V.I. Ferronsky · S.V. Ferronsky

Formation of the Solar System

A New Theory of the Creation and Decay of the Celestial Bodies

Springer

Formation of the Solar System

V.I. Ferronsky • S.V. Ferronsky

Formation of the Solar System

A New Theory of the Creation and Decay of the Celestial Bodies

Springer

V.I. Ferronsky
Water Problems Institute of the
Russian Academy of Sciences
Moscow
Russia

S.V. Ferronsky (deceased)

This is a translation of the book Origin and Evolution of the Solar System (in Russian), published by Scientific World, Moscow, 2012

ISBN 978-94-007-5907-7 ISBN 978-94-007-5908-4 (eBook)
DOI 10.1007/978-94-007-5908-4
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2012956182

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book presents the solution of the problem of origin and evolution of the solar system based on Jacobi dynamics. The work's continuous study on the dynamics was published earlier (Ferronsky et al. 1978, 1979a, b, c, 1981a, b, 1982, 1984, 1987, 1996, 2011, Ferronsky 1983, 1984, 2005; Ferronsky and Ferronsky 2010).

By analysis of orbital motion of the Earth, the Moon, other planets, and their satellites, we discovered a common dynamical effect valid for all the solar system bodies. The effect demonstrates that all the planets and satellites have been orbited by the first cosmic velocity of their protoparents. Namely, the planets move in orbits with the first cosmic velocity of the protosun, the radius of which was equal to the semimajor axis of modern orbit of each planet. The satellites of each planet have mean orbital velocity equal to the first cosmic velocity of the corresponding planet having radius equal to the semimajor axis of modern orbit of each satellite. This effect holds for all the small planets of the asteroid belt and for all the comets.

We can state now that the discovered common dynamical effect of the celestial bodies' orbital motion with the first cosmic velocity of their protoparents demonstrates the nature of the forces, which initiate and govern this motion. The protoparental body originates these forces in the form of an integral effect of its constituting interacted elementary particles, which is the body's inner energy. In fact, this is Newton's gravitational force, which he searched for the solution of Kepler's problem. The Kepler's laws, in particular its third law, follow from the found dynamical effect of celestial bodies' orbital motion.

The found dynamical effect was used as a basis for more-detailed analytical consideration of the solar system's cosmogony. We demonstrate that all the solar system bodies have been formed, separated, and orbited from the upper weightlessness shells of their protoparents during the evolutionary process.

The details of the creation process like differentiation of the initial cloud into the shells, physics of the secondary body formation and first cosmic velocity orbiting, separation of the protosolar cloud itself from the protogalaxy, and other effects of the system origin and evolution are considered in the form of separate tasks solution.

The following basic physical principles were accepted for the problem solution. The Sun and other stars, their planets, and satellites are considered as self-gravitating celestial bodies, which themselves generate the energy for the motion by means of their constituent elementary particle interaction. The particle interaction is considered as a process of their collision and scattering. Because of absence of hydrostatic equilibrium of celestial bodies, found by the artificial satellite studies, the condition of dynamical equilibrium was introduced. This condition is based on the analysis of the artificial satellite orbital motion and also on the observable fact of disagreement with the virial theorem regarding the relationship between the potential and kinetic energy. The condition was accepted not as assumption but proved by derivation of the generalized virial theorem for n-interacted particles as volumetric matter values. This fundamental principle also follows from the Jacobi dynamics. In this case the energy is accepted as the measure of the particle interaction. The energy action is developed in the form of its inner pressure and accomplishes by oscillations of the moment of inertia. The resulting dynamical effect of a self-gravitating body at its dynamical equilibrium results in the periodically repeated oscillations of all the fundamental parameters like the moment of inertia, potential and kinetic energy, and their frequency and period of oscillation. In the other words, the inner energy initiates all the body's dynamical effects. In this connection, for instance, the widespread opinion that the hydrostatic equilibrium of stars (equation of state) results in the form of equality between the gaseous and gravity pressure appears to be a meaningless idea. In the case of a self-gravitating body, its gaseous pressure is the dynamical effect of interaction of the constituting particles, that is, its gravitational pressure. The measure of the body's interaction of mass particles is the energy but not the force being its first derivative. For a celestial body, the gravitational effect of its interacted masses is determined by integration of the interacted particle effects over the whole volume, that is, obtaining its energy.

In contrast to the hydrostatic equilibrium where the outer forces are used for solving the problems of motion under force action, dynamical equilibrium is based on the inner energy or on the inner integral force field. Dynamical equilibrium of celestial bodies opens new possibilities for studying the nature of their motion. Their own inner and outer force field determines dynamics of a celestial body. Earlier, the inner force field was accepted to be the central symmetric field of vector forces, the sum of which is equal to zero. For dynamical equilibrium, the interacted particles form the volumetric field of pressure which cannot be equal to zero by definition. Such a field of pressure can be reduced to a resultant shell of pressure. For a sphere it will be a spherical shell and for an ellipsoid this is an ellipsoidal shell.

We demonstrate that the basic mode of a body motion is its oscillation. Interaction of the uniform in density body mass realizes all its kinetic energy in the form of oscillations. For a nonuniform body, the tangential component of the potential energy appeared. This component is responsible for the body's axial rotation (tangential oscillation). It is assumed up to now that in mechanics of the macroscopic bodies the wave properties of such nature for massive particles are unessential. It is shown in this work that virial oscillations of a body masses represent the main part of kinetic energy. In the theories based on the hydrostatic equilibrium, this energy is ignored. But in this case, the potential energy of celestial bodies by two or more orders exceeds the kinetic one presented by axial inertial rotation of the masses. This effect has a simple physical explanation. In the beginning of the last century, the famous French physicist Louis de Broglie extended on the matter of the wave–particle duality theory of light. Later on, his theory was fully confirmed and becomes the basis for developing the present-day wave mechanics for matter on an atomic scale. The particles of greater mass, which are the subject of classical mechanics, have mainly corpuscular properties. The relationship between oscillation of the gravity field and the Earth moment of inertia, which was proved by artificial satellite data, shows that the interaction of its masses results on the level of elementary particles. The only form of motion of the interacted mass particles is their oscillation. The continuous "tremor" of the Earth's gravity field fixed by changes of the gravity moments is one more conformation of the de Broglie's idea for the mass interaction of celestial bodies.

Finally, the important effect of a body mass interaction is its outer force field. Its potential energy is changing according with the inverse square law (proportionally to the body's surface shell area), and the fundamental parameter of the field is its frequency of oscillation. The outer force field fills in all the space of the universe including galaxies, stars, and other bodies. And the oscillation frequency in a given point of the space indicates the energy emitted by the corresponding celestial body during its stay there and velocity of its elementary particle interaction. The outer force field is an indicator of legitimacy of the energy conservation law for the universe as a whole.

Such are the main physical principles used for the solution of the solar system origin and evolution problem, which follows from our previous studies.

The last chapter of the work considers some aspects of application of the obtained results to the universe problems. In particular, the results are used for interpretation of the "dark matter," "dark energy," and "Big Bang." The conclusion is made that our universe in framework of the accepted geometry is a closed pulsating system. During its expansion (present stage of evolution), the system's decay results up to the matter, becoming like "dark matter" with "dark energy" (weightlessness discrete particle matter). During the contraction stage, the mass particles (electrons, protons, nucleus, atoms, and molecules) and bodies are created in the form of a common galaxy being in the force field ("dark energy") of the universe. During the stage of expansion, the energy is emitted by the decaying bodies. During the stage of contraction, the "dark energy" is bounded into, what we say, "matter," which, in fact, is a form of the compressed mass defect.

V. Ferronsky

References

Ferronsky SV (1983) The solution for seasonal variations in the earth's rate of rotation. Celest Mech 30:71–83

Ferronsky SV (1984) Observation of coherent pulsations in the earth atmosphere with period about one and half hour. Phys Atmos Oceans 20:922–928

Ferronsky VI (2005) Virial approach to solve the problem of global dynamics of the earth. Investigated in Russia. http://zhurnal.ape.relarn/articles/2005/120.pdf, pp 1207–1228

Ferronsky VI, Ferronsky SV (2010) Dynamics of the Earth. Springer, Dordrecht/Heidelberg
Ferronsky VI, Denisik SA, Ferronsky SV (1978) The solution of Jacobi's virial equation for celestial bodies. Celest Mech 18:113–140
Ferronsky VI, Denisik SA, Ferronsky SV (1979a) The virial-based solution for the velocity of gaseous sphere gravitational contraction. Celest Mech 19:173–201
Ferronsky VI, Denisik SA, Ferronsky SV (1979b) The asymptotic limit of the form- factor α and β product for celestial bodies. Celest Mech 20:69–81
Ferronsky VI, Denisik SA, Ferronsky SV (1979c) The solution of Jacobi's virial equation for nonconservative system and analysis of its dependence on parameters. Celest Mech 20: 143–172
Ferronsky VI, Denisik SA, Ferronsky SV (1981a) Virial oscillations of celestial bodies: I. The effect of electromagnetic interactions. Celest Mech 23:243–267
Ferronsky VI, Denisik SA, Ferronsky SV (1981b) On the relationship between the total mass of a celestial body and the averaged mass of its constituent particles. Phys Lett 84A:223–225
Ferronsky VI, Denisik SA, Ferronsky SV (1982) Virial oscillations of celestial bodies: II General approach to the solution of perturbed oscillations problem and electromagnetic effects. Celest Mech 27:285–304
Ferronsky VI, Denisik SA, Ferronsky SV (1984) Virial approach to solution of the problem of global oscillations of the earth atmosphere. Phys Atmos Oceans 20:802–809
Ferronsky VI, Denisik SA, Ferronsky SV (1987) Jacobi dynamics. Reidel, Dordrecht
Ferronsky VI, Denisik SA, Ferronsky SV (1996) Virial oscillations of celestial bodies: V. The structure of the potential and kinetic energies of a celestial body as a record of its creation history. Celest Mech Dyn Astron 64:167–183
Ferronsky VI, Denisik SA, Ferronsky SV (2011) Jacobi dynamics, 2nd edn. Springer, Dordrecht/ Heidelberg

Contents

Chapter 1
Introduction: New Data Related to the Nature of Creation and Orbiting of the Planets and Satellites

Abstract The authors' discovery of a dynamical phenomenon that throws light on the nature of creation and orbiting of the planets, their satellites, and the Sun itself is presented in this chapter. In addition, a short review of the more important cosmogony hypothesis and celestial body motion laws, based on hydrostatics, is presented.

All the authors assume that celestial bodies were created from a gaseous or gas-dusty cloud (nebula). But in order to obtain the observed parameters of motion and matter content of the Sun, the planets, and the satellites, some of the authors consider a common cloud, and the others accept separate clouds for the Sun and the planets with the satellite creation. Philosophic ideas of Rene Descartes and Immanuel Kant about the creation of the universe and the hypotheses of Kant–Laplace, Buffon, Jeans, and others are described briefly.

The heliocentric world system of Copernicus, Kepler's laws of the solar system and planets' motion, Huygens' semicubic parabola law in pendulum motion, and Newton's solution of Kepler's problem, based on hydrostatic equilibrium state, are analyzed.

The main points of the common dynamical effects found for all the solar system bodies are as follows: applying Jacobi dynamics, it was found that the mean orbital velocity and the period of revolution of every planet are equal to the first cosmic velocity and corresponding period of virial oscillation of the protosun, with its radius equal to the semimajor axes of the planet's orbit. The same effect holds for all satellites of the planets and other small bodies.

The first cosmic velocity v_1 of the protosun and protoplanetary bodies and the period of oscillation of the corresponding outer shell $_1$ of the created bodies were calculated by the formulas from which the third Kepler's law follows: $v_1 = \omega R = R\sqrt{Gm/R^3} = \sqrt{Gm/R}$, $T_1 = 2\pi/\omega = 2\pi R/v_1$, and $(2\pi)^2/{T_1}^2 = Gm/R^3$. Here, m is the body's mass; G is the gravity constant; R is the semimajor axis; and

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_1,

$\omega = v_1/R$ is the frequency of virial oscillation of the outer shell, which appears to be equal to the angular velocity of orbital motion of the created body.

This fact opens the way for solving the cosmogony problem as a problem of dynamics.

The problem of the origin of the Earth and the solar system evidently appeared together with the appearance of man himself. The interest was initiated and developed by the natural normal and catastrophic events, which were necessary to be predicted. Step-by-step after accumulation of the observational results, the ancient stargazers started to think about the problem of the creation of the universe. The first thoughts about this have been expressed in ancient myths, legends, and religious writings.

It is obvious that the theory of origin of the celestial bodies should be based first of all on the laws of their motion and on the specific features of the nature of their substantial content and formation of properties. As early as in the fourth century BP, Aristoteles formulated fundamentals of the geocentric system of the world. And two centuries later on, Ptolemaeus in his 13-volume description entitled *The Great Mathematical Construction* with the Arabian name *Almagest* developed further the above work for practical applications. But only in the sixteenth century, after the Copernican heliocentric world system appeared and Kepler's laws of the planets' orbital motion were discovered, the problem of the creation of the solar system and the universe as a whole became one of the main problems in natural sciences and in philosophy. At present, more than 50 million references of cosmologic and cosmogony publications are collected in internet sites, which demonstrate the scale of interest in the problem. But its resolution has not progressed beyond speculative description of possible origin events presented in the style of scientific fantasy. Any solutions, based on the natural laws and facts proving the nature of the body formation, have not yet been obtained.

Let us make a short review of the more important cosmogony hypothesis and consider once more the laws of celestial body motion and also the dynamical phenomenon discovered by the authors, which throws light on the nature of creation and orbiting of the planets, their satellites, and the Sun itself.

1.1 Hypotheses of Celestial Body Creation

There is one common idea regarding the existing numerous hypotheses of the solar system origin. Practically all the authors assume that celestial bodies were created from a gaseous or gas-dusty cloud (nebula). But in order to obtain the observed parameters of motion and matter content of the Sun, the planets, and the satellites, some of the authors consider a common cloud, and the others accept separate clouds for the Sun and the planets with the satellites. And also in some hypotheses, the protoplanet's cloud was captured by the protosun during its motion, and in the others the protoplanet's cloud was taken away from the protosun by another star passing along.

Rene Descartes (1596–1650), the French philosopher, physicist, and mathematician, descendant from the eminent Cartesian family, was the first to draw the general picture of celestial body creation from matter, which fills the universe and is composed of moving elementary particles. Descartes did not recognize indivisible atoms and a void space. In addition to the visible matter, he recognized a form of invisible fine matter. The latter one he used for explaining the effects of heat, gravity, electricity, and magnetism. He assumed that there is no place in physics for the forces, which act through the void space distant. This conception was called Cartesian and remains until now.

To contrast to the finite world, Descartes assumed that the world matter is infinite and uniform. Matter has no voids, and the particles are infinitely divisible. Descartes' principle of velocity conservation, which he understood as the law of the momentum conservation, is interesting because of its cyclic vortex motion of the universe matter. Because of absent voids, he considered any matter motion as cyclic: if one part of matter is moving, then it is replaced by some other and the latter by the next one and so on. As a result, the total universe is penetrated by vortex motion. The motion in the universe and the matter itself are perpetual. All the natural events result in motion of the particles of matter.

The main form of motion he accounted to be inertial motion. The matter is represented by the elementary particles, the local interaction (collision) of which initiates all the natural events. Through Descartes, creation of matter and its first momentum was done by God.

Descartes states the further development of the cosmogony on the basis of his three laws of motion:

1. Any simple and indivisible object remains invariable until some other subject changes it by means of interaction.
2. The initial form of a body motion is the rectilinear motion.
3. At collision of one body with the other that is stronger, the first one loses nothing. At collision with a weaker one, the first body loses as much as it transmits to the other one.

Initially random motion of the uniform infinitesimals and their collision leads to the creation of the multiplicity vortexes. Gradually and sporadically by the above three laws, the chaotic world has been converted into the observed universe. The fine particles which form matter of first generation form the stars and the Sun. Spherical particles of the second generation create the interstellar space (the sky). More coarse and cohesive particles of the third transform their vortex into the Earth and the planets which are revolving about the Sun.

The laws of mechanics and the idea of the vortex motion allowed Descartes to explain the daily rotation and yearly revolution of the Earth about the Sun. He wanted to construct the nature where all the events result by the effect of motion and interaction (collision) of the fine and coarse particles formed from the initial elementary particles.

That is the general picture of the creation of the world by Descartes. His hypothesis of the vortex universe and the creation of celestial bodies on this basis was supported by Fontenelle, Leibniz, Huygens, Bernoulli, and other known philosophers and scientists of that time. But some of them criticized separate positions and even rejected all the philosophy. For example, the well-known French philosopher, mathematician, and astronomer Pierre Gassendi said, “One must be surprised that such a great geometer instead of proving he has proposed sleep dreams.” In this connection it is worth noting that the later cosmogony hypotheses up to the present day cannot demonstrate the proof, including the most popular Kant–Laplace’s hypothesis.

Immanuel Kant (1724–1804), the parent of classical German philosophy, published his fundamental work *Universal Natural History and Theory of Heavens* (1755) when Laplace was only 6 years old. Laplace published his hypothesis in 1796 in the work *The System of the World*, where he has not even mentioned Kant’s name. Nevertheless, their names were joined because both authors used in their works the nebular hypothesis in explaining the origin of the solar system. They used different arguments and approaches to the problem. They applied different assumptions and obtained diverse conclusions and predictions. But both hypotheses have a common basis, namely, Newton’s law of attraction.

Kant considered the matter as a substance dispersed all over the universe and being in the state of a general chaos. He assumed that matter is formed on the basis of known laws of attraction and changes its motion due to collision and repulsion. The matter of all substances, he said, is subjected to specific laws and being free in action must find fine combination. It cannot deviate from the tendency to harmony. Assuming the world to be in the state of chaos by the attraction and repulsion forces, which are equally reliable, he has explained the great order in nature: “Give me the matter and I will construct the World,” said Kant.

Kant’s universe has had the beginning and the end, which becomes its beginning. He expressed and justified the hypothesis of a pulsating universe, in fact, going up to the Big Bang idea.

Pierre-Simon Laplace (1749–1827), the great French astronomer, physicist, and mathematician, is the author of many classical works on celestial mechanics, analytical theory of probability, differential equations, and other sections of sciences. He was the member of many royal societies in Torino and Copenhagen and academies of sciences in Göttingen, Berlin, and Petersburg.

Laplace tends to explain all the visible motion of the celestial bodies by Newton’s law of universal gravitation. He said that the mean acceleration of the Moon’s motion is the only celestial event which up to now has not obeyed the law of gravity. But Laplace also found an explanation for this. For him, this is because the long periodic change of the Earth orbital eccentricity discovered by astronomers leads to changes in the Moon’s velocity.

In 1796, Laplace published the book *The System of the World* in which he expounded in popular form his fundamental five-volume work *Tractate on Celestial Mechanics*. In the appendix, Laplace presented there his cosmogony hypothesis on the origin of the solar system. He assumed that the solar system has been created from the rotating hot gaseous nebula, the diameter of which extended beyond the last planet. While cooling, the nebula has been divided by the centrifugal forces that appeared into gaseous rings from the outer border. The planets and the satellites have formed from the rings. Laplace proved the stability of the motion of the solar system. For this, he used the facts such as the orbital body motion to the same size, small orbital eccentricities, and small inclination of the orbits, which determine the invariability of the distances from the Sun and the narrow limit of changes in the orbital elements.

This hypothesis has captured common imagination. And only a century after the discovery of new regularities, some discrepancies have appeared, which caused doubt in its admissibility. The main doubt was the difference in distribution of the mass and moment of momentum between the Sun and the planets. It will be shown in this work that this obstacle is a scientific misunderstanding.

Georges-Louis Buffon (1707–1788), French naturalist, mathematician, and encyclopedic author, member of honor of the Petersburg Academy of Sciences, in voluminous *Histoire Naturelle*, alongside with a general description of historical development of nature, gives his view on the origin of the Earth and other planets. He assumes that the planets are the fragments of the Sun divided during the falling of the comets. In the Earth's history ($\sim$85,000 years), he derived seven periods during which the cooling, rock formation, mainland development from the oceans, plant, and animal growing, and finally human appearance have taken place.

Frederick Wilhelm Herschel (1738–1822), German astronomer and mathematician, has made a huge amount of astronomical observations and discoveries. In 1781, he discovered the planet Uranium and its two satellites. He also discovered two satellites of Saturn and a number of other solar system bodies, about 400 double stars and a multitude of nebulae. Herschel's catalog includes 2,500 nebulae, some of which are surrounded by gaseous clouds and others without those. Extensive information was obtained about the structure of our galaxy. His sister Caroline and son John were also astronomers. Many authors and researchers of the cosmogony problem used Herschel's observational data for proving own or refuting others' hypotheses.

James Jeans (1877–1946), English theoretician, physicist, mathematician, and astronomer, established the law of energy distribution in the long-wave part of the star radiation spectrum (the Rayleigh–Jeans' law) in 1905, which establishes the relationship between the density of radiation energy of the absolute black body and the source temperature.

Jeans stated that his analysis of the rotating bodies disproves Laplace's hypothesis of the formation of the solar system from a common gaseous cloud. In 1922, he proposed his own tidal theory of planet formation from the matter removed from the Sun by a star passing along. In spite of the absence of astronomical observation (including Herschel's) of wandering stars, the idea of tidal appearance of clouds is still in the arsenal of cosmogony until now.

The idea of the tidal origin of the protoplanetary cloud was put forth in 1905 by the US astronomers T. Chamberlin and F. Moulton. Russian academician O. Yu. Schmidt in 1943 developed the idea of tidal capture by the Sun of a gas-dust cloud, from which the planets were formed (Schmidt 1957). Astronomer G. Russell in 1935 assumed that the Sun was a binary star, the second component of which was destroyed into protoplanetary cloud by tidal effect of a passing third star. Astronomer R. Lyttleton in 1936 proposed that the Sun was an unstable triple star. After two stars separated, they left a part of gaseous matter, from which the planets were formed. H. Alfven in 1942 assumed that the Sun, on its way, met a gaseous cloud, atoms of which were ionized and while moving created the planets. C. F. von Weizsäcker in 1944 proposed that after contraction of the protosolar cloud in its outer shells, the vortexes were formed, from which the planets were created. In the same year, F. Hoyle wrote that the Sun was a binary star, where the second component was a supernova. After it flared, the protoplanetary cloud remained, and the star left the system. Russian academician V. G. Fesenkov in 1953 developed the idea of the origin of the Sun and planets from a common gas and dust cloud within the general dynamical evolutionary process. He assumed that the stars were also created from the interstellar gas and dust matter. A. Cameron in 1962 proposed that during near explosion of a supernova, compaction in a massive cloud of gas has occurred. The cloud compression resulted in its separation on smaller condensations; one of them was the protosolar nebula of the outer planet size. The nebula rotated and has had notable momentum. The centrifugal forces developed a gaseous disk around the cloud, which has separated into rings. Finally, the planets and satellites have formed from the rings (Cameron 1973; Cameron and Pine 1973).

Researches of the hypotheses of the origin of the solar system underline a number of important parameters in the structure and motion of the bodies, which do not find observational conformation or theoretic justification. Some of them are as follows (Kuiper 1951; Spenser 1956):

1. The main dynamical contradiction in distribution of the moment of momentum between the Sun and the planets (the Sun has 0.5% of the total moment of momentum and 99.8% of the total mass) remains unsolved.
2. The Earth-group planets have high density and relatively small mass; at the same time, the Jupiter's are of low density and one- to two-order mass higher.
3. The Jupiter's planet group has many satellites, but the Earth's planets have practically none.
4. There was no explanation of the Titius–Bode law of the planetary distances.
5. The observational data related to roaming stars in the stellar space are absent.
6. There was no physical explanation for the axial inertial rotation of celestial bodies.
7. There is no explanation for the nature of the initial gaseous cloud appearance from which the Sun, other stars, and planets were formed.

Research has given rise to many queries, like the ones just listed, related to individual hypotheses of the solar system's origin. Here, we stress that all hypotheses considered the laws of celestial body motion based on the hydrostatic equilibrium state. In other words, the studied cosmogony task, which is the fundamental problem of dynamics, was considered on the hydrostatic basis. But it was the objective reality of the development of science in the past epochs.

1.2 The Laws of Celestial Body Motion Based on Hydrostatics

In the Aristotelian and Ptolemaic geocentric world system, the Earth was accepted as the center of the universe. The Earth was assumed to be surrounded by the nine “heavens.” They were Moon, Sun, Mercury, Venus, Mars, Jupiter, and Saturn. The stars represented the eighth heaven and, on the ninth one located is “the spirit,” which governed all the heavens’ motion. That is why the term “heavens” has appeared in our vocabulary. In his *Almagest*, Ptolemaios placed the motion of the Sun, the Moon, and the planets and the position of the 1,022 stars from the Hipparchos catalog in the form of tables in an elliptic system of coordinates.

A decisive step toward the modern conception of the world system was made by Polish astronomer *Nicolaus* C*opernicus* (1473–1543). He found from planet observations that when a planet happens to be in the opposite point in the sky from the Sun, its orbit does not remain fixed. The upside line between the perihelion and aphelion points changes its position in comparison with that when it was in the Ptolemaios’ *Almagest*. The Ptolemaios’ theory has not explained many observed astronomical events, for example, the loop-like motion of planets along the firmament. Long before Copernicus, the ancient Greece astronomer Aristarchus of Samos stated that the Earth revolve around the Sun, but he could not confirm it. After many years of hard work and complicated calculations, Copernicus concluded that Ptolemaios’ theory was not correct. He found in his tables a number of contradictories and came to the conclusion that the planets’ motion appears to be more ordinary if their center is the Sun. That was his way of understanding that the planets revolve around the Sun. His year-long observations and calculations were expounded in 1543 in the work *About Revolution of the Sky Spheres*. This work was the start of a new modern conception about the hierarchic relation in the galaxy system–Sun–planets–satellites.

Copernicus’ heliocentric world system became the basis for Kepler’s important discovery of the three laws of the solar system planets’ motion. Johannes Kepler (1571–1630), German astronomer and mathematician, discovered these laws on the basis of Tycho Brahe’s many years of observation of the motion of planet Mars. In 1609, Kepler published his *New Astronomy*, where he exposed the results of evaluation of Brahe’s observation and presented the first and second laws. The first law states that the orbit of the unperturbed planet’s motion is a curve of the second order, in one of the focuses of which the Sun is situated. It follows from the second law that the radius vector joining the unperturbed planet’s motion around the Sun sweeps out equal areas in equal times. In 1619, Kepler published a new work entitled *Harmony of the World*, where he described his third law. In accordance with that law, in unperturbed elliptic motion of the planets, the ratio of the square of their periods of revolution to the cube of the semimajor axes is the same.

After the discovery of the Copernican heliocentric system, it was assumed that the planets move around the Sun along circular orbits. Kepler, being a convinced follower of Copernicus, shared the same idea. But after analysis of Brahe’s data, he

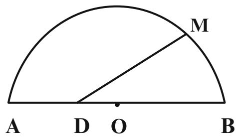


Fig. 1.1 Kepler's problem

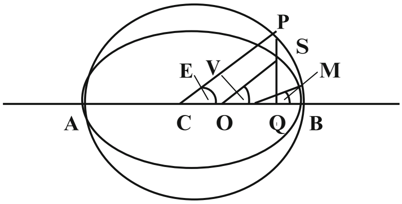


Fig. 1.2 The true *V*, mean , and eccentric anomalies for determination of a body's position on the orbit by Kepler's equation

found that the observational points of the planet's annual motion did not describe circles. The points inscribe the circle but do not form it. However, the results from processing the data of the planet's annual trajectories indicated that they do describe some form of spatial curves. In order to reduce the space coordinates of the planet's motion to mean values and to obtain the plane elliptic figure, Kepler developed a specific method of averaging the observational points based on inscribing polygons into a circle and calculation with infinitesimals. Finally, Kepler succeeded in finding a methodology of reducing the data, which allows one to obtain an elliptic trajectory and to formulate the first two laws of the planets' motion. In *New Astronomy*, Kepler presented this method in the form of the following geometric solution of the problem, which allows one to find elements of a planet's orbit, satisfying his first two laws of the planets' motion. Across point *D* on diameter AB of semicircle AOBM (Fig. 1.1), the straight line DM should be drawn in such a way that it divides the area in the given ratio. The problem was written in the following transcendental equation:

$$y - c \sin y = x, \tag{1.1}$$

which is solved by the given values and , when ($|c| < 1$). Here, is the figure's eccentricity, which at a value less than unit gives an ellipse and at zero gives a circle. The value characterizes the scale of averaging taken as a ratio of the semicircle areas formed by the line DM.

Equation (1.1) represents projections of the reduced space coordinates of a body's motion along its orbit on the plane. With the help of this equation, astronomers could determine the body's position on a point of the orbit at a given moment of time and solve the reverse problem of determination of the time moment of a body passing through a given point of the orbit. To come back from the projections of the trajectory on the plane to space coordinates in the sky, three angles called the true, mean, and eccentric anomalies are used (Fig. 1.2).

Figure 1.2 demonstrates the true anomaly expressed by angle *V* between the pericenter of the orbit and radius vector S of the body in the direction of the body's motion. In accordance with the second Kepler's law, the angle V changes in time faster when the body moves in orbit to the pericenter , and its motion is slower in a direction away from the pericenter .

The mean anomaly is determined by angle , which lies between the direction to the pericenter and radius vector of a fictitious point, but it is assumed to be moving with constant velocity during which that point passes pericenter B and apocenter simultaneously with the real body. Thus, while moving from point to point , the real body precedes the fictitious point, whereas moving from A to B, the real body lags behind it.

The eccentric anomaly is expressed by angle with the point in the center of the orbit and situated between the direction to the pericenter and point P. That point lies on the circle drawn from the geometric center of the orbit and the perpendicular Q, carried out on the diameter, and passes through the point S. The point plays an auxiliary role to determine the mean and true V anomalies by formulas

$$E - e \sin E = M, \tag{1.2}$$

$$M = M_0 + v(t - t_0), \tag{1.3}$$

$$tg\frac{V}{2} = \sqrt{\frac{1+e}{1-e}}tg\frac{E}{2}, \tag{1.4}$$

where is the eccentricity of the orbit; $_0$ is the mean anomaly at some initial moment of time t_0, which is accepted as an element; and v is the mean value of the body's orbit velocity of motion.

It is clear that the meaning of Kepler's problem, presented by Eq. (1.2), is to inscribe into a circle an ellipse, which is the real averaged trajectory of the body motion on the orbit, by applying the mean velocity value of the motion v and the mean anomaly $_0$. Herewith, the inscribed ellipse must touch the circle only in two points of the body's orbit, namely, in the perihelion and aphelion .

Kepler's two first laws and the equation represent the averaged space picture of a planet's motion over the period of revolution around the Sun. They do not describe small variations of the motion parameters within each period of revolution or from one period to another. Those variations of the parameters' motion are smoothed by the mean anomaly , and Kepler's laws and the equation express conditions of the hydrostatic equilibrium of the system. Kepler's equation was solved by Newton in his two-body problem in order to find the force which sets the body in motion. Newton's solution was done in the framework of Kepler's formulation of the problem, that is, for the condition of hydrostatic equilibrium of a planet's motion. This remark is important for understanding the logic of his judgment and geometric construction which Newton used for solution of the two-body problem and the problem of the Earth's oblateness. As to the method of averaging of Kepler's space coordinates by means of the infinitesimals, it served to be the ideological base for development of the differential and integral calculus originally initiated

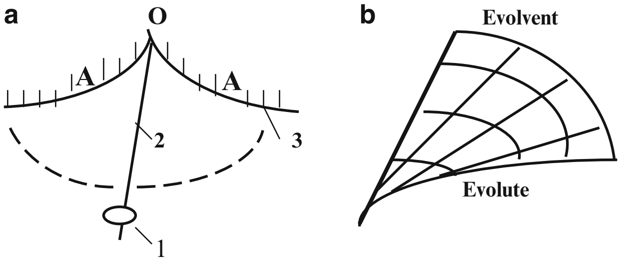


Fig. 1.3 Scheme of the Huygens pendulum clock ( ), evolvent and evolute (**b**): pendulum (*1*); filament of suspension (*2*); cycloid (*3*)

by Newton and Leibniz simultaneously, but obviously not without the influence of Kepler's and Huygens works.

Christian Huygens (1629–1695), the Netherlands physicist, mathematician, mechanic, and astronomer, was the founder of the wave theory of light and the theory of probability, the author of the first pendulum clock, and the investigator of the pendulum laws of motion which synchronously follow the Earth's motion. At 22 years of age, he published his first work about determination of the arc length of circle, ellipse, and hyperbola. And after 3 years, he writes about the ratio of the circle's length to its diameter, which was called π. Then, there was the work *About Calculation of the Bones Game*, where studies of cycloid, logarithmic, and chain lines were undertaken and which became a part of the foundation of the theory of probability. Together with Hooke, he established the points of freezing and boiling of water. At the same time, Huygens actively worked over increasing of luminosity in astronomical telescopes. In 1655, with his own instruments, he discovered the satellite Titan of Saturn, its rings, the nebulae of the constellation Orion, and the poles of Jupiter and Mars.

Astronomical observations always needed precise and easily calculated measurements of time. In 1657, Huygens designed the first pendulum clock to be driven by a trigger mechanism of motion. In the next year, he published a treatise *The Pendulum Cloak*, where his description of the discovery and the study of the pendulum clock motion were presented. It was known that the period of oscillation of a pendulum depends on the amplitude of oscillation. In order to determine the precise motion of the clock, Huygens developed a construction, astonishing even for modern standards, schematically presented in Fig. 1.3.

Figure 1.3 shows, by dashed lines, barriers having a cycloid configuration, which bounds the swings of elastic filament of the suspended pendulum. The filament from a suspension point O up to some point sags to both sides of the cycloid. Below point , the filament is held tight by the weight of the pendulum due to its motion to that point along the cycloid. During that motion the pendulum itself traces the cycloid (shown by dashes). In such a device, the period of the pendulum oscillation does not depend on the amplitude of the oscillation.

The described Huygens' project was not realized because at that time a more suitable design to solve the problem of synchronizing the oscillations was found. The interest in Huygens' technical idea has been lost, and his name is mentioned only in differential geometry in connection with his introduction of the curves known as evolvent and evolute (Fig. 1.3b). In our time, the idea of Huygens is used for the design of geophysical devices like gravitational variometers and gravimeters for measurement of the Earth's gravitational field. Technical solutions for such devices were proposed at the end of the nineteenth century by Hungarian physicist Eötvös. However, Huygens' study on pendulum motion contains much more fruitful, although unrealized, ideas.

Recall that the evolute is a curve which is formed from the locus of the centers of curvature of another plane curve (evolvent). The equation of this curve is a semicubic parabola. The evolvent is an unwound form of a curve perpendicular to a family of tangents to the evolute. The meaning of Huygens' idea is the following: first, the relation between an evolute and evolvent represents a relation between a function and its derivative or between a function and its integral. But these relations exist in the integrated form and are geometrically observable but not in a local form such as in mathematical analysis. Secondly, as it is seen from the drawn family of such unwound curves with different fixed lengths of the pendulum filaments (Fig. 1.3b) in each point of the initial curve, the corresponding evolvent has a peculiarity. And third, an important point for us, the marked peculiarity is always of the same type. It is a semicubic parabola like $x^2 = y^3$ or $\quad = \quad ^{3/2}$. This is a universal law, being a consequence of the simple fact that in each task related to motion, we always have some initial conditions inherited by the moving object. For example, in the case of Huygens' swinging pendulum, its suspension filament winds away from the curve at some fixed point.

If one recalls Kepler's laws, then it is possible to notice their important property. The first two laws determine the trajectory of the same body. The third law relates to the family of trajectories traced by different planets of the same solar system family. According to this law, squares of periods of the revolving planets are proportional to cubes of their semimajor axes. It means that on the plane of time coordinates, this law is expressed by a semicubic parabola. And in turn, this law is an evidence of the fact that if the motion is considered in the space of time, but not in the space of configurations, then Kepler's laws express a universal law of the nature in integral form. Here, constancy of the light velocity plays the role of isochronism of the oscillations.

Huygens applied the design and study of the pendulum motion to description of the elastic wave propagation, including in anisotropic media (double refraction of light beams in crystals) which he considered in *The Treatise on the Light*. He discovered here one more effect. Namely, the line of peculiar points, which was discussed earlier, determines the edge of the region (this is a space edge according to Huygens). The sphere has no such edge. It appears when the waves propagate inside the closing curve. Huygens considered a case with ellipse (Fig. 1.4).

Here, waves propagate with constant velocity to the inner hollow of an ellipse. At the beginning, the curve is transferred equidistantly to the ellipse. After that, a time

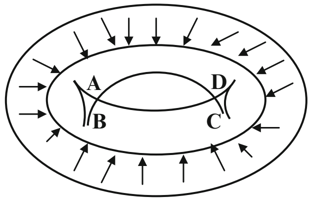


Fig. 1.4 Huygens' peculiar points for a family of evolvents

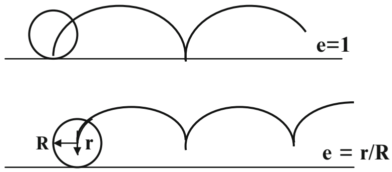


Fig. 1.5 Huygens' solution of Kepler's equation

comes when the peculiar points , , , and *D* appear. That is not a continuum but a limited number of points. If the considered region is not spatial as Huygens discussed but is a space–time region, then the phase transition phenomenon appears, which is described by the van der Waals cubic equation.

Finally, there is one more important element of analysis proposed by Huygens. In fact, he introduced into physics an integral approach to studying the behavior of a system. An example to explain his approach is a straight-rolling wheel. A point on its rim or on the spoke traces the curve, which gives the solution of Kepler's equation (Fig. 1.5).

Some of Huygens' profound ideas are far from being realized. The laws of the pendulum motion of his clock, which in detail and synchronously follows the Earth's motion, could be physically demonstrated by the appropriate technical implementation not only for gravimetry but also for study of the planet's dynamics. As to the theoretical conclusions, we used them in our previous works, and reference will be made later in the book.

Isaac Newton (1643–1727), the genius and the intellectual English mathematician and physicist, is the founder of classical mechanics and astronomy and the originator of the law of gravitation. His merits and contribution to the development of the natural sciences are difficult to be overestimated. The main task of Newton's scientific work became the generalization of the scientific results of Copernicus, Kepler, Galileo, Huygens, Borelli, Hooke, Galley, and other predecessors and contemporaries, all of whose work was presented in his *Philosophiae Naturalis Principia Mathematica* and published in 1686. In that book, a mathematical (geometric) approach was used for solution of the problems of celestial mechanics and dynamics of the Earth. Later on, an analytical basis for such a purpose was developed by Lagrange, Euler, d'Alembert, Hamilton, Jacobi, Cauchy, Bernoulli, and other mathematicians in the seventeenth to nineteenth centuries.

Newton adopted the condition of the Earth's hydrostatic equilibrium state together with the Keplerian laws of motion and his problem. That model of equilibrium comprises the basis for solution of the two-body problem and the problem of the Earth's oblateness.

Newton opens his work with definitions of matter; momentum; innate, applied, centripetal force and with formulation of his three laws of motion. In Book I *The Motion of Bodies*, the solution of the two-body problem is presented. In Book II *The Motion of Bodies (In Resisting Medium)*, the hydrostatics theorems are discussed. And in Book III *The System of the World*, the solution of the Earth's oblateness problem is considered. Let us recall the original Newton's formulations of the more important principles which we cite and discuss later on in this book. For that purpose, we quote from the English translation of Newton's *Principia*, made by Andrew Mott in 1729 (Newton 1934):

Definition I. The quantity of matter is the measure of the same, arising from its density and bulk conjointly.

Definition II. The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.

Definition III. The vis insita, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be rest, or moving uniformly forwards in a right line.

Definition IV. An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.

Definition V. A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the center of the earth; magnetism, by which iron tends to the load stone; and that force, whatever it is, by which the planets are continually drawn aside from the rectilinear motion, which otherwise they would pursue, and made to revolve in curvilinear orbits.

The quantity of any centripetal force may be considered as of three kinds: absolute, accelerative, and motive.

Definition VI. The absolute quantity of a centripetal force is the measure of the same, proportional to the efficacy of the cause that propagates from the centre, through the spaces round about.

Definition VII. The accelerating quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.

Definition VIII. The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.

These quantities of forces, we may, for the sake of brevity, call by the names of motive, accelerative, and absolute forces; and for the sake of distinction, consider them with respect to the bodies that tend to the centre of forces towards which they tend; that is to say, I refer the motive force to the body as an endeavor and propensity of the whole towards a centre, arising from the propensities of the several parts

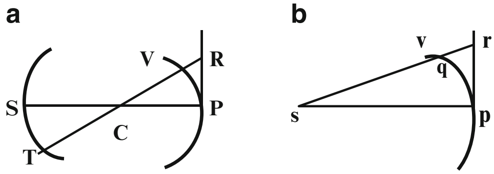


Fig. 1.6 The problem of two bodies mutually attracted

taking together; the accelerative force to the place of the body, as a certain power diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the space round about; whether that cause be some central body (such as is the magnet in the centre of the magnetic force, or the earth in the centre of the gravity force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical cause and seats…

I likewise call attractions and impulses, in the same sense, accelerative and motive; and use the words attraction, impulse, or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore the rider is not to imagine that by those words I anywhere take upon me to define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centers (which are only mathematical points); when at any time I happen to speak as attracting, or as endued with attractive powers.

