Chapter 3
Pose and Displacement

Abstract The homogenous transformation matrix describes either the pose (position and orientation) or displacement (translation and orientation) of an object. The displacement can be performed either with respect to a reference (fixed) coordinate frame or with respect to a relative frame (attached to the object). Perspective transformation can also be described by homogenous transformation matrix.

In the previous chapter we became acquainted with orientation and rotation. There are, however, two other similar terms, namely position and translation. Position is associated with a point in the space, usually in the cartesian coordinate frame. Translation represents a displacement along a line. We have learned that either rotation or orientation can be described by the orthogonal rotation matrices of 3×3 order. In a similar way position and translation are described by a 3×1 vector, having three components along the x, y, and z axes of cartesian coordinate frame [1].

In robotics we are interested into objects more than into points. These are either the segments of robot mechanism or objects manipulated by the robot. When dealing with the objects, we speak about their pose and their displacement. The pose of an object represents its position and orientation. When defining the position of an object in the space, we must select a point on this object. Usually this is the center of mass or some characteristic corner. We already know that orientation of the body can be described either by the use of rotation matrix, RPY or Euler angles or quaternions. An arbitrary displacement of an object can be described by combination of translation and rotation. In this chapter we shall come to know the homogenous transformation matrices of 4×4 order, describing both the pose and the displacement of the objects.

3.1 Homogenous Transformation Matrix

Let us select a reference coordinate frame x_0, y_0, z_0 in the space together with another arbitrary frame x_1, y_1, z_1, as shown in Fig. 3.1. The origins of the frames do not coincide one with another as in Sect. 2.3. Let us select an arbitrary point P, denoted by vector ${}^1\mathbf{p}$ in the frame x_1, y_1, z_1. Our goal is to determine the position of the

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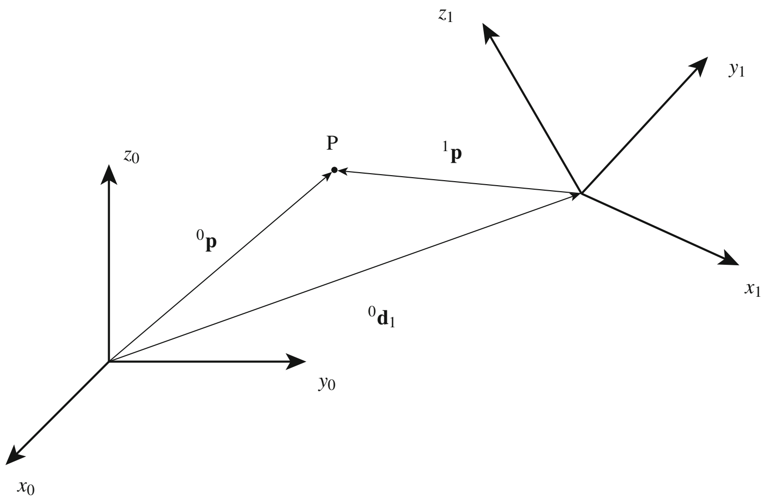

Fig. 3.1 Two arbitrary frames in the space

selected point or corresponding vector in the frame x_0, y_0, z_0. The easiest way to calculate the vector $^0\mathbf{p}$ is when the axes of the frames x_0, y_0, z_0 and x_1, y_1, z_1 are parallel, while the frames are displaced for the distance $^0\mathbf{d}_1$. In the previous chapter we learned that there always exists an equivalent axis about which the frame x_1, y_1, z_1 can be rotated, so that it will be parallel to x_0, y_0, z_0. The point P preserves its position with respect to the reference frame x_0, y_0, z_0, while vector $^1\mathbf{p}$ has new coordinates in the rotated frame x_1, y_1, z_1:

$$^1\mathbf{p}' = {}^0\mathbf{R}_1 {}^1\mathbf{p} \tag{3.1}$$

$^0\mathbf{R}_1$ in equation (3.1) represents the rotation matrix, which aligns the frame x_1, y_1, z_1 with respect to the frame x_0, y_0, z_0. Figure 3.2 shows a bird's-eye view on both coordinate frames after aligning the axes of the frame x_1, y_1, z_1 with respect to the reference frame x_0, y_0, z_0. Let us suppose that we have equal scales on the axes of both frames, so that the components of all three vectors can be simply added:

$$\begin{aligned} {}^0\mathrm{p}_x &= {}^1\mathrm{p}'_x + {}^0\mathrm{d}_{1x} \\ {}^0\mathrm{p}_y &= {}^1\mathrm{p}'_y + {}^0\mathrm{d}_{1y} \\ {}^0\mathrm{p}_z &= {}^1\mathrm{p}'_z + {}^0\mathrm{d}_{1z} \end{aligned}$$

The position of point P in the frame x_0, y_0, z_0 can be written by the following vector equation:

$$^0\mathbf{p} = {}^1\mathbf{p}' + {}^0\mathbf{d}_1 \tag{3.2}$$

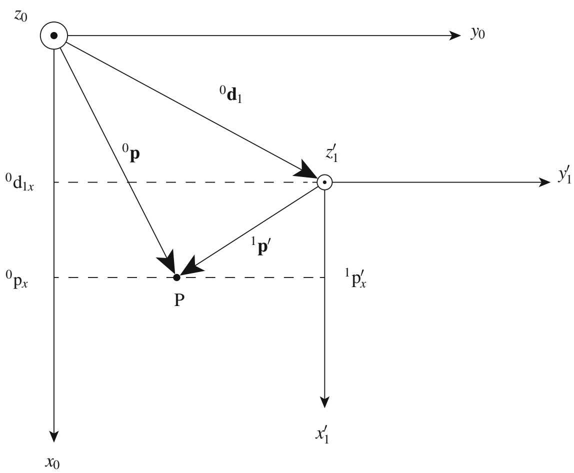

Fig. 3.2 The aligned coordinate frames

We are interested into a general case when the frame x_1, y_1, z_1 is not parallel to the reference frame x_0, y_0, z_0, but arbitrarily rotated. In Eq. (3.2) we must take into account that the frame x_1, y_1, z_1 results from the rotation (3.1):

