Inventory Control Under Parametric Uncertainty of Underlying Models

Nicholas A. Nechval, Konstantin N. Nechval and Maris Purgailis

Abstract A large number of problems in inventory control, production planning and scheduling, location, transportation, finance, and engineering design require that decisions be made in the presence of uncertainty of underlying models. In the present paper we consider the case, where it is known that the underlying distribution belongs to a parametric family of distributions. The problem of determining an optimal decision rule in the absence of complete information about the underlying distribution, i.e., when we specify only the functional form of the distribution and leave some or all of its parameters unspecified, is seen to be a standard problem of statistical estimation. Unfortunately, the classical theory of statistical estimation has little to offer in general type of situation of loss function. In the paper, for improvement or optimization of statistical decisions under parametric uncertainty, a new technique of invariant embedding of sample statistics in a performance index is proposed. This technique represents a simple and computationally attractive statistical method based on the constructive use of the invariance principle in mathematical statistics. Unlike the Bayesian approach, an invariant embedding technique is independent of the choice of priors. It allows one to eliminate unknown parameters from the problem and to find the best invariant decision rules, which have smaller risk than any of the well-known decision rules. A numerical example is given.

N. A. Nechval (✉)
Department of Statistics, EVF Research Institute, University of Latvia,
Raina Blvd 19, Riga 1050, Latvia
e-mail: nechval@junik.lv

K. N. Nechval
Department of Applied Mathematics, Transport and Telecommunication Institute,
Lomonosov Street 1, Riga1019, Latvia
e-mail: konstan@tsi.lv

M. Purgailis
Department of Cybernetics, University of Latvia, Raina Blvd 19, Riga 1050, Latvia
e-mail: marispur@lanet.lv

G.-C. Yang et al. (eds.), *IAENG Transactions on Engineering Technologies*,
Lecture Notes in Electrical Engineering 229, DOI: 10.1007/978-94-007-6190-2_1,

Keywords Demand · Distribution · Inventory · Model · Optimization · Risk · Uncertainty

1 Introduction

Most of the inventory management literature assumes that demand distributions are specified explicitly. However, in many practical situations, the true demand distributions are not known, and the only information available may be a time-series of historic demand data. When the demand distribution is unknown, one may either use a parametric approach (where it is assumed that the demand distribution belongs to a parametric family of distributions) or a non-parametric approach (where no assumption regarding the parametric form of the unknown demand distribution is made).

Under the parametric approach, one may choose to estimate the unknown parameters or choose a prior distribution for the unknown parameters and apply the Bayesian approach to incorporating the demand data available. Scarf [1] and Karlin [2] consider a Bayesian framework for the unknown demand distribution. Specifically, assuming that the demand distribution belongs to the family of exponential distributions, the demand process is characterized by the prior distribution on the unknown parameter. Further extension of this approach is presented in [3]. Application of the Bayesian approach to the censored demand case is given in [4, 5]. Parameter estimation is first considered in [6] and recent developments are reported in [7, 8]. Liyanage and Shanthikumar [9] propose the concept of operational statistics and apply it to a single period newsvendor inventory control problem.

Within the non-parametric approach, either the empirical distribution or the bootstrapping method (e.g. see [10]) can be applied with the available demand data to obtain an inventory control policy.

Conceptually, it is useful to distinguish between “new-sample” inventory control, “within-sample” inventory control, and “new-within-sample” inventory control.

For the new-sample inventory control process, the data from a past sample of customer demand are used to make a statistical decision on a future time period for the same inventory control process.

For the within-sample inventory control process, the problem is to make a statistical decision on a future time period for the same inventory control process based on early data from that sample of customer demand.

For the new-within-sample inventory control process, the problem is to make a statistical decision on a future time period for the inventory control process based on early data from that sample of customer demand as well as on a past data sample of customer demand from the same process.

In this paper, we consider the case of the within-sample inventory control process, where it is known that the underlying distribution function of the customer demand belongs to a parametric family of distribution functions. However, unlike in the Bayesian approach, we do not assume any prior knowledge on the parameter values.