Law I. Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Law II. The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Law III. To every action there is always opposite and equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

The theorem about mutual attraction of two bodies Newton formulates and solves as follows:

Theorem XI. *If two bodies attract each other with forces of any kind, and revolve about the common centre of gravity: I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other.*

Let the bodies S and P (*Fig.* 1.6*a*) *revolve about their common centre of gravity proceeding from S to , and from to Q.*

From the given point s (*Fig.* 1.6*b*) *let there be continually drown sp and sq equal and parallel to SP and TQ; and the curve pqv, which the point p described by point* *at its revolution will be equal and similar to the curves which are described in its revolution round the fixed point S, will be similar and equal to the curve which the bodies S and P describes about each other; and therefore, by Theor. XX, similar to the curves in curves S and QV which the same bodies describe about their common centre of gravity ; and that because the proportions of the lines S , , S or sp, to each other given.*

At the end of Book III, after discussion of the Moon motion, the tidal effects, and the comets' motion, Newton concludes as follows:

> Hitherto we have explained the phenomena of the heavens and our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to very centers of the sun, and planets, without suffering the least diminution of its force, that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes used to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always as the inverse square of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately as the inverse square of the distance as far as the orbit of Saturn, as evidently appears from the quiescence of the aphelion of the planets; nay even to the remotest aphelion of the comets, if those 4 aphelions are also quiescent.
>
> But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea.
>
> And now we might add something concerning a certain most subtle spirit which pervades and lies in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the solid filaments of the nerves, from the outward organs of sense to the brain, and from brain into the muscles. But these are things that cannot be explained in few wards, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic spirit operates.

Lagrange referred to Newton's work as "the greatest creature of a human intellect." It was published in England in Latin in 1686, 1713, and 1725 in his lifetime and many times later on. We reiterate that the passages about are from the translation by Andrew Mott in 1729 that was printed in 1934.

As it follows from Newton's definition of the centripetal innate forces, his understanding of their meaning and action in the nature is very wide. The innate force of matter is the power of resistance. It can develop as the force of body's resistance due to which it remains at rest or moves with constant velocity. It can develop as a body's resistance (reactive) force to an outer effect and as a pressure when the body faces an obstacle. In modern mechanics, this force is understood synonymously as the force of inertia. The resistance force or force of reaction has found its place in the theory of elasticity, and the pressure force is used in hydrodynamics and aerodynamics.

The main meaning of the centripetal force which was introduced by Newton is that each body is attracted to a certain center. He demonstrates this ability of bodies and objects on the Earth to attract to its geometric center by action of the gravity force. Newton distinguishes three kinds of manifestation of the centripetal force, namely, absolute, accelerating, and moving. The absolute value of this force is a measure of the source power of its action from the center to outer space. The body's attraction to the center and emission of the attraction from the center are demonstrated by Newton in Book III *The System of the World*, where in Theorem II, he notes that gravity forces from the planets are directed to the Sun. In Theorem I , he says that attraction of the planets themselves goes from their surfaces to the centers. According to Newton's idea, the planet's surface is an area of formation of absolute value of the centripetal force from where it emits that force upward and down from.

The accelerating value of the centripetal force by Newton's definition is a measure proportional to velocity which it develops over a long time. The moving value of the centripetal force is a measure which is proportional to the momentum, that is, to the mass and velocity.

After such a wide spectrum of functions which Newton attributes to the centripetal force, it becomes clear why after such wide spectrum of functions which Newton endows to the centripetal force, he was unable to understand its physical meaning and acknowledged: "But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called an hypotheses; and hypotheses, whether metaphysical or physical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea."

It is worth noting that mathematicians, to whom Newton expounded the theory, because of complication in analytical operation with the forces, introduced to celestial mechanics and analytical dynamics the force function, that is, energy with its ability to develop pressure. In doing so, they practically generalized the physical meaning of the force effects. As to the centripetal forces then, later on, we shall show that volumetric forces of mass particle interaction in reality generate Newton's physical pressure, which in formulation of practical problems is expressed by the energy. Once more, note that Newton, as he said himself, instead of using the correct physical meaning of the concept "pressure," gave preference to the concept "attraction" to be more understandable to mathematicians.

Newton's problem about the mutual attraction of two bodies, which depict similar trajectories around their common center of gravity and around each other, is based on the geometric solution of Kepler's problem formulated in his first two laws. Newton's solution is founded on his conception of the centripetal and innate forces under which the bodies depict similar trajectories around their common center of gravity and around each other. In celestial mechanics, developed on the basis of Newton's law of attraction, the two-body problem is reduced to the analytical problem of one body, the motion of which takes place in the central field of the common mass. Both Newton's geometric theorem and analytical solution of celestial mechanics are based on the hydrostatic equilibrium state of a body motion due to Kepler's laws. Newton understood this well and expressed it in his hydrostatics laws. But in both cases, the two-body problem was solved correctly in the framework of its formulation. The only difference is that according to Kepler, the planet motion occurs under the action of the Sun's forces, whereas Newton shows that this motion results from the mutual attraction of both the Sun and the planet.

In Section V of Book II *Density and Compression of Fluids: Hydrostatics*, Newton formulates the hydrostatics laws, and on their basis in Book III *The System of the World*, he considers the problem of the Earth's oblateness by applying real values of the measured distances between a number of points in Europe. Applying the found measurements and the hydrostatic approach, he calculated the Earth's oblateness as being 1/230, where in his consideration, the centrifugal force plays the main contraction effect expanding the body along the equator. In fact, the task is related to the creation of an ellipsoid of rotation from a sphere by action of the centrifugal force. Here, Newton applied his idea that the attraction of the planet itself goes from the surface to its center. In this case, the total sum of the centripetal forces and the moments are equal to zero, and rotation of the Earth should be inertial. It means that the planet's angular velocity has a constant value.

Inertial rotation of the Earth is accepted a priori. There is no evidence or other form of justification for this phenomenon. There are also no ideas relative to the mode of planet's rotation, namely, whether it rotates as a rigid body or there is differential rotation of separate shells. In modern courses of mechanics, there is only analytical proof that in the case when the body exists in the outer field of central forces, then the sum of its inner forces and torques is equal to zero. Thus, it follows from here that the Earth rotation should have a mode of rigid body and the velocity of rotation in time should be constant.

That is the arsenal of the motion laws, which was applied and has been applied up to now by researchers for problem solutions in dynamics of celestial bodies including the problem of the origin of the solar system. The basis of these laws is the hydrostatic equilibrium. But the observational data introduce serious corrections into the concepts.

1.3 Inner Energy of Body's Interacted Masses as a Bullet Point of the Solar System's Cosmogony

Applying the Jacobi dynamics, we analyzed orbital motion of the Earth, the Moon, and the other planets and satellites and discovered a dynamical effect common for all the solar system bodies. It appears that the mean orbital velocity and period of revolution of every planet are equal to the first cosmic velocity and corresponding period of virial oscillation of the protosun, with its radius equal to the semimajor axes of the planet's orbit. And also, the mean orbital velocity and periods of revolution of every satellite are equal to the first cosmic velocity and corresponding period of oscillation of the protoplanet, with its radius equal to the semimajor axes of the satellite's orbit. The same effect is valid for the asteroids, comets, and other small bodies. The subsequent body evolution has not broken the above regularity.

The conceptions of "cosmic velocity" became especially popular at the time of development of the artificial satellite techniques. The following three cosmic velocities are defined. The first one has the minimal velocity (~7.9 km/s), with which a satellite can overcome the planet's gravity attraction at its surface. The second, or parabolic, velocity (~11.2 km/s) enables a satellite to escape from the planet's gravity field. And the third one (~16.6 km/s) enables a satellite to escape from the Sun's gravity field at the Earth level.

These velocity figures were obtained on the basis of the energy conservation law and the third Kepler's law in the framework of hydrostatics. Applying Jacobi dynamics on the basis of the same laws in the framework of dynamical equilibrium state, the first cosmic velocity has the same (~7.9 km/s) value. But two other so-called escape velocities, in connection with the considered cosmogony problem, need to be corrected.

The first cosmic velocity v_1 of the protosun and protoplanetary bodies and the period of oscillation of the corresponding outer shell $_1$ of the created bodies were calculated by the formulas (see Chaps. 6 and 7), from which, in fact, the third Kepler's law follows:

$$v_1 = \omega R = R\sqrt{\frac{Gm}{R^3}} = \sqrt{\frac{Gm}{R}}, \tag{1.5}$$

$$T_1 = \frac{2\pi}{\omega} = \frac{2\pi R}{v_1}, \tag{1.6}$$

$$\frac{(2\pi)^2}{T_1^2} = \frac{Gm}{R^3}, \tag{1.7}$$

where m is the body's mass; G is the gravity constant; R is the semimajor axis; and $\omega = v_1/R$ is the frequency of virial oscillation of the outer shell, which appears to be equal to the angular velocity of orbital motion of the created body. Note that the frequency of virial oscillation of the outer weighty shell does not equal its angular velocity because the frequency is the parameter of the force field.

For example, when the protosun's radius R is extended up to the present-day Earth's orbit ($m = 1.99 \cdot 10^{30}$ kg, $R = 1.496 \cdot 10^{11}$ m), then its first cosmic velocity is equal to

$$\nu_1 = \omega R = \sqrt{\frac{Gm_s}{R}} = \sqrt{\frac{6.67 \cdot 10^{11} \cdot 1.99 \cdot 10^{30}}{1.496 \cdot 10^{11}}}$$
$$= 29786.786\,\text{m/s} = 29.786786\,\text{km/s}.$$

This value corresponds to the observed mean orbital velocity of the Earth.

The period of oscillation of the interacted mass particles of the protosun's outer shell ($R = 1.496 \cdot 10^{11}$ m, $\nu_1 = 29786.786$ m/s) was equal to

$$T_1 = \frac{2\pi R}{\nu_1} = \frac{6.28 \cdot 1.496 \cdot 10^{11}}{29786.786} = 3.1540428 \cdot 10^7\,\text{s} = 1\,\text{year},$$

which is equal to the observed period of the planet's orbital revolution.

When the Protoearth's radius R is extended up to the present-day Moon's orbit ($m_e = 5.976 \cdot 10^{24}$ kg, $R = 3.844 \cdot 10^8$ m), then its first cosmic velocity is equal to

$$\nu_1 = \sqrt{\frac{Gm_e}{R}} = \sqrt{\frac{6.67 \cdot 10^{11} \cdot 5.976 \cdot 10^{24}}{3.844 \cdot 10^8}} = 1018.3018\,\text{m} = 1.0183918\,\text{km/s},$$

which is the present-day Moon's mean orbital velocity.

The period of oscillation of the interacted mass particles of the Protoearth's outer shell ($R = 3.844 \cdot 10^8$ m, $\nu_1 = 1018.3018$ m/s) is equal to

$$T_1 = \frac{2\pi R}{\nu_1} = \frac{2 \cdot 3.14 \cdot 3.844 \cdot 10^8}{1018.3018} = 23.706449 \cdot 10^5\,\text{s} = 27.438019\,\text{days},$$

which corresponds to the present-day Moon's period of orbital revolution.

Tables 1.1 and 1.2 demonstrate the observed and calculated values of the orbital periods of revolution of the planets, asteroids (small planets), and satellites obtained by applying first cosmic velocities of the protosun and the protoplanets, which prove these calculations.

The obtained results mean that all the planets and satellites were launched by first cosmic velocity of the self-gravitating protosun and protoplanets after their outer shells acquired weightlessness. As it will be shown below, the process of evolutionary loss of energy by emission led to redistribution and differentiation of the body's mass density: it increases in the inner shells and decreases in the outer one by the light components dilution. In general, due to this process of accumulation of the less dense matter in the outer shell, its density decreases up to the state of weightlessness and creation of the secondary self-gravitating body by the eddy currents results. The process of the outer shell separation appears to be the

Table 1.1 Observed orbital periods of revolution of the planets around the Sun and calculated periods of oscillation of its corresponding outer shell

Planets	Orbital radius, R × 10^{11} (m)	Observable period of revolution (year)	Calculating period of oscillation ${}_{1}$ (year)
Mercury	0.579	0.24	0.2408
Venus	1.082	0.62	0.6153
Earth	1.496	1.0	1.00
Mars	2.28	1.88	1.8823
Vesta	3.53	3.63	3.7594
Juno	3.997	4.37	4.3733
Ceres	4.13	4.6	4.598
Themis	4.68	5.539	5.5397
Jupiter	7.784	11.86	11.8781
Saturn	14, 271	29.48	29.4802
Uranus	28.708	84.01	84.1951
Neptune	44.969	164.8	164.9185
Pluto	59.466	248.09	250.8882

mechanism of contraction (volume decrease and increase in the density) of a body during evolution (Fig. 1.7).

The discovered regularity of creation of the solar system's planets and satellites seems to be valid for the process of separation of the protosun itself and other protostars from the protogalaxy Milky Way. If we accept the known galaxy's astrometrical data (mass $m_g = 2.5 \cdot 10^{41}$ kg, and distance of the Sun from the galaxy center $R_s = 2.5 \cdot 10^{20}$ m), then it is not difficult to calculate that the first cosmic velocity of the protogalaxy, the size of which was limited by the Sun's semimajor orbital axes, is equal to 230 km/s, and the orbital period of revolution is $220 \cdot 10^6$ year. The values are close to those found by observation, namely, mean orbital velocity of the Sun is called as (230–250) km/s and the orbital period of revolution ${}_{s} = (220–250) \cdot 10^6$ year.

The observed picture of the Milky Way today, consisting of a bar-shaped core surrounded by a disc of gaseous matter and stars, which creates two major and four smaller logarithmic spiral arms with a spherical halo of old stars and globular clusters, proves the common mechanism of creation of the galactic system. From the viewpoint of Jacobi's dynamics, the observed picture evidences the generally common vortex mechanism of creation of a hierarchic system from the initial heterogeneous baryonic and nonbaryonic (dark) matter of the compressed universe. During its present-day expansion stage, due to redistribution of mass density and after reaching the state of weightlessness, the protostars creation and separation process will be continued in the spiral arms.

The analogous unified process was repeated for all the planets and their satellites.

The creation of the other small bodies like comets, meteors, and meteorites also found its explanation within the considered mechanism and physics. In fact, the only condition for separation of outer body's shell is its weightlessness (its corresponding mean density relative to the body's mean density) but not a limit of some amount

Table 1.2 Observed orbital periods of revolution of the satellites around the planets and calculated periods of oscillation of their corresponding outer shells

Planets	Satellites	Orbital radius, R × 10^3	Observable period of revolution (day)	Calculated period of revolution $_1$ (day)
Earth	Moon	384.4	27.32	27.4380
Mars	Phobos	9.4	0.319	0.3208
	Deimos	23.5	1.262	1.2604
Jupiter	V	181	0.498	0.4973
	Io	422	1.769	1.7706
	Europa	671	3.551	3.5508
	Ganymede	1, 070	7.155	7.154
	Callisto	1, 880	16.69	16.6709
	XIII	11, 100	240.92	239.0960
	VII	11, 750	259.14	259.5899
	XII	21, 000	620.77	660.7744
	1X	23, 700	758.90	745.1833
Saturn	Janus	151.5	0.7	0.6956
	Mimas	185.6	0.94	0.9431
	Enceladus	238.1	1.37	1.3704
	Tethys	294.7	1.89	1.8869
	Dione	377.4	2.74	2.7366
	Titan	1212.9	15.95	15.7548
	Iapetus	3560.8	79.33	79.2494
	Phoebe	12, 944	548.2	549.2722
Uranus	Cordelia	49.751	0.3350	0.3348
	Cupid	74.8	0.618	0.6172
	Miranda	129.39	1.4135	1.4043
	Ariel	191.02	2.5204	2.5189
	Umbriel	266.3	4.1442	4.1463
	Titania	435.91	8.7058	8.6840
	Oberon	583.52	13.4632	13.4503
Neptune	Triton	354.8	5.877	5.8523
	Nereid	5513.4	360.14	359.8227
Pluto	Charon	19.571	6.387	9.5065
	Nix	48.675	24.856	37.2873
	Hydra	64.780	38.206	54.2482

of mass. In this connection, any volume and amount of mass could probably be separated at any time. For example, we found by calculation that the short-periodic Encke's Comet (1970 I, $T = 3.302$ year) has a semimajor orbital axis $R \approx 1.5{\cdot}10^{11}$ m and has separated from the protosun after small planet Vesta and before Mars. The short-periodic Halley's Comet (1910 II, $T = 76.1$ year) has a semimajor orbital axis $R = 2.7{\cdot}10^{12}$ m and has separated from the protosun after Saturn and before Jupiter. The long-periodic Ikeya–Seki's Comet (1965 III, $T = 874$ year) has a semimajor orbital axis $R = 1.35{\cdot}10^{14}$ m and has separated from the protosun before Pluto. Like the asteroid belt between Jupiter and Mars, the comet belts should definitely

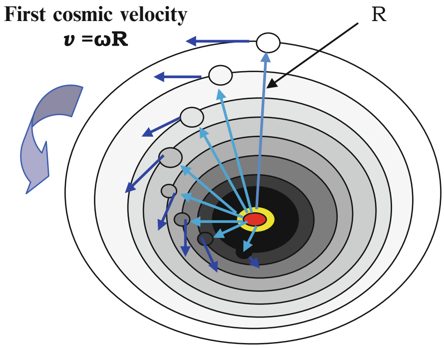


Fig. 1.7 Scheme of successive creation, separation, and orbiting of planets from the upper weightlessness shells of the protosun with its first cosmic velocity

exist between the orbits of all the Jupiter group planets. As to the meteors and meteorites, they all should be separated from the planets by the same way. From the point of view of dynamical equilibrium of their orbital motion, the orbits of all the small bodies (comets, meteors, and meteorites) should have large eccentricities and steep angles of inclination to the equator of their central bodies. This is because of probable oblateness of the protosun body, where its polar regions should have higher values of the first cosmic velocity. Those small bodies and meteorites, which have not reached or have later on lost dynamical equilibrium, fell down on the planet's or satellite's surface.

As shown in Table 1.2, the small planets of the asteroid belt separated from the protosun by the same mechanism. From the point of view of the orbital motion and first cosmic velocities, there are no any features of their separation from a broken planet.

Thus, the bullet point of creation and orbiting of the solar system bodies is the inner energy generation by the elementary particle interaction of the protoparents. The conditions of creation and orbiting of the planets and their satellites look like the conditions of launching of the artificial satellite, which are orbiting upon reaching weightlessness. The indicator of the body's weightlessness is its first cosmic velocity in orbital motion, which represents the energy of the outer force field of the parental body at a given height. So, all the planets appear to be weightless relative to the Sun and move on their orbits by the solar outer gravitational field. All the satellites are also weightless relative to their planets' gravitational fields and move along the orbits by first cosmic velocities of the inner energy of the planets' interacted masses. Dynamical equilibrium of their orbits' motion is guaranteed by their own outer force fields, which are generated by interaction of their own masses.

It is worth noting that in scientific literature, the physical meaning of the term "weightlessness" is defined as a complicated state. In the encyclopedias, one can find that weightlessness is the state of a material body moving in the gravity field by the gravity forces which do not initiate mutual pressure of each other's body's particles. The weightlessness effect in cosmic space is compared with man's feelings in the free fall of the elevator. Unfortunately, such a definition of weightlessness does not contain both the nature of the unique phenomenon and real physical understanding. In Chaps. 3 and 4, we show that the complexities related to understanding the nature of many dynamical events are placed in hydrostatics, which is the basis for solving problems of celestial body dynamics. Here, we just note that the gravitational forces in hydrostatics act as outer forces. To the contrary, at dynamical equilibrium, these forces, including the gravitation, are inner. By Tables 1.1 and 1.2, we can say that the orbital moment of momentum of each planet is not its parameter but the parameter of kinetic energy of the protosun. So, the existing discussion related to the mass and orbital momentum of the planets and the Sun is meaningless.

We have to study and explain the nature, mechanism, and conditions which lead to the creation and decay of the solar system's bodies in the galaxy. These proofs can be found by experimental data of artificial satellites and with the help of dynamical equilibrium introduction.

It was shown earlier (Ferronsky and Ferronsky 2010) by the satellite orbit study, that the Earth and the Moon do not stay in hydrostatic equilibrium. Therefore, the existing results and conclusions based on hydrostatics need to be corrected.

Moreover, we discovered the main discrepancy related to the hydrostatic equilibrium of the planets, the satellites, and the Sun. Namely, the potential energy of the Earth, Mars, Jupiter, Saturn, Uranus, and Neptune exceeds their kinetic energy by about 300 times. And, for Mercury, Venus, Moon, and the Sun, this ratio equals to about 10^4. In fact, all the celestial bodies with their inertial rotation are without the kinetic energy. This is dynamics based on hydrostatics.

This consideration takes off an old misunderstanding about the difference in the orbital planet's and the Sun's moment of momentum. The planet conserves creation energy of the Sun in accordance with the third Kepler's law, and its orbital moment of momentum is the parameter of the Sun's outer force field as well as the first cosmic velocity. As to the direction of a body's axial rotation and orbital revolution, then these parameters enter by the inner and outer force fields, like in electrodynamics, in accordance with Lenz's law. As to the specific (for unit of the mass) orbital moment of momentum of the planets and satellites, which increases with distance from the central body, the explanation of this gives the increasing radius from the central body. Later on, we will come back for the problem discussion.

First cosmic velocity was practically applied by man only in the twentieth century. Nature seems to use it perpetually as the main instrument of the evolution of the universe. Our universe seems to be a pulsating system, and its basic infinitesimal particle, which is 10^{-36} g or less in weight (see calculation in Sect. 7.5), is responsible for the system's equilibrium. Because of the matter evolution and energy conservation law, this process is continuing infinitely for a long time.

So, the inner energy of interacted masses with its weightlessness and weightness (self-gravitation) is the bullet point of the solar system cosmogony and cosmology as a whole. We will try to prove the main aspects of this problem on the basis of Jacobi dynamics with its dynamical (oscillating) equilibrium of natural system state. We will start with the fundamentals.

References

Cameron AGW (1973) Accumulation processes in the primitive solar nebula. Icarus 18:407–450

Cameron AGW, Pine MR (1973) Numerical models of the primitive solar nebula. Icarus 18: 377–406

Ferronsky VI, Ferronsky SV (2010) Dynamics of the earth. Springer, Dordrecht/Heidelberg

Kuiper GP (1951) On the origin of the solar system. In: Hynek JA (ed) Astrophysics. A topical symposium. McGraw Hill, New York, pp 357–424

Newton I (1934) Mathematical principles of natural philosophy and his system of the world (Trans. from Latin by Andrew Mott, 1729). Cambridge University Press, Cambridge

Schmidt OY (1957) Four lectures on origin of the earth. Academy of Sciences of USSR Publishing House, Moscow

Spenser JH (1956) The origin of the solar system. In: Physics and chemistry of the earth, vol 1. Pergamon, London, pp 1–16

Chapter 2
Physical Meaning of Hydrostatic Equilibrium of Celestial Bodies

Abstract It was shown earlier that the fundamentals of classical dynamics, based on hydrostatics, do not satisfy the solution of dynamical problems of celestial bodies (Ferronsky and Ferronsky 2010; Ferronsky et al. 2011). The discovered common dynamical effect of orbiting the creating planets and satellites with the first cosmic velocity proves correct for this purpose Jacobi's dynamical (oscillating) approach. In this connection, in Chaps. 2 and 3, the physical meaning of the hydrostatic and dynamic equilibrium of celestial bodies is discussed in detail.

Newton's model of the hydrostatic equilibrium of a uniform body, Clairaut's model of the hydrostatic equilibrium of a nonuniform body, Euler's model of the hydrostatic equilibrium of a rotating rigid body, Clausius' virial theorem, and the model of hydrostatic equilibrium of elastic and viscous-elastic body are analyzed in this chapter. The main features of the hydrostatic equilibrium are the outer acting forces and the force field and the loss of kinetic energy. As a result, the sum of the inner forces and moments is equal to zero, and the body's equilibrium is not controlled.

Demonstrated evidences obtained by the artificial satellite and other geodetic observation prove that the Earth and the Moon do not stay in hydrostatic equilibrium.

The roots of hydrostatic fundamentals for solution of the problems in dynamics of celestial bodies date back to the distant past and are related to the founders of modern science. But even at that time, these pioneers understood well that the applicability of the hydrostatic equilibrium to a body's dynamic problems is restricted by certain boundary conditions. Thus, Newton in his *Principia* (Sect. 1.5 of Book III), while considering the conditions of attraction in the planets, writes: "The attraction being spread from the surface downwards is approximately proportional to distance of the center. Be the planet's matter uniform in density, then this proportion would have exact value. It follows from here that the error is caused

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_2,

by non-uniformity in density." At that time, the thoughts of scientists were engaged with how to solve the principle problem of a body's orbital motion. Now, we search its correct solution.

Recall briefly the conditions of a body in hydrostatic equilibrium. By definition, hydrostatics is a branch of hydromechanics which studies the equilibrium of a liquid and gas and the effects of a stationary liquid on immersed bodies relative to the chosen reference system. For a liquid equilibrated relative to a rigid body, when its velocity of motion is equal to zero and the field of densities is steady, the equation of state follows from the Eulerian and Navier–Stokes equations in the form (Landau and Lifshitz 1954; Sedov 1970)

$$\mathrm{grad}\, p = \rho F, \tag{2.1}$$

where is the pressure, ρ is the density, and F is the mass force.

In the Cartesian system of reference, Eq. (2.1) is written as

$$\begin{aligned} \frac{\partial p}{\partial x} &= \rho F_x, \\ \frac{\partial p}{\partial y} &= \rho F_y, \\ \frac{\partial p}{\partial z} &= \rho F_z. \end{aligned} \tag{2.2}$$

If the outer mass forces are absent, that is, $F_x = F_y = F_z = 0$, then

$$\mathrm{grad}\, p = 0.$$

In this case, in accordance with Pascal's law, the pressure in all liquid points will be the same.

For the uniform incompressible liquid, when $\rho = \mathrm{const}$, its equilibrium can be only in the potential field of the outer forces. For the general case of an incompressible liquid and the potential field of the outer forces from (2.1), one has

$$\mathrm{d}p = \rho\, \mathrm{d}U, \tag{2.3}$$

where U is the forces' potential.

It follows from Eq. (2.3) that for the equilibrated liquid in the potential force field, its density and pressure appear to be a function only of the potential U.

For a gravity force field, when in the steady-state liquid only these forces act, one has

$$F_x = F_y = 0, \quad F_z = -g, \quad U = -gz + \mathrm{const} \quad \text{and} \quad p = p(z), \quad \rho = \rho(z).$$

Here, the surfaces of the constant pressure and density appear as the horizontal planes. Then Eq. (2.3) is written in the form

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g < 0. \tag{2.4}$$

It means that with elevation, the pressure falls, and with depth, grows. From here it follows that

$$p - p_0 = -\int\limits_{z_0}^{z} \rho g \,\mathrm{d}z = -\rho g\,(z - z_0), \tag{2.5}$$

where g is the acceleration of the gravity force.

If a spherical vessel is filled by an incompressible liquid and rotates around its vertical axis with constant angular velocity ω, then for determination of the equilibrated free surface of the liquid in Eq. (2.2), the centrifugal inertial forces should be introduced in the form

$$\begin{aligned}\frac{\partial p}{\partial x} &= \rho\omega^2 x,\\ \frac{\partial p}{\partial x} &= \rho\omega^2 y,\\ \frac{\partial p}{\partial x} &= -\rho g.\end{aligned} \tag{2.6}$$

From here, for the rotating body with radius $r^2 = x^2 + \mathrm{y}^2$, one finds

$$p = -\rho g z + \frac{\rho\omega^2 r^2}{2} + C. \tag{2.7}$$

For the points on the free surface $r = 0$, $z = z_0$, one has $= {}_0$. Then,

$$C = p_0 + \rho g z_0, \tag{2.8}$$

$$p = p_0 + \rho g\,(z_0 - z) + \frac{\rho\omega r^2}{2}. \tag{2.9}$$

The equation of the liquid free surface, where $= {}_0$, has a paraboloidal shape

$$z - z_0 = \frac{\omega^2 r^2}{2g}. \tag{2.10}$$

These facts determine the principal physical conditions and equations of the hydrostatic equilibrium of a liquid. They remain a basis of the modern dynamics and theory of the planet's figure. Attempts to harmonize these conditions with the planet's motion conditions have failed, as proven by observation. The main obstacle for such harmonization is the condition (2.1), which ignores the planet's inner force field and without which the hydrostatics is unable to provide the equilibrium between the body interacted forces as Newton's third law requires. The Earth and other planets are self-gravitating bodies. Their matter moves in their own force field which is generated by the mass particle interaction. The mass density distribution, rotation, and oscillation of the body shells result from the inner force field. And the orbital motion of the planets is controlled by interaction of the outer force fields of the planets and the Sun in accordance with Newton's theory.

Because any celestial body is de facto self-gravitating systems, we will study equilibrium in its own force field of the interacted masses. It is shown that by action of this field, separation of the masses in density, oscillation, and axial rotation results. In this case, the planet's orbital motion will originate by the Sun's first cosmic velocity of the outer protosun's surface force field. But first, the proposed models of the hydrostatic equilibrium are discussed.

2.1 Newton's Model of Hydrostatic Equilibrium of a Uniform Body

In Section V of Book II *Density and Compression of Fluids: Hydrostatics*, Newton formulates the hydrostatic laws, and on their basis in Book III *The System of the World*, he considers the problem of the Earth's oblateness by applying real values of the measured distances between Landon and York, Amiens and Malvoisine, Collioure and the observatory of Paris, and the Observatory and the Citadelo of Dunkirk.

Taking advantage of measurements, Newton calculated the ratio of the total gravitation force over the Paris latitude to the centrifugal force over the equator and found that the ratio is equal to 289:1. After that, he imagines the Earth in the form of an ellipse of rotation (Fig. 2.1) with axis PQ and channel ACQqca.

If the channel is filled in with water, then its weight in the branch will be related to the water weight in the branch Q q as 289:288 because of the centrifugal

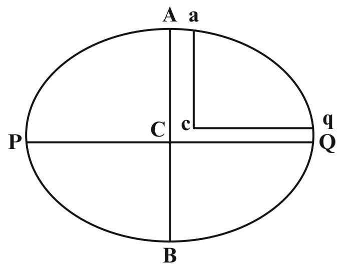


Fig. 2.1 Newton's problem of the Earth's oblateness

force which decreases the water weight in the last branch by the unit. He found by calculation that if the Earth has a uniform mass of matter and has no any motion and the ratio of its axis PQ to the diameter is 100:101, then the gravity force of the Earth at the point Q relates to the gravity force at the same point of the sphere with radius Q or as 126:125. By the same argument, the gravity in point of a spheroid drawn by revolution around axis relates to the gravity in the same point of the sphere drown from center with radius as 125:126. However, since there is one more perpendicular diameter, then this relation should be 126:125$^1/_2$. Having multiplied these ratios, Newton found that the gravity force at point *Q* relates to the gravity force at point as 501:500. Because of daily rotation, the liquid in the branches should be in equilibrium at a ratio of 505:501. So, the centrifugal force should be equal to 4/505 of the weight. In reality, the centrifugal force composes 1/289. Thus, the excess in water height under the action of the centrifugal force in the branch is equal to 1/289 of the height in branch Q q.

After calculation by hydrostatic equilibrium in the channels, Newton obtained that the ratio of the Earth's equatorial diameter to the polar diameter is 230:229, that is, its oblateness is equal to (230−229)/230 = 1/230. This result demonstrating that the Earth's equatorial area is higher than the polar region was used by Newton for explanation of the observed slower swinging of pendulum clocks on the equator than on the higher latitudes.

Thus, applying the found measurements and the hydrostatic approach, he calculated the Earth's oblateness equal to 1/230, where in his consideration the centrifugal force plays the main contraction effect expanding the body along the equator. In fact, the task is related to the creation of an ellipsoid of rotation from a sphere by action of the centrifugal force. Here, Newton applied his idea that the attraction of the planet itself goes from the surface to its center. In this case, the total sum of the centripetal forces and the moments is equal to zero, and rotation of the Earth should be inertial. It means that the planet's angular velocity has a constant value.

Inertial rotation of the Earth is accepted a priori. There is no evidence or other form of justification for this phenomenon. There are also no ideas relative to the mode of the planet's rotation, namely, whether it rotates as a rigid body or there is differential rotation of separate shells. In modern courses of mechanics, there is only analytical proof that in case the body occurs in the outer field of central forces, then the sum of its inner forces and torques is equal to zero. Thus, it follows that the Earth's rotation should have a mode of rigid body, and the velocity of rotation in time should be constant.

The proof of the conclusion that if a body occurs in the field of the central forces, then the sum of the inner forces and torques is equal to zero, and the moment of momentum has a constant value, is directly related to the Earth's dynamics. Let us see it in modern presentation (Kittel et al. 1965).

Write the expression of the moment of momentum $\mathbf{L}$ for a mass point m, the location of which is determined by radius vector $\mathbf{r}$ relative to an arbitrarily selected fixed point in an inertial system of coordinates

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \equiv \mathbf{r} \times m\mathbf{v}, \tag{2.11}$$

where is the moment and $\mathbf{v}$ is the velocity.

The torque $\mathbf{N}$ relative to this point is equal to

$$\mathbf{N} \equiv \mathbf{r} \times \mathbf{F},$$

where $\mathbf{F}$ is the force acting on a particle.

After differentiation of (2.11) with respect to time, one obtains

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \frac{\mathrm{d}r}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{2.12}$$

Since vectorial product

$$\frac{\mathrm{d}r}{\mathrm{d}t}\mathbf{r} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = 0, \tag{2.13}$$

then taking into account the second Newton's law for the inertial reference system, we have

$$\mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F} = \mathbf{N},$$

from where

$$\mathbf{N} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}. \tag{2.14}$$

For the central force $\mathbf{F} = \hat{\mathbf{r}} f(r)$, which acts on the mass point located in the central force field, the torque is equal to

$$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \cdot \hat{\mathbf{r}} f(r) = 0. \tag{2.15}$$

Consequently, for the central forces, the torque is equal to zero, and the moment of momentum $\mathbf{L}$ appears to be constant.

In the case where the mass point presents a body composed of n material particles, then the moment of momentum $\mathbf{L}$ of that system will depend on location of the origin of the reference system. If the reduced vector of the mass center of the system relative to the origin is $\mathbf{R}_\mathrm{c}$, then the equation for the moment of momentum $\mathbf{L}$ is written as

$$\mathbf{L} = \sum_{n=1}^{N} m_n(\mathbf{r}_n - \mathbf{R}_\mathrm{c}) \times \mathbf{v}_n + \sum_{n=1}^{N} m_n \mathbf{R}_\mathrm{c} \times \mathbf{v} = \mathbf{L}_\mathrm{c} + \mathbf{R}_\mathrm{c} \times \mathbf{P}, \tag{2.16}$$

where $\mathbf{L}_c$ is the moment of momentum relative to the system's center of the masses; $\mathbf{P} = \sum m_n \times \mathbf{v}_n$ is the total moment of the system. Here, the term $\mathbf{R}_c \times \mathbf{P}$ expresses the moment of momentum of the mass center and depends on the origin, and the term $\mathbf{L}_c$, on the contrary, does not depend on the reference system.

$$\mathbf{N} = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{F}_n,$$

and the sum of the inner forces is

$$\mathbf{F}_i = \sum_{n=1}^{N} \mathbf{F}_{ij}; \tag{2.17}$$

here and further, the summing is done at condition $i \neq j$.

The torque of the inner forces is

$$\mathbf{N}_{\text{in}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij}. \tag{2.18}$$

Since

$$\sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} = \sum_i \sum_j \mathbf{r}_j \times \mathbf{F}_{ji}, \tag{2.19}$$

then the torque of inner forces can be presented in the form

$$\mathbf{N}_{\text{in}} = \frac{1}{2} \sum_i \sum_j \left(\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}\right). \tag{2.20}$$

Because Newton's forces $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$, then

$$\mathbf{N}_{\text{in}} = \frac{1}{2} \sum_i \sum_j \left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ij}. \tag{2.21}$$

Taking into account that central forces $\mathbf{F}_{ij}$ are parallel to $\mathbf{r}_i - \mathbf{r}_j$,

$$\left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{ij} = 0,$$

from where the torque of the inner forces is equal to zero.

$$\mathbf{N}_{\text{in}} = 0. \tag{2.22}$$

Assuming that the inner forces $\mathbf{F}_{\text{in}} = 0$, then from (2.15), (2.16), and (2.22) one finds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{\sum} = \mathbf{N}_{\text{ex}} \tag{2.23}$$

$$\mathbf{L}_{\sum} = \mathbf{L}_{\text{c}} + \mathbf{R}_{\text{c}} \times \mathbf{P}. \tag{2.24}$$

Here, $\mathbf{L}_{\text{c}}$ is also the moment of momentum relative to the mass center, and $\mathbf{R} \times \mathbf{P}$ is the moment of momentum of the mass center relative to an arbitrarily taken origin.

For practice, it is often convenient to select the geometric center of the mass as an origin. In this case, the derivative from the moment of momentum relative to the mass center is the torque of the outer forces, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}_{\text{c}} = \mathbf{N}_{\text{ex}}. \tag{2.25}$$

It is seen from the previous classical consideration that in the model of two interacted mass points reduced to the common mass center, which Newton used for solution of Kepler's problem, resulting in the planets' motion around the Sun, the inner forces and torques in the central force field are really equal to zero. The torque, which is a derivative with respect to time from the moment of momentum of material particles of the body's material particles, is determined here by the resultant of the outer forces and the planet's orbits in the central force field that exists in the same plane. This conclusion follows from Kepler's laws of the planets' motion.

Passing to the problem of the Earth's dynamics, Newton had no choice for the formulation of new conditions. The main conditions were determined already in the two-body problem where the planet appeared in the central force field of the reduced masses. The only difference here is that the mass point has a finite dimension. The condition of zero equality of the inner forces and torques of the rotating planet should mean that the motion could result from the forces among which the known were only the Galilean inertial forces. Such a choice followed from the inertial motion condition of two-body motion which he had applied. The second part of the problem related to reduction of the two bodies to their common center of masses and to the central force that appeared accordingly as predetermining the choice of the equation of state. Being in the outer uniform central force field, it became the hydrostatic equilibrium of the body state. The physical conception and mathematical expression of hydrostatic equilibrium of an object based on Archimedes' laws (third century BP) and Pascal's law (1663) were well known in that time. This is the story of the sphere model with the equatorial and polar channels filled in by a uniform liquid mass in the state of hydrostatic equilibrium at inertial rotation.

In Newton's time, the dynamics of the Earth in its direct sense had not been found as it is absent up to now. The planet, rotating as an inertial body and deprived of its own inner forces and torques, appeared to be a dead–alive creature. But up to now, the hydrostatic equilibrium condition, proposed by Newton, is the only theoretical concept of the planet's dynamics because it is based on the two-body problem solution which satisfies Kepler's laws and in practice plays the role of Hooke's law of elasticity.

In spite of the discrepancies noted here, the problem of determining the Earth's oblateness was the first step towards the formulation and solution of the very complicated task of determining the planet's shape, an effort on which theoretical and experimental study continues up to the present time. As to the value of the polar oblateness of the Earth, it appears to be much higher than believed before. More recent observations and measurements show that relative flattening has a smaller value, and Newton's solution needs to have further development. And its nature disappeared in the heterogeneous mass density of the body.

2.2 Clairaut's Model of Hydrostatic Equilibrium of a Nonuniform Body

Aleksi Klod Clairaut (1713–1765), a French mathematician and astronomer, continued working on Newton's solution of the problem of the Earth's shape based on hydrostatics (Clairaut 1947). The degree measurements in the equatorial and northern regions made in the eighteenth century by French astronomers proved Newton's conclusion about the Earth's oblateness, which at that time was regarded with scepticism. But the measured value of the relative flattening appeared to be different. In the equatorial zone, it was equal to 1/314, and in the northern region, to 1/214 (Grushinsky 1976). Clairaut himself took part in the expeditions and found that Newton's results are not correct. It was also known to him that the Earth is not a uniform body. Because of that, he focused on taking into account the consideration of this effect. Clairaut's model was represented by an inertia-rotating body filled with liquid of a changing density. In its structure, such a model was closer to the real Earth having a shell structure. But the hydrostatic equilibrium condition and inertial rotation remained to be as previously the physical basis for the problem solution. Clairaut introduced a number of assumptions in the formulation of the problem. In particular, since the velocity of inertial rotation and the value of the oblateness are small, the boundary areas of the shells and their equilibrium were taken as ellipsoidal figures with a common axis of rotation. Clairaut's solution comprised obtaining a differential equation for the shell-structured ellipsoid of rotation relative to geometric flattening of its main section. Such an equation was found in the form (Melchior 1972)

$$\frac{\mathrm{d}^2 e}{\mathrm{d}a^2} + \frac{\mathrm{d}\rho\, a^2}{\int_0^a \rho a^2\,\mathrm{d}a}\frac{\mathrm{d}e}{\mathrm{d}a} + \left(\frac{2\rho a}{\int_0^a \rho a^2\,\mathrm{d}a} - \frac{6}{a^2}\right)e = 0, \tag{2.26}$$

where $e = (b-a)/a$ is the geometric flattening, a and b are the main axes, and ρ is the density.