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_1 {}^1\mathbf{p} + {}^0\mathbf{d}_1 \tag{3.3}$$

The equation where the rotation matrix ${}^0\mathbf{R}_1$ appears together with the position vector ${}^0\mathbf{d}_1$, represents the general description of pose [2]. Equation (3.3) describes the position of point expressed in the frame x_0, y_0, z_0, while knowing its position in the frame x_1, y_1, z_1. Let us now suppose that we have in the space three arbitrary frames $x_0, y_0, z_0,\ x_1, y_1, z_1$, and x_2, y_2, z_2. We have a single point P in the space, which is connected to the origins of the frames with three vectors ${}^0\mathbf{p}$, ${}^1\mathbf{p}$, and ${}^2\mathbf{p}$. Let us write the equation for the position of the point P in the frame x_1, y_1, z_1, while we know its position in the frame x_2, y_2, z_2:

$$ {}^1\mathbf{p} = {}^1\mathbf{R}_2 {}^2\mathbf{p} + {}^1\mathbf{d}_2 \tag{3.4}$$

Now we shall find the position of point P in the frame x_0, y_0, z_0 by inserting the Eq. (3.4) into (3.3):

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_1 {}^1\mathbf{R}_2 {}^2\mathbf{p} + {}^0\mathbf{R}_1 {}^1\mathbf{d}_2 + {}^0\mathbf{d}_1 \tag{3.5}$$

The equation describes the transformation between vectors ${}^2\mathbf{p}$ and ${}^0\mathbf{p}$ and can be therefore adapted to the equation representing the pose (3.3):

$$ {}^0\mathbf{p} = {}^0\mathbf{R}_2 {}^2\mathbf{p} + {}^0\mathbf{d}_2 \tag{3.6}$$

After comparing the Eqs. (3.5) and (3.6), we can see that the following two relations exist:

$$^{0}\mathbf{R}_2 = {}^{0}\mathbf{R}_1 {}^{1}\mathbf{R}_2 \tag{3.7}$$

$$^{0}\mathbf{d}_2 = {}^{0}\mathbf{d}_1 + {}^{0}\mathbf{R}_1 {}^{1}\mathbf{d}_2 \tag{3.8}$$

The first equation is already known from the previous Sect. (2.24). The second equation only tells that two position vectors can be added when expressed in the same coordinate frame. The vector ${}^{1}\mathbf{d}_2$, connecting the origins x_1, y_1, z_1 and x_2, y_2, z_2, must be expressed in the frame x_0, y_0, z_0, which is accomplished by premultiplying the vector ${}^{1}\mathbf{d}_2$ by the rotation matrix ${}^{0}\mathbf{R}_1$. In this way the frame x_1, y_1, z_1 is made parallel to the frame x_0, y_0, z_0.

The Eqs. (3.7) and (3.8) represent a system of equations which can be written in the following matrix form:

$$\begin{bmatrix} {}^{0}\mathbf{R}_1 & {}^{0}\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} {}^{1}\mathbf{R}_2 & {}^{1}\mathbf{d}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} {}^{0}\mathbf{R}_1 {}^{1}\mathbf{R}_2 & {}^{0}\mathbf{R}_1 {}^{1}\mathbf{d}_2 + {}^{0}\mathbf{d}_1 \\ \mathbf{0} & 1 \end{bmatrix} \tag{3.9}$$

As the rotation matrix ${}^{0}\mathbf{R}_1$ is of 3×3 dimension, $\mathbf{0}$ means a row of zeros [0, 0, 0]. The equation shows that the general description of pose (3.3) can be written in the following matrix form:

$$\begin{bmatrix} {}^{0}\mathbf{p} \\ 1 \end{bmatrix} = {}^{0}\mathbf{H}_1 \begin{bmatrix} {}^{1}\mathbf{p} \\ 1 \end{bmatrix} \tag{3.10}$$

where ${}^{0}\mathbf{H}_1$ represents homogenous transformation matrix:

$$^{0}\mathbf{H}_1 = \begin{bmatrix} {}^{0}\mathbf{R}_1 & {}^{0}\mathbf{d}_1 \\ & \\ \mathbf{0} & 1 \end{bmatrix} \tag{3.11}$$

The homogenous transformation matrix is homogenizing or unifying the orientation and position or rotation and translation into a single matrix, what we shall learn in details in the next chapters. The orthogonality of the matrix ${}^{0}\mathbf{R}_1$, which is part of the homogenous matrix ${}^{0}\mathbf{H}_1$, leads to rather simple calculation of inverse matrix ${}^{0}\mathbf{H}_1^{-1}$. Equation (3.3) is multiplied on both sides of the equality sign by ${}^{0}\mathbf{R}_1^{\mathrm{T}}$ and after expressing the column ${}^{1}\mathbf{p}$ we have:

$$^{1}\mathbf{p} = {}^{0}\mathbf{R}_1^{\mathrm{T}} {}^{0}\mathbf{p} - {}^{0}\mathbf{R}_1^{\mathrm{T}} {}^{0}\mathbf{d}_1$$

what can be written in the form of homogenous transformation matrix:

$$^{0}\mathbf{H}_1^{-1} = \begin{bmatrix} {}^{0}\mathbf{R}_1^{\mathrm{T}} & -{}^{0}\mathbf{R}_1^{\mathrm{T}} {}^{0}\mathbf{d}_1 \\ & \\ \mathbf{0} & 1 \end{bmatrix} \tag{3.12}$$