2 Cumulative Customer Demand

The primary purpose of this paper is to introduce the idea of cumulative customer demand in inventory control problems to deal with the order statistics from the underlying distribution. It allows one to use the available statistical information as completely as possible in order to improve statistical decisions for inventory control problems under parametric uncertainty.
Assumptions. The customer demand at the ith period represents a random variable $Y_i, i \in \{1, \ldots, m\}$. For the cumulative customer demand, X, it is assumed that the random variables

$$X_1 = Y_1, \ldots, X_k = \sum_{i=}^{k} Y_i, \ldots, X_l = \sum_{i=1}^{l} Y_i, \ldots, X_m = \sum_{i=1}^{m} Y_i \tag{1}$$

represent the order statistics ($X_1 \leq \ldots \leq X_m$) from the exponential distribution with the probability density function

$$f_\sigma(x) = \frac{1}{\sigma} \exp\left(-\frac{x}{\sigma}\right), \quad x \geq 0, \quad \sigma > 0, \tag{2}$$

and the probability distribution function

$$F_\sigma(x) = 1 - \exp\left(-\frac{x}{\sigma}\right). \tag{3}$$

Theorem 1 Let $X_1 \leq \ldots \leq X_k$ be the first k ordered observations (order statistics) in a sample of size m from a continuous distribution with some probability density function $f_\theta(x)$ and distribution function $F_\theta(x)$, where θ is a parameter (in general, vector). Then the conditional probability density function of the lth order statistics $X_l (1 \leq k < l \leq m)$ given $X_k = x_k$ is

$$\begin{aligned} g_\theta(x_l|x_k) = {} & \frac{(m-k)!}{(l-k-1)!(m-l)!} \left[\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}\right]^{l-k-1} \\ & \times \left[1 - \frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)}\right]^{m-l} \frac{f_\theta(x_l)}{1 - F_\theta(x_k)} \\ = {} & \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \left[\frac{1 - F_\theta(x_l)}{1 - F_\theta(x_k)}\right]^{m-l+j} \\ & \times \frac{f_\theta(x_l)}{1 - F_\theta(x_k)} \end{aligned}$$

$$= \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{m-l} \binom{m-l}{j} (-1)^j \left[\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)} \right]^{l-k-1+j} \times \frac{f_\theta(x_l)}{1 - F_\theta(x_k)} \tag{4}$$

Proof From the marginal density function of X_k and the joint density function of X_k and X_l, we have the conditional density function of X_l, given that $X_k = x_k$, as

$$g_\theta(x_l|x_k) = g_\theta(x_l, x_k)/g_\theta(x_k). \tag{5}$$

This ends the proof.

Corollary 1.1 The conditional probability distribution function of X_l given $X_k = x_k$ is

$$\begin{aligned} &P_\theta\{X_l \le x_l | X_k = x_k\} \\ &= 1 - \frac{(m-k)!}{(l-k-1)!(m-l)!} \times \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \frac{(-1)^j}{m-l+1+j} \\ &\quad \times \left[\frac{1 - F_\theta(x_l)}{1 - F_\theta(x_k)} \right]^{m-l+1+j} \\ &= \frac{(m-k)!}{(l-k-1)!(m-l)!} \sum_{j=0}^{m-l} \binom{m-l}{j} \frac{(-1)^j}{l-k+j} \left[\frac{F_\theta(x_l) - F_\theta(x_k)}{1 - F_\theta(x_k)} \right]^{l-k+j}. \end{aligned} \tag{6}$$

Corollary 1.2 Let $X_1 \le \ldots \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution (2). Then the conditional probability density function of the lth order statistics $X_l (1 \le k < l \le m)$ given $X_k = x_k$ is

$$\begin{aligned} g_\sigma(x_l|x_k) &= \frac{1}{\mathrm{B}(l-k, (m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} (-1)^j \\ &\quad \times \frac{1}{\sigma} \exp\left(-\frac{(m-l+1+j)(x_l - x_k)}{\sigma} \right) \\ &= \frac{1}{\mathrm{B}(l-k, (m-l+1)} \sum_{j=0}^{m-l} \binom{m-l}{j} (-1)^j \\ &\quad \times \frac{1}{\sigma} \left[1 - \exp\left(-\frac{x_l - x_k}{\sigma} \right) \right]^{l-k-1+j} \exp\left(\frac{x_l - x_k}{\sigma} \right), \end{aligned} \tag{7}$$

and the conditional probability distribution function of the lth order statistics X_l given $X_k = x_k$ is

$$P_\sigma \{X_l \le x_l | X_k = x_k\} = 1 - \frac{1}{\mathrm{B}(l-k, (m-l+1)} \sum_{j=0}^{l-k-1} \binom{l-k-1}{j} \times \frac{(-1)^j}{m-l+1+j} \exp\left(-\frac{(m-l+1+j)(x_l - x_k)}{\sigma}\right)$$
$$= \frac{1}{\mathrm{B}(l-k, (m-l+1)} \sum_{j=0}^{m-l} \binom{m-l}{j} \times \frac{(-1)^j}{l-k+j} \left[1 - \exp\left(-\frac{x_l - x_k}{\sigma}\right)\right]^{l-k+j}. \quad (8)$$

