Labeling the Nodes in the Intrinsic Order Graph with Their Weights

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Abstract This chapter deals with the study of some new properties of the intrinsic order graph. The intrinsic order graph is the natural graphical representation of a complex stochastic Boolean system (CSBS). A CSBS is a system depending on an arbitrarily large number n of mutually independent random Boolean variables. The intrinsic order graph displays its 2^n vertices (associated to the CSBS) from top to bottom, in decreasing order of their occurrence probabilities. New relations between the intrinsic ordering and the Hamming weight (i.e., the number of 1-bits in a binary n-tuple) are derived. Further, the distribution of the weights of the 2^n nodes in the intrinsic order graph is analyzed.

Keywords Complex stochastic Boolean systems · Hamming weight · Intrinsic order · Intrinsic order graph · Subgraphs · Subposets

1 Introduction

Consider a system depending on an arbitrary number n of random Boolean variables. That is, the n basic variables, $x_1, \ldots, x_n$, of the system are assumed to be stochastic (non-deterministic), and they only take two possible values (either 0 or 1). We call such a system a complex stochastic Boolean system (CSBS). CSBSs often appear in many different knowledge areas, since the assumption "random Boolean variables" is satisfied very often in practice.

Each one of the possible situations (outcomes) associated to a CSBS is given by a binary n-tuple of 0s and 1s, i.e.,

$$u = (u_1, \ldots, u_n) \in \{0, 1\}^n$$

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and, from now on, we assume that the n random Boolean variables $\{x_i\}_{i=1}^n$ are mutually independent. Hence, denoting

$$\Pr\{x_i = 1\} = p_i, \quad \Pr\{x_i = 0\} = 1 - p_i \quad (1 \leq i \leq n),$$

the occurrence probability of each binary n-tuple, $u = (u_1, \ldots, u_n)$, can be computed as the product

$$\Pr\{(u_1, \ldots, u_n)\} = \prod_{i=1}^{n} \Pr\{x_i = u_i\} = \prod_{i=1}^{n} p_i^{u_i} (1 - p_i)^{1-u_i}, \tag{1.1}$$

that is, $\Pr\{(u_1, \ldots, u_n)\}$ is the product of factors p_i if $u_i = 1$, 1-p_i if $u_i = 0$. Throughout this chapter, the binary n-tuples $(u_1, \ldots, u_n)$ of 0s and 1s will be also called binary strings or bitstrings, and the probabilities $p_1, \ldots, p_n$ will be also called basic probabilities.

One of the most relevant questions in the analysis of CSBSs consists of ordering the binary strings $(u_1, \ldots, u_n)$ according to their occurrence probabilities. For this purpose, in [2] we have established a simple, positional criterion (the so-called *intrinsic order criterion*) that allows one to compare two given binary n-tuple probabilities, $\Pr\{u\}$, $\Pr\{v\}$, without computing them, simply looking at the positions of the 0s and 1s in the n-tuples u, v. The usual representation for the intrinsic order relation is the *intrinsic order graph*.

In this context, the main goal of this chapter is to state and derive some new properties of the intrinsic order graph, concerning the Hamming weights of the binary strings (i.e., the number of 1-bits in each binary n-tuple). Some of these properties can be found in [9], where the reader can also find a number of simple examples that illustrate the preliminary results presented in this chapter.

For this purpose, this chapter has been organized as follows. In Sect. 2, we present some preliminary results about the intrinsic ordering and the intrinsic order graph, in order to make the presentation self-contained. Section 3 is devoted to present new relations between the intrinsic ordering and the Hamming weight. In Sect. 4, we study the distribution of the Hamming weights of the 2^n nodes in the intrinsic order graph. Finally, conclusions are presented in Sect. 5.

2 Intrinsic Ordering in CSBSs

2.1 The Intrinsic Partial Order Relation

The following theorem [2, 3] provides us with an intrinsic order criterion—denoted from now on by the acronym IOC—to compare the occurrence probabilities of two given n-tuples of 0s & 1s without computing them.