In a similar way as successive orientations were written by postmultiplying the rotation matrices, the successive poses are described by postmultiplication of homogenous transformation matrices. Equation (3.9) can be shortly written as:

$$\begin{aligned} {}^0\mathbf{H}_2 &= {}^0\mathbf{H}_1 {}^1\mathbf{H}_2 \\ {}^0\mathbf{H}_n &= {}^0\mathbf{H}_1 {}^1\mathbf{H}_2 \ldots {}^{n-1}\mathbf{H}_n \end{aligned} \tag{3.13}$$

In the next chapter we shall learn that Eq. (3.13) represents the geometric model of robot.

In the case of pure translation the rotation matrix (2.18) becomes a unit matrix:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as the diagonal dot products in Eq. (2.18) are as follows:

$$\begin{aligned} {}^1\mathbf{i} \cdot {}^0\mathbf{i} &= 1 \\ {}^1\mathbf{j} \cdot {}^0\mathbf{j} &= 1 \\ {}^1\mathbf{k} \cdot {}^0\mathbf{k} &= 1 \end{aligned}$$

All the other products of the unit vectors are zero. The homogenous matrix is as follows:

$${}^0\mathbf{H}_1 = \begin{bmatrix} \mathbf{I} & {}^0\mathbf{d}_1 \\ & \\ \mathbf{0} & 1 \end{bmatrix} \tag{3.14}$$

Let us consider a simple example. The vector **a** (represented by the unit vector **i**) is first rotated in the clockwise direction for 90° about the z axis. The new vector is afterwards translated for 2 units into positive y direction. Finally, the vector obtained is rotated in counter clockwise direction for 90° about the x axis. Let us solve this simple example first graphically (Fig. 3.3). After rotating the vector **a** in the clockwise direction for 90° about the z axis, the vector **b** is obtained. It is directed in negative y axis. This is written by the use of homogenous matrix as follows:

$$\mathbf{b} = \mathbf{H}_{z,-90}\mathbf{a} \tag{3.15}$$

Translation for +2 units in y axis brings us from point **b** into the point **c**:

$$\mathbf{c} = \mathbf{H}_{y,+2}\mathbf{b} \tag{3.16}$$

Finally the vector **c** is rotated in the counter clockwise direction about the x axis:

$$\mathbf{d} = \mathbf{H}_{x,90}\mathbf{c} \tag{3.17}$$

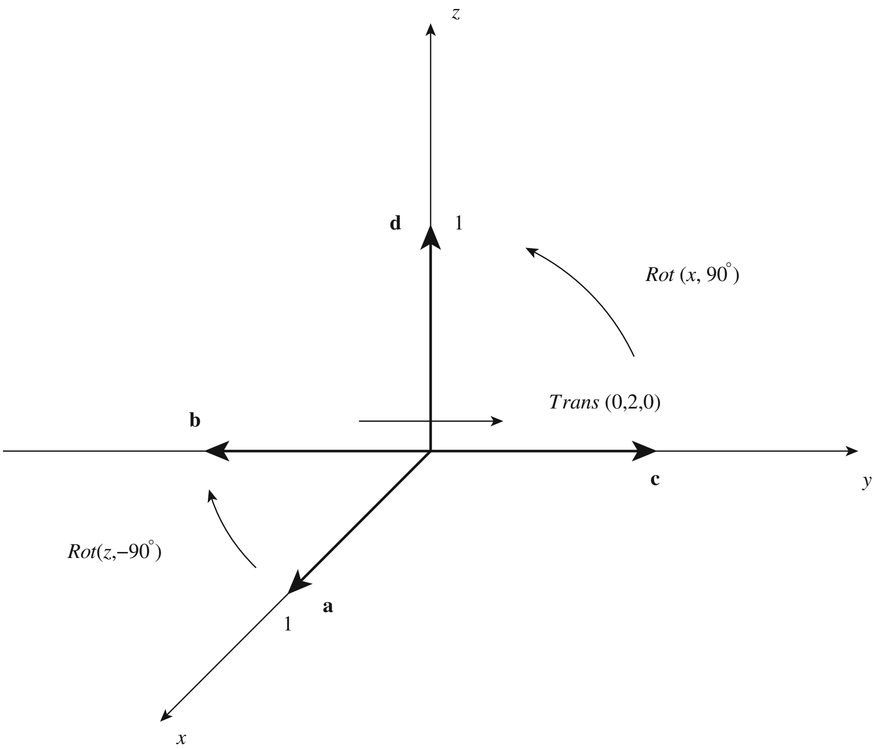

Fig. 3.3 Displacements of a vector in the space

From Fig. 3.3 we can see, that after three displacements the unit vector **k** is obtained. The same result can be obtained through calculations. Equation (3.15) is inserted into (3.16) and the equation obtained into (3.17):

$$\mathbf{d} = \mathbf{H}_{x,90}\mathbf{H}_{y,+2}\mathbf{H}_{z,-90}\mathbf{a}$$

After inserting the numbers, we have:

$$\mathbf{d} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

We obtained the expected result. In continuation we shall be interested more into the displacement of objects than vectors.

3.2 Pose

In the previous chapter we learned that the rotation matrix **R** describes either rotation or orientation. The homogenous transformation matrix **H** has similar double meaning, which is either pose or displacement. When a **H** matrix represents the pose, than the rotation matrix **R** describes the orientation, while the column **d** means the position [1].