Corollary 1.3 If $l = k + 1$,

$$g_\sigma(x_{k+1}|x_k) = (m-k)\frac{1}{\sigma}\exp\left(-\frac{(m-k)(x_{k+1} - x_k)}{\sigma}\right)$$
$$= (m-k)\sum_{j=0}^{m-k-1} \binom{m-k-1}{j} (-1)^j \times \frac{1}{\sigma}\left[1 - \exp\left(-\frac{x_{k+1} - x_k}{\sigma}\right)\right]^j \exp\left(\frac{x_{k+1} - x_k}{\sigma}\right), \quad (9)$$

and

$$P_\sigma \{X_{k+1} \le x_{k+1} | X_k = x_k\} = 1 - \exp\left(-\frac{(m-k)(x_{k+1} - x_k)}{\sigma}\right)$$
$$= (m-k)\sum_{j=0}^{m-k-1} \binom{m-k-1}{j} \frac{(-1)^j}{1+j} \times \left[1 - \exp\left(-\frac{x_{k+1} - x_k}{\sigma}\right)\right]^{1+j}, \quad 1 \le k \le m-1. \quad (10)$$

Corollary 1.4 If $l = k + 1$ and $Y_{k+1} = X_{k+1} - X_k$, then the probability density function of Y_{k+1}, $k \in \{1, \ldots, m-1\}$, is given by

$$g_\sigma(y_{k+1}) = \frac{m-k}{\sigma}\exp\left(-\frac{(m-k)y_{k+1}}{\sigma}\right), \quad y_{k+1} \ge 0, \quad (11)$$

and the probability distribution function of Y_{k+1} is given by

$$G_\sigma\{y_{k+1}\} = 1 - \exp\left(-\frac{(m-k)y_{k+1}}{\sigma}\right). \tag{12}$$

Theorem 2 Let $X_1 \le \ldots \le X_k$ be the first k ordered observations (order statistics) in a sample of size m from the exponential distribution (2), where the parameter σ is unknown. Then the predictive probability density function of the lth order statistics $X_l (1 \le k < l \le m)$ is given by

$$g_{s_k}(x_l|x_k) = \frac{k}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j \times\left[1+(m-l+1+j)\frac{x_l-x_k}{s_k}\right]^{-(k+1)}\frac{1}{s_k}, \quad x_l \ge x_k, \quad s_k > 0, \tag{13}$$

where

$$S_k = \sum_{i=1}^{k} X_i + (m-k)X_k \tag{14}$$

is the sufficient statistic for σ, and the predictive probability distribution function of the lth order statistics X_l is given by

$$P_{s_k}\{X_l \le x_l | X_k = x_k\} = 1 - \frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j} \times \frac{(-1)^j}{m-l+1+j}\left[1+(m-l+1+j)\frac{x_l-x_k}{s_k}\right]^{-k}. \tag{15}$$

Proof Using the technique of invariant embedding [11, 22], we reduce (7) to

$$g_\sigma(x_l|x_k) = \frac{1}{\mathrm{B}(l-k,(m-l+1)}\sum_{j=0}^{l-k-1}\binom{l-k-1}{j}(-1)^j \times v\exp\left(-\frac{(m-l+1+j)(x_l-x_k)}{s_k}v\right)\frac{1}{s_k} = g_{s_k}(x_l|x_k, v), \tag{16}$$

where

$$V = S_k/\sigma \tag{17}$$

is the pivotal quantity, the probability density function of which is given by

$$f(v) = \frac{1}{\Gamma(k)} v^{k-1} \exp(-v), \quad v \geq 0. \tag{18}$$

Then

$$g_{s_k}(x_l|x_k) = E\{g_{s_k}(x_l|x_k, v)\} = \int_0^\infty g_{s_k}(x_l|x_k, v) f(v) dv. \tag{19}$$

This ends the proof.