Theorem 2.1 *Let $n \geq 1$. Let $x_1, \ldots, x_n$ be n mutually independent Boolean variables whose parameters $p_i = \Pr\{x_i = 1\}$ satisfy*

$$0 < p_1 \leq p_2 \leq \ldots \leq p_n \leq \frac{1}{2}. \tag{2.1}$$

Then the probability of the n-tuple $v = (v_1, \ldots, v_n) \in \{0, 1\}^n$ is intrinsically less than or equal to the probability of the n-tuple $u = (u_1, \ldots, u_n) \in \{0, 1\}^n$ (that is, for all set $\{p_i\}_{i=1}^n$ satisfying (2.1)) if and only if the matrix

$$M_v^u := \begin{pmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column in M_v^u there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 2.2 In the following, we assume that the parameters p_i always satisfy condition (2.1). The $\binom{0}{1}$ column preceding to each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position. The term *corresponding,* used in Theorem 2.1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u, there must exist (at least) two *different* $\binom{0}{1}$ columns preceding to each other.

The matrix condition IOC, stated by Theorem 2.1 is called the *intrinsic order criterion*, because it is independent of the basic probabilities p_i and it only depends on the relative positions of the 0s and 1s in the binary n-tuples u, v. Theorem 2.1 naturally leads to the following partial order relation on the set $\{0, 1\}^n$ [3]. The so-called intrinsic order will be denoted by "$\preceq$", and we shall write $u \succeq v$ ($u \preceq v$) to indicate that u is intrinsically greater (less) than or equal to v. The partially ordered set (from now on, poset, for short) $\left(\{0, 1\}^n, \preceq\right)$ on n Boolean variables, will be denoted by I_n.

Definition 2.3 *For all $u, v \in \{0, 1\}^n$*

$$\begin{aligned} v \preceq u \quad &\text{iff} \quad \Pr\{v\} \leq \Pr\{u\} \text{ for all set } \{p_i\}_{i=1}^n \quad \text{s.t. (2.1)} \\ &\text{iff} \quad M_v^u \text{ satisfies IOC.} \end{aligned}$$

2.2 A Picture for the Intrinsic Ordering

Now, the graphical representation of the poset $I_n = \left(\{0, 1\}^n, \preceq\right)$ is presented. The usual representation of a poset is its Hasse diagram (see [12] for more details about these diagrams). Specifically, for our poset I_n, its Hasse diagram is a directed graph (digraph, for short) whose vertices are the 2^n binary n-tuples of 0s and 1s, and whose

0
|
1

Fig. 1 The intrinsic order graph for $n = 1$

edges go upward from v to u whenever u covers v, denoted by $u \rhd v$. This means that u is intrinsically greater than v with no other elements between them, i.e.,

$$u \rhd v \quad \Leftrightarrow \quad u \succ v \text{ and } \nexists \; w \; \in \{0,1\}^n \quad \text{s.t. } u \succ w \succ v.$$

A simple matrix characterization of the covering relation for the intrinsic order is given in the next theorem; see [4] for the proof.

Theorem 2.4 (Covering Relation in I_n) *Let $n \geq 1$ and let $u, v \in \{0,1\}^n$. Then $u \rhd v$ if and only if the only columns of matrix M_v^u different from $\binom{0}{0}$ and $\binom{1}{1}$ are either its last column $\binom{0}{1}$ or just two columns, namely one $\binom{1}{0}$ column immediately preceded by one $\binom{0}{1}$ column, i.e., either*

$$M_v^u = \begin{pmatrix} u_1 \ldots u_{n-1} & 0 \\ u_1 \ldots u_{n-1} & 1 \end{pmatrix} \tag{2.2}$$

or there exists i $(2 \leq i \leq n)$ s.t.

$$M_v^u = \begin{pmatrix} u_1 \ldots u_{i-2} & 0 & 1 & u_{i+1} \ldots u_n \\ u_1 \ldots u_{i-2} & 1 & 0 & u_{i+1} \ldots u_n \end{pmatrix}. \tag{2.3}$$

The Hasse diagram of the poset I_n will be also called the *intrinsic order graph* for n variables, denoted as well by I_n.