Let us consider an arbitrary matrix **H** (3.18). When describing the orientation of an object or coordinate frame by the use of rotation matrix, we already learned that the first three columns of the homogenous matrix describe how the frame x_1, y_1, z_1 is rotated with respect to the reference frame x_0, y_0, z_0:

$$\begin{array}{c} \begin{matrix} x_1 & y_1 & z_1 & \end{matrix} \\ \left[\begin{matrix} \lceil 0 \rceil & \lceil 0 \rceil & \lceil 1 \rceil & 4 \\ 1 & 0 & 0 & -3 \\ \lfloor 0 \rfloor & \lfloor 1 \rfloor & \lfloor 0 \rfloor & 7 \\ 0 & 0 & 0 & 1 \end{matrix}\right] \begin{matrix} x_0 \\ y_0 \\ z_0 \\ \\ \end{matrix} \end{array} \tag{3.18}$$

The fourth column represents the position of the origin of the displaced coordinate frame x_1, y_1, z_1 with respect to the base coordinate frame x_0, y_0, z_0 (2.18). By using this piece of knowledge we can plot the coordinate frame represented by the homogenous matrix in the reference frame (Fig. 3.4). From matrix (3.18) we "read", that the x_1 axis has the same direction as y_0 axis of the reference frame, y_1 axis the same direction as z_0 axis, while z_1 axis is directed in the same way as x_0 axis.

A very simple example was selected to explain the description of the pose by the use of homogenous transformation matrix, where the axes of the frames x_1, y_1, z_1 and x_0, y_0, z_0 are either parallel or antiparallel. Such an example, however is not without sense in robotics. Characteristic property of industrial robot is that the axes of the neighboring joints are either parallel or perpendicular. Also the robots start their movements from the so called "home" pose where the segments are placed either parallel or perpendicular to each other. With robot home pose we encounter the pose of the coordinate frames as shown in Fig. 3.4.

3.3 Displacement

We can explain the pose of the coordinate frame x_1, y_1, z_1 in the reference frame x_0, y_0, z_0 by the displacement of the reference frame. When the matrix **H** represents the displacement, then the rotation matrix **R** describes rotation, while the column **d** belongs to translation. The matrix (3.18) can be considered as a result from three successive steps:

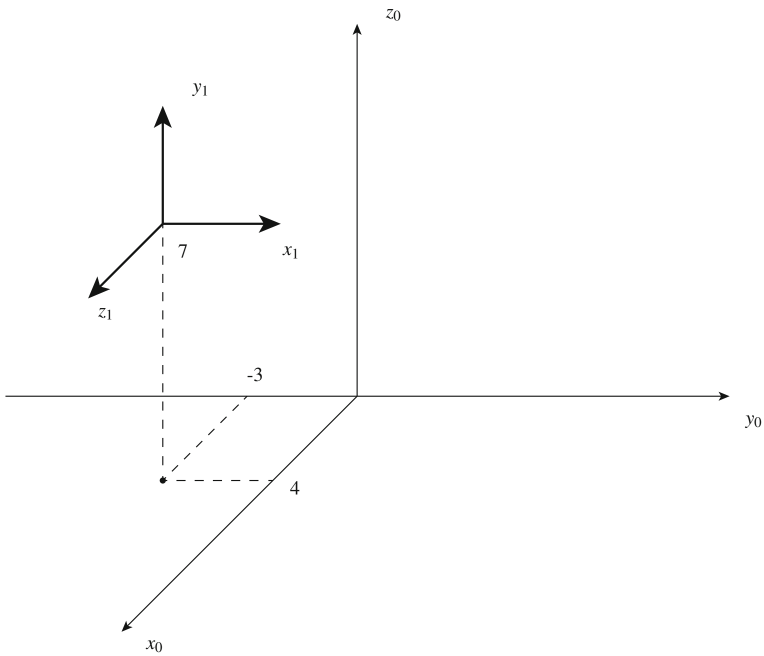

Fig. 3.4 The pose of frame x_1, y_1, z_1 with respect to reference frame x_0, y_0, z_0

$$\mathbf{H} = Trans\,(4, -3, 7)\; Rot\,(y,\ 90^\circ)\; Rot\,(z,\ 90^\circ) \tag{3.19}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When performing the displacements with respect to a relative frame, Eq. (3.19) is read from left to right:

$$\overrightarrow{Trans\,(4, -3, 7)\; Rot\,(y,\ 90^\circ)\; Rot\,(z,\ 90^\circ)}$$

We can examine the correctness of Fig. 3.4 and the homogenous transformation matrix (3.18) by performing the displacements described in Eq. (3.19) which are shown in Fig. 3.5 The coordinate frame from Fig. 3.4 can be obtained by first

