first at Cornell, and next at the Institute of Technology in Rochester, NY, that he joined in 1992. He founded (1978) in the USA the *Journal of Thermal Stresses*, and later on a successful series of international conferences known under the title "*Thermal stresses*". He is the author of a well known monograph (Hetnarski 1986) and the general editor of the formidable *Encyclopedia of Thermal Stresses* (Springer, 2013).

In all, the strong impulse given by a series of remarkable individuals after WWII led the Polish school of mechanics to the forefront of this field in the world, it is true, somewhat out of proportion with the size and population of the country. This is all the more an incredible achievement that we place at the same level as the reconstruction of the old city in Warsaw after WWII.

8.4 Further Reading

On the history of mechanics in Poland in the period 1950-1990, see Germain (1981); Nowacki (1985); Maugin (1988); Olesiak (2004); Biographical notices (1981). Note that the feverish research activity in Warsaw in continuum mechanics fostered the publication of many research monographs (originally in Polish but often with a foreign publication agreement in Western Europe) by the Polish Scientific Publisher PWN. This is illustrated by, among others, the monographs by Kleiber (1989), König (1987), Nowacki (1975, 1983, 1986a), Nowacki WK (1978), Sobczyk (1985), Wilmanski (1974), and Zorski (ed 1979, 1992).

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Chapter 9
German Revival in Continuum Mechanics After WWII

Abstract Contrary to France, Germany is not a centralized country, having lastly taken a federal form but having in the past been made of a variety of smaller states. A consequence of this mosaiclike structure is the multiplicity of scientific and engineering strongholds in friendly competition. This, together with the traditional strength of the German mechanical industry and the success of German scientific giants in the Nineteenth century, explains the status of continuum mechanics in Germany in the second half of the Twentieth century when a revival was necessary after World War II. Before WWII, the strength of the German mechanical community had materialized in a well-organized scientific society (GAMM) and journal (ZAMM) and influential textbooks (Föppl, Hamel). After WWII, the network of celebrated Technical universities was successfully revived and extended, while the Journal known as Ingenieur Archiv won prominence. The chapter exposes the role played by various centres (Munich, Bochum, Hannover, and also Berlin, Darmstadt, Aachen, etc) with the corresponding strong personality of the local leaders. Rather typical interests of German institutions are reported involving problems of plasticity, generalized continuum mechanics, fracture mechanics, and more recently the continuum thermodynamics of complex materials and computational mechanics. A successful blend of modern continuum mechanics and numerical techniques justified in a rigorous mathematical frame has thus emerged.

9.1 Pre-WWII Germany and Mechanics

Germany is the most populated country in Western Europe. Very well equipped with a network of technical universities (e.g., in Aachen, Braunschweig, Darmstadt, Karlsruhe, Hannover, Berlin, Stuttgart, Munich) in the period following WWI, the country was at the time the strongest industrial nation in Europe. Mechanical sciences and more particularly continuum mechanics (then fluid and solid mechanics, separately) strongly benefited from such a favourable

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environment. Also, thanks to luminaries such as Richard von Mises and Ludwig Prandtl, the country succeeded in organizing its scientific community in an efficient framework. This is illustrated by the creation of the Gesellschaft für angewandte Mathematik und Mechanik (*GAMM*) in 1921. The yearly meetings of this society have been—and still are—instrumental in spreading a common spirit and helping the share of new ideas. The good idea—unfortunately not followed in France and the USA—was to combine both applied mathematics and mechanics in the same society, to the benefit of both. In the same line of thought the same people with a future-oriented vision had created a specialised scientific journal with title "Zeitschrift für angewandte Mathematik und Mechanik" (for short *ZAMM*).

Probably the most influential thinker on the foundations of mechanics in the post WWI period was Georg Hamel (1887–1954) who proposed a rather sound axiomatization of mechanics (cf. Hamel 1922) and formed generations of German mechanicians through his textbooks (e.g., Hamel 1949). August Föppl's (1897-1900) books were also much influential. Like in other fields of knowledge, the advent of the Third Reich was a catastrophe for continuum mechanics, as it led to the emigration of many valuable scientists. However, some mixed theoretical and applied sciences such as aerodynamics and combustion studies kept a strong momentum due to military needs. They led to the successful design of both civilian and military airplanes, including of the jet type (not to speak of rockets such as V1 and V2).

9.2 German Revival of Mechanics After WWII

A large number of official buildings, among them educational ones, were destroyed in Germany at the end of WWII. But most technical schools were reopened. In the course of time some of them would change their name from *Technishe Hochshule* to *Technische Universität* (e.g., Darmstadt) while others would simply drop the "Technical" to become standard scientific universities (e.g., Stuttgart). Others, still, kept the "technical" connotation in their name but became multidisciplinary universities offering courses in both sciences and humanities (e.g., TU Berlin). New universities with a technical vocation were also created such as in Kaiserslautern and Bochum (Ruhr Universität). But the strongholds in mechanics remained the older schools such as Aachen, Munich, Hannover, Darmstadt, Berlin and Karlsruhe.

Of course the situation was different in the German Democratic Republic (DDR; until the re-unification of the country in 1990) where technical schools were in Rostock, Magdeburg, Karl-Marx-Stadt (now back to the old name "Chemnitz") where education was very much influenced by the Russian one and several of their students had the opportunity to continue their graduate studies in the Soviet Union. East Germany inherited the editorship of *ZAMM,* while in Federal (West) Germany the Journal titled *Ingenieur Archiv* was the most popular journal dealing with many applied engineering types of problems, e.g., dealing with the strength of materials and more generally the mechanics of structures. This of course corresponded to the specific interest of German engineers for solutions of these problems for which they had developed a remarkable professional knack. This journal published essentially in German, but the journal became a more international forum when it switched to English and changed its title to *Archives of Applied Mechanics* (for short *AAM*) under the head-editorship of Prof. Horst Lippmann in Munich (see below), and then Prof. Reinhold Kienzler in Bremen.

With time the emphasis of research in German continuum mechanics shifted, like in other countries, towards more theoretical and formal subjects under the influence of the American school (see Chap. 5), the introduction of thermo-mechanics, the mechanics of materials, and numerical solutions and simulations, all entangled with good applied mathematics. In particular, elasto-plasticity, problems of symmetry, and generalized continuum mechanics (e.g., Cosserat media; See Chap. 13) were given special attention. However, for some time, the German tradition was kept of a compulsory previous industrial experience before obtaining a chair of applied mechanics.

It is impossible here to screen all of the German achievements in continuum mechanics during the second half of the 20th century. We shall be satisfied—and we hope to satisfy the curiosity of readers—by perusing the career and works of a few remarkable individuals who left a strong print by organizing local schools of international level and forming a large number of disciples. They forcefully contributed to a rebirth of German continuum mechanics.

9.3 Leaders and Organizers

9.3.1 Theodor Lehmann and Bochum

In 1962 faculties for Mechanical Engineering and Civil Engineering were established at the new university known as the *Ruhr-Universität Bochum*. Professor Theodor Johannes Lehmann (1920–1991) became the first holder of the chair for Mechanics. The scientific formation and industrial experience of Th. Lehmann were classical for the time period. He had been formed as an engineer in Mechanical engineering just after WWII—where he practically spent 6 years in the armed forces—at TH Hannover where he obtained his doctoral degree in 1949. He had 4 years of industrial experience and returned to TH Hannover before joining Bochum in 1969. There he exploited his talents—and his kindness—to create a true centre of continuum mechanics dealing with all aspects of this science, but more particularly elasto-plasticity, thermal effects, and fracture. He formed and/or attracted such known scientists as Otto T. Bruhns—who succeeded Lehmann at the Chair of Technical mechanics—, Prof. H. Stumpf, and Prof. K. Chau Le (originally from Vietnam, but formed in Moscow in Leonid Sedov's group).

9.3.2 Horst Lippmann, TU Munich and the AAM

Quite originally in the German engineering community of post WWII, Prof. Horst Lippmann (1931–2008) received a university formation in abstract mathematics and theoretical physics in East Germany with a doctoral degree in mathematics in 1955. He joined the Institute of Mechanics in Hannover in 1957, having developed an interest in mechanical engineering. Together with Oskar Mahrenholtz (see below) he became much involved in so-called *plasto-mechanics* in which Germany had been somewhat surpassed by other countries. The scope of his fruitful research then became extremely large, covering engineering plasticity, the plasticity of granular media, the plasto-mechanics of forming, and the study of rockbursts in particular in underground mines. For these works he received many honours (prizes, medals, honorary doctorates; memberships in scientific societies). His university career was spent successively at TU Braunschweig, the University of Karlsruhe, and finally TU Munich (1975–1996) where he also supervised the State Office of Material Testing for Mechanical Engineering. He was very active in teaching. It is said that more than 10,000 students completed his basic courses in mechanics. He mentored about 50 doctoral students. He did not loose interest in mathematics as is witnessed by his book on the application of tensors published in 1992.

Prof. Lippman, a good natured and ever active person, contributed to both German and European organization in mechanics as he was very instrumental in the programme of the *International Centre of Mechanical Sciences* (*CISM*) in Udine, Italy, as well as being associated for almost 50 years with the *Mathematischen Forschungsintitut* in Oberwolfach (Black Forest) where he co-organized periodical sessions on the mechanics of materials. Finally, as a chief-editor of the *Archives in Applied Mechanics*, he re-oriented that journal to the international community and to more theoretical and mathematical subjects while keeping the engineering spirit alive.

9.3.3 Erwin Stein and TU Hannover

The prevalent role played in mechanics by TU Hannover is already illustrated in the two foregoing cases. This is even truer with the case of Prof. Erwin Stein (born 1931). E. Stein received a civil-engineering and mathematical education at TH Darmstadt and then a doctoral degree dealing with the mechanics of structural elements at the University of Stuttgart (1964) and Habilitation there in 1969. This last diploma involved the recently (at the time) formulated Finite-Element Method (for short *FEM*). Indeed, this was a rather recent discovery since the powerful *FEM* formulation is attributed to a trio, Ray W. Clough (USA), Olgierd Zienkiewicz (UK), and John Hadji Argyris (1913–2004) in the 1960s. The latter was of Greek origin, studied first at the Technical University in Athens, and then TU Munich, and concluded his formal engineering education at the Polytechnicum

(ETH) of Zürich. He taught at the Imperial College in London before moving to TU Stuttgart in 1959. We may assume that the virus of *FEM* was injected to Stein by Argyris himself. Anyway, numerical computations became a favourite subject for Stein for his whole professional career which he mostly spent at TU Hannover (1971–1999) holding the Chair of Structural Mechanics and then of *Computational Mechanics*.

A powerful thinker, hard worker, and a very well educated gentleman with historical and philosophical interests (especially Leibniz, a “Hannover-ian”), Erwin Stein could develop in Hannover a true school of computational mechanics. That school was to blossom all over Germany through his many doctoral students (almost a hundred as first or second supervisor) and co-workers, to name a few: P. Wriggers, C. Miehe, R. Mahnken, V. I. Levitas, P. Steinmann, F. J. Barthold, S. Ohnimus, M. Rüter, E. Kuhl, and A. Mielke. With so many bright disciples the spirit of Hannover computational mechanics was carried over to many of the important German universities. The numerous works of Stein and co-workers touch all aspects of the numerical approach to the mechanics of structures and materials, including the influence of microstructure, the presence of cracks, phase transformations, large plastic deformations, thermo-mechanics, shape-memory effects, and all the technical aspects of the required variational formulations and computations (adaptability, stability, etc.,). Rarely have we seen such an influence in a whole country except perhaps with the more mathematically minded Jacques-Louis Lions in France.

Among his students we must single out Paul Steinmann who became a professor first in Kaiserslautern and then in Erlangen-Nürnberg, and who himself formed many “grand-sons” of Stein, and Ellen Kuhl who successfully applied her training to complex problems of biomechanics. She is now at the University of Stanford, California. These two cases illustrate perfectly the reason for the success of Stein’s school: an efficient combination of nonlinear continuum thermo-mechanics and of an excellent knowledge of the most efficient computational methods.

9.3.4 Oskar Mahrenholtz and Northern Germany

Oskar Mahrenholtz (born 1931) was first educated at the Ingenieurschule in Hamburg. He then studied at the Max-Planck Institut in Göttingen and the TH Hannover. This is where he co-operated with his fellow researcher Horst Lippman on “plasto-mechanik” (cf. Lippmann and Mahrenholtz 1967). He also became professor there and head of the Institute of Applied Mechanics before joining in 1982 the TU Hamburg-Harburg where he remained until he was named emeritus in 1996. Mahrenholtz, a highly educated gentleman, not only contributed to various fields of applied and continuum mechanics, especially in connection with the behaviour of large deformations of metals and polymers, and also with the naturally associated field of ocean engineering (he was the director of the corresponding institute in Hamburg), but he proved to be a remarkable organizer never loosing contact with industry. He brought his gained experience and expertise to many scientific organizations whether in Germany (the powerful German Research Foundation called *DFG = Deutsche Forschungsgemeinschaft*) or at an international level—he was the representative for Germany in the *NATO* Science Committee.

In concluding this section we note the remarkable fact that all four above scientists spent some time of their studies or professional career in Hannover. Furthermore, the last three were born the same year—1931—and with a compulsory age of retirement at 65 or 68, they perfectly fit the second half of the 20th century. We shall now consider younger contributors and smaller, but perhaps as much important, research centres.

9.4 Other Schools and Centres of Research

9.4.1 Berlin

The non-German readers may wonder why Berlin was not cited before as an active research centre. The reason may be found in the special political and geographical location of this city during the period 1945–1990. First there was a hard work of reconstruction of the city, and then the separation in two “Berlin” during the cold war, resulting in a city locked within walls, and the split in universities between West Berlin and East Berlin. The Humboldt University continued as the main educational centre in East Berlin, while TU Berlin on the west side was complemented by the newly created Free University. In so far as we know no substantial continuum mechanics was created or even studied either at the Humboldt University or at the Free University. The burden fell on TU Berlin and the Federal Institute for Testing Materials in Mechanics (so-call *BAM*).

At TU Berlin there was Istvan Szabò (1906–1980) who was an influential professor of applied mechanics in the period 1948–1973. According to Walter Noll’s recollection Szabò’s teaching was very classical in form and much focussed on the strength of materials. However, Szabò had the immense merit to write in German a celebrated *History of the Principles of Mechanics* (Szabò 1977). This became part of the natural curriculum of all students in mechanics in Germany. This has no equivalent for students in the English or the French languages, and this is much wanted. However, TU Berlin became some kind of stronghold for the thermodynamics of continua. This is due to the originality of the works by Ingo Müller (see Chap. 5 for his creative works) in rational and extended thermodynamics (cf. Müller 1973; Müller and Ruggeri 1998), and the synthetic and critical work of Wolfgang Muschik (a disciple of Walter Schottky—cf. Muschik 1990). But this was really achieved outside any department of engineering sciences. Other teachers such as Rudolph Trostel and Arnold Krawietz (1986), although not so much involved in theoretical research, were influential. In particular, they influenced Albrecht Bertram—first at *BAM* in Berlin and then at Magdeburg after the re-unification of Germany who presented an axiomatization of continuum mechanics with a strong flavour of the Coleman-Noll vision—cf. Bertram (1989). Bertram also contributed to the theory of finite deformations in plasticity. He was recently joined by Holm Altenbach from Halle-Wittenberg, a specialist of shell theory who, originally from East Germany, had obtained his PhD in Leningrad/St Petersburg.

9.4.2 Darmstadt

T.H Darmstadt, now called TU, has for a long time been one of the acknowledged schools of mechanics. In recent times, two actors have played there an important role for continuum mechanics. One is Dietmar Gross and the other is Kolumban Hutter.

Prof. Dietmar Gross is Austrian by birth but obtained his doctoral degree in Rostock (then in the DDR) in 1968. He habilitated in Stuttgart. He joined TH Darmstadt in 1976 to stay there until retirement. His main works concern the theory of fracture (see his book on *Bruchdynamik*—revised edition by Gross and Seelig 2001), configurational mechanics (see Chap. 14 below), and the mechanics of materials at macro and micro scales. He is influential throughout Germany with the collection of books he co-authored on *Technische Mechanik*. In research, he formed a real school of successful students, among them Ralf Müller who succeeded Paul Steinmann on the chair at Kaiserslautern.

The other strong personality at TH/TU Darmstadt has been K. Hutter. Originally from Switzerland, but with a PhD obtained at Cornell and a Habilitation presented in Vienna, Hutter contributed to so many branches of continuum mechanics with a long roster of remarkable research papers and also a large number of books (among these, Hutter and Jöhn 2004). His large main field seems to be the thermo-mechanics of continua in which we can feel a strong influence of Ingo Müller's approach. They in fact created together the journal entitled *Continuum Mechanics and Thermodynamics* (for short CMT), now in the hands of other people. As a true Swiss, he contributed much to the modern mechanics of ice and the flow of glaciers for which he is the acknowledged best specialist, introducing there all good elements of non-linear continuum mechanics and of thermodynamics.

9.4.3 Other Contributors and Places

To be complete, we should also cite other people who have been active on the German stage of continuum mechanics. First we note Dieter Weichert—for sometime at Karlsruhe and in France at Rouen—at TU Aachen with his main scientific contributions in the field of the shakedown analysis of plastic structures. Next, we have Reinhold Kienzler in Bremen who has been instrumental in developing some aspects of fracture mechanics and configurational mechanics (see

Chap. 14, also Kienzler 1993) in particular in co-operation with George Herrmann in Stanford and in its applications to the strength of materials (cf. Kienzler and Herrmann 2000). He plays an important role with the editorship of the *Archives in Applied Mechanics*. We also have Wolfgang Bürger (born 1931) in Karlsruhe who co-authored an influential book on continuum mechanics, and Peter Haupt (born 1938) who, in Kassel, creatively dealt with finite deformations, visco-elasticity and visco-plasticity of elastomers. W. Bürger was popular in Germany because of his appearance in a TV programme (*Kopf um Kopf*) popularizing science with illustration by means of mechanical toys, and for its column in the Magazine *Bild der Wissenschaft*. Finally, Prof Bob Svendsen, who presents the originality to be American born, with an initial formation in geophysics (MS, PhD Caltech 1987), a further training in glaciology at ETH Zürich, a stay at *BAM* in Berlin, and then a University position in Dortmund, and finally went to RWTH Aachen on a new chair of material mechanics. His contributions are essentially in the thermo-mechanics of crystal plasticity, granular materials, and, like many others, the transition between macro- and micro scales.

Somewhat on an apart status we cannot avoid mentioning the study of *porous continua*. The main German contributor to this field in the spirit of continuum mechanics was Reint de Boer (1935–2010). Also educated in engineering at TH Hannover with H. Lippman, he obtained both his PhD and Habilitation there, where he remained as a professor before joining the University of Essen in 1977. He stayed in Essen until retirement, having in this period created a true school involved in the thermo-mechanical modelling and the theory of porous media. He was also much interested in the recent history of the subject matter (in particular as concerns the contribution of Paul Fillunger as opposed to that of another pioneer, Karl von Terzaghi, the Austrian civil engineer considered to be the father of the modern theory of soil mechanics—see de Boer's (2000) book on the subject). One of his most successful students was Wolfgang Ehlers, also originally with a diploma in civil engineering from TH Hannover in 1979, who became a Professor of continuum mechanics at TU Darmstadt (1991–1995) before joining his actual position at the University of Stuttgart. Ehlers has developed a general view of continuum mechanics and the theory of porous media (cf. Ehlers 2010) with a mixed interest in modelling, theory and numerical simulations, as also in other fields such as biomechanics and the electro-chemical–mechanical couplings.

9.5 A Peculiarity: German Contributions to Generalized Continuum Mechanics

It seems that Cosserat continua (see Chap. 13) attracted very early German scientists in the 1950–1970s. First among them were W. Günther (1958), H. Neuber (1964) and H. Schaeffer (1967). But this is also true of many of the already mentioned scientists who all at a time—perhaps within a fashionable trend—

contributed to this special case of generalized continuum mechanics. Simultaneously, but not always related to these developments, many advances were reached in the field of structural defects, principally by Ekkehart Kröner (1958) in Stuttgart with the introduction of deep geometrical concepts, his direct students or co-workers, K.-H. Anthony and B.K. Datta, and recent followers such as Markus Lazar now at TU Darmstadt. The *nonlocal theory* of continua was also introduced by these authors (cf. Kröner and Datta 1966). This was more the work of theoretical physicists than that of mechanicians. Recent works on Cosserat media were done in many places, including in Halle-Wittenberg, Saarbrücken, Bochum, Erlangen, and Berlin as proved by the contributions to the celebration of the Centenary of the publication of the Cosserats' book in 2009 (cf. Maugin and Metrikine 2010). The interaction with the theory of configurational forces was also cultivated in many places including Darmstadt, Kaiserslautern and Erlangen.

9.6 Conclusion

According to the author's somewhat external view, the German development of continuum mechanics in the second half of the 20th century is marked by characteristic traits: (1) a continuous interaction with industry, (2) the multiple influence of various technical universities (with a "plus" granted to Hannover in its formative role), (3) the importance of plasticity studies but the fact that all scientists finally joined the bandwagon of thermo-mechanics, (4) the fruitful role played by the special programmes of the *DFG*, (5) the unity enforced by the existence of the *GAMM* and its yearly meetings, the journals such as *Ingenieur Archiv* (and then *AAM*), and the meetings in Olberwolfach, (6) a proliferation of competitive books on basic continuum mechanics (cf. Altenbach and Altenbach 1994; Basar and Weichert 2000; Becker and Bürger 1975; Bertram 2005; Haupt 2002; Hutter and Jöhn 2004; Krawiertz 1986; Lehmann 1975-1994), and (7) the perhaps immoderate attention paid to tensors written in components (see the books by Betten (1987), de Boer (1982), and Lippmann 1993). This does not hinder the obvious existence of networks of relations and lobbies, but maybe not at the scale met in centralized countries like France. In a sense, the federal structure of the country and the co-existence of many technical universities of equivalent level but with a rewarding competition are assets from this standpoint.

Personal touch: The author has had many contacts with German continuum mechanics. He visited most cited places, and delivered a large number of seminars, while having German co-workers and associates, and contributing several times to the annual *GAMM* meeting. It happens that he was selected as a member of the *Wissenschaftskolleg* (Institute for Advanced Studies) in Berlin for the year 1991–1992, he holds an honorary Doctoral degree in Natural Science from TU Darmstadt, and he was awarded a *Max-Planck Prize* for research conferred jointly by the Max-Planck Society and the Humboldt Foundation in 2001.

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Chapter 10
European Miscellanei and Asia

Abstract This chapter mostly concerns European countries that do not receive a separate focus in specific chapters. In spite of the tentative construction of a united Europe, the offered presentation still reflects the print left by History in the Nineteenth century and various zones of influence. Thus apart from the originality of Switzerland, the following large regions are identified: the Benelux with a prevailing role played by the Netherlands, Scandinavia considered as a historical and cultural linguistic region with special strength in Sweden and Denmark, the former Austro-Hungarian Empire, and southern European countries. A case at point is that of the former Austro-Hungarian Empire because this well-organized political structure - doomed to disappear with the two world conflicts - succeeded in building a network of efficient polytechnic schools in its various "provinces". Strong individual personalities could emerge including in former Yugoslavia and Romania. The geometrical theory of dislocations in Serbia and a specific strength in applied mathematics in Romania are witness of this trend. Italy, adorned by a long section, continues to demonstrate its traditional strength in civil engineering and the allied mathematical analysis. India and China receive but a cursory treatment, while immense expectations are to materialize soon. In Japan, two original characters are singled out, K. Kondo and T. Tokuoka. With time, most countries perused have fit in an international view of continuum mechanics that shares similar subjects of interests (e.g. complex mechanical behaviour, plasticity, numerics, thermomechanics, and coupled fields).

10.1 A Word of Caution

As the United Kingdom, France, Poland, Germany and the former Soviet Union receive a separate treatment, here we examine the rest of Europe and some isolated cases from Asia. It must be understood that we are not expressing a judgment on the technological level of the various countries. All countries involved have had

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and still have excellent technical universities and schools of engineering, preparing well for industrial practice of mechanical and civil engineering. For instance, many readers may be surprised to learn that both Czechoslovakia and Hungary had excellent, productive and competitive mechanical and electro-mechanical industries both before and after WWII. Their engineers were formed in efficient technical universities, often inherited from the Austro-Hungarian Empire. Of course, Greece did not benefit from such an heritage and its technical universities (Athens and Thessaloniki) had to develop in a more difficult background. Norway and Finland are in the same situation with a relatively recent political independence (20th century) of these countries and a rather small industrial potential before WWII. Many of the considered countries formed engineering scientists who, either for economic or political reasons, had to emigrate to other places and could realize a true professional career only outside their original country. This we mention from time to time.

What we want to put in evidence here are the contributions to the evolution of ideas in continuum mechanics. However, it will appear somewhat strange to the reader that we selected groups of countries and a distribution that copies old territorial divisions and political influences. We believe that these divisions have left a definite print that the creation of new countries and the recent construct of a united Europe have not fully erased.

10.2 Switzerland and its Originality

If we are not concerned with banking operations we may have a romantic view of Switzerland with beautiful snowy mountains, Lake Leman, charming wooden chalets, accurate clocks, and milk chocolate. It is hard to believe that once upon a time life was not so easy in that country which provided many emigrants, in particular to the USA. However, the development of railways, electricity production and distribution, and the concomitant need for many works of civil engineering (dams, bridges, viaducts, and tunnels) and electric traction caused a development of both renowned civil engineering companies and electromotive industry. We are far from the Bernoullis and Euler who had most of their Swiss career in Basel. We remember that Lausanne, first, and then Zürich saw the creation of Federal Polytechnic schools—respectively in French (1853, now called *EPFL*) and German (1855, now called *ETH*) languages—by alumni from the Paris "Grande Ecoles", *Ecole Polytechnique* and *Ecole Centrale,* in the nineteenth century.

The *ETH* has been instrumental in the expansion of good mechanical and civil engineering although it also formed an elite of physicists, among them Albert Einstein, and benefited from the teaching of high calibre scientists such as Hermann Minkowski and Wolfgang Pauli. The University of Geneva remained more classical in its teaching and form with a long tradition in the humanities having been founded by Jean Calvin in 1559. It is now particularly strong in medical research.

Hans Ziegler (1910–1985), educated at *ETH* and in Germany was a professor at *ETH* from 1942 till his retirement in 1977. Originally producing works in structural mechanics and dynamical stability (Ziegler 1968) (e.g., on gyroscopes), he acquired a taste for plasticity studies after a visit to William Prager in Brown (cf. Chap. 4). In that field he developed the relationship between plasticity and irreversible thermo-dynamics, something that practically did not exist precisely before him. In particular, he enunciated the principle of orthogonality—of generalized velocities with the bounding surface of the elasticity convex set—, and also a nonlinear generalization of Onsager's reciprocity relations. He published a wonderful (our opinion) book on thermo-mechanics in 1977 (Ziegler 1977, 1986: 2nd edition published after his death). He was also the co-editor of an influential journal: the *ZAMP* (*Zeitschrift für angewandte Mathematik und Physik* = Journal of Applied Mathematics and Physics) founded in 1950. During his long stay at *ETH* he formed many engineers. Kolumban Hutter (who left for a PhD at Cornell, USA) and M. Sayir (who became a Professor at *ETH*) were among his students. William Prager came several times to lecture at *ETH* (cf. Prager 1955, 1961). He advised George Herrmann in his doctoral thesis, and his courses were published in German.

The Federal Polytechnic School (EPFL) in Lausanne may have been less successful in mechanics than its German speaking counterpart but its has developed other aspects of the field including the basic formulation of continuum mechanics and the related numerics (with Alain Curnier, a French engineer with an American PhD), the micromechanics of concrete and cementitious composites, and their homogenization (with Christian Huet, a French civil engineer who passed away untimely in 2002), the biomechanics of tissues (with Lalaorina Rakatomanana, now a Professor of Mathematics at the University of Rennes, France), and various problems of applied mathematics related to mechanics (with Bernard Dacorogna).

10.3 The Benelux: Belgium, the Netherlands, Luxemburg

"Benelux" is an acronym formed by the first two or three letters of the three countries: Belgium, Netherlands and Luxemburg. This was an economic community created in the 1950s. We have considered this entity as it seems to stand for a transition between France on the one hand and Germany and the northern Scandinavian countries of Europe (see below) on the other hand.

This transition is materialized by Belgium, officially a tri-lingual country (French, Dutch and German are the official languages). Of course the three languages are not talked everywhere, German being entertained in a very small part, French talked in Wallonia (south and east) and the Capital Brussels, and Dutch (Flemish) being practiced in the western and northern parts known as the Flanders, closest in spirit to the Netherlands. Belgium was artificially created as a buffer state in the 19th century. The economy of the southern part of the country was rich when coal mines and heavy industry flourished in the 19th century and the first half the 20th century.

Furthermore, Belgium benefited from his colonial occupation of (Belgian) Congo until its independence, a country extremely rich in mineral resources. Now the economic balance is more in favour of the Dutch-speaking part.

Luxemburg is squeezed between France, Belgium and Germany and is too small a country to enjoy a developed university and engineering school system. In so far as we know there exists only one research centre dealing with mechanics and material sciences in Luxemburg, the recently opened Public Research Centre *Henri Tudor* with a group devoted to the technology of materials and their modelling and simulation in the spirit of the thermo-mechanics of materials.

The Netherlands—or Low Countries—, known for their religious tolerance, benefit from an opening on the North Sea (Amsterdam, Rotterdam) and a huge commercial traffic of imports to Europe via its harbours and the Rhine river. It also benefited at some time from a rich colonial territory in Indonesia. It has a strong tradition of philosophy (Erasmus, Spinoza) and learning, still materialized in its many editorial and printing houses, among them the famous scientific Editor Elsevier that goes back to the time of the Renaissance and the seventeenth century (the Elzevires published Galileo Galilei).

Now concerning our interest in this book, noting that there exists a relationship between the flourishing economy of a country and its developments in higher education and research, we examine the cases of both Belgium and the Netherlands.

10.3.1 Belgium

Because of the presence of industry and also some historical factors, the two main centres in Belgium are (the French speaking) Liège and Louvain, the latter being bi-lingual and also known as Leuven in Dutch.

The University of Liège (ULg) was created in 1817 under Dutch power. Its school of Mines opened in 1838 in the recently created new state called the *Kingdom of Belgium*. A peculiarity of the Science Faculty of this University is that it delivers a diploma of "Engineer-Physicist" opening many possibilities of career to its alumni. In recent years and with a continuum mechanics connotation, we note the contributions of Baudoin Fraeijs de Veubeke (1917–1976), a pioneer in numerical methods with applications to aeronautics, and Georgy Lebon (born 1937) who, in co-operation with Spanish physicists from Barcelona (cf. Casa-Vasquez et al. 1984, 1996), expanded a neat and fruitful *extended thermodynamics of continua* (with generalized laws of conduction, viscosity, etc.; see Chap. 5, Lebon 1989). The book on the subject that he co-authored was a best seller. Lebon had his whole student and academic career (until retirement in 2002) in Liège. He had obtained his BSc in 1959 and his doctoral degree with a noted physicist, Léon Rosenfeld, in 1966. He created at ULg a course on theoretical thermodynamics.

Louvain is something of an historical place with a catholic university created in 1425 (but closed by the French occupants in 1797). It is said that in the seventeenth century Louvain was one of the best places for making scientific instruments. The catholic university was re-opened in the nineteenth century. But in 1968, a (stupid; this is our opinion) split between the French-speaking and the Dutch-speaking faculties took place. The French-speaking University was relocated in a newly created city called "Louvain-la-Neuve". Fraeijs de Veubeke taught there. Furthermore, the faculty of applied sciences became a highly estimated centre of research in the flows of non-Newtonian fluids in the expert hands of Marcel Crochet, a former PhD student of Paul Naghdi in Berkeley. A most successful alumnus from Louvain was Maurice A. Biot (1905–1985) who obtained his doctorate there in 1931. He is universally known for his dynamic theory of poro-elasticity, and also for his breakthrough works in variational formulations and the incremental approach to finite-strain theory. But he spent most of his professional career in the USA, often in a non-academic context.

Note: Biot's theory of poro-elasticity has been improved by many authors, more particularly Olivier Coussy in France (cf. Chap. 7)—also Sara Quiligotti and G.A. Maugin (Paris)—, Reint de Boer and Wolfgang Ehlers in Germany (cf. Chap. 9), and Krzysztof Wilmanski and Bettina Albers in Germany/Poland (cf. Chap. 8). Progress has been along the consideration of finite strains and some anisotropy, thermal effects, introduction of the notion of tortuosity, improvement of the equation of the porosity parameter, and dynamical properties).

Brussels is also something special as a stronghold of French in the centre of a Dutch-speaking region. A Free University (for short, *ULB*) was created there in 1834 by free-thinkers, having for object to carry the spirit of the enlightenment independently of any religion. It was French speaking at the start. A polytechnic faculty was opened in 1873 and became later on a Faculty of applied sciences. Among its contributors to continua we mention Ilya Prigogine (1917–2003)—world famous for his basic works in thermodynamics—who founded a real school that attracted researchers from the world over. Also important was the contribution of Baron André Jaumotte (1930–2009) to fluid/gas flows and computations in the theory and applications to aeronautics and aerospace science. The Belgian title of "Baron" is equivalent to that of "Sir" in England and the "Legion of Honour" in France. Midway between mechanics and applied mathematics, we also have works dealing with the bases of the electrodynamics of continua by G. Mayné and Philippe Boulanger (cf. Boulanger and Mayné 1974). The last author was also active in nonlinear elasticity and wave propagation, mostly in co-operation with the Irish mathematician Michael Hayes from Dublin (See Boulanger and Haynes 1993). Finally, we note that a separate Free University in the Dutch language was opened in 1969–1970 under the name of *Vrige Universitet Brussel*, or *VUB*.

A "Belgian" linguistic story: When you leave Berlin there are immediately signs indicating the direction of Paris (Frankreich) although it is more than 600 miles away; it is true that the Prussians are used to going to Paris from time to time in both 19th and 20th centuries. But when you leave Brussels by the belt highway and look for an exit indicating Paris, this you do in vain unless you know how Paris is written in Dutch; just the same if you look for the directions of Lille (Riesel/Rijsel), Mons (Bergen) or Liège (Luik).

10.3.2 The Netherlands

The Delft University of Technology was initially founded in 1842 as a Royal Academy. With a rapid industrialization of the country it changed it name to a "Polytechnic school" in 1864, then to the "Institute of Technology" in 1905, and acquired its present name in 1986. Early in the 20th century the most influential teacher in mechanics there was C. B. Biezeno. He was teaching the theory of elasticity and the strength of materials. He organized the first International Congress of Mechanics in Delft in 1924 (cf. Biezeno and Burgers 1925). But from our viewpoint his most successful actions were attracting J. M. Burgers to Delft and mentoring W. T. Koiter.

Johannes M. Burgers (1895–1981) was educated as a theoretical physicist in classical scientific universities (Amsterdam, Leyden) but joined TU Delft in 1918 to stay there until 1955 when he accepted a position at the University of Maryland, USA. In Delft Burgers developed a taste for mechanics, especially fluid mechanics. His name remains attached to three fundamental results or concepts: the *Burgers equation* in fluid mechanics, which provides the cornerstone for modern studies in turbulence, the *Burgers vector* (Burgers 1939) as the fundamental entity in the study of dislocations in crystalline solids, and the *Burgers material* in visco-elasticity. We particularly appreciate the notion of Burgers vector—which he introduced together with his brother W. Burgers, a crystallographer—without which no further studies in the singularity and characterization of a dislocation line would have existed in the present form.

Prof. Biezeno was also the supervisor of W. T. Koiter's PhD thesis during WWII. We all know the tremendous original and powerful vision brought by Koiter in the theoretical study of the stability of structures (Koiter 1945, 1960). But Koiter was also himself an efficient supervisor of students and with age became an active member of international societies, in particular in the *International Union of Theoretical and Applied Mechanics* (*IUTAM*). The Institute was recently renamed after Koiter's name.

Other influential researchers in continuum mechanics at TU Delft have been J. F. Besseling and his younger co-worker Erik van der Giessen (cf. Besseling 1968; Besseling and van der Giessen 1994). The former is known for his works on various aspects of the micromechanics of inelastic/plastic materials including thermodynamic considerations and accounting for the presence of dislocations. Van der Giessen (born 1959) who obtained his PhD at Delft in 1987 was a professor in Delft from 1992 to 2000. He moved to Groningen in 2001. He is much interested in the so-called structure–property relationship for which he exploits numerical means (see van der Giessen 1989). Finally, René de Borst (born 1955) obtained his PhD at the Koiter Institute where he became a professor at the early age of 31, having received a highly appreciated Spinoza prize. His works span the nonlinear analysis of frictional materials, damage, fracture and micromechanics, numerics, and gradient models of materials. He moved to TU Eindhoven in 2007 to become the dean at the school of Mechanical Engineering.

The second technical university in the Netherlands is the one in Eindhoven. It is rather young having been founded as a "*Technische Hogeschool*" in 1956. It changed its name to the TU Eindhoven in 1980. This university benefits from a favourable industrial and research and development environment (e.g., Philips and DAF companies). In so far as we are concerned we note that J. B. Alblas joined the Department of Mathematics and Computing in 1959 and worked in continuum thermodynamics and the mechanics of electromagnetic continua with some success. He mentored Alfons A. F. Van de Ven (cf. Hutter et al. 2006). The latter has brought new results concerning the stability and bucking of magnetized and superconducting structures, a matter of high interest in powerful electromagnets to be used in controlled fusion reactors but also in magnetically levitated transportation. Finally, in the same department we note the important role played by G. A. Kluitenberg—a disciple of de Groot and Mazur in the linear theory of irreversible processes—who did much for the promotion of the thermodynamics with internal variables of state, in the formulation of anelastic behaviours and electromagnetic materials with hysteretic properties, often in co-operation with scientists in Messina (V. Ciancio, L. Restuccia).

10.4 Scandinavia: Sweden, Denmark, Norway, Finland

Scandinavia here is considered as a historical and cultural linguistic region. It has ethno-cultural heritage and related languages, not to speak of the famous Vikings. Of course *Suomi* talked in Finland belongs to a different language group while Danish, Norwegian and Swedish are all related to the Old Norse language (together with Icelandic and the language of the Faroe Island). Furthermore, Denmark and Sweden formed a united kingdom in the past, Norway got its independence from Sweden early in the 20th century, and Finland was a part of Sweden for about seven centuries; Swedish still is the second official language in Finland. We can agree with Hult and Nystrom (1992) on the global view of a Nordic heritage in so far as technology and industry are concerned: these countries share close economic and cultural ties. Denmark, Norway and Finland have about the same population (circa 5.5 millions of inhabitants) while Sweden has about 9.5 Millions. But Finland, Norway and Sweden have a low density of population while Denmark has a density almost ten times higher than these three countries.

Consider first the case of Sweden. This was an emigration land (in particular to the USA in Minnesota) before a true industrial development with iron ore and steel industry. This is lavishly illustrated in the Swedish movie "The Emigrants" (1971). It might not be out of scope to remind the reader that the new king of Sweden in 1810 was a French general, Jean-Baptiste Bernadotte (marshal in Napoleon's army). The same dynasty still is in power. The first technical institution/school in Sweden was created in Stockholm in 1827. It took its actual name, the *Kungliga Tekniska Högskole* (for short *KTH* = Royal Institute of Technology), in 1877, and grants PhDs since 1927. The second technical institution for higher education is the Chalmers University of Technology in Gothenburg (Göteborg). In contrast to *KTH* it is a private institution managed by a Foundation. It was founded through a donation by a rich Mr Chalmers in 1829. These two technical universities have formed a large majority of Swedish engineers and scientists, while the University of Uppsala—founded 1477, the oldest university in the Nordic countries—remains the stronghold of studies in humanities, having been famous for natural sciences in the past (remember Carl Linnaeus) and having now departments of mathematics, physics and engineering sciences.

At *KTH* in the period of interest we should note the role played by Folke Odqvist (1899–1984) who taught there for 30 years (1936–1966). He remarkably contributed to the special fields of plasticity and creep. In the first of these he introduced what is known as the "Odqvist parameter", now related to what we identify as the integrated past history of the plastic strain, as the most representative parameter of the hardening behaviour of metals. In the theory of creep (phenomenon according to which plastic strain grows in time under the application of a relatively small stress) he obtained definite results, in part with his doctoral student Jan Hult (see Odqvist 1966; Odqvist and Hult 1962; Hult 1966). Indeed Jan Hult (born 1927) obtained his PhD at *KTH* in 1958 under the supervision of Odqvist, but he moved to Chalmers in 1962 to stay there until retirement in 1992. At Chalmers Jan Hult was able to create a true group devoted to the mechanics of materials, involving creep but also the transition between micro- and macromechanics considering arguments of thermo-mechanics.

Another smart scientist who studied at *KTH* (PhD 1956) was Bertram Broberg (1925–2005). First professor at *KTH* he moved to the Lund Institute of Technology in 1961 and stayed there until retirement. He is probably one of the foremost contributors to the theory of cracks with pioneering works. He gave his deep and experienced general vision on dynamic fracture in a book published as Broberg 1999. He also nicely contributed to the theory of the cell structure of materials. Late in his life he developed a strong interest in problems of biomechanics such as that of the intervertebral discs (cf. Ståle et al. 2010).

Personal touch. When the writer co-organized a NATO advanced school on surface waves in Moscow, Prof. Broberg, one of the main invited speakers but also a militant pacifist, refused to have his expenses covered by NATO. Since the writer, responsible of NATO funds, could not keep the money, he advised Broberg to receive the money and give it to a charity, what Broberg, then retired and living in Ireland with his Irish wife, dutifully did.

In the late 1970s-early 1980s there was in the Department of Mechanics of *KTH* a burst of active research in the field of micropolar continua (generalized continua of some kind, see Chap. 13; cf. Brulin and Hsieh 1981) and magnetized solids in a group gathering essentially Stig Hjalmars (1918–2007), Inga Fischer-Hjalmars (1918–2008), Olof Brulin (died 2000), K. Berglund, and Richard K. T. Hsieh (PhD *KTH*, 1978; associate professor at *KTH*). This activity disappeared after a re-organization of the department. Brulin and the two Hjamars had been formed in theoretical physics, under the supervision of Oskar Klein (of the Klein-Gordon equation). Lars Söderholm, a physicist, docent at *KTH* since 1980, contributed to the formal structure of relativistic continuum mechanics in the Nollian manner.

A smooth transition from Sweden to Denmark is provided by the case of Frithiof I. Niordson (1922–2009). Of mixed Russian-Swedish origin, Niordson was basically educated as an engineer at *KTH* in Stockholm. But he obtained his doctoral degree in Brown (USA) with William Prager. On his return to Europe he became a professor at the Technical University of Denmark (*DTH* or *DTU*), in Lyngby where he remained from 1958 till his retirement. There he was the driving force behind the creation of the *Danish Centre for Applied Mathematics and Mechanics* (for short, *DCAMM*) which became a world known meeting and co-operative place for many scientists. His personal fields of expertise were computational mechanics and the theory of plates and shells (Niordson 1985), and also structural optimisation. Niordson was also much involved in the world organization of mechanics through the *International Union of Theoretical and Applied Mechanics* (*IUTAM*). He co-organized its 15th International congress (ICTAM) in Lyngby in 1984. Among the very successful engineering students from *DTU* we find Viggo Tvergaard (born 1943) who obtained at the *DCAMM* both his doctoral degree (1971) and his habilitation (1978). Tvergaard is one of the most cited mechanicians in the world, due mostly to his theoretical works on stability, the formation of shear bands, and other critical problems in continuum mechanics, many of which in fruitful collaboration with John W. Hutchinson and Alan Needleman from Harvard [see his scientific résumé in J. Mech. Phys. Solids, 56, 3–4, 2008].

Norway has always been turned towards the sea (fisheries, maritime transport, not to speak of the Vikings). The recent abundance of oil and natural gas in the North Sea just amplified this orientation with the increased role of places such as Stavanger. Second largest university in Norway after Oslo, the *NTNU* (for Norge tecknisk-naturvitenskapalige universitet) in Trondheim was founded in 1910 as the Norvegian Institute of Technology, and took its actual name in 1996. Probably its most famous alumnus is Lars Onsager, graduate of 1925, who received a Nobel Prize in physics (remember the celebrated Onsager's symmetry relations in irreversible thermodynamics). Because of the strong maritime connection, no wonder that the school of mechanical engineering includes naval architecture and ship engine construction. Technical matters related to the sea now are also taught at a recent university in Stavanger.

Finland was not a very industrial country until recently (think that NOKIA started as a factory producing gum boots for peasants in a country made practically of woods and lakes). Nonetheless, a first technical teaching was initiated in 1849 in Helsinki. This transformed in a Polytechnic school in 1876 until 1908 when courses at the university level were opened. The first doctoral degree was granted there in 1912. The new name was Tekkarikylä (for short TKK) and still another new name was adopted in 2010 as Aallo University. The mechanics of fracture and biomechanics are the most advanced fields of research now cultivated at this university. Among the notable universities in Finland we must also count those in Tampere (with technical teaching) and Jyväskylä.

10.5 Former Austro-Hungarian Empire

10.5.1 Austria

Of course we must start by perusing Austria itself. The main university places there are Vienna, Graz, Linz and Leoben. The Technical University of Vienna (TU Wien) was the first technical institution of high level created within the present-day German speaking Europe, in 1815 as the Imperial-Royal Polytechnic Institute (for short in German *KKPI*). It was renamed TH Wien in 1872. It granted its first doctoral degrees in 1902. It formed most of the Austrian mechanical, civil and electrical engineers in the 19th century and still now. Some of its most famous alumni are Christian Doppler (the celebrated Doppler effect in acoustics), and both Joseph and Johann Strauss who, as we know, became famous in another branch of Art. Note that the famous Austrian writer Robert Musil—a man with many qualities (in spite of the title of his celebrated lengthy book)—, was also educated as a civil engineer but at Brno (Moravia), then in the Austro-Hungarian Empire. *Note*: Aeronautical engineering also provided renowned *gens de lettres*, but in the USA, e.g., Norman Mailer and Thomas Pynchon, both of the Nobel prize calibre.

Back to truly materialized careers in engineering, the teacher at TH Wien with the strongest influence in the first part of the 20th century was Ernst Melan (1890–1963)—himself the son of a famous civil engineer, Joseph Melan (1853–1941)—, who was a professor of steel design and construction. This sounds rather mundane. But Melan proved in 1938 a famous theorem concerning the "shakedown" of plastic structures (Melan 1938). We remind the reader than "shakedown" is the asymptotic property that the plastic strain can stabilize in time (sometimes the response may even become elastic). We can say that the structure "adapts" itself to the load; see the mathematical proof in Maugin 1992, pp. 81–84). Melan's result was later on improved by Symonds (1951) at Brown and Koiter (1960) in Delft. But Melan was a man of many interests. In particular he became interested in thermal stresses and wrote a pioneering small book in that field together with his disciple Parkus (Melan and Parkus 1953).

Heinz Parkus (1909–1982) succeeded Melan on the chair at TU Wien. He was also a man of many technical and scientific interests (including the first helicopters when he was in the USA just after WWII). Apart from thermo-elasticity (Parkus 1968), he also contributed to another theory of coupled fields, the mechanics of electromagnetic continua, with personal contributions and the organization of the first course on the subject at the *CISM* in Udine in 1977 [with a chapter by the present author; see Parkus (1979)], and the influence on younger people such as Adalbert Prechtl (who became a professor in electrical engineering at TU Wien).

Another disciple of Parkus was Franz Ziegler (born 1937). The latter was educated at TH Wien (Engineering Diploma 1961, PhD 1964, Habilitation, 1974) and benefited from a 2 years stay at Northwestern University in Evanston (USA). He succeeded Parkus on his chair until his own retirement in 2002. Also a man of broad vision (this seems to be a mark of the institution), Ziegler contributed to many fields such as stochastic processes in mechanics, wave propagation, computational mechanics, etc. He was very active in the International organization of theoretical and applied mechanics as well as in the *GAMM* in Germany. He brought the basically Austrian journal "*Acta Mechanica*" to the highest international level in continuum mechanics. Among others, he mentored Professor Hans Irschik (born 1951, PhD 1981, Habilitation, 1986) who is now at the Johannes Kepler Universität Linz, a specialist of the mechanics of smart materials, configurational mechanics (cf. Irschik 2007), advanced dynamics and control, and mechatronics.

As to the other two Austrian places of interest, in the present context we note that the University of Graz has become a stronghold of modern biomechanics under the leadership of Gerhard A. Holzapfel, while the Bergakademie in Leoben developed intensive research in the field of the thermo-mechanics of phase transformations in deformable solids under the leadership of Franz D. Fischer.