Corollary 2.1 If $l = k + 1$,

$$g_{s_k}(x_{k+1}|x_k) = k(m-k)\left[1 + (m-k)\frac{x_{k+1} - x_k}{s_k}\right]^{-(k+1)} \frac{1}{s_k}, \tag{20}$$

and

$$P_{s_k}\{X_{k+1} \leq x_{k+1}|X_k = x_k\} = 1 - \left[1 + (m-k)\frac{x_{k+1} - x_k}{s_k}\right]^{-k}, \tag{21}$$

Corollary 2.2 If $l = k + 1$ and $Y_{k+1} = X_{k+1} - X_k$, then the predictive probability density function of Y_{k+1}, $k \in \{1, \ldots, m-1\}$, is given by

$$g_{s_k}(y_{k+1}) = k(m-k)\left[1 + (m-k)\frac{y_{k+1}}{s_k}\right]^{-(k+1)} \frac{1}{s_k}, \quad y_{k+1} \geq 0, \tag{22}$$

and the predictive probability distribution function of Y_{k+1} is given by

$$G_{s_k}(y_{k+1}) = 1 - \left[1 + (m-k)\frac{y_{k+1}}{s_k}\right]^{-k}. \tag{23}$$

3 Inventory Control Models

This section deals with inventory items that are in stock during a single time period. At the end of the period, leftover units, if any, are disposed of, as in fashion items. Two models are considered. The difference between the two models is whether or not a setup cost is incurred for placing an order. The symbols used in the development of the models include:

c = setup cost per order,
c_1= holding cost per held unit during the period,
c_2 = penalty cost per shortage unit during the period,
$g_\sigma(y_{k+1})$ = probability density function of customer demand, Y_{k+1}, during the $(k+1)$th period,
σ = scale parameter,

u = order quantity,

q = inventory on hand before an order is placed.

No-Setup Model (Newsvendor Model). This model is known in the literature as the *newsvendor* model (the original classical name is the *newsboy* model). It deals with stocking and selling newspapers and periodicals. The assumptions of the model are:

1. Demand occurs instantaneously at the start of the period immediately after the order is received.
2. No setup cost is incurred.

The model determines the optimal value of u that minimizes the sum of the expected holding and shortage costs. Given optimal $u(= u^*)$, the inventory policy calls for ordering $u^* - q$ if $q < u^*$; otherwise, no order is placed.

If $Y_{k+1} \leq u$, the quantity $u - Y_{k+1}$ is held during the $(k+1)$th period. Otherwise, a shortage amount $Y_{k+1} - u$ will result if $Y_{k+1} > u$. Thus, the cost per the $(k+1)$th period is

$$C(u) = \begin{cases} c_1 \dfrac{u - Y_{k+1}}{\sigma} & \text{if } Y_{k+1} \leq u, \\ c_2 \dfrac{Y_{k+1} - u}{\sigma} & \text{if } Y_{k+1} > u. \end{cases} \tag{24}$$

The expected cost for the $(k+1)$th period, $E_\sigma\{C(u)\}$, is expressed as

$$E_\sigma\{C(u)\} = \frac{1}{\sigma}\left(c_1 \int_0^u (u - y_{k+1}) g_\sigma(y_{k+1}) dy_{k+1} + c_2 \int_u^\infty (y_{k+1} - u) g_\sigma(y_{k+1}) dy_{k+1} \right). \tag{25}$$

The function $E_\sigma\{C(u)\}$ can be shown to be convex in u, thus having a unique minimum. Taking the first derivative of $E_\sigma\{C(u)\}$ with respect to u and equating it to zero, we get

$$\frac{1}{\sigma}\left(c_1 \int_0^u g_\sigma(y_{k+1}) dy_{k+1} - c_2 \int_u^\infty g_\sigma(y_{k+1}) dy_{k+1} \right) = 0 \tag{26}$$

or

$$c_1 P_\sigma\{Y_{k+1} \leq u\} - c_2(1 - P_\sigma\{Y_{k+1} \leq u\}) = 0 \tag{27}$$

or

$$P_\sigma\{Y_{k+1} \leq u\} = \frac{c_2}{c_1 + c_2}. \tag{28}$$

It follows from (11), (12), (25), and (28) that

$$u^* = \frac{\sigma}{m - k} \ln\left(1 + \frac{c_2}{c_1}\right) \tag{29}$$

and

$$E_\sigma\{C(u^*)\} = \frac{1}{\sigma}\left(c_2 E_\sigma\{Y_{k+1}\} - (c_1+c_2)\int\limits_0^{u^*} y_{k+1} g_\sigma(y_{k+1})dy_{k+1}\right)$$
$$= \frac{c_1}{m-k}\ln\left(1+\frac{c_2}{c_1}\right). \tag{30}$$

Parametric uncertainty. Consider the case when the parameter σ is unknown. To find the best invariant decision rule u^{BI}, we use the invariant embedding technique [11–22] to transform (24) to the form, which is depended only on the pivotal quantities V, V_1, and the ancillary factor η.

