Bayesian Inference for the Parameters of Two-Parameter Exponential Lifetime Models Based on Type-I and Type-II Censoring

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Abstract The parameters of the two-parameter exponential distribution are estimated in this chapter from the Bayesian viewpoint based on complete, Type-I and Type-II censored samples. Bayes point estimates and credible intervals of the unknown parameters are proposed under the assumption of suitable priors on the unknown parameters and under the assumption of the squared error loss function. Illustrative example is provided to motivate the proposed Bayes point estimates and the credible intervals. Various Monte Carlo simulations are also performed to compare the performances of the classical and Bayes estimates.

Keywords Bayes estimate · Censored samples · Credible interval · Maximum likelihood estimate · Mean squared error · Squared error loss function

1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a two-parameter exponential distribution with a scale parameter θ and a location parameter λ, denoted by $E(\theta, \lambda)$, where θ and λ are independent. If the lifetime of a component is assumed to follow an exponential life model with parameters θ and λ then the parameter λ represents the component's guarantee lifetime, and the parameter $1/\theta$ represents the component's mean lifetime.

The probability density function (p.d.f) of X at x is given by:

$$f(x|\theta, \lambda) = \theta e^{-\theta(x-\lambda)}; \quad 0 \le \lambda \le x \quad \text{and} \quad \theta > 0 \tag{1}$$

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This distribution plays an important role in survival and reliability analysis; see for example Balakrishnan and Basu [1].

In life testing experiments, it often happens that the experiment is censored in the sense that the experimenter may not be in a position to observe the life times of all items put on test because of time limitations and other restrictions on the data collection. The two most common censoring schemes are Type-I and Type-II censoring schemes. In Type-I censoring scheme, the experiment continue up to a preselected fixed time T but the number of failures is random, whereas in Type-II censoring scheme, the experimental time is random but the number of failures is fixed, k.

The estimation of the parameters of two-parameter exponential distribution based on Types I and II censored samples has been considered by several authors in the literature from the Bayesian point of view. El-Sayyed [2] has derived Bayes estimate and unbiased estimate for θ^{-1}. Singh and Prasad [3, 4] have considered the problem of estimating the scale parameter θ^{-1} from the Bayesian viewpoint when the scale parameter λ is known. Sarhan [5] has studied several empirical Bayes estimates for one parameter exponential distribution. Singh and Kumar [6, 7] proposed Bayes estimates for the scale parameter under multiply Type-II censoring scheme. Singh and Kumar [8] proposed Bayes point estimates for the scale parameter under Type-II censoring by using generalized non-informative prior and natural conjugate prior. Shi and Yan [9] proposed empirical Bayes estimate for the scale parameter under Type-I censored sample assuming known location parameter. Recently, Bayoud [10] has proposed Bayes estimates and credible intervals for the scale and location parameters based on Type-I censored sample under the assumption of squared error loss function.

It is noted that in many practical applications, the value of the parameter λ may not be known. Therefore, it is useful and important to consider the problem of estimating the parameter θ when λ is unknown.

This chapter aims to derive Bayes point estimates and credible intervals for scale and location parameters of a two-parameter exponential distribution in order to estimate the guarantee and the mean life time of that distribution. This will be performed based on complete, Type-I and Type-II censored samples. Bayes point estimates are proposed under the assumption of the squared error loss function. The scale parameter θ is assumed to follow exponential distribution with hyper parameter A, and the location parameter λ is assumed to follow uniform distribution from zero to B. Suggestions for choosing the hyper parameters A and B are provided.

The rest of this chapter is organized as follows: Sect. 2 describes the probability models that are needed in this work. Bayes point estimates for the scale and location parameters are proposed in Sect. 3 based on complete, Type-I and Type-II censored samples separately. Credible intervals are derived for the unknown parameters in Sect. 4. An illustrative example is provided in Sect. 5. Simulation studies are performed in Sect. 6. Finally, the main conclusions are included in Sect. 7.