10.5.2 Hungary

The kingdom of Hungary was between 1867 and 1918 fully integrated to the Austro-Hungarian Empire with—in theory—an equal status with Austria, hence the often met qualification of "imperial-royal" giving rise to the "KK" abbreviation in German. The previous Austrian Empire had been founded in 1804 in response to the "invention" of a French Empire by Napoleon. Both Austro-Hungarian Empire and Ottoman Empire were dissolved by the victors of WWI, giving rise to the birth of modern—sometimes artificial—states such as Czechoslovakia, Yugoslavia, Bulgaria, Romania, Greece and a revived Poland. The case of Hungary is somewhat special in that this country provided many emigrant scientists of very high calibre. First, Hungarians consider that they created the oldest institution of technology in the World in 1782. This, in fact, is not entirely true since the Royal School of *Ponts & Chaussées* (1747) in France was created before and had, together with the *Ecole Polytechnique* (1794), a much more important influence in our field, and science in general, than any Hungarian school. The Hungarian school of technology was reorganized in 1871 to become the Technical University of Budapest when a Faculty of Mechanical Engineering was founded.

Of course the Magyars do not lack some *panache* and they readily count as Hungarian Nobel prizes those people who in fact did their graduate and doctoral studies outside Hungary and achieved a successful professional career also abroad. For instance, among famous alumni of the Technical University of Budapest, they count Denes Gabor (of holography fame), Eugene Wigner, Leo Szillard, and Edmond Teller in physics and Paul Erdös in mathematics. Only Erdös obtained his doctoral degree in Budapest. To these we would generously add John von Neumann (who went to a German-speaking high school with Wigner and Teller) and, closer to our present interest, Theodore von Kármán (1881–1963) who indeed obtained an engineering degree at TU Budapest in 1902.

In more recent times, Hungary has suffered in post-WWII communist era from its isolation behind the Iron curtain. It is only recently (late 1980s) that it came back in the normal network of international relations, and this, obviously, had consequences in the domain of science and technology. At the now called *TUBE* (the *E* standing for "Economics"), an original school of thermo-mechanics was founded in the 1960s–1970s, led by I. Gyarmati and J. Verhás. The first of these scientists developed a variational field theory of irreversible thermodynamics (see Gyarmati 1970). The second author expanded the thermo-mechanics of continua with internal degrees of freedom (Verhás 1977). Another mechanician, J. Béda, contributed to nonlinear continuum thermo-mechanics (Béda et al. 1995).

10.5.3 Czecholovakia

This was one of the "artificial" countries created in 1918 at the fall of the Austro-Hungarian Empire as a result of WWI. It was made of the Czech part (with Prague) and a Slovak part (with Bratislava). The two (friendly) split in 1993 after the end of the communist era giving birth to the Czech Republic on the one hand and Slovakia on the other hand. There was a rather old tradition of technical teaching in Prague (some say going back to 1707). But the Czech Technical University (for short CTU, now CVUT in Czech) in Prague was essentially formed after WWI. Similarly, the Comnenius University in Bratislava was founded in 1919. The first of these provided the country with knowledgeable engineers. The contribution of the Czech school to continuum mechanics was relatively modest in communist times. But we remember the contribution to rheology (European meeting in Prague in 1986), of Jan Kratochvil to the inelasticity of crystals in the 1970s–1980s, and the remarkable contributions in a rather Truesdell-Coleman-Noll mathematical style of Miroslav Šilhavỳ (see his book, Šilhavỳ 1997). In Bratislava mathematically oriented studies were achieved in particular by the analyst Josef Brilla. There was a rather strong emigration of scientists from Czechoslovakia following dramatic political events in 1968. Thus the University of Montreal in Canada welcomed at its Centre of Applied mathematics Miroslav Kranys and Miroslav Grmela. The first of these two scientists dreamed of a hyperbolic world (all phenomena with a finite speed of propagation), rendering equations of physics and continuum mechanics hyperbolic in all cases, e.g., with a Cattaneo-Vernotte kind of heat conduction law. Grmela, on his side, was very much successful in his approach to problems of rheology put in a proper geometric and symplectic framework. Among the very successful immigrant scientists to the USA was Zdeněk P. Bažant, a native from Czechoslovakia and civil engineer formed in Prague who joined North-western University in Evanston in 1969. He is most well known as a prolific author in various fields of continuum mechanics including the creep of concrete, the stability of structures, size effects in solid mechanics (Bažant 2004), and a theory of nonlocal damage co-authored by the French scientist G. Pijaudier-Cabot. He also rapidly provided a spot-on engineering analysis of the collapse of the New York City twin towers on the eleventh of September 2001.

10.5.4 Yugoslavia

This was another one of the artificial countries created at the end of WWI with the most important contributions from Serbia, Croatia and Slovenia. Serbia suffered from Ottoman occupation in the period 1459–1878, but it had a de facto independence as an autonomous principality in the period 1817–1878. A Kingdom of Yugoslavia was established from 1918 to 1941. After WWII, Tito's original communist rule from 1945 to 1980 was based on "brotherhood and unity". But the latter may have been vain words, as shown by the split of Yugoslavia in different smaller states in dramatic circumstances in the 1990s.

We focus attention on Serbia with its capital Belgrade (Beograd). The University in Belgrade was founded in 1808 with an engineering department from the start. It became officially a University in 1905 with faculties of civil and mechanical engineering. It formed most of the Serbian (and other "Yugoslav") engineers when the latter were not educated in neighbouring Austria; this was the case of probably the most well known Serbian scientist-engineer, Nikola Tesla (1856–1943) who was in part educated at TH Graz. In the USA, he was successful in promoting the use of alternating current in opposition to Thomas Edison, who was in favour of direct current. He also designed many original electric machines and his name is forever attached to a magnetic unit. A statue of this great man stands on a pedestal in front of Belgrade's University although he did not study there.

In the period 1965–1990, several lines of research were developed in Serbia. First, applied mathematicians such as Z. Jancović and T .P. Andjelić promoted continuum mechanics in a modern format. Much more original was the line pursued by Rastko Stojanović (1926–1972) who produced seminal works on the differential geometry of polar continua (generalized continua—see Chap. 13 and his lecture notes at the *CISM* in Udine, 1969–1972). This was achieved in the path paved by K. Kondo in Japan and E. Kröner in Germany (see Stojanović 1969, 1970, 1972). Jovo P. Jarić with a PhD in Belgrade (1973) under the supervision of Nathlija Naerlović-Veljković (herself the author of works in thermo-elasticity) produced interesting works on invariant integrals and configurational mechanics (Jarić 1978, 2004). As to Milan V. Mićunović (born 1944), with a Doctoral degree obtained in Warsaw under the supervision of Henryk Zorski, he exploited the multiplicative decomposition of the deformation gradient in the context of thermo-elasticity in finite strains, and then specialized in the theoretical and experimental characterization of visco-plastic materials in co-operation with the ISPRA research centre of EURATOM in Italy (see Mićunović 2009) while teaching in Kragujevac. Jaric supervised the works of Predrag Cvetković and Mirko Vukobrat. The latter contributed to the mechanics of materials with nonlocal response (another generalized type of continuum). Finally, we note the success met in the USA by two mechanical engineers initially formed in Yugoslavia: Dusan Krajcinović (1935–2008) with BSc and MSc from Belgrade and a PhD from Northwestern who became a world renowned specialist of damage mechanics, while Vlado A. Lubarda became a scientifically productive Professor in general thermo-mechanics in San Diego.

Just for memory we record that part of Poland (Krakow) and Ukraine (Lwov) in so-called Galicia also belonged to the Austro-Hungarian Empire (see Chap. 8 devoted to Poland).

10.5.5 Romania

This country is indeed placed at the crossroad of Central Europe (Mittel Europa). Transylvania (Brasov, Timisoara) was in the Austro-Hungarian Empire. The Kingdom of Romania was created in 1859, and became independent of the Ottoman Empire in 1877. Present day Romania is a sovereign country of nearly 20 millions of inhabitants. Somewhat as a gross simplification, we could say that it is a "Danubian" country with a Latin language, which is surrounded by countries with different languages such as Bulgarian (close to Russian), basically Slavic Serbo-croatian, and Hungarian. Probably because of its Latin background Romania entertained strong intellectual links with France and Italy. In the post WWII period Romania developed a special interest and true talent for the mathematics of continuum mechanics in both solid and fluid mechanics with remarkable contributions by L. Solomon in elasticity (cf. Solomon 1968), Caius Jacob (PhD in Paris, 1935) in theoretical fluid mechanics (see also his magesterial book, Jacob 1959), Elie Carafoli (1901–1983, PhD in Paris 1928) in aerodynamics (cf. Carafoli 1956), L. Dragoş (PhD 1964 with C. Jacob) in magnetohydrodynamics (Dragos 1975), and Grigoriu Moisil (1906–1973) considered to be the father of computer science in Romania. Octav Onicescu (1892–1983) was instrumental in developing various aspects of mechanical sciences including probabilistic aspects and as an ambassador of Romanian mechanics in both West and East. He was among those who conceived and effectively founded the *International Centre of Mechanical Sciences* (CISM) in Udine.

In more recent times the centres of Bucarest, Iasi, and Cluj have been active in modern continuum mechanics with studies in the theory and experiments in crystal elasticity and defects (Teodosiu 1982), visco-plasticity (Critescu and Suliçiu 1982), the general thermo-mechanical approach to inelasticity (Cleja-Ţigoiu and Soós 1990), polar media and thermoelasticity (e.g., Ieşan and Scalia 1996), and non-Newtonian fluids (Victor Ţigoiu). All these studies are marked by a special taste for good applied mathematics as entertained by the University of Bucarest and the Romanian Academy of Sciences. This is well illustrated by the many works of P. P. Teodorescu (with various interests, among others, in the dynamics of elasticity and the application of distribution theory and group theory to mechanics (cf. Teodorescu 1972, Teodorescu and Kecs 1974)) and Eugen Soós

(1937–2001). The latter, with a PhD (1972) with C. Jacob, was an applied mathematician who mentored many (some already cited) young researchers in the Department of Mathematics at the University of Bucarest and at the Institute of Mathematics of the Romanian Academy of Sciences. He successfully contributed to a large spectrum of research in both specialized fields (mechanics of composites, anelasticity in finite strains, electromagnetic continua,) and mathematical tools (tensor and spinor algebra), sometimes in co-operation with N. Cristescu and P. P. Teodorescu (see Beju et al. 1983; Critescu et al. 2003). Finally, we note the role played by Horia Ene (born 1941, PhD 1970 in Bucarest) who expanded the theory of porous media with techniques of asymptotic periodic homogenisation (cf. Ene and Polisevshi 1987)and who contributed to the re-organization of science in Romania at the Ministry level after the fall of Nicolae Ceauşescu's dictatorship.

10.5.6 Bulgaria

This offers a smooth transition with some more southern countries like Greece and Turkey. Bulgaria was occupied by the Ottomans from 1396 until practically the end of the 19th century although it belongs to the Slavic-speaking orthodox group with a close cultural connection with Russia. It became a new state in 1878 with sovereignty obtained in 1908. The Technical University in Sofia was founded in 1945 as a "State Polytechnic Institute" and gained a full University status in 1995. Like in other communist ruled countries beyond the Iron curtain, an essential role was also played by the Bulgarian Academy of Sciences, modelled after the Soviet one. Many of its productive mechanicians benefited from a doctoral formation in the Soviet Union. This was the case of Konstantin Z. Markov (1945–2003) in the theory of composites with random properties (formed in Leningrad by A. A. Vakulenko) and Christo I. Christov (1951–2012) for numerical methods (formed in Novosibirsk by N. N. Yanenko). On the other hand, A. Baltov worked in the plasticity of anisotropic bodies together with Polish scientists (in particular A. Sawszuk), while Alexander Rachev successfully developed some aspects of biomechanics (nonlinear elasticity of soft tissues) before moving to the USA.

10.6 Southern European Countries

10.6.1 Italy

A country like modern Italy seen as a single unified political entity is of rather recent history. It is no more than a 150 years old. Still most of us have a feeling that something underlies the common background of this part of Europe and perhaps most of southern Europe up to the wall of Hadrian in the UK. This "something" is nothing but the famous Romans. If we accept the idea that the ancient Greeks invented philosophy and geometry, we must also admit that the Romans invented the Law and civil engineering. This last invention still is a characteristic property of Italian engineering. Concerning educational institutions Italy saw the creation of the oldest universities in the World with Bologna and Ferrara. These universities offered the typical scholastic (Aristotelian) curriculum, where law, music, theology, astronomy and medicine were often studied by the same students from all European countries. Nicolaus Copernicus was such a student. Other universities such as Pisa offered courses in what we may call Natural Philosophy (physics). Galileo Galilei in the early 17th century is the most representative teacher at this university, and obviously the creator of modern mechanics before Descartes, Huygens and Newton. Some times before, Leonardo da Vinci, after Archimedes in Syracuse, is the prototype of engineer-scientist although with apparently no formal scientific education and the fact that we miss material realizations of his designs. Perhaps closer to our modern spirit—but further back in the past—is Filippo Brunelleschi (1377–1446) with the invention of perspective and his formidable construction of the dome of the Cathedral in Florence. The civil engineering spirit was re-actualized in the 19th century with engineers formed at schools such as the Politecnico in Turin or military schools, certainly influenced by the recently created French schools of engineering. Carlo Castigliano (1847–1864) and Luigi Menabrea (1809–1896) are examples of such engineers-scientists who greatly improved the theory of the strength of materials with the expansion of reciprocity and energy theorems. Menabrea was a disciple of Lagrange in making use of variational formulations. So was also the case of Gabrio Piola (1794–1850)—of Piola–Kirchhoff fame in finite strains—but Piola was more of a theoretical mechanician or an applied mathematician formed in a university. This tradition was carried in the 20th century with the brilliant contributions of the Italian school of elasticity to nonlinear elasticity (Signorini, Cattaneo, Grioli, etc.) that we examined in a previous chapter (see Chap. 3). In a general way we must note the enduring influence that mathematicians Ricci and Levi–Civita had on the teaching of rational mechanics in Italy for almost one century.

We shall now focus our attention on the second half of the 20th century. Whether our Italian colleagues like it or not, we are led to viewing Italian continuum mechanics as three regional strongholds, the South with Naples, Sicily, etc., the North with Torino, Milano, Bologna, Padova, etc., and finally the central part with Rome. This also follows the scheme of political influences and traditions, Rome being seen as the imposed administrative centre albeit rich by itself with three state universities. Other specific traits are (1) the existence of two polytechnic schools in Torino and Milano—remnants of the French influence of Napoleonic times (as also the existence of the *Scuola Normale Superiore* in Pisa, copied on the Paris *ENS*; see Chapter on the French masters)-, and (2) the fact that continuum mechanics, even in its most mathematical form, is often cultivated in departments of civil engineering or *Scienza delle Costruzioni* (e.g., in Pisa, Rome I and Roma II, Udine, Palermo).

Perhaps because there were no formal PhD programs in Italy before the 1980s, Italian mechanics, more than in other countries in Europe, was very much influenced by foreign teams, essentially from the USA. A partial reason for this may also be the (somewhat unreasonable; our opinion) attraction exerted on Clifford A. Truesdell by everything Italian (Arts, language, classical science, pre-WWII mechanics) and that exerted by Truesdell on some Italian scientists in search for some kind of father figure. As a consequence, it is in Italy that we find the last fully fledged "Truesdellians".

We start with the northern part of Italy, but this does not indicate any hierarchy. Angelo Morro, educated in Genova and with a long teaching and research career at the local university, is a versatile applied mathematician with interests ranging from the mathematical formulation of hereditary processes and electromagnetism in continua, to inhomogeneous waves and continuum thermo-mechanics. He produced many works in co-operation with G. Caviglia from Genova (cf. Caviglia and Morro 1992) and Mauro Fabrizio from Bologna (Fabrizio and Morro 1992, 2003). This provides an easy transition as Fabrizio in Bologna was certainly influenced by Dario Graffi (1905–1990), himself a mathematician specialist of the materials with memory in the Boltzmann-Volterra-Pérès tradition (Graffi 1928, 1977). A practically legendary figure of mechanics at Bologna is Professor Grioli (a 100 years old in the spring of 2012—born 1912!) with his mathematical studies on elasticity and polar media, also a powerful personality in the Italian National Research Council and at the Academia dei Lincei. Tommaso Ruggeri, nephew of Grioli (no nepotism here, we hope), also at Bologna was instrumental in expanding so-called "rational extended thermodynamics" in co-operation with Ingo Müller from Berlin. In the Department of Physics, Francesco Mainardi (born 1942), interested in phenomena such as visco-elasticity and diffusion, is one of the foremost propagandist of the notion of fractional calculus in continuum mechanics (cf. Mainardi 2010). In Padova, Mario Pitteri (born 1948) and Giovanni Zanzotto worked on the symmetry and transformation properties of elastic crystals as followers of Jerald Ericksen (cf. Pitteri and Zanzotto 1998), while Cesare Davini in Udine dealt with the geometry of defective elastic crystals. Aldo Bressan, also in Padova, contributed much to the foundations of continuum mechanics in both its classical and relativistic settings (see Chap. 15).

In Pisa, we have the combined influence of both the *Scuola Normale Superiore* and the Department of *Scienza delle Costruzioni* at the University. At the latter founded in 1913 as the *Regio Istituto Superiore di Ingegneria* and included as a faculty in the university in 1936 the most famous alumnus probably is Gustavo Colonneti (1886–1968), a follower of Castigliano. Presently, Piero Villagio (born 1932), a most curious and deep thinker—but also a renowned mountain climber—has dealt with original mathematical problems of the mechanics of continua and the strength of materials sometimes with a surprising sense of humour in the selection of its subjects. In applied mathematics, Tristano Manacorda (1920–2008) contributed to the solution of problems of elasticity and thermo-elasticity, while Carmine Trimarco has developed essential aspects of configurational mechanics—especially in electromagnetic continua and its variational formulation (pioneering works published in 1992; see Chap. 14 for these works)—often in collaboration with the author.

In mathematics at Pisa, Gianfranco Capriz has been the most forceful agent of the Truesdellian vision of continuum mechanics. He contributed with some talent to the mechanics of media with internal degrees of freedom (media with a latent microstructure; see Capriz 1989). He also mentored such reputed mechanicians as Epifanio G. Virga and Paolo Podio-Guidugli (now at Roma II). The former collaborated with Walter Noll on the basis foundation of continuum mechanics with fundamental work on the case of bodies with edges (cf. Noll and Virga 1990), and also proved to be a creative contributor to the continuum theory of liquid crystals (Virga 1994). As to Podio-Guidugli, he appears to be an applied mathematician of wide scientific interests (e.g., Podio-Guidugli 2000) with many fruitful works in the mechanics of bodies with internal constraints, the mechanics of magnetized materials, and configurational mechanics, often in collaboration with Morton E. Gurtin from Pittsburgh. It seems that Antonio di Carlo (now at Roma III), Paolo Mariano (now in Florence) Maurizio Brocatto (now in Paris in a school of architecture), and P. Giovine (now in Calabria; works on granular materials) were also mentored by Capriz along different research lines.

This takes us to Rome where, in addition to the University of Roma II (called "Tor-Vergata", with P. Paolo Podio-Guidugli) and the University of Roma III (with A. di Carlo), we have the large university of Roma I (called "La Sapienza"). Presently, the most active and creative contributor to our field seems to be Francesco dell'Isola (born 1962). Formed in Naples with A. Romano and a true mathematician in his style of approach, his interests span many particular fields including phase-transition fronts, variational formulations, porous media, control of piezoelectric vibrations, and the foundations of continuum mechanics (together with Pierre Seppecher from France). A. Romano in Naples has developed extensive studies on the problem of the propagation of phase-transition fronts in deformable solids (cf. Romano 1993, Romano and Marasco 2010).

In Sicily we have the three universities of Palermo, Catania, and Messina. The first two have played a historical role in mathematics and physics. In civil engineering in Palermo, Castrenze Polizzotto has dealt first with problems of plasticity and shakedown and more recently with generalized continuum mechanics involving nonlocality and gradient elasticity in both variational and thermomechanical settings. As to Messina, the Department of mathematics, with V. Ciancio and L. Restuccia, has developed an original formulation of the thermomechanics of continua with internal variables of state in collaboration mostly with G. A. Kluitenberg (from Eindhoven), Wolfgang Muschik (from Berlin), Bogdan Maruszewcki (from Poznan) and a group from Hungary.

We continue our excursion in the Italian landscape with some comments on individuals who do not precisely fit in the above framework. First we note Giuseppe Saccomandi, an applied mathematician of many interests with a long time spent at Lecce, and now in Perugia. His varied interests range from the symmetries of differential equations (with Edvige Pucci), the nonlinear elasticity of rubber-like materials and soft tissues (with Ray W. Ogden, Michael Hayes, C. O. Horgan, and Michel Destrade), nonlinear waves, and various problems in fluid mechanics and applied mathematics. As just documented, his works are marked by a fruitful co-operation with foreign scientists, often in the British-Irish tradition of applied mathematics. Then we return to the North with the University of Torino. Franco Pastrone, in mathematics, completed there research in the mechanics of media of the Mindlin type (generalized continua, see Chap. 13) and the associated wave phenomena in co-operation with local people and J. Engelbrecht from Tallinn, and Alexey V. Porubov from St Petersburg. At the University of Trento, Augusto Visintin (born 1952) provided interesting developments in the mathematics of hysteretic phenomena, free-boundary problems, phase-transition processes and multiscale approaches. He was formed at the CNR Computer Centre of Pavia under E. Magenes—the Italian correspondent of J.-L. Lions in France—in what we may call applied functional analysis. Giuseppe Geymonat [first in Torino and then in France (Cachan, Montpellier)] had the same typical formation.

We conclude with a look at the two "Politecnico" in Milano and Torino. As already mentioned, these two institutions—originally copies of the French *Ecole Polytechnique*—probably remain the best engineering schools of very high scientific quality in Italy. Because of their sufficient internal strength and potential they were not so much touched by the "Truesdellian" fashion but they radiate a remarkable activity and an obvious aura. In recent times, Giulio Maier (born 1930) in the Department of Structural Engineering in Milano (from 1956 till retirement) was responsible for fruitful developments in elasto-plasticity, limit analysis, shakedown theory, and computational mechanics. He is one of the most renowned contemporary Italian mechanicians with many honours conferred on him in Italy and abroad. His younger colleague at Torino, Alberto Carpinteri (born 1952, educated in Bologna) is a prolific contributor—claiming already about 650 papers and many books (most of them edited only, e.g., Carpinteri 1997; Carpinteri and Mainardi 1997) as on 2011—in the fields of crack propagation and catastrophe theory, applications of fractional calculus (with F. Mainardi, see above), stability of cracks, and size effects. He holds the chair of Structural Mechanics.

In a totally different line, Luigi Preziosi (born 1961, PhD in Minnesota with D. D. Joseph) has created an active group working on bio-rheology and more generally in biomechanics. Until recently, he was at the Politecnico in Torino where Davide Ambrosi pursues his own nice works on the dynamics of cell migration and his studies on tumors seen in the context of the elasto-visco-plasticity of growing bodies. Finally, we must mention Carlo Cercignani (1939–2010), a brilliant mathematician at the Politecnico in Milano, who devoted most of his scientific works to the kinetic theory of gases (Cercignani 1990), but had nonetheless a decisive influence on the teaching of rational and continuum mechanics all over Italy. An obvious admirer of Boltzmann, in addition to his scientific books and papers, he wrote a splendid and definitive biography of this Austrian physicist (Cercignani 1998).

10.6.2 Spain and Portugal

Like other European countries, Spain and Portugal have witnessed in due time (eighteen and nineteenth centuries) the creation of technical schools principally devoted to civil and military engineering. Some of the still existing schools and technical universities are the descendents of these institutions. But these schools, until very recently, served only to provide the necessary technicians for the normal functioning of a state. It is practically in vain that one searches for any marked contribution to the foundations and critical evolutionary aspects of our field until the last quarter of the twentieth century. Furthermore, these countries, once relatively rich in reason of their colonial possessions, remained essentially agricultural and finally became lands of emigration. Accordingly, we briefly focus attention on the last 30 years to pinpoint several tendencies. In Spain, the main educational institutions in sciences and technology are to be found in the largest cities such as Madrid, Barcelona, Sevilla, Valencia, and Saragossa (Zagaroza). In Madrid the Civil Engineering school was founded as soon as 1802. The qualification of aeronautical engineer was granted in 1926. A true polytechnic school with a large selection of departments was finally formed in 1971 by merging different schools. In continuum mechanics the creative level is kept by young people such as Jose Merodio with works in nonlinear elasticity and biomechanics in co-operation with foreign scientists (e.g., Ray W. Ogden). The same holds good for research at Saragossa where Manuel Doblaré also developed a successful school of biomechanics. In Barcelona, an original school of continuum thermo-dynamics was born in the expert hands of David Jou and Casas-Vasquez, while other researchers (e.g., R. Quintanilla) expanded some facets of applied mathematics by studying the mathematical properties of continuum models in thermo-elasticity and polar media in the line of Robin Knops (see Chap. 6) and some Romanian co-workers.

What still heavily characterizes mechanics in Spain and Portugal is the need to go abroad in order to obtain a doctoral degree. Furthermore, these two countries remained countries of emigration for brilliant elements. In this line we think of Michael Ortiz (born 1955) and Juan Carlos Simo (1952–1994) who have or had remarkable careers in the USA in the multiscale modeling of continua and computational mechanics. The same is true of Enrique Sanchez-Palencia (born 1941) who went to Paris after obtaining his degree in aeronautical engineering in Madrid and remained in France to gain an internationally acknowledged scientific success with his creative work on homogenization techniques and other asymptotic problems of continuum mechanics. A similar picture applies to Irene Fonseca from Portugal, with a PhD with David S. Kinderlehrer, who found her way in the medium of applied mathematics in the USA after studying in Minnesota.

10.6.3 Greece

The oldest and most prestigious educational institution in Greece in the field of technology is the National Technical University of Athens (*NTUA*). It was created in 1836, a few years after the independence of the southern part of Greece from the Ottoman Empire. It was re-organized in its present form in 1917. Other engineering schools were opened in Thessasloniki, Patras, Thraki (Democritus University of Thrace) and Chania (Crete) much more recently. In particular, the Aristotle University of Thessaloniki, now the largest university in Greece and in the Balkans (with about 95,000 students), was founded in 1925, after the independence of the Northern part of Greece from the Ottoman Empire in the 1910s.

At the *NTUA*, also know as the "Polytechnion", a somewhat legendary figure in mechanics in Greece was Pericles S. Theocaris (1921–1999). He studied at *NTUA* in the dramatic period of 1942–1948 (WWII followed by a civil war in Greece) and obtained doctoral degrees in Brussels (1952) and Paris (1953). A prolific author and an experimentalist (optical methods), he worked mainly in the strength of materials and the field of dynamic fracture. He formed a large number of mechanicians in Greece. Many engineers formed at *NTUA* went to the USA for further studies and some remained there for a brilliant research and academic career (e.g., in the 1970s, Maria Comninou, Xanthippi Markenscoff, and Yannis F. Dafalias, all in continuum mechanics). Theocaris played an important role in the National Academy in Athens and more generally in the politics of Greek science.

At the Aristotle University of Thessaloniki, we must single out Panagiotis (Panos) D. Panagiotopoulos (1950–2002), with a civil engineering diploma and a PhD from this university and a Habilitation obtained at the RWTH Aachen in Germany. He was one of the best specialists in the treatment of unilateral problems in continuum mechanics. He was responsible for a substantial progress in the exploitation of variational inequalities and the solution of non-convex problems (non-smooth mechanics; Panagiotopoulos 1985, 1993). Another successful contributor to continuum mechanics is Elias C. Aifantis (born in 1950; also simultaneously at the University of Michigan in East Lansing) basically educated in Thessaloniki and with a PhD from Minnesota, he has been instrumental in developing efficient models of gradient elasticity and gradient plasticity (generalized continua, See Chap. 13) after his many contributions to diffusion processes and the evolution of the density of dislocations in defective bodies. Georgios M. Lianis, after a brilliant career at Purdue (USA) in the Department of Aeronautics, came back to Greece and became the Greek Minister for Science and Technology (1982–1985), and then Ambassador of Greece to Japan, and back to Thessaloniki with a professorship in Mechanics. He had established definite results in several advanced domains of continuum mechanics including visco-elasticity and relativistic continuum mechanics when he was in Purdue.

10.6.4 Turkey

Modern Turkey provides a bridge in both rhetorical and physical senses between Europe and the Middle East. The Ottoman Empire considered itself the successor of Constantinople and Byzance, themselves the direct inheritors of the Roman Empire. But while Romans "invented" civil engineering (see above), the Ottoman Empire was not in this line of thought and globally did not do much for education, scientific one in particular. It is said that a technical institution for the training of ship builders and cartographers was created in Istanbul in 1773. But the formal recognition as a technical university in Istanbul goes back only to 1926 with a true autonomous university status in 1946. As to the Middle East Technical University (METU) in Ankara, it was founded in 1956 with an engineering faculty opened in 1959. Istanbul is also the place of the English speaking *University of the Bosphorus* (Bogazici universitesi) founded in 1843 as the first American higher education institution outside the USA under the name of *Robert College*. It has formed an elite of the country and its curriculum, including scientific, is much looked after.

We remind the reader that Richard von Mises found refuge in Turkey and taught in Ankara. Still, for many years all those looking for a graduate education in applied mathematics and mechanics had to pursue doctoral studies outside the country, often in Germany and France before WWII, and mostly in the USA after WWII. Thus Ratip Berker, a mathematician specialist of fluid dynamics and non-Newtonian fluids—and the author of a comprehensive lengthy contribution to the *Handbuch der Physik* edited by S. Flügge (cf. Berker 1965)—obtained his doctoral degree in mathematics in France in 1936, and taught in Paris from time to time.

The author had the privilege to take Berker's course on non-Newtonian fluids in Paris in 1967; Berker's lectures were among the most well delivered and refined—up to the point of some mannerism—lectures ever received by the author.

Soon after WWII, some famous American scientists in fact had immigrated from Turkey, among them A. Cemal Eringen (1921–2009; a foremost figure in generalized continuum mechanics, see Chap. 13), Fazil Erdogan (born 1926; a specialist of fracture in homogeneous and inhomogeneous materials and a long time professor at Lehigh in Pennsylvania), and E. Turan Onat (1925–2000; a specialist of anelastic behaviours first at Brown and then for a long time at the University of Yale). In more recent times—e.g., in the 1960s-1970s—Turkey provided brilliant post-doctoral students and graduate students in the USA, to name a few: Erdogan S. Şuhubi (with his original contributions to the mechanics of media endowed with a microstructure, and to wave propagation), Attila Askar (with his work on deformable piezoelectrics—Askar 1986—nonlinear waves, and the application of numerics to quantum problems), and Hilmi Demiray (theory of mixtures, plasmas, biomechanics, nonlinear waves) all at Princeton. They all returned to Turkey where they succeeded to emulate research in their various specialties either at universities or at the Marmara Research Centre.

10.7 Some Countries in Asia

10.7.1 Note on India

India, this immense country with a rich cultural and intellectual past is well equipped with a network of "Indian Institutes of technology" (*IIT*). This network provides a large number of qualified engineers. Many of them immigrate to richer countries with a large potential of jobs in research and development, but also in universities. English-speaking countries, but above all the USA, are the main beneficiaries of this flux. In the field of interest in this book we obviously note the success in research of many Indian-born scientists in particular in the USA, Canada, and Australia (see below). This leaves little room for a development of original research ideas in continuum mechanics on the "subcontinent" as the *BBC* always calls it. Nonetheless, many lines of research work are attempted. But they are marked by the British tradition of applied mathematics of the 19th century—special solutions of accepted set of equations, lengthy analytical solutions sometimes improved by small numerical simulations exploiting available commercial software. Quite often the considered problems deal with linear or linearized theories such as in thermo-elasticity, magnetized or piezoelectric bodies, polar media and magneto-hydrodynamics, and also linear wave problems in the bulk or at a surface. It is difficult in these conditions to point out a specific advance. However, in the period of interest, we observe the singularity presented by Bhoj Raj Seth (1907–1979), with a PhD obtained in London before WWII and who taught at the *IIT* of Kharagpur when the latter was opened in 1951. Seth was one of the very few Indian scientists who dealt with fundamental questions of continuum mechanics, e.g., the definition of finite-strain measures, creep, plasticity, and second-order effects in elasticity (e.g. Seth 1935).

Among the very successful Indian applied mathematicians in the USA we should mention first **Romesh C. Batra**, initially formed at the Punjabi University in India, but with a PhD obtained at Johns Hopkins University with J. L. Ericksen (1972). He is now a professor at the Virginia Polytechnic Institute and State University in Blacksburg with a large variety of scientific interests and works, including in a practically 40 year long career: nonlinear elasticity, the strength of materials, impact problems, material instabilities, shear banding, dynamic fracture, nano-mechanics, MEMS, and functionally graded materials.

The next scientist in this formidable selective group is **Kombakonam R. Rajagopal** (born 1950), initially formed at the Indian Institute of Technology in Madras/Chennai but with a PhD obtained in Minneapolis (1978) with Roger L. Fosdick (born 1936, himself a PhD from Brown in 1963, Editor of the *Journal of Elasticity*). "Raj", as known by his friends, is a never tired prolific author who has already left a definite print on so many fields of continuum mechanics, including non-Newtonian fluids, the theory of mixtures (see Rajagopal and Tao 1995), the mechanics of polymers (see the book by Wineman and Rajagopal 2000), thermo-mechanics in general, phase transitions, granular materials, biomechanics, and electro-rheological materials, all this with an obvious strong influence of Truesdell (with whom he co-authored a book on fluid mechanics, Truesdell and Rajagopal 1999). In addition, “Raj” is a highly cultivated humanist with a continued interest in traditional Indian texts but also in English literature and poetry. He is a professor at the Texas A&M University in College Station, Texas, and became co-editor of the *International Journal of Engineering Science*.

Finally, among the younger generation, we note **Kaushik Bhattacharya**, also initially formed at the Indian Institute of Technology in Madras/Chennai and a PhD from Minneapolis (1991) with R. James. His works of international quality span the interactions between continuum mechanics, materials science, and applied mathematics with a marked interest in the subtle mechanisms of phase transformations (see Bhattacharya 2003). He is a professor at Caltech and Editor-in-Chief of the famous *Journal of the Mechanics and Physics of Solids.*

10.7.2 Note in China

The case of China resembles that of India. The number of engineering and scientific schools has grown considerably in the last 30 years. Emigration was not as important as in the Indian case because of difficulties to leave the country and also the problem of language. We rather witness an organized system of studies and formation abroad with a possibility—or an obligation—to return to the country. Many centres have developed modern programs after the return to China of scientists who had made a career outside. But we will not go further in our analysis as it is difficult for an outside observer to grasp the details of the organization and of the schools of thought in this out-of-normal-scale country. Like in India, many researchers are satisfied—at the time of writing—with linear problems in combined fields, and the generalization to these fields of the solution techniques—involved as they can be—of problems known in elasticity. This is particularly true of problems dealing with cracks and piezoelectricity, and the mechanics of composite materials.

However, we note in Hong Kong a fruitful activity of international level in modern continuum mechanics in two institutions. One of these is the *City University of Hong Kong* at the Liu Bie Ju Centre for Mathematical Sciences. Prof. Hui–Hui Dai, with a PhD obtained in Newcastle-upon-Tyne (UK) with Alan Jeffrey, here is a specialist of nonlinear wave propagation (in particular solitons) in elastic structures, while Prof. Ph. G. Ciarlet (born 1938), originally from Paris, pursues his research in mathematical elasticity and differential geometry. The other institution is the recent *Hong Kong University of Science and Technology* at Clear Water Bay, where Prof. Qinping Sun, with a PhD (1989) from Tsinghua University, has founded an active research team dealing with the mechanics of phase transformations and the micromechanics of materials.

On the island of Formosa (Taiwan), C.-S. Yeh, a former PhD (1971) student of Y.-H. Pao in Cornell, has expanded research in coupled fields, especially in the mechanics of soft ferromagnetic bodies. Other research conducted in Taipei concerns fracture mechanics. This is also the case in South Korea. Both of these countries are very much influenced by American education.

10.7.3 The Case of Japan

This case is different from those of India and China. First the size and quantity of information are quite different. Second, after its opening to the western influence in the 19th century Japan decided to cultivate all aspects of modern science and technology with an unlimited enthusiasm. A network of “Imperial universities” was created. These universities became “National universities” after WWII. These are much looked after by Japanese students as they are supposed to form the Japanese elite. This holds true for all fields of knowledge including the various guises of engineering. Such institutions exist in the most well known cities (Tokyo, Kyoto, Nagoya,…). They are complemented by a long roster of establishments with varying status, e.g., universities related to the division of the country into “prefectures”, universities associated with municipalities, and then now many private universities.

From the viewpoint of continuum mechanics we note a characteristic trait that is shared with other fields of science, that is, a tendency to pragmatism and very few attempts at discussing the bases of our science and implying the creation of new concepts. In the very large production of high scientific level—but of more standard contents—that we shall briefly survey later on, we nonetheless distinguish two original characters.

One of these is **Kazuo Kondo** (1911–2001). Formed as an engineer at Tokyo Imperial University in the 1930s, Kondo first worked as an aeronautical engineer examining classical problems in structural mechanics or fluid mechanics. But in 1952, probably influenced by his many readings in mathematical physics, he proposed the first interpretation of the theory of structural defects (e.g., dislocations), in geometrical terms borrowed from Einstein’s theory of gravitation. With this revolutionary work, the sophistication of Riemannian geometry entered the field of continuum mechanics. The next steps had to be taken by European groups such as those of Bruce A. Bilby in Sheffield and Ekkehart Kröner in Stuttgart. All subsequent works along this geometric line bear the print of Kondo’s innovative work. In time Kondo developed almost single-handedly a series of publications known as the *R.A.A.G* (for *Research Association of Applied Geometry*; see Kondo 1951–1962) memoirs. These involve a mixture of advanced geometrical notions of non-Riemannian spaces, attempts at geometrizing everything, and unfortunately a philosophical vision where poetry, philology, semiotics, and an abuse of neologisms make the whole thing quite fuzzy and most of the time un-understandable by common mortals (cf. Croll 2006).

Another original character was **Tatsuo Tokuoka** (1929–1985) in Kyoto. He was original in the sense that he was one of the rare Japanese “Truesdellians”. In this line of research he worked mostly on so-called *hypo-elasticity* (that was thought in Truesdell's view as a possible approach to plasticity; but this line is now abandoned) and also on induced effects in nonlinear theories such as in acousto-elasticity and birefringence with applications to optical measurements for transparent polymers. He also encouraged studies of coupled fields in electro-magneto-elasticity. Shinya Motogi (in Osaka) was one of his students who worked with the present author on deformable magnetized bodies.

Along more traditional lines, we find a wealth of works in various branches of continuum mechanics. First, we note a special taste for the theory, numerics, and applications of plasticity. This includes rather old works by K. Washizu (1982) on variational methods in elasto-plasticity (the multi-field Hu-Washizu principle), but also more recent works by T. Miyoshi (1985), T. Inoue (in Kyoto, with a strong interest in phase transformations), and S. Murakami (Nagoya; numerics of elasto-plasticity, creep cracks), and others. The study of fracture is also a favourite subject matter with works by Toshihisa Nishioka (Kobe), Y. Murakami (Kyushu University), S. Murakami (Nagoya), and applications in electromagnetic bodies by Y. Shindo (Sendai). Configurational mechanics and the allied theory of material growth were recently approached by Shoji Imatani (Kyoto) in collaboration with the author. Thermo-elasticity and thermo-piezo-electricity, sometimes in gradient materials, have been dealt with in detail by Naotake Noda (born 1944) and his co-workers at the University of Shizuoka (Hammatsu). Finally, the mechanics of conducting and superconducting structures was contributed by Kenzo Miya in Tokyo after visiting Frank Moon in Cornell. Works along the same line were pursued for some time at the Tohoku University in Sendai under the leadership of Professors Junji Tani (born 1940) and Toshiyugi Takagi. Both Miya and Takagi have been active in the edition of a spot-on journal, the *International Journal of Applied Electromagnetics and Mechanics*. We also note the fruitful co-operation of Muneo Hori (Tokyo) with Sia Nemat-Nasser (San Diego) in problems of micromechanics (cf. Hori 1993). As an example of great success of a Japanese mechanician in the USA we have Toshio Mura (1925–2009) who made a career in micromechanics at Northwestern in Evanston after his formative years in Tokyo.

10.8 Concluding Remarks

In the foregoing sections we have examined a variety of countries and areas with different historical traditions in so far as high-level education and research achievements are concerned. Some of them participated directly in the 19th century European development of industry and allied education. These were those closer to the trio formed by England, France, and Germany and thus easily permeable to the spirit of the enlightenment and the upcoming industrialisation. Some others had to catch up with these developments and their contribution level to our science is still marked by this delayed evolution. Still others were at first completely outside this development and, in effect, their involvement in research is necessarily rather new. Some of the countries were weakened by a rather important emigration of people, and more particularly of scientists and certified engineers. Other countries were helped by the remnants of the influence of a well-organized state such as the Austro-Hungarian Empire. Anyway, what we must emphasize is the role played in some cases by networks of institutions and alumni, e.g., formation obtained in certain elitist engineering schools. This may help in finding teaching and/or research positions. In countries like modern Japan, professional societies such as the *Japanese Society of Mechanical Engineers* (founded as a copy of the *A.S.M.E*) are instrumental in maintaining an *esprit de corps*. In previously communist countries we must acknowledge the seminal role played by the local National Academy of Sciences (more on this in the chapter on the former Soviet Union and Russia). Often devoid of any mercantile and application-oriented vision, they favoured the treatment of theoretical problems and the implementation of serious applied mathematics. Finally, in Europe, but with a much broader region of influence, an institution such as the *International Centre of Mechanical Sciences* (*C.I.S.M*, in Udine, Italy) favoured the dissemination of new development trends. Not only this provided occasions to create new international relations, but this often suggested new ideas for research lines to the young participants (at least this is the feeling of the author who taught there eight times in a period of 35 years).

As a summary of this rapid and necessarily partial and biased survey, we record the obvious importance taken by the following fields of study: plasticity and fracture, the mechanics of phase transformations, coupled fields, and generalized continua.

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Chapter 11
The Soviet and Russian Schools

Abstract It is remarked that essentially for ideological reasons and the use of an original language and alphabet, contributions from this immense and powerful country have often been belittled or altogether neglected. This chapter tries to correct this misconception and biased treatment. In particular, one cannot discard some original facts, among them the general high quality of teaching at high-school and university levels, the essential role played by the Academy of Sciences and its various branches, and the friendly rivalry between Moscow and Leningrad/St Petersburg. That is why, after briefly recalling the role of some precursors, attention is focused on these two main centres that host a multiplicity of competing institutions. The former Soviet Union had the chance to foster strong personalities in continuum mechanics, e.g., L.I. Sedov, A.A. Ilyushin, A.Y. Ishlinsky, G.I. Barenblatt, V.V. Novozhilov, Y.N. Rabotnov, L.M. Kachanov, A.I. Lurie, I.A. Kunin, N.I. Mushkeshisvili, S.A. Amsbartsumian and many others. Their contributions in all fields of continuum mechanics and those of their disciples are surveyed albeit much too briefly. Their books, in contrast to their unevenly translated papers, had a world wide influence in the field. Some of the now much cultivated research fields find their origin in this country that experienced different political schemes (Russian Empire, Soviet Union, Russia, and the New Independent States) and went through difficult times.

11.1 Introduction: Slavic and Russian Background

Preliminary note: In what follows the English transliteration (in Latin alphabet) of Russian (in Cyrillic alphabet) is used but the reader must realize that this is a mere convention, transliteration between different alphabets having for purpose to render at the best the original sound of the foreign word but in your own way of pronouncing what you read. Here examples are: Жуковский = Zhukovsky (English) = Joukovski (French); ИЛЬЮШИН = Ilyushin (English) = Iliouchine (French).

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Transliterations in German, Italian, etc., are different. Also, wherever possible, the patronymic name of persons is given in the Russian way.

Imagine your are sitting in the front row of an assembly of perhaps one thousand people, squeezed between your two "godfathers", and waiting to be called on the stage by the University Rector, Academician and Professor of Mathematics Victor Antonovich Sadovsnichy, to receive a beautiful medal and a diploma of Honorary Professor of Mechanics. The room is a large hall in the *Lomonosov Moscow State University*. The day is 25 January, Year 2001, and a massive statue of Igor Pavlov stands behind the authorities wearing academic gowns. The lady acting as the mistress of ceremony has put her academic hat in the way of Gavroche's cap on the barricades in the *Liberty guiding the People* as painted by Eugène Delacroix. Your thoughts are irresistibly drawn to Pavlov's dogs salivating on demand. And then you realize that you know much less about Lomonosov, just that he is someone who, like Humboldt and the French Encyclopedists—in particular d'Alembert and Diderot—of the Enlightenment, has something to do with universal knowledge and humanism. As a matter of fact, 25 January is the anniversary day of the creation of the University of Moscow in 1755 by Empress Elizaveta Petrovna (Elizabeth I of Russia) following the suggestion of the polymath Academician Mikhail Vasilyevich Lomonosov (1711–1765). The latter is considered a physicist, an astronomer, a chemist, a geographer, a mosaicist, and a poet (cf. Pavlova and Fedorov 1980).

> *Personal touch*. In addition to the above personal description of a real happening, the reader will have noticed the particular attachment of the author to Jean Le Rond d'Alembert—whose name he gave to the Institute of Mechanics, Acoustics and Energetics at the Pierre and Marie Curie University in Paris. D'Alembert is not in the same class of mathematicians as Euler or Lagrange, but someone who invented the *wave equation* cannot be that bad (this is an answer to Truesdell's appraisal). In the Russian context, brilliant offers from Catherine the Great to d'Alembert to join the Academy in St Petersburg met no success; the same happened with Frederick the Great's offers as d'Alembert was an immovable Parisian.

This naturally takes us to a short historical review of the development of higher education in Russia—in its various avatars (Russian Empire, Soviet Union, etc.).

Caveat. Here we do not make any difference between what is now understood as Russia, Ukraine, Belarus ("White" Russia), and "new independent states", east and northwest of Russia. For instance, the oldest university in what was the USSR was in Tartu (founded 1632, in Estonia, then under Swedish rule and the second university in the Swedish Empire after Uppsala). Also, we do not make much difference between the various Slavic languages of the area. We just recall that Saints Cyril and Methodius, who were Byzantine Greek brothers and monks, and apparently true philologists (but they probably had a mother of Slavic origin—this may have helped), devised the so-called Glagolitic alphabet suited to match the specific features of the old Slavic language (especially old church Slavonic) in order to provide a good translation of the Bible. This is the origin of the Cyrillic alphabet, clearly derived from the Greek one but enriched to accommodate new sounds such as the hushing-hissing sounds (ts, ch, sh, shch) and used in Russia, Ukraine, Belarus, Bulgaria and Serbia. But the Baltic countries (Estonia, Latvia, and Lithuania, from north to south) belong to different language groups, while Poland, Czech Republic and Slovakia, although Slavic countries, use the—perhaps badly adapted—Latin alphabet. Countries like Georgia and Armenia still have different languages and alphabets.

We assume that the Moscow State University is the oldest standard university within Russia per se although it is discussed whether the University of "Sankt Petersburg" theoretically founded in 1725 under the Academy of Sciences may be considered older. The Saint-Petersburg State Polytechnical University was established in 1899. The latter was modelled after the Paris *Ecole Polytechnique* although, in contrast with its Parisian original, it was not a military establishment. But an Institute of Transportation and Communications modelled after the French *Ecole des Ponts et Chaussées* (See Chap. 8) was also founded in 1809. Another school for civil engineers was opened in 1832. Other higher-educational institutions were created for special purposes in different places. For instance, the University of Kazan goes back to 1804. But the competition was always principally between St Petersburg (the capital since Peter the Great and until the October revolution) and Moscow (an older and more central place that again became the capital within the Soviet Union). This competition is still very much alive. In addition the Soviet regime instituted a series of research centres within the Academy of Sciences which, therefore, became an institution with dual purposes, honorary on the one side with its Academicians elected by their peers, and a network of research laboratories with permanent researchers on the other side. The French *CNRS* (Chap. 8) in 1939 was more or less modelled after this second purpose of the Russian institution. Again the research centres of the Academy in St Petersburg (Leningrad in Soviet times after a short period with the—quite correct—name of Petrograd) and Moscow have been in competition since their creation. But such research centres were also created in many other places (Nizhny Novgorod, etc.) including in republics such as Ukraine (Kiev, Kharkov), Armenia (Yerevan), Georgia (Tbilisi), Estonia (Tallinn), etc.