Transformation of $C(u)$ based on the pivotal quantities V, V_1 is given by

$$C^{(1)}(\eta) = \begin{cases} c_1(\eta V - V_1) & \text{if } V_1 \le \eta V, \\ c_2(V_1 - \eta V) & \text{if } V_1 > \eta V, \end{cases} \tag{31}$$

where

$$\eta = \frac{u}{S_k}, \tag{32}$$

$$V_1 = \frac{Y_{k+1}}{\sigma} \sim g(v_1) = (m-k)\exp[-(m-k)v_1], \quad v_1 \ge 0. \tag{33}$$

Then $E\{C^{(1)}(\eta)\}$ is expressed as

$$E\{C^{(1)}(\eta)\} = \int\limits_0^\infty \left(c_1\int\limits_0^{\eta v}(\eta v - v_1)g(v_1)dv_1 + c_2\int\limits_{\eta v}^\infty (v_1 - \eta v)g(v_1)dv_1\right) f(v)dv. \tag{34}$$

The function $E\{C^{(1)}(\eta)\}$ can be shown to be convex in η, thus having a unique minimum. Taking the first derivative of $E\{C^{(1)}(\eta)\}$ with respect to η and equating it to zero, we get

$$\int\limits_0^\infty v\left(c_1\int\limits_0^{\eta v} g(v_1)dv_1 - c_2\int\limits_{\eta v}^\infty g(v_1)dv_1\right) f(v)dv = 0 \tag{35}$$

or

$$\frac{\int_0^\infty vP(V_1 \le \eta v)f(v)dv}{\int_0^\infty vf(v)dv} = \frac{c_2}{c_1+c_2}. \tag{36}$$

It follows from (32), (34), and (36) that the optimum value of η is given by

$$\eta^* = \frac{1}{m-k}\left[\left(1+\frac{c_2}{c_1}\right)^{1/(k+1)} - 1\right], \tag{37}$$

the best invariant decision rule is

$$u^{\mathrm{BI}} = \eta^* S_k = \frac{S_k}{m-k}\left[\left(1+\frac{c_2}{c_1}\right)^{1/(k+1)} - 1\right], \tag{38}$$

and the expected cost, if we use u^{BI}, is given by

$$E_\sigma\{C(u^{\mathrm{BI}})\} = \frac{c_1(k+1)}{m-k}\left[\left(1+\frac{c_2}{c_1}\right)^{1/(k+1)} - 1\right] = \frac{c_1(k+1)u^{\mathrm{BI}}}{S_k} = E\{C^{(1)}(\eta^*)\}. \tag{39}$$

It will be noted that, on the other hand, the invariant embedding technique [11–22] allows one to transform equation (25) as follows:

$$\begin{aligned} E_\sigma\{C(u)\} &= \frac{1}{\sigma}\left(c_1\int_0^u (u-y_{k+1})g_\sigma(y_{k+1})dy_{k+1} + c_2\int_u^\infty (y_{k+1}-u)g_\sigma(y_{k+1})dy_{k+1}\right) \\ &= \frac{1}{s_k}\left(c_1\int_0^u (u-y_{k+1})v^2(m-k)\exp\left(-\frac{v(m-k)y_{k+1}}{s_k}\right)\frac{1}{s_k}dy_{k+1}\right. \\ &\quad \left. + c_2\int_u^\infty (y_{k+1}-u)v^2(m-k)\exp\left(-\frac{v(m-k)y_{k+1}}{s_k}\right)\frac{1}{s_k}dy_{k+1}\right). \end{aligned} \tag{40}$$