For small values of n, the intrinsic order graph I_n can be directly constructed by using either Theorem 2.1 (matrix description of the intrinsic order) or Theorem 2.4 (matrix description of the covering relation for the intrinsic order). For instance, for $n = 1$: $I_1 = (\{0,1\}, \preceq)$, and its Hasse diagram is shown in Fig. 1. Note that $0 \succ 1$ (Theorem 2.1).

However, for large values of n, a more efficient method is needed. For this purpose, in [4] the following algorithm for iteratively building up I_n (for all $n \geq 2$) from I_1 (depicted in Fig. 1), has been developed.

Theorem 2.5 (Building Up I_n from I_1) *Let $n \geq 2$. The graph of the poset $I_n = \{0, \ldots, 2^n - 1\}$ (on 2^n nodes) can be drawn simply by adding to the graph of the poset $I_{n-1} = \left\{0, \ldots, 2^{n-1} - 1\right\}$ (on 2^{n-1} nodes) its isomorphic copy $2^{n-1} + I_{n-1} = \left\{2^{n-1}, \ldots, 2^n - 1\right\}$ (on 2^{n-1} nodes). This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of I_n. Finally, the edges connecting one vertex u of I_{n-1} with the other vertex v of $2^{n-1} + I_{n-1}$ are given by the set of 2^{n-2} vertex pairs*

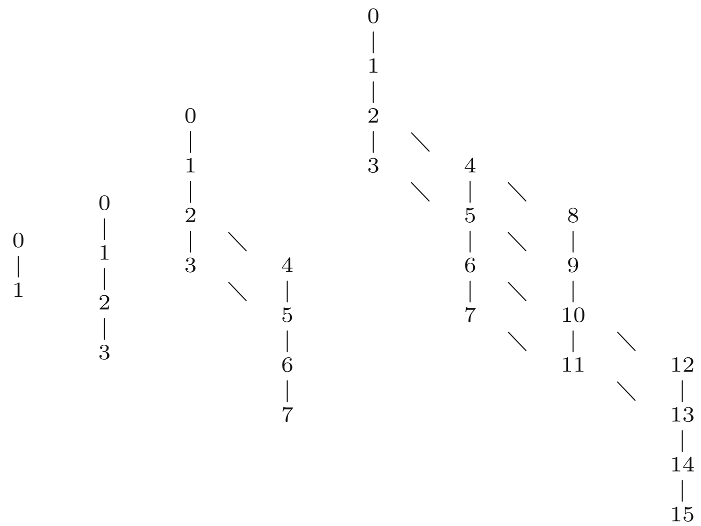

Fig. 2 The intrinsic order graphs for $n = 1, 2, 3, 4$

$$\left\{(u, v) \equiv \left(u_{(10}, 2^{n-2} + u_{(10}\right) \;\middle|\; 2^{n-2} \le u_{(10} \le 2^{n-1} - 1\right\}.$$

Figure 2 illustrates the above iterative process for the first few values of n, denoting all the binary n-tuples by their decimal equivalents.

Each pair (u, v) of vertices connected in I_n either by one edge or by a longer path, descending from u to v, means that u is intrinsically greater than v, i.e., $u \succ v$. On the contrary, each pair (u, v) of non-connected vertices in I_n either by one edge or by a longer descending path, means that u and v are incomparable by intrinsic order, i.e., $u \not\succ v$ and $v \not\succ u$.

The edgeless graph for a given graph is obtained by removing all its edges, keeping its nodes at the same positions [1]. In Figs. 3 and 4, the edgeless intrinsic order graphs of I_5 & I_6, respectively, are depicted.

For further theoretical properties and practical applications of the intrinsic order and the intrinsic order graph, we refer the reader to e.g., [2–11].