In the sequel we use the abbreviations *MSU* (for Moscow State University and its forerunners), *StPbU* (for the University of St Petersburg) and *SPbPU* (for the St Petersburg Polytechnic) since they are the most frequently cited institutions. Research centres of the Academy are simply indicated by the abbreviation *RAS* (for "Russian Academy of Sciences") independently of whether it is in imperial, soviet or actual times.

Before turning to a perusal of research themes and achievements, the following general four remarks may be in order. A general one is that in the Russian Empire and also in the Soviet Union, a difference was made between *nationality* and *citizenship*. Here we call all people "Russian" for the sake of simplicity. Second, when we examine the career of Russian scientists we usually witness very little mobility. We do not analyze the cause of this phenomenon, but a sure result of this was that the notion of curriculum vitae was practically unknown in Russia before scientists tried to emigrate (before, people did not have to apply anywhere; they were locally known!). Third, in many of the institutions where continuum mechanics was cultivated departments or faculties were often of mixed denomination, "mathematics *and* mechanics". This obviously means a certain theoretical bias in the field. However, a remarkable feature is that many "mechanicians" had also acquired an excellent knowledge of mathematical physics (often through the series of books written by L. D. Landau and E. M. Lifshitz). This last remark is corroborated by a certain trend felt in many works by Russian mechanicians.

11.2 Some "Classic" Precursors and Pioneers

Without going back to Lomonosov, we should mention a few Russian scientists, not necessarily mechanicians, who have influenced some active trends in Russian mechanics and engineering in the twentieth century.

Of course the name of **Nikolai Ivanovich Lobachevsky** (1792–1856), one of the inventors of non-Euclidean geometries and rector (1827–1846) of the University in Kazan remains for ever attached to that university. His works strongly influenced many parts of mathematics and physics, including in general relativity. He spent all his life in Kazan.

The second emblematic figure is **Konstantin Eduardovich Tsiolkovsky** (1857–1935)—a teacher and not a professional scientist—who laid the bases of space travel and space propulsion. As we know, this was to bring Soviet aeronautics and astronautics to the forefront of the corresponding engineering developments and research. Himself influenced by the French novelist Jules Verne, he was a source of inspiration for many Russian engineers, not the least, the Soviet rocket engineer Sergey Korolev (1907–1966). The latter was educated at the Kiev Polytechnic Institute and the Bauman Moscow State Technical University (school originally founded by Empress Catherine II in 1763 and therefore considered as the second oldest higher-educational institution in Moscow) which formed many future engineers in aeronautics in the twentieth century. Andrei Tupolev, Nikolay Zhukovsky and Pavel Sukhoi were also alumni of the Bauman University. This university also swarmed in institutions such as the Moscow Aviation Institute, the *Central Aerohydrodynamics Institute* (with Russian abbreviation *TsAGI*, see below), and the Zhukovsky Military Academy of Aviation Engineering (now called Zhukovsky-Gagarin Air Force Academy).

Among the people just mentioned, special attention is to be paid to **Nikolay Yegorovich Zhukovsky** (1847–1921). Famous for his "Zhukovsky transform" in the complex plane, and the Kutta-Zhukovsky circulation theorem that explains the lift of an airfoil, this scientist established the first Aerodynamic Institute in Kachino (near Moscow) and was head of the *TsAGI* starting in 1918.

Sergey Alekseevich Chaplygin (1869–1942) was originally educated in physics and mathematics at MSU. But he achieved his first research work under the guidance of Zhukovsky. This was in the line of analytic works by Clebsch and Kirchhoff (see Sect. 1.6). He wrote famous papers in different areas of mechanics (analytical mechanics, gas streams). It is in one of these works that he proposed an efficient method to study jet flows of a gas at any subsonic speed. This opened the way for the study of high-velocity aerodynamics, providing the basis for the solution of problems of subsonic flows that became of actuality with the crossing of the Mach sound barrier with a jet plane in the late 1940s. The most well known student of Chaplygin was Leonid I. Sedov (see below in Sect. 11.3).

Personal touch. The author had the chance to take with Professor Paul Germain (see Sect. 7.2) one of the few specialized graduate courses offered in the world on the “Hodograph method and Chaplygin transform” in 1967. This is one of the most wonderful teaching experiences I went through since Germain had done much research work on this timely (1950–1955) subject and Tricomi’s equation—that describes the change of mathematical type of partial differential equations between elliptic and hyperbolic regimes. This proves that one never teaches and explains so well as when one gives orally the detail (without notes) of one’s successful research, even though a span of ten to twenty years may have elapsed.

The next three scientists, although not specifically mechanicians, but rather all round physicists, were instrumental in creating institutions and publishing books that fostered many domains of continuum mechanics and the allied applied mathematics, and contributed to the formation of an elite in the field. First of these is **Abram Fedorovich Ioffe** (pronounced Ioffé, sometimes written Joffe; 1880–1960). He was educated at *SpbPU* and worked at the Leningrad Physico-Technical Institute in St Petersburg (now named after him). A scientist with a very large spectrum of interests, he formed several world renowned scientists in various branches of physics, to name a few: Pyotr Kapitsa (low temperature physics), Igor Kurchatov (atomic research), Lev Artsimovich (plasma physics), and Yakov Frenkel. The Ioffe Institute (RAS) in St Petersburg remains one of the best research centres in Russia.

Note. An unexpected connexion with continuum mechanics is as follows. Physics professor Orest Danilovich Chwolson (1852–1924)—also written Khvol’son—in St Petersburg was one of the examiners of Ioffe’s thesis in 1913. He was the author of a five-volume treatise on physics that was translated into French in the period 1906–1928. It is in this translation (mostly done by their brother in law) that the Cosserat brothers first published as a complement their long 1909 paper on what has become known as “Cosserat continua” (see Chap. 13). The world is so small that the following should not come as a surprise. As a young graduate from St Petersburg, Abram Ioffe spent two years in Wilhelm Roentgen’s laboratory in Munich, where he obtained his PhD in 1905. He was asked to assess the possible publication in the Annalen der Physik of a revolutionary paper by an unknown German working in Switzerland (this was Albert Einstein). Ioffe recommended the publication of what was to become the foundational paper in special relativity; See Ioffe AF (1955). “In remembrance of A. Einstein”, Uspekhi Fizicheskikh Nauk, 57(2):187 (in Russian)].

Yakov Il’ich Frenkel (1894–1952), a young prodigy in mathematics, was educated at *StPbU* and did his first research work under Ioffe’s guidance. He later taught at both *StPbU* and *SPbPU*. During WWII he taught for some time at Kazan. He published many works and books, including on Vector and Tensor analysis (1925), Electrodynamics (1926), Analytical mechanics (1935) and Theoretical mechanics based on vector and tensor analysis (1940). It is said that his incredible productivity was a drawback. Like for Ioffe, his Jewish background made him the target of an anti-Semitic campaign in the late 1940s–early 1950s. He is most well know in mechanics for the dynamic model of dislocations that he proposed (1938) together with his younger co-worker Tatyana Abramovna Kontorova (1911–1976)—the Frenkel-Kontorova model. In some sense this was the first ever formulated equation of soliton theory in solid-state physics and mechanics (apart from the Korteweg-de Vries equation in fluid mechanics), now called the sine–Gordon equation. That is, in non-dimensional units, the nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \sin \quad u = 0 \tag{11.1}$$

Here the *sine* function provides a balance between nonlinearity and dispersion effects, hence the phenomenon of solitary waves (here topological solitons called *kinks*).

Note. During the anti-Semitic campaign that attacked physical views—supposedly held mainly by Jews (as "cosmopolitans") in favour of quantum indeterminism—in error compared to the official credo, his former co-worker Kontorova published, willingly or not, a paper negating Frenkel's contributions to physics; no comments!.

The third physicist of interest in the present context is the world renowned **Lev Davidovich Landau** (1908–1968; Nobel Prize in 1962). He was formed at *StPbU*, in Göttingen and in Copenhagen. He worked further in Kharkov and at the Institute of Physical Problems in Moscow, which now bears his name. He left his name in many fields of physics, but probably most famously for his contributions in condensed matter physics and more particularly to the theory of phase transitions. More appropriately for this book, together with his friend and frequent co-worker **Evgeni Mikhailovich Lifshitz** (1915–1985) he conceived and partly wrote a monumental treatise of theoretical physics in ten volumes—the celebrated "Landau curriculum". Of particular interest for mechanicians of the continuum are Volume 1 (Mechanics), Volume 2 (Theory of fields), Volume 6 (Fluid mechanics), Volume 7 (Theory of elasticity) and Volume 8 (Electrodynamics of continuous media). In contrast to Truesdell (see Chap. 5), Landau and Lifshitz favour an approach exploiting the Hamiltonian-Lagrangian variational approach. The above mentioned five volumes have often been inscribed in the curriculum of many Russian specialists of continuum mechanics and physics. This treatise played in the Soviet Union and still today a role equivalent to the Feynman lectures or the Berkeley course of physics in the USA, albeit in a more ambitious and complete, detailed, and advanced form.

Personal touch. As a young student the author benefited in 1965 from the French translation of the first two volumes of the Landau-Lifshitz course. With a typical enthusiasm of the youth I wrote to the MIR Publishers in Moscow to congratulate them and ask when the other volumes would be available in translation. Surprisingly enough (letters form Russia were rare at the time), I received an answer. But the other translations were not ready yet. I had to turn to the English translation of the volume on electrodynamics published in a rather luxurious edition by Pergamon Press to know the rest of the story. This cost half one

> month of my student fellowship. It was less expensive to self teach Russian. I wanted to study other volumes directly in Russian. I ordered the volume on elasticity and other volumes through the Russian bookstore in Paris, a very unreliable shop. I then discovered to my surprise and in truth with some deception that some chapters were *not* written by Landau and Lifshitz. Thus the chapter on dislocations was written in 1962 by **Arnold Markovich Kosevich** (1928–2006), then in Kharkov, Ukraine. I could not imagine that one day we would co-author papers on solitons in crystal physics (cf. Kosevich et al. 2001). Furthermore, having acquired a kind of reputation as an expert on science in the Eastern European countries, in the 1990s I was often called by the *SOROS* foundation in Washington to write expert reports on many Russian applications for financial support in physics and mechanics; just the same with the *INTAS* Foundation of the European Union in Brussels.

In what follows we had the choice between exposing research fields by separate themes or by selecting the institutions. Because of the restricted mobility of people and the consequent formation of durable schools, the second solution has been chosen, even though the same theme can be cultivated in various places but with difference styles.

11.3 Continuum Mechanics at Moscow

The richness of research results in our field in the various institutions of Moscow in the twentieth century is remarkable. As a matter of fact, several institutions seem to be in harsh competition, including between departments of the same university (MSU) or between MSU and local laboratories of the RAS.

11.3.1 MSU and Leonid I. Sedov

Leonid Ivanovich Sedov (1907–1999) is considered an emblematic figure in the landscape of mechanics in Russia. He was educated in mathematics and mechanics at MSU and his early research was achieved under the guidance of Chaplygin at *TsAGY*. His first remarkable result (1934) was in establishing an integral formula while studying by means of complex variables two-dimensional problems of hydrodynamics (see Sedov’s book of 1937). It happens that the same expression is also found in studies of cracks in elastic solids [see Maugin 1992, Eq. (A4.24)]. He made numerous contributions to hydro- and aero-dynamics, noticeably, those concerning the impact of bodies on water, hydroplaning, and aerodynamic forces on deformable wings. One of his favourite tools was the exploitation of dimensional analysis and so-called similarity, a subject on which he wrote a very popular monograph (1944). He became extremely famous in 1946 when he presented at the Landau seminar his “blast-wave” solution, obviously of actuality then with the first nuclear explosions. In more recent times, Sedov focused on the teaching and general formulation of continuum mechanics. His books (1962, 1973) on the subject have met an incredible success in Russian and in their many foreign-language translations. One of his favourite general formulations is now known as "Sedov's variational principle". This formulation (Sedov 1968) is close to one using a general view of the principle of virtual power (cf. Maugin 1980, with comparison to Sedov's formulation in Sect. 8.3). This principle is formulated in space–time in the following form:

$$\delta \int_V \Lambda_1 d\tau + \delta W + \delta W* = 0 \tag{11.2}$$

Here V is a region of space–time, Λ_1 is a generalized Lagrangian function including space–time metric and curvature, deformation, scalar or tensor complexes (e.g., spin effects), various parameters and their gradients. Eventually shock conditions across a discontinuity surface sweeping out V can also be deduced from such a general formulation. The term $\delta W*$stands for an already-varied—not necessarily holonomic (e.g., corresponding to dissipation and defects)—form and contains all constraints applied to the various fields in the bulk and at surfaces. This includes generalized external forces. The term δW is the looked for expression that provides the field equations for any admissible variation of the basic fields. Note that thermal and dislocation density considerations can be included so that (11.2) is much more general than any traditional formulation of the principle of virtual work. Examples of applications of (11.2) have been worked out by Sedov and some of his collaborators, including Berdichevsky (1966a, b), V.V. Lokhin, G.A. Lyubinov, V.A. Zhelnorovich and A.N. Golyubiutnikov. The first of these scientists produced one of the best ever written books on variational principles in continuum mechanics, first in Russian (1983) and then in a much revised and enlarged English version (2009). He also worked out (1979) a homogenization technique based on a variational formulation that elegantly provides the governing equations for slender structures such as shells, plates and rods.

Personal touch. One of the first considerations of Sedov's variational principle in the West was by the present author in his Princeton Doctoral thesis, 1971, Chapter One. The author was in touch with Sedov's group for a long period, starting in 1974 (cf. Maugin 1978). For some time the rumour even spread that he was going to translate Berdichevsky's book.

Among other disciples of Sedov who were or are very productive we note Lev. M. Truskinovsky and K. Chau Le (both in fact mentored by Berdichevsky), and V. Z. Parton. Truskinovsky published early pioneering works on the propagation of phase-transition fronts seen as discontinuity waves (Truskinovsky 1982, 1983, 1985, 1987). Le has applied Berdichevsky's technique to piezoelectric structures and their vibrations (See Le's book Le 1999). As to **Vladimir Zalmanovich Parton** (1940–2000) he became a noted specialist of mathematical methods in elasticity and plasticity. His most well known books are, among several more, on integral equations in elasticity (1982) and the elastic–plastic theory of fracture (1978). He published in 1976 an interesting pioneering paper on fracture in piezoelectrics. Together with Kudryavtsev, he published in 1988 a nice book on electro-magneto-elasticity with an emphasis on piezoelectrics and electricity conductors (Parton and Kudryavtsev 1988). This book was more or less in competition with the author's own book of 1988 that was also translated into Russian (Maugin 1991). Note that Berdichevsky, Truskinovsky, Le and Parton all left the USSR for the USA or Germany. Another disciple of Sedov at MSU and friend of the author was Naïl Sigbatulin—a Tatar who passed away untimely—who was a specialist of non-linear waves (book in 1994). In the same line, A. G. Kulikovsky, a noted specialist of hyperbolic systems, magneto-hydrodynamics and shock waves with works extending from the 1960s is also to be considered in the environment of Sedov (see Kulikovsky et al. 2001).

11.3.2 MSU and A. A. Ilyushin

With Ilyushin's group in Moscow we have a different trend and scientific environment. **Alexey Antonovich Ilyushin** (1911–1998) had a long career in solid mechanics mixed with politically marked organizational roles. One has to consult the biography given in Russian and English in the book (Kiyko et al. 2001) for a complete overview. Here we focus on some aspects that are closer to our own scientific experience (plasticity, general formulation of the thermo-mechanics of continua, electromechanical interactions). The strength of materials and the engineering aspects of elastoplasticity seem to have been a constant preoccupation of Ilyushin, starting with his early works in the mid 1930s. What is quite remarkable is that these most innovative and rewarding developments were made in an obviously difficult period, the 1940s, corresponding also to the strength of maturity reached by this scientist.

In Ilyushin's contributions to elasto-plasticity we like first to emphasize the iterative solution method (Ilyushin 1943). Elasto-plasticity is a nonlinear theory in which the stress-displacement solution appears in an incremental scenario that reflects the evolutionary nature of the related problems with possible loading and unloading phases. In a typical implicit scheme one has to determine the increments in displacement $\Delta\mathbf{u}$, strain $\Delta\mathbf{E}$ and stress $\Delta\Sigma$ such that the increment in displacement equals the increment in the kinematic condition at the boundary at the target time t_{n+1} and the new stress $\Sigma_n + \Delta\Sigma$ is in the target static condition S_{n+1} at target time t_{n+1}. In Ilyushin's method the solution in displacement is approximated by a sequence of elastic solutions taking, at each time, as initial condition, the result at the previous step. The basic idea goes back to Picard's method of successive approximations to nonlinear equations. In a more recent computer era, Ilyushin's method provided the basis for many numerical works (see Chap. 11 in Maugin 1992). In 1948, Ilyushin published one of the best books on plasticity, somewhat in competition with Hill's celebrated book (cf. Chap. 5).

Ilyushin's postulate (1961, 1963) in elasto-plasticity belongs in the flourishing period when minimum (or maximum) principles started to play a fundamental role, directly reflecting the thermodynamic irreversibility of the evolution of the plasticity phenomenon. Ilyushin's postulate can be stated thus: for any strain cycle

$\varepsilon(t)\,,\ t \in [0,1]$ with $\varepsilon(0) = \varepsilon(1)$ in small strains and stress σ, the strain power is positive or zero (semi-colon here means contracted product of two Cartesian tensors)

$$\int_0^1 \sigma : \dot{\varepsilon}(t)dt \geq 0. \tag{11.3}$$

This applies to materials with hardening. Equation (11.3) can be viewed as a *global stability criterion* which says that any closed response loop in a strain (abscissa)-stress (ordinate) diagram is always followed *clockwise*. It can be compared to Drucker's inequality (here ε^p is the plastic strain; see Chap. 4)

$$\dot{\sigma} : \dot{\varepsilon}^p \geq 0, \tag{11.4}$$

that implies that in such a diagram increments in stress and strain must always have the same sign in both loading and unloading, hence angular points at the highest and lowest points of a strain–stress hysteresis curve (hysteresis curves cannot be rounded at their upper and lower extremities). This positive hardening condition provides a *local stability criterion* for the material and, as just explained, gives further information on the shape of the loop considered by Ilyushin. The relationship between Ilyushin's, Drucker's and Hill's principle was discussed by several authors (see Maugin 1992, 2011, 2013; and Marigo in Kiyko et al. 2001).

In his lectures at MSU published in book form (1971) Ilyushin makes a special effort at a general rational presentation of continuum mechanics but not at the level of C.A. Truesdell in the west, and with less international success than Sedov's courses. This matter is emphasized in Brovko (2013) who examines the formal structure of Ilyushin's general approach. One notion that was not acknowledged so much is that of *Ilyushin-Lensky five-dimensional space.* This may seem a trivial idea when we remember the six-dimensional notation for stresses of Voigt used in modern piezoelectricity. But this is not the case. Indeed, we all agree now that the actual state of stresses in a material body depends on the whole past history of the body. This was clearly stated by Ilyushin in his book of 1948, but also continuously emphasized by him in his publications and his textbook for the Lomonosov University (Ilyushin 1971–1990). First implemented in small-strain plasticity, Ilyushin introduced the notion of six-dimensional spaces of strains and stresses (1954 on)—reduced to *five-dimensional spaces of deviatoric quantities* in the case of plasticity. This allows a classification of deformation processes with deformation trajectories with the required degree of complexity. This is particularly well adapted to the description of hardening in elasto-plastic materials, in particular in accounting for the deformation acquired by the yield surface during the evolutionary history of the material. Remarkably enough, testing machines devised in Ilyushin's group reproduce the trajectories in such spaces for complex loading. This led to the introduction of a "*principle of isotropy*" and the subsequent proposal of a "*postulate of macroscopic determinability*". This was often developed in co-operation with V. S. Lensky, a long-time associate of Ilyushin (see Ilyushin and Lensky 1959; Lensky 1960). V. S. Lensky and his son E.V. Lensky (1994)—

the E stands for “Era-Lenina” (no need for a translation!)—have shown the better agreement obtained with experiments by such a description than by accepted standard theories of elasto-plasticity. A modern reference to these works with some extensions is Zyczkowski and Kurtyka (1984).

The complete works of Ilyushin (2003–2009) deal with many more subjects including thermoelasticity, viscoelasticity, thermo-viscoelasticity, penetration problems, stability of structures, and the nonlinear dynamics of continua and structures, all fields in which he formed a number of scientists and mechanical engineers. This was achieved in a difficult period during which Ilyushin was also much involved in academic duties—both lecturing and administration—and also with a serious involvement in national scientific and engineering matters. Historical circumstances (the cold war, the discussed role of Ilyushin as rector of the University of Leningrad in 1950–1952, the secrecy of some of his applied research works, and his friendship with Lavrentiy Pavlovich Beria of *NKVD* fame) did not favour a full international recognition at these times.

11.3.3 MSU and Y. N. Rabotnov

Yuri Nikolaevich Rabotnov (1914–1985) was educated in the Department of Mathematics and Mechanics at MSU. He became the dean of this department in 1938 and organized a chair of plasticity. But he was also closely associated with Akademgorod (“Science town”) of the Siberian branch of the USSR Academy of Sciences. Among the scientists he mentored we know B. D. Annin, A. V. Berezin, A. A. Movchan, V. P. Tamuzh, E. V. Lomakin, and many others. He was back at his chair at MSU in 1965. He shared an equal interest in theoretical and experimental studies, in particular in fracture mechanics and composite materials. Among his original works we note the discovery of edge effects and local buckling in elastic shells. But he is mostly known all over the world for his creation of the *modern theory of creep* in which he duly exploited integral equations with hereditary type kernels in the style of Boltzmann and Volterra. He wrote definite books on this matter (see his books of 1969 and 1980). In particular, he proposed there original mathematical techniques to solve the problem of the inversion of action of these kernels that are of the fractional-exponential type. Such kernels sharing the properties of singularity and exponential nature were introduced by Rabotnov in 1948. Their application was recently discussed by Suvorova (2004). We remind the reader that the phenomenon of creep is of great importance as it is what determines the resistance and life duration of mechanical elements submitted to high temperature. It corresponds to an observed growth with time of plastic strain under the effect of a relatively small stress. Creep may be related to *damage*, a subject matter that interested Rabotnov all along his life and in which he proposed seminal ideas (cf. Rabotnov 1963) almost at the same time as Kachanov (see below).

Other departments at MSU have educated and still welcome scientists who contribute remarkably to continuum mechanics and its applications. This is the case of the Department of Physics, for instance with V. G. Mozhaev and his original mathematical results on elastic surface waves.

11.3.4 Moscow Institutes of the Russian Academy of Sciences

Institute of Mechanics

The *Institute for Problems of Mechanics* of the RAS in Moscow has always been in friendly competition with various departments of the Lomonosov University. But this is more a matter of personalities and sensibilities than of research fields. For a long time its head was **Alexander Yulyevich Ishlinsky** (1913–2003). The latter was also a scientist of many interests including gyroscopic and inertial-guidance systems as well as elasto-plasticity. The Institute took his name after his death. It is no question to review the works done and the results obtained at this Institute during some 50 years. We prefer to mention more particularly two individuals who have been closer to the author's own interests. One is **Robert V. Goldstein** whose fruitful scientific activity embraces fracture mechanics, wave propagation, the mechanics of large scale structures as also the mechanics of materials in arctic conditions (ice and ice cover). His most cited work (1974) deals with the brittle fracture of solids with arbitrary cracks. But he has also been involved in such problems as multi-fractal fracture geometry and scaling effects, and a clever application of invariant integrals to the problem of defect identification. Like many Russian mechanicians of his generation he demonstrates a high dexterity in applied mathematics. The other scientist is **Vladimir N. Kukudzhanov** (born 1931) who is the acknowledged Russian specialist of numerical computations in elasto-plastic and visco-plastic materials and non-linear wave propagation (see his book of 2008). Note that he was originally formed at the Moscow Institute of Physics and Technology and spent almost 20 years (1964–1983) at the Computer Centre of the RAS as a research senior before joining the Institute for Problems in Mechanics as Head of the Department of Mathematical modelling in solid mechanics and retiring in 2004.

Institute of Crystal physics

The Theoretical department of this institute was created in 1966 as an initiative of the famous crystallographer Alexey Vasilyevich Shubnikov (1887–1970), whose name was later on given to the Institute. Among many works dealing with all aspects of crystal physics, we are more specifically concerned by those that deal with the mechanics of defects and wave propagation in crystalline structures, i.e., anisotropic elastic solids. V. L. Indenbom (1915–2007) was the first head of this department and has been the driving force behind its successful researches. One of his interests was the relationship between plasticity and dislocations. His direct disciple **Vladimir I. Alshits** (born 1942) dealt essentially with dislocation problems and surface wave propagation. Together with J. Lothe (from Norway) and D. M. Barnett (from Stanford) and also sometimes with H. O. K. Kirchner (from Austria and Paris-Orsay), Polish scientists, and the present author, he has been one of the most active propagandists of Stroh's formulation (see Chap. 6), especially in stratified elastic, electro-elastic and magneto-elastic structures. His own students A. V. Shuvalov and A. Darinskii have continued along the same line. He also developed a theory—supported by experimental evidence—of magnetic effects in plastic crystals. In one of his old papers he had introduced the notion of Radon transform to represent a given intrinsic distortion in an infinite anisotropic crystal. Examples of works are: Indenbom (1979), Indenbom and Alshits (1974), Indenbom and Orlov (1962), Alshits and Maugin (2005).

Institute of Oceanology

The relevant well known scientist is none other than **Grigory Isaakovich Barenblatt** (born 1927). He is referred to here because he was the Head of the Theoretical Department of this Institute in the period 1975–1992. Although he may be introduced as a specialist of fluid mechanics dealing with such various problems as turbulence, porous flows, and non-Newtonian fluids, he is mostly celebrated for his fundamental works in asymptotic techniques ("intermediate asymptotics") and the theory of similarity in mechanics (cf. Barenblatt 1996). His achievements in the last subject obviously created some competition with Leonid Sedov (see above). Barenblatt graduated from the Department of Mathematics and Mechanics of MSU. He obtained his PhD (1953) there under the supervision of the famous mathematician A. N. Kolmogorov. He worked first at the Petrol Institute of the RAS and then as Head of the Department of Plasticity at the Institute of Mechanics at MSU (1961–1975) before joining the Institute of Oceanology. Starting from 1992 he has been sharing his time between various institutions in the west, essentially the University of Cambridge, UK, and the University of California at Berkeley. Among his many works which brought him international recognition and many scientific honours, we like to single out his work on nonlinear waves in polymers and his *elastic theory of cohesive forces* (Barenblatt 1962). In this theory of cracks it is admitted that ahead of the crack front there exists a zone in which the "atoms" can be pushed aside at a variable distance δ and that this separation leads to cohesion stresses which are opposed to a clear separation. These cohesion stresses vary from zero at $\delta = 0$ as a function $\sigma(\delta)$ according to a law characteristic of the material. J. R. Rice has shown the equivalence of Barenblatt's theory with Griffith's results.

Some of Barenblatt's successful disciples were R. L. Salganik (works on cracks) and **Genady P. Cherepanov** (born 1937). The latter, a very gifted student and researcher, was instrumental in introducing elements of configurational mechanics (see Chap. 14) at an early stage (Cherepanov 1967, 1968) in fracture theory with the notion of invariant Γ-integrals (Cherepanov 1977), of which the J-integral of fracture of Rice is but a special case. He showed how this notion can be used in industrial situations (e.g., machine-tooling and cutting). His book

(Cherepanov 1979, original Russian in 1974) is an unsurpassed marvel (in the same line see also Cherepanov 1985, 1987, 1989, 1998).

Institute of Earth Physics

For a long time at this institute, **Michael A. Grinfeld** is an original thinker who has contributed to various facets of continuum thermo-mechanics in the late 1980s until now. He is most well known for his works on the propagation of thermo-mechanical fronts (e.g., Grinfeld 1980) synthesized in his comprehensive book (Grinfeld 1991), and for his discovery of an elastic instability (so called Grinfeld instability) that may occur during molecular beam epitaxy (Grinfeld 1986). This is manifested when there exists a mismatch between the lattice sizes of the growing film and the supporting crystal, as elastic energy will be accumulated in the growing film. At some critical height, the free energy of the film can be lowered if the film breaks into isolated islands, where the tension can be relaxed laterally. The critical height depends on Young's moduli, mismatch size, and surface tensions. This instability is also known as the Asaro-Tiller-Grinfeld instability, but Grinfeld undoubtedly deserves to have his name alone attached to this effect. This creative scientist also worked on the effects of initial stresses on elastic waves, stress-driven morphological instabilities in rocks, glass and ceramics, the morphology of fractured domains in brittle fracture, the plasticity in monocrystals with limited active slip systems, and the kinetics of dielectric and piezoelectric crystals with lattice defects. He moved to the USA in 1992 but unfortunately he did not find there a well deserved stable university position. He works now on shocks in condensed matter.

Institute of Electronics

This is not a priori related to mechanics. But we include it here because it was the place where a new type of surface acoustic waves—now called Bleustein-Gulyaev waves—after J. L. Bleustein, a co-worker of H. F. Tiersten in the USA (see Sect. 4.5) and Yu. V. Gulyaev in this institute—was invented in 1968–1969. This may sound a small thing if we judge from the short length of the paper. But in fact it is one example of exact—and quite simple—solution that adds up to the short gallery already started with the Rayleigh, Love and Stoneley waves mentioned in Chap. 6. It is a pure shear-horizontally (SH) polarized surface elastic wave of which the existence is allowed by the perturbation of the surface boundary conditions by an electro-elastic coupling at the upper surface of an otherwise free (of mechanical load) linear piezoelectric half-space (Gulyaev 1969). In effect, this coupling plays the same role as the "slow" superimposed layer considered for Love waves. Since this discovery this institute has produced a huge quantity of works that find applications in signal processing.

Personal note. Several of the author's co-workers obtained their doctoral degrees in Moscow: Sanda Cleja-Tigoiu (from Romania) with Ilyushin, Rainaldo Rodriguez-Ramos (from Cuba) with Pobedrya in Ilyushin's environment, Ahmed F. Ghaleb (from Egypt) in Sedov's group, A. Darinskii with Alshits in crystal physics, and Boris A. Malomed (now in Israel) in Barenblatt's institute.

11.4 Continuum Mechanics at Leningrad/St Petersburg

11.4.1 The University of Leningrad/St Petersburg

The university in Leningrad/St Petersburg is usually considered as a successful research centre in mathematical studies related to elasticity and plasticity. Among the great mathematicians who contributed to the study of the partial differential equations of physics and mechanics, Vladimir Ivanovich Smirnov (1887–1974), Solomon Grigor'evich Mikhlin (1908–1990), and Vladimir Gilelevich Maz'ya (born 1937) must be singled out. In particular, Smirnov is the author of a much praised multi-volume course on "Higher mathematics", while Mikhlin wrote essential works on elasticity theory and boundary value problems early in his career, including plane problems (period 1932–1935), shell theory, and the so-called "Cosserat spectrum" (see his book of 1957). Among Smirnov's students, in addition to Mikhlin, we also find such well known mathematicians as Victor Kupradze (1903–1985) from Georgia and Sergei Lvovich Sobolev (1908–1989). The latter, while instrumental in the creation of the Akademgorodok Siberian scientific city together with physicist-mathematician Mikhail Alexeyevich Lawrentyev in the 1950s, is responsible for the introduction in functional analysis of generalized functions (later called "distributions" by the French mathematician Laurent Schwartz). "Sobolev spaces" are named after him. The tradition of studying mathematical problems in elasticity has been pursued until now in Leningrad/St Petersburg. Nowadays this is done around Academician Nikolai F. Morozov (born 1932) in the Department of Elasticity, with main interest focused on mathematical problems arising in fracture mechanics (cf. Morozov 1984, Bratov et al. 2009).

The university in Leningrad/St Petersburg is also considered a source of high level works in *non-linear elasticity*. Examining this point on an international level in the 1940s it is of interest to browse the lectures delivered by **Valentine Valentinovich Novozhilov** (1910–1993) at the University in 1947 (and reproduced in Novozhilov's book of 1948). Remarkably enough, the expression "rubber elasticity"—in contrast with work in the West by people like Rivlin; see Chap. 3—does not appear in this book. Practically the only western reference given is that to the 1937 paper by Murnaghan apart from a few papers by Italian scientists. Novozhilov rightfully explains that his non-linear elasticity is essentially useful for studying problems of stability in elasticity, correctly accounting for initial stresses, and reproducing the elastic–plastic non-linear response in monotonously increasing loading only. Of course he is interested in metals and physical nonlinearity only. The book received a positive appraisal from Truesdell in the review that the latter published in 1953 in the Bulletin of the American Mathematical Society, Vol. 59, pp. 457–473; Truesdell qualifies Novozhilov's treatment as "simple but profound". But Novozhilov is the author of many other papers, including in the theory of thin shells (see his book of 1951). Furthermore, the flag of non-linear elasticity was taken over by A. I. Lurie and then P. A. Zhilin at the Polytechnic University of Leningrad.

Lazar Markovich Kachanov (1914–1993) educated and with a full career in Leningrad is a mechanician who did much work in elasto-plasticity and the theory of creep. He contributed (Kachanov 1942) to variational formulations in both non-linear elasticity and elasto-plasticity. His book (Kachanov 1974) on plasticity may be considered a classic. It was translated into different languages. His greater success from our viewpoint is the introduction of a simple but efficient model for the phenomenon of *damage* (Kachanov 1958, 1960). The latter corresponds to a decrease in elastic properties of a material upon a growth of microcracks and microvoids during successive loading and unloading sequences. Indeed such a growth results locally in a decrease of the areas that can transmit internal forces (stresses; remember Cauchy's introduction of the stress concept). In one-space dimension a scalar D with value between zero and one is sufficient to describe the phenomenon in a sketchy manner. This is well explained by Kachanov in his last book (Kachanov 1986) published a few years before his death. He gave a law of damage accumulation in fatigue process in the form

$$\frac{\delta D}{\delta N} = g(\sigma_m, \sigma_M,\ D) \tag{11.5}$$

where N is the number of cycles, σ_m is the mean stress and σ_M is the maximum stress. This was developed in parallel with studies by Rabotnov. A thermodynamically admissible formulation of damage was formulated by French mechanicians (in particular, Jean Lemaitre and Jean-Louis Chaboche) for damage starting in the mid 1970s from ideas of Kachanov and Rabotnov.

11.4.2 St Petersburg Polytechnical Institute

Anatolii Isakovich Lurie or Luri'e (pronounced Lurié; 1901–1980) was educated at the *SpbPU* where he became the Head of the Department of Theoretical Mechanics before WWII and the Head of the Department of Dynamics and Strength of Machines or Department of Mechanics and Control Processes in the period 1944–1977. He was scientifically active for almost half a century. The numerous works of this scientist-encyclopedist span the hydrodynamics of viscous liquids as well as solid mechanics and control theory. He also formed many scientists through his efficient lectures, published books, and supervision. In his first works he was one of the first contributors to fully exploit the operational calculus of Oliver Heaviside in solutions of problems of fluid mechanics and of linear elasticity. Some of his nice works dealt with analytical mechanics on which he wrote a noted monograph. This seems to have naturally led him to deal with non-linear problems in the theory of automatic control with the notion of absolute stability and the due exploitation of the Lyapunov-function method (applications to spacecraft control).

In analytical mechanics Lurie dealt with great care with the representation of finite rotations (relation of angular-velocity vector with the Rodrigues-Hamilton and Cayley-Klein parameters)—this is useful in describing rotational internal degrees of freedom in generalized continuum mechanics. He also generalized to various friction laws the notion of dissipation potential introduced by Rayleigh for linear viscosity. This was before the exploitation of general dissipation potentials in continuum mechanics by French mechanicians (see Sect. 7.3) or H. Ziegler in Switzerland and D. G. B. Edelen in the USA. Like many Russian authors, e.g., Goldenweizer or Novozhilov—and also Ambarstumian; see below- he was also concerned by the theory of thin elastic shells, using asymptotics and rigorous mathematical analysis.

Compared to many Russian authors who kept alive components or the indicial notation for tensors, Lurie was a propagandist of the direct intrinsic notations for vectors and then for tensors. This brought him more closely to his American competitors, in particular in the Truesdellian School. He in fact translated into Russian Truesdell's "First course in rational mechanics" of 1975. This style is best reflected in what the author considers his most achieved book (Lurie 1980) published in 1980, the year of his death. This book is rather exceptional in the Russian landscape of continuum mechanics at the period, but it is little known, even in translation, in the west, being handicapped by its somewhat unusual notation that makes it difficult reading. This included the consideration of rubber as a good example of incompressible material, variational formulations with restrictions of the elasticity potential, and the problem of superimposition of small motions on a finite strain state.

For completeness we recall that in linear elasticity Lurie introduced an original symbolic formalism for describing the inverse of spatial differential operators. This is sometimes used in applications to the theory of elastic layers and thick plates, but it requires some practice and its formal nature is somewhat puzzling. On Lurie's works see Zhilin (2001).

Anyway, as a consequence of his never tired activity and his ingenuity Lurie created a true Leningrad school of mechanics. Among his direct disciples we find L. M. Zubov (who developed a geometrical theory of defects in finite strains; 1997) and above all **Pavel Andreevich Zhilin** (1942–2005). Also a never tired author like his mentor, Zhilin worked in different branches of continuum mechanics while teaching at the *SpbPU*. In 1994 he became the head of the Department of the Dynamics of Mechanical Systems at the Institute of Problems in Mechanics of the RAS in St Petersburg. He has left a definite print with his investigations on spinor motions in mechanics and physics, phase transitions in inelasticity, electrodynamics within rational mechanics, and the logical foundations of the field. The first of these is related to a rebirth of the notion of Kelvin continuum as a model of generalized continuum mechanics (cf. Chap. 13). Among his students we note Elena Grekova and Holm Achenbach (from East Germany).

He was instrumental in organizing the yearly Summer school on "Advanced Problems in Mechanics" held in St Petersburg/Repino.

Another researcher of this Department produced a pioneering work in the field of generalized continuum mechanics (See Chap. 13). It is Vladimir A. Palmov who proposed in 1964 one of the first modern models of a Cosserat continuum (Palmov 1964). He complemented this with a general model of complex continua in 1969. In more recent times, Palmov has been concerned with vibrations in complex mechanical structures (e.g., elastic media containing oscillators) and elasto-plastic bodies.

11.4.3 Institute for Problems of Mechanics RAS

This Institute has a long tradition and co-operation with Leningrad Polytechnical University, especially with P. A. Zhilin. Many results were obtained in common. But we like to focus on two characters. One is Eron L. Aero (born 1934) who, before even Palmov and long before anybody in the USA, produced (1960) a nice original paper on a model of Cosserat continuum. Remarkably enough Aero is still active at the moment of writing this book and he considers nonlinear effects in media that are also generalized continua.

Personal touch. The author of this book had the privilege to co-author a paper in Physical Review with Eron Aero and a younger colleague, Alexey V. Porubov.

The other researcher of this Institute who contributed much to the advance of continuum mechanics in Russia is A. A. Vakulenko. His main interest is the transition from the micro-level to the macro-level in the mechanics of materials. This is basically the fundamental problem of homogenization. Konstantin Z. Markov (1945–2007) from Sofia made under his supervision a Ph.D dissertation on anisotropy in creeping materials, and became himself a recognized specialist in micromechanics and random media. As a matter of fact, materials with memory, brittle fracture in creep, irreversible thermodynamics, the mechanics of polymeric materials, and the dynamics of liquid crystals all are in the range of interests and contributions of Vakulenko. Many of his papers are co-authored with A. V. Zakharov.

11.4.4 Ioffe Institute

This Institute is not specifically concerned with continuum mechanics. But some of its researchers have contributed to some interesting aspects of our field. This is the case of Alexander M. Samsonov who, with his co-workers, proved first mathematically, and then experimentally, the existence of solitary wave solutions in a transparent elastic rod. The theoretical proof is first based on an approximation of a rod of non-linear elastic material and of finite cross section by a quasi one-dimensional body (this is in the tradition of Kirchhoff, Love, and Mindlin), and then showing that the resulting equation that contains two different wave operators—hence a so-called doubly-dispersive nonlinear wave guide—admits solitary wave solutions. They are then observed by interferogrametry (see Samsonov's book 2001). One of Samsonov's original co-workers, Alexey V. Porubov, has since developed many works on solitary waves in one- and two-dimensional elastic structures after joining Eron Aero at the Institute for Problems in Mechanics in St Petersburg. Some of his works are written in close co-operation with the present writer. This includes new phenomena of amplification and localization of nonlinear waves (see Porubov 2003, 2009). He also developed a strong interest in so-called "rogue" waves, those localized waves of incredibly large amplitude that sometimes appear at sea in front of ships.

11.4.5 USSR Naval Academy in Leningrad

In this military school **Leonid I. Slepyan** (born 1930) first worked on problems related to the interests of the Academy. But he rapidly became a world renowned specialist in the theory of fracture where he examined in detail the propagation of cracks in lattices (discrete viewpoint), visco-elastic fracture, the dynamics of chains with non-monotonous stress–strain relations and phase transitions (cf. Slepyan 2002). In the 1990s he moved to Israel (Tel-Aviv) which he had previously seen from a distance only from aboard a ship of the Russian Navy.

11.5 Continuum Mechanics Elsewhere in Russia and Ukraine

11.5.1 Nizhny-Novgorod/Gorki

Nizhny Novgorod (**Gorki** in Soviet times) saw the burgeoning of a school of applied mathematics devoted to non-linear effects in physics and mechanics. This is mostly due to Aleksandr Aleksandrovich Andronov (1901–1952), who was himself a student of a famous mathematician, L. I. Mandelstam, and who founded in Gorki a scientific school on radio physics at the Gorki State University with a close connection with the Institute of Applied Physics of the RAS. But eminent members of the school became interested in non-linear effects in many fields including fluid mechanics and physical acoustics. Andronov was succeeded by Andrei Viktorovich Gaponov-Grekhov (born 1926) as leader of this school. Mikhail I. Rabinovich (born 1941) did his first steps in research under his supervision. Both have become world renowned in non-linear science. Note that

Rabinovich co-authored chapters on the evolution of turbulence in the volume on fluid mechanics of the Landau-Lifshitz course in theoretical physics. But more interestingly, he discovered stable stationary waves in dissipative nonlinear media, developed asymptotic methods for the analysis of nonlinear processes in distributed systems, discovered a synchronization phenomenon in various chaotic systems, and then turned to neuro-dynamics and dynamical principles of brain activity. In recent times disciples of this school (Lev A. Ostrovsky, A. V. Metrikine, A. I. Potapov, and V. I. Erofeyev) have fruitfully expanded research in non-linear waves in elastic crystals with or without microstructure in the framework of the Mechanical Engineering Institute of the RAS in Nizhny Novgorod, sometimes in co-operation with the author of this book (study of radiation stresses in the line of Leon Brillouin, and multi-wave resonance following phase synchronization evidenced by Rabinovich and Trubetskov 1989 see e.g., Ostrovsky and Potapov 1999; Potapov and Maugin 2001a, b; Potapov et al. 2005, Erofeyev 2003).

11.5.2 Novosibirsk

In **Novisibirsk** and the Siberian Branch of the RAS, there was of course active research in continuum mechanics and the accompanying computational aspects. Concerning this last point, emphasis must be placed on the roles played by Nikolaï Nikolaevich Yanenko with his introduction of the fractional-step scheme in numerics and by Sergei Konstantinovich Godunov (born 1929). The latter was formed in mathematics-mechanics at MSU, was first a researcher in Moscow (1952–1969), and then moved to Novosibirsk to become a professor at the local State University and to hold the chair of Differential Equations (1969–1997). He is an internationally acknowledged authority on hyperbolic systems and conservation laws of continuum physics in conjunction with their numerics (see his book of 1998).

As to continuum mechanics per se, we cannot ignore the beautiful results obtained by **Isaak A. Kunin** (born 1928) before he emigrated to the USA in 1979. He was originally educated at the *StPbPU* (Ph.D 1958). He moved to Novosibirsk in 1956 to become a professor and the Chairman of a Department of Theoretical Physics at the Institute of Thermophysics of the RAS (1963–1974) and then as a professor and Chairman of the Department of Mathematics at the Electrotechnical Institute. Kunin may be considered an all round mathematical physicist with a deep knowledge of geometric methods, an analytical dexterity, and a shared interest between discrete (crystal) and continuum approaches. It is in this rather general frame of mind that we can have a fair appraisal of his definite contributions to crystal lattice dynamics, the theory of point defects, dislocations and disclinations, cracks and the application of group theory and geometric concepts to these specific points. Perhaps that he is mostly known for his pioneering works (1966, 1970) on the *nonlocal theory of elasticity*, at the same time as D. Rogula in Poland and E. Kröner in Germany—one of the possible avenues to generalized continuum mechanics; cf. Chap. 13—and his general approach to the theory of *elastic media with microstructure*, studies that culminated in the publication of an incredibly detailed and powerful book Kunin (1982/1983). In the USA (Texas) he turned to the gauge theory of dislocations, the thermodynamics of vortices, and nonlinear dynamics and the theory of chaos. This versatility is very uncommon.

Two of co-authors or friends of Kunin in the USA have also come from the former USSR. One is Serge Preston (with original name Sergei Prishepionok) who was one of the last (1978) PhD students of Sergei Sobolev at the Steklov Institute of Mathematics in Moscow. He spent some time in Novosibirsk (1975–1976, 1980–1987) before immigrating to the USA in 1988, first for a short time at Berkeley, and then in Portland, Oregon. He is a specialist of contact geometry and the geometrical modelling of inelastic evolution of materials. He also contributed to the theory of material uniformity and attempts at describing the phenomenon of ageing of materials, using a formalism close to a four-dimensional relativistic one. This was done in co-operation with the second person, Alexander Chudnovsky—more a mechanician of materials—responsible for the introduction of a small damaged (processing) zone in front of crack tips in fracture and for a statistical theory of fracture—who became a professor in the Department of Civil and Materials Engineering at the University of Illinois in Chicago.

11.5.3 Ukraine

Now we look at **Ukraine** although now an independent country, but certainly with so much in common with Russia (religion, close language, and a long shared history). The two main cities are Kiev and Kharkov. We cannot ignore that Timoshenko, although with a long successful scientific career in the USA where he really created American mechanical engineering (see Chap. 4), had first been in his native Ukraine. The Institute of Mechanics in Kiev now proudly bears his name. Its present Head at the time of writing is Academician **Aleksandr Nikolaevich Guz'** (born 1939), himself a prolific and never tired author, who has been and remains the driving force behind the high production of scientific results in this institute. The subjects contributed include the mechanics of brittle fracture, the diffraction of elastic waves, the three-dimensional theory of the stability of deformable bodies, the effects of initial stresses on wave propagation and their application to non-destructive testing techniques, weakened shells, the mechanics of rigid bodies, electro-magneto-elasticity and, above all, the mechanics of composites. The Institute regularly publishes the "International Journal of Applied Mechanics" (in Russian) and has published influential series (in Russian) of monographs on composites (twelve volumes), elasticity and plasticity (six volumes), the mechanics of coupled fields (five volumes), etc., all between 1980 and 2000. Most of the books by A. N. Guz' have been originally published in Russian by the publishing firm Naukova Dumka in Kiev, but they also received translations into English.

The second institution of interest for us in Ukraine is the former Landau Institute (now called B. Verkin Institute of Low Temperature physics; Ukrainian Academy of Sciences) in Kharkov. This is indeed where Lev Landau started his remarkable career before joining Moscow. He left a definite print in a place where all physicists claim their (at least intellectual) descent from Landau. This was the case of our friend **Arnold Markovich Kosevich** (1928–2006) who, as one remembers, contributed the chapter on dislocations in the Landau-Lifshitz treatise when he was in his early 30s. Indeed, Kosevich proved to be one of the most important contributors to the theory of structural defects in the former Soviet Union and in independent Ukraine. He first masterly established the dynamical form of the Peach-Koehler force acting on a dislocation line by using an electrodynamic analogy (1962, 1964). Then his contributions to the field multiplied resulting in being invited to write much acclaimed syntheses on this matter (1979, 1988, 1999). He paid a special attention to the dynamics of magnetic spin with his co-worker Alexander S. Kovalev. He was naturally laid to becoming a specialist of solitons in solid-state physics (remember the Frenkel-Kontorova model and the sine–Gordon equation). More recently, together with his student M. M. Bogdan and the author (2001), he considered more general cases (complexes of solitons) with simultaneous addition of dispersive and nonlinear terms which still conserve the solitary-wave behaviour, but not the exact integrability typical of true solitonic systems. He also developed an original theory of crystal plasticity (1991) with his colleagues Boïko and Graber from the Physico-Technical Institute of the Ukrainian Academy of Sciences.