Then it follows from (40) that

$$E\{E_\sigma\{(C(u)\}\} = \int_0^\infty E_\sigma\{C(u)\}f(v)dv = E_{s_k}\{C^{(1)}(u)\}, \tag{41}$$

where

$$\begin{aligned} E_{s_k}\{C^{(1)}(u)\} = \frac{k}{s_k}\left(c_1\int_0^u (u-y_{k+1})g^\bullet_{s_k}(y_{k+1})dy_{k+1}\right. \\ \left. + c_2\int_u^\infty (y_{k+1}-u)g^\bullet_{s_k}(y_{k+1})dy_{k+1}\right) \end{aligned} \tag{42}$$

represents the expected predictive cost for the $(k+1)$th period. It follows from (42) that the cost per the $(k+1)$th period is reduced to

$$C^{(2)}(u) = \begin{cases} c_1 \dfrac{u - Y_{k+1}}{s_k/k} & \text{if } Y_{k+1} \le u, \\ c_2 \dfrac{Y_{k+1} - u}{s_k/k} & \text{if } Y_{k+1} > u, \end{cases} \tag{43}$$

and the predictive probability density function of Y_{k+1} (compatible with (25)) is given by

$$g^{\bullet}_{s_k}(y_{k+1}) = (k+1)(m-k)\left[1 + (m-k)\frac{y_{k+1}}{s_k}\right]^{-(k+2)} \frac{1}{s_k}, \quad y_{k+1} \ge 0. \tag{44}$$

Minimizing the expected predictive cost for the $(k+1)$th period,

$$E_{s_k}\{C^{(2)}(u)\} = \frac{k}{s_k}\left(c_1 \int_0^u (u - y_{k+1}) g^{\bullet}_{s_k}(y_{k+1}) dy_{k+1} + c_2 \int_u^{\infty} (y_{k+1} - u) g^{\bullet}_{s_k}(y_{k+1}) dy_{k+1}\right), \tag{45}$$

with respect to u, we obtain u^{BI} immediately, and

$$E_{s_k}\{C^{(2)}(u^{\mathrm{BI}})\} = \frac{c_1(k+1)}{m-k}\left[\left(1 + \frac{c_2}{c_1}\right)^{1/(k+1)} - 1\right]. \tag{46}$$

It should be remarked that the cost per the $(k+1)$th period, $C^{(2)}(u)$, can also be transformed to

$$C^{(3)}(\eta) = \begin{cases} c_1 k \left(\dfrac{u}{s_k} - \dfrac{Y_{k+1}}{s_k}\right) & \text{if } \dfrac{Y_{k+1}}{s_k} \le \dfrac{u}{s_k} \\ c_2 k \left(\dfrac{Y_{k+1}}{s_k} - \dfrac{u}{s_k}\right) & \text{if } \dfrac{Y_{k+1}}{s_k} > \dfrac{u}{s_k} \end{cases} = \begin{cases} c_1 k(\eta - W) & \text{if } W \le \eta \\ c_2 k(W - \eta) & \text{if } W > \eta, \end{cases} \tag{47}$$

where the probability density function of the ancillary statistic $W = Y_{k+1}/S_k$ (compatible with (25)) is given by

$$g^{\circ}(w) = (k+1)(m-k)\left[1 + (m-k)w\right]^{-(k+2)}, \quad w \ge 0. \tag{48}$$

Then the best invariant decision rule $u^{\mathrm{BI}} = \eta^* S_k$, where η^* minimizes

$$E\{C^{(3)}(\eta)\} = k\left(c_1\int_0^{\eta}(\eta - w)g^{\circ}(w)dw \ + c_2\int_{\eta}^{\infty}(w - \eta)g^{\circ}(w)dw\right). \tag{49}$$

Comparison of statistical decision rules. For comparison, consider the maximum likelihood decision rule that may be obtained from (29),

$$u^{\mathrm{ML}} = \frac{\widehat{\sigma}}{m - k}\ln\left(1 + \frac{c_2}{c_1}\right) = \eta_j^{\mathrm{ML}} S_k, \tag{50}$$

where $\widehat{\sigma} = S_k/k$ is the maximum likelihood estimator of σ,

$$\eta^{\mathrm{ML}} = \frac{1}{m - k}\ln\left(1 + \frac{c_2}{c_1}\right)^{1/k}. \tag{51}$$

Since u^{BI} and u^{ML} belong to the same class,

$$\mathbf{C} = \{u : u = \eta S_k\}, \tag{52}$$

it follows from the above that u^{ML} is inadmissible in relation to u^{BI}.