2 Models

2.1 Complete Sample

Let $X_1, X_2, \ldots, X_n \sim E(\theta, \lambda)$, with p.d.f given in (1). The likelihood function of the complete sample $X_1, X_2, \ldots, X_n$ given θ and λ is given by:

$$L\left(x_1, x_2, \ldots, x_n | \theta, \lambda\right) = \theta^n e^{-\theta \sum\limits_{i=1}^{n} (x_i - \lambda)} \tag{2}$$

Suitable priors on the unknown parameters are assumed in order to derive Bayes estimates and credible intervals.

The parameter θ is assumed to follow exponential distribution with p.d.f given by:

$$g(\theta) = Ae^{-A\theta}; \quad A > 0 \tag{3}$$

where the hyper parameter A is a preselected positive real number that is chosen to reflect our beliefs about the expected value of $1/\theta$, because the expected value of θ equals $1/A$.

The parameter λ is assumed to follow a uniform distribution with p.d.f given by:

$$p(\lambda) = \frac{1}{B}; \quad 0 \le \lambda \le B \tag{4}$$

where the hyper parameter B is a preselected positive real number that is chosen to reflect our beliefs about the lower bound of the x's , which can be easily assumed to equal the minimum observed value, $x_{(1)}$.

The joint posterior p.d.f of θ and λ given $\{x_1, x_2, \ldots, x_n\}$ is given by:

$$h_C\left(\theta, \lambda | x_1, x_2, \ldots, x_n\right) = \frac{L\left(x_1, x_2, \ldots, x_n | \lambda, \theta\right) g\left(\theta\right) p\left(\lambda\right)}{\int\limits_0^B \int\limits_0^\infty L\left(x_1, x_2, \ldots, x_n | \lambda, \theta\right) g\left(\theta\right) p\left(\lambda\right) d\theta \, d\lambda}$$

$$= \frac{n\theta^n e^{-\theta[A + \sum\limits_{i=1}^{n} (x_i - \lambda)]}}{C\Gamma(n)} \tag{5}$$

where $C = \frac{1}{D^n} - \frac{1}{E^n}$ in which $D = A + \sum\limits_{i=1}^{n} (x_i - B)$ and $E = A + \sum\limits_{i=1}^{n} x_i, \theta > 0$ and $0 \le \lambda \le B$.

Therefore,

The marginal posterior p.d.f of θ given $\{x_1, x_2, \ldots, x_n\}$ is given by:

$$h_{\theta,C}(\theta|x_1, x_2, \ldots, x_n) = \int_0^B h(\theta, \lambda|x_1, x_2, ..., x_n)\, d\lambda$$
$$= \frac{\theta^{n-1}}{C\Gamma(n)}\left(e^{-D\theta} - e^{-E\theta}\right) \tag{6}$$

where $\theta > 0$, D, E and C are defined in (5).

The marginal posterior p.d.f of λ given $\{x_1, x_2, \ldots, x_n\}$ is given by:

$$h_{\lambda,C}(\lambda|x_1, x_2, \ldots, x_n) = \int_0^\infty h(\theta, \lambda|x_1, x_2, \ldots, x_n)\, d\theta$$
$$= \frac{n^2}{C}\frac{1}{\left(A + \sum_{i=1}^{n}(x_i - \lambda)\right)^{n+1}} \tag{7}$$

where $0 \le \lambda \le B \le x_{(1)}$ and C is defined in (5).

2.2 Type-I Censored Sample

In Type-I censored scheme a random sample of n units is tested until a predetermined time T at which the test is terminated. Failure times of r units are observed, where r is a random variable. Thus the lifetime $x_i$is observed only if $x_i \le T$; $i = 1, 2, \ldots, n$.

Let $\delta_i = \begin{cases} 0; & x_i > T \\ 1; & x_i \le T \end{cases}$

Therefore, $r = \sum_{i=1}^{n} \delta_i$ which is assumed to be greater than zero.