Unfortunately we have to leave aside many other places in Russia, e.g., Krasnodar (capital of the Kuban region), Voronezh (where the “Concordski” Tupolev Tu-144, was built) and Samara (called Kuybyshev during the Soviet regime, and formerly a closed—i.e., forbidden to foreigners—city like Gorki because of its defence and space industry). Yet, interesting research is conducted in these large cities. In particular, scientists at Samara (N. Kh. Arutyunyan, V. E. Naumov, and Yu. N. Radayev) have proposed an interesting theory of the growth of deformable solids by accretion in the period 1989–1995, unfortunately published only in Russian journals.

11.6 Continuum Mechanics in Peripheral Republics and Countries

Now we consider what we call peripheral republics that became independent in the early 1990s and share the property of having national languages that are not Slavic. This is the case of Georgia, Armenia and the Baltic countries.

11.6.1 Georgia

First, Georgia. The two leading figures active in the birth of a school of applied mathematics and mechanics were **Nikolai Ivanovich Muskhelishvili** (1891–1976) and **Victor Kupradze** (1903–1985). Both were born in Georgia then in the Russian Empire, the first one in the Capital Tbilisi and the second one in a small village. Muskhelishvili was educated at the University of St Petersburg. He did his first research work with Gury V. Kolosov (1867–1936) who had published a breakthrough original work on "An application of the theory of functions of a complex variable to the plane mathematical problems of the theory of elasticity" in 1909. This work may have been inspired by works on the bi-harmonic equation by the French mathematician Edouard Goursat (1858–1936). This was published at the Yuriev (now Tartu in Estonia) Russian University. Kolosov taught in St. Petersburg/Leningrad from 1914 to his death in 1936. Muskhelishvili and Kolosov published a joint paper in 1915. This proposed the first explicit solution of the fundamental boundary-value problem in the planar theory of elasticity for a circular region. Muskhelishvili became a specialist of this kind of problems. He returned to Tbilisi in 1930 facing the duty to teach and participate in the creation a true school of applied mathematics and theoretical mechanics in Georgia, after the first lectures in analysis given there in 1918 by Andrea Razmadze. As we know now, the most successful works of Muskhelishvili dealt with the theory of functions of a complex variable and the theory of singular integral equations (Muskhelishvili 1934-translation in (1953)-and Muskhelishvili 1946). The first paper published by Muskhelishvili in Tbilisi was in French (Muskhelishvili 1922). Courses had to be delivered in Georgian. Muskhelisvili and his colleagues had to establish and refine a scientific mathematical terminology in Georgian, even for such frequent terms as "equations" and "inequalities". Thus Muskhelishvili had to use his gifts as an amateur philologist. To make connection with a preceding chapter (Chap. 6), we remind the reader that Westergaard's famous solution (1939) of the crack singularity problem was obtained by use of the Kolosov-Muskhelishvili approach.

Victor Kupradze graduated from Tbilisi University having taking courses with Rakmadze and Muskhelishvili. He then went to Leningrad to work with the mathematicians Krylov and Smirnov. He returned to Tbilisi in 1935 to become the Head of the Institute of Mathematics. Starting in 1943 he held administrative and political positions. Still his mathematical inheritance includes many contributions to mathematics and mechanics, especially in problems of diffraction and scattering, potential theory, elasticity and thermoelasticity, and boundary-value problems (cf. Kupradze et al. 1976).

A famous alumnus from the Tbilisi State University was Ilia N. Vekua (1907–1977) who worked in the same fields as Muskhelishvili and Kupradze, to which we must add celebrated works on the theory of elastic shells (Vekua 1982), what became also a cultivated specialty of Georgian applied mathematicians.

11.6.2 Armenia

Armenia is theoretically more known throughout the world because of the numerous Armenian natives who fled from Turkey and settled essentially in Lebanon, France and the USA. But here we speak of the former republic of the Soviet Union with capital Yerevan or Erevan. Here also we witness the important role played by some gifted individuals. Our hero in this context is **Sergei A. Ambartsumian** (born 1922—not to be mistaken for his namesake in astrophysics). He was educated at the Polytechnic State University in Yerevan, graduating in 1942. He became a Doctor of Science (Russian system; roughly equivalent to a German Habilitation) in 1952 and Professor in 1953. Ambartsumian was active in different lines of research. Perhaps that he is most well known for this asymptotic approach of slender elastic bodies (plates, shells) and his numerous contributions to the mechanics of anisotropic plates (see Ambartsumian 1970, 1987, 1991). Two other trends that he largely contributed to develop in Armenia and outside are the elasticity of bodies with differing elasticities in tension and compression, as also electro-magneto-elasticity, in particular with its application to plates and shells. This has become the speciality of many Armenian mechanicians. Furthermore, Ambartsumian was responsible for the rational organization of continuum mechanics both at the University of Yerevan and at the Institute of Mechanics of the Armenian Academy. Finally, already much involved in the local academic affairs, he was one of those who played a political role during the final years of the Soviet Union with his intervention in favour of Armenia's independence at the Supreme Soviet assembly in Moscow.

11.6.3 Estonia

Estonia is a rather small country with no more than one and a half million of inhabitants of which only two thirds are native Estonians. The capital is Tallinn (old Reval). The spoken language belongs to the Finno-Ugric group (that includes only Estonian, Finnish, Hungarian and the languages spoken in some small territories in Russia). Furthermore, the history of Estonia has been a rather complicated one with occupations by many different people such as Swedes, Germans, and Russians. This gave a deep national feeling and a special resilience to its nationals confronted to adversity. By chance, if we may say so, it was sometimes decided in centralized Soviet Union that Tallinn would become a city specialized in the emerging robotics, informatics and the design of computers in Soviet times. That explains why an *Institute of Cybernetics* was founded in Tallinn. This became an active, although necessarily modest in size, centre of theoretical mechanics and applied mathematics under the cover of cybernetics. Thanks to Academician Nikolai Alumäe (1915–1992) this transformed into a source of original works in continuum mechanics and non-linear science. Uno Nigul first, and then

Jüri Engelbrecht (born 1939) greatly contributed to this orientation, including studies on wave propagation in visco-elastic materials, and non-linear waves in various classes of elastic materials (e.g., Engelbrecht 1997). Great progress was achieved in the understanding of the solutions of the Korteweg-de Vries (KdV) equation. This equation may be viewed as a one-directional evolution equation associated with a Boussinesq-like equation in elastic crystals by the application of the so-called reductive perturbation method by introducing appropriate scaling and moving coordinate. Generalizations of the KdV equation have been obtained for more complex models and the main properties of the solutions of these equations duly studied. Prof. Andrus Salupere has been very active in this line of research. Another co-worker, Prof. Arkadi Berezovski, formed at the use of advanced computational schemes in Novosibirsk (see Yanenko and Godunov above) has with success devoted much work to the simulation of the propagation of phase-transformation fronts. This is dealt with by constructing a thermodynamically admissible evolutionary scheme (finite volumes) that accounts for the material transformations in an original manner (see the book by Berezovski et al. 2008).

Here again we apologize to the mechanicians/scientists of Latvia (in Riga with works on fracture, composites, micromechanics of polymeric materials, in particular by V. P. Tamuzh), Lithuania (in Vilnius), Belarus (in Minsk with works on polar materials and the electrodynamics of heterogeneous materials, in particular by N. P. Migoun and A. V. Luikov), and other republics in the Caucasus and east of this region (Kazakhstan, Uzbekistan, etc.) for not perusing their interesting contributions.

11.7 Conclusions

From the above given description several general comments can be given and some general conclusions drawn. First, one must emphasize the generally high mathematical quality mixed to a good overall education in physics that transpires from many works. This is a direct consequence of the high quality of secondary and college education in the former USSR and a special kind of curriculum for many engineers. Russians are often seen as dreamers and poets who like to remember and recite thousands of verses from Pushkin or more recent poets. But we note very little speculation in most of the works that are rather distinguished by a formidable dexterity in analytic developments. The socialist system helping, it was possible for many authors to publish books that a capitalistic system of publishing would not have permitted. This remarkably high production of professional books is illustrated by the impressive list that follows in the reference list. For publications in scientific journals, the language used was an obstacle for a direct reading by many foreigners. The most important journals were translated into English, but this process suffered from two difficulties: the often unfaithful translation, and more than often a specific type of redaction—an imposed brevity—that hindered a clear understanding of the original matter. Some fields, principally mathematical problems in elasticity (Goldenblatt 1969), anelasticity (see Golovin 2008) and the theory of fracture (see Kostrov and Nikitin 1970), have received a greater attention than others—but this is not proper to this part of the world; cf. Chaps. 4 and 6—as it is more a question of timeliness.

One must also realize that this evolution of Russian continuum mechanics took place in a more dramatic period ever than everywhere else, including the October revolution, the ensuing civil war, the more or less rational re-organization of all Russian science, the great purges before the second World War, this war itself with its twenty millions of victims, the Gulag, the anti-Semitic campaigns, the Cold war that distracted money from pure science, and the disaggregating of the Soviet Union. Russian quality in teaching survived most of these events. The largest blow on Russian science occurred with the last event because it simultaneously brought a tightening in funds and a required change in the mentality of many scientists. For instance, they discovered the notions of research contract and of justification of expenses. The SOROS foundation—in which the present author was a frequently called expert—helped bringing small amounts of money for immediate survival during the transition. A direct reaction to this, mixed with the attraction of a long desired freedom, led to a true haemorrhage among scientists in general, but particularly among engineers, mechanicians and applied mathematicians who readily found opportunities in research or teaching positions abroad. Suffice it to record the following list among people cited in the foregoing sections: Berdichesvky, Truskinovsky, Le, Parton, Movchan, Shuvalov, Darinskii, Barenblatt, Cherepanov, Grinfeld, Malomed, K. Lurie, L. M. Kachanov, Slepyan, Rabinovich, Ostrovsky, Metrikine, Kunin, Preston, Chudnovsky, all emigrated principally to the USA. This phenomenon was not observed in peripheral countries and new independent states where scientists had the feeling to participate in a new adventure to the benefit of their native country, and this in spite of the evident reduction in available funds.

In both cases—successful immigration to the USA, Western Europe or Israel, or active contribution to the development of science in their own country with new perspectives—these by themselves are a patent recognition of the quality of the involved individuals and of the previously received formation. Now progress in Russia is in the hands of younger generations which are not so enthusiastic for scientific studies and careers.

Personal touch. Personal scientific contact and correspondence with foreigners in the former Soviet Union were something difficult. Original non printed scientific matter could not be sent outside the country (Some foreign students left the USSR without a copy of their own PhD Thesis. This was the case of my friend Ahmed F. Ghaleb from Egypt). Photo-copying was practically impossible. Contacts were usually established during meetings in other communist countries (e.g., Poland, where I met Ilyushin in 1975). Still the USSR organized the International Congress of Theoretical and Applied Mechanics in Moscow in 1972. The present author had the chance to visit several universities and ("non-sensible") research laboratories in 1978. But he returned there only in 1991. The SOROS programmes favoured some contacts in the 1990s. The author co-organized together with R. V. Goldstein a NATO Advanced Workshop in Moscow in 2002 (cf. Goldstein and Maugin 2004). Finally, in 2010, a trilateral seminar (between France, Russia and Germany) on continuum mechanics was organized by Holm Altenbach (an East German with a PhD from Leningrad), V. I. Erofeyev from Nizhny-Novgorod and the author in Lutherstadt-Wittenberg (Germany)—cf. Altenbach et al. (2011). This experience was fruitfully renewed in 2012.

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Chapter 12
Continuum Mechanics and Electromagnetism

Abstract The combination of pure continuum mechanics and electromagnetism cannot be a simple linear superimposition. That explains why it took some time to arrive at a rational formulation of this exemplary coupled-field theory. In spite of the experimental discovery of simple coupled effects in the Nineteenth century (e.g., magnetostriction, piezoelectricity), one practically had to await the second half of the Twentieth century to find a rational theory of deformable magnetized, electrically polarisable and electricity conducting continua. This is due to a small group of mechanicians who possessed a good apprehending of electromagnetic theory. The role of scientists such as R.A. Toupin, R.D. Mindlin, A.C. Eringen, W.F. Brown, H.F. Tiersten, M. Lax, D.F. Nelson, K. Hutter and the author of this book was instrumental in this intellectual construct. This is reported in a vivid manner, without neglecting the constructive works of electrical engineers and some mathematical physicists. After a brief survey of Nineteenth-century developments in electromagnetism the emphasis is placed on the seminal role played by Toupin in the 1950s and 1960s and on the author's own contributions in the period 1970–1990 concerning the fundamentals and the formulation of nonlinear electro- and magneto-elasticity often in the footsteps of H.F. Tiersten. A particular attention is paid to the evolution of the notions of electromagnetic force, momentum and stress tensor, and electro-magneto-mechanical couplings at the energy level.

12.1 Introduction

For electromagnetism, the first half of the 19th century is the time of the land-clearers such as Ampère, Faraday, Gauss, Poisson, and Oersted. The second half of the 19th is the time of unification in a grand scheme involving electricity and magnetism on equal footing, and culminating in the works of Kelvin, Weber, Helmholtz, and above all, Maxwell (1873) and Heaviside (1892) (to whom we owe the presently used form of Maxwell's equations). In parallel, coupled effects

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of the electro-mechanical, magneto-mechanical and galvano-magnetic types were discovered. Among them electric conduction, piezoelectricity (Curie brothers) and magnetostriction (Joule) are still those that steer attention because of the many received applications. Then there followed a long period, early and first part of the 20th century, during which many relevant discussions were devoted to the relativistic framework, while electrical engineering took the front with applications to energy productive or transforming machines and to macroscopic electromechanical devices. It is only in the second part of the 20th century that we witness an in-depth thinking about the continuum representation of multi-physical couplings with works of R.A. Toupin, R.D. Mindlin, A.C. Eringen, W.F. Brown, H.F. Tiersten, M. Lax, and D.F. Nelson, to whom we associate ourselves as we clearly agree with many of these developments, in particular with due consideration of interaction forces, and this in a pre fast-computer age. In parallel one must account for the constructive works of electrical engineers such as Penfield, Haus and Livens, and physicists such as Lorentz and de Groot and Suttorp.

12.2 Prerequisite: 19th Century: Physics Versus Electrical Engineering

The thermomechanics of solely deformable material continua and the electromagnetism of vacuum are two well established bodies of knowledge. The main question arises when material continua and electromagnetic fields co-exist spatially. It is then agreed upon that the relevant Maxwellian fields in matter, magnetic field $\mathbf{H}$ and electric displacement $\mathbf{D}$, differ from the characteristic electromagnetic fields of vacuum, the magnetic induction $\mathbf{B}$ and the electric field $\mathbf{E}$, in such a way that with appropriate electromagnetic units (so-called Lorentz-Heaviside units; neither factor 4π nor coefficients ε_0 and μ_0) we have the equations

$$\mathbf{H} = \mathbf{B} - \mathbf{M},\ \mathbf{D} = \mathbf{E} + \mathbf{P}, \tag{12.1}$$

where $\mathbf{M}$ and $\mathbf{P}$ are the magnetization and electric polarization per unit volume, fields that differ from zero only in magnetized and electrically polarized matter, respectively, i.e., when the celebrated set of Maxwell's equations in a fixed laboratory frame reads in full generality as

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},\ \nabla . \mathbf{B} = 0, \tag{12.2}$$

and

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{J},\ \nabla . \mathbf{D} = q_f, \tag{12.3}$$

where c is the velocity of light in vacuum, $\mathbf{J}$ is the electric current vector, and q_f is the density of free electric charges. The first set (12.2) is valid everywhere and yields the notion of electromagnetic potentials. In general, to close the system of field Eqs. (12.2) and (12.3), we are to be given *electromagnetic constitutive equations*, e.g., to give an idea to the reader, functional relations of the type

$$\mathbf{M} = \mathbf{M}(\mathbf{H}, .), \ \mathbf{P} = \mathbf{P}(\mathbf{E}, .), \ \mathbf{J} = \mathbf{J}(\mathbf{E}, .), \tag{12.4}$$

where the dots stand for some other variables such as temperature or a strain in a deformable solid.

Remark 12.1 The formulation (12.2)–(12.3) indeed is the one given by Oliver Heaviside who is supposed to have said that "Maxwell's theory is none other than Maxwell's equations" (accordingly, no need for explanation!). Maxwell himself proposed a formulation in terms of the potentials (twenty equations for twenty variables), and sometimes even used the formalism of quaternions, fashionable at the time. The first of (12.2) is none other than Faraday's equation; the first of (12.3) is Ampère's equation amended to account for Maxwell's displacement current without which Maxwell could not have forecasted the existence of electromagnetic waves—experimentally checked by Heinrich Rudolph Hertz (1857–1894) in 1888. The second of (12.3) is the Gauss-Poisson's equation. As to the second of (12.2) it means that there are no sources of magnetic induction $\mathbf{B}$ or, in other words, magnetic monopoles do not exist, an assumption valid unless contradicted by some new experimental evidence. In all we may consider that Eqs. (12.2) and (12.3) are the results of collective—but not necessarily coordinated—efforts by scientists, some experimentalists, others more theoreticians or mathematicians, such as Oersted, Ampère, Faraday, Gauss, Poisson, Kelvin, Weber, Helmholtz, and above all, **James Clerk Maxwell** (1831–1879) and **Oliver Heaviside** (1850–1925) over a long stretch of time. Among the crucial steps in that lengthy story we like to single out (1) the discovery by the Danish physicist Hans Christian Oersted (1777–1851) of a link between electricity and magnetism (flowing electricity in a wire could cause the needle of a nearby magnetic compass to be deflected), (2) its mathematical formulation by André-Marie Ampère (1775–1836), (3) the discovery of the solenoid by Arago, (4) that of electromagnets by Sturgeon, and (5) the discovery of electromagnetic induction by Michael Faraday (1791–1867). Without these it would not have been possible to conceive electromagnetic machines to generate electricity (the electric dynamo; alternating current) and vice versa, those to produce motion (the electric motor). The invention of the battery (pile) by Volta was also crucial to have handy a source of electricity (direct current).

While other possibilities exist, the selection (12.4) of dependent variables is not gratuitous. It pertains to the *characteristic* electromagnetic fields of *matter*. Several remarks are in order. First, by taking the divergence of $(12.3)_1$ and accounting for $(12.3)_2$, we obtain the law of *conservation of electric charge*:

$$\frac{\partial q_f}{\partial t} + \nabla.\mathbf{J} = 0, \tag{12.5}$$

a *strict* conservation law. Second, by a usual manipulation, one also deduces from (12.2)–(12.3) an *energy identity* called the "Poynting-Umov theorem", such that

$$\mathbf{H}.\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J}.\mathbf{E} - \nabla.\mathbf{S},\ \mathbf{S} \equiv c\mathbf{E} \times \mathbf{H}, \tag{12.6}$$

without any hypothesis concerning the electromagnetic constitutive equations.

Remark 12.2 Note that (12.6) is **not** the first law of thermodynamics (conservation of energy); it is just an identity relating to electromagnetic fields only, but these may be interacting with other fields as we shall see further down. In the West (12.6) is referred to as Poynting theorem after John Henry Poynting (1852–1914). But the Russian physicist at Moscow University, Nikolay A. Umov (1846–1915), was responsible for introducing the notion of *energy flux* (in liquid and elastic media) in 1874. Early disciples of Maxwell such as Poynting, Heaviside, and Larmor in the UK are called the "Maxwellians" (see Hunt 1991).

If we are in a *vacuum* (for which the three quantities in (12.4) vanish identically), long before the proof of her "invariance" theorem by Emmy Noether in 1918, Maxwell proved the existence of the following *vectorial* strict conservation law:

$$\frac{\partial \mathbf{p}^{\mathrm{em}.f}}{\partial t} - div\, \mathbf{t}^{\mathrm{em}.f} = \mathbf{0}, \tag{12.7}$$

wherein the electromagnetic momentum (in vacuum) and the so-called (symmetric) Maxwell stress tensor (stress tensor of *free* electromagnetic fields) are defined by

$$\mathbf{p}^{\mathrm{em}.f} = \frac{1}{c}\mathbf{E} \times \mathbf{B},\ \mathbf{t}^{\mathrm{em}.f} = \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B} - u^{\mathrm{em}.f}\mathbf{1},\ u^{\mathrm{em}.f} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \tag{12.8}$$

where the last quantity $u^{\mathrm{em}.f}$ is the electromagnetic energy of free fields per unit volume. As a matter of fact, in the same condition this verifies the conservation law (electromagnetic energy in vacuum)

$$\frac{\partial}{\partial t} u^{\mathrm{em}.f} + \nabla.\mathbf{S} = 0. \tag{12.9}$$

Equations (12.7) and (12.9) are peculiar expressions that hold here because of the inherent linearity of the electromagnetic constitutive equations ($\mathbf{H} = \mathbf{B}, \mathbf{D} = \mathbf{E}, \mathbf{J} = \mathbf{0}$) in vacuum. The latter serves as a (nonpolarized) *medium of comparison* for other electromagnetic media (an idea that will be successfully translated into mechanical behaviour by John R. Willis for studying effective properties of composites and deviations from a standard homogeneous elastic model in the 1970s; See Chap. 6).

Dealing with *energy* in a magnetized, electrically polarized, and conducting material in electromagnetism is a much more subtle matter as shown by the Eq. (12.6).

The latter can be integrated in a usual conservation form for a global volume only if the electromagnetic constitutive equations are *linear* and the body is *rigid*. Indeed, with simple constitutive equations $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \varepsilon\mathbf{E}$, for a rigid body occupying volume V bounded by regular boundary ∂V, of unit outward pointing normal $\mathbf{n}$, from (12.6) we would have the global balance of electromagnetic energy

$$\frac{d}{dt}\int_V u^{\mathrm{em}.m} dV = -\int_V \mathbf{J}.\mathbf{E} dV - \int_{\partial V} \mathbf{n}.\mathbf{S} dA, \tag{12.10}$$

with

$$u^{\mathrm{em}.m} = \frac{1}{2}(\varepsilon\mathbf{E}^2 + \mathbf{B}^2/\mu) = \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{B}.\mathbf{H}). \tag{12.11}$$

But this is **generally not true** as an expression for electromagnetic energy in an arbitrary deformable solid where (1) the electromagnetic constitutive equations may be strongly nonlinear and may even be dissipative (e.g., with relaxation, hysteresis), and (2) electromagnetic fields do **not** constitute an isolated thermodynamic system and they are in strong interaction with the deformation field. A consequence of this fact is that, if (12.6) is always true, it does not constitute a local statement of energy conservation for the *whole* mechanical-*plus*-electromagnetic system (sorry, the "plus" may be misleading with a connotation of simple "addition"). Similarly, Eq. (12.7) does not constitute an equation for conservation of so-called canonical momentum for the whole system. Much more work is required to reach this general result. What is remarkable is that, in spite of these words of caution, many authors have a natural tendency to think of an expression such as $(12.11)_2$ as a starting point in any electromagnetic continuum. This is particularly true in relativistically invariant theories where the a priori viewpoint of Minkowski (1908) concerning electromagnetic momentum and electromagnetic stress tensor (there the energy-momentum tensor) has been damaging. But Minkowski's reasoning is not based on a sophisticated physical model of field-matter interactions (Minkowski was a pure mathematician). The same remark also applies concerning another energetic quantity such as a Lagrangian density per unit volume. The density

$$l^{\mathrm{em}.f} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \tag{12.12}$$

strictly applies only to electromagnetic fields in a vacuum although it was proposed by authors such as Voigt, following Thomson (Kelvin) and Maxwell, in analogy with a "mechanical" Lagrangian $L = K - W$ with kinetic and potential contributions. All this clearly means that part of the electromagnetic energy and of Lagrangian densities is stored also in the internal/free energy or "matter" Lagrangian for the combined mechanical-*plus*-electromagnetic medium that includes the missing interaction terms that should be expressed in terms of the essentially material fields (12.4).

One remark about the electric current: for all practical purposes, we note that the Joule term $\mathbf{J}.\mathbf{E}$ can be interpreted as a power expended by an electric force. Indeed, we can write as an example

$$\mathbf{J}.\mathbf{E} = (q\mathbf{v}).\mathbf{E} = (q\mathbf{E}).\mathbf{v} = \mathbf{f}.\mathbf{v}, \tag{12.13}$$

where $\mathbf{f} = q\mathbf{E}$ is seen in statics, according to Lorentz, as the elementary mechanical force acting on a point particle of electric charge q in an electric field $\mathbf{E}$. For a particle moving at velocity $\mathbf{v}$, we have the *Lorentz force*

$$\mathbf{f} = q\mathbf{E} + \frac{q}{c}\mathbf{v}\times\mathbf{B} = q\tilde{\mathbf{E}},\ \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}, \tag{12.14}$$

where the electric field $\tilde{\mathbf{E}}$ is called the *electromotive intensity*.

To close this section of prerequisite we briefly recall the relationship of Maxwell's theory with electrical engineering.

First, Faraday's equation $(12.2)_1$ relates the circuitage voltage that appears when the flux linkage varies in time, as in electrical generators. Indeed, by use of Stokes' theorem applied to a surface element S leaning on a circuit C, one shows that the difference of potential is given by

$$e.m.f = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}, \tag{12.15}$$

or

$$e = \frac{d\lambda}{dt} \tag{12.16}$$

in terms of the flux linkage λ.

Second, Ampère's law $(12.3)_1$ relates the magnetic field that curls around a current flux, corrected for unsteady values of electric fields (this last correction is due to Maxwell; cf. the notion of displacement current). By applying Stokes' theorem to a surface element S leaning on a circuit C, one finds, for a coil of n turns of length l, the relation

$$\int_C \mathbf{H}.dl = nI \; or \; H = nI/l, \tag{12.17}$$

where I is the current.

Third, Gauss-Poisson's equation $(12.3)_2$ tallies the field lines emanating (hence the divergence) from a distribution of charges.

Finally, the last of Maxwell's equation $(12.2)_2$ reflects the circumstance that isolated magnetic poles do not exist. As a consequence a line of magnetic induction closes on itself. It does not "emanate" from a magnetic charge distribution as the latter does not exist.

Ampère's and Gauss-Poisson's equations are not used as such in circuitry, but the law of conservation of charges (12.5) yields, by integration, the circuitry equation

$$I = \frac{dq}{dt}, \tag{12.18}$$

where q is the electric charge. Then the system (12.15)–(12.18) is closed by the well known *constitutive equations* of passive circuit elements:

$$\begin{aligned} \lambda &= L\,I\,(inductor), \\ q &= C\,e\,(capacitor), \\ e &= R\,I\,(resistor), \end{aligned} \tag{12.19}$$

where L, C and R are an *inductance*, a *capacitance*, and a *resistance*, respectively. The last of these represents the celebrated *Ohm law*. There exist nonlinear generalizations of the constitutive Eq. (12.19). Added to Kirchhoff's laws of currents at nodes, the above set Eqs. (12.15) through (12.19) are all what one needs at the macro-scale of *electrical engineering*.

12.3 Passing to a Charged, Magnetized, Electrically Polarized, Deformable Continuum

12.3.1 A True 20th Century Adventure

For a true physicist the generalization of above given equations such as (12.7) and (12.9) is a difficult task that is identified with the evaluation of the forces, couples and energy sources arising from the interaction between a large number of electric charges in moving matter at a microscopic scale. This is in order to avoid any arbitrary or a priori macroscopic expressions that are hard to posit save by divination. Such an approach that we favour over any other methods accounts for the rich information about the interactions between the mechanical system and electromagnetic fields in matter that are gained from a particle model due initially to **Hendrik Anton Lorentz** (1853–1928). It was taken over by Dixon and Eringen (1964), Nelson (1979), and Maugin and Eringen (1977) to whom we owe the present formulation. This analysis belongs in the most rewarding improvements brought to the *electrodynamics of moving media* in the 20th century. It consists in evaluating the total force, couple and power acting on, or developed by, electromagnetic fields on the elementary electric charges contained in a stable cloud or representative volume element of volume ΔV, and introducing the approximations of multipoles, a truncation of these at a certain order, and a volume or phase-space average. Lorentz's vision is essentially that of a free space containing point charged particles. The starting point is the celebrated Lorentz force (Lorentz 1909) that is written as [cf. (12.14)]

$$\delta\mathbf{f}_\alpha = \delta q_\alpha \left(\mathbf{e}\left(\mathbf{r}_\alpha\right) + \frac{1}{c}\dot{\mathbf{x}} \times \mathbf{b}_\alpha(\mathbf{r}_\alpha)\right), \tag{12.20}$$

where $\mathbf{e}$ and $\mathbf{b}$ are the electric field and magnetic induction at the current placement $\mathbf{r}_\alpha$ of the elementary electric charge δq_α contained in ΔV. The computation then consists in evaluating the quantities (force, couple, power of forces)

$$\sum_{\alpha\in\Delta V}\delta\mathbf{f}_\alpha,\ \sum_{\alpha\in\Delta V}(\mathbf{r}_\alpha \times \delta\mathbf{f}_\alpha),\ \sum_{\alpha\in\Delta V}\delta\mathbf{f}_\alpha.\dot{\mathbf{x}}_\alpha, \tag{12.21}$$

and then dividing by ΔV. This may be considered a naive volume average technique. A more advanced one would envisage a statistical average in phase space, and perhaps a formulation in a relativistic framework. Anyway this is the technique followed first by Lorentz and then by Dixon and Eringen (1964), Nelson (1979), and Maugin and Eringen (1977) in a Galilean approximation and by de Groot and Suttorp (1972) in a relativistic framework.

Remark 12.3: On the notation of fields. By way of example, let $\mathbf{M}$ denote the magnetization per unit volume in a fixed laboratory frame. Then derived fields are noted with a superimposed ornament. Thus $\tilde{\mathbf{M}}$ designates the volume magnetization in a frame co-moving with the element of matter; $\bar{\mathbf{M}}$ is the same but reported (i.e., convected back) to the material framework and $\mu = \tilde{\mathbf{M}}/\rho$ is the magnetization per unit mass. Similarly for the electric polarization $\mathbf{P}$, with $\tilde{\mathbf{P}}$, $\bar{\mathbf{P}}$ and $\pi = \tilde{\mathbf{P}}/\rho$. Note that both magnetization and electric polarization relate to matter and are *extensive* quantities, i.e., proportional to the volume of matter. This is most relevant in continuum thermo-mechanics.

12.3.2 Results from the Microscopic Model

From (12.20) and (12.21), expanding the expressions in terms of the internal coordinates $\xi_\alpha = \mathbf{x}_\alpha(t) - \mathbf{x}$, neglecting quadrupole contributions and higher-order multipoles, lengthy calculations lead to the following electromagnetic source terms of force, couple and energy per unit continuous volume:

$$\mathbf{f}^{\mathrm{em}} = q_f\tilde{\mathbf{E}} + \frac{1}{c}(\tilde{\mathbf{J}} + \mathbf{P}^*) \times \mathbf{B} + (\mathbf{P}.\nabla)\mathbf{E} + (\nabla\mathbf{B}).\tilde{\mathbf{M}}, \tag{12.22}$$

$$\mathbf{c}^{\mathrm{em}} = \mathbf{r} \times \mathbf{f}^{\mathrm{em}} + \tilde{\mathbf{c}}^{\mathrm{em}}, \tag{12.23}$$

$$w^{\mathrm{em}} = \mathbf{f}^{\mathrm{em}}.\mathbf{v} + \tilde{\mathbf{c}}^{\mathrm{em}}.\Omega + \rho h^{\mathrm{em}}, \tag{12.24}$$

where **r** refers to the centre of charges of the volume element, ρ is the matter density, and **v** is the physical velocity, Ω is the vorticity $\Omega = (\nabla \times \mathbf{v})/2$, and we have set

$$q_f(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha, \tag{12.25}$$

$$\mathbf{P}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \delta q_\alpha \xi_\alpha(\mathbf{x}, t), \tag{12.26}$$

$$\mathbf{M}(\mathbf{x}, t) = (\Delta V)^{-1} \sum_{\alpha \in \Delta V} \frac{1}{2c} \delta q_\alpha \xi_\alpha \times \dot{\xi}_\alpha, \tag{12.27}$$

Note the lack of symmetry between polarization and magnetization effects. We have also defined the intrinsic electromagnetic sources of couple, energy and stress by (here tr = trace; subscript s stands for the operation of symmetrisation)

$$\tilde{\mathbf{c}}^{em} = \mathbf{P} \times \tilde{\mathbf{E}} + \tilde{\mathbf{M}} \times \mathbf{B}, \tag{12.28}$$

$$\rho h^{em} = \widetilde{\mathbf{J}}\widetilde{\mathbf{E}} + \widetilde{\mathbf{E}}\mathbf{P}^* - \widetilde{\mathbf{M}}.\mathbf{B}^* + tr\left(\widetilde{\mathbf{t}}^{em} (\nabla \mathbf{v})_s\right), \tag{12.29}$$

and

$$\widetilde{\mathbf{t}}^{em} = \mathbf{P} \otimes \widetilde{\mathbf{E}} - \mathbf{B} \otimes \widetilde{\mathbf{M}} + (\widetilde{\mathbf{M}}\mathbf{B})\mathbf{1}, \tag{12.30}$$

where the following fields are those in a co-moving frame (*Galilean approximation*; first of these is the conduction current *per se*):

$$\tilde{\mathbf{J}} = \mathbf{J} - q_f \mathbf{v},\ \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B},\ \tilde{\mathbf{M}} = \mathbf{M} + \frac{1}{c} \mathbf{v} \times \mathbf{P} \tag{12.31}$$

and **E** and **B** are simple volume averages of **e** and **b**. The first contribution in the *r-h-s* of (12.22) is none other than a "Lorentz force" [cf. (12.14)] since

$$\mathbf{f}_L = q_f \mathbf{E} + \frac{1}{c} (q_f \mathbf{v}) \times \mathbf{B} = q_f \tilde{\mathbf{E}}. \tag{12.32}$$

Finally, a left asterisk denotes a so-called convected (Oldroyd) time derivative such that (see Sect. 3.4)

$$\mathbf{P}^* = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) + \mathbf{v}(\nabla.\mathbf{P}) = \frac{d\mathbf{P}}{dt} - (\mathbf{P}.\nabla)\mathbf{v} + \mathbf{P}(\nabla.\mathbf{v}). \tag{12.33}$$

The above given expressions where fields in the laboratory frame and those in a co-moving frame co-exist is only *Galilean invariant*, but this is sufficient for engineering purposes. Note that no $\tilde{\mathbf{P}}$ intervenes here for the good reason that $\tilde{\mathbf{P}} = \mathbf{P}$ in this approximation (Galilean approximation materialized by a lack of symmetry between electric polarization and magnetization).

12.3.3 Contributions in the Macroscopic Balance Laws

In principle, the above obtained *source terms,* once their origin forgotten, have to be carried into the classical balance laws of a continuum (with a possible *non symmetric* Cauchy stress), leaving however the internal/free energy of the medium to depend on the electromagnetic fields. A remarkable fact is that in spite of their farfetched outlook, some may be given a form that reminds us of some standard expression [such as in (12.6)]. For instance, Maugin and Eringen (1977) have shown that (12.24) can also be written as

$$w^{\mathrm{em}} = \mathbf{J}.\mathbf{E} + \mathbf{E}.\frac{\partial \mathbf{P}}{\partial t} - \mathbf{M}.\frac{\partial \mathbf{B}}{\partial t} + \nabla(\mathbf{v}(\mathbf{E}.\mathbf{P})) = -\frac{\partial u^{\mathrm{em}.f}}{\partial t} - \nabla.(\mathbf{S} - \mathbf{v}(\mathbf{E}.\mathbf{P})), \tag{12.34}$$

in which we identify the terms already present in (12.9).

The electromagnetic volume force defined in (12.22) is sometimes called the electromagnetic *ponderomotive force*, $\tilde{\mathbf{c}}^{\mathrm{em}}$ being then the *ponderomotive couple*. In 1974, Collet and Maugin proved the following remarkable identity at any regular material point:

$$\frac{\partial \mathbf{p}^{\mathrm{em}}}{\partial t} - div\,\mathbf{t}^{\mathrm{em}} = -\mathbf{f}^{\mathrm{em}}, \tag{12.35}$$

where

$$\mathbf{p}^{\mathrm{em}} = \mathbf{p}^{\mathrm{em}.f} = \frac{1}{c}\mathbf{E}\times\mathbf{B}, \tag{12.36}$$

$$\mathbf{t}^{\mathrm{em}} = \mathbf{t}^{\mathrm{em}.f} + \tilde{\mathbf{t}}^{\mathrm{em}}. \tag{12.37}$$

Since we are dealing with nonsymmetric second-order tensors, we must specify that their divergence is taken on the first index. Simultaneously, the ponderomotive couple is the axial vector associated with the skew part of $\tilde{\mathbf{t}}^{\mathrm{em}}$. The latter vanishes together with the source terms in (12.28) outside matter, and (12.35) reverts to (12.7) in a vacuum. Because of the source term in its *r-h-s* equation (12.35) is *not* a conservation law for the whole physical system. But it allows one to rewrite the balance law of linear momentum for the whole continuum in a specific form (see Maugin 1988, for these developments). We can also rewrite (12.22) emphasizing the occurrence of an *effective* Lorentz force $\mathbf{f}_L^{\mathrm{eff}}$ in the form

$$\mathbf{f}^{\mathrm{em}} = \mathbf{f}_L^{\mathrm{eff}} + div\,\tilde{\mathbf{t}}^{\mathrm{em}}, \tag{12.38}$$

with [compare to (12.20)]

$$\mathbf{f}_L^{\mathrm{eff}} = q^{\mathrm{eff}}\tilde{\mathbf{E}} + \frac{1}{c}\tilde{\mathbf{J}}^{\mathrm{eff}}\times\mathbf{B}, \tag{12.39}$$

where

$$q^{\text{eff}} = q_f - \nabla.\mathbf{P},\ \tilde{\mathbf{J}}^{\text{eff}} = \tilde{\mathbf{J}} + \mathbf{P}^* + c\nabla \times \tilde{\mathbf{M}}. \tag{12.40}$$

We easily check that there holds the identity

$$\frac{\partial \mathbf{p}^{\text{em}}}{\partial t} - div\,\mathbf{t}^{\text{em}.f} = -\mathbf{f}_L^{\text{eff}}. \tag{12.41}$$

Equations (12.35) and (12.41) are compatible, but they may suggest different ways to combine mechanics and electromagnetism in the balance of linear momentum as it may be tempting to many researchers to consider $\mathbf{f}_L^{\text{eff}}$ as the primitive interaction force because effective charge and electric current appear also in Maxwell's equations (cf. Eringen and Maugin 1990, p. 54) as natural perturbations of the vacuum equations, e.g., (12.3) also read

$$\nabla.\mathbf{E} = q_f - \nabla.\mathbf{P},\ \nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}\left(\mathbf{J} + \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M}\right). \tag{12.42}$$

These can be recast using convected fields and time derivatives yielding source expressions such as in (12.33). Finally, we remark that Eqs. (12.34) and (12.35) reduce to (12.9) and (12.7) in a vacuum, respectively.

Note While the above-given results are obtained, a similar treatment of Maxwell's equations in vacuum with source terms due to the individual electric charges, yields, after space average, the macroscopic equations (12.2) and (12.3)—this was the basic idea of Lorentz [on Maxwell's equations proper, see de Groot (1969) and Tiertsen (1990)].

12.3.4 Postulate of Equations Accounting for Informations from a Microscopic Model

This is the manner *à la* Newton-Cauchy dear to the Truesdellians. Global balance laws are written for linear and angular momenta along with the first and second laws of thermodynamics, in which electromagnetic source terms as recalled above are introduced. This is the viewpoint expanded in Eringen and Maugin (1990) and Maugin (1988), and other authors, in great detail. Of course the result depends on the microscopic model used to obtain the sources or else, on an a priori and somewhat arbitrary choice for these sources (**not** our viewpoint). The full application of the method shows its pertinence, albeit in spite of a complexity arising in the description of stresses. The latter are not symmetric a priori since there exists an applied couple (12.28), something that cannot be denied as otherwise there would not exist such an evident effect as the compass alignment with a magnetic field. But in the end the obtained thermo-mechanics proves to be satisfactory with an energy (internal or free-Helmholtz) containing part of the interactions, a part of constitutive origin. Among the results obtained we note the formula for the stresses $\mathbf{t}$ appearing in the local balance of linear momentum of a continuum (divergence of tensors taken on the first index; $\mathbf{f}$ = body force such as gravity, ρ = actual matter density; $\dot{\mathbf{v}}$ = acceleration)

$$\rho\dot{\mathbf{v}} = \mathbf{f} + \mathbf{f}^{\text{em}} + div\,\mathbf{t}, \tag{12.43}$$

with a *nonsymmetric* Cauchy stress

$$\mathbf{t} = \mathbf{t}^{E} + (\mathbf{t}^{\text{em}.f} - \mathbf{t}^{\text{em}}) = \mathbf{t}^{E} - \tilde{\mathbf{t}}^{\text{em}}, \tag{12.44}$$

or a total *symmetric (Cauchy)* stress τ such that

$$\tau = \mathbf{t} + \mathbf{t}^{\text{em}} = \mathbf{t}^{E} + \mathbf{t}^{\text{em}.f}, \tag{12.45}$$

where $\mathbf{t}^{E}$ is a symmetric "elastic" stress such that, in components (here symmetric and skewsymmetric parts)

$$t^{E}_{(ij)} = t_{(ji)} + \tilde{t}^{\text{em}}_{(ji)},\; t^{E}_{[ji]} \equiv 0. \tag{12.46}$$

To the same degree of generality as (12.43), the local forms of the energy equation and inequality of entropy read (Eringen and Maugin 1990)

$$\rho\dot{e} = tr\left(\mathbf{t}(\nabla\mathbf{v})^{T}\right) - \mathbf{f}^{\text{em}}.\mathbf{v} + w^{\text{em}} - \nabla.\mathbf{q} + \rho h, \tag{12.47}$$

and

$$\rho\dot{\eta} \geq \rho h\theta^{-1} - \nabla.(\mathbf{q}\theta^{-1}), \tag{12.48}$$

where e, η, θ, $\mathbf{q}$ and h are the internal energy per unit actual mass, the entropy per unit actual mass, the thermodynamic temperature, the heat flux vector, and the external heat supply per unit actual mass, respectively. The electromagnetic energy "source" w^{em} is given by (12.24) with expressions (12.28) through (12.30) valid. Equivalent forms were given in (12.34). Another equivalent expression is given by

$$w^{\text{em}} = \mathbf{f}^{\text{em}}.\mathbf{v} + \rho\tilde{\mathbf{E}}.\dot{\pi} - \tilde{\mathbf{M}}.\dot{\mathbf{B}} + \tilde{\mathbf{J}}.\tilde{\mathbf{E}}, \tag{12.49}$$

where $\pi = \mathbf{P}/\rho$ is the electric polarization per unit mass. On introducing the Helmholtz free energy function per unit mass

$$\psi = e - \eta\,\theta, \tag{12.50}$$

and substituting from (12.47), (12.22) and (12.49) in (12.48), one arrives at the so-called *Clausius-Duhem inequality*

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + tr\left(\mathbf{t}(\nabla\mathbf{v})^T\right) + \tilde{\mathbf{J}}.\,\tilde{\mathbf{E}} + \rho\tilde{\mathbf{E}}.\dot{\pi} - \tilde{\mathbf{M}}.\,\dot{\mathbf{B}} - (\mathbf{q}/\theta).\nabla\theta \geq 0. \qquad (12.51)$$

In a now well established tradition, this is conceived as a constraint on the formulation of constitutive equations for the fields $(\psi, \eta, \mathbf{t}, \tilde{\mathbf{J}}, \tilde{\mathbf{E}}, \tilde{\mathbf{M}}, \mathbf{q})$. The formulation (12.51) clearly emphasizes for electromagnetic processes the role of independent variables (causes) played by the pair $(\tilde{\mathbf{E}}, \nabla\,\theta)$, electric polarization and magnetic induction for galvanomagnetic effects.

What is important here is that, in deformable solids, one often prefers to reformulate the theory in terms of so-called *material* fields. To that effect we set the Piola transform or pull-back of electromagnetic fields (cf. Chap. 3)

$$\bar{\mathbf{B}} = J_F\mathbf{F}^{-1}.\mathbf{B},\, \bar{\mathbf{D}} = J_F\mathbf{F}^{-1}.\mathbf{D},\, \bar{\mathbf{P}} = \Pi = J_F\mathbf{F}^{-1}.\,\mathbf{P} = \rho_0\mathbf{F}^{-1}.\pi, \qquad (12.52)$$

and

$$\bar{\mathbf{E}} = \mathbf{E}.\mathbf{F},\, \tilde{M}_K = J_F F^{-1}_{\mathrm{kp}}\tilde{M}_p,\, \bar{M}_K = \tilde{M}_i F_{iK}, \qquad (12.53)$$

with

$$J_F = det\,\mathbf{F}, \rho_0 = \rho J_F,\, \mathbf{F} = \left\{F_{iK} = x_{i,K}\right\}. \qquad (12.54)$$

We then check that the relations (12.1) translate in material components to

$$\bar{D}_K = J_F C^{-1}_{\mathrm{KL}}\bar{E}_L + \Pi_K\,,\, \bar{H}_K = J_F^{-1}C_{\mathrm{KL}}\bar{B}_L - \bar{M}_L, \qquad (12.55)$$

with

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \left\{C_{\mathrm{KL}} = x_{i,L}\,x_{i,L}\right\},\, \mathbf{C}^{-1} = (\mathbf{C})^{-1}. \qquad (12.56)$$

First and second Piola-Kirchhoff stresses are defined by

$$\mathbf{T} = J_F\mathbf{F}^{-1}.\mathbf{t}\,,\, \mathbf{S} = \mathbf{T}.\mathbf{F}^{-T}. \qquad (12.57)$$

Similar definitions hold for Piola-Kirchhoff stresses associated with the stresses $\mathbf{t}^E$ and $\mathbf{t}^{\mathrm{em}.f}$. Thus we write

$$\mathbf{T}^E = J_F\mathbf{F}^{-1}.\mathbf{t}^E,\, \mathbf{S}^E = \mathbf{T}^E.\mathbf{F}^{-T},\, \mathbf{T}^F = J_F\mathbf{F}^{-1}.\mathbf{t}^{\mathrm{em}.f}. \qquad (12.58)$$

Then it is proved that Eqs. (12.43) and (12.51) can be rewritten as (here no body force)

$$\frac{\partial}{\partial t}\mathbf{p}^t_R\bigg|_X - div_R(\mathbf{T}^E + \mathbf{T}^F) = \mathbf{0},\, \mathbf{p}^t_R \equiv \rho_0\left(\mathbf{v} + \frac{1}{\rho c}\,\mathbf{E}\times\mathbf{B}\right), \qquad (12.59)$$

and

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2}S^E_{\mathrm{KL}}\dot{C}_{\mathrm{KL}} + \bar{E}_K\dot{\Pi}_K - \bar{M}_K\dot{\bar{B}}_K + J_F(\tilde{\mathbf{J}}.\tilde{\mathbf{E}} - (\mathbf{q}/\theta).\nabla\theta) \geq 0, \quad (12.60)$$

where we have set

$$W = \rho_0 \psi,\ N = \rho_0\ \eta. \tag{12.61}$$

Once we have established constitutive equations for S^{E}_{KL}, $\bar{E}_K$ and $\bar{M}_K$, we can return to the original Eulerian fields, including the Cauchy stress $\mathbf{t}$ which is the stress present in the mechanical boundary condition. In order to complement Eq. (12.59) we also need Maxwell's equations expressed in the material framework. A first hint of this form of Maxwell equations was given by Walker et al. (1965), and McCarthy (1968). But the final form was definitely set by Lax and Nelson (1976).

Equation (12.60) is perfectly equipped to treat both reversible and irreversible coupled electro-magneto-deformable properties. Irreversible properties have been dealt with in particular by Maugin and co-workers (electric relaxation, hysteresis). However, both Eqs. (12.59) and (12.60), although fully dynamical in the Galilean approximation, are *not* equipped to treat the case of electromagnetic materials endowed with electromagnetic *internal degrees* of freedom (see Sects. 12.4 and 12.5 for these).