The difficulty in solving the previous equation was in the absence of the density radial distribution law of the Earth. Later on, by application of seismic data, researchers succeeded in obtaining a picture of the planet's shell structure. But quantitative interpretation of the seismic observations relative to the density appeared to be possible again, based on the same idea of hydrostatic equilibrium of the body masses. In spite of that, as a result of analysis of the Clairaut's equation, a number of dynamic criteria for a rotating Earth were obtained. In particular, the relationship between the centrifugal and the gravity force on the equator was found, the ratio between the moments of inertia of the polar and equatorial axes (dynamical oblateness) was obtained, and also the dependence of the gravity force on the latitude of the surface area was derived. That relationship is as follows:

$$g = g_{\mathrm{a}}\left(1 - \beta \sin^2\varphi\right), \tag{2.27}$$

where φ is the latitude of the observation point; g_{e} is the acceleration of the gravity force: $\beta = 5/2q-e$; $q = \omega^2 a/g_{\mathrm{e}}$ is the ratio of the centrifugal force to the gravity force on the equator; ω is the angular velocity of the Earth's rotation; e is the geometric oblateness of the planet; and is the semimajor axis.

The solutions obtained by Clairaut and further developed by other authors became a theoretical foundation for practical application in the search for the planet's shape, for interpretation of seismic observation relative to the structure and density distribution of the Earth, and also for analysis of the observed natural dynamic processes.

Later on, the quantitative values of the geometric and dynamic oblateness of the Earth and the Moon, different in values, were obtained by Clairaut's equation and with the use of satellite data. This fact underlies the conclusion that the Earth and the Moon do not stay in hydrostatic equilibrium.

2.3 Euler's Model of Hydrostatic Equilibrium of a Rotating Rigid Body

Leonard Euler (1707–1783), a prominent Swiss mathematician, mechanic, and physicist, possessed a great capacity for work, fruitful creativity, and extreme accuracy and strictness in problem solution. There are about 850 titles in the list of his publications, and their collection comprises 72 volumes. Half of them were prepared in Russia. He was twice invited to work in the St. Petersburg Academy of Sciences, where he spent more than 30 years. The spectrum of Euler's scientific interests was very wide. In addition to mathematics and physics, they included the theory of elasticity, theory of machines, ballistics, optics, shipbuilding, theory of music, and even insurance business. But 3/5 of the work were devoted to mathematics problems.

In mechanics, Euler developed a complete theory of motion of the rigid (nondeformable) body. His dynamic and kinematics equations became the main mathematical instrument in the solution of the rigid body problems. These equations, with the use of the known law of a body rotation, enable the determination of the acting forces and torques. And vice versa, by the applied outer forces, one may find the law of motion (rotation, precession, nutation) of a body.

On the basis of Newton's equations of motion for rotational motion of a rigid body whose axes of coordinates *x, y,* and *z* in the rotating reference system are matched with the main axes connected with the body, Euler's dynamical equations have the form:

$$\begin{aligned} I_x\dot{\omega}_x + \left(I_z - I_y\right)\omega_y\omega_z &= N_x, \\ I_y\dot{\omega}_y + \left(I_x - I_z\right)\omega_x\omega_z &= N_y, \\ I_z\dot{\omega}_z + \left(I_y - I_x\right)\omega_x\omega_y &= N_z, \end{aligned} \tag{2.28}$$

where I_x, I_y, and I_z are the moments of inertia of the body relative to the main axes; ω_x, ω_y, ω_z, are the components of the instantaneous angular velocities on the axes; N_x, N_y, and N_z are the main torques of the acting forces relative to the same axes; and $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ are the derivatives with respect to time from the angular velocities.

Euler's kinematic equations are written as follows:

$$\begin{aligned} \omega_x &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \\ \omega_y &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \\ \omega_z &= \dot{\varphi} + \dot{\psi}\cos\theta. \end{aligned} \tag{2.29}$$

The Eulerian angles φ, ψ, and θ determine the position of a rigid body that has a fixed point relative to the fixed rectangular axes of coordinates. At hard linkage of the axes with the body and specification of the line of crossed planes of corresponding angles, they fix the rotation angle, the angle of precession, and the angle of nutation of the rotation axis.

For a uniform sphere, such as the Earth is according to Newton, $I_x = I_y = I_z$. Then the Eulerian equations of motion (2.28) acquire the form

$$\begin{aligned} I\dot{\omega}_x &= N_x, \\ I\dot{\omega}_y &= N_y, \\ I\dot{\omega}_z &= N_z. \end{aligned} \tag{2.30}$$

At free (inertial by Newton) rotation of the uniform Earth, which is not affected by the torque, $N_x = N_y = N_z = 0$. In that case, it follows from (2.30) that the components of the instantaneous velocities of their axes become constant and the angular velocity $\omega = \text{const}$. Thus, angular velocity of a body at nonperturbed rotation is equal to a constant value.

Newton found that the Earth is flattened relative to the polar axis by centrifugal inertial force, and Clairaut has agreed with that. Then from the symmetry of the body having the form of an ellipsoid of rotation, it is found that $N_x = N_y \neq N_z$ and only $\omega_z = \text{const}$. From this in the case of absence of the outer torque, Eq. (2.28) is reduced to

$$\dot{\omega}_x + \Omega\omega_y = 0, \tag{2.31}$$

$$\dot{\omega}_y - \Omega\omega_x = 0, \tag{2.32}$$

where Ω is the angular velocity of free rotation, which at $I_x = I_y$ is equal to

$$\Omega = \frac{I_z - I_x}{I_x}\omega_z \tag{2.33}$$

After transformation of Eqs. (2.31) to (2.32), one obtains their solution in the form of ordinary equations of the harmonic oscillation

$$\omega_x = A\cos\,\Omega t, \tag{2.34}$$

$$\omega_z = A\sin\,\Omega t, \tag{2.35}$$

where is the constant value representing the amplitude of oscillation.

Thus, the component ω_z of the angular velocity along the body's axis of rotation is a constant value, and the component perpendicular to the axis is rotating with angular velocity Ω. So the whole body, while rotating by inertia relative to the geometric axis with angular velocity ω_z, in accordance with (2.33) is wobbling with the frequency Ω. The oscillations described by Eqs. (2.34) and (2.35) are observed in reality and are called nutation of the rotating axis or a variation of latitude. The numerical value of the ratio of inertia moments (2.34) for the Earth is known and equal to

$$I_z - I_x/I_z = 0.0032732,$$

and the value of the angular velocity (free precession) is

$$\Omega = \omega_z/305.5.$$

For the known value $\omega_z = 7.29 \cdot 10^{-5}\ \text{s}^{-1}$, the period of Euler's free precession is equal to 305 days or about 10 months. But analysis of the results of the long series of observations done by the American researcher Chandler has shown that, together with the annual component of the forced nutation, there is one more component having a period of about 420 days, which was called as free wobbling of the rotation axis. This component differs substantially from Euler's free precession. The nature of the latter has not been understood up to now.

Euler also developed a complete theory of motion of the perfect liquid in hydromechanics, where differential equations in his variables become the basis for solution of hydrodynamic problems. Euler's hydrodynamic equations for the perfect liquid in the rectangular Cartesian reference system x, y, z based on Newton's equations of motion have the form

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x},\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y},\\
\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + \omega\frac{\partial \omega}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z},
\end{aligned} \tag{2.36}$$

where u, v, and ω are the components of the velocity of liquid particles; p is the liquid pressure; ρ is the density; and X, Y, and Z are the components of the volumetric forces.

Solution of the hydrodynamic problems is reduced to determination of the components of velocities u, v, ω, the pressure and the density as a function of the coordinates with known values of X, Y, Z, and the given boundary conditions. For that purpose, in addition to Eq. (2.36), the equation of continuity is written in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho \omega)}{\partial z} = 0. \tag{2.37}$$

If the density of liquid depends only on pressure, then the extra equation of state will be presented by the relation $\rho = f(p)$, and for the incompressible liquid, it is $\rho = \text{const}$.

Because the Earth is a system with continuous distribution of its masses, we will use the Eulerian hydrodynamic equations repeatedly.

2.4 Clausius' Virial Theorem

Rudolf Clausius (1822–1888), a German physicist, is one of the founders of thermodynamics and the molecular kinetic theory of heat. Simultaneously with W. Thomson (Lord Kelvin), he has formulated the second law of thermodynamics in the following form: "Heat cannot be transferred by any continuous, self-sustaining process from a cold to a hotter body" without some changes, which should compensate that transfer. Clausius introduced the conception of entropy to thermodynamics.

In 1870, based on the study of the process and mechanism of Carnot's thermal machine work, Clausius proved the virial theorem, according to which for a closed system the mean kinetic energy of the perfect gas particles' motion is equal to half of their potential energy. The virial relation between the potential and kinetic energy was found to be a universal condition of the hydrostatic equilibrium for describing dynamics of the natural systems in all branches of physics and mechanics.

That equation was used first of all in the kinetic theory of gases for derivation of an equation of state for the perfect gases in the outer force field of the Earth; we assume that a specific perfect gas is found in a vessel of volume V and consists from N uniform particles (atoms or molecules). The mean kinetic energy of a particle of that gas at temperature T_0 is equal to $3k\ _0/2$, where k is the Boltzmann's constant. Then the virial theorem is written in the form

$$-\frac{1}{2}\overline{\sum_i F_i \cdot r_i} = \frac{3}{2}NkT_0. \tag{2.38}$$

In this case, the effect of interaction of the gas atoms and molecules between themselves is negligibly small, and all the gas energy is realized by its interaction with the vessel's wall. The gas pressure p inside the vessel appears only because of the walls, the elastic reaction of which plays the role of the inertial forces. The pressure is expressed through the energy of the molecules and atoms' motion in the vessel, and expression (2.38) is written as the Clapeyron–Mendeleev equation of state for a perfect gas in the form

$$\frac{3}{2}pV = \frac{3}{2}NkT_0,$$

or

$$pV = NkT_0. \tag{2.39}$$

Equation (2.39) is the generalized expression of the laws of Boyle and Mariotte, Gay-Lussac, and Avogadro and represents the averaged virial theorem. Its left-hand side represents the potential energy of interaction of the gas particles, and the right-hand side is the kinetic energy of the gas pressure on the walls. In astrophysics, this equation is used as the equation of hydrostatic equilibrium state of a star, which is accepted as a gas and plasma system, where the gas pressure is equilibrated by the gravity forces of the attracted masses. In this case, the gravity forces play the role of the vessel's walls or the outer force field, where the kinetic energy of motion of the interacted particles is not taken into account. Later on, it will be shown that for the natural gaseous and plasma self-gravitating systems, the only generalized virial equation can be used as the equation of state.

For celestial bodies including the Earth, other planets, and satellites, whose mass particles interact by the reverse square law and the forces of interaction are characterized by the potential $U(r)$ as the uniform function of coordinates, the averaged virial theorem is reduced to the relation between the potential and kinetic energy in the form (Goldstein 1980)

$$T = \frac{1}{2} \sum_i \overline{\nabla U \cdot r_i}. \tag{2.40}$$

For a particle moving in the central force field expression (2.40) is

$$T = \frac{1}{2} \overline{\frac{\partial U}{\mathrm{d}r}} r. \tag{2.41}$$

If U is the force function of r^n, then

$$T = \frac{n+1}{2} \overline{U}.$$

Or, taking into account Euler's theorem about the uniform functions and Newton's law of interaction, when $n = -2$, one has

$$T = -\frac{1}{2} \overline{U}. \tag{2.42}$$

Relationship (2.42) is valid only for a system that is found in the outer uniform force field. It expresses only mean values of the potential and kinetic energy per the period τ without effect of the inner kinetic energy of the interacting particles.

For a uniform sphere in outer uniform force field ρF at inner isotropic pressure ρ, relation (2.42) represents the condition of hydrostatic equilibrium written by means of Euler's equation in the form

$$\frac{\partial p}{\partial r} = \rho F_r.$$

Here, the left-hand side of the equation is the potential energy, and the right-hand side represents the kinetic energy of the sphere in the framework of the averaged virial theorem.

It is clear that the averaged virial theorem in evolutionary dynamics is restricted by closed systems of a perfect gas, which corresponds to their hydrostatic equilibrium state.

The general model of hydrostatic equilibrium based on a body's outer central force field is shown in Fig. 2.2.

The problems of dynamics in the framework of hydrostatics cover also the tasks that consider a body under action of outer forces in elastic and viscous media of Hooke and Newton–Maxwell.

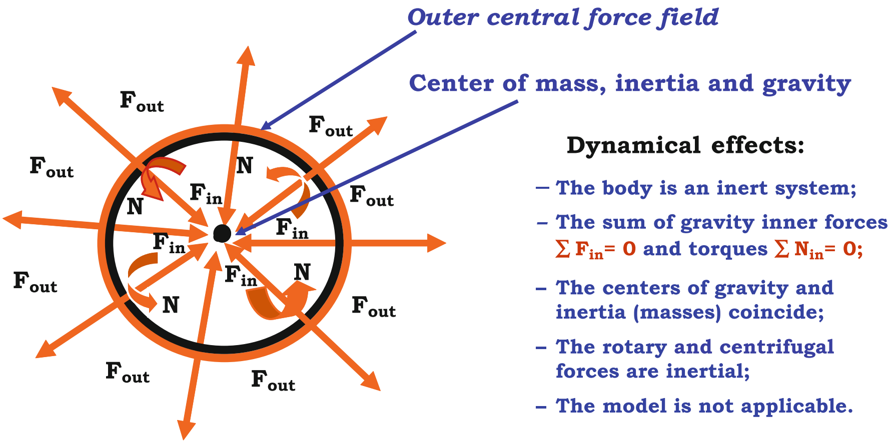


Fig. 2.2 Conditions of hydrostatic equilibrium of a body based on the outer central gravity force field

2.5 The Model of Hydrostatic Equilibrium of Elastic and Viscous-Elastic Body

The model of the Earth proposed by Newton and developed by Clairaut was in the form of a spheroid, rotating on inertia and filled in with uniform and nonuniform liquid, the mass of which resides in hydrostatic equilibrium in the outer force field. This model became generally accepted, commonly used, and in principal has not changed up to now. Its purpose was to solve the problem of the planet's shape, that is, the form of the planet's surface, and this goal was reached in the first approximation. Moreover, the equation obtained by Clairaut on surface changes in the acceleration of the gravity force as function of the Earth's latitude opened the way for the experimental study of the oblateness of spheroid of rotation by means of measuring the outer gravity force field. Later on, in 1840, Stokes solved the direct and reverse task of determining the surface gravity force for a rotating body and above its level, applying the known parameters, namely, the mass, radius, and angular velocity. These parameters uniquely determined the gravity force at surface level, which is taken as the quiet ocean's surface, and in all outer space. By that task, the relation between the Earth's shape and the gravity force was determined. In the middle of the last century, Molodensky (1961) proposed the idea of considering the real surface of the Earth as a reduced surface and solved the corresponding boundary problem. The doctrine of the spheroidal figure of the Earth has found common understanding, and researchers, armed with theoretical knowledge, started to refine the dimensions and other details of the ellipsoid of rotation and to derive corresponding corrections.

Many publications were devoted to analysis of the observed inaccuracies in the Earth's rotation together with explanation of their possible causes, based on experimental data and theoretical solutions. The most popular review work in the twentieth century was the book authored by the well-known English geophysicist Harold Jeffreys *The Earth: Its Origin, History and Physical Constitution.* The first publication of the book happened in 1922, and later four more editions appeared, including the last one in 1970. Jeffreys was a great expert and a direct participant of the development of the most important geophysical activities. The originality of his methodological approach in describing the material lies in that, after the formulation and theoretical consideration of the problem, he writes a chapter devoted to the experimental data and facts on the theme, the comparison with analytical solutions, and discussion.

Maintaining his position on Newton's and Clairaut's models, Jeffreys considers the planet as an elastic body and describes the equation of the force equilibrium from the hydrostatic pressure, which appears from the outer uniform central force field and exhibits strengths at a given point in the form

$$\rho f_i = \rho X_i + \sum_{k=1,2,3} \frac{\partial p_{ik}}{\partial x_k}, \tag{2.43}$$

where ρ is the density, f_i is the acceleration component, $p_{ik} = p_{ki}$ is the stress component from the hydrostatic pressure, and $\;_i$ is the gravity force on the unit mass from the outer force field.

Additionally, the equation of continuity (like the continuity equation in hydrodynamics) is written as the condition of equality of velocity of the mass inflow and outflow from elementary volume in the form

$$\frac{\partial \rho}{\partial t} = -\sum_i \frac{\partial}{\partial x_i} (\rho v_i), \tag{2.44}$$

where v_i is the velocity component in the direction of x_i.

Further, applying the laws of elasticity theory, he expresses elastic properties of matter by the Lame coefficients and writes the basic equations of the strength state of the body, which links the strengths and the deformations in the point as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \nabla^2 u_i, \tag{2.45}$$

where u_i is the displacement component, λ and μ are the Lame coefficients, Δ is the component of the relative displacement, and ∇ is the Laplacian operator.

The author introduces a number of supplementary physical ideas related to the properties of the Earth's matter, assuming that it is not perfectly elastic. With the development of stresses, matter reaches its limit of resistance and passes to the stage of plastic flow with a final effect of break in the matter's continuity. This break leads to a sharp local change in the strength state, which, in turn, leads to the appearance of elastic waves in the planet's body, causing earthquakes. For this case, Equation (2.45) after the same corresponding transformations is converted into the form of plane longitudinal and transversal waves, which propagate in all directions from the break place. Such is the physical basis of earthquakes, which was a starting point of development of seismology as a branch of geophysics studying the propagation of elastic longitudinal and transversal waves in the Earth's body. By means of seismic study, mainly by strong earthquakes and based on difference in velocity of propagation of the longitudinal and transversal waves through the shells having different elastic properties, the shell-structured body of the planet was identified.

Jeffreys has analyzed the status of study in the theory of the shapes of the Earth and the Moon following Newton's basic concepts. Namely, the planet has an inner and outer gravitational force field. The gravitational pressure is formed on the planet's surface and affects the outer space and the planet's center. The Earth's shape is presented as an ellipsoid of rotation, which is perturbed from the side of inaccuracies in the density distribution, as well as from the side of the Moon's perturbations. The problem is to find the axes of the ellipsoid under action of both perturbations which occur because of a difference in the gravity field for the real Earth and the spherical body. It is accepted that the oceans' level is close to the spherical surface with deviation by a value of the first order of magnitude, and the geometric oblateness of the ellipsoid is close to the value of $e \approx 1/297$. But the value squares of deviation cannot always be ignored because the value e^2 differs substantially from the value e. The observed data cannot be compared with theoretical solutions because the formulas depending on the latitudes give precise expressions neither for the radius vector from the Earth center to the sea level nor for the value of the gravity force. The problem of the planet's mass density distribution finds its resolution from the condition of the hydrostatic pressure at a known velocity of rotation. The value of oblateness of the outer spheroid can be found from the observed value of the precession constant with a higher accuracy than one can find from the theory of the outer force field. A weak side of such approach is the condition of the hydrostatic stresses, which however are very small in comparison with the pressure at the center of the Earth. The author also notes that deviation of the outer planet's gravitational field from spherical symmetry does not satisfy the condition of the inner hydrostatic stresses. Analysis of that discrepancy makes it possible to assess errors in the inner strengths related to the hydrostatics. Because of the Earth's ellipticity, the attraction of the Sun and the Moon creates a force couple applied to the center, which forces the instantaneous axis of rotation to depict a cone around the pole of the ecliptic and to cause the precession phenomenon. The same effect initiates an analogous action on the Moon's orbit

These are the main physical fundamentals that Jeffreys used for the analysis and theoretical consideration of the planet's shape problem and for determination of its oblateness and of semimajor axis size. The author has found that the precession constant $= 0.00327293 \pm 0.00000075$ and the oblateness $1/e = 297.299 \pm 0.071$. He assumes that these figures could be accepted as a result that gives the hydrostatic theory. But in conclusion, he says that the theory is not correct. If it is correct, then the solid Earth would be a benchmark of the planet's surface covered by oceans. There are some other data confirming that conclusion. But this is the only and the most precise method for determining the spheroid flattening, which needs nonhydrostatic corrections to be found. Analogous conclusion was made by the author relative to the Moon's oblateness, where the observed and calculated values are much more contrast.

The other review works on the irregularity of rotation and the pole motion of the Earth are the monographs of W. Munk and H. MacDonald (1960), P. Melchior, (1972–1973), and P. Sabadini and B. Vermeertsen (2004). The authors analyze there the state of the art and geophysical causes leading to the observed incorrectness in the planet's rotation and wobbling of the poles. They draw the attention of the readers to the practical significance of the two main effects and designate about ten causes for their initiation. Among them are seasonal variations of the air masses, moving of the continents, melting and growing of the glaciers, elastic properties of the planet, and convective motion in the liquid core. The authors stressed that solution of any part of this geophysical task should satisfy the dynamical equations of motion of the rotating body and the equations, which determine a relationship between the stresses and deformations inside the body. The theoretical formulation and solution of a task should be considered on the hydrostatic basis, where the forces, inducing stresses, and deformations are formed by the outer uniform force field and the deformations occur in accordance with the theory of elasticity for the elastic body model, and in the framework of rheology laws for the elastic and viscous body model. The perturbation effects used are the wind force, the ocean currents, and the convective flows in the core and in the shells.

The causes of the axis rotation wobbling and the pole motion are considered in detail. The authors find that the problem of precession and nutation of the axis of rotation has been discussed for many years and does not generate any extra questions. The cause of the phenomena is explained by the Moon and the Sun perturbation of the Earth, which has an equatorial swelling and obliquity of the axis to the ecliptic. Euler equations for the rigid body form a theoretical basis for the problem's solution. In this case, the free nutation of the rigid Earth according to Euler is equal to 10 months.

2.6 Evidences that the Earth and the Moon Move Being Not in Hydrostatic Equilibrium State

The effects of the Earth's oblateness and the related problems of irregularity in the rotation and the planet's pole motion and also the continuous changes in the gravity and electromagnetic field have a direct relation to the solution of a wide range of scientific and practical problems in the Earth dynamics, geophysics, geology, geodesy, oceanography, physics of the atmosphere, hydrology, and climatology. In order to understand the physical meaning and regularities of these phenomena, regular observations are carried out. Newton's first attempts to find the quantitative value of the Earth's oblateness were based on degree measurements done by

Table 2.1 Parameters of the Earth's oblateness by degree measurement data

Author	Year	*a, m*	e	e_e	λ
D'Alambert	1800	6, 375, 553	1/334.00		
Valbe	1819	376, 895	1/302.78		
Everest	1830	377, 276	1/300.81		
Eri	1830	376, 542	1/299.33		
Bessel	1841	377, 397	1/299.15		
Tenner	1844	377, 096	1/302.5		
Shubert	1861	378, 547	1/283.0		
Clark	1866	378, 206	1/294.98		
Clark	1880	378, 249	1/293.47		
Zhdanov	1893	377, 717	1/299.7		
Helmert	1906	378, 200	1/298.3		
Heiford	1909	378, 388	1/297.0		
Heiford	1909	378, 246	1/298.8	1/38,000	38°
Krasovsky	1936	378, 210	1/298.6	1/30,000	10°
Krasovsky	1940	378, 245	1/298.3		
International	1967	378, 160	1/298.247		

Here, is the oblateness of the polar axis, is the semimajor axis, $_e$ is the equatorial oblateness, and
λ is the longitude of the maximal equatorial radius

Norwood, Pikar, and Kassini. As mentioned previously, by his calculation of the Paris latitude, the oblateness value appears to be 1/230. Very soon, some analogous measurements were taken in the equatorial zone in Peru and in the northern zone in Lapland Clairaut, Mopertui, and Buge, and other known astronomers also took part in these works. They confirmed the fact of the Earth's oblateness as calculated by Newton. The degree of the arc in the northern latitudes appeared to be maximal, and the oblateness was equal to 1/214. In the equatorial zone, the arc length was minimal, and the oblateness was equal to 1/314. So the Earth pole axis from these measurements was found to be shorter of the equatorial approximately by 20 km.

As of the end of the first part of the twentieth century, more than 20 large degree measurements were done from which the values of the oblateness and dimension of the semimajor axes were found. The data of the measurements are presented in Table 2.1, and in Table 2.2, the parameters of the triaxial ellipsoid are shown (Grushinsky 1976).

It is worth noting that in geodesy, a practical application of the triaxial ellipsoid has not been found, because it needs more complicated theoretical calculations and more reliable experimental data. In the theory, this important fact is ignored, because it is not inscribed into the hydrostatic theory of the body.

In addition to the local degree measurements, which allow determination of the Earth's geometric oblateness, more precise integral data can be obtained by observation of the precession and nutation of the planet's axis of rotation. It is assumed that the oblateness depends on deflection of the body's mass density

Table 2.2 Parameters or the Earth's equatorial ellipsoid

uthor	Year	$_1 - {}_2$, m	λ
Helmert	1915	230 ± 51	17°W
Berrot	1916	150 ± 58	10°W
Heyskanen	1924	345 ± 38	18°
Heyskanen	1929	165 ± 57	38°
Hirvonen	1933	139 ± 16	19°W
Krasovsky	1936	213	10°
Isotov	1948	213	15°

Here, $_1$ and $_2$ are the semimajor and semiminor axes of the equatorial ellipsoid

distribution from spherical symmetry and is initiated by a force couple that appeared to be an interaction of the Earth with the Moon and the Sun. The precession of the Earth's axis is proportional to the ratio of the spheroid's moments of inertia relative to the body's axis of rotation in the form of the dynamical oblateness ε:

$$\varepsilon = \frac{C - A}{C}$$

At the same time, the retrograde motion of the Moon's nodes (points of the ecliptic intersection by the Moon orbit) is proportional to the second spherical harmonics coefficient J_2 of the Earth's outer gravitational potential in the form

$$J_2 = \frac{C - A}{Ma^2}.$$

It is difficult to obtain a rigorous value of geometric oblateness from its dynamic expression because we do not know the radial density distribution. Moreover, the Moon's mass is known up to a fraction of a percent, but it is inconvenient to calculate analytically the joint action of the Moon and the Sun on the precession. In spite of that, some researchers succeeded in making such calculations, assuming that the Earth's density is increasing proportionally to the depth. Their data are the following:

By Newcomb	$\varepsilon = 1/305.32 = 0.0032753$; $e = 1/297.6$;
By de Sitter	$\varepsilon = 1/304.94 = 0.0032794$; $e = 1/297.6$;
By Bullard	$\varepsilon = 1/305.59 = 0.00327236$; $e = 1/297.34$;
By Jeffreys	$\varepsilon = 1/305.54 = 0.00327293$; $e = 1/297.3$.

After appearance of the Earth's artificial satellites and some special geodetic satellites, the situation with observation procedures has in principle changed. The satellites made it possible to determine directly, by measuring of the even zonal moments, the coefficient J_n in expansion of the Earth gravitational potential by spherical functions. In this case at hydrostatic equilibrium, the odd and all the tesseral moments should be equal to zero. It was assumed before the satellite era that the correction coefficients of a higher degree of J_2 will decrease and the main expectations to improve the calculation results were focused on the coefficient J_4. But it has appeared that all the gravitational moments of higher degrees are the values proportional to square of oblateness, that is, $\sim(1/300)^2$ (Zharkov 1978).

On the basis of the calculated harmonic, the coefficients of the expanded gravitational potential of the Earth published by Smithsonian Astrophysical observatory and the Goddard cosmic center of the USA, the fundamental parameters of the gravitational field, and the shape of the so-called standard Earth were determined. Among them are coefficient of the second zonal harmonic $J_2 = 0.0010827$, equatorial radius of the Earth ellipsoid $= 6378160$ m, angular velocity of the Earth's rotation $\omega_3 = 7.292 \cdot 10^{-5}$ rad/s, equatorial acceleration of the gravity force $\gamma\ = 978031.8$ mgl, and oblateness $1/\ = 1/298.25$ (Grushinsky 1976; Melchior 1972). At the same time, if the Earth stays in hydrostatic equilibrium, then, applying the solutions of Clairaut and his followers, the planet's geometric oblateness should be equal to $e' = 1/299.25$. On the basis of that contradiction, lchior (1972) concluded that the Earth does not stay in hydrostatic equilibrium. It represents either a simple equilibrium of the rigid body, or there is equilibrium of a liquid and not static but dynamic with an extra hydrostatic pressure. Coming to interpretation of the density distribution inside the Earth by means of the Williamson–Adams equation, lchior (1972) adds that in order to eliminate there the hydrostatic equilibrium, one needs a supplementary equation. Since such an equation is absent, we are obliged to accept the previous conditions of hydrostatics.

The situation with the absence of hydrostatic equilibrium of the Moon is much more striking. The polar oblateness of the body is (Grushinsky 1976)

$$e_{\mathrm{p}} = \frac{b - c}{r_0} = 0.94 \cdot 10^{-5},$$

and the equatorial oblateness is equal

$$e_{\mathrm{e}} = \frac{a - c}{r_0} = 0.375 \cdot 10^{-4},$$

where , b, and are the equatorial and polar semiaxes and r_0 is the body mean radius.

It was found by observation of the Moon libration that

$$e_{\mathrm{p}} = 4 \cdot 10^{-4} \text{ and } e_{\mathrm{e}} = 6.3 \cdot 10^{-4}.$$

The calculation of the ratio of theoretical values of the dynamic oblateness $e_{\mathrm{d}} = e_{\mathrm{p}}/e_{\mathrm{e}} = 0.25$ substantially differs from the observation, which is $0.5 \le e_{\mathrm{d}} \le 0.75$. At the same time, the difference of the semiaxes is $a_1 - a_3 = 1.03\,\mathrm{km}$ and $a_2 - a_3 = 0.83\,\mathrm{km}$, where a_1 and a_2 are the Moon's equatorial semiaxes.

After the works of Clairaut, Stokes, and Molodensky, on the basis of which the relationship between the gravity force change at the sea level and on the real Earth surface with the angular velocity of rotation was established, one more problem has risen. During measurements of the gravity force at any point of the Earth's surface, two effects are revealed. The first is an anomaly of the gravity force, and the second is a declination of the plumb line from the normal in a given point.

Analysis of the gravity force anomalies and the geoid heights (a conventional surface of a quiet ocean) based on the existent schematic maps, compiled from the calculated coefficients of expansion of the Earth's gravity potential and ground level gravimetric measurements, allows derivation of some specific features related to the parameter forming the planet. As Grushinsky (1976) notes, elevation of the geoid over the ellipsoid of rotation with the observed oblateness reaches 50–70 m only in particular points of the planet, namely, in the Bay of Biscay, North Atlantic, and near the Indonesian Archipelago. In the case of triaxial ellipsoid, the equatorial axis is passed near those regions with some asymmetry. The maximum of the geoid heights in the western part is shifted towards the northern latitudes and maximum in the eastern part remains in the equatorial zone. The western end of the major radius reaches the latitudes of 0–10° to the west of Greenwich, and the western end falls on the latitudes of about 30–40° to the west of a meridian of 180°. This also indicates asymmetry in distribution of the gravity forces and the forming masses. And the main feature is that the tendency to asymmetry of the northern and the southern hemispheres as a whole is observed. The region of the geoid's northern pole rises above the ellipsoid up to 20 m, and the Antarctic region is situated lower by the same value. The asymmetry in planetary scale is traced from the northwest of Greenland to the southeast through Africa to the Antarctic with positive anomalies and from Scandinavia to Australia through the Indian Ocean with negative anomalies up to 50 mgl. Positive anomalies up to 30 mgl are fixed within the belt from Panama to Fiery Land and to the peninsula Grechem in the Antarctic. The negative anomalies are located on both sides, which extend from the Aleut bank to the southeast of the Pacific Ocean and from Labrador to the south of the Atlantics. The structure of the positive and negative anomalies is such that their nature can be interpreted as an effect of spiral curling of the northern hemisphere relative to the southern one.

As to the plumb line declination, this effect is considered only in geodesy from the point of view of practical application in the corresponding geodetic problems. Physical aspects of the problem are not touched. Later on, we will discuss this problem.

The problem of the Earth's rotation has been discussed at the NATO workshop (azenave 1986). It was stated that both aspects of the problem still remain unsolved. The problems are variations in the day's duration and the observed Chandler's wobbling of the pole with the period of 14 months in comparison with 10 months, given by the Euler rigid body model. Chandler's results are based on the analysis of 200-year observational data of motion of the Earth's axis of rotation, done in the USA in the 1930s. He found that there is an effect of free wobbling of the planet's axis with the period of about 420 days. Since that time, the discovered effect remains the main obstacle in the explanation of the nature and theoretical justification of the pole's motion.

Summing up this short excursion to the problem's history, we found the situation as follows. The majority of researchers dealing with the dynamics of the Earth and its shape come to the unanimous conclusion that the theories based on hydrostatics do not give satisfactory results in comparison with the observations. For instance, Jeffreys straightforwardly says that the theories are incorrect. Munk and MacDonald more delicately note that dozens of the observed effects can be called that do not satisfy the hydrostatic model. It means that the dynamics of the Earth as a theory is absent. This state of art and the conclusion motivated the authors to search for a novel physical basis for the dynamics and creation of the Earth, planets, and satellites. The first step in that direction was presented in their publication (Ferronsky and Ferronsky 2010).

References

azenave A (ed) (1986) Earth rotation: solved and unsolved problems. In: Proceeding of the NATO advanced research workshop. Reidel, Dordrecht

Clairaut AK (1947) Theory of the earth figure based on hydrostatics (transl. from French). Publishing House of the USSR Academy of Science, Moscow/Leningrad

Ferronsky VI, Ferronsky SV (2010) Dynamics of the earth. Springer, Dordrecht/Heidelberg

Ferronsky VI, Denisik SA, Ferronsky SV (2011) Jacobi dynamics, 2nd edn. Springer, Dordrecht/Heidelberg

Goldstein H (1980) Classical mechanics, 2nd edn. Addison-Wesley, Reading

Grushinsky NP (1976) Theory of the earth figure. Nauka, Moscow

Kittel C, Knight WD, Ruderman MW (1965) Mechanics, Berkeley physics course, vol 1. McGraw Hill, New York

Landau LD, Lifshitz EM (1954) Mechanics of continuous media. Gostechizdat, Moscow

Melchior P (1972) Physique et dynamique planetaires. Vander-Editeur, Bruxelles

Molodensky MS, Kramer MV (1961) The Earth's tidals nutation of the planet. Nauka, Moscow

Munk W, MacDonald G (1960) Rotation of the Earth (transl. from English). Mir, Moscow

Sabadini R, Vermeertsen B (2004) Global dynamics of the earth. Kluwer, Dordrecht

Sedov LI (1970) Mechanics of continuous media, vol 2. Nauka, Moscow

Zharkov VN (1978) Inner structure of the earth and planets. Nauka, Moscow

Chapter 3
Physical Meaning of Dynamical Equilibrium of a Celestial Body

Abstract Interpretation of the artificial satellite data proving the absence of the hydrostatic equilibrium state of the Earth and the Moon are discussed in detail. It is shown that the Earth and the Moon are triaxial bodies, and their axial rotation is not an inertial effect. Observational data from earthquakes demonstrate the planet's oscillating dynamics with periods from 8.4 to 57 min. Two general modes of the Earth's oscillation were found, namely, spectral with a vector of radial direction and torsion with a vector perpendicular to the radius.

The main problem of a celestial body's equilibrium state, which is a ratio of the kinetic and potential energies, is discussed thoroughly. It is shown that the ratio of Earth's kinetic and potential energy is equal to $\sim$1/300. The other planets, the Sun, and the Moon, the hydrostatic equilibrium for which is also accepted as a fundamental condition, stay in analogous situation. This is because the hydrostatic approach does not take into account the kinetic energy of the interacted elementary mass particles, which is, in fact, Newton's energy of gravity (and force). As a result, celestial body dynamics have been left without kinetic energy.

In order to correct this situation, the generalized virial theorem was derived by introduction of the volumetric forces and moments into the classic one. As a result, the oscillating mode of the body motion has appeared in the form of Jacobi's virial equation in the form $\ddot{\Phi} = 2E - U$ (where Φ is the Jacobi's function; E and U are the total and potential energy, respectively). In addition, the inner and outer force fields and the energy as the measure of interacted mass particles of a celestial body were revealed. The reduced inner gravitational (weighting) field was obtained.

Let us come back to the fact of absence of the Earth's hydrostatic equilibrium found by satellite data. The initial factual material for the problem study is presented by the observed orbit elements of the geodetic satellites, which move on perturbed Kepler's orbits. The satellite motion is fixed by means of observational stations located within zones of a visual height range of 1,000–2,500 km, which is optimal for the planet's

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_3,

gravity field study. It was found that the satellite's perturbed motion at such a close distance from the Earth surface is connected with the nonuniform distribution of mass density, the consequences of which are the nonspherical shape of the figure and the corresponding nonuniform distribution of the outer gravity field around the planet. These nonuniformities cause corresponding changes in trajectories of the satellite's motion, which are fixed by the tracking stations. Thus, distribution of the Earth's mass density determines the adequate equipotential trajectory in the planet's gravity field, which follows the satellite. The main goal of the geodetic satellites, launched under different angles relative to the equatorial plane, is the measurement of all deviations in the trajectory from the unperturbed Kepler's orbit.

3.1 Relationship of Gravitational Field and Moment of Inertia by Satellite Data

The satellite orbits data for solving the problem of the nature of the Earth's oblateness are interpreted on the basis of the known (in celestial mechanics) theory of expansion of the gravity potential of a body, the structure and the shape of which do not much differ from the uniform sphere. The expression of the expansion, by spherical functions, recommended by the International Union of Astronomy, is the following equation (Grushinsky 1976):

$$U(r,\varphi,\lambda) = \frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\varphi) + \sum_{n=2}^{\infty}\sum_{m=1}^{n} \left(\frac{R_e}{r}\right)^n P_{nm}(\sin\varphi)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right], \tag{3.1}$$

where r, φ, and λ are the heliocentric polar coordinates of an observation point; G is the gravity constant; M and R_e are the mass and the mean equatorial radius of the Earth; P_n is the Legendre polynomial of n order; $P_{nm}(\sin\varphi)$ is the associated spherical functions; and J_n, C_{nm}, and S_{nm} are the dimensionless constants characterizing the Earth's shape and gravity field.

The first terms of Eq. (3.1) determine the zero approximation of Newton's potential for a uniform sphere. The constants J_n, C_{nm}, and S_{nm} represent the dimensionless gravitational moments, which are determined by analyzing the satellite orbits. The values J_n express the zonal moments, and C_{nm} and S_{nm} are the tesseral moments. In the case of hydrostatic equilibrium of the Earth as a body of rotation, in the expression of the gravitational potential (3.1), only the even n-zonal moments J_n are rapidly decreased with growth, and the odd zonal and all tesseral moments turn into zero, that is,

$$U = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\,\theta) - \sum_{n=3}^{\infty} J_n\,\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\,\theta)\right], \tag{3.2}$$

where θ is the angle of the polar distance from the Earth's pole.

Here, the constant J_2 represents the zonal gravitational moment, which characterizes the axial planet's oblateness and makes the main contribution to correction of the unperturbed potential. That constant determines the dimensionless coefficient of the moment of inertia relative to the polar axis and equal to

$$J_2 = \frac{C - A}{MR_e^2}, \tag{3.3}$$

where and are the Earth moments of inertia with respect to the polar and equatorial axes, accordingly, and R_e is the equatorial radius.

For expansion of the Earth's gravity forces potential by spherical functions, the rotation of which is taken to be by action of the outer inertial forces, but not by the own force field, the centrifugal force potential is introduced into Eq. (3.2). Then, for the hydrostatic condition with the even zonal moments J_n, one has

$$U = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\,\theta) - \sum_{n=3}^{\infty} J_n\,\left(\frac{R_e}{r}\right)^{n+1} P_n(\cos\,\theta)\right] + \frac{\omega^2 r^2}{3}\left[1 - P_2(\cos\,\theta)\right], \tag{3.4}$$

where W is the potential of the body rotation and $\omega^2 r$ is the centrifugal force.