The likelihood function of the Type-I censored data is given by:

$$L_I(x_1, x_2, \ldots, x_n|\theta, \lambda, T) = \prod_{i=1}^{n} [f(x_i|\theta, \lambda)]^{\delta_i} [1 - F(T|\theta, \lambda)]^{1-\delta_i}$$
$$= \theta^r e^{-\theta \sum_{i=1}^{n} x_i\, \delta_i} e^{-\theta[T(n-k) - n\lambda]} \tag{8}$$

The joint posterior p.d.f of θ and λ based on the Type -I censored sample is given by:

$$h_I\left(\theta,\lambda|x_1,x_2,\ldots,x_n,T\right)=\frac{L_I\left(x_1,x_2,...,x_n|\lambda,\theta\right)g\left(\theta\right)p\left(\lambda\right)}{\int_0^B\int_0^\infty L_I\left(x_1,x_2,...,x_n|\lambda,\theta\right)g\left(\theta\right)p\left(\lambda\right)d\theta\,d\lambda}$$

$$=\frac{ne^{-\theta\left[\sum_{i=1}^{n}x_i\,\delta_i+T(n-k)-n\lambda+A\right]}}{C_1\Gamma(r)} \tag{9}$$

where $\theta>0, 0\le\lambda\le B$ and $C_1=\frac{1}{D_1^r}-\frac{1}{E_1^r}\ne 0$ in which $D_1=\sum_{i=1}^{n}x_i\,\delta_i+A+T\,(n-r)-nB$ and $E_1=\sum_{i=1}^{n}x_i\,\delta_i+A+T\,(n-r)$.

The marginal posterior p.d.f of θ given Type-I censored data is given by:

$$h_{\theta,I}(\theta|x_1,x_2,\ldots,x_n,T)=\int_0^B h_I\left(\theta,\lambda|x_1,x_2,...,x_n,T\right)d\lambda$$

$$=\frac{\theta^{r-1}}{C_1\Gamma(r)}\left(e^{-D_1\theta}-e^{-E_1\theta}\right) \tag{10}$$

where $\theta>0$, D_1, E_1 and C_1 are defined in (9).

The marginal posterior p.d.f of λ given Type-I censored data is given by:

$$h_{\lambda,I}(\lambda|x_1,x_2,\ldots,x_n,T)=\int_0^\infty h_I\left(\theta,\lambda|x_1,x_2,...,x_n,T\right)d\theta$$

$$=\frac{nr}{C_1}\frac{1}{\left[\sum_{i=1}^{n}x_i\,\delta_i+T\,(n-r)-n\lambda+A\right]^{r+1}} \tag{11}$$

where $0\le\lambda\le B\le x_{(1)}$ and C_1 is defined in (9).

2.3 Type-II Censored Sample

In Type-II censored scheme the number of failures k is determined at the beginning of the experiment, the time needed to observe those k failures equals $x_{(k)}$, the kth order statistic. The likelihood function of the Type-II censored data is given by:

$$
\begin{aligned}
L_{II}\left(x_{(1)}, x_{(2)}, \ldots, x_{(k)}|\theta, \lambda\right) &= \frac{n!}{(n-k)!}\prod_{i=1}^{k} f\left(x_{(i)}|\theta, \lambda\right)\left[1 - F\left(x_{(k)}\right)\right]^{n-k} \\
&= \frac{n!}{(n-k)!}\theta^{k} e^{-\theta\left[\sum_{i=1}^{k} x_{(i)} + (n-k)x_{(k)} - n\lambda\right]} \qquad (12)
\end{aligned}
$$

The joint posterior p.d.f of θ and λ based on the Type-II censored sample is given by:

$$
\begin{aligned}
h_{II}\left(\theta, \lambda|x_{(1)}, x_{(2)}, \ldots, x_{(k)}\right) &= \frac{L_{II}\left(x_{(1)}, x_{(2)}, \ldots, x_{(k)}|\lambda, \theta\right) g(\theta)\, p(\lambda)}{\int_0^B \int_0^\infty L_{II}\left(x_{(1)}, x_{(2)}, \ldots, x_{(k)}|\lambda, \theta\right) g(\theta)\, p(\lambda)\, d\theta\, d\lambda} \\
&= \frac{n\theta^{k} e^{-\theta\left[\sum_{i=1}^{k} x_{(i)} + (n-k)x_{(k)} - n\lambda + A\right]}}{C_2 \Gamma(k)} \qquad (13)
\end{aligned}
$$

where $\theta > 0, 0 \le \lambda \le B$ and $C_2 = \frac{1}{D_2^k} - \frac{1}{E_2^k} \ne 0$ in which $D_2 = \sum_{i=1}^{k} x_{(i)} + A + (n-k)\, x_{(k)} - nB$ and $E_2 = \sum_{i=1}^{k} x_{(i)} + A + (n-k)\, x_{(k)}$