In this section we have presented the continuum dynamic theory—with the restrictions just mentioned—in its achieved form. Of course the path to this final form (reported in the author's formalism and its obvious shortcomings) was long and paved by many researchers, using various methods of approach. In particular, we must single out the enlightening book of Livens (1962)—George Henry Livens (1886–1950) was a Cambridgian whose main work was in effect electrical theory, the work of Tiersten and Tsai (1972) at the Rensselaer Polytechnic, the theory developed by Kolumban Hutter and Y.-H. Pao at Cornell in the early 1970s—see the book by Hutter and Van de Ven (1978), the variational approach by electrical engineers (Penfield and Haus 1967) at M.I.T, the treatise of Truesdell and Toupin (1960) with its appropriate sections, the papers by Alblas (1974) in the Netherlands, the remarkable works of physicists Lax and Nelson synthetized in Nelson's (1979) book, and the often unjustly ignored works by L.I. Sedov and his co-workers at Moscow State University in the 1960s–1970s (see Sedov's books on continuum mechanics; Chap. 11). We shall shortly deal with the fundamental role played by the pioneers such as Toupin, Brown, Eringen, Mindlin and Tiersten. In more recent works overlapping the early 21st century we find the variational formulations of Trimarco and Maugin (2001)—also in Maugin (1993a), Chap. 8, and Dorfmann and Ogden (see all these in the Udine course of 2009 published by Ogden and Steigmann 2011). These formulations require the addition of an interaction term between matter and fields to the Lagrangian density (12.12). These continuum-mechanics approaches supersede all presentations by well-known physicists, even the classic books on electrodynamics by Jackson and Landau and Lifshitz.

12.4 Theory of Elastic Dielectrics and Generalizations

12.4.1 Toupin's Theory

In the above reported developments a fundamental role was played by a beautiful work published by Toupin in 1956. A few words of gossip may be exceptionally introduced at this point. In his autobiographic notes (reprinted in his collected works edited by Barenblatt and Joseph (1997), Rivlin tells that he met Toupin at the National Research Laboratory in Maryland where both were visiting in 1953. Toupin was then working on a PhD with Melvin Lax at the University of Syracuse (Lax left Syracuse to join Bell Telephone Laboratories and ended his brilliant career of solid-state physicist at the City College of New York where we visited him). Rivlin advised Toupin to change his research subject to the theory of deformable dielectrics where he foresaw some promising developments in the finite-strain framework of continuum mechanics, what Toupin did with the success we know. Indeed, Toupin's publication of his "Elastic dielectric" paper in the *J.R.M.A.* in 1956 proved to be a true milestone. This was followed by another paper about dynamics in 1963. Of course, this is now contained in the general presentation given in the foregoing section.

It must be realized that before Toupin's landmark work, the only well formulated and very much applied theory of *electro-mechanical interactions* in continuum mechanics was the standard theory of *linear* piezoelectricity. This went back to the original discovery of the effect by the Pierre and Jacques Curie in Paris in 1881. It was recognized that this required the consideration of crystals having no centre of symmetry [in order to allow a *linear* relation between a second-order tensor (e.g., strain) and a vector (electric field or polarization)]. The relevant Cartesian tensor formulation was established and the effect became popular through its exploitation in "sonars" (cf. the pioneering work by Paul Langevin during WWI). In the 1940s and 1950s, the importance of piezoelectric couplings was duly recognized in devices of *signal processing* exploiting the short wavelength of piezoelectric waves compared to electromagnetic ones, and using the properties of piezoelectric vibrations of structures (e.g., plates). The collaboration between the US Army Signal Corps Laboratory in Fort-Monmouth, New Jersey, and Raymond Mindlin culminated in a beautiful lengthy technical Army report by Mindlin that was only recently published in book form (Mindlin 2006). Harry Tiersten, a former student of Mindlin, put some of these in Lagrangian-Hamiltonian variational form in a small monograph (Tiersten 1969). Now back to Toupin.

Toupin's theory can be extracted from the contents of Sect. 12.3 by discarding magnetic effects and all forms of dissipation. Thus we obtain the following reduction of (12.60) to an equality for *hyperelastic dielectric solids* ($q_f = 0$, $\tilde{\mathbf{M}} = \mathbf{0}$, $\tilde{\mathbf{J}} = \mathbf{0}$):

$$-(\dot{W} + N\dot{\theta}) + \frac{1}{2} S^{E}_{\mathrm{KL}} \dot{C}_{\mathrm{KL}} + \bar{E}_K \dot{\Pi}_K = 0, \tag{12.62}$$

from which there follows the constitutive equations

$$S^E_{\mathrm{KL}} = 2\frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}},\ \bar{E}_K = \frac{\partial \hat{W}}{\partial \Pi_K},\ N = -\frac{\partial \hat{W}}{\partial \theta}, \tag{12.63}$$

wherein the free energy per unit undeformed volume is given by

$$W = \hat{W}(C_{KL}, \Pi_K,\ \theta). \tag{12.64}$$

Accordingly, the following constitutive equations are obtained for the "elastic" stress and the material electric field

$$t^E_{ji} = 2J_F^{-1} F_{jK} F_{iL} \frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}},\ \bar{E}_K = \frac{\partial \hat{W}}{\partial \Pi_K}, \tag{12.65}$$

Then, after (12.14),

$$\mathbf{t} = \mathbf{t}^E - \mathbf{P} \otimes \tilde{\mathbf{E}} = \mathbf{t}^E - J_F^{-1}\mathbf{F}.\Pi \otimes \tilde{\mathbf{E}}, \tag{12.66}$$

hence in components for the Cauchy stress

$$t_{ji} = J_F^{-1} F_{jK}\left(2\frac{\partial \hat{W}}{\partial C_{\mathrm{KL}}} - \Pi_K \frac{\partial \hat{W}}{\partial \Pi_L}\right) F_{iL}. \tag{12.67}$$

Toupin's theory is not exactly this because temperature effects are not included and, astutely, Toupin makes a difference between the Maxwellian field $\mathbf{E}$ and a *local* electric field, noted $\mathbf{E}^L$ provided by a constitutive equation, so that we in fact have a kind of *local balance law for electric fields*:

$$\mathbf{E} + \mathbf{E}^L = \mathbf{0},\ \bar{E}^L_K = E^L_i F_{iK} = -\frac{\partial \hat{W}}{\partial \Pi_K}. \tag{12.68}$$

Contrary to the theory of linear piezoelectricity where all nonlinear terms in the fields are discarded, Toupin's theory still includes nonzero ponderomotive force and couple (hence a non symmetric Cauchy stress) given by (compare to the general expression in (12.22) and (12.28))

$$\mathbf{f}^{\mathrm{em}} = (\mathbf{P}.\nabla)\mathbf{E} = (\nabla\mathbf{E}).\mathbf{P},\ \tilde{\mathbf{c}}^{\mathrm{em}} = \mathbf{P} \times \mathbf{E}, \tag{12.69}$$

where the transformation of the first of these follows from the quasi-static electric equation $\nabla \times \mathbf{E} = \mathbf{0}$.

Toupin's theory is potentially rich of many effects and generalizations. First, it can include electro-elastic interactions at any order (piezoelectricity, electrostriction, and higher order effects in the electric field). Second, it is not limited to small deformations, and can therefore be applied in modern technology to finitely deformable polymeric dielectrics. Finally, the writing of the first of (12.68) that looks somewhat artificial and unnecessary, is gross of further generalizations that we are going to examine. Toupin's theory was presented in a variational form by Eringen (1963). For sure, it influenced all works after 1956.

12.4.2 Generalizations

It was discovered by Mindlin (1968; see also Herrmann 1974) and others that in the presence of a centre of symmetry—that forbids the existence of linear piezoelectricity, there still exist a possibility of a linear coupling between deformation and a gradient of electric polarization, for nonuniformly polarized materials. This is rare but possible for ionic crystals such as alkali halides (e.g., *NaCl*, *KCl*). In this theory the generalization of (12.68) reads

$$\mathbf{E} + \mathbf{E}^L + \rho^{-1} div\, \hat{\mathbf{E}}^L = \mathbf{0}, \tag{12.70}$$

where the new tensor $\hat{\mathbf{E}}^L$ is principally determined by the gradient $\nabla\mathbf{P}$, while the vector $\mathbf{E}^L$ remains determined by the electric polarization itself. In a nonlinear theory, both of these quantities will contribute to the skew symmetric part of the Cauchy stress (Collet and Maugin 1974). In this theory called "the theory of polarization gradients", Eq. (12.70) has the true status of a field equation on the same footing as the standard equation of equilibrium. This theory is entirely corroborated by the appropriate approach from lattice dynamics, as shown by Askar et al. (1970) [P.C.Y. Lee also was a PhD student of Mindlin]. The full formulation of this theory with applications to nontrivial physical effects is to be found in Mindlin's synthesis of 1972, but above all in Chap. 7 of our book (Maugin 1988) with a generous relevant bibliography.

In his original work of 1963, Toupin alludes to the possibility of having an inertial (polarization) term in the right-hand side of the above given Eq. $(12.68)_1$. In a successful attempt at a dynamical theory of ferroelectric crystals, Maugin and Pouget (1980) formulated a complete theory in the finite-strain framework of continuum thermo-mechanics in which a field equation governing the electric polarization is obtained in a form that looks like (12.70) but with a polarization inertia in its right-hand side, i.e.,

$$\mathbf{E} + \mathbf{E}^L + \rho^{-1} div\, \hat{\mathbf{E}}^L = d_E\, \ddot{\mathbf{P}} \tag{12.71}$$

where the tensor field $\hat{\mathbf{E}}^L$ is related to the interaction between neighbouring permanent electric dipoles. This theory is also justified by a lattice-dynamics approach as shown by Pouget et al. in 1986 for ferroelectrics of the molecular-group type (e.g., $NaNO_2$). More on this model and wave propagation (including the structure and motion—as solutions—of ferroelectric domain walls is to be found in the book of Maugin et al. (1992)—also Bassiouny et al. 1988. The theory was extended to the case of elastic antiferroelectrics by Soumahoro and Pouget (1994).

12.5 Theory of Magneto-Elastic Continua

The magneto-elastic coupling called *magnetostriction* was discovered by Joule in the 19th century. It results in a very small strain upon the application of a longitudinal magnetic field to a bar, and this independently of the direction of the field; Hence it intensity varies like the square of that field and it is, basically, a nonlinear effect. In spite of its smallness this effect is important because a corresponding linear effect—linear piezomagnetism—is much more rare than piezoelectricity in natural conditions. However it can appear as linearized magnetostriction about an intense magnetic field. Note also that with the discovery of "giant" magnetostriction in some compounds the effect is improved by two orders of magnitude so that magnetostriction may be envisaged in competition with some piezoelectric devices (for the physical viewpoint on magnetostriction see the book by du Trémolet 1993).

After many works in the field, William F. Brown Jr proposed a serious continuum theory of magneto-elastic interactions in his book of 1966, in a collection edited by Truesdell. Simultaneously, Tiersten (1964, 1965)—he had been a PhD student of Mindlin and worked at Bell Labotatories for sometimes before joining the Rensselaer Polytechnic—following works by the solid-state physicist Kittel (1958) dealing with the interaction between elastic and magnetic-spin waves, proposed in 1964 a theory of elastic hard ferromagnets in the finite-strain framework that accounts for the presence of a density of magnetic spin and the interaction between neighbouring spins (or magnetic dipoles). This he complemented with an astute variational formulation in Tiersten (1965). This modelling was taken over by Maugin (1971) in his Princeton PhD thesis and in papers by Maugin and Eringen (1972). It was also exposed by Akhiezer et al. (1968) in a famous book on spin waves. What is remarkable is that in this theory the equation governing the magnetic spin density has the following form:

$$\gamma^{-1}\dot{\mu} = \mu \times (\mathbf{B} + \mathbf{B}^{L} + \rho^{-1}\, div\, \hat{\mathbf{B}}^{L}), \tag{12.72}$$

which guarantees that the magnetization per unit mass has a prescribed modulus (condition of saturation). Here γ is the so-called gyromagnetic ratio of the material, and the "local" fields $\mathbf{B}^{L}$ and $\hat{\mathbf{B}}^{L}$ are primarily determined by the magnetization and its gradient, respectively, reflecting in the continuum framework the effects of magnetic anisotropy (preferential directions of magnetization) and Heisenberg exchange forces between neighbouring spins. In principle, both may have dissipative contributions associated with them. It was shown by Maugin (1972, 1975) that the first yields a correct formulation of the effect of spin-lattice relaxation in deformable ferromagnets (using the notion of Jaumann co-rotational time derivative). Later on the theory was extended to the case of deformable ferrimagnets and antiferromagnets (Maugin 1976; Maugin and Sioké-Rainaldy 1983) by adopting the idea of the French physicist Louis Néel of the co-existence of multiple magnetic sub-lattices. For all these and a rather complete presentation of dynamical processes (coupled waves), we refer the reader to Chap. 7 in Maugin (1988) and Chap. 9 in Eringen and Maugin (1990). At the same time, Sabir and Maugin (1990) provided a phenomenological theory of irreversible magnetic hysteresis coupled to stresses by applying the thermomechanical framework using internal variables of state (compare Sect 5.6 and Maugin 1993b) by analogy with plasticity and visco-plasticity. This concurs with Néel's theory of the 1940s.

We note the resemblance of the expression within parentheses in the right-hand side of (12.72) with the left-hand side of (12.70) and (12.71). As a matter of fact, whenever exchange interactions and gyromagnetic effects are discarded, Eq. (12.72) reduces to a balance equation for the Maxwellian and local magnetic inductions in the form

$$\mathbf{B} + \mathbf{B}^L = \mathbf{0}, \tag{12.73}$$

a form entirely analogous to that of Toupin's equation (12.68). The resulting theory for soft ferromagnets and paramagnets could be derived from the theory exposed in Sect. 12.3. But the more general theory contained in both Eqs. (12.71) and (12.72) was shown to be derivable from a modern formulation of the *principle of virtual power* (d'Alembert's principle) by Collet and Maugin (1974) by considering that electric polarization and magnetization provide additional *internal degrees of freedom*, on equal footing with the classical deformation motion [general theory in Maugin (1980)]. The coupling between these internal degrees of freedom and stresses then appear naturally in writing the power expended by internal forces upon the constraint of being objective. This safe and powerful method was further used in all models of electro-magneto-mechanical interactions, including in complex modellings such as that of deformable semi-conductors (Daher and Maugin 1986; Maugin and Daher 1986), after initial studies on piezoelectric semiconductors by Ancona and Tiersten (1983).

12.6 Concluding Remarks

In the above given survey we emphasized the evolution in the very bases of the theory of electro-magneto-elastic interactions, noting the seminal role played by Richard A. Toupin and Raymond D. Mindlin. This should be complemented by a description of the many applications treated during the period 1960–2010 and witnessed by the present writer in a very active position. This would prove to be a formidable task, perhaps not as instructive as imagined. What we can notice is that, apart from the general principles already scrutinized, many of the applicative developments have more or less followed the trends of the corresponding pure mechanical developments in the same period.

Concerning *wave propagation*, the very new ingredient in the linear theory of piezoelectricity was the discovery in 1968 of the so-called Bleustein-Gulyaev piezoelectric surface wave simultaneously by J.L. Bleustein (a co-worker of Tiersten) in the USA and Yu. V. Gulyaev in the USSR (cf. Chap. 11). This is a shear horizontal surface wave (like the celebrated Love surface mode) of which the propagation is allowed by the perturbation created at the surface by a piezoelectric coupling. Many other wave problems including bulk waves, surface waves, shock waves, and solitary waves have been treated in particular—for complex models of interactions—by the writer in Paris in collaboration with many researchers among whom we must single out Bernard Collet, Joel Pouget, Anaclet Fomethe, and Naoum Daher (see Maugin et al. 1992; Maugin 1988). Other centres of study of some of these waves were in Besançon (France) with J.-J. Gagnepain and M. Planat, but also in Tiersten's environment at the Rensselaer Polytechnic in Troy (USA), with David F. Parker in Nottingham (UK), and in the active team of A.N. Guz in Kiev (Ukraine). Special attention was paid to superconducting deformable solids by S.A. Zhou (1999).

The stability of magnetoelastic structures received much attention in relation to the fantastic strength of electromagnets used in magnetically-levitating trains and in some thermo-nuclear technologies. This culminated in the splendid book of Moon (1984). Other studies along the same line were conducted in Japan (with K. Miya in Tokyo and J. Tani in Sendai), and in Europe (G.A. Maugin and C. Goudjo in Paris, A.A.F Van de Ven in Eindhoven).

The mechanics of slender structures (plates, shells) coupled to electromagnetic properties was perfected to a high degree of analysis by Academician S.A. Ambartsumian and co-workers in Yerevan (Armenia) applying the asymptotic integration method introduced in pure mechanics by Golden'veizer and Ambartsumian himself—see Ambartsumian et al. (1977), while the zoom technique of P.G. Ciarlet and P. Destuynder was exploited by Attou and Maugin (1990) for piezoelectric plates.

Homogenization techniques first applied in pure continuum mechanics in the 1980s were rapidly applied to electro- and magneto-elasticity. Here we must cite the much original work in dynamics of Turbé and Maugin (1991) using a Bloch expansion of waves, and the application to nonlinear electroelasticity by Rodriguez-Ramos et al. (2004). Homogenization schemes have also been introduced in ferromagnetic bodies in order to account for the influence of their microstructure in industrial applications, for instance, by the group of René Billardon in Cachan (France) including Laurent Hirsinger, Nicolas Buiron, Olivier Hubert, and Laurent Daniel, and also in Metz and Besançon (France).

The theory of structural defects (e.g., dislocations) in elastic dielectrics has been carefully approached especially by V.I. Alshits (Moscow), A. Radowicz (Kielce, Poland) and J.P. Nowacki (Warsaw)—see Nowacki's book (2006). The corresponding theory of configurational forces—acting on defects, shock waves, phase-transition fronts—has also been extensively expanded in electro- ad magneto-elasticity (see Chap. 14 below). The works of R.M. McMeecking et al. should also be noted in conjunction with crack studies.

The relationship between complex models of electromagnetic deformable materials (e.g., ferroelectrics, ferromagnets) and *generalized continuum mechanics* will be briefly discussed in Chap. 13 [see also Chap. 1 in the book edited by Altenbach and Eremeyev (2012)].

Numerically convenient variational formulations and applications of the nonlinear electroelasticity and magnetoelasticity have recently been given by several groups of authors, e.g., R.W. Ogden, A. Dorfmann and R. Bustamante on the one hand, D.J. Steigmann et al. on the other, and also Paul Steinmann and co-workers, and N. Triantafyllidis et al.

Finally, we mention that a specific scientific journal entitled the *International Journal of Applied Electromagnetics and Mechanics* was founded in 1991 by Kenzo Miya (Tokyo), Richard Hsieh (Stockholm) and G.A. Maugin (Paris), simultaneously with a successful series of technical volumes called "*Applied Electromagnetics and Mechanics*" published by I.O.S, in The Netherlands and Japan.

Personal touch: In addition to his doctoral students, co-workers in Paris, and visiting research associates, the author has had, or still entertain, friendly relations with many of the strongly involved actors: A.C. Eringen, R.D. Mindlin, H.F. Tiersten, M. Lax, D.F. Nelson, F.C. Moon, W. Nowacki, J.P. Nowacki, A. Askar, L.I. Sedov, S.A. Ambartsumian, S.R. de Groot, K. Hutter, D.F. Parker, R.W. Ogden, V.I. Alshits, A. Dorfmann, E. du Trémolet, R. Billardon, K. Miya and R.K.T. Hsieh. Unfortunately, he never met Richard Toupin.

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Chapter 13
Generalized Continuum Mechanics: Various Paths

Abstract This chapter focuses on a field of continuum mechanics that belongs almost entirely to the twentieth century, so called generalized continuum mechanics. First, a special effort is produced to define this term which essentially means going beyond the traditional view of Cauchy—with the notion of stress introduced by this early nineteenth-century scientist. Three possible paths to such a generalization are discussed with the related mention of main scientific contributors: involving an additional microstructure at each material point in addition to the traditional translational degree of freedom (e.g., micromorphic media, Cosserat continua, in modern times works by Eringen and others), or a better analytic description of the displacement field at each material point by introducing higher order gradients of this displacement in the energy density (e.g., in a theory mostly expanded by Mindlin), or else calling for a truly nonlocal theory that leads to considering spatial functionals for the constitutive equations—this follows contributors such as Kröner, Rogula, Kunin, and Eringen. A more drastic "generalization" started in the mid 1950s involves a loss of the Euclidean nature of the material manifold, as may apply in a densely defective crystal. In each case, the pioneers are mentioned and the most recent formulations are briefly sketched out.

13.1 Introduction

A natural question is posed at the outset: what do we understand by the expression "Generalized Continuum Mechanics" (for short, GCM)? The simple understanding we give is that a generalized continuum mechanics is one that goes beyond the standard Euler-Cauchy definition that involves only a *symmetric* "Cauchy" stress, the later being defined by the celebrated Cauchy tetrahedron argument (see Chap. 1). This definition, contrary to what is commonly thought by many, includes, mutatis mutandis, a "constitutive" argument based on geometry: only the first order description of the geometry of a facet cut in a body, the local

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unit normal, is involved in the argument. A consequence of this is that only so-called "simple" materials (in the vocabulary of Walter Noll—cf. Truesdell and Noll 1965), may rely on such a description via the emerging notion of stress tensor and the energetically dual notion of deformation gradient. Furthermore, the notion of displacement so useful in elasticity requires a certain (Euclidean) structure of the material manifold that makes up the considered material. Thus the main points in our conspectus are the appearance of non-symmetric stresses, the notion of couple stresses, internal degrees of freedom and microstructure, the introduction of strain gradient theories, and material inhomogeneities with a length scale, non locality of the weak and strong types, the loss of Euclidean geometry to describe the material manifold, and finally the loss of classical differentiability of basic operations as can occur in a deformable fractal material object.

We claim that in a structured overview generalization can obviously be presented through the successive abandonment of the basic working hypotheses of standard continuum mechanics of Cauchy (cf. Maugin 2010): that is, introduction of a rigidly rotating microstructure and *couple stresses* (Cosserat continua or *micropolar* bodies, nonsymmetric stresses), introduction of a truly deformable microstructure (*micromorphic* bodies), "weak" *non localization* with *gradient theories* and the notion of *hyperstresses*, and the introduction of characteristic lengths, "strong non localization" with space functional constitutive equations and the loss of the Cauchy notion of stress, and finally giving up the Euclidean and even Riemannian material background. In recent times this evolution was paved by landmark papers and timely scientific gatherings (e.g., Freudenstadt in 1967; Udine in 1970, Warsaw in 1977) to which the Paris colloquium of 2009 celebrating the centennial of the publication (cf. Maugin and Metrikine 2010) of the Cosserats' book (see Chap. 2; Cosserat and Cosserat 1909) must now be added. This is examined in some detail in the following sections. Here we simply recall that the publication of the book of the Cosserat brothers in 1909 was a true initial landmark, although at the time noticed by very few people—among them Elie Cartan and Ernst Hellinger. Thus a true "generalized continuum mechanics" developed first slowly and rather episodically and then with a real acceleration in the 1960s. Accordingly, a new era was born in the field of continuum mechanics.

As a preliminary we recall that Cauchy's expression (1.1) that introduces the standard Cauchy stress tensor was applied in the 19th century and by most engineers in the 20th century to a *symmetric* tensor σ. That is, in the two classical intrinsic and Cartesian tensor notations (an upper T denotes the operation of transposition),

$$\underline{\sigma} = \underline{\sigma}^T i.e., \sigma_{ji} = \sigma_{ij}. \tag{13.1}$$

This results from the application of the balance of angular momentum. Isotropy, homogeneity, and small strains are further hypotheses but they are not so central to our argument. For the validity of (13.1) two working hypotheses are essential: (1) there are no applied couples in both volume and surface, and (2) there exists no "microstructure" described by additional internal degrees of freedom at each material point. We admit that imagining an applied couple per unit volume or mass is not obvious while the notion of a force per unit volume or mass comes naturally to mind with the at-a-distance effects of gravitation and electromagnetic fields.

Then generalizations of various degrees consist in relaxing more or less these different points above, hence the notion of *generalized continuum*. But this notion of "generalization" depends also on the culture and physical insight of the concerned scientist. For instance, generalizations such as: (1) considering the so-called "generalized" Hooke law (linear, homogeneous, but *anisotropic* medium), (2) envisaging the linear Hooke-Duhamel law in thermo-elasticity in a simple coupled-field theory, or (3) applying the scheme of linear homogeneous piezoelectricity in obviously anisotropic media (no centre of symmetry—see Chap. 12), are "weak" generalizations because they do not alter the main mathematical properties of the system. Of course, thermo-elasticity and linear piezoelectricity require adding new independent variables (e.g., temperature θ or scalar electric potential ϕ). In some sense, the problem becomes four-dimensional for the basic field (elastic displacement and temperature in one case, elastic displacement and electric potential in the other). The latter holds in this mere simplicity under the hypothesis of weak electric fields, from which there follows the neglect of the so-called ponder motive forces and couples, e.g., the couple (Cf. Chap. 12; Maugin 1988)

$$(\mathbf{P} \times \mathbf{E})_i = \varepsilon_{ijk} P_j E_k, \tag{13.2}$$

and this will yield (square brackets denote anti-symmetrization; ε_{ijk} is the permutation symbol in Cartesian tensor index notation)

$$\sigma_{[ji]} = C_{ji},\ e.g.,\ C_{ji} = P_{[j}E_{i]}, \tag{13.3}$$

when electric field $\mathbf{E}$ and electric polarization $\mathbf{P}$ are not necessarily aligned. Such theories, just like standard elasticity, do not involve a *length scale*. But classical linear *inhomogeneous* elasticity presents a higher degree of generalization because a characteristic length intervenes necessarily. Now we can deal with what we refer to as *true* generalizations.

13.2 First True Generalization: Cosserat Continua et al.

The Cauchy stress tensor may become *asymmetric* for various reasons. This may be due to

(1) the existence of body couples (e.g., just as above in electromagnetism: $\mathbf{P} \times \mathbf{E}$ *or/and* $\mathbf{M} \times \mathbf{H}$ if $\mathbf{M}$ and $\mathbf{H}$ denote volume magnetization and magnetic field; case of intense electromagnetic fields or linearization about intense bias fields; cf. Eringen and Maugin 1990);
(2) the existence of surface couples (by Cauchy's argument, introduction of "internal forces" of a new type: so-called *couple stresses*); the medium possesses internal degrees of freedom that modify the balance of angular momentum;

(3) the existence of internal degrees of freedom (of a nonmechanical nature in origin, e.g., polarization inertia in ferroelectrics, intrinsic spin in ferromagnetics; see Maugin 1988);
(4) the existence of internal degrees of freedom of "mechanical" nature.

This is where the Cosserats' model comes into the picture. The first example in this class pertains to a *rigid microstructure* (three additional degrees of freedom corresponding to an additional rotation at each material point, independently of the vorticity). Examples of media of this type go back to the early search for a continuum having the capability to transmit transverse waves (as compared to acoustics in a pure fluid), i.e., in relation to optics. The works of James McCullagh (1839) and Lord Kelvin (already cited in Chap. 1) must be singled out (cf. an historical view in the celebrated—but controversial—book of Whittaker 1951). As we remember from Chap. 2, Pierre Duhem (1893) proposed to introduce a triad of three rigidly connected *directors* (unit vectors) to represent the required rotation. In modern physics there are other tools for this including Euler's angles (not very convenient), quaternion's (after Hamilton) and spinors (after E. Cartan) and, obviously orthogonal transformations. It is indeed the Cosserats who, among other studies in elasticity, really introduced internal degrees of freedom of the rotational type—these are *micropolar continua* in the sense of Eringen—and the dual concept of *couple stress*. Hellinger (1914), in his brilliant essay, recognized at once the new potentialities offered by this generalization but did not elaborate on these. It seems that this kind of approach laid dormant for a few decades. The resulting theory of continua is christened under different names: Cosserat theory, polar media, oriented media, micro polar media, asymmetric elasticity.

A modern rebirth of the field had to await works in France by crystallographers (e.g., J. Laval 1957a, b, c and Y. Le Corre (1956) at the University of Paris—future UPMC). Mechanicians took over in the early 1960s with works in Russia [Aero and Kuvshinskii (1960) and Palmov (1964)], in Germany [Schaeffer (1967), Günther (1958), and Neuber (1964)], and in Italy [Grioli (1960, 1962), and Capriz (1989)]. But the best formulations are those obtained by considering a field of orthogonal transformations (rotations) and not the directors themselves: Eringen (1968), Kafadar and Eringen (1971), Nowacki (1986), although we note some obvious success of the "director" representation, e.g., in *liquid crystals* by Ericksen (Ericksen 1959/60) and Leslie (1968) and in the kinematics of the deformation of slender bodies (works by Ericksen, Truesdell, Naghdi).

But in the mid 1960s a complete revival of continuum mechanics took place which, by paying more attention to the basics, favoured the simultaneous formulation of many more or less equivalent theories of generalized continua in the line of thought of the Cosserats (works by Mindlin, Tiersten, Eshel, Green and Rivlin (1964), Green and Naghdi (1967), Toupin (1962, 1964), Truesdell and Toupin (1960), and Eringen and Suhubi (1964), etc.). All these works are listed in the bibliography of the present chapter. Among them the fundamental paper of

Toupin (1964) must be singled out as one of the most influential ones. The contribution of Eringen (1968) stands out for its clear simple presentation while the papers of Eringen and Suhubi (1964)—and Suhubi and Eringen (1964)—and that of Mindlin also in 1964 are truly creative landmarks, and Grioli's book of (1960) is important from the mathematical viewpoint. Also, we must mention the formidable work done by the Polish school around Witold Nowacki (See Chap. 8) in the field with many worked out static and dynamic solutions (see the synthesis in Nowacki's book of 1986). Romanian mathematicians were also very active along this line.

More precisely, in the case of a *deformable microstructure* at each material point, the vector triad of directors of Duhem-Cosserats becomes deformable and the additional degree of freedom at each point, or *micro-deformation*, is akin to a general linear transformation (nine degrees of freedom). These are micromorphic continua in Eringen's classification. Mindlin's 1964 is somewhat equivalent although in a different wording. A particular case is that of continua with microstretch (Eringen 1969). A truly new notion for Cosserat media here is that of the existence of a conservation law of *micro-inertia* (Eringen 1966). We illustrate these various generalizations by giving the relevant form of the local equation of moment of momentum in quasi-statics:

Micromorphic bodies (Eringen, Mindlin; Years 1962–1966) [Notation: μ_{kji} is the hyperstress tensor, s_{ji} is the so-called symmetric micro-stress, and l_{ij} is the body-moment tensor of which the skew part represents a body couple $C_{ji} = -C_{ij}$]:

$$\mu_{kij,k} + \sigma_{ji} - s_{ji} + l_{ij} = 0, \; \sigma_{ji} = \sigma_{(ji)} + \sigma_{[ji]}, \; s_{[ji]} = 0, \; l_{ji} = C_{ji} + l_{(ji)}. \tag{13.4}$$

Micropolar bodies (Cosserat brothers, etc.) [Notation: $\mu_{k[ji]}$ is the couple-stress tensor; C_i is the axial vector uniquely associated with C_{ji} while m_{ji} is associated in the same way with $\mu_{k[ji]}$]:

$$\mu_{k[ji],k} + \sigma_{[ji]} + C_{ij} = 0 \, or \, m_{ji,j} + \varepsilon_{ikj}\sigma_{kj} + C_i = 0. \tag{13.5}$$

Bodies with microstretch (Eringen 1969) [Notation: m_k denotes the intrinsic dilatational stress or microstretch vector; l is the body microstretch force such that $l_{(ij)} = (l/3)\delta_{ij}$, and σ and s are intrinsic and micro scalar forces]:

$$\mu_{klm} = \frac{1}{3} m_k \delta_{lm} - \frac{1}{2} \varepsilon_{lmr} m_{kr} \tag{13.6}$$

so that

$$m_{kl,k} + \varepsilon_{lmn}\sigma_{mn} + C_l = 0, \; m_{k,k} + \sigma - s + l = 0. \tag{13.7}$$

Note that an additional natural boundary condition involving the new higher-order stresses μ_{kij} and m_{ji} must complement the standard Cauchy condition (1.1) of Chap. 1, e.g.,

$$n_k \mu_{kij} = M^d_{ij} \, or \, n_j m_{ji} = M^d_i, \tag{13.8}$$

where M^d_i is akin to a surface couple.

Dilatational elasticity (Cowin and Nunziato 1983) [only the second of (13.7) is relevant]:

$$m_{k,k} + \sigma - s + l = 0. \tag{13.9}$$

Here the additional natural boundary condition will be of the form

$$n_k m_k = M^d, \tag{13.10}$$

where M^d is akin to a tension.

In these equations given in Cartesian components in order to avoid any misunderstanding (note that following a convention in mathematical physics, the divergence is always taken on the first index of the tensorial object to which it applies), μ_{kij} is a new internal force having the nature of a third–order tensor. It has no specific symmetry in Eq. (13.4) and it may be referred to as a *hyperstress*. In the case of Eq. (13.5) this quantity is skew symmetric in its last two indices and a second order tensor—called a *couple stress*—of components m_{ji} can be introduced having *axial* nature with respect to its second index. The fields s_{ji} and l_{ij} are, respectively, a symmetric second-order tensor and a general second-order tensor. The former is an *intrinsic interaction stress*, while the latter refers to an external source of *both* stress and couple according to the last of Eq. (13.4). Only the skew part of the later remains in the special case of micro polar materials [Eq. (13.5) in which C_i represents the components of an *applied couple*, an axial vector associated with the skew symmetric C_{ji}]. The latter can be of electromagnetic origin, and more rarely of pure mechanical origin. Equations (13.6) and (13.7) represent a kind of intermediate case between micromorphic and micro polar materials. The case of dilatational elasticity in Eq. (13.8) appears as a further reduction of that in Eq. (13.7). This will be useful in describing the mechanical behaviour of media exhibiting a distribution of holes or cavities in evolution.

Concerning the micromorphic case, a striking example is due to Drouot and Maugin (1983) while dealing with fluid solutions of macromolecules, while Pouget and Maugin (1983) have provided a fine example of truly micromorphic solids with the case of piezoelectric powders treated as continua.

Remark 13.1 Historical moments in the development of this avenue of generalization have been the IUTAM symposium organized by E. Kröner in Freudenstadt in 1967 (see Kröner 1968) and the CISM Udine summer course of 1970. Were present: Mindlin, Eringen, Nowacki, Stojanovic, Sokolowski, Maugin, Jaric, Micunovic, etc.

Remark 13.2 Strong scientific initial motivations for the studies of generalized media at the time (1960s–1970s) were (1) the expected elimination of field singularities in many problems with standard continuum mechanics, (2) the continuum description of real existing materials such as granular materials, suspensions, blood flow, etc. But further progress was hindered by a notorious lack of knowledge of new (and too numerous) material coefficients despite trials at estimates of such coefficients e.g., by Gauthier and Jashman (1975) at the Colorado School of Mines by building artificially micro structured *solids.*

Remark 13.3 The intervening of a rotating microstructure allows for the introduction of wave modes of rotation of the "optical" type with an obvious application to many solid crystals (e.g., crystals equipped with a polar group such as $NaNO_2$; cf. Pouget and Maugin 1989).

Remark 13.4 In some physical theories (Micromagnetism, cf. Maugin 1971–also Chap. 12 above), an equation such as the first of (13.5) can be obtained in full dynamics:

$$m_{kij,k} + \sigma_{[ji]} + C_{ij} = \dot{S}_{ij}, \tag{13.11}$$

where m_{kij} (Heisenberg exchange-force tensor that is skewsymmetric in its last two indices), C_{ij} (interaction couple between material and electronic-spin continua) and S_{ij}(magnetic spin) all have a magnetic origin.

The full thermo-mechanical theories corresponding to these various cases can be developed along the now admitted general lines (first and second laws complementing the field equations, thermodynamical admissibility, Clausius–Duhem inequality) for both fluid and solid types of behaviour. The fluid type was applied in a multitude of papers dealing with a variety of materials—suspensions, liquid crystals, blood flow, etc. [on this subject matter the books of Stokes (1984) and Eringen (2001) may be consulted and the (James D.) Lee-Eringen theory of liquid crystals (cf. Lee and Eringen 1971, 1973) is of high interest]. In the case of solids, special attention was paid to the study of field singularities (at corners, at a crack tip, along a dislocation line), a pregnant idea being that accounting for couple stresses would reduce the singularity order.

Of course the theory would not be complete without expending the relevant kinematics and theory of deformation. In the sufficiently illustrative case of polar media, the usual motion mapping (1.3) must be complemented by the time evolution of χ that accounts for the micro-motion:

$$\chi = \bar{\chi}(\mathbf{X}, t), \chi^T = \chi^{-1}, \det \chi = +1. \tag{13.12}$$

This can be represented with the help of a formula established by Gibbs (1901):

$$\chi = (\cos \phi)\mathbf{1} + (1 - \cos \phi)\mathbf{d} \otimes \mathbf{d} + (\sin \phi)\mathbf{d} \times \mathbf{1}, \tag{13.13}$$

where ϕ is the angle of rotation about the axis of eigenvector $\mathbf{d}$ corresponding to the real eigenvalue +1of χ (i.e., χ**.d = +d**). Expression (13.13) can also be written in terms of a vectorial angle of components ϕ_k, which often is a preferred representation, in particular in the small strain *and* small micro-rotation case. Then from (13.13) we have in this approximation

$$\chi_{ji} = \delta_{ji} + \Omega_{ji} = \delta_{ji} - \varepsilon_{jik}\phi_k,\ \Omega_{ji} = -\Omega_{ij}. \tag{13.14}$$

The resulting *infinitesimal* measures of generalized deformation are then given by

$$\bar{e}_{ji} = u_{i,j} - \varepsilon_{jik}\phi_k = e_{ij} + \left(\omega_{ij} - \Omega_{ij}\right), \gamma_{ji} = \phi_{i,j}, \tag{13.15}$$

where e_{ji} is the usual (symmetric) infinitesimal strain, ω_{ji} is the accompanying macro-rotation tensor, and Ω_{ji} is the micro-rotation tensor. The set of deformations (13.15) is the one used in "asymmetric" elasticity (cf. Eringen 1968, 1999; Nowacki 1986). Stress and couple stress for elasticity then are derived from an energy density $W = \hat{W}\left(e_{ji}, \gamma_{ji}\right)$ by the constitutive equations

$$\sigma_{ji} = \frac{\partial \hat{W}}{\partial \bar{e}_{ji}},\ m_{ji} = \frac{\partial \hat{W}}{\partial \gamma_{ji}}. \tag{13.16}$$

The first of Eq. (13.15) exhibits the special case when micro-rotation is slaved to the macro-rotation. With the definition of γ_{ji} this boils down to an energy

$$W = \tilde{W}\left(e_{ij}, u_{i,jk}\right), \tag{13.17}$$

which enters the framework of the next section. This is referred to as the theory of *constrained Cosserat continua*. This may be unsound in dynamics as the rotational kinetic energy would provide terms including the time derivative of the gradient of the displacement.

13.2.1 Finite-Strain Formulation

We understand by this both finite deformation and finite micro-rotation. Hence the Gibbs representation (13.13) plays its full role. This is a very technical subject which was beautifully addressed by Kafadar and Eringen (1971), and more thoroughly by Eremeyev and Pietraszkiewicz (2012) [where they solve the problem of the exact representation of the energy density in terms of invariants, from the original form $W(\mathbf{F}, \chi, \nabla_R \chi)$]. We refer to these authors for these developments.

13.3 Second True Generalization: Gradient Theories

This occurs with the loss of validity of the traditional Cauchy postulate. Then the geometry of a cut in a body intervenes at a higher order than one (variation of the unit normal, role of the curvature, edges, apices and thus capillarity effects). We may consider two different cases referred to as the *weakly nonlocal theory* and the *strongly nonlocal theory* (distinction introduced by the author at the Warsaw meeting of 1977; cf. Maugin 1979; also in Kunin's book of 1982). Only the first type does correspond to the exact definition concerning a cut and the geometry of the cut surface. This is better referred to as *gradient theories of the n-th order,* it being understood that the standard Cauchy theory in fact is a *theory of the first gradient* (meaning by this the first gradient of the displacement or a theory involving just the strain and no gradient of it in the constitutive equations).

Now, as a matter of fact, gradient theories abound in physics, starting practically with all continuum theories in the 19th century. Thus, Maxwell's electromagnetism is a first-gradient theory (of the electromagnetic potentials); the Korteweg (1901) theory of fluids is a theory of the first gradient of density (equivalent to a second-gradient theory of displacement in elasticity); Einstein's (1916) theory of gravitation (general relativity; cf. Einstein 1956) is none other than a second-gradient theory of the metric of curved space–time, and Le Roux (see Chap. 2) seems to be the first public exhibition of a second-gradient theory of (displacement) elasticity in small strains (using a variational formulation). There was a renewal of such theories in the 1960s with the works of Casal (1963) on capillarity, and of Toupin (1962), Mindlin (1964), Mindlin and Eshel (1968), Mindlin and Tiersten (1962), and Grioli (1960, 1962) in elasticity.

However, it is with a neat formulation basing on the *principle of virtual power* that some order was imposed in these formulations with an unambiguous deduction of the—sometimes tedious—boundary conditions and a clear introduction of the notion of *internal forces* of higher order, i.e., *hyperstresses* of various orders (see, Germain 1973a, b; Maugin 1980)—as a result of the duality between generalized internal forces and generalized measures of deformation and deformation rates. Phenomenological theories involving gradients of other physical fields than displacement or density, coupled to deformation, were envisaged consistently by the author in his Princeton doctoral thesis (1971) dealing with typical ferroïc electromagnetic materials. This is justified by a microscopic approach, i.e., the continuum approximation of a crystal lattice with medium-range interactions, with distributed magnetic spins or permanent electric dipoles. This also applies to the pure mechanical case (see, for instance, the Boussinesq paradigm in Christov et al. 2007). The following are examples of such theories illustrated by the dependence of the potential energy W per unit volume for small strains:

Le Roux (1911, 1913; *see* Chap. 2):

$$W = W\left(u_{i,j}, u_{i,jk}, \ldots\right), \tag{13.19}$$

where $u_{i,j}$ denotes the displacement gradient, and $u_{i,jk}$ is the second gradient of the displacement.

Modern form (Mindlin, Toupin, Sedov, Germain, etc.; in the period 1962–1973):

$$W = W\left(e_{ij}, ei_{j,k}\right). \tag{13.20}$$

In the last case, *the symmetric first-order stress* $\bar{\sigma}_{ji}$ and the second-order stress or *hyperstress* (symmetric in its last two indices) are given by

$$\bar{\sigma}_{ji} = \frac{\partial W}{\partial e_{ij}} = \bar{\sigma}_{ij}, m_{kji} = \frac{\partial W}{\partial e_{ij,k}} = m_{kij}, \tag{13.21}$$

where e_{ij} is the symmetric small strain, and $e_{ij,k}$ denotes its first gradient. Then the symmetric Cauchy stress reads

$$\sigma_{ji} = \bar{\sigma}_{ji} - m_{kji,k} = \frac{\delta W}{\delta e_{ij}} = \sigma_{ij}. \tag{13.22}$$

That is, it is none other than the functional derivative of the energy W.

Very interesting features of these models are:

F1. The inevitable introduction of characteristic lengths;
F2. The appearance of so-called capillarity effects (surface tension) due to the explicit intervening of curvature of surfaces;
F3. Correlative boundary layers effects;
F4. Dispersion of waves with a possible competition and balance between non-linearity and dispersion, and the existence of solitonic structures (see Maugin 1999, Maugin and Christov 2002);
F5. Intimate relationship with the Ginzburg–Landau theory of phase transitions and, for fluids, van der Waals' theory.

A rather unpleasant feature of this modelling is that the mathematical problems become more *stiff* than before with its higher-order space derivatives, creating potential difficulties in dynamical computations unless one constructs appropriate finite-difference schemes (as done by Christov and Maugin 1995).

Indeed, regarding F1, a typical characteristic length l is introduced by the ratio

$$l = \frac{|m_{kji}|}{|\bar{\sigma}_{ji}|}, \tag{13.23}$$

and this is obviously supposed to be much smaller than a typical macroscopic length L, i.e., $l << L$.

Features F2 and F3 above are typically illustrated by the following set of boundary conditions (Tiersten, Germain; $\Omega = -\frac{1}{2} D_j n_j$ is the mean curvature)

$$n_j \sigma_{ji} + \left(n_j D_p n_p - D_j\right)\left(n_k m_{kji}\right) = T_i^d \, at \, \partial B - \Gamma \uparrow, \tag{13.24}$$

$$n_k m_{kji} n_j = R_i \, at \, \partial B - \Gamma \uparrow, \tag{13.25}$$

$$\varepsilon_{ipq} \tau_p \left[n_k m_{kjq} n_j\right] = E_i along \, \Gamma \uparrow, \tag{13.26}$$

where $\Gamma \uparrow$ is an oriented edge, τ_p denotes unit tangent, D_j indicates a tangential gradient, and the symbolism [...] stands for the jump of its enclosure. Here T_i^d, R_i and E_i are, respectively, an applied surface traction, a prescribed double-normal force, and a lineal force density.

Remark 13.5 The principle of virtual power here is an interesting tool to obtain the set (13.24)–(13.26) unambiguously. But it also shows in agreement with Eq. (13.13) that the power of internal forces can be written either as

$$p_{(\text{int})}(\sigma) = -\sigma : \nabla\dot{\mathbf{u}}, \tag{13.27}$$

or as

$$p_{(\text{int})}(\bar{\sigma}, \mathbf{m}) = -(\bar{\sigma} : \nabla\dot{\mathbf{u}} + \mathbf{m} : \nabla\nabla\dot{\mathbf{u}}), \tag{13.28}$$

so that

$$p_{(\text{int})}(\sigma) = p_{(\text{int})}(\bar{\sigma}, \mathbf{m}) + \nabla.(\mathbf{m} : \nabla\dot{\mathbf{u}}). \tag{13.29}$$

Repeated use of the divergence theorem will then directly leads to the set (13.24)–(13.26).

Truly sophisticated examples of the application of these gradient theories are found in

(1) the coupling of a gradient theory (of the carrier fluid) and consideration of a microstructure in the study of the inhomogeneous diffusion of microstructures in polymeric solutions (Drouot and Maugin 1983).
(2) the elimination of singularities in the study of structural defects (dislocations, disclinations) in elasticity combining higher-order gradients and polar microstructure (cf. Lazar and Maugin 2007).

Remark 13.6 Insofar as general mathematical principles at the basis of the notion of gradient theory are concerned, we note the fundamental works of Noll and Virga (1990) and dell'Isola and Seppecher (1995), the latter with a remarkable economy of thought (and space!).

Remark 13.7 The reader may wonder about the large number of material coefficients needed to write down explicit expressions of the stress and hyperstress tensors, even in the isotropic case. This problem was emphasized by the original contributors to this type of modelling, e.g., Mindlin, Tiersten and Eshel in the 1960s. This is a recurring problem with GCM modelling. Faced with this difficulty but having in mind practical considerations, Aifantis (1992) astutely suggested considering a simplified expression of the energy of isotropic elastic materials of the form

$$W = \frac{1}{2}\left(\lambda e_{ii}e_{jj} + 2\mu e_{ij}e_{ij}\right) + c^2\left(\frac{1}{2}\lambda e_{ii,k}e_{jj,k} + \mu e_{ij,k}e_{ij,k}\right) \tag{13.30}$$

where only *one* extra coefficient, a characteristic length c, is involved in addition to the usual Lamé moduli of linear isotropic elasticity. An expression such as (13.30), rustic as it is, allows one to provide exemplary solutions to many problems such as those involving dislocations (cf. Lazar and Maugin, 2007). Note that the Cauchy stress σ now reads

$$\sigma = \left(1 - c^2\nabla^2\right)\sigma_c, \tag{13.31}$$

where σ_c is the usual stress tensor. We can say that this expression contains a Helmholtz operator in factor of σ_c.

Second-gradient models in finite strain have been devised by authors like Gurtin (for dislocation theory), Cleja-Tigoiu (for plasticity), and Ciarletta and Maugin (for the bio-mechanics of soft tissues). In this case, on must replace (13.19) by a more general expression

$$W = \bar{W}(\mathbf{F}, \nabla_R \mathbf{F}), \tag{13.32}$$

per unit reference volume.

Remark 13.8 Most recent works consider the application of the notion of gradient theory in elastoplasticity for nonuniform plastic strain fields (works by Aifantis, Fleck, Hutchinson, and many others)—but see the thermodynamically admissible formulation in Maugin (1990).

Remark 13.9 **Relation with discrete models and crystal-lattice dynamics**.