The first two terms and the term of the centrifugal force in Eq. (3.4) express the normal potential of the gravity force:

$$W = \frac{GM}{r}\left[1 - J_2\left(\frac{R_e}{r}\right)^2 P_2(\cos\,\theta) + \frac{\omega^2 r^2}{3}\left[1 - P_2(\cos\,\theta)\right].\right. \tag{3.5}$$

The potential (3.5) corresponds to the spheroid's surface, which coincides with the ellipsoid of rotation with accuracy up to its oblateness. Rewriting term $_2(\cos\theta)$ in this equation through the sinus of the heliocentric latitude and the angular velocity – through the geodynamic parameter q – one can find the relationship of the Earth's oblateness ε with the dynamic constant J_2. Then equation of the dynamic oblateness ε is obtained in the form (Grushinsky 1976; Melchior 1972)

$$\varepsilon = \frac{3}{2}J_2 + \frac{q}{2}, \tag{3.6}$$

where the geodynamic parameter q is the ratio of the centrifugal force to the gravity force at the equator

$$q = \frac{\omega^2 R}{GM/R^2}. \tag{3.7}$$

Geodynamic parameter J_2 found by satellite observation in addition to the oblateness calculation is used for determination of a mean value of the Earth's moment of inertia. For this purpose, the constant of the planet's free precession is also used, which represents one more observed parameter expressing the ratio of the moments of inertia in the form

$$H = \frac{C - A}{C}. \tag{3.8}$$

This is the theoretical base for interpretation of the satellite observations. But its practical application gave very contradictory results (Grushinsky 1976; Melchior 1972; Zharkov 1978). In particular, the zonal gravitation moment calculated by means of observation was found to be $J_2 = 0.0010827$, from where the polar oblateness $\varepsilon = 1/298.25$ appeared to be shorter of the expected value and equal to 1/297.3. All zonal moments J_n, starting from J_3, which relate to the secular perturbation of the orbit, were close to constant value and equal, by the order of magnitude, to the square of the oblateness, that is, $\sim(1/300)^2$, and slowly decreasing with an increase in n. The tesseral moments C_{nm} and S_{nm} appeared to be not equal to zero, expressing the short-term nutational perturbations of the orbit. In the case of hydrostatic equilibrium of the Earth at the found value of J_2, the polar oblateness ε should be equal to 1/299.25. On this basis, the conclusion was made that the Earth does not stay in hydrostatic equilibrium. The planet's deviation from the hydrostatic equilibrium evidenced that there is a bulge in the planet's equatorial region with an amplitude of about 70 m. It means that the Earth body is forced by normal and tangential forces that develop corresponding stresses and deformations. Finally, by the measured tesseral and sectorial harmonics, it was directly confirmed that the Earth has an asymmetric shape with reference to the axis of rotation and to the equatorial plane.

Because the Earth does not stay in hydrostatic equilibrium, the above described initial physical fundamentals for interpretation of the satellite observations should be recognized as incorrect, and the related physical concepts cannot explain the real picture of the planet's dynamics.

The question is raised how to interpret the obtained actual data and where the truth should be sought. First of all, we should verify the correctness of the oblateness interpretation and the conclusion about the Earth's equatorial bulge. It is known from observation that the Earth is a triaxial body (see Table 2.2). Theoretical application of the triaxial Earth model was not considered because it contradicts to hydrostatic equilibrium hypothesis. But after it was found that the hydrostatic equilibrium is absent, the triaxial Earth alternative should be considered first.

Let us analyze Eq. (3.7). It is known from the observation data that the constant of the centrifugal oblateness q is equal to

$$q = \frac{\omega^2 R}{GM/R^2} = \left(\frac{1}{17.01}\right)^2 = \frac{1}{289.37}. \tag{3.9}$$

Determine a difference between the centrifugal oblateness constant q and the polar oblateness ε' found by the satellite orbits, assuming that the desired value has a relationship with the perturbation caused by the equatorial ellipsoid

$$\begin{aligned}\varepsilon' &= \frac{a-b}{a} = \frac{a-c}{a} - \frac{b-c}{a} = \frac{1}{289.37} - \frac{1}{298.25} = \frac{1}{9,720}\\ &= \left(\frac{1}{98.59}\right)^2 = 1.713\left(\frac{1}{289.37}\right)^2,\end{aligned} \tag{3.10}$$

where a, b, and c are the semiaxes of the triaxial Earth.

The differences between the major and minor equatorial semiaxes can be found from Eq. (3.10). If the major semiaxis is taken in accordance with recommendation of the International Union of Geodesy and Geophysics as = 6,378,160 m, then the minor equatorial semiaxis b can be equal to

$$a - b = 6{,}378{,}160/9{,}720 = 656 \text{ m}; \quad b = 6{,}377{,}504 \text{ m}.$$

One can see that the second semiaxes are shorter of the first one by 656 m. There is a reason now to assume that the value of equatorial oblateness $\varepsilon' = 1/9{,}720$ is a component in all the zonal gravitation moments J_n, related to the secular perturbations of the satellite orbits including J_2. They are perturbed both by the polar and the equatorial oblateness of the Earth. This effect ought to be expected because it was known long ago from observation that the Earth is a triaxial body. If our conclusion is true, then there is no ground for discussion about the equatorial bulge. And also the problem of the hydrostatic equilibrium is closed automatically because in this case the Earth is not a figure of rotation, and the nature of the observing fact of rotation of the Earth should be looked for rather in the action of the own inner force field but not in the effects of the inertial forces. As to the nature of the Earth's oblateness, then for its explanation later on, the effects of perturbation arising during separation of the Earth's shells by mass density differentiation and separation of the Earth itself from the protosun will be considered. In particular, the effect of heredity in the creation of the body's oblateness is evidenced by the ratio of kinetic energy of the Sun and the Moon expressed through the ratio of square frequencies of oscillation ε'' of their polar moments of inertia, which is close to the planet's equatorial oblateness:

$$\varepsilon'' = \frac{\omega_c^2}{\omega_\pi^2} = \frac{\left(10^{-4}\right)^2}{(0.96576 \cdot 10^{-2})^2} = \left(\frac{1}{96.576}\right)^2 = 1,713\left(\frac{1}{289.37}\right)^2,$$

where $\omega_c = 10^{-4}$ s^{-1} and $\omega_\pi = 0.96576 \cdot 10^{-2}$ s^{-1} are the frequencies of oscillation of the Sun's and the Moon's polar moment of inertia correspondingly.

The most prominent effect, which was discovered by investigation of the geodetic satellite orbits, is the fact of a physical relationship between the Earth's mean (polar) moment of inertia and the outer gravity field. That fact without exaggeration can be called as a fundamental contribution to understanding the nature of the planet's self-gravity. The planet's moment of inertia is an integral characteristic of the mass density distribution. Calculation of the gravitational moments based on the measurement of elements of the satellite orbits is the main content of satellite geodesy and geophysics. Short-periodic perturbations of the gravity field fixed at revolution of a satellite around the Earth, the period of which is small compared to the planet's period, provide evidence about oscillation of the moment of inertia or, to be more correct, about oscillating motion of the interacted mass particles. It will be shown that oscillating motion of the interacting particles forms the main part of a body's kinetic energy and the moment of inertia itself is the periodically changing value.

Oscillation of the Earth's moment of inertia and also the gravitational field is fixed not only during the study by artificial satellites but both parameters have also been registered by surface seismic investigations. Consider briefly the main points of those observations.

3.2 Earthquakes' Observational Data

The study of the Earth's eigenoscillation started with Poisson's work on oscillation of an elastic sphere, which was considered in the framework of the theory of elasticity. At the beginning of the twentieth century, Poisson's solution was generalized by Love in the framework of the problem solution of a gravitating uniform sphere of the Earth's mass and size. The calculated values of periods of oscillation were found to be within the limit of some minutes to one hour.

In the middle of the twentieth century during the powerful earthquakes in 1952 and 1960 in Chile and Kamchatka, an American team of geophysicists headed by Beneoff, using advanced seismographs and gravimeters, reliably succeeded in recording an entire series of oscillations with periods from 8.4 up to 57 min. Those oscillations in the form of seismograms have represented the dynamical effects of the interior of the planet as an elastic body, and the gravimetric records have shown the "tremor" of the inner gravitational field (Zharkov 1978). In fact, the effect of the simultaneous action of the potential and kinetic energy in the Earth's interior was fixed by these experiments.

About 1,000 harmonics of different frequencies were derived by expansion of the line spectrum of the Earth's oscillation. These harmonics appear to be integral characteristics of the density, elastic properties, and effects of the gravity field, that is, of the potential and kinetic energy of separate volumetric parts of the nonuniform planet. As a result, two general modes of the Earth's oscillations were found by this spectral analysis, namely, spherical with a vector of radial direction and torsion with a vector perpendicular to the radius.

From the point of view of the existing conception about the planet's hydrostatic equilibrium, the nature of the observed oscillations was considered to be a property of the gravitating nonuniform (regarding density) body in which the pulsed load of the earthquake excites elementary integral effects in the form of elastic gravity quanta (Zharkov 1978). Considering the observed dynamical effects of earthquakes, geophysicists came close to a conclusion about the nature of the oscillating processes in the Earth's interior. But the conclusion itself still has not been expressed because it continues to relate to the position of the planet's hydrostatic equilibrium.

Now we move to one of the main problems related to the Earth's equilibrium or, more correctly, to the absence of the Earth's equilibrium if it is considered on the basis of hydrostatics.

3.3 Oscillating Kinetic Energy of a Celestial Body's Interacted Masses

We discovered the most likely serious cause, for which even formulation of the problem of the body's dynamics based on the hydrostatic equilibrium is incorrect. The point is that the ratio of the kinetic energy to the potential one of a celestial body is too small. For example, this ratio for the Earth is equal to ~1/300, that is, the same as its oblateness. Such a ratio does not satisfy the fundamental condition of the virial theorem, the equation of which expressed the hydrostatic equilibrium state. According to that condition, the ratio of the considered energies should be equal to 1/2. Taking into account that the kinetic energy of the Earth is presented by the planet's inertial rotation and assuming it to be a rigid body rotating with the observed angular velocity $\omega_r = 7.29\cdot10^{-5}\ s^{-1}$, the mass $= 6\cdot10^{24}$ kg, and the radius $R = 6.37\cdot10^6$ m, the energy is equal to

$$T_e = 0.6 M R^2 \omega_r^2 = 0.6 \cdot 6 \cdot 10^{24} \cdot \left(6.37 \cdot 10^6\right)^2 \cdot \left(7.29 \cdot 10^{-5}\right)^2$$
$$= 7.76 \cdot 10^{29}\,\text{J} = 7.76 \cdot 10^{36}\,\text{erg}.$$

The potential energy of the Earth at the same parameters is

$$U_e = 0.6 \cdot GM^2/R = 0.6 \cdot 6.67 \cdot 10^{-11} \cdot \left(6 \cdot 10^{24}\right)^2/6.37 \cdot 10^6$$
$$= 2.26 \cdot 10^{32}\,\text{J} = 2.26 \cdot 10^{39}\,\text{erg}.$$

The ratio of the kinetic and potential energy comprises

$$J_2 = \frac{T_e}{U_e} = \frac{7.76 \cdot 10^{29}}{2.26 \cdot 10^{32}} = \frac{1}{292}.$$

One can see that the ratio is close to the planet's oblateness. It does not satisfy the virial theorem and does not correspond to any condition of equilibrium of a really existing natural system because in accordance with the third Newton's law, the equality between the acting and the reacting forces should be satisfied. The other planets, such as Mars, Jupiter, Saturn, Uranus, and Neptune, exhibit the same behavior, but for Mercury, Venus, the Moon, and the Sun, the equilibrium states of which are also accepted as hydrostatic, their potential energy exceeds their kinetic energy by 10^4 times. Since the bodies in reality exist in equilibrium and their orbital motion strictly satisfies the ratio of the energies, the question arises where the kinetic energy of the body's own motion has disappeared. Otherwise, the virial theorem for the celestial bodies is not valid. Moreover, if one takes into account that the energy of inertial rotation does not belong to the body, then the celestial body equilibrium problem appears to be out of discussion. And the celestial body dynamics is left without kinetic energy.

Thus, we came to the problem of a body equilibrium from two positions. From one side, it does not stay in hydrostatic equilibrium, and from the other side, it does not stay in general mechanical equilibrium because there is no reaction forces to counteract to the acting potential forces. The answer to both questions is given in the following section while deriving an equation of the dynamical equilibrium of a body by means of a generalization of the classical virial theorem.

3.4 Generalized Classical Virial Theorem as Equation of Dynamical Equilibrium of a Body

The main methodological question arises: in which state of equilibrium does the Earth exist? The answer to the question results from the generalized virial theorem for a self-gravitating body, that is, the body that generates the energy for motion by interaction of the constituent particles having the innate moments. The guiding effect that we use here is the motion observed by an artificial satellite, which is the functional relationship between changes in the gravity field of the Earth and its mean (polar) moment of inertia. The deep physical meaning of this relationship is as follows: the planet's mean (polar) moment of inertia is an integral (volumetric) parameter, which does not represent the location of the interacting mass particles but expresses changes in their motion under the constrained energy. The virial theorem of Clausius for a perfect gaseous cloud or a uniform body is presented in its averaged form. In order to generalize the theorem for a nonuniform body, we introduce there the volumetric moments of interacted particles, taking into account their volumetric nature. Moreover, the interacted mass particles of a continuous medium generate volumetric forces (pressure or capacity of energy) and volumetric momentums, which, in fact, generate the motion in the form of oscillation and rotation of masses. The oscillating form of motion of the Earth and other celestial bodies is the dominating part of their kinetic energy, which up to now has not been taken into account. We wish to fill in this gap in dynamics of celestial bodies by applying the volumetric forces of their gravitational interaction.

The virial theorem is an analytical expression of the hydrostatic equilibrium condition and follows from Newton's and Euler's equations of motion. Let us recall its derivation in accordance with the classical mechanics (Goldstein 1980).

Consider a system of mass points, the location of which is determined by the radius vector $\mathbf{r}_i$ and the force $\mathbf{F}_i$ including the constraints. Then, equations of motion of the mass points through their moments $\mathbf{p}_i$ can be written in the form

$$\dot{\mathbf{p}}_i = \mathbf{F}_i . \tag{3.11}$$

The value of the moment of momentum is

$$Q = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i ,$$

where the summation is done over all masses of the system. The derivative with respect to time from that value is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i . \tag{3.12}$$

The first term on the right-hand side of (3.12) is reduced to the form

$$\sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i = \sum_i m_i \cdot \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T,$$

where T is the kinetic energy of particle motion under action of forces $\mathbf{F}_i$.

The second term in Eq. (3.12) is

$$\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i .$$

Now Eq. (3.12) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i . \tag{3.13}$$

The mean values in (3.13) within the time interval τ are found by their integration from 0 to t and division by τ:

$$\frac{1}{\tau} \int_0^t \frac{\mathrm{d}Q}{\mathrm{d}t} \mathrm{d}t = \overline{\frac{\mathrm{d}Q}{\mathrm{d}t}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$$

or

$$\overline{2T} + \overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = \frac{1}{\tau} \left[Q\left(\tau\right) - Q(0) \right] . \tag{3.14}$$

For the system in which the coordinates of mass point motion are repeated through the period τ, the right-hand side of Eq. (3.14) after its averaging is equal to zero. If the period is too large, then the right-hand side becomes a very small quantity. Then, the expression (3.14) in the averaged form gives the following relation:

$$-\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i} = 2T, \tag{3.15}$$

or in mechanics, it is written in the form

$$-U = 2T.$$

Equation (3.15) is known as the virial theorem, and its left-hand side is called the virial of Clausius (German *virial* is from the Latin *vires* which means forces). The virial theorem is a fundamental relation between the potential and kinetic energy and is valid for a wide range of natural systems, the motion of which is provided by the action of different physical interactions of their constituent particles. Clausius proved the theorem in 1870 when he solved the problem of work of the Carnot thermal machine, where the final effect of the water vapor pressure (the potential energy) was connected with the kinetic energy of the piston motion. The water vapor was considered as a perfect gas. And the mechanism of the potential energy (the pressure) generation at the coal burning in the firebox was not considered and was not taken into account.

The starting point in the previously presented derivation of virial theorem in mechanics is the moment of the mass point system, the nature of which is not considered either in mechanics or by Clausius. By Newton's definition, the moment "is the measure of that determined proportionally to the velocity and the mass." The nature of the moment by his definition is "the innate force of the matter." By his understanding, that force is an inertial force, that is, the motion of a mass continues with a constant velocity.

The observed (by satellites) relationship between the potential and the kinetic energy of the gravitation field and the Earth's moment of inertia provides evidence that the kinetic energy of the interacted mass particle motion, which is expressed as a volumetric effect of the planet's moment of inertia, is not taken into account. The evidence of this was given in the previous section in the quantitative calculation of a ratio between the kinetic and potential energies, equal to $\sim$1/300.

In order to remove that contradiction, the kinetic energy of motion of the interacting particles should be taken into account in the derived virial theorem. Because any mass has volume, the moment should be written in volumetric form:

$$\mathbf{p}_i = \sum_i m_i \dot{\mathbf{r}}_i. \tag{3.16}$$

Now the volumetric moment of momentum acquires the wave nature and is presented as

$$Q = \sum_i \mathbf{p}_i \cdot \mathbf{r}_i = \sum_i m_i \cdot \dot{\mathbf{r}}_i \cdot \mathbf{r}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{m_i \mathbf{r}_i^2}{2}\right) = \frac{1}{2}\dot{I}_\mathrm{p}, \tag{3.17}$$

where I_p is the polar moment of inertia of the system (for the sphere it is equal to 3/2 of the axial moment).

The derivative from that value with respect to time is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{2}\ddot{I}_\mathrm{p} = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i. \tag{3.18}$$

The first term in the right-hand part of (3.18) remains without change:

$$\sum_i \dot{\mathbf{r}}_i \cdot \mathbf{p}_i = \sum_i m_i \cdot \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T. \tag{3.19}$$

The second term represents the potential energy of the system:

$$\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i = U. \tag{3.20}$$

Equation (3.18) is written now in the form

$$\frac{1}{2}\ddot{I}_\mathrm{p} = 2T + U. \tag{3.21}$$

Expression (3.21) represents a generalized equation of the virial theorem for a mass point system interacting by Newton's law. Here on the left-hand side of (3.21), the previously ignored inner kinetic energy of interaction of the mass particles appears. Solution of Eq. (3.21) gives a variation in the polar moment of inertia within the period τ. For a conservative system averaged expression (3.18) by integration from 0 to t within time interval, τ gives

$$\frac{1}{\tau}\int_0^t \frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{d}t = \overline{\frac{\mathrm{d}Q}{\mathrm{d}t}} = \overline{2T} + \overline{U} = \ddot{I}_\mathrm{p} = 0. \tag{3.22}$$

Equation (3.22) at $\ddot{I}_\mathrm{p} = 0$ gives $\dot{I}_\mathrm{p} = E = \text{const.}$, where E is the total system's energy. It means that the interacting mass particles of the system move with constant velocity. In the case of dissipative system, Eq. (3.22) is not equal to zero, and the interacted mass particles move with acceleration. Now the ratio between the potential and kinetic energy has a value in strict accordance with Eq. (3.21). The kinetic energy of the interacted mass particles in the form of oscillation of the polar moment of inertia in that equation is taken into account. And now in the frame of the law of energy conservation, the ratio of the potential to kinetic energy of a celestial body has a correct value.

Expression (3.21) appears to be an equation of dynamical equilibrium of the self-gravitating planets with the force field of the Sun and the self-gravitating moons with the force fields of their planets. Here, static equilibrium is absent because interacting particles continuously move and generate energy due to their inner potential. The integral effect of the moving particles is fixed by the satellite orbits in the form of changing zonal, sectorial, and tesseral gravitational moments. We used the resulting energy of the initial moment (3.16) for derivation of the generalized virial theorem. The initial moments form the inner, or "innate" by Newton's definition, energy of the body, which has an inherited origin. The nature of Newton's centripetal forces and the mechanism of their energy generation will be discussed in some detail in Chap. 8.

Thus, we obtained a differential equation of the second order (3.21), which describes a celestial body dynamics and its dynamical equilibrium.

The virial equation (3.21) was already obtained by Jacobi one and a half centuries ago from Newton's equations of motion in the form (Jacobi 1884)

$$\ddot{\Phi} = U + 2T, \tag{3.23}$$

where Φ is Jacobi's function (the polar moment of inertia).

At that time, Jacobi was not able to consider the physical meaning of the equation. For that reason, he assumed that as there are two independent variables Φ and U in the equation, it cannot be resolved. We succeeded in finding an empirical relationship between the two variables and at first obtained an approximate and later on a rigorous solution of the equation (Ferronsky et al. 1978, 1987, 2010; Ferronsky 2005). The relationship is proved now by means of the satellite observation.

We can now explain the cause of discrepancy between the geometric (static) and dynamic oblateness of the Earth. The reason is as follows: the planet's moments of inertia with respect to the main axes and their integral form of the polar moment of inertia do not stay in time as constant values. The polar moment of inertia of a self-gravitating body has a functional relationship with the potential energy, the generation of which results by interaction of the mass particles in the regime of periodic oscillations. The hydrostatic equilibrium of a body does not express the picture of the dynamic processes from which, as it follows from the averaged virial theorem, the energy of the oscillating effects was lost from consideration. Because of that, it was not possible to understand the nature of the energy. The main part of the body's kinetic energy of the body's oscillation was also lost. As to the rotational motion of the body shells, it appears only in the case of the nonuniform radial distribution of the mass density. The contribution of rotation to the total body's kinetic energy comprises a very small part.

The cause of the accepted incorrect ratio between the Earth's potential and kinetic energy is the following: Clairaut's equation (2.26) derived for the planet's hydrostatic equilibrium state and applied to determine the geometric oblateness, because of the previously stated reason, has no functional relationship between the force function and the moment of inertia. Therefore, for the Earth dynamical problem, the equation gives only the first approximation. In the formulation of the

Earth's oblateness problem, Clairaut accepted Newton's model of action of the centripetal forces from the surface of the planet to its geometric center. In such a physical conception, the total effect of the inner force field becomes equal to zero. In Sect. 3.6, it will be shown that the force field of the continuous body's interacting masses represents volumetric pressure but not a vector force field. That is why the accepted postulate relating to the planet's inertial rotation appears to be physically incorrect.

The question was raised about how it happened that geodynamic problems and first of all the problem of stability of the Earth's motion up to now were solved without knowing the planet's kinetic energy. The probable explanation for that seems to lie in the history of development of science. In Kepler's problem and in Newton's two-body problem solution, the transition from the averaged parameters of motion to the real conditions is implemented through the mean and the eccentric anomalies, which by geometric procedures indirectly take into account the energy of motion. In the Earth's shape problem, this procedure of Kepler is not applied. Therefore, the so-called inaccuracies in the Earth's motion appear to be the regular dynamic effects of a self-gravitating body, and the hydrostatic model in the problem is irrelevant. The hydrostatic model was accepted by Newton for the other problem, where just this model allowed discovery and formulation of the general laws of the planets' motion around the Sun. Newton's centripetal forces in principle satisfy Kepler's condition when the distance between bodies is much more than their size accepted as mass points. Such a model gives a first approximation in the problem solution.

Kepler's laws express the real picture of the planets' and satellites' motion around their parent bodies in averaged parameters. All the deviations of those averaged values related to the outer perturbations are not considered as it was done in Clausius' virial theorem for the perfect gas.

Newton solved the two-body problem, which has already been formulated by Kepler. The solution was based on the heliocentric world system of Copernicus, on the Galilean laws of inertia and free fall in the outer force field and on Kepler's laws of the planet's motion in the central force field considered as a geometric plane task. The goal of Newton's problem was to find the force in which the planet's motion resulted. His centripetal attraction and the inertial forces in the two-body problem satisfy Kepler's laws.

As was mentioned, Newton understood the physical meaning of his centripetal or attractive forces as a pressure, which is accepted now like a force field. But by his opinion, for mathematical solutions, the force is a more convenient instrument. And in the two-body problem, the force pressure is acting from the center (of point) to the outer space.

It is worth discussing briefly Newton's preference for the force but not for the pressure. In mechanics, the term *mass point* is understood as a geometric point of space that has no dimension but possesses a finite mass. In physics, a small amount of mass is called by the term *particle*, which has a finite value of size and mass. But very often, physicists use models of particles that have neither size nor mass. A body model like mass point is known since ancient times. It is simple and convenient for mathematical operations. The point is an irreplaceable geometric symbol of a reference point. The physical point, which defines inert mass of a volumetric body, is also suitable for operations. But the interacting and physically active mass point creates a problem. For instance, in field theory, the point value is taken to denote the charge, the meaning of which is no better understood than is the gravity force. But it is considered often there that the point model for mathematical presentation of charges is not suitable because operations with it lead to zero and infinite values. Then for resolution of the situation, the concept of charge density is introduced. The charge is presented as an integral of density for the designated volume, and in this way, the solution of the problem is resolved.

The point model in the two-body problem allowed reduction of it to the one-body problem and for a spherical body of uniform density to write the main seven integrals of motion. In the case when a body has a finite size, then not the forces but the pressure becomes an effect of the body particle interaction. The interacting body's mass particles form a volumetric gravitational field of pressure, the strength of which is proportional to the density of each elementary volume of the mass. In the case of a uniform body, the gravitational pressure should also be uniform within the whole volume. The outer gravitational pressure of the uniform body should also be uniform at the given radius. The nonuniform body has a nonuniform gravitational pressure of both inner and outer field, which has been observed in studying the real body field. Interaction of mass particles results appears in their collision, which leads to oscillation of the whole body system. In general if the mass density is higher, then the frequency of body oscillation is also higher.

It was known from the theory of elasticity that in order to calculate the stress and the deformation of a beam from a continuous load, the latter can be replaced by the equivalent lumped force. In that case, the found solution will be approximate because the beam's stress and deformation will be different. The question is what degree of approximation of the solution and what kind of the error is expected. Volumetric forces are not summed up by means of the parallelogram rule. Volumetric forces by their nature are not to be reduced for application either to a point or to a resultant vector value. Their action is directed to the 4π space, and they form inner and outer force fields. The force field by its action is proportional to the action of the energy. This is because the force is the derivative of the energy.

The centrifugal and Coriolis forces are also proved to be inertial forces as a consequence of the inertial rotation of the body. And the Archimedes force has not found its physical explanation, but it became an observational fact of hydrostatic equilibrium of a body mass immersed in a liquid.

Such is the short story of the appearance and development of the hydrostatic equilibrium of the celestial bodies in the outer uniform gravity field. The force of gravity of a body mass is an integral value. In this connection, Newton's postulate about the gravity center as a geometric point should be considered as a model for presentation of two interacting bodies, when their mutual distance is much more than the body size. It is shown in the next section that the reduced physical, but not geometrical, gravity center of a volumetric body is represented by an envelope of the figure, which draws an averaged value of the radial density distribution of the body.

Therefore, the theorem of the classical mechanics, cited in Sect. 2.1 and stated that if a body is found in the central force field, then the sum of their inner forces and torques is equal to zero, from the mathematical point of view is correct in the frame of the given initial conditions. As in case of the derived virial theorem, the moment of momentum L in expression (2.11) can be presented by the first derivative from the polar moment of inertia. And then the torque equal to zero in the central field will be presented by oscillation of the polar moment of inertia not equal to zero.

The problem of dynamics of a self-gravitating body, including its shape problem, in its formulation and solution needs a higher degree of approximation. The generalized virial theorem (3.21) satisfies the condition of a body's dynamical equilibrium state and creates a physical and theoretical basis for further development of theory. It follows from the theorem that in hydrostatic equilibrium state there is the particular case of the dynamics. The solution of the problem of the body's dynamics based on the equation of dynamical equilibrium appears to be the next natural and logistic step from the hydrostatic equilibrium model to a more perfect method without loss of the previous preference.

In the next section, we consider the problem of "decentralization" of the own force field for a self-gravitating body.

3.5 Jacobi's *n*-Body Problem

In 1842–1843, when Jacobi was a professor at Königsberg University, he delivered a special series of lectures on dynamics. The lectures were devoted to the dynamics of a system of n mass points, the motion of which depends only on the mutual distance between them and is independent of velocities. In this connection, by deriving the law of conservation of energy, where the force function is a homogeneous function of space coordinates, Jacobi gave an unusual form and a new content to this law. In transforming the equations of motion, he introduced an expression for the system's center of mass. Then, following Lagrange, he separated the motion of the center of mass from the relative motion of the mass points. Making the center of mass coincident with the origin of the coordinate system, he obtained the following equation (Jacobi 1884):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum m_i \mathbf{r}_i^2\right) = -(2k+4)\,U + 4E,$$

where m_i is the mass point i, $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$ is the distance between the points and the center of mass, k is the degree of homogeneity of the force function, U is the system's potential energy, and E is its total energy.

When $k = -1$, which corresponds to the interaction of mass points according to Newton's law, and denoting

$$\frac{1}{2}\sum m_i \mathbf{r}_i^2 = \Phi,$$

Jacobi obtained

$$\ddot{\Phi} = U + 2T = 2E - U,$$

where Φ is the Jacobi function (the polar moment of inertia).

This is Jacobi's generalized (nonaveraged) virial equation. In the Russian scientific literature, it is known as the Lagrange–Jacobi equation since Jacobi derived it by applying Lagrange's method of separation of the motion of the mass center from the relative motion of mass points.

On the right-hand side of the virial equation, there is a classical expression of the virial theorem, that is, relation between the potential and kinetic energy. In the case of constancy of its left-hand side, when motion of the system happens with a constant velocity, the equation acquires conditions of hydrostatic equilibrium of a system in the outer force field. The left-hand side of the equation, that is, the second derivative with respect to the Jacobi function, expresses oscillation of the polar moment of inertia of the system, which, in fact, is kinetic energy of the inner volumetric torques of the interacted mass points moving in accordance with Kepler's laws.

Jacobi has not paid attention to the physics of his equation, which expresses kinetic energy of the interacted volumetric particles in the form of their oscillation. He used the equation for a quantitative analysis of stability of the solar system and noted that the system's potential and kinetic energies should always oscillate within certain limits. In the contemporary literature of celestial mechanics and analytical dynamics, Jacobi's virial equation is used for the same purposes (Whittaker 1937; Duboshin 1975). Since this equation contains two independent variables, it found no other practical applications. The functional relationship between the potential (kinetic) energy and the polar moment of inertia was disclosed in our works. On that basis, the rigorous solution of the equation will be found and applied to study the dynamics of a self-gravitating body (Ferronsky and Ferronsky 2010; Ferronsky et al. 2011).

3.6 Reduction of Inner Gravitational Field to Resultant Envelope of Pressure

Consider a planet as a self-gravitating sphere with uniform and one-dimensional interacting media. The motion of the body proceeds both in its own and in the Sun's force fields. It is known from theoretical mechanics that any motion of a body can be represented by a translation motion of its mass center, rotation around that center, and motion of the body mass related to its changes in the shape and structure (Duboshin 1975). In the two-body problem, the last two effects are neglected due to their smallness.

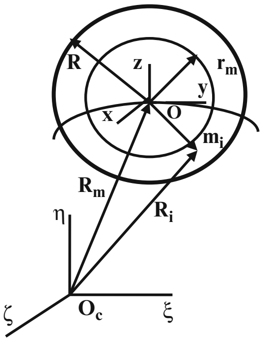


Fig. 3.1 Body motion in its own force field

In order to study the planet's motion in its own force field, the translational (orbital) motion relative to the fixed point (the Sun) should be separated from the two other components of motion. After that, one can consider the rotation around the geometric center of the planet's masses under the action of the own force field and changes in the shape and structure (oscillation). Such separation is required only for the moment of inertia, which depends on what frame of reference is selected. The force function depends on a distance between the interacting masses and does not depend on selection of a frame of reference (Duboshin 1975). The moment of inertia of the planet relative to the solar reference frame should be split into two parts. The first is the moment of the body mass center relative to the same frame of reference, and the second is the moment of inertia of the planet's mass relative to the own mass center.

So, set up the absolute Cartesian coordinates $O_c\xi\eta\zeta$ with the origin in the center of the Sun, and transfer it to the system $Oxyz$ with the origin in the geometrical center of the planet's mass (Fig. 3.1).

Then, the moment of inertia of the Earth in the solar frame of reference is

$$I_c = \sum m_i R_i^2, \tag{3.24}$$

where m_i is the planet mass of particle and R_i is its distance from the origin in the same frame.

The Lagrange method is applied to separate the moment of inertia (3.24). The method is based on his algebraic identity

$$\left(\sum_{1\le i\le n} a_i^2\right)\left(\sum_{1\le i\le n} b_i^2\right) = \left(\sum_{1\le i\le n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n}\left(a_i b_j - b_i a_j\right)^2, \tag{3.25}$$

where a_i and b_i are whichever values and n is any positive number.

Jacobi in his *Vorlesungen über Dynamik* was the first who performed the mathematical transformation for separation of the moment of inertia of n-interacting mass points into two algebraic sums (Jacobi 1884; Duboshin 1975; Ferronsky et al. 1987, 2011). It was shown that if we denote (Fig. 3.1)

$$\xi_i = x_i + A; \quad \eta_i = y + B; \quad \zeta_i = z + C;$$

$$\sum m_i = M; \quad \sum m_i \xi_i = MA; \quad \sum m_i \eta_i = MB; \quad \sum m_i \zeta_i = MC; \tag{3.26}$$

where A, B, and C are the coordinates of the mass center in the solar frame of reference.

Then, using identity (3.25), one has

$$\begin{aligned}\sum m_i r_i^2 &= \sum m_i \xi_i^2 + \sum m_i \eta_i^2 + \sum m_i \zeta_i^2 \\ &= \sum m_i x_i^2 + 2A \sum m_i x_i A^2 \sum m_i \\ &\quad + \sum m_i y_i^2 + 2B \sum m_i y_i + B^2 \sum m_i + \sum m_i z_i^2 \\ &\quad + 2C \sum m_i z_i + C^2 \sum m_i .\end{aligned}$$

Since

$$MA = \sum m_i \xi_i = \sum m_i x_i + \sum m_i A = \sum m_i x_i + MA,$$

then

$$\sum m_i x_i = 0, \text{ and also } \sum m_i y_i = 0, \sum m_i z_i = 0.$$

Now, the moment of inertia (3.24) acquires the form

$$\sum m_i R_i^2 = M\left(A^2 + B^2 + C^2\right) + \sum m_i \left(x_i^2 + y_i^2 + z_i^2\right), \tag{3.27}$$

where

$$M\left(A^2 + B^2 + C^2\right) = MR_{\mathrm{m}}^2, \tag{3.28}$$

$$\sum m_i \left(x_i^2 + y_i^2 + z_i^2\right) = M r_{\mathrm{m}}^2, \tag{3.29}$$

M is the planet's mass, and R_{m} and r_{m} are the radii of inertia of the planet in the Sun's and the planet's frame of reference.

Thus, we separated the moment of inertia of the planet, rotating around the Sun in the inertial frame of reference, into two algebraic terms. The first one (3.28) is the planet's moment of inertia in the solar reference system $O\ \xi\eta\zeta$. The second term (3.29) presents the moment of inertia of the planet in its own frame of reference *Oxyz*. The planet mass here is distributed over the spherical surface with the reduced radius of inertia r_{m}. In the literature, the geometrical center of mass O in the planet reference system is erroneously identified with the center of inertia and center of gravity of the planet.

For further consideration of the problem of the Earth's dynamics, we accept the polar frame of reference with its origin at center O. Then, expression (3.29) for the planet's polar moment of inertia I_{p} acquires the form

$$I_{\mathrm{p}} = \sum m_i \left(x_i^2 + y_i^2 + z_i^2\right) = \sum m_i {r_i}^2 = M r_{\mathrm{m}}^2. \tag{3.30}$$

Now the reduced radius of inertia r_{m}, which draws a spherical surface, is

$$r_{\mathrm{m}}^2 = \frac{\sum m_i r_i^2}{M}. \tag{3.31}$$

Here, $M = \sum m_i$ is the planet's mass relative to own frame of reference.

Taking into account the spherical symmetry of the uniform and one-dimensional planet, we consider the sphere as a concentric spherical shell with the mass $dm(r) = 4\pi r^2 \rho(r)\mathrm{d}r$. Then, the expression (3.31) in the polar reference system can be rewritten in the form

$$r_{\mathrm{m}}^2 = \frac{1}{M}\int\limits_0^R r^2 4\pi r^2 \rho(r)\mathrm{d}r = \frac{4\pi R^2}{M R^2}\int\limits_0^R r^4(r)\mathrm{d}r, \tag{3.32}$$

or

$$\frac{r_{\mathrm{m}}^2}{R^2} = \frac{4\pi \int_0^R r^4 \rho(r)\mathrm{d}r}{M R^2} = \frac{\beta^2 M R^3}{M R^2} = \beta^2 \tag{3.33}$$

from where

$$r_{\mathrm{m}} = \beta\ R,$$

where $\rho(r)$ is the law of radial density distribution, R is the radius of the sphere, and β is the dimensionless coefficient of the reduced spheroid (ellipsoid) of inertia $\beta^2 MR^2$.

The value of β depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \beta > 0$. Earlier (Ferronsky et al. 1987, 2011), it was defined as a structural form factor of the polar moment of inertia.

Analogously, the reduced radius of gravity r_{g} is expressed as a ratio of the potential energy of interaction of the spherical shells with density $\rho(r)$ to the potential energy of interaction of the body mass distributed over the shell with radius R. The potential energy of the sphere is written as

$$U = 4G\pi \int_0^R r\rho(r)\, m(r)\, \mathrm{d}r = \alpha \frac{GM^2}{R},$$

from where

$$\alpha = \frac{4G\pi \int_0^R r\rho(r)\, m(r)\, \mathrm{d}r}{\frac{GM^2}{R}} = \frac{r_g}{R}. \tag{3.34}$$

The form factor α of the inner force field, which controls its reduced radius, can be written as

$$\alpha = \frac{r_\mathrm{g}}{R} = \frac{4G\pi \int_0^R r\rho(r)\, m(r)\, \mathrm{d}r}{\frac{GM^2}{R}}, \tag{3.35}$$

where in expressions (3.34) and (3.35) $m(r) = 4\pi \int_0^r r^2\rho(r)\, \mathrm{d}r$, and $r_\mathrm{g} = \alpha R$.

The value of β depends on the density distribution $\rho(r)$ and is changed within the limits of $1 \geq \alpha > 0$. Earlier (Ferronsky et al. 1987, 2011), it was defined as a structural form factor of the force function.

Numerical values of the dimensionless form factors α and β for a number of density distribution laws $\rho(r)$, obtained by integration of the numerators in Eqs. (3.33) and (3.34) for the polar moment of inertia and the force function, are presented in Table 3.1 (Ferronsky et al. 1987, 2011). Note that the value of the polar I_p and axial I_a moments of inertia of the one-dimensional sphere is related as $I_\mathrm{p} = 3/2 I_\mathrm{a}$.

It follows from Table 3.1 that for a uniform sphere with $\rho(r) = \text{const.}$, its reduced radius of inertia coincides with the radius of gravity. Here, both dimensionless structural coefficients α and β^2 are equal to 3/5, and the moments of gravitational and inertial forces are equilibrated, and because of that, the rotation of the mass is absent (Fig. 3.2). Thus,

$$\frac{r_\mathrm{m}^2}{R^2} = \frac{r_\mathrm{g}}{R} = \frac{3}{5}. \tag{3.36}$$

from where

$$r_\mathrm{m} = r_\mathrm{g} = \sqrt{3/5R^2} = 0{,}7745966\, R. \tag{3.37}$$

Table 3.1 Numerical values of form factors α and β^2 for radial distribution of mass density and for politropic models

Distribution law Index of politrope	α	$\beta_{\perp}^2$	β^2
Radial distribution of mass density			
$\rho(r) = \rho_0$	0.6	0.4	0.6
$\rho(r) = \rho_0(1-r/R)$	0.7428	0.27	0.4
$\rho(r) = \rho_0(1-r^2/R^2)$	0.7142	0.29	0.42
$\rho(r) = \rho_0 \exp(1-kr/R)$	0.16k	$8/k^2$	$12/k^2$
$\rho(r) = \rho_0 \exp(1-kr^2/R^2)$	$\sqrt{k/2\pi}$	$1/k$	$1.5/k$
$\rho(r) = \rho_0\, \delta(1-r/R)$	0.5	0.67	1.0
Politrope model			
0	0.6	0.4	0.6
1	0.75	0.26	0.38
1.5	0.87	0.20	0.30
2	1.0	0.15	0.23
3	1.5	0.08	0.12
3.5	2.0	0.045	0.07

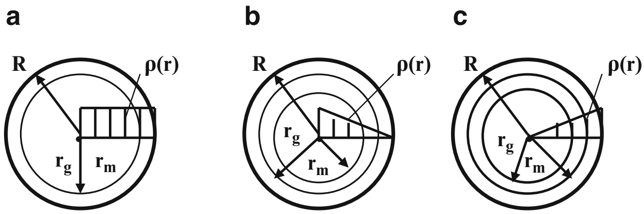


Fig. 3.2 Radius of inertia r_m and radius of gravity r_g as a function of radial density distribution $\rho = f(r)$

For a nonuniform sphere at $\rho(r) \neq$ const. from Eqs. (3.33), (3.34), and (3.35), one has

$$0 < \frac{r_m^2}{R^2} < \frac{3}{5} < \frac{r_g}{R} < 1. \tag{3.38}$$

It follows from the inequality (3.38) and Table 3.1 that in comparison with the uniform sphere, the reduced radius of inertia of the nonuniform body decreases and the reduced gravity radius increases (Fig. 3.2b). Because of $r_m \neq r_g$ and $r_m < 0.77\mathrm{R} < r_g$, the torque appears as a result of an imbalance between gravitational and inertial volumetric forces of the shells. Then from Eq. (3.38), it follows that

$$r_m = r_{m\,0} - \delta r_{mt} \quad \text{and} \quad r_g = r_{g\,0} + \mathrm{d}r_{gt}, \tag{3.39}$$

where subscripts 0 and t relate to the uniform and nonuniform sphere.