The marginal posterior p.d.f of θ given Type-II censored data is given by:

$$
\begin{aligned}
h_{\theta,II}(\theta|x_{(1)}, x_{(2)}, \ldots, x_{(k)}) &= \int_0^B h_{II}\left(\theta, \lambda|x_{(1)}, x_{(2)}, \ldots, x_{(k)}\right) d\lambda \\
&= \frac{\theta^{k-1}}{C_2 \Gamma(k)}\left(e^{-D_2\theta} - e^{-E_2\theta}\right) \qquad (14)
\end{aligned}
$$

where $\theta > 0$, D_2, E_2 and C_2 are defined in (13).

The marginal posterior p.d.f of λ given Type-II censored data is given by:

$$
\begin{aligned}
h_{\lambda,II}(\lambda|x_{(1)}, x_{(2)}, \ldots, x_{(k)}) &= \int_0^\infty h_{II}\left(\theta, \lambda|x_{(1)}, x_{(2)}, \ldots, x_{(k)}\right) d\theta \\
&= \frac{nk}{C_2} \frac{1}{\left[\sum_{i=1}^{k} x_{(i)} + (n-k)\, x_{(k)} - n\lambda + A\right]^{k+1}} \qquad (15)
\end{aligned}
$$

where $0 \le \lambda \le B \le x_{(1)}$ and C_2 is defined in (13).

3 Classical and Bayes Point Estimates

In this section the maximum likelihood and Bayesian estimates (MLE and BE) are proposed for the unknown parameters based on the complete, Type-I and Type-II censored samples. The BE are derived under the assumption of the squared error loss function (SELF). However, the BE of a parameter equals the posterior mean of that parameter if the SELF is assumed.

3.1 Based on Complete Sample

In the case of complete sample, the BE of the unknown parameters θ and λ are respectively given by:

$$\hat{\theta}_C = E_{h_\theta,C}(\theta) = \int_0^\infty \theta h_{\theta,C}(\theta|x_1, x_2, \ldots, x_n)d\theta$$
$$= \frac{n}{C}\left(\frac{1}{D^{n+1}} - \frac{1}{E^{n+1}}\right) \tag{16}$$

$$\hat{\lambda}_C = E_{h_{\lambda,C}}(\lambda) = \int_0^B \lambda h_{\lambda,C}(\lambda|x_1, x_2, \ldots, x_n)d\lambda$$
$$= \frac{1}{C}\left(\frac{B}{D^n} + \frac{1}{n(1-n)}\left(\frac{1}{D^{n-1}} - \frac{1}{E^{n-1}}\right)\right) \tag{17}$$

The MLE of θ and λ based on the complete sample are respectively: $\hat{\theta}_{MLE,C} = \frac{n}{\sum_{i=1}^{n}(x_i - x_{(1)})}$ and $\hat{\lambda}_{MLE,C} = x_{(1)}$

3.2 Based on Type-I Censored Sample

In the case of Type-I censored sample, the BE of the unknown parameters θ and λ are respectively given by:

$$\hat{\theta}_I = E_{h_{\theta,I}}(\theta) = \int_0^\infty \theta h_{\theta,I}(\theta|x_1, x_2, \ldots, x_n)d\theta$$
$$= \frac{r}{C_1}\left(\frac{1}{D_1^{r+1}} - \frac{1}{E_1^{r+1}}\right) \tag{18}$$
$$\hat{\lambda}_I = E_{h_{\lambda,I}}(\lambda) = \int_0^B \lambda h_{\lambda,I}(\lambda|x_1, x_2, \ldots, x_n)d\lambda$$
$$= \frac{1}{C_1}\left(\frac{B}{D_1^k} + \frac{1}{n(1-r)}\left(\frac{1}{D_1^{r-1}} - \frac{1}{E_1^{r-1}}\right)\right) \tag{19}$$