The idea of further gradients—of order higher than one—in the approximation of the displacement field rings a bell and reminds us of the discrete definition of the second-order, fourth-order, and so on, space derivatives of successive order, in particular in one dimension. This is practically isomorphic to the theory of crystal lattices as beautifully inaugurated by Born and von Kármán in the early 20th century, where interactions with next neighbours and next–next neighbours would be taken into account in the potential of interactions. The Boussinesq model of elasticity that includes a fourth-order space derivative in the field equation in the continuum framework [cf. the "Boussinesq" paradigm in Christov et al. (2007)], is an example of this outcome. The sign of the higher-order interactions and those of the fourth-order derivatives play an important role in the discussion of stability of the resulting continuum model in the sense of Hadamard. In particular, some models may yield an *anomalous* dispersion of elastic waves in the crystal, although it is not reasonable to stretch the model to too small wavelengths. This is thoroughly debated in papers by Mülhaus and Oka (1996) and Askes et al. (2008), to which we refer the curious reader.

13.4 Strongly Nonlocal Theory (Spatial Functionals)

Initial concepts in this framework were etablished by Kröner and Datta (1966), Kunin (1966), Rogula (1965), and Eringen and Edelen (1972). Synthesis works on the subject are by Kunin (1982) and Eringen (2002). Concerning these concepts but stretching a little the historical perspective, the basic idea permeating this notion of non locality—that the mechanical response at a material point may depend on a larger spatial domain than the immediate neighbourhood of the considered point—may also be traced back to Duhem (1893). Technically, the Cauchy construct does *not* apply anymore. In principle, only the case of infinite bodies should be considered as any cut would destroy the prevailing long-range ordering. Constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with distance of the spatial kernel. This, of course, inherits from the action-at-a-distance dear to the Newtonians, while adapting the disguise of a continuous framework. This view is justified by the approximation of an infinite crystal lattice: the relevant kernels can be justified through this discrete approach. But this approach raises the matter of solving integro-differential equations instead of partial-differential equations. What about boundary conditions that are in essence foreign to this representation of matter–matter interaction? There remains a possibility of the existence of a "weak-non-local" limit by the approximation by gradient models. Typically one would consider in the linear elastic case a stress constitutive equation in the form

$$\sigma_{ji}(\mathbf{x}) = \int_{all\,space} C_{jikl}(|\mathbf{x}-\mathbf{x}'|)e_{kl}(\mathbf{x}')d^3\mathbf{x}', \tag{13.33}$$

where the constitutive functions C_{jikl} decreases markedly with the distance between material points **x**' and **x**. In space of one dimension, an inverse to (13.33) may be of the form

$$\sigma - K\nabla^2\sigma \equiv \left(1 - K\nabla^2\right)\sigma = Ee \tag{13.34}$$

with coefficients K and E, a model that we call "Helmholtz" one because of the presence of the operator $(1 - K\nabla^2)$—cf. (13.31). It is this kind of relation that allows one to compare the effects of "weakly" and "strongly" nonlocal theories in so far as the degree of singularity of some quantities is concerned (cf. Lazar and Maugin 2007).

Note that standard local linear elasticity follows from (13.33) by considering the special case

$$C_{jikl}(|\mathbf{x}-\mathbf{x}'|) = C^0_{jikl}\delta(|\mathbf{x}-\mathbf{x}'|),$$

where δ is Dirac's delta generalized function, and the tensorial coefficient C^0_{jikl} depends at most on the point **x** alone (for inhomogeneous materials).

The historical moment in the recognition of the usefulness of strongly nonlocal theories was the EUROMECH colloquium on nonlocality organized by Dominik Rogula in Warsaw in 1977 (cf. Maugin 1979). A now standard reference is Eringen's book (2002). A recent much publicized application of the concept of non locality is that to *damage* by Pijaudier-Cabot and Bazant (1987).

Note in conclusion to this point that any field theory can be generalized to a nonlocal one while saving the notions of linearity and anisotropy; but loosing the usual notion of flux. Also, it is of interest to pay attention to the works of Lazar and Maugin (2004a, b, 2007) for a comparison of field singularities in the neighbourhood of structural defects in different "generalized" theories of elasticity (micro polar, gradient-like, strongly non local or combining these).

13.5 Loss of the Euclidean Nature of the Material Manifold

Indeed the basic relevant problem emerges as follows. How can we represent geometrically the fields of structural defects (such as dislocations associated with a loss of continuity of the elastic displacement, or disclinations associated with such a loss for rotations)? A similar question is raised for vacancies and point defects. One possible answer stems from the consideration of a non-Euclidean material manifold, e.g., a manifold without curvature but with affine connection, or an Einstein-Cartan space with both torsion and curvature, etc. With this one enters a true "geometrization" of continuum mechanics of which conceptual difficulties compare favourably with those met in modern theories of gravitation. Pioneers in the field in the years 1950–1970 were Kondo (1955) in Japan, Kröner (1958) in Germany, Bilby in the UK, Stojanovic (1969) in what was then Yugoslavia, Noll (1967) and Wang (1967) in the USA. Modern developments are due to, among others, M. Epstein and Maugin (1990, 1997), M. Elzanowski and S. Preston (see the theory of material inhomogeneities by Maugin 1993). Main properties of this type of approach are (1) the relationship to the multiple decomposition of finite strains (Bilby, Kroener, E. H. Lee) and (2) the generalization of theories such as the theory of volumetric growth (Epstein and Maugin 2000) or the theory of phase transitions within the general theory of local structural rearrangements (local evolution of reference; see Maugin 2003b, examining Kröner's inheritance and also the fact that true material inhomogeneities (dependence of material properties on the material point) are then seen as pseudo-plastic effects—Maugin 2003a. All local structural rearrangements and other physical effects (e.g., related to the diffusion of a dissipative process)) are reciprocally seen as pseudo material inhomogeneities (Maugin 2003b). Many of these advances are first-hand critically expanded in a recent book (Maugin 2011). We shall deal in greater detail with this type of generalization in the next chapter. An antiquated forerunner work of all this may be guessed in Burton (1891), but only with obvious good will by a perspicacious reader.

We end our panoramic tour of GCM by mentioning an original geometric solution as presented in the book of Rakotomanana (2003), which offers a representation of a material manifold—that is everywhere dislocated–with the appropriate generalized gradient operator.

Introduction of the notion of *fractal sets*—as many real materials may be compared to—opens new horizons [cf. Li and Ostoja-Starzewski 2010; and recent works by Michelitsch et al. (2009, 2012), by a careful limit definition of the needed differential operators (gradient, Laplacian, D'alembertian)]. It is shown that *fractional* *derivatives* are necessarily involved in the continuum limit and this gives rise to a new kind of non-locality in elasticity as shown by Atanakovic and Stankovic (2009) and Carpinteri et al. (2011).

13.6 Conclusion

Since the seminal work of the Cosserats, three more or mess successful paths haven been taken towards the generalization of continuum mechanics. These were recalled above. An essential difference between the bygone times of the pioneers and the present time is that artificial materials can now be man-made that are indeed generalized continua. In addition, mathematical methods have been developed (homogenization techniques) that allow one to show that generalized continua are deduced as macroscopic continuum limits of some structured materials. This is illustrated by the book of Forest (2006).

In conclusion, we can answer three basic questions that are clearly posed: (1) Do we need GCM at all? (2) Do we find the necessary tools in what exists nowadays? (3) What is the relationship between discrete and continuous descriptions if there must exist a consistent relationship between the two? The first two questions are positively answered in view of the above described developments. The third question is of a different nature because, in principle, continuum theories can be developed independently of any precise microscopic vision, being judged essentially on their inherent logical structure, the possibility to have access to the material constants they introduce through appropriate experiments, and finally their efficiency in solving problems. However, in contrast to some hard-line continuum theoreticians, the author personally believes that any relationship that can be established with a sub-level degree of physical description is an asset that no true physicist can discard.

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Chapter 14
Configurational Mechanics

Abstract Starting with pioneering works by Peach, Koehler and Eshelby, an original branch of continuum physics has developed in the period 1950—2010 that consists in providing means of evaluating the evolution of particular material zones of bodies under the action of external loadings. These zones are essentially more or less localized regions of the bodies in which irreversible changes of properties occur through a reorganization of material components of which fracture is the most drastic form. This is interpreted as changes of local configuration in the accepted view of the continuum mechanics of deformable solids. The present conspectus reviews the formidable progress achieved in this "configurational mechanics" from an historical and somewhat personal perspective. In this general view phenomena such as fracture, phase transformations, the presence of material heterogeneities, and more generally the expansion of structural defects of different types find a natural unified frame work. Here the emphasis is placed on the original works, the various breakthroughs and their contributors, the connection with the notion of "material" force, the modern —but often unfamiliar — concept of mechanics on the material manifold, a strategy of post-processing to evaluate driving forces or to improve numerical schemes, and a methodology imported from mathematical physics. Unavoidable ingredients are those of Eshelby stress tensor, material momentum in dynamics, and material forces of inhomogeneity.

14.1 A Long but Useful Historical Introit

As indicated by the title of this chapter, here we are concerned with relationships between *configurations*. The nineteenth century has seen the appearance of the material configuration along with the more traditional actual configuration (See Chap. 1). More precisely, we are interested in the transition, back and forth, between these two configurations as also with possible changes in the reference

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configuration. This is the most typical development of continuum mechanics in the second half of the twentieth century. *Configurational mechanics* is the most recent and fruitful avatar of this development to which the author had the chance to contribute forcefully. This, as a consequence, means that this historical introit is not to be entirely objective so that we beg the reader to forgive us for this privileged but certainly first-hand vision.

Amateur historians of science such as the author of this book have a tendency to search in the remote past the origin—or at least traces pointing to a future full emergence—of presently acknowledged concepts. In our case, nothing sounds better than a reference to Nicole (Nicholas) Oresme (French philosopher, mathematician, theologian, economist; ca 1320–1382, Doctorate at Paris University 1356) who could be our oldest precursor with his well named "Tractatus de configuratione qualitatum et motuum". In this opus he introduced a graphic representation of material inhomogeneity by plotting the variation of a characteristic material property along a direction, and then generalizing this to three dimensions, inventing by the same token rectangular coordinate geometry long before Descartes. Indeed, he developed a universal theory explaining physical phenomena via the notion of geometrical configuration (cf. Duhem 1909; Taschow 2003). It is indeed true that the modern notion of material inhomogeneity is related to configurational mechanics (see the author's book of 1993 with its spot on title). A second possible allusion is to a work by Burton (1891) who spoke in evasive terms about the representation of continuous matter which may be related to the actual notion of material manifold. Burton also contributed, albeit belatedly, to the theory of the ill-fated "aether" (or ether), the supposed substratum of light/electromagnetic waves.

More seriously, we must invoke the "avant-garde" vision of Piola (cf. Chap. 1). However, if both Piola and Kirchhoff did introduce a reference configuration and helped put the basic local equations of balance in a useful format closer to the notion of strict conservation laws (in a mathematical sense), they did not go as far as *fully* projecting the field and balance equations onto the material. This would make these equations no longer contingent on the actual configuration. But this is the object of most of this chapter. Nonetheless, the introduction of the finite deformation gradient **F**—Eq. (1.5)—is tantamount to viewing the action of a *regular* local change of reference configuration $K_R \rightarrow K'_R$ by the differential rule

$$\mathbf{F}' = \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{X}'} = \mathbf{F} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{X}'}. \tag{14.1}$$

Such a transformation (multiplicative decomposition) based on standard analysis was known to rheologists in the 1950s (e.g., M. S. Green and A. V. Tobolsky). Simultaneously, engineers working in small-strain elasto-plasticity were commonly using the additive decomposition

$$\varepsilon = \varepsilon^e + \varepsilon^p = (\nabla \mathbf{u})_S, \tag{14.2}$$

where ε^e and ε^p refer to elastic and plastic infinitesimal strains, both not integrable in a true gradient since only the total strain possesses this property as symbolized in the last expression in (14.2) via the symmetrised displacement gradient. The so-called incompatibility of each constituent in (14.2) led Ekkehart Kröner (1958) to the formulation of a beautiful geometric theory of incompatibility. Pondering the matter of generalizing (14.2) to the case of finite strains, the idea arose in mechanical circles that a formula almost like (14.1) would be the looked for answer, i.e., formally

$$\mathbf{F} = \mathbf{F}^e.\mathbf{F}^p = \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{X}}, \tag{14.3}$$

where none of the two contributors in the product, although commonly called "elastic" and "plastic" gradients, actually are gradients separately integrable in a motion mapping. Mathematically, they are Pfaffian forms. The first manifestation of a multiplicative decomposition of this type seems to be in the proceedings of the Brussels ICTAM congress of 1956, by a group led by Bruce A. Bilby (Bilby et al. 1957). Such a decomposition was also advocated by Kröner at the same period. Strangely enough most works nowadays refer to (14.3) as due to Lee (1969). This is not fair. But it is true that (14.3) became popular only after Lee's work because its need had become more timely. Furthermore, it can also be remarked that the decomposition (14.3) that clearly involves not only the actual and reference configurations, but also an *intermediate configuration*—also called the *elastically-relaxed configuration*—is not unique as, introducing an orthogonal transformation $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{1}, \mathbf{Q}^{-1} = \mathbf{Q}^T$, we can also write

$$\mathbf{F} = \hat{\mathbf{F}}^e.\hat{\mathbf{F}}^p, \hat{\mathbf{F}}^e = \mathbf{F}^e.\mathbf{Q}, \hat{\mathbf{F}}^p = \mathbf{Q}^T.\mathbf{F}^p. \tag{14.4}$$

There is need to be more specific by fixing some privileged directions in the intermediate configuration. This was done by Mandel (1971, 1973) who introduced at this point so-called isoclinic directions and a director triad. This applies to the case of crystals in particular.

The idea contained in (14.3) was a fruitful one as it applies in many types of *anelastic* behaviours. It is possible to introduce a succession of—or multiple—intermediate configurations. This allows one to represent complex rheological behaviours in finite strains (e.g., work by Sidoroff 1975) in visco-elasticity. More generallly, all thermo-mechanical behaviours exhibiting so-called *internal stresses*—and thus called *quasi-plastic* processes—can also make good use of a multiplicative decomposition. This was done in thermo-elasticity by Milan V. Mićunović—from Serbia—in his Polish doctoral thesis published in 1974.

A second way in which expressions of the type of a multiplicative decomposition of the deformation gradient can appear belongs in the theory of *material inhomogeneities* and the accompanying notion of local structural rearrangements. It was introduced by Epstein and Maugin (1990a, b) in a kind of thought experiment. This configurational definition directly yields a relationship to the so-called

Eshelby material stress tensor. This, in turn, leads us to examining briefly the short "history" of that tensor.

The short story starts in the early 1950s with works devoted to the driving force acting on a dislocation line and a somewhat similar force acting on a general material inhomogeneity. This notion of driving force is not obvious as a dislocation line is not a material object. Still, we can observe its "motion" through the crystal and, by the inherent duality of mechanics, there should be a (driving) "force" causing this "motion", so that the product of this force and the velocity of the observed motion should be a dissipated power. The first quantity in this line is the force on a dislocation computed by physicists Peach and Koehler in 1950. It is important for the sequel to understand the *modus operandi*. The evaluation of the said force follows from a knowledge of the elastic solution in the neighbourhood of the dislocation. We can say in modern terms that this is a kind of post-processing computation. This methodology permeates the whole of this chapter.

The second force in the same class was the force acting on a material inhomogeneity (small region of the body occupied by a foreign material and fitting exactly in the elastic body) was computed by Eshelby (1951) in a pioneering work. It did not take long to Eshelby, also a specialist of dislocations, to realize the relationship of his force with the divergence of a second-order tensor which he called the "Maxwell elastic stress" or—but this was not happy as a coinage—the "energy–momentum tensor" of elasticity because of its resemblance to the Maxwell electromagnetic stress (See Chap. 12 for this concept). Accordingly, it is quite natural to consider Eshelby as the ≪ founding father ≫ of our field and, more appropriately than the name given by Eshelby, to call the relevant stress tensor the *Eshelby material stress tensor* (Maugin and Trimarco 1992).

The remarkable feature of these developments in a half century, but accelerated in the years 1980s–2000s, has been the new interrelation of continuum mechanics with recent fields of mathematical physics, in particular in so far as invariance properties are concerned, and with some fields of solid-state physics and materials science where the relationship between the two antagonistic views of the continuum and the discrete are concerned. It was soon uncovered for non-dissipative materials of which the mechanics follows from a variational principle in the manner of Lagrange and Hamilton, that the appearance of the Eshelby stress was a direct consequence of the application of a famed theorem of mathematical physics, a subtle theorem proved by Emmy Noether in 1918. In essence this theorem says that with each parameter of a group under which a physical theory (here the considered action) is invariant, there is associated a *conservation law*. Such laws apply to the whole physical system under consideration and, therefore, should not be mistaken for the standard field equations (balance in continuum mechanics), one for each degree of freedom. The energy equation is such a canonical equation of conservation (invariance under time translation for non-dissipative processes).

This difference between the standard balance equations (e.g., for linear—physical—momentum and the moment of momentum) and canonical conservation laws is made transparent in solid mechanics in finite deformations in which we are able to clearly distinguish between the dependent variables (e.g., the actual placement and the displacement) and the space–time parametrization (e.g., Newtonian time and material coordinates). This brings us back to the relationship between actual and reference configurations. For the classical balance equations in pure mechanics we thus need both the equations in the actual configuration (as they are usually presented in courses and used in the solution of boundary-value problems) and the equivalent equations but entirely projected onto the material manifold. Thus, the format of Piola–Kirchhoff is not sufficient for this purpose. One must go one step further, a step strangely not taken by the most prominent actor of modern continuum mechanics (Walter Noll) in his theory of material inhomogeneities. We remind the reader that the material manifold M^3 is the set of material points constituting the body in a more or less smooth manner.

This is directly related to the notion of material heterogeneity since that feature describes the dependency of the material properties on the material point (not the point occupied in physical space), hence on the local reference configuration. With this we are practically done. In particular Dominik Rogula in Warsaw considered the balance laws of continuum mechanics in both physical and material frameworks as soon as 1977. His co-worker Alicia Gołebiewska-Lasota, later Gołebiewska-Herrmann (1981, 1982), having moved from Poland to the USA, strongly influenced engineering scientists at Stanford University, the late George Herrmann and his co-workers (e.g., Pak and Herrmann 1986a, b; Eischen and Herrmann 1987). This was taken over very successfully by Reinhold Kienzler from Bremen, in collaboration with G. Herrmann, in their brilliant application of the concept of configurational forces to engineering mechanics (Kienzler and Herrmann 1986) and the strength of materials culminating in a book (Herrmann and Kienzler 1999).

In parallel, a more traditional school of mechanical engineering, following along the path opened by German scientists such as Günther (1962), studied in depth various path-independent integrals (Knowles and Sternberg 1972; Fletcher 1976; Bui 1978) with a repeated interest in fracture and other problems of which the solution exploits these integrals (see, Buggisch et al. 1981). Pioneers such as Cherepanov (1967, see also the collection of papers in the book edited by Cherepanov 1998), and Rice (1968) must be cited as having been instrumental in the application of some path-independent integrals (e.g., the celebrated *J*-integral of fracture). We may say that the works of Abeyaratne and Knowles (1990) on interfaces follow also this line keeping simultaneously close contact with physical features (see their book, 2001). Along another line, it was natural for M. E. Gurtin, a pioneer in good mathematical approaches to fracture (Gurtin 1979a, b) and moving interfaces (Gurtin 1993), to enter the domain of configurational forces with an original view (1995, his book of 1999) which we cannot share for reasons repeatedly explained in our book of (2011): Gurtin claimed to have introduced a new law of physics, but here there is **no** new law of physics.

Indeed, the present author entered the field in a pedestrian way by establishing first the relationship between the general geometric considerations of K. Kondo, E. Kröner, W. Noll and C.C. Wang (e.g., Noll 1967; Wang 1967) and their theory of *material uniformity* with the notion of *Eshelby stress tensor* (his "energy–momentum" tensor). This was achieved in collaboration with Marcelo Epstein

(Epstein and Maugin 1990a, b), while, in co-operation with Carmine Trimarco (Maugin and Trimarco 1992), we revisited the relationship of the notion of Eshelby stress with that of variational principle (obviously in the absence of dissipation in matter *per se*). Generalizations to electromagnetic materials of different types were to follow rapidly by the same group of three authors, while a small book (Maugin 1993) written while the author was a member of the *Wissenschaftskolleg in Berlin* (1991–1992) set forth in a few pages general ideas on the subject with special emphasis on the differences between Newtonian mechanics and Eshelbian mechanics. This was complemented by a long review (Maugin 1995). Both book and review were instrumental in attracting the attention of several researchers to the field. Particularly noteworthy was the important remark made by Braun (1997) from Duisburg on the possibility to exploit the material momentum equation or its equilibrium version to the benefit of *finite-element computations*. This was to generate a series of works by Ralf Mueller and co-workers (in Darmstadt), including the present author (e.g., Mueller and Maugin 2002; Maugin 2002) and the very active group of Paul Steinmann then in Kaiserslautern (e.g., Steinmann 2000, 2002a, b; Steinmann et al. 2001). Steinmann and co-workers cleverly introduced in a systematic way the so-called Cauchy and Eshelby formats of a stress tensor, and treated a number of numerical applications in particular in the field of large-strain biomechanics. In the meantime we established a kind of universality of the *canonical thermomechanics of continua*, including in most intrinsically dissipative cases (Maugin 1998b, 2006). This was an answer to those critics who said that we were relying too much on variational formulations. But the case of dissipative materials could be treated, as we have shown (Maugin 2006), by mimicking Noether's identity.

Before that, Epstein and the author had introduced the notion of *thermal material force* in heat conductors (Epstein and Maugin 1995a) of which a simple form had been formulated by Bui (1978). We also examined the geometrical definition of the Eshelby stress in the case of finite-strain plasticity (Epstein and Maugin 1995b) as well as in the theory of material growth (Epstein and Maugin 2000). In this line the author identified effects of pseudo-inhomogeneity and pseudo-plasticity by their resemblance to the Eshelbian type of inhomogeneity effects (Maugin 2003). Many generalizations to the cases of electromagnetic materials of different classes and to generalized continuum mechanics were given by the author and co-workers between 1991 and 2005. An original approach to dissipative interfaces such as phase-transition fronts and shocks was given in 1997–1998 (e.g., Maugin 1997, 1998a). Applications to the conservation laws and perturbation of soliton-like solution were performed in co-operation with Christo I. Christov (e.g., Maugin 1999b; Maugin and Christov 2002)—after a work of 1992 published in the *Journal of the Mechanics and Physics of Solids* (Maugin 1992a, b)—while an original thermodynamically admissible numerical scheme of the finite-volume type was conceived together with Arkadi Berezovski with a special interest in moving interfaces (see the book by Berezovski et al. 2008) leaning heavily on the notion of material framework and configurational force.

In all, the author and co-workers succeeded in unifying three lines of research in continuum mechanics:

1. The finite deformation line that considers the multiplicative decomposition of the deformation gradient as fundamental (it was proved that Mandel's stress that appears in this line is none other than one part of the Eshelby stress tensor expressed in the intermediate configuration);
2. The geometric line created by scientists such as Kondo, Bilby, Kröner and others (the canonical equation of momentum of the third line indeed is the basic equation in that approach);
3. The configurational-force line—or theory of material forces—following Eshelby (see the chart flow in p. 164 in Maugin's book of 2011, originally given in 2003 to commemorate Kröner's legacy) in which it is shown that the "force" on singularities (e.g., dislocation line, crack tip, singular surface) is intimately related to the limit behaviour of the divergence of the Eshelby stress in the appropriate setting.

In the remainder of this chapter we shall critically examine the most important points and results of this approach. But we like to mention a fact that is connected both with the theory of configurational forces and with human behaviour. It shows that even excellent scientists can be prejudiced and wear blinkers. Here we refer to the complete ignorance shown by some pundits (Truesdell et al.) of continuum mechanics for the pioneering work of Eshelby, perhaps because the latter often referred to variational formulations! In thermodynamics we find the misunderstanding demonstrated by Ingo Müller. As to certain fluid dynamicists, it seems that they cannot—or do not make the required effort to—understand the bases of solid mechanics (various configurations and the reference configuration in particular).

14.2 Simple Formulas to Help the Reader in his Appraisal of the Sequel

In order to facilitate the reading of the next section we give here without explanation illustrative equations in finite- and small-strain elasticities.

14.2.1 Finite-Strain Inhomogeneous Elasticity

With the field equations of motion (Piola–Kirchhoff format)

$$\frac{\partial}{\partial t}(\rho_0 \mathbf{v}) - div_R \mathbf{T} = 0, \quad \mathbf{T} = \frac{\partial}{\partial \mathbf{F}} W(\mathbf{F}; \mathbf{X}), \mathbf{T}.\mathbf{F} = \mathbf{F}^T.\mathbf{T}^T, \tag{14.5}$$

there are associated the following conservation laws of energy and material momentum:

$$\frac{\partial}{\partial t}(K+W)-\nabla_R.(\mathbf{T}.\mathbf{v})=0, \tag{14.6}$$

$$\frac{\partial}{\partial t}\mathbf{P}+div_R\mathbf{b}=\mathbf{f}^{inh}, \tag{14.7}$$

where in

$$K=\frac{1}{2}\rho_0\mathbf{v}^2, \mathbf{P}:=-\rho_0\mathbf{v}.\mathbf{F}, \tag{14.8}$$

$$\mathbf{b}=-(L\mathbf{1}+\mathbf{T}.\mathbf{F}), L=K-W, \mathbf{f}^{inh}:=\frac{\partial}{\partial \mathbf{X}}L(\mathbf{v},\mathbf{F};\mathbf{X})\Big|_{\mathbf{v},\mathbf{F}\,fixed} \tag{14.9}$$

with the identity

$$\left(\frac{\partial}{\partial t}(\rho_0\mathbf{v})-div_R\mathbf{T}\right).\mathbf{F}+\left(\frac{\partial}{\partial t}\mathbf{P}-\left(div_R\mathbf{b}+\mathbf{f}^{inh}\right)\right)\equiv\mathbf{0}. \tag{14.10}$$

The latter is none other than *Noether's identity* for the variational problem associated with the Lagrangian density L, (14.5) provides the corresponding Euler–Lagrange equations, and (14.7) and (14.6) are the conservation laws that follow from the application of Noether's theorem for invariance under translations of the space–time parametrization $(\mathbf{X}, t)$. $\mathbf{P}$ is called the material (or canonical) momentum. Tensor $\mathbf{b}$ is the so-called Eshelby material stress tensor; $\mathbf{f}^{inh}$ is called the material force of inhomogeneity (Maugin and Trimarco 1992).

14.2.2 Small-Strain Inhomogeneous Elasticity

Then in place of Eqs. (14.5) through (14.10), we have the following equations in Cartesian tensor notation:

$$\frac{\partial}{\partial t}(\rho_0\dot{u}_i)-\frac{\partial}{\partial x_j}\sigma_{ji}=0, \quad \sigma_{ji}=\frac{\partial}{\partial e_{ij}}W\left(e_{pq};x_k\right)=\sigma_{ij}, \tag{14.11}$$

$$\frac{\partial}{\partial t}(K+W)-\frac{\partial}{\partial x_j}\left(\sigma_{ji}\dot{u}_i\right)=0, \tag{14.12}$$

$$\frac{\partial}{\partial t}P_i-\frac{\partial}{\partial x_j}b_{ji}=f_i^{inh}, \tag{14.13}$$

$$K = \frac{1}{2}\rho_0 \dot{u}_i \dot{u}_i, \quad P_i = -\rho_0 \dot{u}_j u_{j,i}, \tag{14.14}$$

$$b_{ji} = -\left(L\delta_{ji} + \sigma_{jk} u_{k,i}\right), L = K - W, f_i^{inh} = \left.\frac{\partial L}{\partial x_i}\right|_{\underline{\mathbf{u}},\mathbf{e}\,fixed}, \tag{14.15}$$

$$\left(\frac{\partial}{\partial t}(\rho_0 \dot{u}_i) - \frac{\partial}{\partial x_j}\sigma_{ji}\right) u_{i,k} + \left(\frac{\partial}{\partial t} P_k - \frac{\partial}{\partial x_j} b_{jk} - f_k^{inh}\right) = 0. \tag{14.16}$$

14.3 Landmark Results in Configurational Mechanics

Here we follow a practically chronological order.

14.3.1 Driving Force on a Material Inhomogenity (Eshelby 1951)

$$F_n^{inh} = \int_B \left((W)_{,n} - \left(\sigma_{mk} u_{k,n}\right)_{,m} + \sigma_{mk,m} u_{kn}\right) dv = \int_{\partial B} (\mathbf{n}.\mathbf{b})_n da, \tag{14.17}$$

with a quasi-static Eshelby elastic stress given by the first of (14.15) with K set equal to zero.

14.3.2 Driving Force on the Tip of a Crack (Cherepanov 1967, Rice 1968)

$$J_k = \lim_{\Gamma \to 0} \int_\Gamma (\mathbf{n}.\mathbf{b})_k d\Gamma, \tag{14.18}$$

In particular the contour-independent, so-called, J-integral (as named by Rice; projection of (14.18) in the direction of extension of the crack, here axis x_1) of linear fracture theory:

$$J = J_1 = \int_\Gamma \left(W n_1 - n_k \sigma_{ki} u_{i,1}\right) d\Gamma. \tag{14.19}$$

Here Γ is a contour in the anti-clockwise direction around the crack tip, with end points on the two load-free faces of the crack. Other contour-independent integrals, so-called L and M integrals were introduced by Günther (1962), Knowles and Sternberg (1972), Fletcher (1976), Budiansky and Rice (1973) and are associated with rotational and dilatational defects.

14.3.3 Dual I-Integral of the Linear Theory of Fracture (Bui 1973)

$$I = \lim_{\Gamma \to 0} \int_{\Gamma} \left(-W_c(\sigma)\, n_1 + n_k \frac{\partial \sigma_{ki}}{\partial x_1} u_i \right) d\Gamma, \tag{14.20}$$

where W_c is the complementary elastic energy.

14.3.4 The Analytical Theory of Brittle Fracture (Dascalu and Maugin 1993)

This holds in finite strains and full dynamics and treats simultaneously the driving force and its associated dissipated power. With the notation of the previous Sect. 14.3.1 we have the following two compatible expressions for the force on the crack (generalization of the J-integral) and the associated energy-release rate:

$$F_1^{crack} = \int_{\Gamma} \big((\mathbf{N}.\mathbf{b})_1 + P_1 (\mathbf{V}.\mathbf{N}) \big) dA - \frac{\partial}{\partial t} \int_G P_1 dV \tag{14.21}$$

and

$$G^{crack} = \int_{\Gamma} (H(\mathbf{V}.\mathbf{N}) + \mathbf{N}.\mathbf{T}.\mathbf{v}) dA - \frac{\partial}{\partial t} \int_G H dV, \tag{14.22}$$

where $\bar{\mathbf{V}}$ is the velocity of progress of the crack tip in the material, G is the volume enclosed in the contour Γ, and $H = K + W = \mathbf{P}.\mathbf{V} - L$ is the Hamiltonian density. It is the simultaneous presence of the Lagrangian (in the force) and Hamiltonian (in the energy) densities and the fact that the two are related by the Legendre transformation $H = \mathbf{P}.\mathbf{V} - L$ which made us call this the *analytical* theory of fracture. We should also mention here the approach of the J, $\boldsymbol{L}$ and M integrals and their associated dissipated energy by means of the theory of generalized functions (distributions) by the same authors (Dascalu and Maugin 1994a, b). That approach leads to considering a total conserved energy as the sum of the elastic energy and the energy of the studied defect (moving dislocation, growing hole). Previously, dynamical fracture was more conventionally studied in depth by Freund (1972).

14.3.5 Contour Integrals in Electroelasticity and Electromagnetic Materials

It was not long before some of the above results were generalized to the case of piezo-electricity, yielding, for instance, the following "piezo-electric" J-integral (Suo et al. 1992)

$$J = \int_{\Gamma} \left(WN_1 - \mathbf{n}.\sigma.\mathbf{u}_{,1} - \mathbf{n}.\mathbf{D}\varphi_{,1} \right) d\Gamma, \tag{14.23}$$

where $\mathbf{D}$ is the electric displacement vector and φ is the electrostatic potential. Here W is jointly quadratic in the deformation and the electric field such that $\mathbf{E} = -\nabla\varphi$. However, more exact results in the nonlinear linear framework had been given before by Pak and Herrmann (1986a, b), with an expression in the reference configuration such as

$$J(\Gamma) = \int_{\Gamma} \left\{ \tilde{W}N_1 - \mathbf{N}.\left(\mathbf{T}^E + \mathbf{T}^F\right).\frac{\partial \mathbf{u}}{\partial X_1} - \left(\hat{\mathbf{D}}.\mathbf{N}\right)\frac{\partial \hat{\varphi}}{\partial X_1} \right\} dS, \tag{14.24}$$

where $\mathbf{T}^E$ and $\mathbf{T}^F$ are "elastic" and "electric-field "contributions to the Piola–Kirchhoff stress and $\hat{\mathbf{D}}$ is the material electric displacement. Another approach was proposed by Maugin and Epstein (1991) on the basis of their theory of material forces. The case of soft magnetic and paramagnetic bodies was similarly formulated by Sabir and Maugin (1996) and the much more complicated case of elastic ferromagnets (exhibiting a magnetic-spin internal degree of freedom) was settled by Fomethe and Maugin (1998).

14.3.6 Material Inhomogeneity and Local Structural Rearrangements

Here we take one step back in the past to report briefly on the works of Epstein, Trimarco and Maugin. In the winter of 1989, Epstein and Maugin (reported in 1990a, b) conceived of a thought experiment that provided for the first time a relation between the notion of *local material rearrangement* and the Eshelby material stress. In effect, considering the case of finite-strain inhomogeneous elasticity with energy density $\bar{W}(\mathbf{F};\mathbf{X})$, they considered a local change $\mathbf{K}$ (at each material point $\mathbf{X}$) of reference contribution such that the material looks like a purely elastic one, but in a kind of prototypical local configuration (what they call the *reference crystal*). This yields a new energy density accounting for the related change in volume:

$$W = \overline{W}(\mathbf{F};\mathbf{X}) = J_{\mathbf{K}}^{-1} W(\mathbf{F}\mathbf{K}(\mathbf{X})) = \tilde{W}(\mathbf{F},\mathbf{K}). \tag{14.25}$$

We check that

$$\mathbf{T} = \frac{\partial \bar{W}(\mathbf{F};\mathbf{X})}{\partial \mathbf{F}}, \quad \mathbf{b} = -\frac{\partial \tilde{W}(\mathbf{F},\mathbf{K})}{\partial \mathbf{K}}\mathbf{K}^T = W\mathbf{1}_R - \mathbf{TF}. \tag{14.26}$$

Thus there definitely exits a relationship between the notion of material inhomogeneity and that of configurational (or Eshelby) stress. Furthermore, computing the material divergence of **b**, we find that

$$div_R \mathbf{b} = -\mathbf{f}^{inh}, \tag{14.27}$$

with cf. the last of (14.9)

$$\mathbf{f}^{inh} = -\left.\frac{\partial \overline{W}}{\partial \mathbf{X}}\right|_{\mathbf{F}\,fixed} = \mathbf{b}.\Gamma(\mathbf{K}), \Gamma(\mathbf{K}) := \left(\nabla_R \mathbf{K}^{-1}\right).\mathbf{K}. \tag{14.28}$$

The geometric quantity $\boldsymbol{\Gamma}$ is a *connection* based on the local structural rearrangement **K**. Equation (14.27) with $\Gamma(\mathbf{K})$ is the equation missed by Noll and Wang in their (nonetheless) beautiful works. We finally note that the symmetry of the Cauchy stress results in the symmetry of **b** with respect to the finite Cauchy strain **C**, i.e., (Epstein and Maugin 1990a, b)

$$\mathbf{C}.\mathbf{b} = \mathbf{b}^T.\mathbf{C}. \tag{14.29}$$

Also, we note that

$$\mathbf{M} := \mathbf{T}.\mathbf{F} = \mathbf{S}.\mathbf{C} = W\mathbf{1}_R - \mathbf{b}. \tag{14.30}$$

This establishes the relation between the *Mandel stress* **M** and the Eshelby material stress. This is corroborated by the study of the role played by the Eshelby stress—in the intermediate configuration—in elasto-plasticity in finite strains (Maugin 1994; also Epstein and Maugin 1995b). Le (1999) has provided an interpretation of the Eshelby stress as the resolved shear stress in the activation of slip systems in crystal plasticity.

The fully dynamical equation of material momentum (14.7) was deduced from a variational formulation by Maugin and Trimarco (1992). The same authors have constructed the whole dynamical theory for magnetized and electrically polarized materials in the exact material frame work.

14.3.7 Application to Materials Endowed with a Microstructure

Here we think in terms of the contents of the foregoing Chap. 13. The first case consists in envisaging the action of a higher-order gradient of the deformation. The basic equations were deduced by Maugin and Trimarco (1992) for a second-gradient theory involving the gradient of the deformation gradient. Such a formulation, in spite of its complexity and its subtle geometrical background (studied by authors such as Marcelo Epstein and Marek Elzanowski), proves to be useful in the biomechanics of soft tissues and morphogenetics (see recent works by Ciarletta and Maugin 2011). In this applicative framework we also find the fundamental work of Epstein and Maugin (2000) on the thermo-mechanical theory of material growth.

In the case of a microstructure of the Cosserat type (micropolar materials) it seems that the first proposal of a J-type of integral was by Atkinson and Leppington (1974) with an expression of the type

$$J = \int_{\Gamma} \left(W N_1 - \mathbf{t}.\frac{\partial \mathbf{u}}{\partial X_1} - \mathbf{m}.\frac{\partial \phi}{\partial X_1} \right) dL, \tag{14.31}$$

where $\mathbf{t} = \mathbf{N}.\mathbf{T}$ and $\mathbf{m} = \mathbf{N}.\mathbf{M}$ along the contour Γ, while the Eshelby stress has the following quasi-static canonical definition:

$$b_{ji} = W\delta_{ji} - \sigma_{jk}u_{k,i} - m_{jk}\phi_{k,i}, \tag{14.32}$$

where m_{ji} are the components of the reduction of $\mathbf{M}$ to this linearized case (see the notation in Chap. 13). The J-integral (14.31) as also the corresponding $\boldsymbol{L}$ and M integrals were discussed by Jaric (1978) while Lubarda and Markenscoff (2000) rediscovered some of these results. But the general theory of configurational forces in polar materials together with its application in fracture and the progress of phase-transition fronts was given by Maugin (1998c) in a lengthy memoir of the Philosophical Transactions of the Royal Society.

The case of complex media involving dissipative internal variables of state is considered together with thermal effects in the next item.

14.3.8 Thermal Effects

In this case—e.g., in thermo-elasticity—the energy density, whence the *free energy*, W must depend on the thermodynamical temperature $\theta > 0, \inf \theta = 0$. That is, in a materially inhomogeneous thermoelastic material,

$$W = \overline{W}(\mathbf{F}, \theta; \mathbf{X}). \tag{14.33}$$

Epstein and Maugin (1995a) have shown that Eq. (18.8) is now replaced by

$$\frac{\partial}{\partial t}\mathbf{P} - div_R \mathbf{b} = \mathbf{f}^{inh} + \mathbf{f}^{th},\ \mathbf{f}^{th} := S\nabla_R \theta, S = -\frac{\partial \overline{W}}{\partial \theta}. \tag{14.34}$$

The newly introduced material force $\mathbf{f}^{th}$ we called the *thermal material force* while S is none other than the *entropy* density. The thermal material force—not a usual physical force—vanishes only once the body has reached a state of spatially uniform temperature. Until that moment it plays the same role as the material force of inhomogeneity $\mathbf{f}^{inh}$ on the material manifold. Accordingly, it may also be referred to as a material force of pseudo-inhomogeneity. There are other cases like this one. Indeed, each time we will add an extra variable in the functional dependence of W, we will create a new such pseudo-inhomogeneity force. This is what happens in materials where dissipative processes (e.g., plasticity, visco-elasticity, damage) are described by means of a set of internal variables of state, here collectively denoted by α and that will eventually be governed by an evolution equation subject to the second law of thermodynamics. Thus, keeping the thermal effects but discarding the true material inhomogeneity (dependence on $\mathbf{X}$), we have to replace (14.33) by

$$W = \overline{W}(\mathbf{F}, \theta, \alpha). \tag{14.35}$$

We no longer have at hand a variational formulation, but we can proceed by mimicking the Noether identity (14.10) by applying $\mathbf{F}$ to the right to all contributions in the first of Eq. (14.5). We obtain thus a "non-conservation" of material momentum in the form

$$\frac{\partial}{\partial t}\mathbf{P} - div_R\mathbf{b} = \mathbf{f}^{th} + \mathbf{f}^{\mathrm{intr}}, \mathbf{f}^{\mathrm{intr}} := A.(\nabla_R \alpha)^T, A := -\frac{\partial \bar{W}}{\partial \alpha}, \tag{14.36}$$

where A is the *thermodynamic force* associated with α, all other quantities being formally defined just as before. The dot in the definition of the material force $\mathbf{f}^{intr}$ due to the presence of internal variables of state—the *intrinsic* material force—stands for the appropriate inner product in the space of the α's. What is most remarkable is that, simultaneously, we can associate with the first of Eq. (14.36) a canonical equation of energy in the form (cf. Maugin 2006, but announced at symposia and seminars since 2000)

$$\frac{\partial}{\partial t}(S\theta) + \nabla_R.\mathbf{Q} = h^{th} + h^{intr}, \tag{14.37}$$

where $\mathbf{Q}$ is the material heat flux, and we have set

$$h^{th} := S\dot{\theta},\ h^{intr} := A.\dot{\alpha}. \tag{14.38}$$

It is easily realized that the first of (14.36) and (14.38) are but the spatial and temporal components of a unique four-dimensional "non-conservation" law of energy and momentum.

14.3.9 General Theory of Shocks, Transition Fronts, Inelastic Discontinuities

In his early works of the 1950s Eshelby realized that at a fixed boundary Σ between two different elastic bodies there could exist a surface "inhomogeneity force" related to the normal jump of his "Eshelby" stress, i.e., in the quasi-statics where the first of (14.15) holds with $K = 0$, a force

$$\mathbf{f}^{inh}(\Sigma) = -\mathbf{n}.[\mathbf{b}], \tag{14.39}$$

where the square brackets capture the jump of their enclosure. Again, this is not a force (here traction) of the Newtonian type. Rather, it serves as a *directional indicator* of the change in the elastic properties as we traverse Σ. With appropriate sign convention, it is oriented from the "harder" side to the "softer" side. In 1997–1998, Maugin and his Japanese co-workers, Tatsuo Inoue and Shoji Imatani have dealt with this matter when temperature is also involved (results included in the author's book of 2011). However, the most interesting matter here is provided by the theory of moving discontinuity surfaces $\Sigma(t)$in a full thermo-dynamical framework where discontinuity equations associated with the regular canonical conservation laws complement the usual discontinuity relations. These new jump relations in fact govern the irreversible progress of $\Sigma(t)$. Classical works on the subject are the historical contributions of Rankine, Maxwell, Gibbs, Duhem, Hugoniot, Hadamard and Kotchine. In recent times and in relation with Eshelbian concepts the most active contributors have been Morton E. Gurtin and co-workers, Rohan Abeyaratne and James K. Knowles, Claude Stolz, and Maugin and co-workers. In the formalism of the author, the standard jump relations (in Piola–Kirchhoff format) relating to mass, physical (linear) momentum, energy and entropy are given by

$$\bar{V}_N[\rho_0] = 0, \tag{14.40}$$

$$\bar{V}_N[\mathbf{p}_R] + \mathbf{N}.[\mathbf{T}] = \mathbf{0}, \tag{14.41}$$

$$\bar{V}_N[H_R] + \mathbf{N}.[\mathbf{T}.\mathbf{v} - \mathbf{Q}] = 0, \tag{14.42}$$

$$\bar{V}_N[S_R] - \mathbf{N}.[\mathbf{S}] = \sigma_\Sigma \geq 0, \tag{14.43}$$

where $\bar{V}_N$is the normal velocity of $\Sigma(t)$, $\mathbf{p}_R = \rho_0 \mathbf{v}$, $\mathbf{S}$ is the material entropy flux, and H_R and S_R are the total energy (Hamiltonian) density and entropy density per unit reference volume. In the presence of thermal and intrinsic dissipative processes (e.g., described by internal variables of state), the jump relations associated with the regular equations (14.36) and (14.37) are given by

$$\bar{V}_N[\mathbf{P}] + \mathbf{N}.[\mathbf{b}] = -\mathbf{f}_\Sigma, \tag{14.44}$$

and

$$\bar{V}_N[S\theta] - \mathbf{N}.[\mathbf{Q}] = h_\Sigma, \tag{14.45}$$

where the three unknown quantities $\mathbf{f}_\Sigma$, h_Σ and σ_Σ have to be estimated consistently. Of course, in physical reality these correspond to integrated source terms throughout the thickness of a transition zone that is not mathematically of zero thickness (i.e., the discontinuity possesses a structure; such is the case of shock waves that exist only because of dissipation occurring through such a transition region).

The looked for consistency is obtained by evaluating the power dissipated by $\mathbf{f}_\Sigma$ and comparing (14.45) and (14.43) while taking account of (14.40) through (14.42) and of the formal expressions of **P** and **b**. The case of phase-transition fronts is beautifully treated by Abeyaratne and Knowles, especially in their synthetic book of 2001. Suffice it to notice that the driving force on such (homothermal) fronts is given by the scalar quantity (formalism of the author)

$$Hugo_{PT} := [W - \langle \mathbf{N}.\mathbf{T} \rangle.\mathbf{F}.\mathbf{N}], \tag{14.46}$$

the symbolism $< .. >$ indicating the mean value at Σ. When inertia is fully disregarded from the start, then $\mathbf{N}.[\mathbf{T}] = \mathbf{0}$ across Σ. Using the Maxwell-Hadamard condition for **F**, it is then shown that (14.46) can also be written as

$$Hugo_{PT}(quasi\text{-}statics) = [W - tr(\langle \mathbf{T} \rangle.\mathbf{F})]. \tag{14.47}$$

This contributes to the following *balance of* $\ll$ *material* $\gg$ *forces* at $\Sigma(t)$

$$f_\Sigma + Hugo_{PT} = 0, \tag{14.48}$$

This surface $\ll$ balance $\gg$ equation is written down just to emphasize the different roles of $Hugo_{PT}$—a field quantity that is known once we know the field solution by any means on both sides of Σ—and the driving force f_Σ that is the thermodynamic conjugate of the normal speed $\bar{V}_N$since it is shown that the second law at Σ requires that

$$p_\Sigma = f_\Sigma \bar{V}_N = \theta_\Sigma \sigma_\Sigma \geq 0. \tag{14.49}$$

The expression of $\bar{V}_N$ in terms of $f_\Sigma = \mathbf{N}.\mathbf{f}_\Sigma$ is the *kinetic law* for normal progress of which examples basing on a more microscopic approach can be found in Truskinovsky (1994).

In comparison with (14.46), the driving force on shock waves is given by the scalar quantity

$$Hugo_{SW} := [E(\mathbf{F}, S, \alpha) - \langle \mathbf{N}.\mathbf{T} \rangle.\mathbf{F}.\mathbf{N}] \equiv 0 \, at \, \Sigma, \tag{14.50}$$

where it is the *internal energy E* per unit reference volume that is involved. The classical theory of shock waves imposes that $Hugo_{SW} = 0$ (the well known Rankine-Hugoniot condition) although $\bar{V}_N \neq 0$ in general! Accordingly, the entropy growth condition across such shockwaves—that is supposed to give the direction of propagation—comes out of the blue as

$$m_\Sigma [S/\rho_0] \geq 0, \; m_\Sigma := \rho_0 \bar{V}_N. \tag{14.51}$$

In order to correct this inconsistency, the author (1997, 1998a), following an earlier work of Stolz (1994), has shown that for both truly dissipative shock-wave and phase-transition fronts, the quantities σ_Σ and $p_\Sigma$could be derived from a common entity, a so-called (Massieu) generating function M_Σ by the relations

$$\sigma_\Sigma = [M_\Sigma] \geq 0 \tag{14.52}$$

and

$$p_{\Sigma} := \mathbf{V}.\mathbf{f}_{\Sigma} = [\theta M_{\Sigma}]. \tag{14.53}$$

The notion of generating function was already used in hydrodynamics and magnetohydrodynamics by Paul Germain in shock wave studies. It is a linear weighted combination of the invariants (continuous fields) across Σ (see Chap. 7 in Maugin 2011).