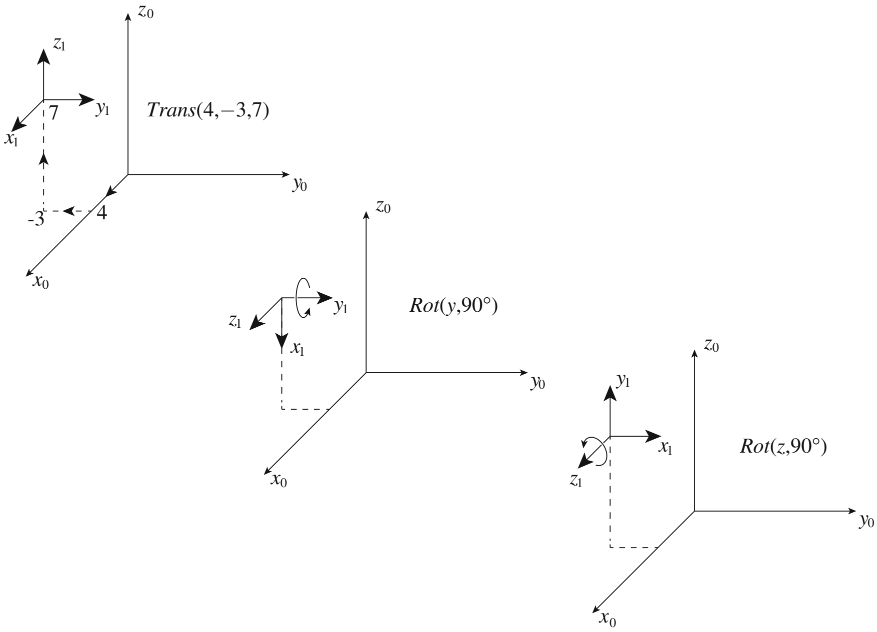

Fig. 3.5 The displacements of a frame with respect to a relative coordinate frame

translating the reference frame x_0, y_0, z_0 for $[4, -3, 7]^T$, then rotating it for 90° about the new y axis and finally for 90° about again new z axis.

When performing the displacements with respect to the reference frame, Eq. (3.19) is read from right to left:

$$\overleftarrow{\underleftarrow{Trans\,(4,-3,7)\;Rot\,(y,\ 90^\circ)\;Rot\,(z,\ 90^\circ)}}$$

Now all the displacements are made with respect to the x_0, y_0, and z_0 axes, as shown in Fig. 3.6.

Let us examine the displacements somewhat closer. We already learned that multiplying a vector **p**, representing position of a point in space, by homogenous matrix **H** displaces the vector into a new position described by the product **Hp**. In the continuation we will be interested into objects, which are represented by the coordinate frames attached to those objects. The pose of a free object having 6 degrees of freedom can be described by homogenous transformation matrix. Homogenous transformation matrix, however, describes also displacement of an object. Therefore, we shall in continuation of this chapter denote a homogenous matrix representing the pose of an object by **P**, while the homogenous matrix describing the displacement will be written as **D**. When dealing with points we had product of a matrix and a

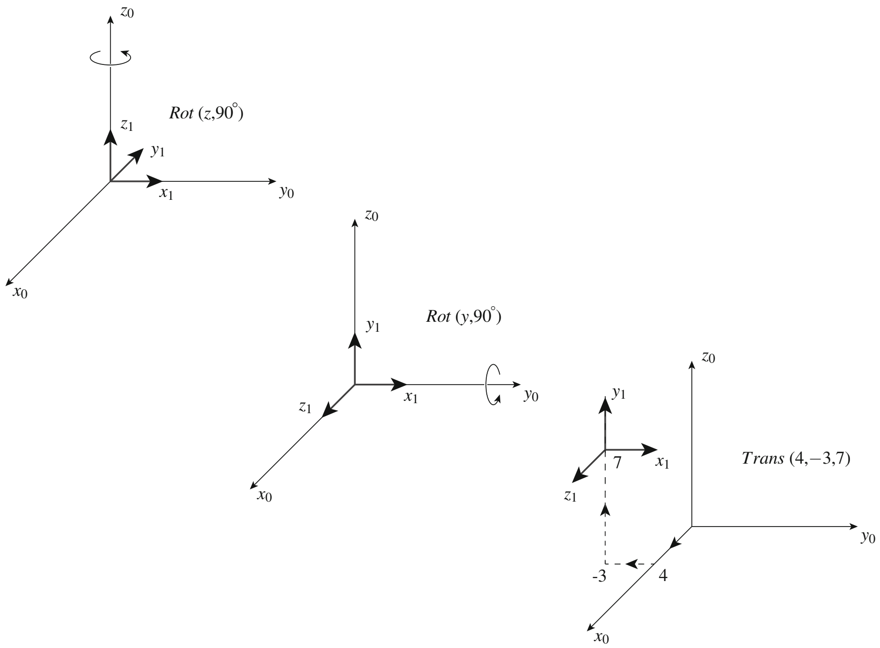

Fig. 3.6 The displacements of a frame with respect to a reference frame

column, while with objects we have two matrices. The pose of an object **P** can be either premultiplied by the displacement **D**:

$$\mathbf{X} = \mathbf{DP} \tag{3.20}$$

or it can be postmultiplied:

$$\mathbf{Y} = \mathbf{PD} \tag{3.21}$$

The new poses of the object **X** and **Y** are different. Premultiplication (3.20) represents a displacement with respect to the reference frame, while postmultiplication (3.21) describes a displacement with respect to the relative coordinate frame. Let us examine both displacements by the help of simple example.

Let us select an initial pose of a coordinate frame:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.22}$$

The displacement consists from translation and rotation:

$$\mathbf{D} = Trans(0, 20, 0)\,Rot(z, 90^\circ) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.23}$$

After premultiplication (3.20) the new pose is obtained:

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is shown in Fig. 3.7. The expression for displacement (3.23) was read in the reverse order, which means, that translation with respect to the reference frame was performed after rotation.

Postmultiplication of a pose by displacement **D** means a displacement with respect to the relative coordinate frame. After multiplication (3.21) the following new pose is obtained:

$$\mathbf{Y} = \begin{bmatrix} 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is shown in Fig. 3.8. Here, the expression for displacement (3.23) was read in usual order (from left to right), which means that rotation was performed after translation.