3 Weights and Intrinsic Ordering

Now, we present some new relations between the intrinsic ordering and the Hamming weight. Let us denote by $w_H(u)$ the Hamming weight—or weight, simply—of u (i.e., the number of 1-bits in u), i.e.,

$$w_H(u) := \sum_{i=1}^{n} u_i.$$

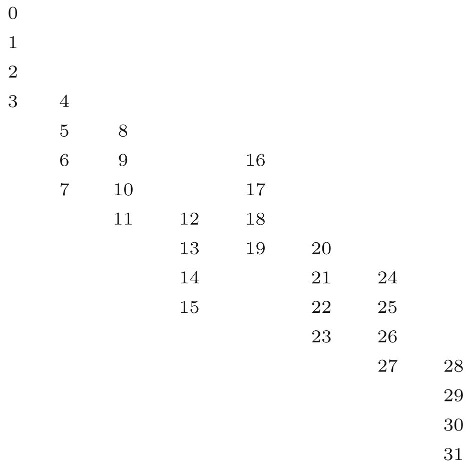

Fig. 3 The edgeless intrinsic order graph for $n = 5$

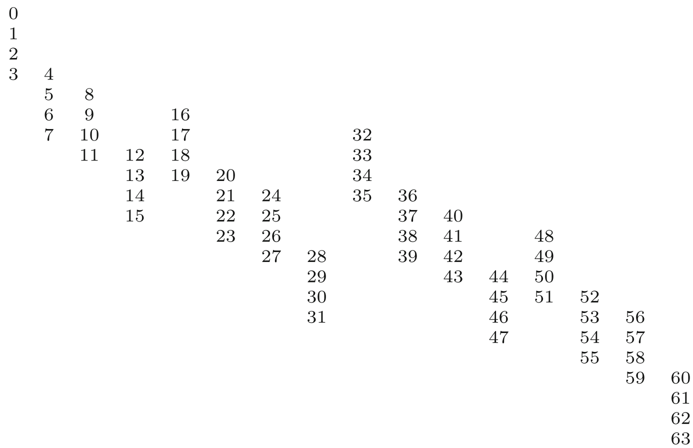

Fig. 4 The edgeless intrinsic order graph for $n = 6$

Our starting point is the following necessary (but not sufficient) condition for intrinsic order (see [3] for the proof).

$$u \succeq v \quad \Rightarrow \quad w_H(u) \leq w_H(v) \quad \text{for all } v \in \{0, 1\}^n . \tag{3.1}$$

However, the necessary condition for intrinsic order stated by Eq. (3.1) is not sufficient. That is,

$$w_H(u) \leq w_H(v) \nRightarrow u \succeq v,$$

as the following simple counter-example (indeed, the simplest one that one can find!) shows.

Example 3.1 For

$$n = 3, \quad u = 4 \equiv (1, 0, 0), \quad v = 3 \equiv (0, 1, 1),$$

we have (see the digraph of I_3 in Fig. 2)

$$w_H(4) = 1 < 2 = w_H(3).$$

However $4 \not\succeq 3$, since matrix

$$M_3^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

does not satisfy IOC.

In this context, two dual questions naturally arise. They are posed in the two subsections of this section. First, we need to set the following notations.

Definition 3.2 *For every binary n-tuple u, C^u (C_u, respectively) is the set of all binary n-tuples v whose occurrence probabilities* $\Pr\{v\}$ *are always less (greater, respectively) than or equal to* $\Pr\{u\}$*, i.e., those n-tuples v intrinsically less (greater, respectively) than or equal to u, i.e.,*

$$\begin{aligned} C^u &= \left\{ v \in \{0,1\}^n \;\middle|\; \Pr\{u\} \geq \Pr\{v\}, \forall \{p_i\}_{i=1}^n \text{ s.t. } (2.1) \right\} \\ &= \left\{ v \in \{0,1\}^n \mid u \succeq v \right\}, \end{aligned}$$

$$\begin{aligned} C_u &= \left\{ v \in \{0,1\}^n \;\middle|\; \Pr\{u\} \leq \Pr\{v\}, \forall \{p_i\}_{i=1}^n \text{ s.t. } (2.1) \right\} \\ &= \left\{ v \in \{0,1\}^n \mid u \preceq v \right\}. \end{aligned}$$