In accordance with (3.38) and (3.39), the rotation of shells of a one-dimensional body should be hinged-like and asynchronous. In the case of increasing mass density

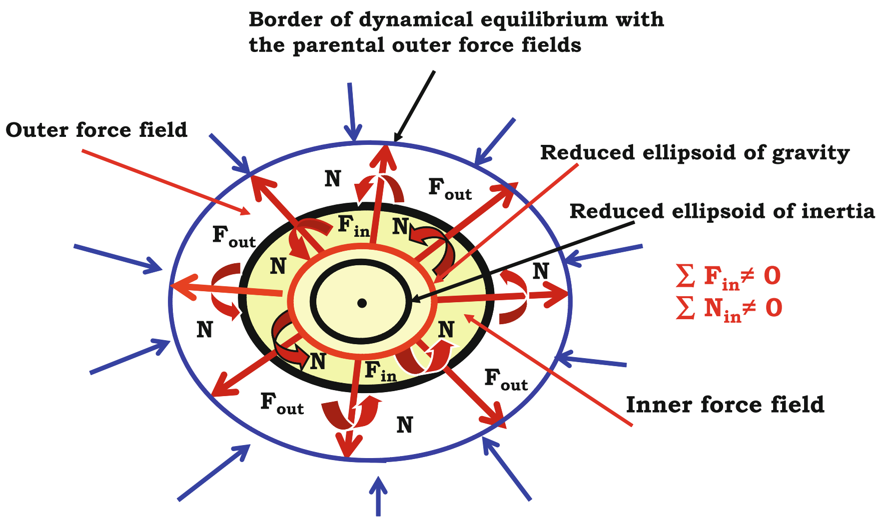


Fig. 3.3 Scheme of dynamic (oscillating) equilibrium of a body based on its inner energy of the mass particles interaction

towards the body surface, then the signs in (3.38) and (3.39) are reversed (Fig. 3.2c). This remark is important because the direction of rotation of a self-gravitating body is a function of its mass density distribution.

The main conclusion from this consideration is that the inner force field of a self-gravitating body is reduced to a closed envelope (spheroid, ellipsoid, or more complicated shape) of gravitational pressure, but not to a resulting force passing through the geometric center of the masses. In the case of a uniform body, the envelopes have a spherical shape and both gravitational and inertial radii coincide. For a nonuniform body, the radius of inertia does not coincide with the radius of gravity; the reduced envelope is closed but has nonspherical (ellipsoidal or any other) shape. Analytical solutions done in the following paragraphs justify this.

So, we accept the force pressure as an effect of mass particle interaction, which is the property producing work in the form of motion. In the other words, the pressure of interacted masses appears to be the force function or a flux of the potential energy.

The scheme of forces defining conditions of dynamical equilibrium of a body based on inner energy of the mass particles interaction is shown in Fig. 3.3.

Now we pass to derivation of the equation of dynamical equilibrium (Jacobi virial equation) for the well-known physical interaction models of natural systems. The only restriction here is the requirement of uniformity of the potential energy function of the system relative to the frame of reference. But that requirement appears to be not always obligatory. A specific physical model which is used for description of the system's dynamics in classical mechanics, hydrodynamics, statistic mechanics, quantum mechanics, and theory of relativity in that case will be not an important factor.

References

Duboshin GN (1975) Celestial mechanics: the main problems and the methods. Nauka, Moscow
Ferronsky VI (2005) Virial approach to solve the problem of global dynamics of the earth. Investigated in Russia. http://zhurnal.ape.relarn/articles/2005/120.pdf, pp 1207–1228
Ferronsky VI, Ferronsky SV (2010) Dynamics of the earth. Springer, Dordrecht/Heidelberg
Ferronsky VI, Denisik SA, Ferronsky SV (1978) The solution of Jacobi's virial equation for celestial bodies. Celest Mech 18:113–140
Ferronsky VI, Denisik SA, Ferronsky SV (1987) Jacobi dynamics. Reidel, Dordrecht
Ferronsky VI, Denisik SA, Ferronsky SV (2011) Jacobi dynamics, 2nd edn. Springer, Dordrecht/Heidelberg
Goldstein H (1980) Classical mechanics, 2nd edn. Addison-Wesley, Reading
Grushinsky NP (1976) Theory of the earth figure. Nauka, Moscow
Jacobi CGJ (1884) Vorlesungen über Dynamik. Klebsch, Berlin
Melchior P (1972) Physique et dynamique planetaires. Vander-Editeur, Bruxelles
Whittaker ET (1937) A treatise on the analytical dynamics of particles and rigid bodies. Cambridge University Press, Cambridge
Zharkov VN (1978) Inner structure of the earth and planets. Nauka, Moscow

Chapter 4
Derivation of Jacobi's Virial Equation for Description of Dynamics of a Self-Gravitating Body

Abstract In order to demonstrate the universality of Jacobi's virial equation for the description of the dynamics of natural systems, including their origin and evolution, it was derived from the main existing equations, describing a wide range of physical models of the systems. In particular, Jacobi's virial equation was derived from the equations of motion of Newton, Euler, Hamilton, Einstein, and quantum mechanics.

The derived equation represents not only formal mathematical transformation of the initial equations of motion. Physical quintessence of the mathematical transformation of the equations of motion involves changes in the vector forces and moment of momentums by the volumetric forces or pressure and the oscillation of the interacted mass particles (inner energy) expressed through the energy of oscillation of the polar moment of inertia of a body. Here the potential (kinetic) energy and the polar moment of inertia of a body have a functional relationship and within the period of oscillation are inversely changed by the same law. Moreover, the virial oscillations of a body represent the main part of the body's kinetic energy, which is lost in the hydrostatic equilibrium model.

The change in the vector forces and moment of momentums by the force pressure and the oscillation of the interacting mass particles disclose the physical meaning of the gravitation and mechanism of generation of the gravitational and electromagnetic energy and their common nature. The most important advantage given by Jacobi's virial equation is its independence from the choice of the coordinate system, the transformation of which, as a rule, creates many mathematical difficulties.

It was shown in Sect. 1.3 that the bullet point of the solar system cosmogony and cosmology as a whole is the inner energy of interaction of the elementary particles, which leads to weightlessness and self-gravitation of the system's upper shell of matter. It means that the body's matter and its force field (inner and outer) are the principal participants in the origin and evolution processes. Jacobi's virial equation, in fact, appears to be the generalized virial theorem, and its solution compiles fundamentals of the theory of dynamics of a self-gravitating body.

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_4,

Let us begin by deriving Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein, and also from the equations of quantum mechanics. By doing so we can show that Jacobi's virial equation appears to be a unified instrument for the description of the dynamics of natural systems in the framework of the various physical models of the matter interaction employed. Jacobi's virial equation for a system moving in its own force field and establishing a relationship between the potential and kinetic energy of the oscillating polar moment of inertia is defined as the generalized (nonaveraged) virial theorem or the equation of the dynamical equilibrium of a body.

The theory presented in this book can be applied to study the body that, by its structure, presents a system that includes gaseous, liquid, and solid shells. For this purpose, derivation of Jacobi's virial equation from the equations of Newton, Euler, Hamilton, Einstein, and also from the equations of quantum mechanics is presented. In this part of the work, we justify physical applicability of this fundamental equation for the study of the dynamics and structure of stars, planets, satellites, and their shells. For this purpose the volumetric forces and moments are introduced into the transformed equations, as was done by Eqs. (3.17) and (3.18). In this case the energy becomes the measure of the matter interaction as it is observed in nature.

4.1 Derivation of Jacobi's Virial Equation from Newtonian Equations of Motion

Throughout this section, the term "system" is defined as an ensemble of material mass points m_i $(i = 1, 2, 3, \ldots, n)$ that interact by Newton's law of universal attraction. This physical model of a natural system forms the basis for a number of branches of physics, such as classical mechanics, celestial mechanics, and stellar dynamics.

We shall not present the traditional introduction in which the main postulates are formulated; we shall simply state the problem (see, e.g., Landau and Lifshitz 1973). We start by writing the equations of motion of the system in some absolute Cartesian coordinates ξ, η, ζ. In accordance with the conditions imposed, the mass point m_i is not affected by any force from the other $n-1$ points except that of gravitational attraction. The projections of this force on the axes of the selected coordinates ξ, η, ζ can be written (Fig. 4.1) as

$$\begin{aligned}
\Xi_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j\left(\xi_j-\xi_i\right)}{\Delta_{ij}^3},\\
H_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j\left(\eta_j-\eta_i\right)}{\Delta_{ij}^3},\\
Z_i &= -Gm_i \sum_{1\le i<j\le n} \frac{m_j\left(\zeta_j-\zeta_i\right)}{\Delta_{ij}^3},
\end{aligned} \tag{4.1}$$

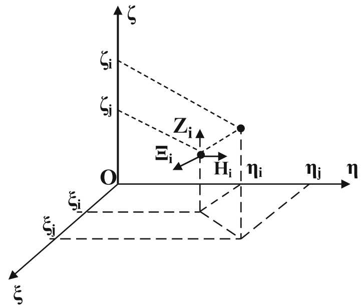


Fig. 4.1 Absolute Cartesian coordinate system $O\xi\ \eta\ \zeta$

where G is the gravitational constant and

$$\Delta_{ji} = \sqrt{\left(\xi_j - \xi_i\right)^2 \left(\eta_j - \eta_i\right)^2 \left(\zeta_j - \zeta_j\right)^2}$$

is the reciprocal distance between points i and j of the system.

It is easy to check that the forces affect the ith material point of the system and are determined by the scalar function U, which is called the potential energy function of the system and is given by

$$U = -G \sum_{1 \le i < j \le n} \frac{m_i m_j}{\Delta_{ij}}. \tag{4.2}$$

Now Eqs. (4.1) can be rewritten in the form

$$\Xi_i = -\frac{\partial U}{\partial \xi_i},$$
$$H_i = -\frac{\partial U}{\partial \eta_i},$$
$$Z_i = -\frac{\partial U}{\partial \zeta_i}.$$

Then Newton's equations of motion for the ith point of the system take the form

$$m_i \ddot{\xi}_i = \Xi_i,$$
$$m_i \ddot{\eta}_i = H_i,$$
$$m_i \ddot{\zeta}_i = Z_i, \tag{4.3}$$

or

$$
\begin{aligned}
m_i \ddot{\xi}_i &= -\frac{\partial U}{\partial \xi_i}, \\
m_i \ddot{\eta}_i &= -\frac{\partial U}{\partial \eta_i}, \\
m_i \ddot{\zeta}_i &= -\frac{\partial U}{\partial \zeta_i},
\end{aligned}
\tag{4.4}
$$

where dots over coordinate symbols mean derivatives with respect to time.

The motion of a system is described by Eqs. (4.4) and (4.5) and is completely determined by the initial data. In classical mechanics, the values of projections ξ_{i0}, η_{i0}, $\zeta_{\mathrm{i}0}$ and velocities $\dot{\xi}_{i0}$, $\dot{\eta}_{i0}$, $\dot{\zeta}_{i0}$ at the initial moment of time $t = t_0$ may be known from the initial data.

The study of motion of a system of n material points affected by self-forces of attraction forms the essence of the classical many-body problem. In the general case, ten classical integrals of motion are known for such a system, and they are obtained directly from the equations of motion.

Summing all the Eqs. (4.4) for each coordinate separately, it is easy to be convinced of the correctness of the expressions:

$$
\begin{aligned}
\sum_{1 \le i \le n} \Xi_i &= 0, \\
\sum_{1 \le i \le n} H_i &= 0, \\
\sum_{1 \le i \le n} Z_i &= 0.
\end{aligned}
$$

From those equations, it follows that

$$
\begin{aligned}
\sum_{1 \le i \le n} m_i \ddot{\xi}_i &= 0, \\
\sum_{1 \le i \le n} m_i \ddot{\eta}_i &= 0, \\
\sum_{1 \le i \le n} m_i \ddot{\zeta}_i &= 0.
\end{aligned}
\tag{4.5}
$$

Equations (4.5), appearing as a sequence of equations of motion, can be successively integrated twice. As a result, the first six integrals of motion are obtained:

$$\sum_{1 \le i \le n} m_i \dot{\xi}_i = a_1,$$

$$\sum_{1 \le i \le n} m_i \dot{\eta}_i = a_2,$$

$$\sum_{1 \le i \le n} m_i \dot{\zeta}_i = a_3.$$

$$\sum_{1 \le i \le n} m_i (\xi_i - \dot{\xi}_i t) = b_1,$$

$$\sum_{1 \le i \le n} m_i (\eta_i - \dot{\eta}_i t) = b_2,$$

$$\sum_{1 \le i \le n} m_i (\zeta_i - \dot{\zeta}_i t) = b_3, \tag{4.6}$$

where $_1$, $_2$, $_3$, b_1, b_2, b_3 are integration constants.

These integrals are called integrals of motion of the center of mass. The integration constants $_1$, $_2$, $_3$, b_1, b_2, b_3 can be determined from the initial data by substituting their values at the initial moment of time for the values of all the coordinates and velocities.

Let us obtain one more group of first integrals. To do this, the second of Eqs. (4.3) can be multiplied by $-\zeta_i$, and the third by η_i. Then all expressions obtained should be added and summed over the index i. In the same way, the first of Eqs. (4.3) should be multiplied by ζ_i, and the third by $-\xi_i$ added and summed over index i. Finally, the second of Eqs. (4.3) should be multiplied by ξ_i, and the first by $-\eta_i$ added and summed over index i. It is easy to show directly that the right-hand sides of the expressions obtained are equal to zero:

$$\sum_{1 \le i \le n} (Z\eta_i - H\zeta_i) = 0,$$

$$\sum_{1 \le i \le n} (\Xi\zeta_i - Z\xi_i) = 0,$$

$$\sum_{1 \le i \le n} (H\xi_i - \Xi\eta_i) = 0.$$

Consequently, their left-hand sides are also equal to zero:

$$\sum_{1 \le i \le n} m_i (\ddot{\zeta}_i \eta_i - \ddot{\eta}_i \zeta_i) = 0,$$

$$\sum_{1 \le i \le n} m_i (\ddot{\xi}_i \zeta_i - \ddot{\zeta}_i \xi_i) = 0,$$

$$\sum_{1 \le i \le n} m_i (\ddot{\eta}_i \xi_i - \ddot{\xi}_i \eta_i) = 0. \tag{4.7}$$

Integrating Eqs. (4.7) over time, three more first integrals can be obtained:

$$\sum_{1 \le i \le n} m_i \left(\dot{\xi}_i \eta_i - \dot{\eta}_i \xi_i \right) = c_1,$$

$$\sum_{1 \le i \le n} m_i \left(\dot{\zeta}_i \xi_i - \dot{\xi}_i \zeta_i \right) = c_2,$$

$$\sum_{1 \le i \le n} m_i \left(\dot{\eta}_i \zeta_i - \dot{\zeta}_i \eta_i \right) = c_3. \tag{4.8}$$

The integrals (4.8) are called area integrals or integrals of moments of momentum. Three integration constants ${}_1$, ${}_2$, ${}_3$ are also determined from the initial data by changing over from the values of all the coordinates and velocities to their values at the initial moment of time.

The last of the classical integrals can be obtained by multiplying the three Eqs. (4.4) by $\dot{\xi}_i$, $\dot{\eta}_i$, and $\dot{\zeta}_i$, respectively, and adding and summing all the expressions obtained. As a result, the following equation is obtained:

$$\sum_{1 \le i \le n} m_i \left(\ddot{\xi}_i \dot{\xi}_i + \ddot{\eta}_i \dot{\eta}_i + \ddot{\zeta}_i \dot{\zeta}_i \right) = - \sum_{1 \le i \le n} \left(\frac{\partial U}{\partial \xi_i} \dot{\xi}_i + \frac{\partial U}{\partial \eta_i} \dot{\eta}_i + \frac{\partial U}{\partial \zeta_i} \dot{\zeta}_i \right). \tag{4.9}$$

It is not difficult to see that the right-hand side of Eq. (4.9) is the complete differential over time of the potential energy function U of the system as a whole. The left-hand side of the same equation is also the complete differential of some function   called the kinetic energy function of the system and equal to

$$T = \frac{1}{2} \sum_{1 \le i \le n} m_i \left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2 \right). \tag{4.10}$$

Equation (4.9) can then be written finally in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(T) = -\frac{\mathrm{d}}{\mathrm{d}t}(U),$$

from which, after integration, one finds that

$$E = T + U, \tag{4.11}$$

where   is the integration constant, determined from the initial conditions.

Equation (4.11) is called the integral of motion or the integral of living (kinetic) forces.

To derive the equation of dynamic equilibrium, or Jacobi's virial equation, each of the Eqs. (4.4) should be multiplied by ξ_i, η_i, and ζ_i, respectively; then, after summing all the expressions, one can obtain

$$\sum_{1\le i\le n} m_i\left(\xi_i\ddot{\xi}_i + \eta_i\ddot{\eta}_i + \zeta_i\ddot{\zeta}_i\right) = -\sum_{1\le i\le n}\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right). \tag{4.12}$$

We can take farther advantage of the obvious identities:

$$m_i\xi_i\ddot{\xi}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\xi_i^2\right) - m_i\dot{\xi}_i^2,$$

$$m_i\eta_i\ddot{\eta}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\eta_i^2\right) - m_i\dot{\eta}_i^2,$$

$$m_i\zeta_i\ddot{\zeta}_i = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(m_i\zeta_i^2\right) - m_i\dot{\zeta}_i^2$$

from the Eulerian theorem concerning the homogenous functions. For the interaction of the system points, according to Newton's law of universal attraction, the degree of homogeneity of the potential energy function of the system is equal to -1, and hence

$$-\sum_{1\le i\le n}\left(\xi_i\frac{\partial U}{\partial \xi_i} + \eta_i\frac{\partial U}{\partial \eta_i} + \zeta_i\frac{\partial U}{\partial \zeta_i}\right) = U.$$

Substituting these expressions into the right- and left-hand side of Eq. (4.12), one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{1}{2}\sum_{1\le i\le n} m_i\left(\xi_i^2+\eta_i^2+\zeta_i^2\right)\right] - 2\sum_{1\le i\le n}\frac{1}{2}m_i\left(\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2\right) = U.$$

For a system of material points, we now introduce the Jacobi function expressed through the moment of inertia of the system and presented in the form

$$\Phi = \frac{1}{2}\sum_{1\le i\le n} m_i\left(\xi_i^2+\eta_i^2+\zeta_i^2\right).$$

Then taking into account (4.11), the previous equation can be rewritten in a very simple form as follows:

$$\ddot{\Phi} = 2E - U. \tag{4.13}$$

This is the equation of dynamic equilibrium or Jacobi's virial equation describing both the dynamics of a system and its dynamic equilibrium using integral (volumetric) characteristics Φ and U or T.

Let us now derive another form of Jacobi's virial equation where the translational moment of the center of mass of the system is separated and all the characteristics depend only on the relative distance between the mass points of the system. For this purpose, the Lagrangian identity can be used:

$$\left(\sum_{1\le i\le n} a_i^2\right)\left(\sum_{1\le i\le n} b_i^2\right) = \left(\sum_{1\le i\le n} a_i b_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n}\left(a_i b_j - b_i a_j\right)^2, \tag{4.14}$$

where a_i and b_i may acquire any values and n is any positive number.

Let us now put $a_i = \sqrt{m_i}$, and b_i equal to $\sqrt{m_i}\xi_i$, $\sqrt{m_i}\eta_i$, and $\sqrt{m_i}\zeta_i$, respectively. Then three identities can be obtained from (4.14):

$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\xi_i^2\right) = \left(\sum_{1\le i\le n} m_i\xi_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\xi_j - \xi_i\right)^2,$$

$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\eta_i^2\right) = \left(\sum_{1\le i\le n} m_i\eta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\eta_j - \eta_i\right)^2,$$

$$\left(\sum_{1\le i\le n} m_i\right)\left(\sum_{1\le i\le n} m_i\zeta_i^2\right) = \left(\sum_{1\le i\le n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\left(\zeta_j - \zeta_i\right)^2.$$

In summing up one finds

$$2m\Phi = \left(\sum_{1\le i\le n} m_i\xi_i\right)^2 + \left(\sum_{1\le i\le n} m_i\eta_i\right)^2 + \left(\sum_{1\le i\le n} m_i\zeta_i\right)^2 + \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\Delta_{ij}^2.$$

Using now Eqs. (4.6), the last equality can be rewritten in the form

$$2m\Phi = \frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j\Delta_{ij}^2 + (a_1 t + b_1)^2 + (a_2 t + b_2)^2 + (a_3 t + b_3)^2, \tag{4.15}$$

where

$$m = \sum_{1\le i\le n} m_i$$

is the total mass of the system.

Let us put

$$\Phi_0 = \frac{1}{4m} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \Delta_{ij}^2.$$

The value Φ_0 does not depend on the choice of the coordinate system and coincides with the value of the Jacobi function in the barycentric coordinate system. Moreover, from Eq. (4.15) it follows that

$$\ddot{\Phi} = \ddot{\Phi}_0 + \frac{a_1^2 + a_2^2 + a_3^2}{m}.$$

Excluding the value Φ from Jacobi's equation (4.13) with the help of the last equality, the same equation can be obtained in the barycentric coordinate system:

$$\ddot{\Phi}_0 = 2E_0 - U, \tag{4.16}$$

where $\;_0 = \;_0 + U_0$ is the total energy of the system in the barycentric coordinate system equal to

$$E_0 = E - \frac{a_1^2 + a_2^2 + a_3^2}{2m}.$$

We can now show that the value of $\;_0$ does not depend on the choice of the coordinate system. For this purpose, we can again use the Lagrangian identity (4.14). In this case, $a_i = \sqrt{m_i}$ and $b_i = \sqrt{m_i}\dot{\xi}_i$, $\sqrt{m_i}\dot{\eta}_i$, and $\sqrt{m_i}\dot{\zeta}_i$. Then the following three identities can be justified:

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\xi}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\xi}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\xi}_j - \dot{\xi}_i\right)^2,$$

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\eta}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\eta}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\eta}_j - \dot{\eta}_i\right)^2,$$

$$\left(\sum_{1 \le i \le n} m_i\right)\left(\sum_{1 \le i \le n} m_i \dot{\zeta}_i^2\right) = \left(\sum_{1 \le i \le n} m_i \dot{\zeta}_i\right)^2 + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left(\dot{\zeta}_j - \dot{\zeta}_i\right)^2.$$

After summing and using (4.6), one obtains

$$\begin{aligned} 2mT &= \left(a_1^2 + a_2^2 + a_3^2\right) \\ &\quad + \frac{1}{2} \sum_{1 \le i \le n} \sum_{1 \le j \le n} m_i m_j \left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right], \end{aligned}$$

or

$$T = \frac{\left(a_1^2 + a_2^2 + a_3^2\right)}{2m} + \frac{1}{2m}\left\{\frac{1}{2}\sum_{1\le i\le n}\sum_{1\le j\le n} m_i m_j \left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right]\right\}. \tag{4.17}$$

Here the second term on the right-hand side of Eq. (4.17) coincides with the expression for the kinetic energy T_0 of a system.

Substituting (4.17) into an expression for $_0$, one obtains

$$E_0 = T_0 + U = \frac{1}{2m}\sum_{1\le i\le j\le n} m_i m_j \left[\left(\dot{\xi}_i - \dot{\xi}_j\right)^2 + \left(\dot{\eta}_i - \dot{\eta}_j\right)^2 + \left(\dot{\zeta}_i - \dot{\zeta}_j\right)^2\right] - G\sum_{1\le i\le j\le n}\frac{m_i m_j}{\Delta_{ij}}. \tag{4.18}$$

Thus, the total energy of the system $_0$ depends only on the distance between the points of the system and on the velocity changes of these distances. But Jacobi's equation (4.16) appears to be invariant with respect to the choice of the coordinate system.

We can now show that the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not always obligatory. For this purpose we consider two examples.

4.2 Derivation of Jacobi's Virial Equation for Dissipative Systems

Let us derive Jacobi's virial equation for a nonconservative system. We consider a system of n material points, the motion of which is determined by the force of their mutual gravitation interaction and the friction force. It is well known that the friction force always appears in the course of evolution of any natural system. It is also known that there is no universal law describing the friction force (Bogolubov and Mitropolsky 1974). The only general statement is that the friction force acts in the direction opposite to the vector of velocity of a considered mass point.

Consider as an example the simplest law of Newtonian friction when its force is proportional to the velocity of motion of the mass:

$$\begin{aligned} \Xi_{\mathrm{f}} &= -k m_i \dot{\xi}_i, \\ H &= -k m_1 \dot{\eta}_i, \\ Z &= -k m_i \dot{\zeta}_i, \end{aligned} \tag{4.19}$$

where $\dot{\xi}_i$, $\dot{\eta}_i$, and $\dot{\zeta}_i$ are the components of the radius vector of the velocity of the ith mass point in the barycentric coordinate system, k is a constant independent of i, and $k > 0$.

Sometimes the friction force is independent of the velocity of the mass point. There are also some other laws describing the friction force.

We derive the equation of dynamical equilibrium for a system of n material points using the equations of motion (4.4) and taking into account the friction force expressed by Eqs. (4.19):

$$
\begin{aligned}
m_i \ddot{\xi}_i &= -\frac{\partial U}{\partial \xi_i} - k m_i \dot{\xi}_i, \\
m_i \ddot{\eta}_i &= -\frac{\partial U}{\partial \eta_i} - k m_i \dot{\eta}_i, \\
m_i \ddot{\zeta}_i &= -\frac{\partial U}{\partial \zeta_i} - k m_i \dot{\zeta}_i,
\end{aligned}
\tag{4.20}
$$

where the value of the system's potential energy is determined by Eq. (4.2).

Multiplying each of Eqs. (4.20) by ξ_i, η_i, and ζ_i, respectively, and summing through all i, one obtains

$$
\sum_{1 \le i \le n} m_i \left(\xi_i \ddot{\xi}_i + \eta_i \ddot{\eta}_i + \zeta_i \ddot{\zeta}_i \right) = - \sum_{1 \le i \le n} \left(\frac{\partial U}{\partial \xi_i} \xi_i + \frac{\partial U}{\partial \eta_i} \eta_i + \frac{\partial U}{\partial \zeta_i} \zeta_i \right) - k \sum_{1 \le i \le n} m_i \left(\xi_i \dot{\xi}_i + \eta_i \dot{\eta}_i + \zeta_i \dot{\zeta}_i \right). \tag{4.21}
$$

Transforming the right- and left-hand sides of Eq. (4.21) in the same way as in deriving Eq. (4.13), one obtains

$$
\ddot{\Phi} = 2E - U - k\dot{\Phi}. \tag{4.22}
$$

Let us show that the total energy of the system is a monotonically decreasing function of time. For this purpose we multiply each of the Eqs. (4.20) by the vectors $\dot{\xi}_i$ $\dot{\eta}_i$ and $\dot{\zeta}_i$, respectively, and sum over all from 1 to n, which results in

$$
\sum_{1 \le i \le n} m_i \left(\xi_i \ddot{\xi}_i + \eta_i \ddot{\eta}_i + \zeta_i \ddot{\zeta}_i \right) = - \sum_{1 \le i \le n} \left(\frac{\partial U}{\partial \xi_i} \xi_i + \frac{\partial U}{\partial \eta_i} \eta_i + \frac{\partial U}{\partial \zeta_i} \zeta_i \right) - k \sum_{1 \le i \le n} m_i \left(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2 \right).
$$

The last expression can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(T) = -\frac{\mathrm{d}}{\mathrm{d}t}(U) - 2kT$$

or

$$\mathrm{d}E = -2kT\mathrm{d}t. \tag{4.23}$$

Since the kinetic energy T of the system is always greater than zero, $\mathrm{d}E \leq 0$, that is, the total energy of a gravitating system is a monotonically decreasing function of time. Thus, the expression for the total energy $\;(t)$ of the system can be written as

$$E(t) = E_0 - 2k\int_{t_0}^{t} T(t)\mathrm{d}t = E_0\left[1 + q(t)\right],$$

where $q(t)$ is a monotonically increasing function of time.

Finally, the equation of dynamical equilibrium for a nonconservative system takes the form

$$\ddot{\Phi} = 2E_0\left[1 + q(t)\right] - U - k\dot{\Phi}. \tag{4.24}$$

The second example where the requirement of homogeneity of the potential energy function for deriving Jacobi's virial equation is not obligatory is as follows. We derive Jacobi's virial equation for a system whose mass points interact mutually in accordance with Newton's law and move without friction in a spherical homogenous cloud whose density ρ_0 is constant in time. Let, also, the geometric center of the cloud coincide with the center of mass of the considered system. The equations of motion for such a system can be written in the form

$$\begin{aligned} m_i\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\xi_i - \frac{\partial U}{\partial \xi_i},\\ m_i\frac{\mathrm{d}^2\eta_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\eta_i - \frac{\partial U}{\partial \eta_i},\\ m_i\frac{\mathrm{d}^2\zeta_i}{\mathrm{d}t^2} &= -\frac{4}{3}\pi G\rho_0 m_i\zeta_i - \frac{\partial U}{\partial \zeta_i}, \end{aligned} \tag{4.25}$$

where $i = 1, 2, \ldots, n$.

It is obvious that this system of equations possesses the ten first integrals of motion and that Jacobi's virial equation written in the form

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} = 2E - U - \frac{8}{3}\pi G\rho_0\Phi. \tag{4.26}$$

is valid for it.

The equation in the form (4.26) was first obtained by Duboshin et al. (1971). Equations (4.24) and (4.26) can be written in a more general form:

$$\ddot{\Phi} = 2E - U + X\left(t, \Phi, \dot{\Phi}\right), \tag{4.27}$$

where $X\left(t, \Phi, \dot{\Phi}\right)$ is a given function of time t, the Jacobi function Φ, and first derivative $\dot{\Phi}$. Moreover, we can call Eq. (4.27) a generalized equation of dynamical equilibrium.

The above examples prove justify the statement that for conditions of homogeneity of the potential energy function, required for the derivation of Jacobi's virial equation, is not always necessary. This condition is required for description of the dynamics of conservative systems but not for dissipative systems or for systems in which motion is restricted by some other conditions.

4.3 Derivation of Jacobi's Virial Equation from Eulerian Equations

We now derive Jacobi's virial equation by transforming of the hydrodynamic or continuum model of a physical system. As is well known, the hydrodynamic approach to solving problems of dynamics is based on the system of differential equations of motion supplement, in the simplest case; by the equations of state and continuity; and by the appropriate assumptions concerning boundary conditions and perturbations affecting the system.

In this section, we understand by the term "system" some given mass of ideal gas localized in space by a finite volume V and restricted by a closed surface S. Let the gas in the system move by the forces of mutual gravitational interaction and of baric gradient. In addition, we accept the pressure within the volume to be isotropic and equal to zero on the surface S bordering the volume V. Then for a system in some Cartesian inertial coordinate system ξ, η, ζ, the Eulerian equations can be written in the form

$$\begin{aligned}
\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial}{\partial \xi}u + \rho v\frac{\partial}{\partial \eta}u + \rho w\frac{\partial}{\partial \zeta}u &= -\frac{\partial p}{\partial \xi} + \rho\frac{\partial U_G}{\partial \xi},\\
\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial}{\partial \xi}v + \rho v\frac{\partial}{\partial \eta}v + \rho w\frac{\partial}{\partial \zeta}v &= -\frac{\partial p}{\partial \eta} + \rho\frac{\partial U_G}{\partial \eta},\\
\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial}{\partial \xi}w + \rho v\frac{\partial}{\partial \eta}w + \rho w\frac{\partial}{\partial \zeta}w &= -\frac{\partial p}{\partial \zeta} + \rho\frac{\partial U_G}{\partial \zeta},
\end{aligned} \tag{4.28}$$

where $\rho(\xi, \eta, \zeta, t)$ is the gas density; u, v, w are components of the velocity vector $\bar{v}$ (ξ, η, ζ, t) in a given point of space; $p(\xi, \eta, \zeta, t)$ is the gas pressure; and U_G is Newton's potential in a given point of space.

The value U_{G} is given by

$$U_{\mathrm{G}} = G \int\limits_{(V)} \frac{\rho\,(x, y, z, t)}{\Delta} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{4.29}$$

where G is the gravity constant and $\Delta = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ is the distance between system points.

The potential energy of the gravitational interaction of material points of the system is linked to the Newtonian potential (4.29) by the relation

$$U = -\frac{1}{2} \int\limits_{(V)} U_{\mathrm{G}} \rho\,(\xi, \eta, \zeta, t)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta.$$

To supplement the system of equations of motion, we write the equation of continuity

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial \xi}(\rho u) + \frac{\partial}{\partial \eta}(\rho v) + \frac{\partial}{\partial \zeta}(\rho w) = 0, \tag{4.30}$$

and the equation of state

$$p = f\,(\rho), \tag{4.31}$$

assuming at the same time that the processes occurring in the system are barotropic.

Let us obtain the ten classical integrals for the system whose motion is described by Eqs. (4.28).

We derive the integrals of the motion of the center of mass by integrating each of the Eqs. (4.28) with respect to all the volume filled by the system. Integrating the first equation, we obtain

$$\int\limits_{(V)} \rho \frac{\mathrm{d}u}{\mathrm{d}t} \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta + \int\limits_{(V)} \rho \left(u \frac{\mathrm{d}u}{\mathrm{d}\xi} + v \frac{\mathrm{d}u}{\mathrm{d}\eta} + w \frac{\mathrm{d}u}{\mathrm{d}\zeta} \right) \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$
$$= -\int\limits_{(V)} \frac{\mathrm{d}p}{\mathrm{d}\xi} \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta + G \int\limits_{(V)} \rho\,(\xi, \eta, \zeta, t) \left[\int\limits_{(V)} \rho\,(x, y, z, t)\, \frac{x-\xi}{\Delta^3} \mathrm{d}x \mathrm{d}y \mathrm{d}z \right] \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{4.32}$$

The second term on the right-hand side of Eq. (4.32) disappears because of the symmetry of the integral expression with respect to and ξ. In accordance with the Gauss–Ostrogradsky theorem, the first term on the right-hand side of Eq. (4.32) turns to zero. In fact,

$$\int\limits_{(V)} \frac{\mathrm{d}p}{\mathrm{d}\xi}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta = \int\limits_{(S)} p\,\mathrm{d}\eta\,\mathrm{d}\zeta = 0 \tag{4.33}$$

as pressure p on the border of the considered system is equal to zero owing to the absence of outer effects.

Bearing in mind the possibility of passing to a Lagrangian coordinate system and taking into account the law of the conservation of mass $\mathrm{d}m = \rho\,\mathrm{d}V = \rho_0\mathrm{d}V_0 = \mathrm{d}m_0$, we get

$$\int\limits_{(V)} \rho\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta + \int\limits_{(V)} \rho\left(u\frac{\mathrm{d}u}{\mathrm{d}\xi} + v\frac{\mathrm{d}u}{\mathrm{d}\eta} + w\frac{\mathrm{d}u}{\mathrm{d}\zeta}\right)\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$$
$$= \int\limits_{(V)} \rho\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}V = \int\limits_{(V_0)} \rho_0\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V_0)} u\rho_0\mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} \rho u\,\mathrm{d}V,$$

where V_0 and ρ_0 are the volume and the density in the initial moment of time t_0.

Finally, Eq. (4.32) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} \rho u\,\mathrm{d}V = 0. \tag{4.34}$$

Integrating (4.34) with respect to time and writing analogous expressions for two other equations of the system (4.28), we obtain the first three integrals of motion:

$$\int\limits_{(V)} \rho u\,\mathrm{d}V = a_1,$$
$$\int\limits_{(V)} \rho v\,\mathrm{d}V = a_2,$$
$$\int\limits_{(V)} \rho w\,\mathrm{d}V = a_3. \tag{4.35}$$

Equations (4.35) represent the law of conservation of the system moments. Integration constants $_1$, $_2$, $_3$ can be obtained from the initial conditions.

We consider the first equation of the system (4.35) using again the law of conservation of mass. Then it is obvious that

$$\int\limits_{(V)} \rho u\,\mathrm{d}V = \int\limits_{(V)} \frac{\mathrm{d}\xi}{\mathrm{d}t}\rho\,\mathrm{d}V = \int\limits_{(V)} \frac{\mathrm{d}\xi}{\mathrm{d}t}\rho_0\,\mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} \xi\rho_0\,\mathrm{d}V_0 = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} \xi\rho\,\mathrm{d}V = a_1. \tag{4.36}$$

Analogous expressions can be written for the two other equations (4.35). Integrating them with respect to time, we obtain integrals of motion of the center of mass of the system in the form

$$\int_{(V)} \xi\rho\,\mathrm{d}V = a_1 t + b_1,$$

$$\int_{(V)} \eta\rho\,\mathrm{d}V = a_2 t + b_2,$$

$$\int_{(V)} \zeta\rho\,\mathrm{d}V = a_3 t + b_3. \tag{4.37}$$

We now derive three integrals of the moment of momentum of motion. For this purpose, we multiply the second of Eqs. (4.28) by $-\zeta$, the third by η, and then sum and integrate the resulting expressions with respect to volume V occupied by the system. We obtain

$$\int_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}v}{\mathrm{d}t}\right)\mathrm{d}V = -\int_{(V)} \left(\eta\frac{\partial p}{\partial\zeta} - \zeta\frac{\partial p}{\partial\eta}\right)\mathrm{d}V + \int_{(V)} \rho\left(\eta\frac{\partial U_\mathrm{G}}{\partial\zeta} - \zeta\frac{\partial U_\mathrm{G}}{\partial\eta}\right)\mathrm{d}V. \tag{4.38}$$

Analogously, multiplying the first of Eqs. (4.28) by ζ, the third by $-\xi$, then summing and integrating with respect to volume V, we obtain

$$\int_{(V)} \rho\left(\zeta\frac{\mathrm{d}u}{\mathrm{d}t} - \xi\frac{\mathrm{d}w}{\mathrm{d}t}\right)\mathrm{d}V = -\int_{(V)} \left(\zeta\frac{\partial p}{\partial\xi} - \xi\frac{\partial p}{\partial\zeta}\right)\mathrm{d}V + \int_{(V)} \rho\left(\zeta\frac{\partial U_\mathrm{G}}{\partial\xi} - \xi\frac{\partial U_\mathrm{G}}{\partial\zeta}\right)\mathrm{d}V. \tag{4.39}$$

Multiplying the second of Eqs. (4.28) by ξ, the first by $-\eta$, and summing and integrating as above, the third equality can be written as

$$\int_{(V)} \rho\left(\xi\frac{\mathrm{d}v}{\mathrm{d}t} - \eta\frac{\mathrm{d}u}{\mathrm{d}t}\right)\mathrm{d}V = -\int_{(V)} \left(\xi\frac{\partial p}{\partial\eta} - \eta\frac{\partial p}{\xi}\right)\mathrm{d}V + \int_{(V)} \rho\left(\xi\frac{\partial U_\mathrm{G}}{\partial\eta} - \eta\frac{\partial U_\mathrm{G}}{\partial\xi}\right)\mathrm{d}V. \tag{4.40}$$

We write the second integral on the right-hand side of Eq. (4.38) in the form

$$\int\limits_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}\nu}{\mathrm{d}t}\right)\mathrm{d}V = G\int\limits_{(V)} \rho\left(\xi,\eta,\zeta,t\right)\eta\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\int\limits_{(V)} \rho\left(x,y,z,t\right)\frac{z-\zeta}{\Delta^3}\mathrm{d}x,\mathrm{d}y,\mathrm{d}z$$
$$-\,G\int\limits_{(V)} \rho\left(\xi,\eta,\zeta,t\right)\zeta\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\int\limits_{(V)} \rho\left(x,y,z,t\right)\frac{y-\zeta}{\Delta^3}\mathrm{d}x,\mathrm{d}y,\mathrm{d}z.$$

The integral is equal to zero owing to the asymmetry expressed by the integral expressions with respect to z, ζ and y, η. Because the pressure at the border of the domain S is equal to zero, the first term on the right-hand side of Eq. (4.38) is also equal to zero. Actually,

$$\int\limits_{(V)} \left(\eta\frac{\partial p}{\partial\zeta} - \zeta\frac{\partial p}{\partial\eta}\right)\mathrm{d}V = \int\limits_{(V)} \left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\xi p\right) - \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\eta p\right)\right]\mathrm{d}V$$
$$= \int\limits_{(V)} \left[\xi p\,\mathrm{d}\xi\,\mathrm{d}\zeta - \eta p\,\mathrm{d}\eta\,\mathrm{d}\zeta\right] = 0$$

Taking into account the law of mass conservation, the left-hand side of Eq. (4.38) in the Lagrange coordinate system can be rewritten as

$$\int\limits_{(V)} \rho\left(\eta\frac{\mathrm{d}w}{\mathrm{d}t} - \zeta\frac{\mathrm{d}\nu}{\mathrm{d}t}\right)\mathrm{d}V = \int\limits_{(V)} p\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta w - \zeta\nu\right)\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} p\left(\eta w - \zeta\nu\right)\mathrm{d}V = 0. \tag{4.41}$$

Integrating this equation with respect to time, the first of the three integrals is obtained:

$$\int\limits_{(V)} p\left(\eta w - \zeta\nu\right)\mathrm{d}V = C_1.$$

The other two integrals can be obtained analogously. Thus, the system of integrals of the moment of momentum has the form

$$\int\limits_{(V)} p\left(\eta w - \zeta\nu\right)\mathrm{d}V = C_1,$$
$$\int\limits_{(V)} p\left(\zeta u - \xi w\right)\mathrm{d}V = C_2,$$
$$\int\limits_{(V)} p\left(\xi\nu - \eta u\right)\mathrm{d}V = C_3. \tag{4.42}$$

To derive the tenth integral of motion representing the law of energy conservation, we multiply each of the system of equations (4.28) by u, ν, and w accordingly and then sum and integrate the equality obtained with respect to the system volume:

$$\int\limits_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}u + \frac{\mathrm{d}\nu}{\mathrm{d}t}\nu + \frac{\mathrm{d}w}{\mathrm{d}t}w\right)\mathrm{d}V = -\int\limits_{(V)}\left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}\nu + \frac{\mathrm{d}p}{\mathrm{d}\zeta}w\right)\mathrm{d}V$$
$$+\int\limits_{(V)} \rho\,(\xi,\eta,\zeta,t)\left(\frac{\partial U_G}{\partial \xi}u + -\frac{\partial U_G}{\partial \eta}\nu + \frac{\partial U_G}{\partial \zeta}w\right)\mathrm{d}V. \tag{4.43}$$

Applying the law of mass conservation for an elementary volume, it can easily be seen that the left-hand side of Eq. (4.43) expresses the change in the velocity of kinetic energy of the system:

$$\int\limits_{(V)} \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}u + \frac{\mathrm{d}\nu}{\mathrm{d}t}\nu + \frac{\mathrm{d}w}{\mathrm{d}t}w\right)\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int\limits_{(V)}\left(u^2 + \nu + w^2\right)\mathrm{d}V\right] = \frac{\mathrm{d}}{\mathrm{d}t}(T).$$

The first integral on the right-hand side of Eq. (4.43) can be transferred into

$$-\int\limits_{(V)}\left(\frac{\partial p}{\partial \xi}u + \frac{\partial p}{\partial \eta}\nu + \frac{\mathrm{d}p}{\mathrm{d}\zeta}w\right)\mathrm{d}V = 3\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{(V)} p\,\mathrm{d}V$$

and gives the change of velocity of the internal energy of the system.