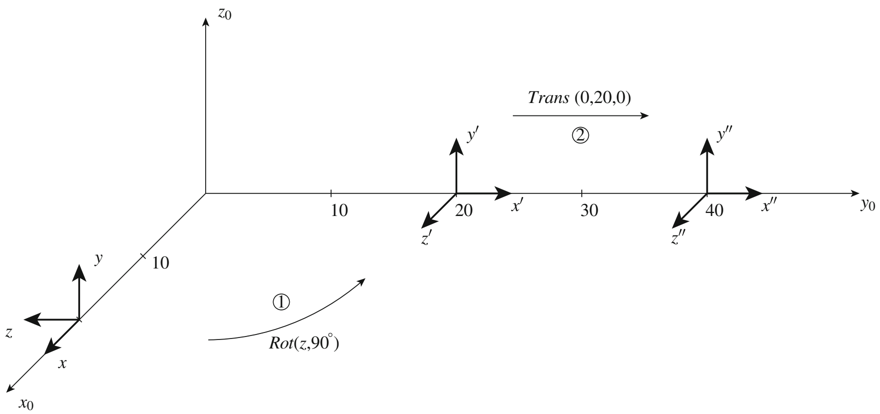

Fig. 3.7 Displacement with respect to reference coordinate frame

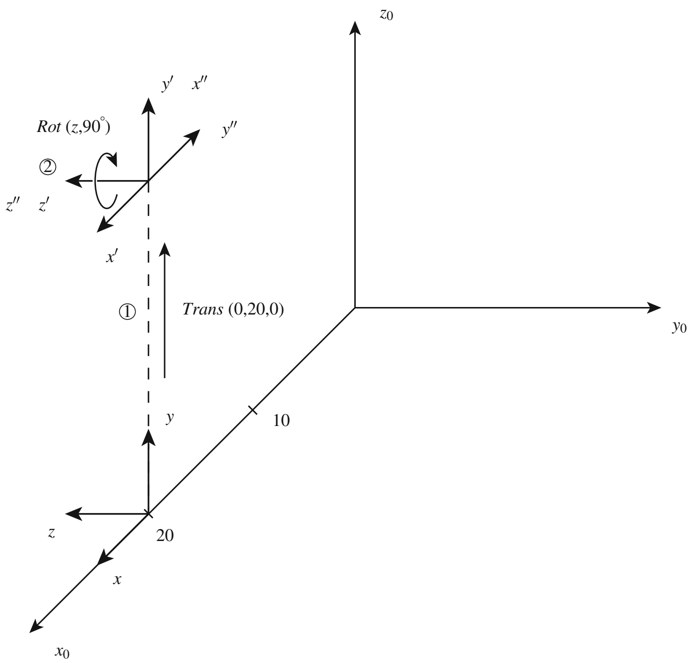

Fig. 3.8 Displacement with respect to relative coordinate frame

3.4 Displacement of Objects in Space

The displacements of two neighboring robot segments will play important role when we shall in next chapters study the geometric model of robot mechanism. We shall therefore look more closely to the description of the displacements of rigid bodies in the space by the use of homogenous transformations.

Let us consider the pose of the objects A and B in space, as shown in Fig. 3.9. The goal is to displace object B into a new pose $\mathbf{B}'$ on the object A, so that both objects are connected. Let us first perform the displacement with respect to the reference coordinate frame. We shall select an arbitrary sequence of displacements, where the object B is first rotated for 180° about the x_0 axis of the reference frame. A new pose of the object B is obtained, denoted as $\mathbf{B}''$. This intermediate pose is shown in Fig. 3.10.

Now, it is not difficult to realize, that we shall reach the final pose $\mathbf{B}'$ by the use of translations only. The object in the pose $\mathbf{B}''$ is first lifted for at least 1 unit in the z_0 direction, in order not to collide with the object A. Afterwards we slide over the object A for 3 units in the x_0 direction. After displacing the object for two units in the y_0 direction, the objects A and B are connected. As we are dealing with a displacement in a reference frame, the individual displacements are written in reverse order:

$$\mathbf{D} = Trans(3, 2, 1)Rot(x, 180°)$$

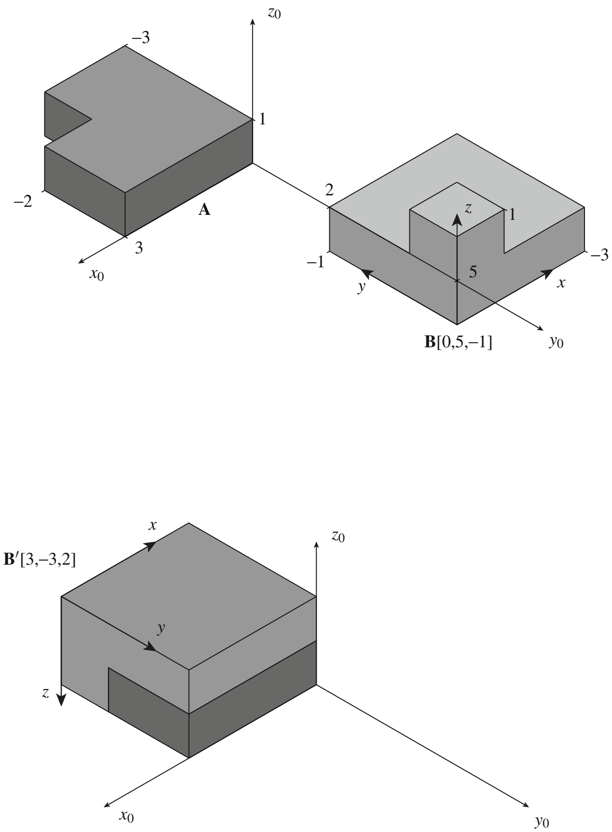

Fig. 3.9 Initial and final pose of objects A and B in space

The displacement can be written by the use of corresponding homogenous transformation matrices:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When calculating the final pose of the object $\mathbf{B}'$, a coordinate frame must be attached to the object B. A relative coordinate frame is attached to the object B in the corner $[0, 5, -1]^T$, as shown in Fig. 3.9. The pose of the object B is described by the following homogenous transformation:

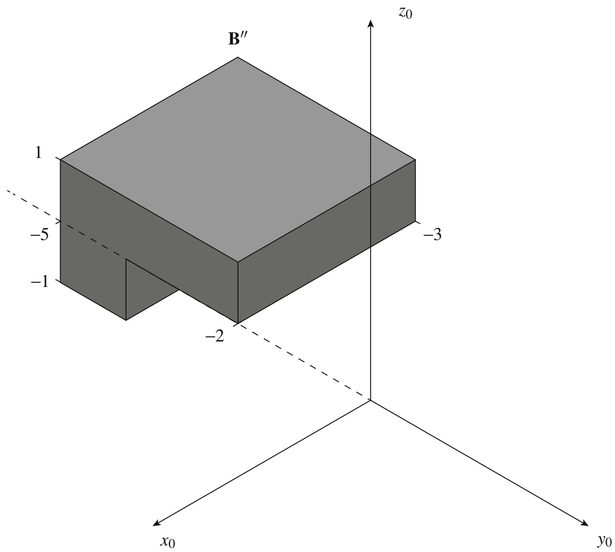

Fig. 3.10 The rotation of object B about the x_0 axis of reference coordinate frame

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final pose $\mathbf{B}'$ is obtained by premultiplication with matrix $\mathbf{D}$:

$$\mathbf{B}' = \mathbf{D}\mathbf{B} \tag{3.24}$$

$$\mathbf{B}' = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can check the correctness of the obtained final pose by the use of Fig. 3.9.

The task can be solved also by calculating the total displacement from the known initial and final pose, without decomposing the displacement into particular rotations and translations. After attaching a relative coordinate frame to the object B, the matrix $\mathbf{B}$ can be determined from Fig. 3.9, describing the initial pose, and the matrix $\mathbf{B}^{'}$, describing the final pose of the object. When postmultiplying Eq. (3.24) by $\mathbf{B}^{-1}$ on the right and the left side of the equals sign, we calculate the transformation $\mathbf{D}$ in the following form:

$$\mathbf{D} = \mathbf{B}^{'}\mathbf{B}^{-1}$$

The inverse matrix $\mathbf{B}^{-1}$ is obtained by Eq. (3.12):

$$\mathbf{B}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The desired displacement of the object in the reference frame is calculated as product of following matrices:

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We obtained the same homogenous matrix as in the first example.

Let us now place the object B over the object A with respect to relative coordinate frame attached to the object B. We now rotate the object B for 180° about the x axis, which is aligned along the edge of the object. The new pose of the object $\mathbf{B}''$ is shown in Fig. 3.11.

We can reach the final pose $\mathbf{B}'$ from the pose $\mathbf{B}''$ with only translational displacements. To avoid the object A, the object $\mathbf{B}''$ must be lifted for at least 4 units. Therefore, we first perform a translation for -4 units along the z axis. Then we slide with the object for -8 units along the y axis and finally translate it for -3 units along the x axis. Finally we drop the object for 1 unit, i.e. translate it for 1 along the z axis.

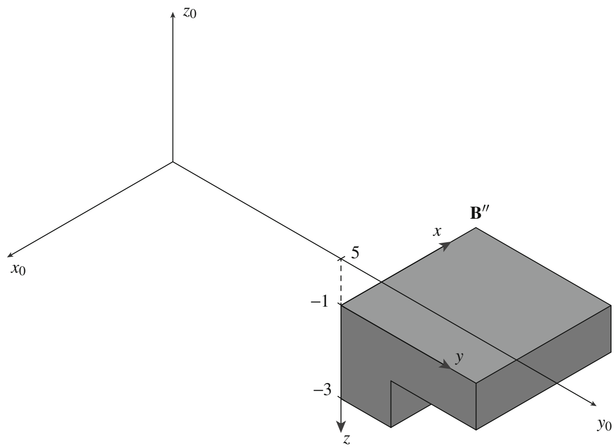

Fig. 3.11 The rotation of object B about the x axis of relative coordinate frame

We are dealing with the following displacement, written in the same order in which particular displacements were performed:

$$\mathbf{D} = Rot(x, 180^\circ)Trans(-3, -8, -3)$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final pose of the object $\mathbf{B}'$ is obtained by postmultiplication of the initial pose $\mathbf{B}$ by the transformation matrix $\mathbf{D}$:

$$\mathbf{B}' = \mathbf{BD} \tag{3.25}$$

$$\mathbf{B}' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can check by use of Fig. 3.9 that the final pose of the object was attained. The task can be solved in the same way as in previous example by only knowing the initial $\mathbf{B}$ and final pose $\mathbf{B}'$, which can be found from Fig. 3.9. By premultiplying Eq. (3.25) on both sides of the equals sign by $\mathbf{B}^{-1}$, we obtain:

$$\mathbf{D} = \mathbf{B}^{-1} \cdot \mathbf{B}'$$

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The same homogenous matrix was obtained as in the example when selected rotation and translation were performed with respect to the relative coordinate frame. The last problem can be solved without any calculations. The displacement $\mathbf{D}$ is equal to the pose of $\mathbf{B}'$ with respect to $\mathbf{B}$, which can be determined directly from Fig. 3.9 without considering the reference frame.