Definition 3.3 *For every binary n-tuple u, H^u (H_u, respectively) is the set of all binary n-tuples v whose Hamming weights are less (greater, respectively) than or equal to the Hamming weight of u, i.e.,*

$$H^u = \left\{ v \in \{0,1\}^n \mid w_H(u) \geq w_H(v) \right\},$$

$$H_u = \left\{ v \in \{0,1\}^n \mid w_H(u) \leq w_H(v) \right\}.$$

3.1 When Greater Weight Corresponds to Less Probability

Looking at the implication (3.1), the following question immediately arises.

Question 3.1: We try to characterize the binary n-tuples u for which the necessary condition (3.1) is also sufficient, i.e.,

$$u \succeq v \quad \Leftrightarrow \quad w_H(u) \leq w_H(v), \text{ i.e., } C^u = H_u.$$

The following theorem provides the answer to this question, in a very simple way.

Theorem 3.4 *Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0, 1\}^n$ with Hamming weight $w_H(u) = m$ $(0 \leq m \leq n)$. Then*

$$C^u = H_u$$

if and only if either u is the zero n-tuple $(m = 0)$ or the m 1-bits of u $(m > 0)$ are placed at the m right-most positions, i.e., if and only if u has the general pattern

$$u = \left(0, \overset{n-m}{\smile\ldots}, 0, 1, \overset{m}{\smile\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n, \tag{3.2}$$

where any (*but not both*!) *of the above two subsets of bits grouped together can be omitted.*

Proof.

Sufficient condition. We distinguish two cases:

(i) If u is the zero n-tuple $0 \equiv \left(0, \overset{n}{\smile\ldots}, 0\right)$, then u is the maximum element for the intrinsic order (see, e.g., [9]). Then

$$C^0 = \left\{v \in \{0, 1\}^n \mid 0 \succeq v\right\} = \{0, 1\}^n$$
$$= \left\{v \in \{0, 1\}^n \mid w_H(0) = 0 \leq w_H(v)\right\} = H_0.$$

(ii) If u is not the zero n-tuple, then u has the pattern (3.2) with $m > 0$. Let $v \in H_u$, i.e., let v let a binary n-tuple with Hamming weight greater than or equal to m (the Hamming weight of u). We distinguish two subcases:

(a) Suppose that the weight of v is

$$w_H(v) = m = w_H(u).$$

Then v has exactly m 1-bits and $n - m$ 0-bits. Call r the number of 1-bits of v placed among the m right-most positions ($\max\{0, 2m - n\} \leq r \leq m$). Obviously, v has r 1-bits and $m - r$ 0-bits placed among the m right-most positions, and also it has $m - r$ 1-bits and $n - 2m + r$ 0-bits placed among the $n - m$ left-most positions. These are the positions of the

$$r + (m - r) + (m - r) + (n - 2m + r) = m + (n - m) = n$$

bits of the binary n-tuple v.
Hence, matrix M_v^u has exactly $m - r$ $\binom{1}{0}$ columns (all placed among the m right-most positions) and exactly $m - r$ $\binom{0}{1}$ columns (all placed among the $n - m$ left-most positions). Thus, M_v^u satisfies IOC and then $u \succeq v$, i.e., $v \in C^u$.
So, for this case (a), we have proved that

$$\left\{v \in \{0,1\}^n \mid w_H(v) = w_H(u) = m\right\} \subseteq C^u \tag{3.3}$$

(b) Suppose that the weight of v is

$$w_H(v) = m + p > m = w_H(u) \quad (0 < p \leq n - m).$$

Then define a new binary n-tuple s as follows. First, select any p 1-bits in v (say, for instance, $v_{i_1} = \cdots = v_{i_p} = 1$). Second, s is constructed by changing these p 1-bits of v into 0-bits, assigning to the remainder $n - p$ bits of s the same values as the ones of v. Formally, $s = (s_1, \ldots, s_n)$ is defined by

$$s_i = \begin{cases} 0 & \text{if } i \in \left\{i_1, \ldots, i_p\right\}, \\ v_i & \text{if } i \notin \left\{i_1, \ldots, i_p\right\}. \end{cases}$$