The MLE of θ and λ based on the Type-I censored sample are respectively:

$$\hat{\theta}_{MLE,I} = \frac{r}{\sum_{i=1}^{n} x_i\delta_i + T(n-r) - nx_{(1)}} \text{ and } \hat{\lambda}_{MLE,I} = x_{(1)}$$

3.3 Based on Type-II Censored Sample

In the case of Type-II censored sample, the BE of the unknown parameters θ and λ are respectively given by:

$$\hat{\theta}_{II} = E_{h_{\theta,II}}(\theta) = \int_0^\infty \theta h_{\theta,II}(\theta|x_{(1)}, x_{(2)}, \ldots, x_{(k)})d\theta$$
$$= \frac{k}{C_2}\left(\frac{1}{D_2^{k+1}} - \frac{1}{E_2^{k+1}}\right) \tag{20}$$
$$\hat{\lambda}_{II} = E_{h_{\lambda,II}}(\lambda) = \int_0^B \lambda h_{\lambda,II}(\lambda|x_{(1)}, x_{(2)}, \ldots, x_{(k)})d\lambda$$
$$= \frac{1}{C_2}\left(\frac{B}{D_2^k} + \frac{1}{n(1-k)}\left(\frac{1}{D_2^{k-1}} - \frac{1}{E_2^{k-1}}\right)\right) \tag{21}$$

The MLE of θ and λ based on the Type-II censored sample are respectively:

$$\hat{\theta}_{MLE,II} = \frac{k}{\sum_{i=1}^{k} x_{(i)} + x_{(k)}(n-k) - nx_{(1)}} \quad \text{and} \quad \hat{\lambda}_{MLE,II} = x_{(1)}$$

4 Credible Intervals

4.1 Based on Complete Sample

Based on the complete sample $x_1, x_2, ..., x_n$ and by using the posterior density function of θ that is defined in (6), the equal- tailed $(1-\alpha)100\,\%$ credible interval for θ denoted by (θ_L, θ_U) can be obtained numerically by solving the following integral equations:

$$\int_0^{\theta_L} \frac{\theta^{n-1}}{C\Gamma(n)}\left(e^{-D\theta} - e^{-E\theta}\right) d\theta = \alpha/2 \quad \text{and} \quad \int_{\theta_U}^{\infty} \frac{\theta^{n-1}}{C\Gamma(n)}\left(e^{-D\theta} - e^{-E\theta}\right) d\theta = \alpha/2$$

Similarly, by using the posterior density function of λ that is defined in (7), the equal-tailed $(1-\alpha)100\,\%$ credible interval for λ can be easily derived as:

$$\left(-\frac{1}{n}\left[\left(\frac{C\alpha}{2} + F^{-n}\right)^{-1/n} - F\right],\ -\frac{1}{n}\left[\left(\frac{2}{C\alpha} - (F - nB)^{-n}\right)^{-1/n} - F\right]\right)$$

in which $F = \sum_{i=1}^{n} x_i + A$.

4.2 Based on Type-I Censored Sample

Based on a Type-I censored sample and by using the posterior density function defined in (10), the equal- tailed $(1-\alpha)100\,\%$ credible interval for θ denoted by $\left(\theta_{L,I}, \theta_{U,I}\right)$ can be obtained numerically by solving the following integral equations:

$$\int_0^{\theta_{L,I}} \frac{\theta^{r-1}}{C_1\Gamma(r)}\left(e^{-D_1\theta} - e^{-E_1\theta}\right) d\theta = \alpha/2 \quad \text{and}$$

$$\int_{\theta_{U,I}}^{\infty} \frac{\theta^{r-1}}{C_1\Gamma(r)}\left(e^{-D_1\theta} - e^{-E_1\theta}\right) d\theta = \alpha/2$$

Similarly, by using the posterior density function of λ that is defined in (11), the equal- tailed $(1-\alpha)$ $100\,\%$ credible interval for λ can be easily derived as:

$$\left(-\frac{1}{n}\left[\left(\frac{C_1\alpha}{2}+F_1^{-r}\right)^{-1/r}-F_1\right],-\frac{1}{n}\left[\left(\frac{2}{C_1\alpha}-[F_1-nB]^{-r}\right)^{-1/r}-F_1\right]\right)$$

in which $F_1 = A + \sum_{i=1}^{n} x_i\,\delta_i + T\,(n-r)$.