For completeness we mention the works of Cherkahoui and Berveiller (2000) in inelastic discontinuities and martensitic phase transitions, and of Schmidt and Gross (1997) on the equilibrium shapes of precipitates.

14.3.10 Involvement in Numerical Schemes

Finite-volume scheme

Berezosvki and Maugin have elaborated a special thermodynamically admissible *finite-volume scheme* to treat the dynamics of phase-transition fronts in the Eshelbian formalism (see the synthesis in the book of Berezovski et al. 2008).

Finite-difference scheme

In order to treat the dynamics of media with microstructure able to transmit strongly localized nonlinear waves and the associated quasi-particles (by means of the exploitation of canonical conservation laws), Christov and Maugin (1995) had to developed specially designed finite-difference schemes dealing with steep systems of partial differential equations.

Finite-element scheme

This is where the very structure of the Eshelbian mechanics influenced most the conception of numerical schemes. The reason for this is due to the co-existence of two equations for momentum, in the actual and material framework, respectively. Indeed, while the first of these is necessarily used to solve numerically the considered problem by means of an FEM scheme, the second can be exploited in a post-processing procedure to improve the said scheme. This technique aims at minimizing the energy in both actual and reference frameworks simultaneously. For the sake of example, consider a homogeneous nonlinear elastic problem governed in the bulk by the static equation [cf. Eq. (14.5)]:

$$div_R \mathbf{T} = \mathbf{0}. \tag{14.54}$$

Imagine that this is solved by means of an FEM with a certain discretization grid. Thus, we can know a solution from which we can evaluate the quantity $div_R \mathbf{b}$ which should vanish in the absence of material inhomogeneities. But we cannot check that exactly because $\mathbf{b}$ is of a higher degree in the fields than $\mathbf{T}$. Thus, we generally obtain the inhomogeneous equation

$$div_R \mathbf{b} = \mathbf{f}^{comp} \neq \mathbf{0}. \tag{14.55}$$

The minimization of the field of spurious material forces $\mathbf{f}^{comp}$ will cause a coordinated displacement of the initial computational points. This can be achieved in several iterative steps. This constitutes the essence of the creative remark made by Braun (1997) in a short but spot on paper. This was followed by many papers by Ralf Müller and Maugin in Paris, Dietmar Gross and his co-workers in Darmstadt and Paul Steinmann and his co-workers in Kaiserslautern, and other people in different countries. This has become a very technical matter so that we do not elaborate further on this line of development.

14.4 Note on Wave Studies

Before concluding we want to mention briefly two unexpected applications of configuration mechanics in wave processes. The first one is related to the so-called kinematic-wave theory of Lighthill and Witham (cf. Whitham 1974). This is based on the remark that, as clearly shown by the usual notion of phase $\varphi = \mathbf{K}.\mathbf{X} - \omega t$ in terms of the wave vector $\mathbf{K}$ (here material) and the frequency ω, the couple of variables $(\mathbf{K}, \omega)$ is *dual* to the space-time parametrization $(\mathbf{X}, t)$ via the four-dimensional phase "scalar" product defining φ. This duality can be exploited to formulate canonical conservation equations associated—in the sense of Noether's theorem—with a Lagrangian density $\tilde{L}$ for waves, re-expressed in terms of $(\mathbf{K}, \omega)$. Thus, a conservation of "wave momentum" with a corresponding flux as a material "wave Eshelby stress" can be formulated. In parallel, for the timelike component, an equation of energy conservation holds expressed in terms of $\tilde{L}$ and a "wave action". This elegant formalism can be used in the study of nonlinear dispersive waves in elastic crystals as shown by the author (Maugin 2007, 2011; Appendix to Chap. 12).

The second application is close to the notion of *phonons* in quantum solid-state physics. It concerns the possibility to associate the notion of "quasi-particle" with continuous wave processes such as in elasticity theory. This was already mentioned in Sect. 14.1 for the remarkable but rare dynamic phenomenon of solitons. But here we like to point out the possibility to associate quasi-particles in more or less stationary motion with surface acoustic waves (SAWs) as proposed by the author and Martine Rousseau in order to tackle the problem of non-destructive testing techniques by SAWs with an original viewpoint (cf. Rousseau and Maugin 2011; Maugin and Rousseau 2012). The general idea consists in the following. Consider the "field" or "wave" momentum defined in the second of Eq. (14.14) in agreement with Brenig (1955). Integrate the associated conservation law over a representative volume element (RVE) for the considered wave process while substituting in it the known wave solution (e.g., the Rayleigh SAW solution mentioned in Chap. 6) obtained on the basis of the standard equation of motion in the bulk, the boundary condition at the surface, and the asymptotic vanishing condition far from the surface, in the substrate. The RVE typically is one wavelength in the propagation direction, from the surface to infinity away from it in the depth direction, and one length unit in the remaining transverse direction. The result of this calculation is a Newtonian-like equation for a "quasi-particle" generally propagating along the limiting surface, and of which the "mass" is a measure of the wave energy that simultaneously accounts for all parameters of the continuous solution. In parallel, applying the same procedure to the energy equation of the continuum framework, a conservation of energy (in the form of kinetic energy with the just introduced virtual "mass") is obtained for the said quasi-particle completing the point-like picture. This is ready for the study of interactions with an obstacle on the path of the wave (e.g., a layered structure with various elastic components). Various types of SAWs were studied in this framework including in the presence of dispersion, nonlinearity, viscosity (then with a virtual dragging force and non-conservation of energy), surface energy, surface inertia, and coupling with electromagnetic fields.

14.5 Conclusion

There is no doubt that the inception of Eshelbian continuum mechanics and its many applications of both theoretical and practical interest constitute one of the last great and original advances in continuum thermo-mechanics at the time of writing of this book. This is shared by the development of various techniques of homogenisation. It happens that both relates to the notion of homogeneity, or its negation, inhomogeneity. This is not mere coincidence. It is just because homogeneity was the last of the three basic tenets of 19th century physics -linearity, isotropy, and homogeneity—that were finally successively rejected due to physical necessity and timely applications. However, whereas homogenisation techniques are more akin to methods of applied mathematics (without altering basic principles and equations), the subject matter of this chapter is more related to the foundations of physics and touches more closely its basic principles. This is reflected in the close relationship of the canonical conservation laws with Lie group theory via Noether's theorem or its generalizations to dissipative processes. Nonetheless, the practical side should not be overlooked, not only in the already mentioned applications but also in problems such as taking off material (in machining or wear) or adding material at a surface (e.g., by accretion). The first of these (machining) was first envisaged by Cherepanov (1987). Wear was studied by Dragon-Luiset (2001), and accretion received the attention of Gurtin (1999) and also of Russian scientists (see Chap. 11).

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Chapter 15
Relativistic Continuum Mechanics: A 20th Century Adventure

Abstract Relativity theory as understood by Einstein is a true Twentieth century development. After the introduction of the four-dimensional version of special relativity by Minkowski and that ofenergy-moment tensor, to which must be added the fact that general relativity is per se a continuum theory, there was need for a true relativistic theory of the continuum. The present chapter reports in a critical manner the progress made in this theory in two distinct periods, one extending before World War II, and the second in the rough time interval 1950−1980, when solutions were finally proposed in an inclusive way. The first period dealt with attempts at discussing the ad hoc introduction of classical concepts in this new landscape. Thisincluded the notion of perfect fluids and a debated discussion of the possible generalization of the notion of rigid-body motion—without which the notion of elasticity could not be introduced. A breakthrough is represented by Eckart's introduction of a systematic covariantspace-and-time resolution of four-dimensional objects and of early elements of continuum thermodynamics. This, combined with the natural influence of the then new trends in classicalcontinuum mechanics (rationalization à la Truesdell), then led to a modern, more axiomatic, formulation that allowed a rational construct of relativistic elasticity, and its generalization to more complex thermomechanical schemes (including generalized continua) andelectromagnetic deformable bodies, a development in which the author has been more than a passive witness.

15.1 Historical Introit and the Need for a Relativistic Continuum Mechanics

The theory of relativity was practically born with the 20th century. Prepared by the discovery of characteristic space–time transformations by Fitzerald, Voigt and Lorentz and with a group structure uncovered later on by Henri Poincaré, the special theory of relativity was formulated in his *annus mirabilis* (1905) by

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Einstein. It was essentially a theory of fast motion of point particles with a velocity bounded by the velocity c of light in vacuum. Not only it views mass as varying with velocity, but it shows the equivalence of mass and energy, and its realm is the *electrodynamics of moving bodies*. Replacing for large velocities Newton's theory and the allied Galilean invariance, it puts the mechanical motion and Maxwell's equations under an unique invariance umbrella governed by the Lorentz-Poincaré transformations of space–time. The lack of symmetry noticed in Chap. 12 between electric and magnetic phenomena is then resolved. Minkowski (1908) introduced the powerful notion of four-dimensional space–time manifold M^4 providing thus an unsurpassed elegance to this theory. But this formulation, elegant as it was, was also pregnant of further developments. The idea emerged that the generalization to intense gravitational fields of Newton's potential of gravitation ϕ governed by the Laplace-Poisson equation (k = Newton gravitational constant; ρ= mass density).

$$\nabla^2\phi = 4\pi k\rho, \tag{15.1}$$

was to be found in the space–time varying metric $g_{\alpha\beta}$ (α, $\beta = 1,2,3,4$; index 4 timelike) of a four-dimensional space–time manifold V^4. The generalization of the second-order space operator ∇^2 had to be related to the *curvature* of space–time (that involves second-order space–time gradients of the metric). As to the mass density present in the right-hand side of (14.1), it had to be generalized in the notion of *energy–momentum* (space–time) *tensor* $T_{\alpha\beta}$ accounting for sources of mass and energy other than gravitational—e.g., electromagnetic, deformational, chemical. The genius of Einstein, helped by some friendly geometers such as Grossman, and perhaps almost simultaneously with David Hilbert and Emmy Noether, was to arrive in November 1916 at the space–time generalization of (15.1) in the celebrated form

$$A_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -\frac{8\pi k}{c^4}T_{\alpha\beta}, \tag{15.2}$$

where $R_{\alpha\beta}$ is the Ricci curvature and R is the scalar curvature deduced from the Riemannian curvature $R^{\alpha}_{.\beta\gamma\delta}$. Tensor $A_{\alpha\beta}$is now called the Einstein tensor. Fortunately, the geometrical path had been paved by Bernhard Riemann and the Italian mathematicians Gregorio Ricci-Cubrastro (1853–1925; for short, Ricci) and Tullio Levi–Civita (1873–1941), both also very active in the Italian community of mechanics, see Sect. 10.6. Solving (15.2) for $g_{\alpha\beta}$in terms of the sources of mass and energy contained in the space–time energy–momentum tensor $T_{\alpha\beta}$ obviously is a formidable task.

Remark 15.1 Several remarks are in order concerning the beautiful Eq. (15.2). First, from the formal point of view, we may consider Einstein's theory of general relativity as a generalized continuum theory of the *second gradient* of the space–time metric (cf. Chap. 13). Second, the Maxwell stress and other such electromagnetic stress tensors introduced in Chap. 12 are none other than purely spatial parts of the space–time energy–momentum tensor. So is the case of the Eshelby stress tensor introduced in Chap 14. Finally, while we could have some doubt with special relativity, it is clear that general relativity is a *continuum theory* from the start. This is more than enough to ponder the formulation of continuum thermomechanics in its framework. The ultimately good invariance of electromagnetic entities is another incentive for this endeavour. At least these are the reasons why the present author became much involved in these developments in the 1960s at the contact of the traditional Princetonian relativistic scientific community.

In the next section we will survey the pioneering period of time before WWII. In Sect. 15.3 we examine the inevitable influence of the new general vision of classical continuum mechanics created by some authors such as Truesdell and others on the essential formulation of the basic principles of relativistic continuum mechanics. Relativistic elasticity is paid special attention in Sect. 15.4 because of its intimate relationship with the definition of rigid-body motion and the need for a good invariant kinematic description of the continuum. Other advances are noted in Sect. 15.5. Finally, the possibility to formulate various theories of generalized continua in the relativistic framework is the object of the short Sect. 15.6. The given bibliography is generous.

15.2 The Early Years: 1908–1940

This period extends between Minkowski's proposal (1908) of a space–time four-dimensional representation of relativistic effects and WWII. It is marked by three essential developments. To describe these some inevitable reminder of notation is necessary (this is a more or less standard notation as used, for instance, in Misner et al. 1970, and Chap. 15 in the book of Eringen and Maugin 1990).

In special relativity the space–time metric is reduced to

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = diag(+1, +1, +1, -1), \alpha, \beta = 1, 2, 3, 4 \tag{15.3}$$

with

$$x^{\alpha}(\alpha = 1, 2, 3, 4) = \left\{x^{i}(i = 1, 2, 3), x^{4} timelike\right\}. \tag{15.4}$$

In general relativity, the space–time metric $g_{\alpha\beta}$of signature $(+, +, +, -)$ is reducible to Minkowski's metric at any space–time point (as if we got rid of gravitation). In some kind of Eulerian view motion is represented by the world velocity of space–time component u^{α} normalized in such a way that

$$g_{\alpha\beta} u^{\alpha} u^{\beta} + c^{2} = 0 \tag{15.5}$$

where c is the velocity of light in vacuum. The four vector u^{α}/c is the unit tangent to a "particle" space–time trajectory C_X. This "particle" (noted X for reasons to become clear later on) is equipped with its proper time τ.

A. ***Perfect fluids***

The first development of relativistic continuum mechanics could consider only the simplest of material behaviours, that of a perfect fluid. That is why the fourth velocity is the only required kinematical ingredient. Such a perfect fluid here is completely characterized by its rest-frame energy density, that we note ϖ and its (thermodynamic) pressure p. The conservation of energy and momentum is written in special relativity as

$$\partial_\beta T^{\beta\alpha} = 0, \partial_\beta = \partial/\partial x^\beta. \tag{15.6}$$

The energy–momentum tensor of this perfect fluid is accordingly written as

$$T^{\alpha\beta} = (\varpi + p)u^\alpha u^\beta + p\eta^{\alpha\beta}. \tag{15.7}$$

The relationship $f(\varpi, p) = 0$ is *the law of state* of this fluid. In spite of its multiple uses, this case is too simple to help us make some progress in the bases of relativistic continuum mechanics.

B. ***The energy–momentum tensor of electromagnetic fields***

Because of its intimate relationship with the formulation of Maxwell's equations, the form of the energy–momentum tensor for electromagnetic fields in moving magnetized and electrically polarized matter received due attention in the years following Minkowski's proposal. In the absence of consideration of a true model of interactions between electromagnetic fields and matter (although the basic model of Lorentz—see Chap. 12—was available), proposals for such a tensor were done on the basis of specious arguments: a priori expression for the field energy, and the associated momentum and energy flux, imposed symmetry. To illustrate this point we recall the expression of the purely spatial part of this energy–momentum tensor according to the main actors of the play in the period 1908–1910 (with the notation of Chap. 12; remember that the divergence of tensors has to be taken on the first index):

Minkowski (1908):

$$\mathbf{t}_M^{em} = \mathbf{D} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}(\mathbf{E}.\mathbf{D} + \mathbf{H}.\mathbf{B})\mathbf{1}; \tag{15.8}$$

Einstein and Laub (1908):

$$\mathbf{t}_{EL}^{em} = \mathbf{D} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\mathbf{1}; \tag{15.9}$$

Abraham (1909–1910) (T stands for transposition):

$$\mathbf{t}_A = \frac{1}{2}\left(\mathbf{t}_{EL}^{em} + \left(\mathbf{t}_{EL}^{em}\right)^T\right). \tag{15.10}$$

The electromagnetic energy present in the isotropic part of (15.8) smells good the undue influence from a purely linear theory of constitutive equations (It is as if in general continuum mechanics we always wrote the deformation energy as half the inner product of stress and deformation, what is true only in linear elasticity!). The symmetrization effected in the definition is not necessary in general. Equation (15.9) could be a good candidate but in fact too much symmetric in electric and magnetic effects. This is to be compared to the expression deduced from a microscopic modelling as the following expression (see Chap. 12 above):

Eringen-Grot-Maugin-Collet-Tiersten (1966–1973):

$$\mathbf{t}^{em} = \mathbf{D} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{H} - \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2 - 2\mathbf{M}.\mathbf{B}\right)\mathbf{1}. \tag{15.11}$$

It is only in the late 1960s that the corresponding four-dimensional relativistic energy–momentum tensor was deduced by de Groot and Suttorp (see these authors 1972) and exploited by Grot and Eringen (1966a, b); Maugin (1971, 1978) and Eringen and Maugin (1990, Chap. 15).

C. ***The problem of the definition of rigid-body motions***

In order to progress further one had to envisage a mechanical behaviour more complex than pure fluidity, for instance, elasticity. Elasticity is meaningful only if defined with respect to a standard (a comparison medium) that is normally rigid-body motion. But here there is a caveat. The inherent limitation to physical speeds imposed by relativity makes it inadmissible to have a physical object of arbitrary large dimension in rigid-body rotation. Still this was the object of intense discussions in the early 1910s and it was shown by several authors (Born 1911; Herglotz 1911; Noether 1911; Ignatoswky 1911) that the condition of rigid-body motion could be defined locally by (this is Killing's theorem in geometry)

$$\partial_\alpha u_\beta + \partial_\beta u_\alpha = 0, \tag{15.12}$$

in a local spatial section of space–time. Attempts at a relativistic theory of elasticity were then given by various authors (e.g., de Donder and Dupont 1932–1933). Then, probably because of the absence of applications of interest, the subject laid dormant for some time, waiting for a simultaneous revival of the basic studies of nonrelativistic continuum mechanics. Most relativists were not equipped for such an approach.

D. ***Space and time resolution of four-dimensional objects***

However, a remarkable work was produced in Eckart 1940 by Carl H. Eckart. Although a quantum physicist by formation and a future successful contributor to problems of geophysics, he gave this opus as some kind of interlude, but a decisive one in the evolution of both classical and relativistic continuum *thermo*-mechanics. This he achieved in a real pearl in a completely covariant format by using systematically the resolution of any space–time object into "proper" components, in particular for the energy–momentum tensor. For this one has to introduce the spatial projector $P_{\alpha\beta}$ such that

$$P_{\alpha\beta} := g_{\alpha\beta} + c^{-2}u_\alpha u_\beta = P_{\beta\alpha}, u_\alpha P^\alpha_{.\beta} = 0, P^\alpha_{.\beta}P^\beta_{.\gamma} = P^\alpha_{..\gamma}. \tag{15.13}$$

These are, respectively, a definition, an orthogonality property, and the condition of idempotence.

A general energy–momentum tensor then decomposes canonically as (in our notation)

$$T^{\alpha\beta} = c^{-2}\omega u^\alpha u^\beta + c^{-2}u^\alpha q^\beta + p^\alpha u^\beta - t^{\alpha\beta} \tag{15.14}$$

where

$$\omega \equiv c^{-2}u_\alpha T^{\alpha\beta}u_\beta, q^\beta \equiv -u_\alpha T^{\alpha\gamma}P^\beta_{.\gamma}, p^\alpha \equiv -c^{-2}P^\alpha_{.\gamma}T^{\gamma\beta}u_\beta, t^{\beta\alpha} \equiv -P^\beta_{.\gamma}P^\alpha_{.\delta}T^{\delta\gamma}. \tag{15.15}$$

These four elements are but "spatial" covariant forms of the energy density, energy (heat) flux, momentum density, and (Cauchy) stress. Clearly, (15.7) is but a special form of (15.14) with $t^{\beta\alpha} = -pP^{\alpha\beta}$. Eckart implemented decompositions such as (15.14) for all tensorial and vectorial four-dimensional objects in presenting relativistic generalizations of the notion of viscous fluid, Fourier's heat conduction law, and Ohm's law. For heat conduction, he proposes the isotropic law

$$q^\alpha = -\kappa P^{\alpha\beta}\left(\frac{\partial\theta}{\partial x^\beta} + \frac{\theta}{c^2}Du_\beta\right), D := u^\alpha\partial_\alpha, \tag{15.16}$$

where the acceleration contribution is a purely relativistic effect due to the inertia of energy. Eckart seems to have been the first author to have written the two laws of thermodynamics in local covariant form.

15.3 The Influence of New Trends in Classical Continuum Mechanics

Of course most knowledgeable authors adopted Eckart's viewpoint and a systematic exploitation of the space and time resolution of space–time objects. Among these authors we count Ehlers (1961); Cattaneo (1962), Grot and Eringen (1966a); and Maugin (1975). But more than that, it was natural in the 1960s to try to frame relativistic continuum mechanics in the new framework provided by the revival and axiomatization of classical continuum thermo-mechanics due to Truesdell, Toupin, Noll, Rivlin, Eringen, etc. This was achieved mainly by Bressan (see his synthesis Bressan 1978); Schöpf (1964); Grot and Eringen (1966a); Lianis (1973a, b) and Maugin (1975), with extensions to the relativistic electrodynamics of deformable continua by the same authors (e.g., Bressan 1963; Grot and Eringen 1966b; Maugin 1971, 1972a, 1978b). The particular points that received special attention were: (1) the relativistic version of the principle of material-frame indifference (or objectivity); (2) the formulation of the second law of thermodynamics (generalization of the classical Clausius–Duhem inequality, and (3) the relativistic theory of elasticity, the first and third of these in view of the difficulty brought in by the relativistic notion of rigid-body motion.

The direction of the solution of the second point that was just emphasized was shown by Eckart. The first point was more difficult to deal with and several solutions were offered. J.G. Oldroyd, himself a brilliant contributor to the modern theory of the rheology of non-Newtonian fluids (see Chap. 3) proposed (Oldroyd 1970) to write relativistic constitutive equations in the convected frame (this is equivalent to writing them in the (space–time) invariant material framework). In a sense this was a relativistic replica of his 1950 landmark paper in classical rheology. Objectivity then is automatically satisfied (that is why we formulated all kinematic and physical variables of interest in this framework in our Doctoral dissertation—Habilitation—of 1975). Lianis (1973a, b) and also Maugin (1975) proposed to project the relevant variables and constitutive equations on a moving Fermi-Walker four-dimensional frame. This is an interesting proposal because such a frame is dragged along the material particle trajectory in space–time while rotating about it in a local spatial section (and thus practically reproducing the invariance under time-dependent rotation of an actual observer in classical objectivity). The third point (Elasticity) was the object of vivid discussions and thus deserves a section just by itself.

15.3.1 Relativistic Elasticity

The fever with which this matter was discussed in the 1960–1970s is somewhat surprising. It may be due to a new interest related to the "discovery" of solid-like stars and the (now ill-fated) search for the evidence of gravitational waves by means of an elastic detector (see, e.g., Maugin 1973, and the bibliography therein). Here we must unfortunately distinguish between physicists with no previous knowledge of modern continuum mechanics and who did their best, and mechanicians appropriately trained in both this new framework and mathematical physics. Most relevant papers are, in alphabetic order, by Bennoun (1964); Cattaneo and Gerardi (1975); Carter and Quintana (1972); Glass and Winicour (1972, 1973); Hernandez (1970); Maugin (1971, 1978a, d); Papapetrou (1972); Rayner (1963); Souriau (1958, also 1964), and Synge (1959).

The problem and solution reside in the choice of the proper geometrical object to describe the deformation in four-dimensional relativity (on this matter see Appendix I in Maugin 1978d). First we note that it would be difficult in the general relativistic framework (curved space–time) to introduce a priori the idea of displacement like in classical linear elasticity in Euclidean space. Next, in order to avoid a discussion about the notion of reference configuration, its could be a good idea to avoid this altogether by formulating a kind of *hypo-elasticity*, i.e., a linear relationship between the appropriately defined relativistic covariant time-rate (e.g., Lie derivative in following the relativistic motion) of the spatial covariant stress

($t^{\beta\alpha}$) and the covariant strain rate (half the quantity appearing in the left-hand side of Eq (15.12)—itself a Lie derivative. This was discussed by Maugin (1977) in his study of wave propagation. Some (physicists) were tempted to directly select the projector $P_{\alpha\beta}$ as the relevant "spatial metric" to represent the strain, making the internal energy density to depend on it. Unfortunately, if this "metric" obviously accounts for the effect of the space–time metric $g_{\alpha\beta}$ in general relativity, it does not account for the deformation itself, so that this reduces to nothing in special relativity. This is the case of Bennoun (1964). More properly, an excellent idea is to envisage the canonical differentiable projection of the relativistic motion x^α of a "particle" X onto a three-dimensional material manifold M^3 of coordinates $X^K, K = 1, 2, 3$ in a space–time parametrization (X^K, τ) where τ is the proper time of X in its motion along its space–time trajectory C_X. Thus we write

$$X^K = \bar{X}^K(x^\alpha), \tau = \bar{\tau}(x^\alpha). \tag{15.17}$$

This defines the whole relevant kinematics in a vision that we called the "inverse motion" view. In classical mechanics (15.17) reduces to the only equation $X^K = \bar{X}^K(x^i, t)$ where t is the absolute Newtonian time. Since $u^\alpha \nabla_\alpha \equiv D = \partial/\partial\tau$, we have

$$DX^K = u^\alpha X^K_{.,\alpha} = 0 \, and \, X^K_\alpha = X^K_{.,\alpha} = \frac{\partial \bar{X}^K}{\partial x^\alpha}. \tag{15.18}$$

The last quantity may be called the relativistic *inverse motion gradient*.

[*Remark* The condition $DX^K = 0$—independence of X^K and τ in the space–time parametrization—reads as

$$X^K_{.,4} = -\left(v^i/c\right) X^K_{.,i} \tag{15.19}$$

in an inertial frame. But this is none other than the relation $\mathbf{V} + \mathbf{F}^{-1}.\mathbf{v} = 0$ between "material" velocity $\mathbf{V}$ and "physical" velocity $\mathbf{v}$ of the "inverse" and "direct" descriptions of classical motion considered in configurational mechanics (cf Chap. 14) after Maugin and Trimarco (1992); $\mathbf{V}$ relates to the "Eshelby" (intrinsically material) format of the balance of momentum].

A natural relativistic measure of deformation is built from the inverse motion gradient as

$$\left(C^{-1}\right)^{KL} := g^{\alpha\beta} X^K_\alpha X^L_\beta = P^{\alpha\beta} X^K_\alpha X^L_\beta. \tag{15.20}$$

This is the relativistic version of the *Piola finite strain* of classical elasticity: It is the canonical projection of $P^{\alpha\beta}$ onto M^3 by the motion. General relativistic constitutive equations for elasticity based on this measure were considered by the author in 1969 (in his Pre-general seminar for the PhD at Princeton in the spring of 1969; published later on only in 1971). As a matter of fact it was recognised later that such a proposal went back to a paper—unknown to Maugin at the time—published by Souriau in his own formalism and in a journal of limited distribution (Souriau 1958, incorporated in Souriau 1964). [Jean-Marie Souriau (1922–2012) was an original character, a mathematician having taught first in North Africa and then in Marseille. His most fruitful works are in symplectic geometry]. Furthermore, this relativistic formulation of elasticity appears to be a relativistic generalization of a formulation given by Murnaghan in 1937 in a paper published in the American Journal of Mathematics.

The Born-Herglotz condition of rigid body motion then takes on the form (**C** is the inverse of $\mathbf{C}^{-1}$on M^3)

$$DC_{KL} = 0. \tag{15.21}$$

For further use it is possible to define a kind of relativistic "Eulerian" tensor of deformation by

$$E_{\alpha\beta} = \frac{1}{2}\left(P_{\alpha\beta} - G_{KL}X_\alpha^K X_\beta^L\right), \tag{15.22}$$

where $G_{KL}$is the background metric of M^3. This shows that $P_{\alpha\beta}$ itself is not sufficient to define the whole deformation in general relativity. Equivalently to (15.21) local rigid-body motion can be covariantly defined by the condition

$$L_u E_{\alpha\beta} = 0, \tag{15.23}$$

where L_u indicates the Lie derivative in following the four-velocity field.

Other formulations of relativistic elasticity are as follows. First, Hernandez (1970) took an original viewpoint in making use of the (3 + 1)—dimensional formalism of Arnowitt, Deser and Misner for gravitation studies instead of the above introduced space–time covariant decomposition (see Misner et al. 1970). This had no followers. Rayner (1963) essentially considers a kind of Hooke constitutive equation by taking the stress $t^{\beta\alpha}$ linear in $E_{\alpha\beta}$. With more precautions Carter and Quintana (1972)—and subsequent works—consider a general thermodynamically admissible elastic constitutive equation in terms of the strain tensor $E_{\alpha\beta}$ with a view to applications of high-pressure elasticity theory as may prevail in dense astrophysical objects. Such a theory is also proposed by Maugin (1978b) with extension to magnetoelasticity. Achilles Papapetrou (1907–1997)—a specialist of gravitation theory who had a rather mobile career in Greece, the UK, East Germany, and finally in Paris for a long period—has shown in 1972 that a good measure of *small* deformations could be given by a spatial tensor of the form

$$E_{ij} = \frac{1}{2}\left(h_{ij} + \xi_{i,j} + \xi_{j,i}\right), \tag{15.24}$$

where ξ_i are the components of a displacement and $h_{ij}$is the perturbation in the spatial part of the space–time metric due to some gravitational perturbation (e.g., incident wave on a gravitational wave detector). This can be deduced from (15.22) by performing an infinitesimal variation of all implied fields. To end with we note the remarkable work of Cattaneo and Gerardi (Cattaneo and Gerardi 1975) offering a solution by iterations of an elastic equilibrium of a body in general relativity.

15.4 Other Developments in the Period 1950–1980

Apart from elasticity most developments in relativistic continuum mechanics in the period 1950–1980 concerned fluid mechanics and magneto-hydrodynamics, with a specific interest in the propagation of discontinuity waves and applications to astrophysics and cosmology. Without entering any detailed description, we simply note the following works for the sake of completeness:

- waves in relativistic elasticity: Bressan (1963); Carter (1973); Maugin (1977, 1978e, 1979);
- singular hypersurfaces in relativistic continuum mechanics (some of these studies with the use of generalized functions): Bressan (1963); Taub (1957); Lichnerowicz (1955); O'Brien and Synge (1953); Maugin (1976a);
- Shock waves and other discontinuity waves: Taub (1948); Lichnerowicz (1967, 1971, 1976); Grot (1968); Maugin (1971, 1977, 1979, 1981);
- Variational principles in general relativistic continuum mechanics: Maugin (1971, 1972b).

15.5 Relativistic Generalized Continuum Mechanics

While dealing with symmetries for generalized continua (e.g., the Cosserat model) we have noticed that some invariances and group-theoretic arguments are in fact related to special types of space–time transformations. Therefore, a natural question is whether a four-dimensional formulation would not be advisable. However, to many engineers the introduction of the notion of generalized continua in the classical Newtonian framework is already a farfetched matter. But there is much more for the true aficionados: the 4D framework of relativistic continuum mechanics. This was expanded of necessity to incorporate the notion of spin in a generalized version of Einstein's theory of gravitation. This practically goes back to early developments in the 1930s. As an initial remark, we note that general relativity was from the start a continuum theory in Einstein's own view. Then original contributions dealing with a spin density in 1939 are due to Polish scientists. But for obvious reasons these works were published only after WWII (Weissenhoff and Raabe 1947). This was taken over by a group of active researchers around Louis de Broglie, in particular in the nice thesis of Halbwach (1960).

Kafadar and Eringen (1972) formulated a covariant theory of polar materials while A. Bressan (1978) in Italy considered strain-gradient effects. Maugin and Eringen (1972), gave a different approach proposing to generalize the notion of *triad* of rigid *directors* dear to Pierre Duhem, to that of *tetrad* in which the three mutually orthogonal spatial directors are complemented in a 4-tuple by the non-dimensionalized world velocity (which is orthogonal to the local spatial section).

They hinted at the relationship of the nonsymmetric energy–momentum with the notion of Einstein-Cartan space–time with torsion. F.W. Hehl, one of the rare students of Ekkehart Kroener when the latter was at Clausthal, also proposed—in analogy with the theory of dislocations—a variational formulation based on the notion of Einstein-Cartan space–time in his habilitation (Hehl 1969). This author became the internationally acknowledged specialist of this approach to gravitation theory (cf. Hehl et al. 1976). In the mean time, Maugin developed a consistent relativistic theory of spinning fluids (Maugin 1974, Maugin 1976a, b) along with a general approach in relativistic electrodynamics in Maugin (1978c). Here we must also mention the works of Herrmann et al. (2004) on an admissible thermodynamic framework. From then on, the field escaped mechanicians and became a fully fledged field of research in generalized gravitation studies.

Personal touch: In Paris in the period 1950–1980 the spirit of relativistic studies was kept alive mainly by André Lichnerowicz (1915–1998) at *Collège de France* (together with Yvonne Choquet-Bruhat at the University of Paris 6), and Olivier Costa de Beauregard (1911–2007) and Marie-Antoinette Tonnelat (1912–1980) at the *Institut Henri Poincaré*, then the Temple of French mathematical physics in the line of Louis de Broglie (1892–1987). Lichnerowicz was an internationally reputed geometer and inveterate pipe smoker, while Choquet-Bruhat was an analyst, the first woman to be elected to the French Academy of Sciences in Paris, something that Madame Curie could not obtain in spite of her two Nobel prizes! Tonnelat worked in the tradition of the (rather vain) unitary gravitational theory of Einstein, but she was also an excellent historian of science. Costa de Beauregard, from an old noble French family, was a deep thinker about the principles of both quantum and relativity theories. American readers will be surprised or happy to learn that a certain General de Beauregard from New Orleans was involved in the American Civil War on the confederate side. The present author had the chance to have André Lichnerowicz, Abraham H. Taub (1911–1999) from Berkeley, and Sybren R. de Groot from Amsterdam in his State Doctor of Science (1975)—equivalent to an Habilitation—committee in Mathematics in addition to Paul Germain and Maurice Roseau (see Chap. 7) for the aspects related to continuum mechanics. For his Ph.D at Princeton (1971) both John A. Wheeler and Martin Kruskal were on the Defence committee.

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Chapter 16
Epilogue

Abstract This concluding chapter first summarizes the historical developments exposed in a critical manner in all preceding chapters. It emphasizes the various nonlinear generalizations proposed in the Twentieth century as also the role played by remarkable schools and individuals in the fantastic progress reached in this period. This concerns more realistic material behaviors (accounting for micro-structures, involving coupled fields), a moreaxiomatic and thermodynamically justified approach, and a clear internationalization of engineering science. Simultaneously, progress in other collateral branches of sciences, both theoretical and experimental, has fostered a rapid, sometimes unexpected, progress in the-science of continuum mechanics. The latter has become more a mechanics of materials while developing tremendously its applicable side with performing numerical schemes andrequiring new developments in applied analysis and the interpretation in terms of advanced geometrical concepts. Final remarks points at the new marked interest of continuummechanics for living matter and the unavoidable relationship, both intellectually and numerically, between different scales of description, a trademark at the dawn of the Twenty first century.

16.1 What was Achieved

If we compare the main ideas and queries formulated at the dawn of the twentieth century—as recalled in Chap. 2—with the developments exposed in Chaps. 3–15, we witness a rather good fulfilment of the proposed programme.

First of all, the existing linear theories of elasticity and viscous fluids have been extended to a true and applicable theory of nonlinear elasticity—essentially for incompressible materials of the rubber-like type and more recently for bio-materials such as soft tissues—and non-Newtonian fluids (Chap. 3). The last case, because of its involvement of time, has necessitated a reflection that led to seriously accounting for the notion of objectivity in order to define properly invariant time derivatives.

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The perspicacious views and works of luminaries such as Rivlin, Oldroyd and others have been instrumental in this intellectual construct corroborated by appropriate experiments and fostered by socio-economical needs (industry of rubber and artificial textiles, paints, food industry, and all strongly viscous products).

In the linear theory of elasticity, two main ingredients have been introduced (Chap. 6). First anisotropy has established a better contact between classical continuum mechanics and the physics of materials, which is the realm of anisotropic crystals. But new materials are also anisotropic (e.g., fibre-reinforced materials). The second ingredient is the necessary consideration of the possible singularities of the elastic field. The way was paved by scientists such as A.E.H. Love. But there was a long way between the theoretical—kind of thought-experimental—notion of dislocation introduced by Vito Volterra and the real physical considerations on dislocations by G.I. Taylor and others. Similarly, the now obvious need to envisage the occurrence of cracks and their catastrophic expansion (in particular in aeronautics and nuclear-power industry) was dutifully answered by the formidable work achieved by the British school (Sneddon, Eshelby, Stroh, etc.) and then by teams both in the west (e.g., USA, France) and the east (Russia). The importance of some mathematical methods such as the application of the technique of complex variables by Kolosov and Muskhelishvili cannot be ignored in this context. This trend of research culminated in the theory of configurational forces (Chap. 14) with the seminal contributions of Eshelby, Cherepanov, Rice and others. Works of a more experimental nature and engineering-type such as those of Griffith and Irwin were of utmost importance in these developments.

Simultaneously, a necessary examination of the mathematical properties of the systems principally deduced from elasticity has led to a definite progress in the proof of theorems of uniqueness and existence. This is not gratuitous as there is no need to look for a classical solution (analytical or numerical) if we know in advance that the considered problem is ill-posed and a standard solution cannot exist. This progress was mostly based on the creation of a true applied functional analysis in the expert hands of mathematicians such as Sobolev, Leray, Schwartz, Lions, Magenes, etc., and more recently on its application in the UK, by Knops, and then John Ball in nonlinear elasticity (Chap. 6).

Of the three "real" mechanical behaviours mentioned in Sect. 2.2, friction and plasticity are certainly those which have commanded the largest number of works in the twentieth century. Plasticity and its application to the mechanics of structures made immense progress among those cultivating the ASME spirit, especially in Stanford and Brown (Chap. 4), but also in the UK with Rodney Hill and his disciples (Chap. 6), Poland (Chap. 8), and Russia with Ilyushin, Rabotnov and others (Chap. 11). We can say that, to the posthumous satisfaction of Pierre Duhem, elastoplasticity, but also allied theories of creep and damage (Odqvist, Hull, Kachanov, Rabotnov, Lemaitre) were given a good thermodynamic foundation thanks to the works of Hill, Mandel, Ziegler and the French school of continuum thermomechanics (Chap. 7). This definitely included one of Duhem's "nonsensical" branches of mechanics into a rational framework.

The attempts of Duhem to organize the “energetics” of many processes in a common frame were completed by the school of linear irreversible thermodynamics in Belgium and the Netherlands. But it is with Truesdell and his partners, Noll, Coleman, Toupin, that the whole field was re-organized in a more mathematical and axiomatized manner (Chap. 5). This fulfils one point in the prospective programme proposed by Hilbert. The offered thermodynamic formulation was audacious but with a certain efficacy. Amendments or generalizations (extended thermodynamics, introduction of internal variables of state, satisfaction of the causality of solutions) were advanced that much improved on the too much corseting original proposal. By the same token stringent conditions of invariance (e.g., objectivity) were duly enforced in continuum mechanics, probably under the influence of the flourishing of such principles in mathematical physics.

Another of Duhem’s “nonsensical” branches of mechanics was electromagnetism. With the pioneering works of Toupin, Mindlin, Eringen and others, this was successfully incorporated in modern continuum mechanics (Chap. 12). This provided a possibility to couple deformation and all types of electric and magnetic behaviours and to treat a large number of applications at the crossing point of mechanics, materials science, and electrical engineering with the same rationality as pure problems of continuum mechanics.

Poro-elasticity (in particular by Biot) and a theory of consolidation of soils allowed a fruitful co-operation between continuum mechanics and an emerging science of geo-materials to the benefit of civil-engineering applications. At the same time, thermo-elasticity, one of the first theories of coupled fields thanks to the pioneering work of Duhamel, developed tremendously to include couplings with electric fields such as in thermo-piezo-electricity, with many contributions from Japan and China. This will particularly apply to new microscopic electro-magneto-mechanical components known as MEMS.

Other thermo-mechanical couplings are those that necessarily play a fundamental role in phase transformations of deformable solids. The invaluable contributions of mechanicians of the continuum (from all over the world, but particularly from the USA, Russia, France, Germany, and Japan) to the mathematical description of such phenomena have brought this community in useful co-operational contact with the community of metallurgists and condensed-matter physics. No doubt that the way of approach and tools favoured by mechanicians—exploitation of balance laws, jump equations at moving discontinuities, considerations of configurational mechanics and variational formulations, mathematical refinements with special classes of functions—have permitted a rational but physically justified description that would have largely escaped the traditional tools of metallurgists. Works by applied mathematicians such as Jerald Ericksen, D.S. Kinderlehrer, John Ball and Richard D. James have been essential in such developments. Here the role played by the University of Minnesota should be underlined.

If we now recall the original works of Duhem and the Cosserats on elastic materials with a microstructure, then after a rest period of some 56 years, their original ideas developed into a real “industry” materializing in various paths to a truly generalized continuum mechanics as exposed in Chap. 13. Three essential lines have expanded, being represented by polar and micromorphic materials, materials described by higher order gradients of deformation, and the so-called non-local theory of materials. In all these we must acknowledge the leading role played by mechanicians such as Toupin, Mindlin, Eringen, Kröner, Kunin, Edelen, not to forget the German pioneers and their followers. What was again instrumental in the most recent developments of these research paths was a now obvious relation of such schemes of deformable matter with real materials, whether of natural origin (crystals of various types) or man-made new materials (composites, cellular materials, etc.). Of necessity this has led to considering representative length scales, and scale effects in general.

16.2 The Influence of New Experimental Equipments and Computational Means

What could not be guessed at the dawn of the twentieth century were "things to come", future developments in both experimental and numerical means that would often be the consequence of progress in small-scale physics, especially wave and quantum mechanics that revolutionized solid-state physics. As a matter of fact, mechanics in the early twentieth century is still based on (1) standard observational means (e.g., the naked eye and optical microscopes) and testing machines in a most elementary—entirely mechanical—form, and (2) the search for analytical solutions, if not of graphical ones by hand. It is at this gauge that we must appreciate the extraordinary achievements of some people in analytical solutions, often based on astute Ansatze that reflect a gifted capture of the physics and symmetries of the looked for solutions. Still, practically only "simple" academic problems could be solved (e.g., in elasto-plasticity where problems are free-boundary ones).

With progress in atomic physics and the applications of modern physics to electronics, experimental means progressed at giant steps (think of electronic microscopy, atomic-force microscopy, image processing, etc.) with a tremendous decrease in scale of observation, while new means of computations were created (electronic computers, miniaturization) with easy access by the common user only in the nineteen seventies. These two facts created a new situation in which large computations of complicated real structures made of materials with complex constitutive equations (think of plastic-forming in large deformations with visco-elasto-plastic constitutive equations) could be performed in finite strains. Scientists like Juan Simo in the USA and people around Erwin Stein in Germany, who combined an excellent knowledge of continuum thermo-dynamics, performing computational methods, and mathematical results, had the most efficient background to realize such wonderful computations. This also applies to large computations in the bio-mechanics of soft tissues such as the practically complete mechanical simulation of the human heart structure with its multi-layered envelope made of variously oriented fibres (see Humphrey 2002).

Personal touch: In the same way as they cannot remember Bakelite telephones and the desperate look for a telephone booth with the requirement to carry dimes in your pocket to get in touch with the phone operator—see old black-and-white mystery US movies of the 1950s, (so-called "*films noirs*" in the jargon of aficionados); also remember the inenarrable sequence in "Dr Strangelove" when the British officer tries to enter in contact with the White House -, young readers may have difficulties to imagine a time (1950–1960s) at which only electro-mechanical desk computers existed. These were essentially used to make boring astronomical calculations, or to help tracing a curve starting from a painstakingly obtained analytical solution. Just to illustrate this, the author recalls that he first did some computations of fluid mechanics on an analogue computer in 1965. He had his first experience on a cumbersome—but extremely weak—digital electronic computer doing only simple algebraic operations in 1966, with programming in machine language. It is only going to the United States that he met more powerful large computers but with programming in Fortran language. One had to bring a thick batch of prepared punched cards to the computer centre and collect the results on large printed output sheets one or two days later. Only finite-difference schemes were available to treat problems of fluid mechanics. The finite-element method was invented only in 1965–1966 to make large computations on aeronautical structures; it took some time to become a commonly used tool.

Another consequence of this drastic development in both experimental and computational means was the rapid transformation of part of the mechanics of structures into a real *mechanics of materials*, that is, the due consideration of the intimate structure of the material with its inherent inhomogeneity, multi-component contributions, and transformations. It is only with modern fast computations that the mathematical theory of homogenization could be applied, delivering the effective coefficients of the replacement material by solving a set of exemplary problems on the relevant basic cell. Simultaneously, the new experimental means produced the appropriate images and measurements to confirm the numerical simulations. From these emerged this new science of the *mechanics of materials*, the last avatar of continuum mechanics. This gives a rather good visual perspective of developments to come in a near future.

A particular point to be emphasized is that while continuum mechanics was for a long time reserved to the study of *inert matter*, this new mechanics of materials now dares to attack the landscape of *living matter* in the framework of biomechanics and mechano-biology for the study of growth, resorption, ageing, remodelling, and morphogenesis. If we already mentioned that non-linear elasticity was in some sense saved from oblivion by its useful applications in biomechanics (Chap. 3; many works by Odgen and Holzapfel), most recent developments in biomechanics involve all new ingredients and ideas introduced in thermo-mechanics within the last 50 years: multiplicative decomposition of the finite deformation gradient, theory of mixtures, notion of internal variables of thermodynamic state, higher-order gradient theory, configurational forces, non-linear waves, and homogenisation techniques, all to a high degree of sophistication [see, for instance, in alphabetic order, Ciarletta and Maugin (2011), Ciarletta et al. (2012), Cowin (1996), Cowin and Hegedus (1976), Epstein (2012; Chap. 7), Epstein and Maugin (2000), Ganghoffer (2005), Humphrey (2002), Maugin (2011; Chap 10), Porubov and Maugin (2011), Rodriguez et al. (1994), Taber (1995)].

16.3 Towards Interactions Between Scales

For a long time continuum mechanics benefited in its simplest form—Hooke's law with two Lamé coefficients—to the evaluation of the strength of large structures. By this we mean human scale and above. The main trait that clearly emerges from the above reminder is a complexification of the modelling allied to a focus on smaller scales with a neat tendency towards the crystal size, microstructure, and an approach to the discrete description. Already mentioned examples relate to the fields of dislocations and phase transformations. Thanks to the power of present-day computational tools, it is now possible to simulate the movement of a large ensemble of interacting dislocations, as also to relate meso- and macro-scopic mechanical responses to it. It is this mutual enrichment between scales that is most characteristic of the developments in the beginning of the twenty first century. The new *multi-scale techniques* involving matching between continuum (finite-elements) and atomistic computations vividly illustrate this tendency (see, e.g., Tadmor and Miller 2011).

Along a somewhat different line one may wonder if the exploitation of direct simulation techniques such as *molecular dynamics*—with an appropriate choice of interaction potentials (see Rapaport 1995)—does not relegate the very concept of continuum to the "dark ages" of phenomenological physics. But if this technique yields spectacular results in some cases (e.g., propagation of cracks and other local structural rearrangements) there is no obvious proof that this may replace the continuum approach—appropriately discretized—in the computed response of structural elements at any scale.

Another question is whether the development of *nano-mechanics* brings a true revolution in the field (see Bhushan 2007; Liu et al. 2006). Of course one has to account for scale effects, noticeably for the natural enhancement of surface effects. Surprisingly enough, many mechanical engineers approach this mechanics with a rather simple adaptation of tools used in macroscopic physics. Progress will necessarily be done in this field.

Also, we cannot avoid a return to geometrical concepts. No doubt that geometry is the basis on which the kinematics and deformation theory of continuum mechanics rely. Until recently only geometry as made analytical by René Descartes and considering the three-dimensional Euclidean space as the normal arena of continuum mechanics was acknowledged as the standard background. Differential geometry as formulated by nineteenth century mathematicians (above all Gauss and Riemann) was the tool that introduced the notions of metric and eventually curvature (and therefore, by negation, a good definition of flatness; think of the Navier-Saint–Venant equations of compatibility). Two facts have complicated the picture. One was the influence of the consideration of non-Euclidean spaces in gravitational theory following Einstein and others. The second, in fact related to the first, was the recognition that taking account of the presence of many structural defects requires abandoning the peculiarity of the Euclidean nature of material space in favour of more general concepts introduced by modern differential geometers such as Elie Cartan: non-Riemannian spaces and the allied incursion of group theoretical concepts. A fundamental question is raised for the future of such developments that have already reached an incredible level of sophistication which unfortunately drastically reduces the potential readership to a happy few while of course becoming extremely strict from a mathematical viewpoint. A similar problem appears in the geometric approach to unified theories of physics that is apprehended by a very few. In mechanics, this will require from a selected group of scientists an education of equivalently high level in modern differential geometry, materials science, and continuum mechanics. Some published books go in that direction; see Epstein 2010; Frankel 2004.