Numerical example. If, say, $k = 1$ and $c_2/c_1 = 100$, we have that

$$\text{Rel.eff.}\{u^{\mathrm{ML}}, u^{\mathrm{BI}}, \sigma\} = E_\sigma\{C(u^{\mathrm{BI}})\}/E_\sigma\{C(u^{\mathrm{ML}})\} = 0.838. \tag{53}$$

Thus, in this case, the use of u^{BI} leads to a reduction in the expected cost of about 16.2 % as compared with u^{ML}. The absolute expected cost will be proportional to σ and may be considerable.

Setup Model (s-S Policy). The present model differs from the one in (24) in that a setup cost c is incurred. Using the same notation, the total expected cost per the $(k + 1)$th period is

$$\begin{aligned} E_\sigma\{\bar{C}(u)\} &= c + E_\sigma\{C(u)\} \\ &= c + \frac{1}{\sigma}\left(c_1\int_0^{u}(u - y_{k+1})g_\sigma(y_{k+1})dy_{k+1} + c_2\int_u^{\infty}(y_{k+1} - u)g_\sigma(y_{k+1})dy_{k+1}\right). \end{aligned} \tag{54}$$

As shown above, the optimum value u^* must satisfy (28). Because c is constant, the minimum value of $E_\sigma\{\bar{C}(u)\}$ must also occur at u^*. In this case, $S = u^*$, and the value of $s(<S)$ is determined from the equation

$$E_\sigma\{C(s)\} = E_\sigma\{\bar{C}(S)\} = c + E_\sigma\{C(S)\}, \quad s < S. \tag{55}$$

This equation yields another value $s_1(>S)$, which is discarded.

Assume that q is the amount on hand before an order is placed. How much should be ordered? This question is answered under three conditions: (1) $q < s$; (2) $s \le q \le S$; (3) $q > S$.

Case 1 ($q < s$). Because q is already on hand, its equivalent cost is given by $E_\sigma\{C(q)\}$. If any additional amount $u - q$ ($u > q$) is ordered, the corresponding cost given u is $E_\sigma\{\bar{C}(u)\}$, which includes the setup cost c, and we have

$$\min_{u>q} E_\sigma\{\bar{C}(u)\} = E_\sigma\{\bar{C}(S)\} < E_\sigma\{C(q)\}. \tag{56}$$

Thus, the optimal inventory policy in this case is to order $S - q$ units.

Case 2 ($s \le q \le S$). In this case, we have

$$E_\sigma\{C(q)\} \le \min_{u>q} E_\sigma\{\bar{C}(u)\} = E_\sigma\{\bar{C}(S)\}. \tag{57}$$

Thus, it is not advantageous to order in this case and $u^* = q$.

Case 3 ($q > S$). In this case, we have for $u > q$,

$$E_\sigma\{C(q)\} < E_\sigma\{\bar{C}(u)\}. \tag{58}$$

This condition indicates that, as in Case 2, it is not advantageous to place an order—that is, $u^* = q$.

The optimal inventory policy, frequently referred to as the $s - S$ policy, is summarized as

$$\begin{array}{l} \text{if } q < s, \ \text{order } S - q, \\ \text{if } q \ge s, \ \text{do not order.} \end{array} \tag{59}$$

The optimality of the $s - S$ policy is guaranteed because the associated cost function is convex.

Parametric uncertainty. In the case when the parameter σ is unknown, the total expected predictive cost for the $(k + 1)$th period,

$$\begin{aligned} &E_{s_k}\{\bar{C}^{(1)}(u)\} = c + E_{s_k}\{C^{(1)}(u)\} \\ &= c + \frac{k}{s_k}\left(c_1 \int_0^u (u - y_{k+1}) g^{\bullet}_{s_k}(y_{k+1}) dy_{k+1} + c_2 \int_u^\infty (y_{k+1} - u) g^{\bullet}_{s_k}(y_{k+1}) dy_{k+1} \right), \end{aligned} \tag{60}$$

is considered in the same manner as above.

4 Conclusion and Future Work

In this paper, we develop a new frequentist approach to improve predictive statistical decisions for inventory control problems under parametric uncertainty of the underlying distributions for the cumulative customer demand. Frequentist probability interpretations of the methods considered are clear. Bayesian methods are not considered here. We note, however, that, although subjective Bayesian prediction has a clear personal probability interpretation, it is not generally clear how this should be applied to non-personal prediction or decisions. Objective Bayesian methods, on the other hand, do not have clear probability interpretations in finite samples. For constructing the improved statistical decisions, a new technique of invariant embedding of sample statistics in a performance index is proposed. This technique represents a simple and computationally attractive statistical method based on the constructive use of the invariance principle in mathematical statistics.