The second integral on the right-hand side of the same equation expresses the velocity of the potential energy change:

$$\int\limits_{(V)} \rho\,(\xi,\eta,\zeta,t)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\left(\frac{\partial U_{\mathrm{G}}}{\partial \xi}\frac{\mathrm{d}\xi}{\mathrm{d}t} + -\frac{\partial U_{\mathrm{G}}}{\partial \eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\partial U_{\mathrm{G}}}{\partial \zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{1}{2}\int\limits_{(V)} \rho\,(\xi,\eta,\zeta,t)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\;U_{\mathrm{G}}\right] = -\frac{\mathrm{d}}{\mathrm{d}t}(U).$$

Finally, the law of energy conservation can be written in the form

$$T + U = W = E = \text{const}, \tag{4.44}$$

where W is the internal energy of the system.

We now derive Jacobi's virial equation for a system described by Eqs. (4.28), (4.29), (4.30), and (4.31). For this purpose we multiply each of Eqs. (4.28) by ξ, η, and ζ, respectively, summing and integrating the resulting expressions with respect to the volume of the system:

$$\int\limits_{(V)} \rho \left(\frac{\mathrm{d}u}{\mathrm{d}t}\xi + \frac{\mathrm{d}v}{\mathrm{d}t}\eta + \frac{\mathrm{d}w}{\mathrm{d}t}\zeta \right) \mathrm{d}V = - \int\limits_{(V)} \left(\frac{\partial p}{\partial \xi}\xi + \frac{\partial p}{\partial \eta}\eta + \frac{\mathrm{d}p}{\mathrm{d}\zeta}\zeta \right) \mathrm{d}V + \int\limits_{(V)} \rho \left(\frac{\partial U_{\mathrm{G}}}{\partial \xi}\xi + -\frac{\partial U_{\mathrm{G}}}{\partial \eta}\eta + \frac{\partial U_{\mathrm{G}}}{\partial \zeta}\zeta \right) \mathrm{d}V. \tag{4.45}$$

Using the obtained identities considered in the previous section, we have

$$\frac{\mathrm{d}u}{\mathrm{d}t}\xi = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\xi^2\right) - u^2 \right),$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}\eta = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\eta^2\right) - v^2 \right),$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}\zeta = \left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\zeta^2\right) - w^2 \right).$$

Taking into account the law of conservation of mass for elementary volume, we transform the left-hand side of Eq. (4.45) as follows:

$$\int\limits_{(V)} \rho \left(\frac{\mathrm{d}u}{\mathrm{d}t}\xi + \frac{\mathrm{d}v}{\mathrm{d}t}\eta + \frac{\mathrm{d}w}{\mathrm{d}t}\zeta \right) \mathrm{d}V = \frac{1}{2}\int\limits_{(V)} \rho \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\xi^2 + \eta^2 + \zeta^2\right) \mathrm{d}V - \int\limits_{(V)} \rho \left(u^2 + v^2 + w^2\right) \mathrm{d}V = \ddot{\Phi} - 2T, \tag{4.46}$$

where

$$\ddot{\Phi} = \frac{1}{2}\int\limits_{(V)} \rho \left(\xi^2 + \eta^2 + \zeta^2\right) \mathrm{d}V$$

is the Jacobi function and

$$T = \frac{1}{2}\int\limits_{(V)} \rho \left(u^2 + v^2 + w^2\right) \mathrm{d}V$$

is the kinetic energy of the system.

We now transform the first integral on the right-hand side of Eq. (4.45). Using the Gauss–Ostrogradsky theorem and the equality with zero pressure at the border of the system, we can write

$$-\int\limits_{(V)}\left(\frac{\partial p}{\partial \xi}\xi + \frac{\partial p}{\partial \eta}\eta + \frac{\partial p}{\partial \zeta}\zeta\right)\mathrm{d}V = -\int\limits_{(V)}\left[\frac{\partial}{\partial \xi}(p\xi) + \frac{\partial}{\partial \eta}(p\eta) + \frac{\partial}{\partial \zeta}(p\zeta)\right]\mathrm{d}V$$

$$+\,3\int\limits_{(V)} p\,\mathrm{d}V = 3\int\limits_{(V)} p\,\mathrm{d}V. \tag{4.47}$$

The obtained equation expresses the doubled internal energy of the system.

The second integral on the right-hand side of Eq. (4.45) is equal to the potential energy of the gravitational interaction of mass particles within the system:

$$\int\limits_{(V)} \rho\left(\frac{\partial U_{\mathrm{G}}}{\partial \xi}\xi + -\frac{\partial U_{\mathrm{G}}}{\partial \eta}\eta + \frac{\partial U_{\mathrm{G}}}{\partial \zeta}\zeta\right)\mathrm{d}V = U. \tag{4.48}$$

Substituting Eqs. (4.46), (4.47), and (4.48) into (4.45), Jacobi's virial equation is obtained in the form

$$\ddot{\Phi} - 2T = 3\int\limits_{(V)} p\,\mathrm{d}V + U. \tag{4.49}$$

Taking into account the law of conservation of energy (4.44), we rewrite Eq. (4.49) in a form which will be used farther:

$$\ddot{\Phi} = 2E - U, \tag{4.50}$$

where $\quad = \quad + U + W$ is the total energy of the system.

4.4 Derivation of Jacobi's Virial Equation from Hamiltonian Equations

Let the system of material points be described by Hamiltonian equations of motion. Let also the considered systems consist of n material points with masses m_i. Its generalized coordinates and moments are q_i and $p_i = m_i(\mathrm{d}q_i/\mathrm{d}t)$. Hamiltonian equations for such a system can be written as

$$\begin{aligned}\dot{p}_i &= -\frac{\partial H}{\mathrm{d}q_i},\\ \dot{q}_i &= \frac{\partial H}{\mathrm{d}p_i},\end{aligned} \tag{4.51}$$

where $H\,(p, q)$ is the Hamiltonian and $i = 1, 2, \ldots, n$.

Using values q_i and p_i, we can construct the moment of momentum:

$$\sum_{i=1}^{n} p_i q_i = \sum_{i=1}^{n} m_i q_i \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} \frac{m_i q_i^2}{2}\right).$$

Now the Jacobi function may be introduced:

$$\sum_{i=1}^{n} p_i q_i = \dot{\Phi}. \tag{4.52}$$

Differentiating Eq. (4.52) with respect to time, Jacobi's virial equation is obtained in the form

$$\ddot{\Phi} = \sum_{i=1}^{n} \dot{p}_i q_i + \sum_{i=1}^{n} p_i \dot{q}_i. \tag{4.53}$$

Substituting expressions for $\dot{p}_i$ and $\dot{q}_i$ taken from the Hamiltonian equations (4.51) into the right-hand side of (4.52), we obtain Jacobi's virial equation written in Hamiltonian form

$$\ddot{\Phi} = \sum_{i=1}^{n} \left(-\frac{\partial H}{\partial q_i} q_i + \frac{\partial H}{\partial p_i} p_i\right). \tag{4.54}$$

The Hamiltonian of the system of material points interacting according to the law of the inverse squares of distance is a homogeneous function in terms of moments p_i with a degree of homogeneity of the function equal to 2 and in terms of coordinates q_i with a degree of homogeneity equal to -1. It follows from this

$$H(p, q) = T(p) + U(q)$$

and hence

$$\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} = 2T,$$

$$\sum_{i=1}^{n} q_i \frac{\partial H}{\partial q_i} = -U.$$

Taking these relationships into account, Eq. (4.54) acquires the usual form of Jacobi's virial equation (4.50) for the system of mass points interacting according to the law of inverse squares of distance.

Equation (4.54) is more general than Eq. (4.50). The use of generalized coordinates and moments as independent variables permits us to obtain the solution of Jacobi's virial equation, taking into account gravitational and electromagnetic perturbations as well as quantum effects, both in the framework of classical physics and in terms of the Hamiltonian written in an operator form. In the general case, Eq. (4.54) can be reduced to (4.50) as the potential energy of interaction of the system's points is a homogenous function of its coordinates.

4.5 Derivation of Jacobi's Virial Equation in Quantum Mechanics

It is known that in quantum mechanics some physical value L by definition takes the linear Hermitian operator $\hat{L}$. Any physical state of the system takes the normalized wave function ψ. The physical value of L can take the only eigenvalues of the operator $\hat{L}$. The mathematical expectation $\hat{L}$ of the value L at state ψ is determined by the diagonal matrix element

$$\bar{L} = \left\langle \psi | \hat{L} | \psi \right\rangle. \tag{4.55}$$

The matrix element of the operators of the Cartesian coordinates $\hat{x}_i$ and the Cartesian components of the conjugated moments $\hat{p}_k$ calculated within wave functions f and g of the system satisfy Hamilton's equations of classical mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f | \hat{p}_i | g \rangle = -\left\langle f \left| \frac{\partial \hat{H}}{\mathrm{d}\hat{x}_i} \right| g \right\rangle, \tag{4.56}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f | \hat{x}_i | g \rangle = -\left\langle f \left| \frac{\partial \hat{H}}{\partial \hat{p}_i} \right| g \right\rangle, \tag{4.57}$$

where $\hat{H}$ is the operator which corresponds to the classical Hamiltonian.

Operators $\hat{p}_i$ and $\hat{x}_k$ satisfy the commutation relations

$$\begin{aligned} [\hat{p}_i, \hat{x}_k] &= i\hbar\delta_{ik}, \\ [\hat{p}_i, \hat{p}_k] &= 0, \\ [\hat{x}_i, \hat{x}_k] &= 0, \end{aligned} \tag{4.58}$$

where $\hbar$ is Planck's constant, δ_{ik} is the Kronecker's symbol, $\delta_{ik} = 1$ at $i = k$, and $\delta_{ik} = 0$ at $i \neq k$.

Operator components of momentum $\hat{p}_i$ for the functions whose arguments are Cartesian coordinates $\hat{x}_i$ have the form

$$\hat{p}_i = i\hbar \frac{\partial}{\partial x_i} \tag{4.59}$$

and reverse vector

$$\hat{p} = -i\hbar\nabla.$$

The derivative taken from the operator with respect to time does not depend explicitly on time; it is defined by the relation

$$\hat{L} = -\frac{i}{\hbar}\left[\hat{L}, \hat{H}\right], \tag{4.60}$$

where $\hat{H}$ is the Hamiltonian operator that can be obtained from the Hamiltonian of classical mechanics in accordance with the correspondence principle.

We have already noted that in the classical many-body problem, the translational motion of the center of mass can be separated from the relative motion of the mass points if only the inertial forces affect the system. We can show that in quantum mechanics the same separation is possible.

The Hamiltonian operator of a system of n particles which is not affected by external forces in coordinates is

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{i=1}^{n}\frac{\nabla_i^2}{m_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{i=1}^{n} U_{ik}\left(x_i - x_k, y_i - y_k, z_i - z_k\right). \tag{4.61}$$

Let us replace in (4.61) the three n coordinates x_i, y_i, z_i by coordinates X, Y, Z of the center of mass and by coordinates $\xi_\lambda, \eta_\lambda, \zeta_\lambda$, which determine the position of a particle λ ($\lambda = 1, 2, \ldots, n{-}1$) relative to particle n. We obtain

$$X = \frac{1}{M}\sum_{i=1}^{n} m_i x_i,$$

$$M = \sum_{i=1}^{n} m_i,$$

$$\xi_\lambda = x_\lambda - x_n, \tag{4.62}$$

where $\lambda = 1, 2, \ldots, n{-}1$.

Analogously the corresponding relations for Y, Z, η_λ, ζ_λ are obtained.

It is easy to obtain from (3.62) the following operator relations:

$$\frac{\mathrm{d}}{\mathrm{d}x_p} = \frac{m_p}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial \xi_p}, \quad p = 1, 2, \ldots, n-1,$$

$$\frac{\partial}{\partial X_n} = \frac{m_n}{M}\frac{\partial}{\partial X} - \sum_{\lambda=1}^{n-1}\frac{\partial}{\partial \xi_\lambda},$$

$$\sum_{\lambda=1}^{n-1}\frac{1}{\partial x_i}\frac{\partial^2}{\partial x_i^2}=\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\left(\frac{m_\lambda^2}{M^2}\frac{\partial^2}{\partial X^2}+2\frac{m_\lambda}{M}\frac{\partial^2}{\partial X\partial\xi_\lambda}+\frac{\partial^2}{\partial\xi_\lambda^2}\right)$$
$$+\frac{1}{m_n}\left(\frac{m_n^2}{M^2}\frac{\partial^2}{\partial X^2}-2\frac{m_n}{M}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial X\partial\xi_\lambda}+\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial\xi_\mu\partial\xi_\lambda}\right)$$
$$=\frac{1}{m_n}\frac{\partial^2}{\partial X^2}+\left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\frac{\partial^2}{\partial\xi_\lambda^2}+\frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\frac{\partial^2}{\partial\xi_\mu\partial\xi_\lambda}\right),$$

where summing on the Greek index is provided from 1 to $n-1$. It is seen that all the combined derivatives $\partial^2/\partial\times\partial\xi_\lambda$ were mutually reduced and do not enter into the final expression. This allows the Hamiltonian to be separated into two parts:

$$H=H_0+H_r,$$

where, on the right-hand side, the first term

$$H_0=\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial Z^2}\right)$$

describes the motion of the center of mass, and the second term

$$H_r=-\frac{\hbar^2}{2}\left(\sum_{\lambda=1}^{n-1}\frac{1}{m_\lambda}\nabla_\lambda^2+\frac{1}{m_n}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}\nabla_\lambda\nabla\mu\right)+U \tag{4.63}$$

describes the relative motion of the particles.

The potential energy in (4.63), which is

$$U=\frac{1}{2}\sum_{\mu=1}^{n-1}\sum_{\lambda=1}^{n-1}U_{\lambda\mu}\left(\xi_\lambda-\xi_\mu,\eta_\lambda-\eta_\mu,\zeta_\lambda-\zeta_\mu\right)+\sum_{\lambda-1}^{n-1}U_{\lambda\mu}\left(\xi_\lambda,\eta_\lambda,\zeta_\lambda\right), \tag{4.64}$$

also certainly does not depend on the coordinates of the center of mass.

Now the Schrödinger's equation

$$\left(H_0+H_\mathrm{r}\right)\psi=E\psi \tag{4.65}$$

permits the separation of variables.

Assuming $\psi=\varphi\,(X,Y,Z)$ and $(\xi_\lambda,\eta_\lambda,\zeta_\lambda,)$ we obtain

$$-\frac{\hbar^2}{2V}\nabla^2\varphi=E_0\varphi, \tag{4.66}$$

$$H_r u = E_r u, \tag{4.67}$$

$$E_0 + E_r = E. \tag{4.68}$$

The solution of Eq. (4.66) has the form of a plane wave:

$$\varphi = e^{i\mathbf{kR}},$$
$$E_0 = \frac{\hbar^2 k^2}{2m}, \tag{4.69}$$

where R is a vector with coordinates X, Y, Z.

The result obtained is in full accordance with the classical law of the conservation of motion of the center of mass. This means that the center of mass of the system moves like a material point with mass m and momentum $\hbar\mathbf{k}$. The mode of relative motion of the particles is determined by Eq. (4.67), which does not depend on the motion of the center of mass.

The existence on the right-hand side of Eq. (4.63) of the third term restricts further factorization of the function $u\left(\xi_\lambda, \eta_\lambda, \zeta_\lambda\right)$. Only in the two-body problem, where $n = 2$ and at $\lambda = \mu = 1$, a part of the Hamiltonian connected with the relative motion is simplified and takes the form

$$H_r = -\frac{\hbar^2}{2}\left(\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right) + U_{12}\left(\xi_1, \eta_1, \zeta_1\right). \tag{4.70}$$

It seems that choosing the corresponding system of coordinates can lead us to an approach for separating the motion of the center of mass to the many-body problem.

Introducing into Eq. (4.70) the reduced mass m^*, which is determined as in classical mechanics by the relation

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m^*}, \tag{4.71}$$

and omitting indices in the notation for relative coordinates and potential energy U_{12}, we come to

$$-\frac{\hbar^2}{2m^*}\nabla^2 u + U\left(\xi, \eta, \zeta\right) u = E_r u. \tag{4.72}$$

This is Schrödinger's equation for the equivalent one-particle problem.

Considering the hydrogen atom in the framework of the one-particle problem, it is assumed that the nucleus is in ground state. In accordance with Eq. (4.72), the normalized mass of the nucleus and electron m^* should be introduced. No changes which account for the effect of the nucleus on the relative motion should be introduced. Because of the nucleus, mass m is much heavier than electron mass m_e; instead of Eq. (4.71), we can use its approximation

$$m^* = m\left(1 - \frac{m}{m_e}\right).$$

Comparing, for example, the frequency of the red line H_α ($n = 3$–$n = 2$) in the spectrum of a hydrogen atom

$$\omega\left(H_\alpha\right) = \frac{5}{36}\frac{m_H^* e^4}{2\hbar^2 h}$$

with the frequency of the corresponding line in the spectrum of a deuterium atom

$$\omega\left(D_\alpha\right) = \frac{5}{36}\frac{m_D^* e^4}{2\hbar^2 h},$$

and taking into account that $m_D \approx 2m_H$, for the difference of frequencies, we obtain

$$\omega\left(D_\alpha\right) - \omega\left(H_\alpha\right) = \frac{m_D^* - m_H^*}{m_H^*}\omega\left(H_\alpha\right) \approx \frac{m}{2M_H}\omega\left(H_\alpha\right).$$

This difference is not difficult to observe experimentally. At wavelength 6,563 Å, it is equal to 4.12 m^{-1}. Heavy hydrogen was discovered by Urey et al. (1932), who observed a weak satellite D_α in the line H_α of the spectrum of natural hydrogen. This proves the practical significance of even the first integrals of motion.

We now show that the virial theorem is valid for any quantum mechanical system of particles retained by Coulomb (outer) forces:

$$2\overline{T} + U = 0.$$

We prove this by means of scale transformation of the coordinates keeping unchanged normalization of wave functions of a system.

The wave function of a many-particle system with masses m_i and electron charge e_i satisfies the Schrödinger's equation

$$-\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\nabla_i^2\psi + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-1}\frac{e_i e_k}{r_{ik}}\psi = E\psi \tag{4.73}$$

and the normalization condition

$$\int d\tau_1 \ldots \int \psi^* \psi d\tau_n = 1. \tag{4.74}$$

The mean values of the kinetic and potential energies of a system at stage ψ are determined by the expressions

$$T = -\frac{\hbar^2}{2}\sum_{i=1}^{n-1}\frac{1}{m_i}\int d\tau_1 \dots \int \psi^* \nabla_i^2 \psi d\tau_n \tag{4.75}$$

$$U = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{i=1}^{n-1} e_i e_k \int d\tau_1 \dots \int d\tau_n \frac{\psi^* \psi}{r_{ik}} d\tau_n. \tag{4.76}$$

The scale transformation

$$\bar{r}_i' = \lambda \bar{r}_i, \tag{4.77}$$

keeps in force the condition (4.74) and means that the wave function

$$\psi\left(\bar{r}_i, \dots, \bar{r}_n\right) \tag{4.78}$$

is replaced by the function

$$\psi_\lambda = \lambda^{3n/2}\psi\left(\lambda \bar{r}_i, \dots, \bar{r}_n\right). \tag{4.79}$$

Substituting (4.79) into Eqs. (4.76) and (4.75), passing to new variables of integration (4.77), and taking into account that

$$\nabla_i^2 = \lambda^2 \nabla_i'^2,$$

$$\frac{1}{r_{ik}} = \lambda \frac{1}{r_{ik}'},$$

instead of the true value of the energy, $\overline{E} = \overline{T} + \overline{U}$, we obtain

$$\overline{E}(\lambda) = \lambda^2 \overline{T} + \lambda \overline{U}. \tag{4.80}$$

Equation (4.80) should have a minimum value in the case when the function which is the solution of the Schrödinger's equation is taken from the family of functions (4.79), that is, when $\lambda = 1$. So, at $\lambda = 1$ the expression

$$\frac{\partial \overline{E}(\lambda)}{\partial \lambda} = 2\lambda^2 \overline{T} + \overline{U}$$

should turn into zero, and thus

$$2\overline{T} + \overline{U} = 0,$$

which is what we want to prove.

We now derive Jacobi's virial equation for a particle in the inner force field with the potential $U(q)$ and fulfilling the condition

$$-q\nabla U(q) = U \tag{4.81}$$

using the quantum mechanical principle of correspondence. We shall also show that in quantum mechanics, Jacobi's virial equation has the same form and contents as in classical mechanics, the only difference being that its terms are corresponding operators.

In the simplest case, the Hamiltonian of a particle is written as

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{U}, \tag{4.82}$$

and its Jacobi function is

$$\hat{\Phi} = \frac{1}{2}m\hat{q}^2. \tag{4.83}$$

It is clear that the following relations are valid:

$$\nabla\hat{\Phi} = m\hat{q},$$
$$\nabla^2\hat{\Phi} = m.$$

Following the definition of the derivative with respect to time from the operator of the Jacobi function of a particle (4.60), we can write

$$\dot{\Phi} = -\frac{1}{h}\left[\hat{\Phi}, \hat{H}\right],$$

where, after corresponding simplification, quantum mechanical Poisson brackets can be reduced to the form

$$\left[\hat{\Phi}, \hat{H}\right] = \frac{h^2}{2m}\left\{\nabla^2\hat{\Phi} + 2\left(\nabla\hat{\Phi}\right)\nabla\right\} = \frac{h^2}{2m}\left(m + 2mq\nabla\right). \tag{4.84}$$

The second derivative with respect to time from the operator of the Jacobi function is

$$\ddot{\Phi} = -\frac{1}{h^2}\left\{\left[\hat{\Phi}, \hat{H}\right], \hat{H}\right\}. \tag{4.85}$$

Substituting the corresponding value of $\left[\hat{\Phi}, \hat{H}\right]$ and $\hat{H}$ from (4.84) and (4.82) into the right-hand side of (4.85), we obtain

$$\ddot{\Phi} = -\frac{h^2}{2m}\frac{1}{h^2}\left[(m + 2m\hat{q}\nabla)\,, \left(-\frac{h^2}{2m}\nabla^2 + \hat{U}\right)\right]. \tag{4.86}$$

After simple transformation, the right-hand side of (4.86) will be

$$\ddot{\Phi} - \frac{1}{2m}\left\{2h^2\nabla^2 + 2m\hat{q}\left(\nabla\hat{U}\right)\right\} = -\frac{2h^2}{2m\nabla^2} + \hat{U}, \tag{4.87}$$

where, in writing this expression on the right-hand side, we used condition (4.81).

Add and subtract the operator $\hat{U}$ from the right-hand side of Eq. (4.87), and following the definition of the Hamiltonian of the system (4.82), we obtain the quantum mechanical Jacobi virial equation (equation of dynamical equilibrium of the system), which has the form

$$\ddot{\Phi} = 2\hat{H} - \hat{U}. \tag{4.88}$$

From Eq. (4.88), by averaging with respect to time, we obtain the quantum mechanical analogue of the classical virial theorem (equation of hydrostatic equilibrium of the system). In accordance with this theorem, the following relation is kept for a particle performing finite motion in space:

$$2\hat{H} = \hat{U}. \tag{4.89}$$

Analogously, one can derive Jacobi's virial equation and the classical virial theorem for a many-particle system, the interaction potential for which depends on the distance between any particle pair and is a homogeneous function of the coordinates. In particular, Jacobi's virial equation for Coulomb interactions will have the form of Eq. (4.88).

4.6 General Covariant Form of Jacobi's Virial Equation

Jacobi's initial equation

$$\ddot{\Phi} = 2E - U,$$

which was derived in the framework of Newtonian mechanics and is correct for the system of material points interacting according to Newton and Coulomb laws, includes two scalar functions Φ and U relates to each other by a differential relation. We draw attention to the fact that neither function, in its structure, depends explicitly on the motion of the particles constituting the body. The Jacobi function Φ is defined by integrating the integrand $\rho(r)r^2$ over the volume (where $\rho(r)$ is the mass density and r is the radius vector of the material point) and is independent in explicit form of the particle velocities. The potential energy U also represents the integral of

$m(r)\mathrm{d}m(r)/r$ over the volume (where $m(r)$ is the mass of the sphere's shell of radius r and $\mathrm{d}m(r)$ is the shell's mass) independent of the motion of the particles for the same reason.

Let us derive Jacobi's equation from Einstein's equation written in the form

$$\nabla G = 2\pi T, \tag{4.90}$$

where ΔG and T are the Einstein tensor and energy–momentum tensor accordingly.

In fact, since the covariant divergence of Einstein's tensor is equal to zero, we consider the covariant divergence of the energy–momentum tensor T only of substance and fields (not gravitational). Moreover, the ordinary divergence of the sum of the tensor T and pseudotensor t of the energy momentum of the gravitational field can be substituted for the covariant divergence of the tensor T. This ordinary divergence leads to the existence of the considered quantities.

Let us define the sum of the tensor T and pseudotensor t through T_{ij} and derive Jacobi's equation in this notation.

The equation for ordinary divergence of the sum $T_{ij} = (\quad + t)_{ij}$ can be written as

$$T_{0k,k} - T_{00,0} = 0, \tag{4.91}$$

$$T_{jk,k} - T_{j0,0} = 0. \tag{4.92}$$

We multiply Eq. (4.92) by x^j and integrate over the whole space (assuming the existence of a synchronous coordinate system). Integrating by parts, neglecting the surface integrals (they vanish at infinity), and transforming to symmetrical form with respect to indices, we obtain

$$\int T_{ij}\mathrm{d}V = \frac{1}{2}\left[\int \left(T_{i0}x^j + T_{j0}x^i\right)\mathrm{d}V\right] = 0, \tag{4.93}$$

where i, j are spatial indices.

Similarly, multiplying (3.91) by $x^i x^j$ and integrating over the whole space, it follows that

$$\left[\int T_{00}x^i x^j \mathrm{d}V\right]_{,0} = -\int \left(T_{i0}x^j + T_{j0}x^i\right)\mathrm{d}V. \tag{4.94}$$

From (3.93) and (3.94), we finally get

$$\int T_{ij\,\mathrm{d}V} = \frac{1}{2}\left[\int T_{00}x^i x^j\right]_{,0,0}. \tag{4.95}$$

It is worth recalling that $_{00}$ also includes the gravitational defect of the mass due to the pseudotensor t by definition.

The integral $\int T_{00} x^i x^j \mathrm{d}V$ represents the generalization of the Jacobi function $\Phi = \frac{1}{2}\int \rho r^2 \mathrm{d}V$ introduced earlier, if we take the spur (also commonly known as the trace) of Eq. (4.95). Let us clarify this operation.

In Eq. (4.95), the spur is taken by the spatial coordinates. It is therefore necessary either to represent the total zero spur by four indices, as happens in the case of a transverse electromagnetic field, or to represent the relationship between the reduced spur with three indices and the total spur, as happens in the case of the energy–momentum tensor of matter.

Special care should be taken while representing the spur of the pseudotensor of the energy momentum t. Consider the post-Newtonian approximation. In this approximation, assuming the value of $2u$ to be $-g \quad -1$, the components of the pseudotensor t are written in the form

$$t^{00} = -\frac{7}{8\pi} u_{,j,i},$$

$$t^{ij} = -\frac{1}{4\pi}\left(u_{,j,i} - \frac{1}{2}\delta_{ij} u_{,k} u_{,k}\right),$$

so that

$$S_{\mathrm{p}} t = t^{00} + S_{\mathrm{p}}\left(t^{ij}\right) = -\frac{1}{\pi} u_{,i} u_{,j} = \frac{1}{7} t^{00},$$

$$S_{\mathrm{p}}\left(t^{ij}\right) = \frac{6}{7} t^{00}.$$

The spur on the left-hand side of Eq. (4.95) can therefore be reduced to the energy of the Coulomb field, the total energy of the transverse electromagnetic field, and the gravitational energy (when it can be separated, i.e., post-Newtonian approximation).

Finally, it follows in this case that the scalar form of Jacobi's equation holds:

$$\Phi_{,0,0} = mc^2, \tag{4.96}$$

where m is the mass, accounting for the baryon defect of the mass and the total energy of the electromagnetic radiation. We do not take into account the radiation of the gravitational waves.

The result obtained by Tolman for the spherical mass distribution (Tolman 1969) is of interest:

$$m = 4\pi \int \hat{\varepsilon} r^2 \mathrm{d}r, \tag{4.97}$$

where r is the radius and $\hat{\varepsilon}$ is the energy density.

The integral (4.97) acquires a form which is also valid in the case of flat space–time. This result can be explained as follows. The curvature of space–time is exactly compensated by the mass defect. This probably explains the fact that Jacobi's virial equation, derived from Newton's equations of motion which are valid in the case of nonrelativistic approximation for a weak gravitational field, becomes more universal than the equations from which it was derived.

We shall not study the general tensor of Jacobi's virial equation, since in the framework of the assumed symmetry for the considered problems, we are interested only in the scalar form of the equation as applied to electromagnetic interactions. As follows from these remarks, in this case, Jacobi's equation remains unchanged, and the energy of the free electromagnetic field is accounted for in the term defining the total energy of the system. Total energy enters into Jacobi's equation without the electromagnetic energy irradiated up to the considered moment of time. Moreover, for the initial moment of time, we take the moment of system formation. This irradiated energy appears also to be responsible for the growth of the gravitational mass defect in the system, as was mentioned previously.

4.7 Relativistic Analogue of Jacobi's Virial Equation

Let us derive Jacobi's virial equation for asymptotically flat space–time. We write the expression of a 4-moment of momentum of a particle:

$$p^i x_i, \tag{4.98}$$

where $p^i = mcu^i$ is the 4-momentum of the particle, is the velocity of light, $u^i = \mathrm{d}x^i/\mathrm{d}s$ is the 4-velocity, x^i is the 4-coordinate of the particle, s is the interval of events, and i is the running index with values 0, 1, 2, 3.

In asymptotically flat space–time, we write

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\frac{\mathrm{d}}{\mathrm{d}s}\left(u^i x_i\right) = mc\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{x^i x_i}{2}\right) \tag{4.99}$$

since

$$x^i x_i = c^2t^2 - r^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s} = \frac{\gamma}{c}\frac{\mathrm{d}}{\mathrm{d}t},$$

where $\gamma = 1/\sqrt{1-(v^2/c^2)}$ and r is the radius of mass particle.

Then we continue the transformation of Eq. (4.99):

$$mc\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{x^i x_i}{2}\right) = mc\frac{\gamma^2}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{c^2t^2 - r^2}{2}\right) = mc\gamma^2 - \frac{\gamma^2}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{mr^2}{2}\right),$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi}, \tag{4.100}$$

where

$$\ddot{\Phi} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{mr^2}{2}\right)$$

is the Jacobi function.

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc\frac{\mathrm{d}}{\mathrm{d}s}\left(u^i x_i\right) = mcu^i u_i + mc\frac{\mathrm{d}u^i}{\mathrm{d}s}x_i. \tag{4.101}$$

Using the identity $u_i u^i \equiv 1$ and the geodetic equation

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} = -\Gamma^i_{k\ell}u^k u^\ell,$$

where

$$\Gamma^i_{k\ell} = \frac{1}{2}g^{im}\left(\frac{\partial g_{km}}{\partial x^\ell} + \frac{\partial g_{\ell m}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^m}\right)$$

are the Christoffel's symbols, the Eq. (4.101) will be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p^i x_i\right) = mc - mcx_i\Gamma^i_{k\ell}u^k u^\ell. \tag{4.102}$$

The metric tensor g_{ik} for a weak stationary gravitational field is

$$g_{ik} = \eta_{ik} + \xi_{ik}, \tag{4.103}$$

where in our notation η_{ik} is the Lorentz tensor with signature $(+, -, -, -)$.

For the Schwarzschild metric tensor ξ_{ik}, we write

$$\begin{aligned} &\xi_{00} = -\frac{r_g}{r}; \quad \xi_{11} = -\frac{1}{1 - r_g/r} + 1 \approx -\frac{r_g}{r}; \\ &\xi_{ik} = 0 \quad \text{if } i \neq k \text{ and } I \neq 0.1. \end{aligned} \tag{4.104}$$

Here, $r_g = 2GV/c^2$ is the Schwarzschild gravitational radius of the mass m'.

Now we can rewrite the second term on the right-hand side of Eq. (4.102) using (4.103) and (4.104)

$$mcx_i\Gamma^i_{k\ell}u^k u^\ell = mcx^m u^k u^\ell\left(\frac{\partial\xi_{km}}{\partial x^m} - \frac{1}{2}\frac{\partial\xi_{k\ell}}{\partial x^m}\right)$$
$$= mc\left(x^0u^0u^1\frac{\partial\xi_{00}}{\partial x^1} + x^1u^1u^1\frac{\partial\xi_{11}}{\partial x^1} - \frac{1}{2}x^1u^0u^0\frac{\partial\xi_{00}}{\partial x^1} - x^1u^1u^1\frac{\partial\xi_{11}}{\partial x^1}\right). \quad (4.105)$$

But $u^1 << u^0 = \gamma$ and $x^1 = r$.

We therefore obtain for Eq. (4.105)

$$mcx_i\Gamma^i_{k\ell}u^k u^\ell = -\frac{mc}{2}x^1u^0u^0\frac{\partial\xi_{00}}{\partial x^1}$$
$$= -\frac{mc}{2}r\gamma^2\frac{r_g}{r^2} = \frac{mc}{2}\gamma^2\frac{2Gm'}{c^2 r} = -\frac{\gamma^2}{c}\frac{Gm'm}{r}. \quad (4.106)$$

Finally, we see that

$$\frac{d}{ds}\left(p^i x_i\right) = mc - \frac{\gamma^2}{c}U, \quad (4.107)$$

where U is the potential energy of the mass in the gravitational field of the mass m'.

Identification of the expression $(d/ds)(p^i x_i)$ obtained from Eqs. (4.100) and (4.107) gives

$$mc\gamma^2 - \frac{\gamma^2}{c}\ddot{\Phi} = mc - \frac{\gamma^2}{c}U. \quad (4.108)$$

It is easy to see that

$$mc\left(\gamma^2 - 1\right) = mc\left(\frac{1}{1 - v^2/c^2} - 1\right) = mc\frac{v^2}{c^2}\frac{1}{1 - v^2/c^2} = \frac{\gamma^2}{c}mv^2 = \frac{\gamma^2}{c}2T.$$

We then obtain

$$\frac{\gamma^2}{c}\ddot{\Phi} = \frac{\gamma^2}{c}U + \frac{\gamma^2}{c}2T,$$

which gives

$$\ddot{\Phi} = U + 2T$$

or

$$\ddot{\Phi} = 2E + U, \quad (4.109)$$

where   is the kinetic energy of the particle m and $= U + T$ is its total energy.

Equations (4.109) are known as classical Jacobi's virial equations, and the expression (4.102) represents its relativistic analogue for asymptotically flat space–time.

4.8 Derivation of Jacobi's Virial Equation in Statistical Mechanics

Statistical mechanics accepts the considered system in equilibrium state a priori at the stage of the problem formulation. Let us derive the virial theorem also for this branch of mechanics.

Denote by r_i generalized moment $p_i, \ldots, p_f$ or generalized coordinate $q_i, \ldots, q_f$ of the system points. Assume also that the value r_i of a physical system is changing from a to b and there is equality $H(a) = \infty$, or $a = 0$, or there are both effects, or also $H(b) = \infty$, or $b = 0$, or both effects. Let symbol $< \ldots >$ denote the mean value of the classical canonic distribution. Then it is possible to show the correctness of the following statement:

$$\left\langle r_i \frac{\partial H}{\partial r_i} \right\rangle = kT, \tag{4.110}$$

where k is the Boltzmann's constant, T is the temperature, and H is the system's Hamiltonian.

In fact, the normalization integral for the canonical distribution is

$$1 = A \int \ldots \int e^{-\frac{H}{kT}} \mathrm{d}q_1 \ldots \mathrm{d}p_f. \tag{4.111}$$

Integrating expression (4.111) by parts on q_1, one has

$$1 = A \int \ldots \int (q_1 e^{-\frac{H}{kT}}) \big|_a^b \, .\mathrm{d}q_2 \ldots \mathrm{d}p_f + \frac{A}{kT} \int \ldots \int q_1 \ldots \mathrm{d}p_f. \tag{4.112}$$

According to this limitation, the first integral contributes nothing and, from this expression, follows the correctness of equation (4.110), which is called the theorem of the uniform distribution.

We can derive now the virial theorem in classical statistical mechanics. For this we assume that a particle i occurs in the point $r_i = (q_{ix}, q_{iy}, q_{iz})$, and it is acted by the force $\bar{F}_i = \frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$, where $\bar{p}_i = (p_{ix}, p_{iy}, p_{iz})$. By definition the system's virial of n particles is the expression $C = -\frac{1}{2}\sum_{i=1}^{n} \bar{F}_i \cdot \bar{r}_i$ which is averaged in time. Assuming that the motion of particles is described by the Hamiltonian equations of motion ($\mathrm{d}q_{ij} / \mathrm{d}t = \mathrm{d}H \big/ \mathrm{d}p_{ij}$, $\mathrm{d}p_{ij} / \mathrm{d}t = -\mathrm{d}H \big/ \mathrm{d}q_{ij}$; $i = 1, 2, \ldots, n$; $j = x, y, z$) and for the system the ergodic hypothesis system is correct, according to which the averaging over the ensemble and on time leads to the same results, we can show that the system virial C is equal to

$$C = \frac{3}{2}nkT. \tag{4.113}$$

According to expression (4.110),

$$\left\langle r_i \frac{\partial H}{\partial r_i} \right\rangle = \langle -q_{ij} F_{ij} \rangle = kT. \tag{4.114}$$

Now, the correctness of expression (4.113) follows from Eq. (4.114) and from the definition of the virial system.