4.3 Based on Type-II Censored Sample

Based on a Type-II censored sample and by using the posterior density function defined in (14), the equal- tailed $(1-\alpha)100\,\%$ credible interval for θ denoted by $\left(\theta_{L,II}, \theta_{U,II}\right)$ can be obtained numerically by solving the following integral equations:

$$\int_0^{\theta_{L,II}} \frac{\theta^{k-1}}{C_2\Gamma(k)}\left(e^{-D_2\theta}-e^{-E_2\theta}\right)d\theta = \alpha/2 \quad \text{and}$$

$$\int_{\theta_{U,II}}^{B} \frac{\theta^{k-1}}{C_2\Gamma(k)}\left(e^{-D_2\theta}-e^{-E_2\theta}\right)d\theta = \alpha/2$$

Similarly, by using the posterior density function of λ that is defined in (15) the equal- tailed $(1-\alpha)100\,\%$ credible interval for λ can be easily derived as:

$$\left(-\frac{1}{n}\left[\left(\frac{C_2\alpha}{2}+F_2^{-k}\right)^{-1/k}-F_2\right],-\frac{1}{n}\left[\left(\frac{2}{C_2\alpha}-[F_2-nB]^{-k}\right)^{-1/k}-F_2\right]\right)$$

in which $F_2 = A + \sum_{i=1}^{k} x_{(i)} + (n-k)\,x_{(k)}$.

5 Numerical Example

Using Mathematica 5, if U has a Uniform(0,1) distribution, then x that satisfies $U = 1 - e^{-\theta(x-\lambda)}$ follows $E(\theta, \lambda)$. Let Data I = {9.25012, 9.67048, 9.98415, 8.35142, 8.26661, 11.1222, 8.79416, 8.16523, 11.3372, 8.68471, 10.478, 11.0089} be a random sample generated from $E(0.5, 8)$.

Table 1 summarizes the values of MLE, BE and credible interval for the scale and location parameters. Those estimates were computed based on the complete, Type-I and Type-II censored samples. The hyper parameters A and B were assumed to equal one over of the available sample's mean and the minimum observation respectively.

Table 1 MLE, BE and 95 % credible interval for θ and λ based on data I

	Complete sample ($n = 12$)			Type-I CS ($T = 11$)			Type-II CS ($k = 6$)		
	MLE	BE	95 %CI	MLE	BE	95 %CI	MLE	BE	95 %CI
$\theta = 0.5$	0.70	0.70	(0.40, 1.06)	0.36	0.54	(0.28, 0.86)	0.66	0.66	(0.29, 1.15)
$\lambda = 8$	8.17	8.04	(7.76, 9.60)	8.17	7.99	(7.61, 9.56)	8.17	8.01	(7.67, 8.92)

It becomes apparent from Table 1 that the MLEs and BEs give almost the same results for estimation the scale parameter θ based on the complete and Type-II censored samples. It can be also seen from Table 1 that BE performs, in terms of the mean square error MSE, better than the MLE for estimation the scale parameter θ based on Type-I censored sample. On another hand, BE dominates, in terms of MSE, the MLE for estimation the location parameter λ based on complete, Type-I and Type-II censored samples. Moreover, the proposed credible interval gives reasonable results for estimation the parameters θ and λ in all cases.

6 Simulation Studies

In this section, the performance of the MLEs and the proposed BEs of λ and θ is investigated through various simulation studies based on complete, Type-I (with arbitrary $T = 5$) and Type-II (with arbitrary $r = 3$) censored schemes. Simulation studies are carried out on various exponential distributions with $(\theta, \lambda) = (0.5, 2), (3, 0.3), (1, 1)$ and $(2, 0)$. The hyper parameters A and B are assumed to equal one over the available sample's mean and the minimum observed value, $x_{(1)}$ respectively. The main reason for doing this is to allow us to compare the proposed