On one hand, $u \succeq s$ since

$$w_H(s) = w_H(v) - p = m = w_H(u)$$

and then we can apply case (a) to s.
On the other hand, $s \succeq v$ since matrix M_v^s has p $\binom{0}{1}$ columns (placed at positions $i_1, \ldots, i_p$), while its $n - p$ reminder columns are either $\binom{0}{0}$ or $\binom{1}{1}$.
Hence M_v^s has no $\binom{1}{0}$ columns, so that it satisfies IOC.
Finally, from the transitive property of the intrinsic order, we derive

$$u \succeq s \text{ and } s \succeq v \quad \Rightarrow \quad u \succeq v, \text{ i.e., } v \in C^u.$$

So, for this case (b), we have proved that

$$\left\{v \in \{0,1\}^n \mid w_H(v) > w_H(u) = m\right\} \subseteq C^u \tag{3.4}$$

From (3.3) and (3.4), we get

$$\left\{v \in \{0,1\}^n \mid w_H(v) \geq w_H(u) = m\right\} \subseteq C^u,$$

i.e., $H_u \subseteq C^u$, and this set inclusion together with the converse inclusion $C^u \subseteq H_u$ (which is always satisfied for every binary n-tuple u; see Eq. 3.1) leads to the set equality $C^u = H_u$. This proves the sufficient condition.

Necessary condition. Conversely, suppose that not all the m 1-bits of u are placed at the m right-most positions. In other words, suppose that

$$u \neq \left(0, \overset{n-m}{\smile}{\ldots}, 0, 1, \overset{m}{\smile}{\ldots}, 1\right).$$

Since, by assumption, $w_H(u) = m$ then simply using the necessary condition we derive that

$$\left(0, \overset{n-m}{\smile}{\ldots}, 0, 1, \overset{m}{\smile}{\ldots}, 1\right) \succ u,$$

and then

$$\left(0, \overset{n-m}{\smile}{\ldots}, 0, 1, \overset{m}{\smile}{\ldots}, 1\right) \in H_u - C^u$$

so that,

$$H_u \nsubseteq C^u.$$

This proves the necessary condition. □

Corollary 3.5 *Let $n \geq 1$ and let*

$$u = \left(0, \overset{n-m}{\smile}{\ldots}, 0, 1, \overset{m}{\smile}{\ldots}, 1\right) \equiv 2^m - 1, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically less than or equal to u is

$$\left|C^u\right| = \binom{n}{m} + \binom{n}{m+1} + \cdots + \binom{n}{n}.$$

Proof. Using Theorem 3.4, the proof is straightforward. □

3.2 *When Less Weight Corresponds to Greater Probability*

Interchanging the roles of u & v, (3.1) can be rewritten as follows. Let u be an arbitrary, but fixed, binary n-tuple. Then

$$v \succeq u \quad \Rightarrow \quad w_H(v) \leq w_H(u) \quad \text{for all } v \in \{0, 1\}^n. \tag{3.5}$$

Looking at the implication (3.5), the following dual question of Question 3.1, immediately arises.

Question 3.2: We try to characterize the binary n-tuples u for which the necessary condition (3.5) is also sufficient, i.e.,

$$v \succeq u \quad \Leftrightarrow \quad w_H(v) \leq w_H(u), \text{ i.e., } C_u = H^u.$$

The following theorem provides the answer to this question, in a very simple way. For a very short proof of this theorem, we use the following definition.