Farther, it is easy to show that in the case when the force is defined by the potential W, that is,

$$F_{ij} = -\frac{\partial W}{\partial q_{ij}},$$

and the moment enters only to the kinetic energy $k = \sum_{i=1}^{n} \frac{p_i^2}{2m}$, then the following equality is correct:

$$\bar{k} = \frac{1}{2} \sum_{i=1}^{n} \bar{\nabla} \bar{W} \bar{r}_i = \frac{3}{2} knT = C. \tag{4.115}$$

If the forces interacting between the gas particles are $f\left(\left|\bar{r}_j - \bar{r}_k\right|\right) = f\left(r_{jk}\right)$ and depend only from distance between the particles, then these forces contribute to virial as

$$-\frac{1}{2} \sum_{1 \le j < k \le n} r_{jk} f_{jk}\left(k_{jk}\right), \tag{4.116}$$

where the summation is done over all particle pairs. In fact, taking the force $f(r_{jk})$, like in the case of repulsion, as positive value, the force affecting on j-particle in the form

$$\bar{F}_j = \frac{\bar{r}_j - \bar{r}_k}{r_{jk}} f\left(r_{jk}\right)$$

and the force affecting on k-particle in the form

$$\bar{F}_{\mathrm{k}} = \frac{\bar{r}_k - \bar{r}_j}{r_{jk}} f\left(r_{jk}\right),$$

then the contribution to virial from the pair (j, k) is equal to

$$-\frac{1}{2}\left(\bar{r}_j\bar{F}_j+\bar{r}_k\bar{F}_k\right)=-\frac{1}{2}\left(\bar{r}_j-\bar{r}_k\right)\frac{\bar{r}_j-\bar{r}_k}{\bar{r}_{jk}}=-\frac{1}{2}\bar{r}_{jk}f\left(\bar{r}_{jk}\right),$$

from where the correctness of (4.116) follows.

If the gas occurs in a vessel of v volume, then the force affecting from the side on the gas of p pressure contributes to virial by $3/2pv$. In fact, the force is acting from the vessel side on an element da of the surface to $-p\bar{n}da$, where n is the unit vector of the outer normal. The contribution to virial here is

$$\frac{1}{2}p\int\limits_{(S)}\bar{n}\cdot\bar{r}\,da=\frac{1}{2}\mathrm{div}\,\bar{r}\,\mathrm{d}v=\frac{3}{2}pv,\tag{4.117}$$

where the Gauss theorem and equality $\mathrm{div}\,\bar{r}=3$ were used.

Let us show now that for classical nonideal gas of n particle volume at temperature T, the following expression is correct:

$$pv=nkT+\frac{1}{3}\sum_{1\le j<k\le n}r_{jk}f\left(r_{jk}\right).\tag{4.118}$$

In fact, applying expressions (4.115) and (4.117), we can write

$$C=\bar{k}=\frac{3}{2}nkT=\frac{3}{2}pv\ -\frac{1}{2}\sum_{1\le j<k\le n}r_{jk}f\left(r_{jk}\right),$$

from where the required result follows.

For the gas, where the energy and its potential result by interaction of gas particles, the virial theorem follows in the form

$$(u+2)\bar{k}=u\bar{E}+3pv.\tag{4.119}$$

Really, in this case

$$\sum_{1\le j<k\le n}r_{jk}f\left(r_{jk}\right)=-r_{jk}\left(\frac{\partial W}{\partial r_{jk}}\right)=-uU,$$

from where one has that

$$\bar{k}=\frac{3}{2}pv+\frac{1}{2}u\bar{U}.$$

Multiplying both parts of this expression by 2 and adding nk to both sides, expression (4.119) appears, where E is the total energy of the system equal to

$$E=T+U.$$

The generalized virial theorem derived in Sect. 3.4 or the Jacobi's virial equation, is valid for the considered physical system in the framework of statistical mechanics.

4.9 Universality of Jacobi's Virial Equation for Description of Dynamics of Natural Systems

It follows from the derivation of Jacobi's virial equation, where the linear forces and momentums were substituted by their volumetric values, that it appears to be a universal mathematical expression for the description of the dynamics of natural bodies in framework of the existing physical models their total, potential, and kinetic energy and the polar moment of inertia. As is seen, the body's energy and moment of inertia are in functional relationship and are changing by oscillating motion. Moreover, the second derivative of the moment of inertia $\ddot{\Phi}$ expresses the potential and kinetic energy of the body's interacted particles, which in fact is sought by Newton force function. This is a unique property of the virial equation (4.1).

At averaging of virial equation $\ddot{\Phi} = 2E + U$, when the first derivative from the system's moment of inertia Φ has a constant value ($\dot{\Phi} = 2E + U = \text{const.}$), it can represent the classical virial theorem like 2 $= U$ or $-U = 2$, which determines the condition of the hydrostatic equilibrium state.

The starting point for derivation of the virial theorem is the particle momentum. By Newton's definition, this value "is a certain measure determined proportionally to the velocity and the mass." This value is defined or it is found experimentally. All the other force parameters are obtained by transformation of the initial momentum, and those actions are explained by physical interaction of the mass particles, which are the carriers of the momentum. In fact, we recognize the momentum to be "innate" value, according to Newton's terminology, that is, the hereditary value. Under the "innate" value, Newton understood "both the resistance and the pressure of the mass" and finally the effect acquires its status of the inertial force. But the essence does not change, because the momentum appears together with the mass. Thus, the circle of the philosophical speculations is locked by the momentum, that is, by the mass and its oscillation. All other attributes of the motion are formed by mathematical transformations.

References

Bogolubov NN, Mitropolsky YA (1974) Asymptotic methods in the theory of non-linear oscillations. Nauka, Moscow

Duboshin GN, Rybakov AI, Kalinina EN, Kholopov PN (1971) Reports of Sternberg Astron. Institute. Moscow State University Publication, Moscow

Landau LD, Lifshitz EM (1973) Mechanics. Nauka, Moscow

Tolman RC (1969) Relativity, thermodynamics and cosmology. Clarendon, Oxford

Urey HC, Brickwedde FG, Murphy GM (1932) A hydrogen isotope of mass 2. Phys Rev 39:1645

Chapter 5
Solution of Jacobi's Virial Equation for Conservative and Dissipative Systems

Abstract It is shown in this chapter that Jacobi's virial equation provides, first of all, a solution for the models of natural systems that have explicit solutions in the framework of the classical many-body problem. A particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The parallel solutions for both the classical and dynamical approaches are given, and in doing so, we show that, with the dynamical approach, the solution acquires a new physical meaning, namely, oscillating motion. That solution appeared to be possible because of existing relationship of $|U|\sqrt{\Phi} = B = \text{const}$. It was also done for the solution of Jacobi's virial equation in hydrodynamics, in quantum mechanics for dissipative systems, for systems with friction, and in the framework of the theory of relativity.

The above solutions acquire a new physical meaning because the dynamics of a system is considered with respect to new parameters, that is, its Jacobi function (polar moment of inertia) and potential (kinetic) energy. The solution, with respect to the Jacobi function and the potential energy, identifies the evolutionary processes of the structure or redistribution of the mass density of the system. Moreover, the main difference of the two approaches is that the classical problem considers motion of a body in the outer central force field. The virial approach considers motion of a body both in the outer and in the own force field applying, instead of linear forces and moments, the volumetric forces (pressure) and moments (oscillations).

Finally, analytical solution of the generalized equation of perturbed virial oscillations in the form $\ddot{\Phi} = -A + B/\sqrt{\Phi} + X\left(t, \Phi, \dot{\Phi}\right)$ was done.

Derivation of the equation of dynamical equilibrium and its solution for conservative and dissipative systems shows that dynamics of celestial bodies in their own force field puts forward wide class of geophysical, astrophysical, and geodetic problems that can be solved by the methods of celestial mechanics and with new physical concepts we considered.

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_5,

In the previous chapter, we derived Jacobi's virial equation of dynamics and dynamical equilibrium in the framework of various physical models that are used for describing the motion of natural systems. We showed that, instead of the traditional description of such systems, like the Sun, planets, and satellites, based on hydrostatics, the problem of dynamics can be studied from more correct physical position, which appears to be dynamical equilibrium.

By transforming the linear forces and momentums into their volumetric values, we obtain the equation of dynamics of a celestial body applying the fundamental integral characteristics, namely, the energy and moment of inertia. Moreover, such a form of equation description does not depend on the choice of the reference system and becomes universal for solving dynamical problems in the framework of any physical models. In addition, the nature of the force field source becomes understood, which is the effect of interaction of the body's elementary particles expressed through the moment of inertia. In this case, we succeeded in restoring the kinetic energy lost at the hydrostatic approach.

The problem is now to find the general solution of Jacobi's virial equation relative to oscillation and rotation of a body and to apply the solution to study its dynamics, origin, and evolution. This application is valid for studying the Sun, the Earth, the Moon, and other celestial bodies.

In this chapter, we show that Jacobi's virial equation provides, first of all, a solution for the models of natural systems that have explicit solutions in the framework of the classical many-body problem. We shall give parallel solutions for both the classical and dynamical approaches, and in doing so, we shall show that, with the dynamical approach, the solution acquires a new physical meaning. We shall also consider a general case of the solution of Jacobi's virial equation for conservative and dissipative systems.

5.1 Solution of Kepler's Problem in Classical and Virial Approach

The many-body problem is known to be the key problem in classical mechanics and especially in celestial mechanics. A particular example of this is the unperturbed problem of Keplerian motion, when the system consists of only two material points interacting by Newtonian law. The explicit solution of the problem of unperturbed Keplerian motion permits the many-body problem to be solved with some approximation by varying arbitrary constants. In this case, the problem of dynamics, for example, that of the solar system, is transferred into the solution of the problem of dynamics of nine pairs of bodies in each of which one body is always the Sun and the second is each of the nine planets forming the system. Considering each planet–Sun subsystem, the influence of the other eight planets of the system is taken into account by introducing the perturbation function. By the virial approach, we can obtain for the Sun one characteristic period of circulation with respect to the center of mass of the system, which will not coincide with any period of the planets.

The dynamical approach evidences that the planet's orbital motion is performed by the central body, that is, by the Sun, by the energy of its outer force field, or by the field of the energy pressure. Each planet interacts with the solar force field by the energy of its own outer force field. The planet's orbit is the certain curve of its equilibrium motion that results from the two interacting fields of pressure. The planet's own oscillation and rotation perform by action of its inner fields of pressure.

Following these brief physical comments on the dynamical equilibrium motion of a planet, we now present two approaches of solving the Keplerian problem: the classical and the integral.

5.1.1 The Classical Approach

The traditional way of solving the unperturbed Keplerian problem is excellently described in the university courses for celestial mechanics found in Duboshin (1978). Here we present only the principal ideas. The method consists in transforming the two-body problem described by the system of Eq. (4.3) into the one-body problem using six integrals of motion of the center of mass (4.6). The system of equations obtained is of sixth order and expresses the change of barycentric coordinates of one point with respect to the center of mass of the system as a whole. Let us write it in the form

$$\begin{aligned} \ddot{x} &= -\frac{\mu x}{r^3}, \\ \ddot{y} &= -\frac{\mu y}{r^3}, \\ \ddot{z} &= -\frac{\mu z}{r^3}, \end{aligned} \tag{5.1}$$

where μ is the constant depending on the number of the point and for which the second point is equal to

$$\mu = \frac{G m_1^3}{(m_1 + m_2)^2}.$$

We then pass on from that Cartesian system of coordinates $OXYZ$ to orbital $\xi\eta\zeta$, using first integrals of the system of Eq. (5.1). Those are three integrals of the area,

$$\begin{aligned} y\dot{z} - z\dot{y} &= c_1, \\ z\dot{x} - x\dot{z} &= c_2, \\ x\dot{y} - y\dot{x} &= c_3, \end{aligned} \tag{5.2}$$

the energy integral,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2\mu}{r} + h, \tag{5.3}$$

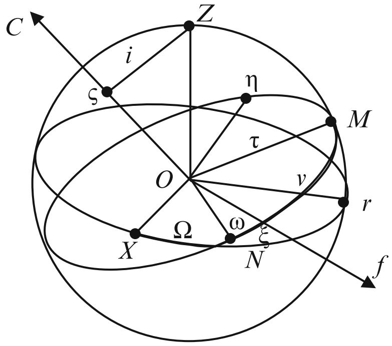


Fig. 5.1 Transition from Cartesian coordinate system *OXYZ* to orbital $\xi\eta\zeta$

and the Laplacian integrals,

$$
\begin{aligned}
-\frac{\mu x}{r} + c_3\dot{y} - c_2\dot{z} &= f_1,\\
-\frac{\mu y}{r} + c_1\dot{z} - c_3\dot{x} &= f_2,\\
-\frac{\mu z}{r} + c_2\dot{x} - c_1\dot{y} &= f_3.
\end{aligned}
\tag{5.4}
$$

As these seven integrals are not independent, we conclude that they cannot form a general solution of the system (5.1). In fact, there are two relations for these integrals:

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0,$$

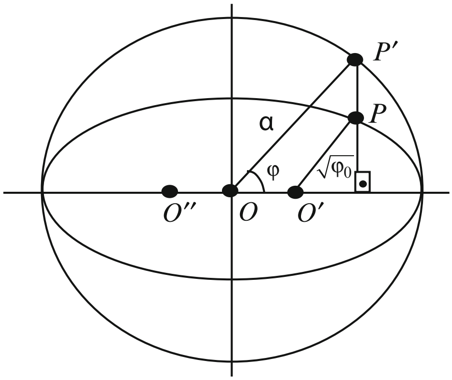
$$f_1^2 + f_2^2 + f_3^2 = \mu^2 + h\left(c_1^2 + c_2^2 + c_3^2\right),$$

showing that only five of them are independent. But the last integral needed can be found by simple quadrature. Using these integrals, we can pass on to the system of orbital coordinates $\xi\eta\zeta$ using the transformation relations (see Fig. 5.1):

$$
\begin{aligned}
\xi &= \frac{f_1}{f}x + \frac{f_2}{f}y + \frac{f_3}{f}z,\\
\eta &= \frac{C_2 f_3 - C_3 f_2}{Cf}x + \frac{C_3 f_1 - C_1 f_2}{Cf}y + \frac{C_1 f_2 - C_3 f_1}{Cf}z,\\
\zeta &= \frac{C_1}{C}x + \frac{C_2}{C}y + \frac{C_3}{C}z.
\end{aligned}
\tag{5.5}
$$

The equation of the curve along which the point moves in accordance with (5.1) has the simplest form in the system of initial coordinates. The equation is

Fig. 5.2 Relationship between the polar and the rectangular coordinates



$$\zeta = 0,$$

$$\mu r = C^2 - f\xi. \tag{5.6}$$

Finally, introducing the polar orbital coordinates r and v, which are related to the rectangular orbital coordinates ξ and η by the expressions (see Fig. 5.2)

$$\xi = r\cos\nu$$

and

$$\eta = r\sin\nu$$

and using the integral of areas

$$r^2 v = C,$$

we come to the equation

$$C(t - r) = \left(\frac{C^2}{\mu}\right)^2 \int_0^{\nu} \frac{d\nu}{\left(1 + \frac{f}{\mu}\cos\nu\right)^2}. \tag{5.7}$$

The solution of Eq. (5.7) gives the change of function v with respect to time. Repetition of the transformation in the reverse order leads to solution of the problem. In doing this, we obtain the expression for the change of coordinates of the material point with respect to the initial data $\xi_{10}, \eta_{10}, \zeta_{10}, \xi_{20}, \eta_{20}, \zeta_{20}, \dot{\xi}_{10}, \dot{\eta}_{10}, \dot{\zeta}_{10}, \dot{\xi}_{20}, \dot{\eta}_{20}$, and $\dot{\zeta}_{20}$. It is remarkable that if the total energy (5.3) has a negative value, then the solution of Eq. (5.7) leads to the Keplerian equation

$$E - e\sin E = n(t - \tau), \tag{5.8}$$

where the function v is related to the variable $\ '$ by the expression

$$tg\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}tg\frac{E}{2},$$

$$e = \frac{f}{\mu}, \quad n = \frac{\sqrt{\mu}}{a^{3/2}}, \quad p = \frac{C^2}{\mu} = a\left(1-e^2\right).$$

Because energy by definition is the property to do work (motion) and can be only a positive value, then the physical meaning of negative total energy, which defines the elliptic orbit of a body moving in the central field of the two-body problem, should be revealed. In the presented solution of the two-body problem, the left-hand side of the energy integral (5.3) expresses the kinetic energy, and the right-hand side means the potential energy of the mass interaction. The integral of energy (5.3) as a whole, in the coordinates and in the velocities, represents the averaged virial theorem, where the potential energy has formally a negative value. Here the physical meaning of the total energy determination consists in comparison of magnitude of the potential and kinetic energies. A negative value of the total energy means that the potential energy exceeds the kinetic one by that value. As it follows from the analysis of the inner force field of a self-gravitating body presented in Chaps. 2 and 3, the potential energy exceeds the kinetic energy only in the case of nonuniform distribution of the mass density and cannot be less than that. In the case of equality of both energies, the total potential energy is realized into oscillating motion. The excess of the potential energy is used for rotation of the masses and in the dissipation. The last case is discussed later.

5.1.2 The Dynamic Approach

Let us consider the solution of the problem of unperturbed motion of two material points on the basis of Jacobi's virial equation, which in accordance with Eq. (4.16) is written in the form

$$\ddot{\Phi}_0 = 2E_0 - U,$$

where $\ _0 = \ _0 + U = \text{const.}$ is the total energy of the system in a barycentric coordinate system.

The Jacobi function Φ_0 is expressed by (4.15)

$$\ddot{\Phi}_0 = \frac{m_1 m_2}{2\left(m_1+m_2\right)}\left[\left(\xi_1-\xi_2\right)^2 + \left(\eta_1-\eta_2\right)^2 + \left(\zeta_1-\zeta_2\right)^2\right],$$

and the potential energy U in accordance with (4.2) is

$$U = \frac{G m_1 m_2}{\sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}}.$$

It is easy to see that between the Jacobi function Φ_0 and the potential energy U, the relationship exists in the form

$$|U|\sqrt{\Phi} = \frac{G(m_1 m_2)^{3/2}}{\sqrt{2\,(m_1 m_2)}} = \frac{G}{\sqrt{2}} m\mu^{3/2} = B = \text{const.}, \tag{5.9}$$

where μ is the generalized mass of the two bodies, m is the total mass of the system, and is a constant value.

The relationship (5.9) is remarkable because it is independent of the initial data, that is, of its coordinates and velocities. Being an integral characteristic of the system and dependent only on the total mass and the generalized mass of the two points, the relationship permits Jacobi's virial equation to be transformed to an equation with one variable as follows:

$$\ddot{\Phi}_0 = 2E_0 + \frac{B}{\sqrt{\Phi_0}}. \tag{5.10}$$

We consider the solution of Eq. (5.10) for the case when total energy $_0$ has a negative value. Introducing $= -2\ _0 > 0$, Eq. (5.10) can be rewritten as

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}. \tag{5.11}$$

We apply the method of change of variable for the solution of Eq. (5.11) and show that the partial solution of two linear equations (Ferronsky et al. 1984),

$$\left(\sqrt{\Phi_0}\right)'' + \sqrt{\Phi_0} = \frac{B}{A}, \tag{5.12}$$

$$t'' + t = \frac{4B\lambda}{\left(\sqrt{2A}\right)}, \tag{5.13}$$

which include only two integration constants, is also the solution of Eq. (5.11).

We now introduce the independent variable λ into Eqs. (5.12) and (5.13), where primes denote differentiation with respect to λ. Note that time here is not an independent variable. This allows us to search for the solution of two linear equations instead of solving one nonlinear equation. The solution of Eqs. (5.12) and (5.13) can be written in the form

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 - \varepsilon\cos(\lambda - \psi)\right], \tag{5.14}$$

$$t = \frac{4B}{(2A)^{3/2}} \left[1 - \varepsilon \sin\, (\lambda - \psi)\right]. \tag{5.15}$$

Let us prove that the partial solutions (5.14) and (5.15) of differential equations (5.12) and (5.13) are the solution of Eq. (5.10), which is sought. For this purpose, we express the first and second derivatives of the function $\sqrt{\Phi_0}$ with respect to λ through corresponding derivatives with respect to time using Eq. (5.15). From (4.15), it follows that

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{4B}{(2A)^{3/2}} \left[1 - \varepsilon \sin (\lambda - \psi)\right]. \tag{5.16}$$

We can replace the right-hand side of the obtained relationship by $\sqrt{\Phi_0}$ from (5.14)

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \sqrt{\Phi_0}\sqrt{\frac{2}{A}}. \tag{5.17}$$

Transforming the derivative from $\sqrt{\Phi_0}$ with respect to λ into the form

$$\frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}\lambda} = \frac{\mathrm{d}\sqrt{\Phi_0}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\dot{\Phi}_0}{2\sqrt{\Phi_0}}\frac{\mathrm{d}t}{\mathrm{d}\lambda}$$

and taking into account (5.17), we can write

$$\left(\sqrt{\Phi_0}\right)' = \frac{\dot{\Phi}_0}{\sqrt{2A}}.$$

The second derivative can be written analogously:

$$\left(\sqrt{\Phi_0}\right)'' = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\Phi}_0}{\sqrt{2A}}\right) = \frac{\ddot{\Phi}_0}{\sqrt{2A}}\sqrt{\Phi_0}\sqrt{\frac{2}{A}} = \frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A}. \tag{5.18}$$

Putting Eq. (5.18) into (5.12), we obtain

$$\frac{\ddot{\Phi}_0\sqrt{\Phi_0}}{A} + \sqrt{\Phi_0} = \frac{B}{A}.$$

Dividing the above expression by $\sqrt{\Phi_0}\,/A$, we can finally write

$$\ddot{\Phi}_0 = -A + \frac{B}{\sqrt{\Phi_0}}.$$

This shows that the partial solution of the two linear differential equations (5.12) and (5.13) appears to be the solution of the nonlinear equation (5.11).

5.2 Solution of n-Body Problem in the Framework of Conservative System

After solving Jacobi's virial equation for the unperturbed two-body problem, we come to the dynamics of a system of n material particles where $n \to \infty$.

Let us assume that an external observer studying the dynamics of a system of n particles in the framework of classical mechanics has the following information. He has the mass of the system and its total and potential energies, and can determine the Jacobi function and its first derivative with respect to time in any arbitrary moment. Then he can use Jacobi's virial equation (4.9) and, making only the assumption needed for its solution that $|U|\sqrt{\Phi} = B = \text{const.}$, may predict the dynamics of the system, that is, the dynamics of its integral characteristics at any moment of time. The assumption $|U|\sqrt{\Phi_0} = \text{const.}$ will be considered separately in Chap. 9.

If the total energy $\ _0$ of the system has a negative value, the external observer can immediately write the solution of the problem of the Jacobi function change with respect to time in the form of (5.14) and (5.15)

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 - \varepsilon\cos(\lambda - \psi)\right],$$

$$t = \frac{4B}{(2A)^{3/2}}\left[1 - \varepsilon\sin(\lambda - \psi)\right],$$

where $\ = -2\ _0$ and ε and ψ are constants depending on the initial values of the Jacobi function Φ_0 and its first derivative $\dot{\Phi}_0$ at the moment of time t_0.

Let us obtain the values of constants ε and ψ in explicit form expressed through the values Φ_0 and $\dot{\Phi}_0$ at the initial moment of time t_0. For convenience, we introduce a new independent variable φ connected to λ by the relationship $\lambda - \psi = \varphi$. Then, Eqs. (5.14) and (5.15) can be rewritten as

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 - \varepsilon\cos\right], \tag{5.19}$$

$$t - \frac{4B}{(2A)^{3/2}}\psi = \frac{4B}{(2A)^{3/2}}\left[\varphi - \varepsilon\sin\varphi\right]. \tag{5.20}$$

Using Eq. (5.19), we write the expression for φ:

$$\varphi = \arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}, \tag{5.21}$$

and taking into account the equality

$$\frac{d\sqrt{\Phi_0}}{d\lambda} = \frac{d\sqrt{\Phi_0}}{d\varphi},$$

substitute Eq. (5.21) into the expression

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon \sin \varphi.$$

The last equation can be finally rewritten in the form

$$\frac{\dot{\Phi}_0}{\sqrt{2A}} = \frac{B}{A}\varepsilon \sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}. \tag{5.22}$$

Equation (5.22) allows us to determine the first constant of integration ε as a function of the initial data Φ_0 and $\dot{\Phi}_0$ at $t = t_0$. Solving Eq. (5.22) with respect to ε after simple algebraic transformation, we obtain

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}\,|_{t=t_0} = \text{const.} \tag{5.23}$$

The second constant of integration ψ can be expressed through the initial data after solving Eq. (5.20) with respect to ψ and change of value φ by its expression from Eq. (5.21). Defining

$$t - \frac{4B}{(2A)^{3/2}}\psi = \tau,$$

we obtain

$$-\tau \left\{\frac{4B}{(2A)^{2/3}}\left[\arccos\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon\varepsilon\sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon}\right)^2}\right] - t\right\}|_{t=t_0} = \text{const.} \tag{5.24}$$

The physical meaning of the integration constants ε, τ, and the parameter $T_v = 8\pi B \big/ (2A)^{3/2}$ can be understood after the definitions

$$T_v = \frac{8\pi B}{(2A)^{3/2}},$$

$$n = \frac{2\pi}{T_v} = \frac{(2A)^{3/2}}{4B},$$

$$a = \frac{B}{A}$$

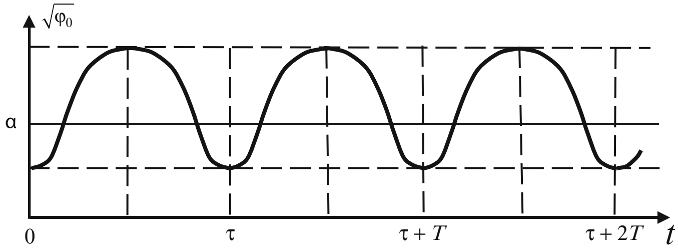


Fig. 5.3 Changes of the Jacobi function over time

and rewriting Eqs. (5.19) and (5.20) in the form

$$\sqrt{\Phi_0} = a\left[\varphi - \varepsilon \sin\varphi\right], \tag{5.25}$$

$$M = \varphi - \varepsilon \sin\varphi, \tag{5.26}$$

where $M = n(t-\tau)$.

The value $\sqrt{\Phi_0}$ draws an ellipse during the period of time $_{\rm v} = 8\pi B/(2A)^{3/2}$ (see Fig. 5.3). The ellipse is characterized by a semimajor axis equal to / and by the eccentricity ε, which is defined by expression (5.23). In the case considered where $E_0 < 0$, the value ε is changed in time from 0 to 1. The value τ characterizes the moment of time when the ellipse passes the pericenter.

Let us obtain explicit expressions with respect to time for the functions $\sqrt{\Phi_0}$, Φ_0, and $\dot{\Phi}_0$. For this purpose, we write Eq. (5.24) in the form of a Lagrangian:

$$F(\varphi) = \varphi - \varepsilon \sin\varphi - M = 0. \tag{5.27}$$

It is known (Duboshin 1978) that by application of Lagrangian formulas, we can write, in the form of a series, the expressions for the root of the Lagrange equation (5.27) and for the arbitrary function *f*, which is dependent on φ:

$$\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[\sin^k M\right] = M + \varepsilon \sin M + \frac{\varepsilon^2}{1\cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[\sin^2 M\right] + \cdots, \tag{5.28}$$

$$f(\varphi) = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[f'(M)\sin^k M\right] = f(M) + \varepsilon f'(M)\sin M + \frac{\varepsilon^2}{1\cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[f(M)\sin^2 M\right] + \cdots \tag{5.29}$$

Using Eq. (5.29), we write expressions for $\cos\varphi$, $\cos^2\varphi$, and $\sin\varphi$ in the form of a Lagrangian series of parameter ε power:

$$\cos\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[(-1)\sin M \sin^k M\right] = \cos M + \varepsilon\,(-1)\sin M \sin(M)$$
$$+ \frac{\varepsilon^2}{1\cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[(-1)\sin(M)\sin^2 M\right] + \cdots = \cos M - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\cos 2M$$
$$- \frac{3}{4}\varepsilon^3 \cos M + \frac{3}{8}\varepsilon^2 \cos 3M + \cdots \tag{5.30}$$

$$\cos^2\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[(-2)\sin M \cos M \sin^k M\right] = \cos^2 M$$
$$+\varepsilon\,(-2)\sin M \cos M \sin M + \frac{\varepsilon^2}{1\cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[(-2)\sin M \cos M \sin^2 M\right] + \cdots$$
$$= \cos^2 M - 2\varepsilon \sin^2 M \cos M + \frac{\varepsilon^2}{2}(-2)\left(3\sin^2 M \cos^2 M - \sin^4 M\right) + \cdots \tag{5.31}$$

$$\sin\varphi = \sum_{k=0}^{\infty} \frac{\varepsilon^{k-1}}{k!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}M^{k-1}} \left[\cos M \sin M\right] = \sin M + \varepsilon \cos M \sin M$$
$$+ \frac{\varepsilon^2}{1\cdot 2} \frac{\mathrm{d}}{\mathrm{d}M} \left[\cos M \sin^2 M\right] + \cdots = \sin M + \varepsilon \cos M \sin M$$
$$+ \frac{\varepsilon^2}{1\cdot 2} \left[2\sin M \cos^2 M - \sin^3 M\right] + \cdots \tag{5.32}$$

We write the expressions for $\sqrt{\Phi_0}$, Φ_0, and $\dot{\Phi}_0$ using Eqs. (5.25) and (5.26) in the form

$$\sqrt{\Phi_0} = a\,(1 - \varepsilon\cos\varphi)\,, \tag{5.33}$$

$$\Phi_0 = a^2\left(1 - 2\varepsilon\cos\varphi + \varepsilon^2\cos^2\varphi\right), \tag{5.34}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\varepsilon B \sin\varphi. \tag{5.35}$$

Substituting into (5.33), (5.34), and (5.35) the expressions for $\cos\varphi$, $\cos^2\varphi$, and $\sin\varphi$ in the form of the Lagrangian series (5.30), (5.31), and (5.32), we obtain

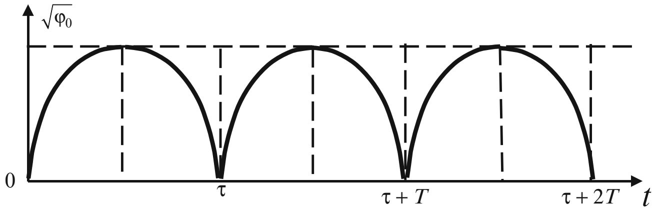


Fig. 5.4 Changes of the value $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{3}{8}\varepsilon^3\cos 3M + \cdots\right], \tag{5.36}$$

$$\Phi_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right], \tag{5.37}$$

$$\dot{\Phi}_0 = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M + \frac{1}{2}\varepsilon\sin 2M + \frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M - \sin^2 M\right) + \cdots\right]. \tag{5.38}$$

The series of Eqs. (5.36), (5.37), and (5.38) obtained are put in the order of increased power of parameter ε and are absolutely convergent at any value of in the case when the parameter ε satisfies the condition

$$\varepsilon < \bar{\varepsilon} = 0.6627\ldots, \tag{5.39}$$

where $\bar{\varepsilon}$ is the Laplace limit.

In some cases, it is convenient to expand the values $\sqrt{\Phi_0}$, Φ_0, and $\dot{\Phi}_0$ in the form of a Fourier series, using conventional methods (see, e.g., Duboshin 1978). Figure 5.4 demonstrates the changes of $\sqrt{\Phi_0}$ in time at $\varepsilon = 1$.

It is also possible to consider the case solution of Jacobi's virial equation for $_0 = 0$ and $_0 > 0$. Readers can find here without difficulty a full analogy of these results as well as the solution of the two-body problem.

5.3 Solution of Jacobi's Virial Equation in Hydrodynamics

Let us consider the solution of the problem of dynamics of a homogeneous isotropic gravitating sphere in the framework of traditional hydrodynamics and the virial approach we have developed.

5.3.1 The Hydrodynamic Approach

The sphere is assumed to have radius R_0 and be filled by an ideal gas with a density ρ_0. We assume that at the initial time the field of velocities that has the only component is described by equation

$$u = H_0 r, \tag{5.40}$$

where u is the radial component of the velocity of the sphere's matter at a distance r from the center of mass and H is independent of the quantity r and equal to H_0 at time t_0.

We also assume that the motion of the matter of the sphere goes on only under the action of the forces of mutual gravitational interaction between the sphere particles. In this case, the influence of the pressure gradient is not taken into account, assuming that the matter of the sphere is sufficiently diffused. Then, the symmetric spherical shells will move only under forces of gravitational attraction and will not coincide. In this case, the mass of the matter of any sphere shell will keep its constant value, and the condition (5.40) will be satisfied at any moment of time, and constant should be dependent on time.

Under those conditions, the Eulerian system of Eq. (4.28) can be written in the form

$$\rho \frac{\partial u}{\partial t} + \rho\,(u\nabla)\,u = \rho \frac{\partial U_{\mathrm{G}}}{\partial r}, \tag{5.41}$$

where $\rho(t)$ is the density of the matter of the sphere at the moment of time t, u is the radial component of the velocity of matter at distance r from the sphere's center, and U_{G} is the Newtonian potential for the considered point of the sphere.

The expression for the Newtonian potential U_{G} (4.29) can be written as follows:

$$U_{\mathrm{G}} = G \frac{4}{3} \pi \rho r^2, \tag{5.42}$$

and the continuity equation will be

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial U}{\partial r} = 0. \tag{5.43}$$

Within the framework of the traditional approach, the problem is to define the sphere radius R and the value of the constant at any moment of time, if the radius R_0, the density ρ_0, and the value of the constant $_0$ at the initial moment of time t_0 are given. If we know the values (t) and $R(t)$, we can then obtain the field of velocities of the matter within the sphere that is defined by Eq. (5.40) and also the density ρ of matter at any moment of time using the relationship

$$\frac{4}{3} \pi R_0^3 \rho_0 = \frac{4}{3} \pi R^3 \rho = m.$$

Hence, the formulated problem is reduced to identification of the law of motion of a particle that is on the surface of the sphere and within the field of attraction of the entire sphere mass $m = 4/3\pi\ \rho_0 R_0{}^3$.

The equation of motion for a particle on the surface of the sphere, which follows from Eq. (5.41) after transforming the Eulerian coordinates into a Lagrangian, has the form

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -G\frac{m}{R^2}. \tag{5.44}$$

It is necessary to determine the law of change of $R(t)$, resolving Eq. (5.44) at the initial data:

$$\begin{aligned} R\,(t_0) &= R_0, \\ \left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t=t_0} &= H_0 R_0. \end{aligned} \tag{5.45}$$

We reduce the order of Eq. (5.44). To do so, we multiply it by $\mathrm{d}R/\mathrm{d}t$

$$\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{\mathrm{d}R}{\mathrm{d}t}\frac{Gm}{R^2}$$

and integrate with respect to time:

$$\int_{t_0}^{t}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{R}\right)^2 = \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Gm}{R}\right)\mathrm{d}t.$$

After integration, we obtain

$$\frac{1}{2}\dot{R}^2 - \frac{1}{2}\dot{R}_0^2 = \frac{Gm}{R} - \frac{Gm}{R_0}$$

or

$$\frac{1}{2}\dot{R}^2 = \frac{Gm}{R} + k, \tag{5.46}$$

where the constant k is determined as

$$\begin{aligned} k &= \frac{1}{2}\dot{R}_0^2 - \frac{Gm}{R_0} = \frac{1}{2}H_0^2 R_0^2 - G\frac{4\pi}{3}\rho_0\frac{R_0^3}{R_0} \\ &= \frac{1}{2}H_0^2 R_0^2\left[1 - \frac{8\pi}{3}\frac{G\rho_0}{H_0^2}\right] = \frac{1}{2}H_0^2 R_0^2\,[1 - \Omega] = \text{const}. \end{aligned} \tag{5.47}$$

Here the quantity $\Omega = \rho_o/\rho_{cr}$, where $\rho_{cr} = 3H_0{}^2/8\pi G$.

Note that Eq. (5.46) obtained after reduction of the order of the initial equation (5.44) is in its substance the energy conservation law. Equation (5.46) permits the variables to be divided and can be rewritten in the form

$$\int_{R_0}^{R} \frac{\mathrm{d}R}{\sqrt{\frac{2Gm}{R} + 2k}} = \int_{t_0}^{t} \mathrm{d}t. \tag{5.48}$$

The plus sign before the root is chosen assuming that the sphere at the initial time is expanding, that is, $\ _0 > 0$.

The differential equation (5.46) has three different solutions at $k = 0$, $k > 0$, and $k < 0$ depending on the sign of the constant k, which is in its turn defined by the value of the parameter Ω at the initial moment of time. First, we consider the case when $k = 0$, which relates, by analogy with the Keplerian problem, to the parabolic model at $k = 0$. Equation (5.46) is easily integrated, and for the expression case, that is, $\dot{R} > 0$, we obtain

$$\dot{R}^2 = \frac{2Gm}{R},$$

$$\dot{R} = \frac{(2Gm)^{1/2}}{R^{1/2}},$$

from which it follows that

$$R^{1/2}\mathrm{d}R = (2Gm)^{1/2}\mathrm{d}t$$

or

$$\frac{2}{3}R^{3/2} = (2Gm)^{1/2}t + \text{const.} \tag{5.49}$$

We choose as initial counting time $t = 0$, the moment when $R = 0$. In this case, the integration constant disappears:

$$R = \left(\frac{9}{2}Gm\right)^{1/3} t^{2/3}. \tag{5.50}$$

The density of the matter changes in accordance with the law

$$\rho(t) = \frac{m}{\frac{4}{3}\pi R^3} = \frac{1}{6\pi G t^2}, \tag{5.51}$$

and the quantity $\ (t)$, as a consequence of (5.50), has the form

$$H(t) = \frac{\dot{R}}{R} = \frac{2}{3}\frac{1}{t}. \tag{5.52}$$

For the case when $k > 0$, which corresponds to so-called hyperbolic motion, the solution of Eq. (5.46) can be written in parametric form (Zeldovich and Novikov 1967):

$$\begin{aligned} R &= \frac{Gm}{2k}\,(ch\eta - 1)\,, \\ t &= \frac{Gm}{(2k)^{3/2}}\,(sh\eta - \eta)\,, \end{aligned} \tag{5.53}$$

where the constants of integration in (5.53) have been chosen so that $t = 0$, $\eta = 0$ at $R = 0$.

Finally, we consider the case when $k < 0$, which corresponds to elliptic motion. At $k < 0$, the expansion of the sphere cannot continue for unlimited time and the expansion phase should be changed by attraction of the sphere.

The explicit solution of Eq. (5.46) at $k < 0$ can be written in parametric form (Zeldovich and Novikov 1967):

$$\begin{aligned} R &= \frac{Gm}{2|k|}\,(1 - ch\eta)\,, \\ t &= \frac{Gm}{(2|k|)^{3/2}}\,(\eta - sh\eta)\,. \end{aligned} \tag{5.54}$$

The maximum radius of the sphere is determined from Eq. (5.46) on the condition $\mathrm{d}R/\mathrm{d}t = 0$ and equals

$$R_{max} = \frac{Gm}{|E|}. \tag{5.55}$$

The time needed for expansion of the sphere from $R_0 = 0$ at $t_0 = 0$ to $R_{\max}$ is

$$t_{max} = \frac{\pi Gm}{(2|k|)^{3/2}}. \tag{5.56}$$

So the sphere should make periodic pulsations with period equal to

$$T_{\mathrm{p}} = \frac{2\pi Gm}{(2|k|)^{3/2}}. \tag{5.57}$$

The considered solution has important cosmologic applications.

5.3.2 *The Virial Approach*

We shall limit ourselves by formal consideration of the same problem in the framework of the condition of the dynamical equilibrium of a self-gravitating body based on the solution of Jacobi's virial equation, which we discussed earlier.

As shown in Chap. 4, Jacobi's virial equation (4.50), derived from Eulerian equations (4.28), is valid for the considered gravitating sphere. It was written in the form

$$\ddot{\Phi} = 2E - U, \tag{5.58}$$

where Φ is the Jacobi function for a homogeneous isotropic sphere and is defined by

$$\Phi = \frac{1}{2}\int_0^R 4\pi r^2 \rho r^2 \, \mathrm{d}r = \frac{2\pi\rho R^5}{5} = \frac{3}{10} m R^2. \tag{5.59}$$

The potential gravitational energy of the matter of the sphere is expressed as

$$U = -4\pi G \int_0^R r\rho(r)\, m(r)\, \mathrm{d}r = -\frac{16\pi^2}{15} G\rho^2 R^2 = -\frac{3}{5} G \frac{m^2}{R}. \tag{5.60}$$

The total energy of the sphere will be equal to the sum of the potential U and kinetic energies.