Table 2 Expected MLE and BE along with their MSE based on complete sample

θ	λ	n	$\hat{\theta}_{MLE,C}$	$\hat{\theta}_{B,C}$	$\hat{\lambda}_{MLE,C}$	$\hat{\lambda}_{B,C}$
	1	5	0.84 (0.41)	0.77 (0.22)	1.38 (0.30)	1.08 (0.15)
		30	0.53 (0.01)	0.53 (0.01)	1.07 (0.01)	1.00 (0.01)
0.5	4	5	0.85 (0.46)	0.82 (0.36)	4.40 (00.32)	4.01 (0.20)
		30	0.54 (0.01)	0.54 (0.01)	4.07 (0.01)	4.00 (0.00)
	1	5	4.93 (15.42)	2.56 (0.65)	1.07 (0.01)	0.97 (0.01)
		30	3.20 (0.45)	2.95 (0.29)	1.01 (0.00)	1.00 (0.00)
3.0	4	5	4.88 (13.38)	3.76 (3.50)	4.07 (0.01)	3.99 (0.01)
		30	3.21 (0.41)	3.13 (0.35)	4.01 (0.00)	4.00 (0.00)
	1	5	8.59 (33.72)	3.25 (5.37)	1.03 (0.00)	0.96 (0.00)
		30	5.87 (1.28)	5.01 (0.81)	1.01 (0.00)	1.00 (0.00)
5.5	4	5	9.25 (51.13)	5.97 (5.00)	4.04 (0.00)	4.00 (0.00)
		30	5.81 (1.06)	5.55 (0.79)	4.01 (0.00)	4.00 (0.00)

Table 3 Expected MLE and BE along with their MSE based on Type-I censored sample

θ	λ	n	$\hat{\theta}_{MLE,I}$	$\hat{\theta}_{B,I}$	$\hat{\lambda}_{MLE,I}$	$\hat{\lambda}_{B,I}$
	1	5	0.84 (0.42)	0.77 (0.23)	1.38 (0.30)	1.07 (0.15)
		30	0.54 (0.01)	0.53 (0.01)	1.07 (0.01)	1.00 (0.01)
0.5	4	5	0.80 (0.37)	0.77 (0.30)	4.40 (00.32)	3.97 (0.18)
		30	0.52 (0.01)	0.52 (0.01)	4.07 (0.01)	4.00 (0.01)
	1	5	4.93 (15.4)	2.56 (0.65)	1.07 (0.01)	0.97 (0.01)
		30	3.20 (0.45)	2.95 (0.29)	1.01 (0.00)	1.00 (0.00)
3.0	4	5	4.83 (13.61)	3.73 (3.36)	4.07 (0.01)	3.99 (0.01)
		30	3.18 (0.39)	3.10 (0.33)	4.01 (0.00)	4.00 (0.00)
	1	5	8.59 (33.72)	3.25 (5.37)	1.03 (0.00)	0.96 (0.00)
		30	5.87 (1.28)	5.01 (0.81)	1.01 (0.00)	1.00 (0.00)
5.5	4	5	9.49 (69.27)	5.97 (5.79)	4.04 (0.00)	3.99 (0.00)
		30	5.83 (1.45)	5.56 (1.1)	4.01 (0.00)	4.00 (0.00)

Table 4 Expected MLE and BE along with their MSE based on Type-II censored sample

θ	λ	n	$\hat{\theta}_{MLE,II}$	$\hat{\theta}_{B,II}$	$\hat{\lambda}_{MLE,II}$	$\hat{\lambda}_{B,II}$
	1	5	1.38 (3.09)	0.95 (0.49)	1.38 (0.30)	1.10 (0.16)
		30	1.54 (6.23)	0.86 (0.38)	1.07 (0.01)	0.99 (0.01)
0.5	4	5	1.35 (4.19)	1.12 (1.37)	4.40 (00.32)	4.03 (0.22)
		30	1.47 (7.25)	1.16 (1.44)	4.07 (0.01)	4.00 (0.01)
	1	5	9.80 (615.3)	2.09 (1.06)	1.07 (0.01)	0.93 (0/01)
		30	8.67 (244.9)	1.95 (1.32)	1.01 (0.00)	0.98 (0.00)
3.0	4	5	9.76 (774.9)	4.31 (5.98)	4.07 (0.01)	3.98 (0.01)
		30	9.39 (1170)	4.10 (4.94)	4.01 (0.00)	4.00 (0.00)
	1	5	13.5 (214.2)	2.40 (9.76)	1.03 (0.00)	0.92 (0.01)
		30	14.2 (375.1)	2.27 (10.5)	1.01 (0.00)	0.98 (0.00)
5.5	4	5	14.4 (426.3)	5.63 (3.76)	4.04 (0.00)	3.98 (0.00)
		30	15.2 (499.6)	5.60 (3.40)	4.01 (0.00)	4.00 (0.00)