Definition 3.6 *(i) The complementary n-tuple of a given binary n-tuple $u \in \{0,1\}^n$ is obtained by changing its* 0s *into* 1s *and its* 1s *into* 0s

$$u^c = (u_1, \ldots, u_n)^c = (1 - u_1, \ldots, 1 - u_n).$$

Obviously, two binary n-tuples are complementary if and only if their decimal equivalents sum up to

$$\left(1, \overset{n}{\smile\!\!\!\ldots}, 1\right)_{(10} = 2^n - 1.$$

(ii) The complementary set of a given subset $S \subseteq \{0,1\}^n$ of binary n-tuples is the set of the complementary n-tuples of all the n-tuples of S

$$S^c = \left\{u^c \mid u \in S\right\}.$$

Theorem 3.7 *Let $n \geq 1$ and $u = (u_1, \ldots, u_n) \in \{0,1\}^n$ with Hamming weight $w_H(u) = m$ $(0 \leq m \leq n)$. Then*

$$C_u = H^u$$

if and only if either u is the zero n-tuple $(m = 0)$ or the m 1-bits of u $(m > 0)$ are placed at the m left-most positions, i.e., if and only if u has the general pattern

$$u = \left(1, \overset{m}{\smile\!\!\!\ldots}, 1, 0, \overset{n-m}{\smile\!\!\!\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n, \tag{3.6}$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted.

Proof. Using Theorem 3.4 and the facts that (see, e.g., [5, 7])

$$(C_u)^c = C^{u^c}, \qquad \left(H^u\right)^c = H_{u^c},$$

we get

$$C_u = H^u \Leftrightarrow (C_u)^c = \left(H^u\right)^c \Leftrightarrow C^{u^c} = H_{u^c}$$
$$\Leftrightarrow u^c \text{ has the pattern (3.2)} \Leftrightarrow u \text{ has the pattern (3.6)},$$

as was to be shown. □

Corollary 3.8 *Let $n \geq 1$ and let*

$$u = \left(1, \overset{m}{\smile\!\!\!\ldots}, 1, 0, \overset{n-m}{\smile\!\!\!\ldots}, 0\right) \equiv 2^n - 2^{n-m}, \quad 0 \leq m \leq n,$$

where any (but not both!) of the above two subsets of bits grouped together can be omitted. Then the number of binary n-tuples intrinsically greater than or equal to u is

$$|C_u| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}.$$

Proof. Using the fact that $(C_u)^c = C^{u^c}$ and Corollary 3.5, the proof is straightforward. □

4 Nodes and Weights in the Intrinsic Order Graph

The results derived in Sect. 3, and more precisely those stated by Theorems 3.4 and 3.7, can be illustrated by labeling the nodes of the intrinsic order graph with their respective Hamming weights. In this way, due to Theorem 3.4 (Theorem 3.7, respectively), for a given binary n-tuple u with weight m whose m 1-bits are all placed among the right-most (left-most, respectively) positions, the set of nodes v with Hamming weight greater (less, respectively) than or equal to m will be exactly the set of nodes v connected to vertex u by a descending (ascending, respectively) path from u to v.

This suggests the analysis of the distribution of the Hamming weights of the 2^n nodes in the intrinsic order graph. The following Theorem provides only some basic consequences of such analysis.

Theorem 4.1 *Let $n \geq 2$. Label each of the 2^n nodes in the intrinsic order graph I_n, with its corresponding Hamming weight. Then*

(i) The weights (labels) of the 2^n nodes are (with repetitions): $0, 1, \ldots, n$.
(ii) The weights (labels) of the 4 nodes in each of the saturated chains $4k \rhd 4k + 1 \rhd 4k + 2 \rhd 4k + 3$ *are:* $w_H(k)$, $w_H(k) + 1$, $w_H(k) + 1$, $w_H(k) + 2$.
(iii) The set of weights (labels) of the 2^n nodes of the graph $I_n = \{0, 1\}^n$ can be partitioned into the following two subsets: (a) The weights of the nodes of the top subgraph $\{0\} \times \{0, 1\}^{n-1}$ of I_n, which one-to-one coincide with the respective weights of the nodes of the graph $I_{n-1} = \{0, 1\}^{n-1}$. (b) The weights of the nodes of the bottom subgraph $\{1\} \times \{0, 1\}^{n-1}$ of I_n, which one-to-one coincide with 1 plus the respective weights of the nodes of the graph $I_{n-1} = \{0, 1\}^{n-1}$.