The methodology described here can be extended in several different directions to handle various problems that arise in practice. We have illustrated the proposed methodology for location-scale distributions (such as the exponential distribution). Application to other distributions could follow directly.

Acknowledgments This research was supported in part by Grant No. 06.1936, Grant No. 07.2036, Grant No. 09.1014, and Grant No. 09.1544 from the Latvian Council of Science and the National Institute of Mathematics and Informatics of Latvia.

References

1. Scarf H (1959) Bayes solutions of statistical inventory problem. Ann Math Stat 30:490–508
2. Karlin S (1960) Dynamic inventory policy with varying stochastic demands. Manage Sci 6: 231–258
3. Azoury KS (1985) Bayes solution to dynamic inventory models under unknown demand distribution. Manage Sci 31:1150–1160
4. Ding X, Puterman ML, Bisi A (2002) The censored newsvendor and the optimal acquisition of information. Oper Res 50:517–527
5. Lariviere MA, Porteus EL (1999) Stalking information: Bayesian inventory management with unobserved lost sales. Manage Sci 45:346–363
6. Conrad SA (1976) Sales data and the estimation of demand. Oper Res Quart 27:123–127
7. Agrawal N, Smith SA (1996) Estimating negative binomial demand for retail inventory management with unobservable lost sales. Naval Res Logist 43:839–861
8. Nahmias S (1994) Demand estimation in lost sales inventory systems. Naval Res Logist 41: 739–757
9. Liyanage LH, Shanthikumar JG (2005) A practical inventory control policy using operational statistics. Oper Res Lett 33:341–348
10. Bookbinder JH, Lordahl AE (1989) Estimation of inventory reorder level using the bootstrap statistical procedure. IIE Trans 21:302–312
11. Nechval NA, Nechval KN, Vasermanis EK (2003) Effective state estimation of stochastic systems. Kybernetes (An International Journal of Systems & Cybernetics) 32:666–678
12. Nechval NA, Berzins G, Purgailis M, Nechval KN (2008) Improved estimation of state of stochastic systems via invariant embedding technique. WSEAS Trans Math 7:141–159

13. Nechval NA, Nechval KN, Purgailis M (2011) Prediction of future values of random quantities based on previously observed data. Eng Lett 9:346–359
14. Nechval NA, Purgailis M, Nechval KN, Strelchonok VF (2012) Optimal predictive inferences for future order statistics via a specific loss function. IAENG Int J Appl Math 42:40–51
15. Nechval NA, Purgailis M, Cikste K, Berzins G, Nechval KN (2010) Optimization of statistical decisions via an invariant embedding technique. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2010, WCE 2010, London, 30 June–2 July 2010, pp 1776–1782
16. Nechval NA, Purgailis M, Cikste K, Nechval KN (2010) Planning inspections of fatigued aircraft structures via damage tolerance approach. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2010, WCE 2010, London, 30 June–2 July 2010, pp 2470–2475
17. Nechval NA, Purgailis M, Nechval KN, Rozevskis U (2011) Optimization of prediction intervals for order statistics based on censored data. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2011, WCE 2011, London, 6–8 July 2011, pp 63–69
18. Nechval NA, Nechval KN, Purgailis M (2011) Statistical inferences for future outcomes with applications to maintenance and reliability. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2011, WCE 2011, London, 6–8 July 2011, pp 865–871
19. Nechval NA, Purgailis M, Nechval KN, Bruna I (2012) Optimal inventory control under parametric uncertainty via cumulative customer demand. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2012, WCE 2012, London, 4–6 July 2012, pp 6–11
20. Nechval NA, Purgailis M, Nechval KN, Bruna I (2012) Optimal prediction intervals for future order statistics from extreme value distributions. In: Lecture notes in engineering and computer science: Proceedings of the world congress on engineering 2012, WCE 2012, London, 4–6 July 2012, pp 1340–1345
21. Nechval NA, Nechval KN, Purgailis M (2011) Inspection policies in service of fatigued aircraft structures. In: Ao S-I, Gelman L (eds) Electrical engineering and applied computing, vol 90. Springer, Berlin, pp 459–472
22. Nechval NA, Nechval KN, Purgailis M (2013) Weibull prediction limits for a future number of failures under parametric uncertainty. In: Ao S-I, Gelman L (eds) Electrical engineering and intelligent systems, vol 130, LNEE. Springer, Berlin, pp 273–284