Personal touch. It was the idea of the author to initiate in 1997 a series of International seminars on the subject of *Geometry, Continua and Microstructure* with a view to gather informally geometers, mechanicians of the continuum, and materials scientists. Eight such seminars were held in various European countries since 1997.

As a final but trivial remark, like in all scientific fields, the second part of the twentieth century has witnessed an internationalisation in notation and research themes. All involved personnel now read the same scientific journals that have gained a true international public, and they practically all have access to a stupendous flux of information by electronic means. Although local scientific traditions and masters are still active, the "provincial" print that we highlighted in some chapters is fading away, giving a chance to all, even in remote or more recently scientifically developed regions, to participate in the marvellous adventure of science of which, obviously, continuum mechanics is only a very small part, but often one at the meeting point of various scientific disciplines.

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Appendix
Selected Biographies of Mechanicians

Here we call "mechanicians" those academics and researchers who worked in the field of applied and/or theoretical mechanics in departments such as those of Mechanical Engineering, Applied Mechanics, Theoretical Mechanics, Engineering sciences, Applied Mathematics, Aerospace and Aeronautics, etc.

Writing about Darwin in the *London Review of Books* of January 07, 2010, p. 5, the Harvard historian of sciences Steven Shapin says: "The very idea of paying homage to the great scientists of the past is problematic. Scientists are not widely supposed either to be heroes or to have heroes. Modern sensibilities insist on scientists' moral equivalence to anyone else, and notions of an impersonal Scientific Method, which have gained classical dominance over ideas of scientific genius, make the personalities of scientists irrelevant". Of course, scientists, if they do not see themselves as heroes, do have heroes. This was the case of all scientists I met, sometimes with the proud posting of their heroes' portraits on the walls of their office. Now the exercise of writing a short biographical note of contemporary scientists is even more perilous, in particular when speaking about living persons whose personality and ego are often quite strong. There is a prime difficulty in choosing the few words authorized by the exercise. I have been careful in this choice, avoiding any negative bias of the practice or personality of persons who are just human beings not devoid of such a trait. But the result is not exactly hagiography, and I tried to be as neutral as possible except in a few cases where my enthusiasm went much over my caution. All the people cited I have met or, for the older ones, they have had a strong influence on my own works or they left a definite print in my memory. I am obviously parsimonious with my own age class where the choice may seem arbitrary to many, especially to those who are not cited in a list that is necessarily limited. I have remedied this shortcoming to some extent by listing names of younger scientists who have been influenced or mentored by our elders. The list concerns uniquely people active in general continuum mechanics and solid mechanics, so that pure fluid mechanicians, excellent as they obviously are and so close to our community, are not cited unless

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they also contributed to general continuum mechanics and/or solid mechanics. Only a very few scientists born after 1950 are listed, and most of those cited in this class died unexpectedly in their fifties or sixties.

Achenbach Jan D. (born 1935): Dutch/American mechanical engineer. Ph.D. Stanford (1962). Spent most of his career at Northwestern University nearby Chicago. An internationally known and recognized specialist of wave propagation in solids and non-destructive evaluation techniques: waves and vibrations in viscoelastic solids, dynamics of composite materials, fracture, acoustic microscopy, quantitative ultrasonics, elasto-dynamics. Author of the classic book "Wave propagation in elastic solids" (1973). Founder and first editor of the influential WAVE MOTION journal. Has received many awards including the National Medal of Science of the USA. Mentored Andrew Norris, Yves Angel, and others.

Aifantis (Αϋφαντής) **Elias C**. (born 1950): Greek-American physicist educated in Greece and Minnesota, with a marked interest in diffusion processes and the motion of defects, in particular that of dislocations, and dissipative structures (under the influence of PRIGOGINE's group). One of the main exponents and developers of the theories of elasticity and plasticity with strain gradients. Professor at the Aristotle University of Thessaloniki and also in Michigan.

Ball John Macleod (**Sir**) (born 1948): British applied mathematician, FRS, Professor at the Mathematical Institute, Oxford University after B.Sc. at Cambridge, Ph.D. (1972) Sussex University with David Eric Edmunds, and a professorship of Applied Analysis at Herriot-Watt University in Edinburgh (1982–1996). Recipient of many honours. Specialist of large strain elasticity and the mathematics of materials with phase transitions. Author of fundamental theorems in the subject. President of the International Mathematical Union (2003–2006).

Barenblatt (БАРЕНВЛАТ) **Grigory I**. (born 1927): Russian mechanician-applied mathematician, Doctoral degree under A. N. Kolmogorov in Moscow, internationally recognized for his contributions to fracture mechanics (the Barenblatt-Dugdale model), the theory of fluid and gas flows in porous media, the mechanics of non-classical deformable solids, turbulence, self-similarity and intermediate asymptotics, nonlinear waves. Honored by many awards. Influenced, among others, Genady P. Cherepanov.

Bažant Zdeněk P. (born 1937): American civil engineer native of Czechoslovakia, formed in Prague. At Northwestern University in Evanston, USA, since 1969. A prolific author of many papers and books with a large number of co-authors. Supervised many Ph.D. Theses including the one of Gilles Pijaudier-Cabot from France. Both a theoretician and an experimentalist with works on the creep of concrete, the stability of structures, and above all scale effects in solid mechanics and a nonlocal theory of damage (with G. Pijaudier-Cabot). A much cited and honoured author in the field.

Berdichevsky (БЕРДИЧЕВСКИЙ) **Victor L**. (born 1944): Russian-American applied mathematician-mechanical engineer. Initially formed under the leadership of Academician Leonid D. Sedov in Moscow. After immigration to the USA, professor at Georgia Tech and then at Wayne State University in Michigan. Best known for his proposal of general variational principles, his works on dislocation theory, and the formulation of asymptotic homogenization for slender structures. Later interested in the foundations of thermodynamics and stochastic processes. Author of a remarkable book on variational principles in mechanics, physics and thermodynamics (Russian version, 1983; English much enlarged version, 2009).

Bingham Eugene C. (1878–1945): American scientist, who coined the term "rheology" together with Markus Reiner. Was a professor at, and Head of, the Department of chemistry at Lafayette College, Pennsylvania. Bingham viscoplastic fluids are named after him. Both a theoretician and experimentalist in rheology. Founded the *Society of Rheology* in 1929. Considered to be the father of the science of rheology.

Biot Maurice A. (1905–1985): Belgian-American physicist-geophysicist. Educated at the French speaking University of Louvain (Belgium; D.Sc. 1931), also Ph.D. at Caltech (1932). Worked at various American universities (Harvard, Columbia, Brown) and for a number of agencies and companies. An original but somewhat lonely researcher, he is famous for his theory of poro-elasticity (so-called "Biot theory"), but also for his various works on variational principles, the incremental theory of deformable solid mechanics, and irreversible thermodynamics.

Bowen Ray M. (born 1936): American mechanical engineer. Ph.D. Texas A&M University (1961). Taught at Rice University (1967–1983), University of Kentucky (1983–1989), and became President of Texas A&M University (1994–2002). Known for his theory of fluid mixtures in continuum mechanics. A gifted administrator, he was instrumental in expanding Texas A&M University.

Bridgman Percy W. (1882–1961): American physicist who studied at Harvard (AB, AM, PhD) and was a Professor of mathematics and natural philosophy there until his retirement. A specialist of high-pressure deformation and flow (plasticity), he was the one who experimentally proved that plastic behaviour is essentially due to slip (or shear) strain (for nonporous metals). He is the author of a famous book "The nature of thermodynamics" (1941) that poses correctly the interpretation of irreversible thermodynamics in continua, in fact proposing the consideration of internal variables of state. He received the Nobel prize in Physics for 1946, probably the only "mechanician", together with Lord Rayleigh (NP, 1904), to have been honoured by this prize.

Budiansky Bernard (1925–1999): American mechanician, Ph.D. Brown 1950, With NACA at Langley (1950–1955) and then Professor at Harvard University from 1955. Author of seminal contributions to the mechanics of solids and materials, and micromechanics. Influenced the whole school of mechanical engineering in the USA.

Bui Hui Dong (born 1937): French mechanician of Vietnamese origin, educated at Ecole Polytechnique, Paris, student of Jean Mandel. Best known for his extensive creative works in the theory of fracture. Research career spent at Electricité de France and Laboratoire de Mécanique des Solides, Ecole Polytechnique, first in Paris and then in Palaiseau. Member of the French Academy of Sciences (Paris).

Caratheodory (Καραθεοδωρή) **Constantin** (1873–1950): Born in Berlin of Greek parents. Educated in Belgium (Lycée, Engineering military school). Worked as a civil engineer in Egypt while educating himself in mathematical analysis. Completed his formal education in mathematics in Berlin and then Göttingen under the supervision of Herrmann MINKOWSKI. Published in 1909 a celebrated axiomatics of thermodynamics introducing the notion of thermodynamic adiabatic accessibility, a work acclaimed by Max Planck and Max Born. Professor (1908–1920) in Bonn, Hannover, Breslau, Göttingen, and Berlin. Then taught in Smyrna, Athens, Munich and finally Berlin until 1950. Published famous mathematical works in analysis with many theorems and conjectures bearing his name.

Cattaneo Carlo (1911–1979): Italian mathematical physicist. A student of Antonio Signorini. Main works in elasticity, thermoelasticity and relativistic continuum mechanics. The heat conduction equation called the Cattaneo-Vernotte equation yielding a finite velocity of heat disturbances is named after him and the French engineer Vernotte. Was a professor at the University or Rome (now Roma I—La Sapienza).

Cherepanov (ЧЕРЕПАНОВ) **Genady P**. (born 1937): Russian born American mechanician. Ph.D., Moscow, 1962, Dr of Sc. Moscow 1964 (the youngest ever in Mechanics in the former Soviet Union); a student of G. I. Barenblatt. Internationally known for his seminal work in the theory of deformation and fracture of materials and structures; one of the creators of configurational mechanics with the original introduction of invariant and path-independent integrals in fracture science. Immigrated to the USA in 1990 and taught at Florida International University before retirement.

Christov (Христов) **Christo I**. (1951–2012): Imaginative Bulgarian/American applied mathematician formed in Sofia and Novosibirsk (PhD with N. N. Yanenko in 1980). Successively at the Meteorological Institute in Sofia, scientific visitor in Madrid, Paris, Brussels, and Stanford before settling down in Lafayette, Louisiana, USA as a university professor of mathematics. Multiple interests in both fluid and solid mechanics with a special knack for numerical simulations. Developed analytical techniques in turbulence (random point approximation), introduced dissipative soliton, models of elastic crystals with high-degree dispersion (generalized Boussinesq systems), constitutive equations yielding hyperbolicity. Works in co-operation with K. Z. Markov, Manuel Velarde and G. A. Maugin.

Ciarlet Philippe G. (born 1938): French applied mathematician in the French "Lions" line. Educated at Ecole Polytechnique and School of Ponts & Chaussées (ENPC), Paris; Ph.D., Cleveland (1966). Renowned specialist of finite-element techniques and the mathematics of elasticity with an interest in plates and shells, and the so-called "zoom" technique (with Ph. Destuynder), allowing passing from 3d to 2d or 1d schemes for structural elements. Developed more recently an interest in differential geometry. Professor at the University of Paris 6 and then at the City University of Hong Kong, after retirement.

Coleman Bernard D. (born 1930): American chemical engineer turned "rational mechanician" at the contact of Clifford A. Truesdell and Walter Noll; Ph.D. Yale 1954. Author of most influential fundamental works in rational continuum thermomechanics and modern rheology (e.g., media with fading memory, the "Coleman-Noll" thermodynamics of continua, etc). Recently interested in biological structures such as DNA. First at Carnegie-Mellon in Pittsburg and then at Rutgers University in New Jersey.

Cosserat Eugène (1866–1931) and **Cosserat François** (1852–1914): Respectively, French mathematician-astronomer (Professor at the University of Toulouse) and French civil engineer ("Corps des Ponts et Chaussées"), brothers, authors of the celebrated book "Théorie des corps déformables" (1909) considered to be a pioneers' vision of generalized continua (introduction of couple stresses). "Cosserat media" and "Cosserat spectrum" (in 2d elasticity) are named after them. Among the first scientists to have introduced the notion of groups in continuum mechanics (see their "Euclidean action"), and thus much acclaimed by Elie Cartan, the famous geometer.

Coussy Olivier (1953–2010): French civil engineer (Dipl. Ing. ENPC, Paris, 1975), PhD 1978, DSc (Habilitation) 1985, both at the UPMC, Paris. From 1979 researcher at the Laboratoire Central des Ponts et Chaussées, Paris, while teaching mechanics in various « grandes écoles » (Polytechnique, ENPC). Head of the Navier Institute of civil engineering (2003). Author of original creative works in the thermo-mechanics of porous and chemically active continua; Biot Medal (2003). Has written a remarkable book on the "Mechanics of porous continua" (Wiley, 1995).

Drucker Daniel C. (1918–2001): American applied mechanician. Taught at Brown University, the University of Illinois, and the University of Florida. A foremost authority on the theory of plasticity (e.g., Drucker's postulate).

Duhem Pierre (1861–1916): Prolific French mathematician and historian of sciences, epistemologist, who pioneered many aspects of the rational mechanics and thermomechanics of continua (a precursor of the Truesdellian School). Considered as an "energetist" as opposed to "atomist". A friend of Henri Poincaré and Jacques Hadamard. Spent most of his career in Bordeaux, having definitively alienated himself from Parisian authorities after his justified but premature criticisms of the theories of Marcelin Berthelot (a "republican hero" of science, but also an excellent scientist).

Duvaut Georges (born 1934): French applied mathematician formed at Ecole Normale Supérieure (Paris) and University of Paris with Paul Germain as mentor. Co-author with J. -L. Lions of a landmark pioneering book on variational inequalities in mechanics and physics (1972); applications to plasticity and viscoplasticity. Works on periodic homogenization. Professor at the University of Paris 6 until retirement. For some time scientific director of O.N.E.R.A (Office National d'Etudes et de Recherches Aéronautiques).

Eckart Carl H. (1902–1973): American physicist. Ph.D. at Princeton (1925). Post-doctoral stay in Munich with Arnold Sommerfeld. Author of known works in quantum mechanics. Professor in Chicago (1928–1941) and then at the University of California in San Diego (1941–1971). Became involved in oceanography and underwater acoustics. Published in 1940 and 1948 a series of four papers that anticipated many developments in modern (classical and relativistic) continuum thermo-mechanics.

Edelen Dominic G. **B**. (1929–2010): American mathematician, Ph.D. Johns Hopkins 1956, Worked first as a researcher at the Rand Corporation, Santa Monica, and then taught mathematics at Lehigh University and mechanics at Texas A&M university. An original and powerful thinker with many works in general relativity, astrophysics, geometry, exterior calculus, the mathematical theory of defects, gauge theory, the nonlocal theory of elasticity, thermomechanics, and transformation methods for nonlinear partial differential equations. Author or co-author of many books in these fields.

Epstein Marcelo (born 1944): Canadian mechanician of Argentine origin and applied mathematician interested in both applications (structural members, biomechanics) and the abstract rigorous framework of continuum mechanics with a strong interest in modern differential geometry. Civil Engineer (Buenos Aires, 1967). Ph.D. at the Technion in Haifa (1972), and professor at the University of Calgary, Alberta, since 1976. Visited many research centres in the world. An excellent amateur musician, and an intellectual in the best sense interested in languages and humanities. Seminal works in co-operation with Marek Elzanowski, Manuel de Leon and Gerard A. Maugin (Differential geometry, Material inhomogeneities, Eshelby stress, configurational forces, theory of material growth).

Ericksen Jerald Laverne (born 1924): American mechanician/physicist. Was a professor at the Johns Hopkins University, Department of Mechanics (1957–1982)—after war service in the US Navy and Ph.D. at Bloomington (1951) and spending some time at the US. Naval Research Laboratory (NRL)—and then joined the University of Minnesota (1982–1999) before retirement. Made important contributions to the fields of mechanics and elasticity. He is best known for his work on anisotropic fluids and liquid crystals, plates and shells, solid crystals and their phase transitions viewed in thermomechanics. An original thinker; somewhat outside main chapels. Many results and objects bear his name—e.g., Rivlin-Ericksen tensors, Rivlin-Ericksen fluids, Baker-Ericksen inequalities, Doyle-Ericksen tensor, Ericksen identity, Leslie-Ericksen theory of liquid crystals. Influenced C. M. Dafermos, F. M. Leslie, R. C. Batra, R. D. James, M. Pitteri, and G. Zanzotto. One of the most influential mechanicians in the second part of the 20th century.

Eringen A. Cemal (1921–2009): Turkish born American engineering scientist. Founded the *Society of Engineering Sciences* (SES) and created the Journal "International Journal of Engineering Sciences". Internationally known for his many seminal contributions to various generalized continua (micropolar fluids and solids, micromorphic continua, media with microstretch, nonlocal theory of continua, media with chemical reactions, interactions with electromagnetism, etc). Ph.D. (1948) at the Brooklyn Polytechnic Institute under the supervision of N. J. Hoff. Professor at the Illinois Institute of Technology (1948–1953), then in Purdue (1953–1966), and finally at Princeton University until retirement. Mentored and/or influenced, among others, J. C. Samuels, S. L. Koh, R. C. Dixon, J. D. Ingram, J. W. Dunkin, Richard A. Grot, Charles B. Kafadar, Robert Twiss, W. D. Clauss Jr, T. S. Chang, James D. Lee, Hilmi Demiray, Gerard A. Maugin, Charles Speziale, Patrick O'Leary, Geneviève Segol, Leslie E. Hajdo, Naşit ARI, Byoung Song Kim, etc.

Eshelby John ("**Jock**") **Douglas** (1916–1981): British physicist educated in Bristol, worked in Cambridge, and taught in Sheffield (first as a reader and then as a professor in 1971), Faculty of (the theory of) materials. Best known for his original work on dislocation motion, the driving force on a material inhomogeneity and on a field singularity, the continuum theory of lattice defects, and the "Eshelby" inclusion problem. The material Eshelby stress tensor, the spatial part of the energy-momentum tensor, is named after him (coinage by G. A. Maugin and C. Trimarco, 1989–1992).

Föppl August (1854–1924): German physicist-civil engineer, Professor of Technical Mechanics and statical graphics at TU Munich (1893–1922). Interested in mathematical physics. Introduced Heaviside's Maxwell electrodynamics to Germany in 1894 in a book that influenced Albert Einstein. Arnold Sommerfeld highly valued him. Ludwig Prandtl was one of his first students. Influenced several generations of mechanicians in Germany through his books.

Germain Paul (1920–2009), French mathematician (ENS alumnus) with early successful works in various branches of theoretical fluid mechanics (transonic flows, flows around delta-wings, structure of shock waves in fluids and MHD; consideration of generalized functions and asymptotic methods in problems of fluid mechanics), introduced a curriculum in continuum mechanics that influenced the teaching of the matter in all institutions in France. Then turned to general continuum thermomechanics and various applications in dissipative solids. Revived the interest for the formulation using the principle of virtual power in the modelling of complex continua and structures. Has shown a remarkable open-mindedness towards various theories. Influenced, among others, Jean-Pierre Guiraud, Georges Duvaut, Patrick Muller, Francois Sidoroff, Monique Piau, Gérard A. Maugin, Alain Gérard, Raymonde Drouot, and Pierre Suquet. Professor at the Sorbonne and then University of Paris 6 (now Université Pierre et Marie Curie) and the celebrated Ecole Polytechnique. One of the founders of the "Journal de Mécanique (Paris)" that was to become the "European Journal of Mechanics A/B". He also created the *Laboratoire de Mécanique Théorique* in association with CNRS (1975), to become the *Laboratoire de Modélisation en Mécanique* (1985), and then integrated in the *Institut Jean Le Rond d'Alembert* by G. A. Maugin (2007). Président of IUTAM (1984–1988). Member of main National Academies of Sciences (Paris, USA, USSR, Poland, Royal Society, Lincei, Pontifical Academy).

Gurtin Morton E. (born 1934): American mechanician-applied mathematician, Ph.D. Brown University (1961) with Eli Sternberg. Taught at Brown University and then Carnegie-Mellon. Author of seminal works in nonlinear continuum mechanics, thermomechanics of continua, dynamical phase transitions, evolving phase boundaries, configurational forces, dislocations and plasticity. Influenced, among others, Ian Murdoch, Paolo Podio-Guidugli and Eliot Fried.

Green Albert E. (1912–1999): English applied mathematician. Ph.D. (1937) in Cambridge under Sir Geoffrey I. Taylor, Professor at Oxford University (1968–1977), FRS, Numerous works in linear and nonlinear elasticity, important contributions to continuum mechanics including generalized continuum mechanics (multipolar theory, thermoelasticity, theory of shells and rods, elastic-plastic behavior, etc). Fruitful co-operation with Ronald S. Rivlin and Paul M. Naghdi.

Grioli, **Giuseppe** (born 1912, reached a hundred in the spring of 2012): Italian mathematician, Long time Professor of Mathematics (Rational mechanics) at the University of Padova. A follower of Antonio Signorini. Specialist of mathematical problems in elasticity and media with couple stresses.

Hamel Georg (1887–1954): German mechanician-applied mathematician, professor in Berlin. Proposed an axiomatization of mechanics and formed many German mechanicians through his influential books.

Hellinger Ernst (1883–1950): German mathematician. Author of a noted (Felix Klein) Encyclopedia article on continuum mechanics (1914) and another article with O. Toeplitz on analysis. Also known in mechanics for the Hellinger-Reissner (two-field, displacement and stress) variational principle. Educated in Heidelberg, Breslau and Göttingen with David Hilbert. He was professor in Frankfurt am Main but left for the USA in 1939 and then taught at Evanston. Most works in integral and spectral theories.

Herrmann George (1921–2007): Swiss/American mechanical engineer with Russian as one of his native tongues (was born in Moscow which he left in his teens). Educated at ETH Zurich; Doctoral degree in 1949 with William Prager. Left Switzerland for North America in 1949, first in Montreal, then at Columbia University, New York, and Northwestern University, Evanston. Professor of

Mechanical engineering at Stanford (1970–1984). Created the journal "International Journal of Solids and Structures" and was editor of the English translation of P.M.M. Works in shell theory, stability of structures, vibrations of elastic bodies, wave propagation, fracture, and the theory of material forces (configurational mechanics).

Hetnarsky Richard B. (born 1928): Polish/American applied mathematician. Fundamental contributions to problems of thermoelasticity. Created the "Journal of Thermal Stresses" and founded a series of international conferences under the title of "Thermal stresses". Author of encyclopaedic books on thermoelasticity. In the USA was a professor at Rochester, New York State, before retirement.

Hill Rodney M. (1921–2011): English applied mathematician. Education in mathematics at Cambridge, Ph.D. 1949, and research career in Sheffield, Nottingham (1953–1963) and at Cambridge (1963 on). One of the main contributors to the modern theory of elastoplasticity and the theory of homogenization of solids. Precocious author of a remarkable book on plasticity (Oxford, 1950). He was the founding editor of the "Journal of the Mechanics and Physics of Solids" in 1952.

Hutchinson John W. (born 1939): American mechanical engineer, BS Lehigh 1960, Ph.D. Harvard 1963 with Bernard Budiansky; Professor at Harvard, author of seminal works in solid and fracture mechanics, and elasto-plasticity.

Hutter Kolumban (born 1941): Swiss theoretical mechanician, Dipl. Civil Engineer Zurich (1964), Ph.D. Cornell (1973, with Y. -H. Pao). Habilitation in Vienna with Heinz Parkus. Worked first at the Hydraulics, Hydrology and Glaciology Research Laboratory of ETH Zurich, and then as a Professor of Mechanics at TU Darmstadt (1987–2006). Retired in Zurich. Prolific author of papers and books, with a marked interest in geophysical mechanics with applications in the dynamics of glaciers and ice sheets, the mechanics of granular materials, avalanching flows of snow, debris and mud, physical limnology, but also in the foundations of continuum mechanics and thermodynamics, and even electrodynamics of continua. Founder and first Editor-in-Chief of "Continuum Mechanics and Thermodynamics". Max Planck Prize (1994), Alexander von Humboldt Prize (1998). Recognized as one of the most creative and successful applicants of modern continuum mechanics to glaciology.

Ilyushin (ИЛЬЮШИН) **A. A**. (1911–1995): Russian mechanical engineer. Author of fundamental works in elasto-plasticity. Rector of the University of St Petersburg (then Leningrad) after WWII and then Professor of continuum mechanics at the Lomonosov State University of Moscow, Chair of elasticity. Introduced Ilyushin's principle in plasticity.

Kachanov (КАЧАНОВ) **Lazar M**. (1914–1993): Russian mechanician at Leningrad/St Petersburg, noted for his pioneer's works in the theory of damage and creep (1958, 1961), works in viscoelasticity and rate-dependent plasticity.

Kestin Joseph (1913–1993): Polish-American thermodynamicist in the UK and then the USA, Brown University. The most knowledgeable specialist on all aspects of thermodynamics. Contributed fundamentally to the modern vision of the thermodynamics with internal state variables (one of the possible avenues to the description of many dissipative processes).

Knops Robin J. (born 1932): British applied mathematician (B.Sc. Nottingham, 1955; Ph.D. with Rodney HILL, 1960). Then visitor to the USA (Brown), lecturer and reader in Newcastle (1962–1971), and finally Professor of Mathematics and Head of Department at Heriot-Watt University in Edinburgh until his retirement. Both an efficient organizer and a highly productive applied mathematician with many works on the mathematics of elasticity (uniqueness theorems, ill-posed problems, stability, Saint-Venant's principle). Many works co-authored with L. E. Payne from Cornell University.

Knowles James K. (1931–2009): American mechanical engineer, studied with Eric Reissner at MIT. Professor at Caltech since 1965. Many seminal works in elasticity and phase-transition problems in solids with Eli Sternberg, Cornelius O. Horgan and Rohan Abeyaratne.

Koiter Warner Tjardus (1914–1997): Influential Dutch mechanical engineer. Landmark Ph.D. Thesis Delft (1945) on the "Stability of elastic equilibrium" acknowledged internationally after its English translation by NACA in 1960. Works on the asymptotic theory of initial post-buckling stability, the theory of shells, plasticity. Professor of Applied Mechanics at Delft Technical University (1949–1979).

Kröner Ekkehart (1919–2000): German mathematical physicist who studied Physics in Stuttgart in 1948–1954 after the Second World War where he was a long time prisoner of war in the Soviet Union. Professor in Clausthal and then at the university of Stuttgart. A deep thinker and pioneer in the geometric approach to defective crystals introducing there notions such as the incompatibility tensor and the Einstein tensor. Definite works in elasto-plasticity of crystals (multiplicative decomposition of deformation gradient), materials with stochastic properties, homogenization techniques. Mentored K. H. Anthony (in defect theory) and F. W. Hehl (in modern gravitation theory). Influenced many others, including W. Noll, I. A. Kunin, M. Berveiller, and G. A. Maugin.

Kruskal Martin D. (1925–2006): American applied mathematician, not exactly a mechanician, but with so many fields of interest. Formed at the University of Chicago and then in New York (NYU, Courant Institute: Ph.D. 1952). Worked on the Matterhorn project and controlled thermonuclear fusion. Internationally known for his seminal work on plasma instabilities and on soliton theory (he coined the word; with co-workers he introduced the inverse-scattering method in this field) and asymptotic methods; also, "Kruskal coordinates" in general relativity in the study of black holes. Long time professor of astrophysics and applied mathematics at Princeton University (1951–1989) and then at Rutgers University. National Medal of Science of the USA.

Kunin (КУНИН) **Isaac A**. (born 1928): Russian/American scientist, Ph.D. 1958 Polytechnical Institute Leningrad. At Novosibirsk (1952–1979) and then Professor at the University of Houston (1979–2003). All round physicist and mechanician with works in dislocation theory, nonlocal theory of continua, media with microstructure, mathematical physics, dynamical systems. Author of a famous book "Media with Microstructure" in two volumes (English translation, 1982, edited by E. Kröner).

Lee Erastus H. (1916–2006): English-American mechanical engineer. Education at Cambridge University, UK, and Ph.D. at Stanford (1940) with S. Timoshenko. Spent WWII in the UK. Moved definitively to the USA in 1948. Taught at Brown University (1948–1962) and then Stanford (1962–1982), and finally moved to Rensselaer Polytechnic for ten years. Contributions to plasticity of metals, viscoelasticity and plastic wave propagation. Is often attributed the multiplicative decomposition of the finite total deformation gradient in elasto-plasticity.

Lemaitre Jean (born 1934): French mechanical engineer with D.Sc. from the University of Paris. Became professor of mechanics at this University while creating the *Laboratoire de Mécanique et Technologie* at the *Ecole Normale Supérieure de Cachan* (suburb of Paris) after applied research on fatigue and viscoelasticity at O.N.E.RA. One of the main contributors to the continuum theory of damage in a thermomechanical framework basing on original ideas of Kachanov and Rabotnov. Co-author with Jean-Louis Chaboche of a pioneering book on damage mechanics (1985).

Leslie Frank M. (1935–2000): British (Scottish) applied mathematician, educated in Dundee and Manchester (Ph.D. 1961 with James Lighthill), FRS, Professor in Newcastle and then in Glasgow at Strathclyde University. Especially known for his theory of dissipative liquid crystals (1968, with Jerald L. Ericksen).

Mandel Jean (1907–1982): French engineer ("Corps des Mines"), professor of mechanics at the celebrated Ecole Polytechnique, founder of a true school of research in solid and soil mechanics; most influential in introducing thermomechanics in France in the 1960–1970s, with applications to anelastic solids. Influenced, among others, Hui D. Bui, Nguyen Quoc Son, Joseph Zarka, Claude Stolz, and Bernard Halphen.

Marsden Jerald E. (1942–2010): Canadian applied mathematician. A prolific author of books and papers in mechanics. B.Sc. Toronto, Ph.D. Princeton (1968). A world leading authority in mathematical and theoretical classical mechanics with a marked interest in differential geometry, symplectic topology, chaos, etc. Professor at Caltech. Max Planck Research Award (2000), Foreign member of the Royal Society of London. Main co-authors: A. E. Fischer, A. Weinstein, T. S. Ratiu, P. J. Holmes, J. C. Simo, T. J. R. Hughes, A. J. Chorin, etc. One of the most highly cited scientists in the field.

Maugin Gerard A. (born 1944): French mechanical-aeronautical engineer with an American education (Ph.D. Princeton, 1971) and a marked interest in mathematical physics; D.Sc. in Mathematics, Paris, 1975. Successive works in relativistic continuum mechanics, foundations of the electrodynamics of continua, the mechanics of ferroic states (ferromagnetism and ferroelectricity), nonlinear waves in lattices and continuum models of solids, such as shock waves and solitons, surface waves on structures, configurational mechanics of defects, growth of biological tissues, and dynamic materials.

Mindlin Raymond D. (1906–1978): American mechanical engineer, educated and then Professor at Columbia University, New York, where he mentored many students, among them Y. -H. Pao, Harry F. Tiersten, P. C. Y. Lee, and Raymond Parnes. Internationally recognized scientist for his works in structural mechanics, photo-mechanics, vibrations of plates, piezoelectricity and its dynamic applications to signal processing (cf. the celebrated US Army monograph on the vibrations of plates), and the mechanics of continua with microstructure including granular materials.

Moreau Jean-Jacques (born 1924): French mathematician-mechanician, Educated at Ecole Normale Supérieure, Paris; Thesis in Mathematics, Université de Paris, (1949). Professor at University of Montpellier, France. An analyst, with first works in hydrodynamics and theoretical fluid mechanics, and then in convex analysis, and a strong interest in numerical simulations for problems with unilateral constraints for which he developed special algorithms. Was instrumental in introducing convex analysis in problems of solid mechanics (friction, plasticity, viscoplasticity, flow of granular materials) in the 1960–1980s. Influenced Bernard Nayroles, Michel Fremond, Michel Jean, Pierre Suquet and many others.

Müller Ingo (born 1936): German thermodynamicist, a student of J. Meixner in Aachen. Taught at Johns Hopkins University and the Technical University of Berlin. Best known for his exploitation of the notion of coldness, and as founder of rational extended thermodynamics. Co-created the Journal “Continuum Mechanics and Thermodynamics”.

Muschik Wolfgang (born 1936): German mathematical physicist educated and Professor at the Technical University of Berlin. A disciple of Walter Schottky (thermodynamics of discrete systems). Author of critical studies of the bases of thermodynamics. Main author of the theory of mesoscopic continuum mechanics with applications to liquid crystals. Together with Joseph Kestin one of the best analysts of the science of thermodynamics.

Naghdi Paul Mansour (1924–1994): Iranian born American mechanical engineer. Came to the USA in 1945. B.Sc. Cornell, Ph.D. University of Michigan 1951. Joined Berkeley in 1958. Professor of mechanical engineering at the University of Berkeley for some thirty years. Long time co-operation with A. E. Green. A specialist of elasto-plasticity, polar materials using the director theory (Cosserat surfaces), the theory of plates and shells, he influenced, among others, James Casey, Miles B. Rubin, Marcel J. Crochet, and A. R. Srinivasa.

Nelson Donald F. (born 1929): American physicist. A co-worker of C. H. Townes in the development of LASERS at Bell Telephone Laboratories; co-developer of the first continuously operating ruby laser (1961). Co-author (1970s) with Melvin LAX of definite works on piezoelectric-pyroelectric crystals in the spirit of modern continuum mechanics.

Noll Walter (born 1925): German/American scientist who, with Clifford A. Truesdell and Bernard D. Coleman, formulated the bases of the modern thermomechanics of continua. Also author of a famous encyclopaedia article (with C. A. Truesdell in 1965) and a theory of uniformity of materials that influenced some later works by C. C. Wang, M. Epstein and G. A. Maugin. Formed in Berlin, Paris and Bloomington, Indiana. Professor at Carnegie Mellon, Pittsburgh.

Nowacki Witold (1911–1996): Polish engineer-mathematician, who, after WWII, contributed to the creation of a successful Polish school of continuum mechanics working in elasticity, thermoelasticity, structural mechanics, Cosserat solids (asymmetric elasticity), plasticity, and electroelasticity.

Nguyen Quoc Son (born 1944), French cvil engineer of Vietnamese origin, a student of Jean Mandel at Ecole Polytechnique in Paris. Seminal contributions to fracture mechanics, the thermomechanics of continua, modelling and numerics of elasto-plasticity, and the stability of continua.

Ogden Ray W. (born 1943): English Applied mathematician, Education at Cambridge (BA, Ph.D. with Rodney Hill), FRS. Professor of mathematics at the University of Glasgow. Best known for his works in nonlinear elasticity with applications to elastomers and biological tissues.

Odqvist Folke K. G. (1899–1984): Swedish mechanical engineer, known for his work on creep and plasticity, 1934, (Odqvist parameter, now identified with the hardening parameter that is the past history of the magnitude of the plastic strain). Was Professor at the Royal Institute of Technology (K.T.H) in Stockholm (1936–1966).

Oldroyd James H. (1921–1982): British mathematician and noted rheologist. FRS. Educated at Cambridge University. Worked at Courtaulds Research Laboratory after WWII before teaching mathematics at Swansea (1953–1965) and then at the University of Liverpool (1965 until retirement). A rather parsimonious writer, he published in 1950 a landmark paper in theoretical rheology, introducing the celebrated Oldroyd model of visco-elasticity of a non-Newtonian fluid.

Pao, Yih-Hsing (born 1930): Chinese-American mechanical engineer, a student of Raymond D. Mindlin, Ph.D. Columbia 1959. Professor at Cornell University and then in Taiwan (from 1984) and Mainland China. Specialist of physical acoustics and wave propagation in solids. Among his Ph.D. students: Francis C. Moon (1967) and Kolumban Hutter (1973).

Parkus Heinz (1909–1982): Austrian mechanical–aeronautical engineer. Main works on helicopter mechanics, thermoelasticity and the electrodynamics of deformable solids. A long-time professor of mechanics at TU Wien after professional engineering experience in Austria and the USA. Mentored, among others, Franz Ziegler, his successor at TU Wien. Kolumban Hutter (Switzerland) took his Habilitation under his supervision.

Podio-Guidugli Paolo (born 1939): Italian civil engineer with a marked interest in applied mathematics. Educated in Pisa. Many works of high standards in continuum thermomechanics, often in cooperation, or in the line of, Morton Gurtin. Professor of Civil Engineering at University of Roma-II.

Prager William (1903–1980): German born American applied mathematician and mechanician. Educated at TU Darmstadt (Ph.D. 1926), he became the Director of the Institute of applied mathematics in Göttingen at the early age of 26. Then a professor at TU Karlsruhe. Left Germany in 1934 and first taught in Istanbul before immigrating to the USA and joining Brown University in 1941 to stay there until his retirement in 1973. He established there the Division of Applied Mathematics in 1946 and founded the Quarterly Journal of Applied Mathematics in 1943. He was the driving force behind the incredible success of mechanics at Brown. One of the prominent figures in the theory of plasticity.

Reiner Markus (1886–1976): Polish/Israeli civil engineer (TH Vienna) who coined the term “rheology” together with Eugene C. Bingham and co-created the Society of Rheology. Moved to Palestine after WWI. Became a Professor at the TECHNION, Haifa, after the independence of Israel. Bear his name: the Buckingham-Reiner Equation and the Reiner-Rivlin Equation. Introduced the Deborah number as measuring the characteristic relaxation time of flows of viscous fluids. For ever one of the creators of the science of rheology.

Reissner Eric (1913–1996): German born (the son of an eminent physicist working in general relativity and gravitation—cf. the celebrated Reissner-Nordström metric) American applied mathematician. Originally Educated at TU Berlin (Doctoral degree, 1935 in Applied Mechanics). Immigrated to the USA in 1937. Ph.D. at MIT (1938) where he conducted his research, becoming Professor of Mathematics there (1949–1969), and then at the university of California at San Diego (from 1969). Published more than 300 papers in scientific journals, many dealing with the elastic theory of beams, plates and shells (e.g., shear-deformation plate theory) that led to significant advances in civil and aeronautical engineering. Much professional recognition.

Rice James R. (born 1940): Education at Lehigh University, Ph.D. in applied mathematics 1964. Professor of Theoretical and Applied Mechanics at Brown University (1964–1981) and then Professor of Engineering Sciences and Geophysics at Harvard University (since 1981). One of the most creative, reputed and honoured American mechanician. Seminal works in theoretical mechanics, civil-environmental engineering and materials physics. Known for his works in crack propagation in elastic-plastic metals, path-independent integrals in elasticity (the celebrated J-integral of fracture), the structure of inelastic constitutive equations, microscopic mechanisms of cleavage and ductile or creep rupture, deformation localization into shear zones, landslides, with applications to geophysics, earthquake studies, fault systems in geology, etc.

Rivlin Ronald S. (1915–2005): British born, later American citizen, applied mathematician, Education at Cambridge University, Ph.D. 1952. First worked as a physicist for the British Rubber Producers Research Association, and then Professor of Applied Mathematics at Brown University (1953–1967), and Director of the Centre for the Application of Mathematics at Lehigh University (1967–1980). Developed the basic mathematical theory of large elastic deformations which became the foundation of the mechanics of rubber elasticity. Are named after him: the Reiner-Rivlin fluids, the Rivlin-Ericksen fluids, the Mooney-Rivlin energy formula for incompressible solids. Influenced a full generation of researchers in continuum mechanics.

Sanchez-Palencia Enrique/Evariste (born 1941): Spanish/French applied mathematician. Originally formed as an Aeronautical Engineer (Madrid), D.Sc. in Mathematics, Paris, 1969. One of the creators of the asymptotic technique of homogenization of periodic structures. Also mathematical works on magnetohydrodynamics and slendered elastic structures (plates, shells). Member of the French Academy of Sciences (Paris).

Schottky Walter H. (1886–1976): German physicist, Ph.D. Berlin 1912 under Max PLANCK. Taught at Jena, Würzburg and Rostock and then joined Siemens Research Laboratories until retirement. Best known for his works in quantum physics, thermodynamics, and above all semi-conductors. Book on Thermodynamics, Berlin 1929.

Sedov (СЕДОВ) **Leonid I**. (1907–1999): Leading and powerful Russian mechanician; Specialist of continuum mechanics, theoretical fluid mechanics (explosions, hydrodynamics, hydrofoils), solid mechanics, general principles of continuum physics, gravitational field, asymptotic and similarity methods. Developed also a genuine interest in variational formulations on basic principles. Author of classic textbooks on two-dimensional problems in fluid mechanics, similarity and dimensional analysis, and on general continuum mechanics in Russian with many influential translations. Mentored, among many, V. Z. Parton, Zhelnorovich, Victor L. Berdichevsky, and Lev M. Truskinovsky, etc. During WWII he devised the so-called *SedovSimilarity Solution* for a blast wave (also attributed to G. I. Taylor in the West). He was also the first chairman of the USSR Space Exploration program. President of the International Astronautical federation (1959–1961). Until recently, it had been thought that L. I. Sedov was the principal Soviet scientist behind the Sputnik project. He admitted to the author that he was just placed there as a figure head (“every great national project needs an official representative”). Nonetheless a true great scientist.

Sidoroff François (born 1943): French mechanical engineer with D.Sc. from Paris University (1976). One of the scientists much influenced by Paul Germain. A specialist of anelastic materials, large deformations and thermomechanics. Formed with Patrick Muller, Raymonde Drouot, Monique Piau and Gérard A. Maugin the initial group of continuum thermomechanics under the leadership of Paul Germain at Paris 6. Became a professor of mechanics at the Ecole Centrale de Lyon (Mechanical engineering) until his retirement.

Signorini Antonio (1888–1963): Italian mathematician, specialist of mathematical problems in elasticity (cf. the celebrated Signorini problem (1959) involving boundaries with unilateral contact and the first appearance of a variational inequality). Works in finite–strain elasticity (e.g., Signorini's perturbation method). Professor in Rome. Had an enduring influence on the Italian rational mechanics of continua and applied mathematics.

Simo Juan C. (1952–1994): Spanish mechanical engineer educated first in Madrid (B.Sc. M.Sc.) and then at Berkeley (M.Sc., 1980, Ph.D., 1982). A foremost authority on computational mechanics in finite strains. Rapidly gained an international recognition to become one of the highest cited and most influential scientists in the field. Professor in the Applied Mechanics Division at Stanford University from 1985 to his untimely death in 1994.

Soós Eugen (1937–2001): Highly productive Romanian applied mathematician. PhD 1972 with Caius JACOB in Bucarest. Worked as a Professor in the Department of Mathematics of the University of Bucarest and the Institute of Mathematics of the Romanian Academy of Sciences. Marked interest in many facets of continuum mechanics including anelasticity, the mechanics of composites, electromagnetism, the structure of mechanics, tensor and spinor algebra.

Spencer Anthony J. M. (1929–2008): English applied mathematician with Cambridge education, Ph.D. with Ian Sneddon, FRS. Author of fundamental works in the theory of invariants for anisotropic bodies, the elasticity and elastoplasticity of anisotropic bodies, and the mechanics of solids with inextensible fibres.

Stroh Alan Neil (1926–1962): Formed (B.Sc., M.Sc.) initially in his native South Africa, moved to the UK in 1951, and obtained his Ph.D. (1953) in Bristol under the supervision first of J. D. Eshelby and then of Sir Nevill Mott. Then spent one year at Cavendish Laboratory in Cambridge. Taught at Sheffield (1955–1958), and moved to M.I.T. (USA) in 1958. Killed in a car accident in 1962 while joining his new position on the West coast of the USA. A very original thinker with creative works in the dynamics of dislocations, cracks and plasticity. The inventor of the rightly celebrated "Stroh" formalism in anisotropic elasticity that greatly helps the formulation of boundary and transmission conditions.

Suquet Pierre (born 1954): French mathematician-mechanician, formed at Ecole Normale Supérieure, Paris, and the University of Paris 6 under the influence of Georges Duvaut and Paul Germain. Seminal works in mathematical plasticity (existence of solutions, functions with bounded variations), nonlinear homogenization, and others. Member of the French Academy of Sciences (Paris).

Szabò Istvan (1906–1980): Hungarian/German mechanician, influential Professor of Applied Mechanics at TU Berlin (1948–1973). Author of a celebrated History of the Principles of Mechanics (Birkhäuser, 1977). Walter Noll was his assistant in the early 1950s.

Tiersten Harry F. (1936–2006): American mechanical engineer, a student of Raymond D. Mindlin at Columbia. Worked at Bell Labs and then became Professor of Mechanical Engineering at the Rensselaer Polytechnic Institute. Author of many creative works on polar continua, linear piezoelectricity, and more generally coupled fields and the electrodynamics of deformable solids with applications to electro-mechanical devices and signal processing.

Toupin Richard A. (born 1926): Ph.D. Thesis at Syracuse with Melvin LAX. A co-worker of Jerald L. Ericksen and Clifford A. Truesdell who spent most of his career at IBM. Co-author of the celebrated Handbuch article on the classical theory of fields with C. A. Truesdell (1960). Also works in generalized continuum mechanics (gradient theory, couple stresses) and a pioneer in the study of nonlinear elastic electrically polarized materials. Acousto-elasticity (Bernstein-Toupin), and fundamental problems of continuum mechanics.

Truesdell Clifford A. (1919–2000): American applied mathematician and historian of science. Ph.D. Princeton, 1943. The most well known and active contributor to the renewal of continuum mechanics in the years 1940–1970, "godfather" of modern continuum thermomechanics. Co-author of celebrated Encyclopaedia articles (Handbuch der Physik). Created the influential "Journal of Rational Mechanics and Analysis", and then the "Archives of Rational Mechanics and Analysis". Taught at the University of Indiana in Bloomington and then Johns Hopkins University, Baltimore. Mentored W. Noll, R. A. Toupin, etc. Prolific author. Never tired editor of Euler's works.

Willis John R. (born 1937): English applied mathematician, B.Sc. (1961) and Ph.D. (1964) at the Imperial College, London. Professor of Applied Mathematics at Bath (1972–1994, 2000–2001), Professor of Theoretical Solid Mechanics at Cambridge University (1994–2000, 2001–2007), FRS. Editor of the "Journal of the Mechanics and Physics of Solids" (1982–2006). Best known for his numerous works in the mathematical investigation of problems arising mostly in the mechanics of solids, including the statics and dynamics of composite materials, dislocation theory, nonlinear fracture mechanics, elastodynamics of crack propagation, and ultrasonic nondestructive evaluation. Recipient of many honors.

Wilmanski Krzysztof (1940–2012): Internationally renowned Polish/German applied mathematician with an initial formation in Łodź (PhD 1965) and a Habilitation in Warsaw (1970). At the I.P.P.T. of the Polish Academy of Sciences (1966–1986) and then in various places in Germany (Berlin, Paderborn, Hamburg-Harburg, Essen, Weierstrass Institute in Berlin), and finally in Zielona-Gora in Poland.

Multiple scientific interests including early the axiomatics of thermodynamics and more recently poroelasticity in which he introduced new modellings accounting for finite strains, thermal effects, and tortuosity, and considered wave propagation and applications in geophysics. Works in thermomechanics in the line of the Trusdellian school and Ingo Müller.

Ziegler Hans (1910–1985): Swiss mechanical engineer educated (mechanics, physics) at the Swiss Federal Institute of Technology, ETH Zurich. D.Sc. with E. Meissner (Switzerland) and R. Grammel (Germany). Professor at ETH from 1942 to his retirement in 1977. A well known specialist of structural and dynamical stability (1948–1956). Switched to the plasticity of solids under the influence of William Prager during a one-year visit at Brown University—cf. the Prager-Ziegler hardening rule. Then developed a strong and creative interest in irreversible thermodynamics and the generalization of Onsager's reciprocity relations to the nonlinear case; introduction of a principle of orthogonality. His deep thoughts on the matter are exposed in his book entitled "An introduction to thermomechanics" (1986).

Zorski Henryk (1927–2003): Polish scientist with various interests in mathematical problems of continuum mechanics, the theory of defects, and nonlinear waves. Refugee in the Soviet Union during WWII, he also studied in the UK, and then back in Poland. Like many other Polish scientists of the period, worked first at the Military Academy and then at the Institute of Fundamental Technical Research (IPPT) of the Polish Academy of Sciences. A rather parsimonious writer of papers, but with a large knowledge of mathematical physics and an original thinker, he nonetheless influenced many scientists in theoretical mechanics and materials science, both in Poland and outside, among them Dominik Rogula, and Milan V. Mićunović from Serbia.

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