3.5 Perspective Transformation Matrix

When defining the homogenous transformation matrix (3.11), three zeros and a one were written into the fourth line. It appears that their aim is only to make the homogenous matrix quadratic. In this section we shall learn that the last line of the

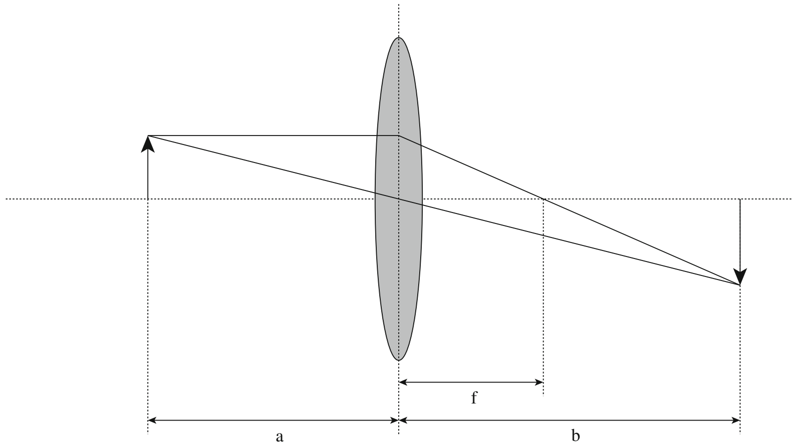

Fig. 3.12 Image formation by the lens

matrix means perspective transformation. The perspective transformation [3] has no meaning in robotics, it is however interesting in computer graphics and designing of virtual environments. The perspective transformation can be explained by formation of the image of an object through the lens with focal length f (Fig. 3.12). The lens equation is:

$$\frac{1}{\mathrm{a}} + \frac{1}{\mathrm{b}} = \frac{1}{\mathrm{f}} \tag{3.26}$$

Let us place the lens into the x, z plane of cartesian coordinate frame (Fig. 3.13). The point with coordinates $[x, y, z]^{\mathrm{T}}$ is imaged into the point $[x', y', z']^{\mathrm{T}}$. The lens equation is in this particular situation as follows:

$$\frac{1}{y} - \frac{1}{y'} = \frac{1}{\mathrm{f}} \tag{3.27}$$

The rays passing through the center of the lens remain undeviated:

$$\frac{z}{y} = \frac{z'}{y'} \tag{3.28}$$

Another equation for undeviated rays is obtained by exchanging z and z' with x and x' in Eq. (3.28). When rearranging the equations for deviated and undeviated rays, we can obtain the relations between the coordinates of the original point x, y, and z and its image x', y', z':

$$x' = \frac{x}{1 - \frac{y}{\mathrm{f}}} \tag{3.29}$$

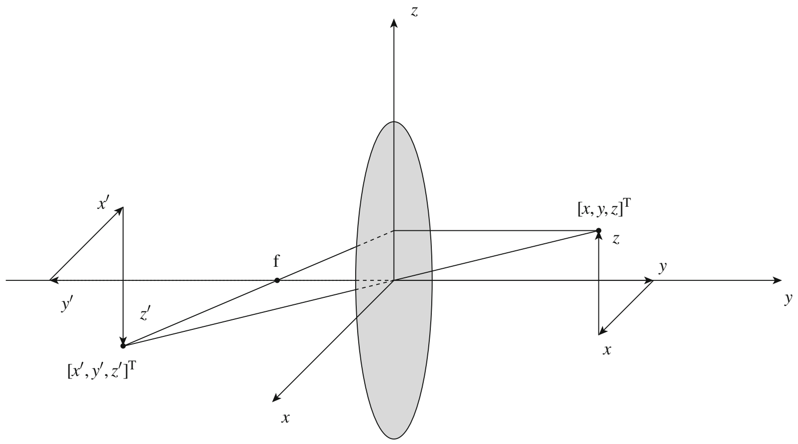

Fig. 3.13 Image of a point through the lens

$$y' = \frac{y}{1 - \frac{y}{\mathrm{f}}} \tag{3.30}$$

$$z' = \frac{z}{1 - \frac{y}{\mathrm{f}}} \tag{3.31}$$

The same result is obtained by the use of homogenous matrix **P**, describing the perspective transformation:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{\mathrm{f}} & 0 & 1 \end{bmatrix} \tag{3.32}$$

The coordinates of the imaged point x', y', z' are obtained by multiplying the coordinates of the original point x, y, z by the matrix **P**:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{\mathrm{f}} & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 - \frac{y}{\mathrm{f}} \end{bmatrix} = \begin{bmatrix} \frac{x}{1-\frac{y}{\mathrm{f}}} \\ \frac{y}{1-\frac{y}{\mathrm{f}}} \\ \frac{z}{1-\frac{y}{\mathrm{f}}} \\ 1 \end{bmatrix} \tag{3.33}$$

The same relation between the imaged and the original coordinates was obtained as in Eqs. (3.29–3.31). When the element $-1/\mathrm{f}$ is at the bottom of the first column, we are dealing with perspective transformation along the x axis, when it is at the bottom of the third column, we have projection along the z axis.

As an example let us solve the inverse problem. Let us consider the lens with the focal length $\mathrm{f} = 2$, which is placed into the x, z plane of cartesian coordinate frame (Fig. 3.13), so that the center of the lens coincides with the origin of the frame. A point $[x, y, z]^T$ is imaged into the point $[-1, -3, -2]^T$. It is our aim to calculate the coordinates of the original point. We need the inverse perspective matrix $\mathbf{P}^{-1}$. Knowing that the product $\mathbf{P}\,\mathbf{P}^{-1}$ equals the unit matrix, it is not difficult to realize:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{f} & 0 & 1 \end{bmatrix} \tag{3.34}$$

In this way the following numerical solution of simple example is obtained:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

The correctness of the solution can be checked by the use of Eqs. (3.29–3.31).

References

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