BEs with the MLEs directly. 1000 simulated datasets are generated from $E(\theta, \lambda)$ by using Mathematica 5. For the purpose of comparison, the average value of the MLE and the proposed BE along with the mean squared error (MSE) in parentheses are reported by assuming $n = 5$ and 30 based on complete, Type-I and Type-II censored samples in Tables 2, 3 and 4 respectively. Estimators with the smallest MSE values are preferred.

It becomes apparent from Tables 2, 3 and 4 that the proposed BEs behave better than the existing MLEs based on the complete, Type-I and Type-II censored samples as the MSE values of the proposed BEs are less than those of the MLEs. It has been shown in Table 4 that the MSE values of the MLEs of θ are so high whereas the MSE values of the proposed BEs are so small relatively to those of the MLEs, this motivates the using the proposed BE based on Type-II censored samples. It can be also observed that when n increases, the MSE of the proposed BEs and of the MLEs decreases, which is expected.

7 Conclusions

In this chapter, Bayes procedures for estimating the scale and location parameters, θ and λ, of a two parameter exponential distribution were developed based on complete, Type-I and Type-II censored samples. Prior probability distributions for the parameters θ and λ were assumed to be exponential and uniform distributions respectively. Bayes point estimates and credible intervals for θ and λ were proposed in the cases of complete, Type-I and Type-II censored samples under the squared error loss. It was shown from a random dataset that, the MLE and Bayes estimates gave excellent and almost equivalent results for estimation the parameter θ in the case of complete and Type-II censored samples. Furthermore, and based on a random dataset, excellent results were obtained from the proposed credible intervals for estimation the scale and the location parameters.

Bayes estimates are highly recommended to estimate the scale and location parameters of two-parameter exponential distribution based on Type-I and Type-II samples as simulation studies showed that the MSE values of the proposed Bayes estimates are much less than those of the existing MLEs .

References

1. Balakrishnan N, Basu AP (1995) The exponential distribution: theory, methods and applications. Gordon and Breach Publishers, Newark, NJ
2. El-Sayyed GM (1967) Estimation of the parameter of an exponential distribution. J Roy Stat Soc: Ser B (Stat Methodol) 29:525–532
3. Singh RS, Prasad B (1989) Uniformly strongly consistent prior distribution and empirical Bayes estimators with asymptotic optimality and rates in a non-exponential family. Sankhya A51:334–342
4. Prasad B, Singh RS (1990) Estimation of prior distribution and empirical Bayes estimation in a non-exponential family. J Stat Plan Infer 24:81–86
5. Sarhan AM (2003) Empirical Bayes estimates in exponential reliability model. Appl Math Comput 135:319–332
6. Singh U, Kumar A (2005) Shrinkage estimators for exponential scale parameter under multiply Type II censoring. Aust J Stat 34:39–49
7. Singh U, Kumar A (2005) Bayes estimator for one parameter exponential distribution under multiply-II censoring. Indian J Math Math Sci 1:23–33
8. Singh U, Kumar A (2007) Bayesian estimation of the exponential parameter under a multiply Type-II censoring scheme. Austrian J Stat 36(3):227–238
9. Shi Y, Yan W (2010) The EB estimation of scale-parameter for two-parameter exponential distribution under the Type-I censoring life test. J Phys Sci 14:25–30
10. Bayoud HA (2012) Bayesian analysis of Type-I censored sample from two-parameter exponential distribution. In: Lecture notes in engineering and computer science: proceedings of the world congress on engineering, WCE (2012), 4–6 July 2012 U.K , London, pp 291–296