Fig. 5 Weights in the edgeless intrinsic order graph for $n = 5$

0							
1							
1							
2	1						
	2	1					
	2	2		1			
	3	2		2			
		3	2	2			
			3	3	2		
			3		3	2	
			4		3	3	
					4	3	
						4	3
							4
							4
							5

Proof.

(i) Trivial.
(ii) Use the fact that for all $k \equiv (u_1 \ldots, u_{n-2}) \in \{0, 1\}^{n-2}$:

$$4k \equiv (u_1 \ldots, u_{n-2}, 0, 0)\,, \quad 4k+1 \equiv (u_1 \ldots, u_{n-2}, 0, 1)\,,$$
$$4k+2 \equiv (u_1 \ldots, u_{n-2}, 1, 0)\,, \quad 4k+3 \equiv (u_1 \ldots, u_{n-2}, 1, 1)\,.$$

(iii) Use Theorem 2.5 and the fact that for all $(u_1 \ldots, u_{n-1}) \in \{0, 1\}^{n-1}$:

$$w_H\,(0, u_1 \ldots, u_{n-1}) = w_H\,(u_1 \ldots, u_{n-1})$$
$$w_H\,(1, u_1 \ldots, u_{n-1}) = w_H\,(u_1 \ldots, u_{n-1}) + 1.$$

as was to be shown. □

Figure 5 illustrates Theorem 4.1, by labeling (and substituting) all the 32 nodes of the graph I_5 (depicted in Fig. 3) with their corresponding Hamming weights.

5 Conclusions

It is well-known that if a binary n-tuple u is intrinsically greater (less, respectively) than or equal to a binary n-tuple v then necessarily the Hamming weight of u must be less (greater, respectively) than or equal to the Hamming weight of v. We have characterized, by two dual, simple positional criteria, those n-tuples u for which each of these necessary conditions is also sufficient. Further, motivated by these questions, we have presented some basic properties concerning the distribution of weights of the 2^n nodes in the intrinsic order graph. For future researches, additional properties of such distribution worth to be studied.

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References

1. Diestel R (2005) Graph theory, 3rd edn. Springer, New York
2. González L (2002) A new method for ordering binary states probabilities in reliability and risk analysis. Lect Notes Comput Sci 2329:137–146
3. González L (2003) N-tuples of 0s and 1s: necessary and sufficient conditions for intrinsic order. Lect Notes Comput Sci 2667:937–946
4. González L (2006) A picture for complex stochastic Boolean systems: the intrinsic order graph. Lect Notes Comput Sci 3993:305–312
5. González L (2007) Algorithm comparing binary string probabilities in complex stochastic Boolean systems using intrinsic order graph. Adv Complex Syst 10(Suppl 1):111–143
6. González L (2010) Ranking intervals in complex stochastic Boolean systems using intrinsic ordering. In: Rieger BB, Amouzegar MA, Ao S-I (eds) Machine learning and systems engineering. Lecture notes in electrical engineering, vol 68. Springer, New York, pp 397–410
7. González L (2012) Duality in complex stochastic Boolean systems. In: Ao S-I, Gelman L (eds) Electrical engineering and intelligent systems. Lecture notes in electrical engineering, vol 130. Springer, New York, pp 15–27
8. González L (2012) Edges, chains, shadows, neighbors and subgraphs in the intrinsic order graph. IAENG Int J Appl Math 42:66–73
9. González L (2012) Intrinsic order and Hamming weight. Lecture notes in engineering and computer science: proceedings of the world congress on engineering 2012, WCE 2012, U.K., pp 783–788, 4–6 July 2012
10. González L (2012) Intrinsic ordering, combinatorial numbers and reliability engineering. Appl Math Model (in press)
11. González L, García D, Galván B (2004) An intrinsic order criterion to evaluate large, complex fault trees. IEEE Trans Reliab 53:297–305
12. Stanley RP (1997) Enumerative combinatorics, vol 1. Cambridge University Press, Cambridge