The kinetic energy is expressed as

$$T = \frac{1}{2}\int_0^R 4\pi u^2 \rho r^2 \, \mathrm{d}r = \frac{1}{2}\int_0^R 4\pi H^2 r^2 \rho r^2 \, \mathrm{d}r = \frac{4\pi\rho H^2 R^5}{10} = \frac{3}{10} m H^2 R^2. \tag{5.61}$$

For a homogeneous isotropic gravitating sphere, the constancy of the relationship between the Jacobi function (5.59) and the potential energy (5.60) can be written as

$$|U| \sqrt{\Phi} = B = \frac{3}{5} G \frac{m^2}{R} \sqrt{\frac{3}{10} m R^2} = \frac{1}{\sqrt{2}} \left(\frac{3}{5}\right)^{3/2} G m^{3/2}, \tag{5.62}$$

where has constant value because of the conservation law of mass m of the considered sphere.

The total energy of the sphere also has a constant value:

$$E = T + U = \frac{A}{2}. \tag{5.63}$$

Then, if the total energy of the sphere has a negative value, Jacobi's virial equation can be written in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \tag{5.64}$$

Let us consider the conditions under which the total energy of the system will have a negative value. For this purpose, we write it explicitly:

$$E = T + U = -\frac{16}{15}\pi^2 G\rho^2 R^5 + \frac{2\pi\rho H^2 R^5}{5} = \frac{2}{5}\pi\rho H^2 R^5 \left[1 - \frac{8\pi G\rho}{3H^2}\right]. \tag{5.65}$$

It is clear from Eq. (5.65) that the total energy has a negative value, when $\rho > \rho_c$, where $\rho_c = 3\ \ ^2/8\pi G$.

The general solution of Eq. (5.64) has the form of Eqs. (5.14) and (5.15):

$$\sqrt{\Phi_0} = \frac{B}{A}\left[1 - \varepsilon\cos\left(\lambda - \psi\right)\right], \tag{5.66}$$

$$t = \frac{4B}{(2A)^{3/2}}\left[\lambda - \varepsilon\sin\left(\lambda - \psi\right)\right], \tag{5.67}$$

where ε and ψ are constants dependent on the initial values of the Jacobi function Φ_0 and its first derivative Φ_0 at the moment of time t_0. The constants ε and ψ are determined by Eqs. (5.23) and (5.24) accordingly.

If we express all the constants in Eq. (5.23)

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2}\left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0} - 2A\Phi_0\right)}\,|_{t=t_0} \tag{5.68}$$

through mass m of the system, it is not difficult to see that

$$-\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0} - 2A\Phi_0 = 0.$$

Then the constant ε will be equal to zero. Hence, the solutions (5.28) and (5.29) coincide with the solution (5.54), which was obtained in the framework of the traditional hydrodynamic approach. In this case, the period of eigenpulsations of the Jacobi function (the polar moment of inertia) of the sphere $\ \ = 8\pi R/(2A)^{3/2}$ will be equal to the period of change of its radius $\ \ = 2\pi Gm/(2|k|)^{3/2}$ obtained from Eq. (5.54).

5.4 The Hydrogen Atom as a Quantum Mechanical Analogue of the Two-Body Problem

Let us consider the problem concerning the energy spectrum of the hydrogen atom, which is a unique example of the complete conformity of the analytical solution with experimental results. The problem consists of a study of all forms of motion using the postulates of quantum mechanics and based on the solution of Jacobi's virial equation.

The classical Hamiltonian in the two-body problem is written as

$$H = \frac{p_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} + U\left(|\bar{r}_1 - \bar{r}_2|\right), \tag{5.69}$$

where

$$\bar{p}_1 = \frac{\partial H}{\partial \dot{\mathbf{r}}_1} = m_1 \dot{\bar{r}}_1,$$

$$\bar{p}_2 = \frac{\partial H}{\partial \dot{\mathbf{r}}_2} = m_2 \dot{\bar{r}}_2,$$

which, after separation of the center of mass, can be transformed into the form

$$H = \frac{\bar{P}^2}{2M} + \frac{\bar{p}^2}{2m} + U(r), \tag{5.70}$$

where $r = |\bar{r}_1 - \bar{r}_2|$ is the distance between two particles and

$$\bar{P} = M\dot{\bar{R}}, \quad \bar{p} = m\dot{\bar{r}}, \quad M = m_1 + m_2,$$

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}, \quad m = \frac{m_1 m_2}{m_1 + m_2}.$$

We obtain the Hamiltonian operator for the quantum mechanical two-body problem through changing the pulses and radii by the corresponding operators with the communication relations

$$[\hat{p}_\mathrm{i}, \hat{p}_\mathrm{k}] = -i\hbar\delta_\mathrm{ik},$$

$$[\hat{p}_\mathrm{i}, \hat{r}_\mathrm{k}] = -i\hbar\delta_\mathrm{ik}.$$

Then,

$$\hat{H} = -\frac{\hbar^2}{2M}\Delta_R - \frac{\hbar^2}{2m}\Delta_r + \hat{U}(r).$$

The wave function $u(\bar{r}_1, \bar{r}_2) = \varphi(\bar{R})\psi(\bar{r})$, which satisfies the Schrödinger equation,

$$\hat{H}u = \varepsilon u,$$

describes the motion of the inertia center (free motion of the particle of mass m_c is described by the function $\varphi(R)$, and the motion of the particle of mass m in the $U(r)$ is described by the wave function $\Psi(\bar{r})$). Subsequently, we consider only the wave function of the motion of particle m.

The Schrödinger equation,

$$\Delta\Psi + \frac{2m}{\hbar^2}\left[E - U(r)\right]\Psi = 0$$

written here for the stationary state in a central symmetrical field in spherical coordinates, has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial\Psi}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2\Psi}{\partial\varphi^2}\right] + \frac{2\text{m}}{\hbar^2}\left[E - U(r)\right]\psi = 0. \tag{5.71}$$

Using the Laplacian operator $\hat{\ell}^2$,

$$\hat{\ell}^2 = \left[\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\frac{\partial}{\partial\Theta}\right) + \frac{1}{\sin^2\Theta}\frac{\partial^2}{\partial\varphi^2}\right],$$

we obtain

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\hat{\ell}^2}{r^2}\Psi\right] + U(r)\Psi = E\Psi.$$

The operators $\hat{\ell}^2$ and $\hat{\ell}_z$ ($\hat{\ell}_z = -i\,\partial/\partial\varphi$) commutate with the Hamiltonian $\hat{H}(r)$, and therefore, there are common eigenfunctions of the operators $\hat{H}$, $\hat{\ell}^2$, and $\hat{\ell}_z$. We consider only such solutions of Schrödinger equations. This condition determines the dependence of the function Ψ on the angles

$$\Psi(r, \Theta, \varphi) = R(r)Y_{\ell k}(\Theta, \varphi),$$

where the quantity $Y_{\ell k}(\Theta, \varphi)$ is determined by the expression

$$Y_{\ell k}(\Theta, \varphi) = \frac{1}{\sqrt{2\pi}}e^{ik\varphi}(-1)^k i^\ell\sqrt{\frac{(2\ell+1)(\ell-k)!}{2(\ell+k)!}}P_\ell^k(\cos\Theta),$$

and $P_\ell^k(\cos\Theta)$ is the associated Legendre polynomial, which is

$$P_\ell^k(\cos\Theta) = \frac{1}{2^\ell \ell!}\sin^k\Theta \frac{\mathrm{d}^{r+\ell}}{\mathrm{d}\cos\Theta^{r+\ell}}\left(\cos^2\Theta - 1\right)^\ell.$$

Since

$$\hat{\ell}^2 Y_{\ell k} = \ell(\ell+1) Y_{\ell k},$$

we obtain for the radial part of the wave function $R(r)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}\left[E - U(r)\right]R = 0. \tag{5.72}$$

Equation (5.72) does not contain the value $\ell_z = m$, that is, at the given ℓ, the energy level corresponds to $2\ell + 1$ states differing by the value ℓ_z.

The operator

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right)$$

is equivalent to the expression

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR),$$

and, thus, it is convenient to make the change of variables, assuming that

$$X(r) = rR(r).$$

So that Eq. (5.71) can be rewritten in the form

$$\frac{\mathrm{d}^2X}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2}X + \frac{2m}{\hbar^2}\left[E - U(r)\right]X = 0. \tag{5.73}$$

We now consider the demand following from the boundary conditions and related to the behavior of the wave function $X(r)$. At $r \to 0$ and the potentials satisfying the condition

$$\lim_{r\to 0} U(r)r^2 = 0, \tag{5.74}$$

only the first two terms play an important role in Eq. (5.73). $X(r) \sim r^{\mathrm{v}}$ and we obtain

$$v(v-1) = \ell(\ell+1).$$

This equation has roots $v_1 = \ell + 1$ and $v_2 = -\ell$

The requirement of normalization of the wave function is incompatible with the values $v = -\ell$ at $\ell \neq 0$ because the normalization integral

$$\int_0^\infty |X_r^2(r)\mathrm{d}r|$$

will be divergent for the discrete spectrum, and the condition

$$\int \Psi(\lambda, \xi)\ \Psi(\lambda, X)\ \mathrm{d}\lambda = \delta(X - \xi)$$

does not hold for the continuous spectrum.

At $\ell = 0$, the boundary conditions are determined by the demand for the finiteness of the mean value of the kinetic energy, which is satisfied only at $v = 1$. So when the condition (5.74) is satisfied, the wave function of a particle is everywhere finite and at any ℓ

$$X(0) = 0.$$

Let us consider the energy spectrum and the wave function of the bounded states of a system of two charges. The bounded states exist only in the case of the attracted particles. Such a system defines the properties of the hydrogen atom and hydrogen-like ions.

The equation for the radial wave function is

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}R + \frac{2m}{\hbar^2}\left(E + \frac{\alpha}{r}\right)R = 0, \tag{5.75}$$

where $\alpha = Ze^2$ is a constant, characterizing the potential, is the electron charge, and Z is the whole number equal to the nucleus charge in the charge units.

The constants e^2, m, and $\hbar$ allow us to construct the value with the dimension of length

$$a_0 = \frac{\hbar^2}{me^2} = 0.529 \cdot 10^{-8}\ \mathrm{cm},$$

known as the Bohr radius and the time

$$t_0 = \frac{\hbar^3}{me^4} = 0.242 \cdot 10^{-11}\ \mathrm{c}.$$

These quantities define the typical space and time scale for describing a system, and it is therefore convenient to use these units as the basic system of atomic units.

Equation (5.75) in atomic units (at $Z = 1$) takes the form

$$\frac{d^2R}{dt^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0. \tag{5.76}$$

At $\quad < 0$, the motion is finite and the energy spectrum is discrete. We need the solutions (5.76) quadratically integrable with r^2. Let us introduce the specification

$$n = \frac{1}{\sqrt{-2E}}, \quad \rho = \frac{2r}{n}.$$

Equation (5.76) can be written as

$$\frac{d^2R}{dt^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left[\frac{n}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0. \tag{5.77}$$

We find the asymptotic forms of the radial function $R(r)$. At $\rho \to \infty$ and omitting the terms $\sim\rho^{-1}$ and $\sim\rho^{-2}$ in (5.77), we obtain

$$\frac{d^2R}{d\rho^2} = \frac{R}{4}.$$

Therefore, at high values of ρ, $R \propto e^{\pm\rho/2}$. The normalization demand is satisfied only by $R(\rho) \propto e^{-\rho/2}$. The asymptotic forms at $r \to 0$ have already been determined.

Substituting

$$R(\rho) = \rho^{\ell} e^{-\rho/2}\omega(\rho),$$

Equation (5.77) is reduced to the form

$$\rho\frac{d^2\omega}{d\rho^2} + (2\ell + 2 - \rho)\frac{d\omega}{d\rho} + (n - \ell - 1)\omega = 0. \tag{5.78}$$

To solve this equation in the limit of $\rho = 0$, we substitute $\omega(\rho)$ in the form of a power series

$$\omega(\rho) = 1 + \frac{(0-\nu)}{(0+\lambda)}\rho + \frac{(0-\nu)(1-\nu)}{(0+\lambda)(1+\lambda)}\frac{\rho^2}{2!} + \frac{(0-\nu)(1-\nu)(2-\nu)}{(0+\lambda)(1+\lambda)(2+\lambda)}\frac{\rho^3}{3!} + \cdots \tag{5.79}$$

where $\lambda = 2\ell + 2$ and $-\nu = -n + \ell + 1$.

At $\rho \to \infty$, the function $\omega(\rho)$ should increase, but not faster than the limiting power ρ. Then, $\omega(\rho)$ has to be a polynomial of ν power. So $-n + \ell + 1 = -k$ and

$n = \ell + 1 + k$ ($k = 0, 1, 2, \ldots$) at a given value of ℓ. Hence, using the definition for n, we can find the expression for the energy spectrum

$$E_n = -\frac{1}{2n^2}. \tag{5.80}$$

The number n is called the principal quantum number. In general units, it has the form

$$E = -Z^2 \frac{me^4}{2\hbar^2 n^2}. \tag{5.81}$$

This formula was obtained by Bohr in 1913 on the basis of the old quantum theory, by Pauli in 1926 from matrix mechanics, and by Schrödinger in 1926 by solving the differential equations.

Let us solve the problem of the spectrum of the hydrogen atom using the equation of dynamical equilibrium of the system. In Chap. 3, we obtained Jacobi's virial equation for a quantum mechanical system of particles whose interaction is defined by the potential being a homogeneous function of the coordinates. This equation in the operator form is

$$\ddot{\hat{\Phi}} = 2\hat{H} - \hat{U}. \tag{5.82}$$

where $\hat{\Phi}$ is the operator of the Jacobi function, which, for the hydrogen atom, is written as

$$\hat{\Phi} = \frac{1}{2} m \hat{r}^2. \tag{5.83}$$

The Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta_r + \hat{U}, \tag{5.84}$$

and the operator of the function of the potential energy for the hydrogen atom is

$$\hat{U} = -\frac{e^2}{r}. \tag{5.85}$$

We solve the problem with respect to the eigenvalues of Eq. (5.82), using the main idea of quantum mechanics. For this, we use the Schrödinger equation

$$\hat{H}\Psi = E\Psi$$

and rewrite Eq. (5.82) in the form

$$\ddot{\hat{\Phi}} = 2E - \hat{U}. \tag{5.86}$$

This equation includes two (unknown in the general case) operator functions $\hat{\Phi}$ and $\hat{U}$. In the case of the interaction, the potential is determined by the relation (5.85), and we can use a combination of the operators $\hat{\Phi}$ and $\hat{U}$ in the form

$$|\hat{U}|\sqrt{\hat{\Phi}} = \frac{e^2 m^{1/2}}{\sqrt{2}} = B. \tag{5.87}$$

We now transform (5.86) into the form that was considered in classical mechanics:

$$\ddot{\hat{\Phi}} = 2E + \frac{B}{\sqrt{\hat{\Phi}}}. \tag{5.88}$$

Equation (5.88) is a consequence of Eq. (5.86) when the Schrödinger equation and the relationship (5.87) are satisfied. Its solution for the bounded state, that is, when total energy is determined in parametric form, can be written as

$$\sqrt{\Phi} = \frac{B}{2|E|}\left(1 - \varepsilon\cos\varphi\right), \tag{5.89}$$

$$\varphi - \varepsilon\sin\varphi = M, \tag{5.90}$$

where the parameter is defined by the relation

$$M = \frac{(4|E|)^{3/2}}{4B}\left(t - \tau\right), \tag{5.91}$$

where ε and τ are integration constants and where

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}},$$

$$C = -\dot{\hat{\Phi}}_0^2 + 4B\sqrt{\hat{\Phi}_0 - 2A\hat{\Phi}_0}.$$

Moreover, the solution can be written in the form of Fourier and Lagrange series. Thus, the expression (5.37) describes the expansion of the operator $\hat{\Phi}$ into a Lagrange series including the accuracy of ε^3 and has the form

$$\hat{\Phi}_0 = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos M - \frac{\varepsilon^2}{2}\cos 2M - \frac{\varepsilon^3}{4}\cos 3M + \cdots\right]. \tag{5.92}$$

Using the general expression for the mean values of the observed quantities in quantum mechanics

$$\left\langle \Psi|\hat{\Phi}|\Psi \right\rangle = \bar{\Phi}$$

and taking into account that the mean value of the Jacobi function of the hydrogen atom should be different from zero, we find that our system has multiple eigenfrequencies $\nu_n = n\nu_0$ with respect to the basic ν_0, which corresponds to the period

$$T_0 = \frac{8\pi B}{(4|E|)^{3/2}}. \tag{5.93}$$

In accordance with the expression

$$E_n = \hbar\omega_n = \frac{\hbar 2\pi n}{T_0}, \tag{5.94}$$

each of these frequencies corresponds to the energy level $_n$ of the hydrogen atom. We substitute expression (5.93) for T_{v} into Eq. (5.94) and resolve it in relation to $_n$:

$$|E_n| = \frac{\hbar 2\pi n(4\,|E_n|)^{3/2}}{8\pi B} = \frac{\hbar n(4\,|E_n|)^{3/2}}{\frac{4e^2 m^{1/2}}{\sqrt{2}}} = \frac{\hbar n 2\sqrt{2\,|E_n|}}{e^2 m^{1/2}}. \tag{5.95}$$

The expression obtained by Bohr follows from (5.95)

$$E_n = \frac{e^4 m}{2\hbar^2 n^2}. \tag{5.96}$$

This equation solves the problem.

5.5 Solution of a Virial Equation in the Theory of Relativity (Static Approach)

We consider now the solution of Jacobi's virial equation in the framework of the theory of relativity, showing its equivalence to Schwarzschild's solution.

Let us write down the known expression for the radius of curvature of space–time as a function of mass density:

$$\frac{1}{R^2} = \frac{8\pi}{3}\frac{G\rho}{c^2}, \tag{5.97}$$

where R is the curvature radius, ρ is the mass density, G is the gravitational constant, and c is the velocity of light.

Equation (5.97) can also be rewritten in the form

$$\rho R^2 = \frac{3}{8\pi}\frac{c^2}{G}. \tag{5.98}$$

If the product ρR^2 in Eq. (5.98) is the Jacobi function ($\Phi = \rho R^2$ is the density of the Jacobi function), then from (5.98)

$$\Phi = \frac{3}{8\pi}\frac{c^2}{G}, \tag{5.99}$$

and it follows that the Jacobi function is a fundamental constant for the universe. (In general relativity, the spatial distance does not remain invariant. Therefore, instead of this, the Gaussian curvature is used, which has the dimension of the universe distance and is the invariant or, more precisely, the covariant.)

The constancy of the Jacobi function in this case reflects the smoothness of the description of motion in general relativity. The oscillations relative to this smooth motion described by Jacobi's equation are the gravitational waves and horizons, in particular the collapse and all types of singularity up to the process of condensation of matter in galaxies, stars, and so on.

Now we can show that Schwarzschild's solution in general relativity is equivalent to the solution of Jacobi's equation when $\ddot{\Phi} = 0$. Let us write the expression for the energy–momentum tensor

$$T_{\mathrm{i}}^{k} = (\rho + p)\, u_{\mathrm{i}} u^{k} + p\delta_{\mathrm{i}}^{k}. \tag{5.100}$$

In the corresponding coordinate system, we obtain

$$u^{i} = \left(0, 0, 0, \frac{1}{\sqrt{-g_{00}}}\right), \tag{5.101}$$

where $\rho = \rho(r)$ and $p = p(r)$.

The independent field equations are written as

$$G_1^1 = T_1^1, \quad G_0^0 = T_0^0,$$
$$R^{-2} = \frac{1}{3} G\rho c^2. \tag{5.102}$$

The expression for the metric is written in the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2(\mathrm{d}\Omega)^2 - \left\{A - B\sqrt{1 - \frac{r^2}{R^2}}\right\}^2 c^2 r^2, \tag{5.103}$$

where

$$\frac{\mathrm{d}r^2}{1 - \frac{r^2}{R^2}} + r^2(\mathrm{d}\Omega)^2$$

is the spatial element.

In this case, the expression for the volume occupied by the system is written as

$$V = \int_0^r \int_0^\pi \int_0^{2\pi} \frac{r^2 \sin \Theta}{\sqrt{1 - \frac{r^2}{R^2}}} \mathrm{d}r \, \mathrm{d}\Theta \, \mathrm{d}\Psi = \frac{4\pi R^3}{3} \left[\arcsin \frac{r}{R} - \frac{r}{R} \sqrt{1 - \frac{r^2}{R^2}} \right]. \quad (5.104)$$

It can be easily verified that the right-hand side of Eq. (5.104) coincides with solutions (5.14) and (5.15) of the equation of virial oscillations (5.11) at $\ddot{\Phi} = 0$, that is,

$$\arcsin x - x\sqrt{1 - x^2} = \arccos \left(\frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}} \right) - \sqrt{1 - \frac{AC}{2B^2}} \sqrt{1 - \left(\frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}} \right)}. \quad (5.105)$$

In fact, Eq. (5.105) is satisfied for

$$x = \frac{\frac{A}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{AC}{2B^2}}} \quad \text{and} \quad x = \sqrt{1 - \frac{AC}{2B^2}}, p$$

that is,

$$\frac{A}{B}\sqrt{\Phi} - 1 = -1\frac{AC}{2B^2}, \quad \text{or} \quad \frac{AC}{2B^2} + \frac{A\sqrt{\Phi}}{B} = 2.$$

At $\ddot{\Phi} = 0$, the parameter of virial oscillations

$$e = \sqrt{1 - \frac{AC}{2B^2}} \quad \text{and} \quad \sqrt{\Phi} = \frac{B}{A}.$$

so the last condition is satisfied.

Schwarzschild's solution is rigorous and unique for Einstein's equation for a static model of a system with spherical symmetry.

Since this solution coincides with the solution of virial oscillations at the same conditions, the solutions (5.14) and (5.15) of Eq. (5.11), obtained in this chapter, should be considered rigorous. Thus, we can conclude that the constancy of the product $U\sqrt{\Phi}$ in the framework of the static system model is proven. In Chap. 9, we will come back to this condition and will obtain another proof of the same very important relationship.

5.6 General Approach to Solution of Virial Equation for a Dissipative System

In the previous chapter, we have considered a number of cases of explicitly solved problems in mechanics and physics for the dynamics of *n*-body system and have shown that all those classical problems have also explicit solutions in the framework of the virial approach. But in the latter case, the solutions acquire a new physical meaning because the dynamics of a system is considered with respect to new parameters, that is, its Jacobi function (polar moment of inertia) and potential (kinetic) energy. In fact, the solution of the problem in terms of coordinates and velocities specifies the changes in location of a system or its constituents in space. The solution, with respect to the Jacobi function and the potential energy, identifies the evolutionary processes of the structure or redistribution of the mass density of the system. Moreover, the main difference of the two approaches is that the classical problem considers motion of a body in the outer central force field. The virial approach considers motion of a body both in the outer and in the own force field applying, instead of linear forces and moments, the volumetric forces (pressure) and moments (oscillations).

It appears from the cases considered that the existence of the relationship between the potential energy of a system and its Jacobi function written in the form

$$U\sqrt{\Phi} = B = \text{const.} \tag{5.106}$$

is the necessary condition for the resolution of Jacobi's equation.

This is the only case when the scalar equation

$$\ddot{\Phi} = 2E - U$$

is transferred into a nonlinear differential equation with one variable in the form

$$\ddot{\Phi} = 2E + \frac{B}{\sqrt{\Phi}}. \tag{5.107}$$

It was shown earlier that if the total energy of a system $_0 = - \quad /2 < 0$, then the general solution for Eq. (5.107) can be written as

$$\begin{aligned} \sqrt{\Phi_0} &= \frac{B}{A}\left[1 - \varepsilon \cos(\lambda - \psi)\right] \\ t &= \frac{4B}{(2A)^{3/2}}\left[\lambda - \varepsilon \sin(\lambda - \psi)\right], \end{aligned} \tag{5.108}$$

where ε and ψ are integration constants, the values of which are determined from initial data using Eqs. (5.23) and (5.24).

Equation (5.107) is called the equation of virial oscillation because its solution discovers a new physical effect—periodical nonlinear change of the Jacobi function and hence the potential energy of a system around their mean values determined by the virial theorem. Thus, in addition to the static effects determined by the hydrostatic equilibrium, in the study of dynamics of a system, the effects, determined by the condition of dynamical equilibrium expressed by the Jacobi function, are introduced.

The equation of virial oscillations (5.107) reflects physics of motion of the interacted mass particles of a body or masses of bodies themselves by the inverse square law. Its application opens the way to study the nature and the mechanism of generation of the body's energy, which performs its motion, and search the law of change in the system's configuration, that is, mutual change location of particles or the law of redistribution of the mass density for the system matter during its oscillations. This problem was considered earlier in our work (Ferronsky et al. 1987, 2011). We continue its study in Chap. 8.

As described above, cases of solution of Eq. (5.107) relate to unperturbed conservative systems. But in reality, in nature, all systems are affected by internal and external perturbations, which, from a physical point of view, are developed in the form of dissipation or absorption of energy. In this connection, as shown in the right-hand side of the equation of virial oscillations (5.107), an additional term appears, which is proportional to the Jacobi function Φ (indicating the presence of gravitational background or the existence of interaction between the system particles in accordance with Hook's law) and its first derivative $\dot{\Phi}$ depending on time t (indicating the existence of energy dissipation). All these and other possible cases can be formally described by a generalized equation of virial oscillations (4.27):

$$\ddot{\Phi} = 2E + \frac{B}{\sqrt{\Phi}} + X(t, \Phi, \dot{\Phi}), \tag{5.109}$$

where $X\left(t, \Phi, \dot{\Phi}\right)$ is the perturbation function, the value of which is small in comparison with the term $B \big/ \sqrt{\Phi} \neq \text{const}$.

In this chapter, we consider general as well as some specific approaches to the solution of Eq. (5.109) in the framework of different physical models of a system.

5.7 Analytical Solution of Generalized Equation of Virial Oscillations

The equation of perturbed virial oscillations is generalized in the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right), \tag{5.110}$$

where $= -2$, is constant, and $X\left(t, \Phi, \dot{\Phi}\right)$ is the perturbation function that we assume is given and dependent in general cases on time t, the Jacobi function Φ, and its first derivative $\dot{\Phi}$.

We consider two ways for the analytical construction of the solution of Eq. (5.110). In addition, let the function $X\left(t, \Phi, \dot{\Phi}\right)$ in Eq. (5.110) depend on some small parameter in relation to which the function can be expanded into absolute convergent power series of the form

$$X\left(t, \Phi, \dot{\Phi}\right) = \sum_{r=1}^{\infty} e^k X^k\left(t, \Phi, \dot{\Phi}\right). \tag{5.111}$$

Let the series be convergent in some time interval t absolute for all values of , which are satisfied to condition $|e| < \bar{e}$. Then, Eq. (5.110) can be rewritten in the form

$$X\left(t, \Phi, \dot{\Phi}\right) = -A + \frac{B}{\sqrt{\Phi}} \sum_{r=1}^{\infty} e^k X^k\left(t, \Phi, \dot{\Phi}\right). \tag{5.112}$$

We look for the solution of Eq. (5.112) also in the form of the power series of parameter . For this purpose, we write the function $\Phi(t)$ in the form of a power series, the coefficients of which are unknown:

$$\Phi(t) = \sum_{k=1}^{\infty} e^k \Phi^k(t). \tag{5.113}$$

Putting (5.113) into (5.112), the task can be reduced to the determination of such function $\Phi^{(\mathrm{k})}(t)$, which identically satisfies Eq. (5.112). In this case, the coefficient $\Phi^{(\ )}(t)$ becomes the solution of the unperturbed oscillation equation (5.107), which can be obtained from (5.112) by putting $= 0$.

One can consider the series (5.113) as a Taylor series expansion in order to determine all the other coefficients $\Phi^{(\mathrm{k})}(t)$, that is,

$$\Phi^{(k)} = \frac{1}{k!}\left(\frac{\mathrm{d}^k \Phi}{\mathrm{d}e^k}\right)\Big|_{e=0},$$
$$\dot{\Phi}^{(k)} = \frac{1}{k!}\left(\frac{\mathrm{d}^k \dot{\Phi}}{\mathrm{d}e^k}\right)\Big|_{e=0}. \tag{5.114}$$

Accepting the series (5.113) for the introduction into Eq. (5.112), it becomes identical with respect to the parameter . Thus, we have justified the differentiation of the identity with respect to the parameter several times assuming that the identity remains after repeated differentiation.

We next obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}e}\right) = \frac{1}{2}\frac{B}{\Phi^{3/2}}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}e}\right) + \sum_{k=1}^{\infty} k e^{k-1} X^{(k)} + \sum_{k=1}^{\infty} e^k \left(\frac{\mathrm{d}X^{(k)}}{\mathrm{d}e}\right), \tag{5.115}$$

where $\mathrm{d}X^{(\mathrm{k})}/\mathrm{d}e$ is the total derivative of the function $X^{(\mathrm{k})}$ with respect to parameter , expressed by

$$\frac{\mathrm{d}X^{(k)}}{\mathrm{d}e} = \frac{\partial X^{(k)}}{\partial \Phi}\left(\frac{\mathrm{d}\Phi}{\mathrm{d}e}\right) + \frac{\partial X^{(k)}}{\partial \dot{\Phi}}\left(\frac{\mathrm{d}\dot{\Phi}}{\mathrm{d}e}\right).$$

Now let $= 0$ in (5.115). Then, by taking into account (5.113) and (5.114), we obtain

$$\frac{\mathrm{d}^2\Phi^{(1)}}{\mathrm{d}t^2} + p_1\Phi^{(1)} = X_1, \tag{5.116}$$

where

$$p_1 = \frac{1}{2}\frac{B}{\Phi^{3/2}}\,|_{e=0} = \frac{1}{2}\frac{B}{\Phi^{(0)3/2}}, \quad X_1 = X^1\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}\right)$$

are known functions of time, since the solution of the equation in the zero approximation (unperturbed oscillation equation (5.108)) is known.

Carrying out differentiation of Eq. (5.112) with respect to parameter for the second, third, and so on $(k-1)$ times and assuming after each differentiation that $= 0$, we will step-by-step obtain equations determining second, third, and so on approximations. It is possible to show that in each succeeding approximation, the equation will have the same form and the same coefficient p_1 as in Eq. (5.116). If so, the equation determining the functions $\Phi^{(k)}$ and $\dot{\Phi}^{(k)}$ has the form

$$\frac{\mathrm{d}^2\Phi^{(k)}}{\mathrm{d}t^2} + p_1\Phi^{(k)} = X_k\left(t, \Phi^{(0)}, \dot{\Phi}^{(0)}, \ldots, \Phi^{(k-1)}, \dot{\Phi}^{(k-1)}\right), \tag{5.117}$$

where the function $_k$ depends on $\Phi^{(0)}$, $\dot{\Phi}^{(0)}$, …, $\Phi^{(\mathrm{k}-1)}$, $\dot{\Phi}^{(k-1)}$, which were determined earlier and are the functions of t and unknown functions $\Phi^{(0)}$ and $\dot{\Phi}^{(0)}$.

It is known that there is no general way of obtaining a solution for any linear differential equation with variable coefficients, but in our case, we can use the following theorem of Poincare (Duboshin 1975). Let the general solution of the unperturbed virial oscillation equation be determined by the function $\Phi^{(0)} = f(t, C_1, C_2)$, where C_1 and C_2 are, for instance, arbitrary constants ε and Ψ in the solution (5.108) of Eq. (5.107). Then, Poincare's theorem confirms that the function determined by the equalities

$$\Phi_1 = \frac{\partial f}{\partial C_1},$$

$$\Phi_2 = \frac{\partial f}{\partial C_2},$$

satisfies the linear homogeneous differential equation reduced by omission of the right-hand side of Eq. (5.117).

Thus, the general solution of the linear homogeneous equation

$$\frac{d^2\Phi^{(k)}}{dt^2} + p_1\Phi^{(k)} = 0$$

has the form

$$\Phi_1 C_1^{(k)} + \Phi_2 C_2^{(k)} = \Phi^{(k)}, \tag{5.118}$$

and the general solution of Eq. (5.117) can be obtained by the method of variation of arbitrary constants, that is, assuming that $C_2^{(k)}$ are functions of time. Then, using the key idea of the method of variation of arbitrary constants, we obtain a system of two equations:

$$\begin{aligned} \dot{C}_1^{(k)}\Phi_1 + \dot{C}_2^{(k)}\Phi_2 &= 0_k, \\ \dot{C}_1^{(k)}\dot{\Phi}_1 + \dot{C}_2^{(k)}\dot{\Phi}_2 &= X_k. \end{aligned} \tag{5.119}$$

Solving this system with respect to $\dot{C}_1^{(k)}$ and $\dot{C}_2^{(k)}$ and integrating the expression obtained, we write the general solution of Eq. (5.117) as follows:

$$\Phi^{(k)}(t) = \Phi_2 \int_{t_0}^{t} \frac{\Phi_1 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1} - \Phi_1 \int_{t_0}^{t} \frac{\Phi_2 X_k \mathrm{d}t}{\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1},$$

where

$$\Phi_1 = \frac{\partial f\ (t, C_1, C_2)}{\partial C_1}$$

and

$$\Phi_2 = \frac{\partial f\ (t, C_1, C_2)}{\partial C_2}.$$

Thus, we can determine any coefficient of the series (5.113), reducing Eq. (5.112) into an identity, and therefore write the general solution of Eq. (5.110) in the form

$$\Phi = \sum_{k=0}^{\infty} e^k \Phi^{(k)} = \sum_{k=0}^{\infty} e^k \left[\Phi_2 \int_{t_0}^{t} \frac{\Phi_1 X_k \mathrm{d}t}{\Phi_1 \dot{\Phi}_2 - \Phi_2 \dot{\Phi}_1} - \Phi_1 \int_{t_0}^{t} \frac{\Phi_2 X_k \mathrm{d}t}{\Phi_1 \dot{\Phi}_2 - \Phi_2 \dot{\Phi}_1} \right]. \tag{5.120}$$

Let us consider the second way of approximate integration of the perturbed virial equation (5.110), based on Picard's method (Duboshin 1975). It is convenient to apply this method of integrating the equations that were obtained using the Lagrange method of variation of arbitrary constants.

Assuming that the first integrals (5.23) and (5.24)

$$\varepsilon = \sqrt{1 - \frac{A}{2B^2} \left(-\dot{\Phi}_0 + 4B\sqrt{\Phi_0 - 2A\Phi_0} \right)}, \tag{5.121}$$

$$-\tau = \left\{ \frac{4B}{(2A)^{3/2}} \left[\arccos \frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} - \varepsilon \sqrt{1 - \left(\frac{1 - \frac{A}{B}\sqrt{\Phi_0}}{\varepsilon} \right)^2} \right] - t \right\} \tag{5.122}$$

of the unperturbed virial oscillation equation (5.107) are also the first integrals of the perturbed oscillation equation (5.110). But constants ε and τ are now unknown functions of time. Let us derive differential equations that are satisfied by these functions using the first integrals (5.121) and (5.122). For convenience, we replace the integration constant ε by , using the expression

$$\varepsilon = \sqrt{1 - \frac{AC}{2B^2}}.$$

Now we rewrite Eq. (5.121) in the form

$$C = -\dot{\Phi}_0^2 + 4B\sqrt{\Phi_0 - 2A\Phi_0}. \tag{5.123}$$

Then using the main idea of the Lagrange method, after variation of the first integrals (5.122) and (5.123) and replacement of $\ddot{\Phi}$ by

$$\left(-A + \frac{B}{\sqrt{\Phi}} + X\left(t, \Phi, \dot{\Phi}\right) \right),$$

we write

$$\dot{C} = -2\dot{\Phi} X\left(t, \Phi, \dot{\Phi}\right), \tag{5.124}$$

$$\dot{\tau} = \Psi(\Phi, C)\,\dot{C} = -2\dot{\Phi} X\left(t, \Phi, \dot{\Phi}\right)\Psi(\Phi, C), \tag{5.125}$$

where

$$\Psi\left(\Phi, C\right) = -\frac{4B}{(2A)^{3/2}}\frac{\mathrm{d}}{\mathrm{d}C}\left[\arccos\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}}-\sqrt{1-\frac{AC}{2B^2}}\sqrt{1-\left(\frac{1-\frac{A}{B}\sqrt{\Phi}}{\sqrt{1-\frac{AC}{2B^2}}}\right)^2}\,\right].$$

We now express Φ and $\dot{\Phi}$ in explicit form through , τ, and t, using, for example, the Lagrangian series:

$$\Phi(t) = \frac{B^2}{A^2}\left[1+\frac{3}{2}\varepsilon^2+\left(-2\varepsilon+\frac{\varepsilon^3}{4}\right)\cos M-\frac{\varepsilon^2}{2}\cos 2M-\frac{\varepsilon^3}{4}\cos 3M+\cdots\right], \tag{5.126}$$

$$\dot{\Phi}(t) = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin M+\frac{1}{2}\varepsilon\sin 2M+\frac{\varepsilon^2}{2}\sin M\left(2\cos^2 M-\sin^2 M\right)+\cdots\right]. \tag{5.127}$$

Thus, taking into account Eqs. (5.126) and (5.127) for the functions Φ and $\dot{\Phi}$, Eqs. (5.124) and (5.25) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}t} &= F_1\left(t, C, \tau\right),\\ \frac{\mathrm{d}\tau}{\mathrm{d}t} &= F_2\left(t, C, \tau\right).\end{aligned} \tag{5.128}$$

To solve the system of differential equations (5.128), we use Picard's successive approximation method, obtained in the kth approximation expressions for $C^{(k)}$ and $\tau^{(k)}$ in the form

$$C^{(k)} = C^{(0)} + \int_{t_0}^{t} F_1\left(t, C^{(k-1)}, \tau^{(k-1)}\right)\mathrm{d}t, \tag{5.129}$$

$$\tau^{(k)} = \tau^{(0)} + \int_{t_0}^{t} F_2\left(t, C^{(k-1)}, \tau^{(k-1)}\right)\mathrm{d}t, \tag{5.130}$$

where $C^{(0)}$ and $\tau^{(0)}$ are the values of arbitrary constants C and τ at initial time t_0, and $k = 1, 2, \ldots$.

Then, in the limit of $k \to \infty$, we obtain the solution of the system (5.128):

$$\begin{aligned}C &= \lim_{k\to\infty} C^{(k)},\\ \tau &= \lim_{k\to\infty} \tau^{(k)}.\end{aligned} \tag{5.131}$$

Consider now two possible cases of the perturbation function behavior. First, assume that the perturbation function does not depend explicitly on time. Then, since it is possible to expand functions Φ and $\dot{\Phi}$ into a Fourier series in terms of sine and cosine of argument , the right-hand sides of the system (5.128) can also be expanded into a Fourier series in terms of sine and cosine of M.

Finally, we obtain

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \left[A_0 + \sum_{k=1}^{\infty} \left(A_k \cos kM + B_k \sin kM \right) \right], \tag{5.132}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left[a_0 + \sum_{k=1}^{\infty} \left(a_k \cos kM + b_k \sin kM \right) \right], \tag{5.133}$$

where A_0, A_k, B_k, a_0, a_k, and b_k are the corresponding coefficients of the Fourier series that are

$$A_0 = \frac{2}{\pi} \int_0^{2\pi} F_1 \left(M, C \right) \mathrm{d}M,$$

$$A_k = \frac{2}{\pi} \int_0^{2\pi} F_1 \left(M, C \right) \cos kM \; \mathrm{d}M,$$

$$B_k = \frac{2}{\pi} \int_0^{2\pi} F_1 \left(M, C \right) \sin kM \; \mathrm{d}M,$$

$$a_0 = \frac{2}{\pi} \int_0^{2\pi} F_2 \left(M, C \right) \; \mathrm{d}M,$$

$$a_k = \frac{2}{\pi} \int_0^{2\pi} F_2 \left(M, C \right) \cos kM \; \mathrm{d}M,$$

$$b_k = \frac{2}{\pi} \int_0^{2\pi} F_2 \left(M, C \right) \sin kM \; \mathrm{d}M,$$

$$a_k = \frac{2}{\pi} \int_0^{2\pi} F_2 \left(M, C \right) \cos kM \; \mathrm{d}M,$$

$$b_k = \frac{2}{\pi} \int_0^{2\pi} F_2 \left(M, C \right) \sin kM \; \mathrm{d}M.$$

Following Picard's method, in order to solve Eqs. (5.132) and (5.133) in the first approximation, we introduce the values of arbitrary constants C and τ corresponding to the initial time t_0 into the right-hand side of the equations.

Then, we obtain

$$C^{(1)}(t) = \mathrm{C}^{(0)} + A_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ A_k^{(0)} \left[\sin kM - \sin kM_0 \right] + B_k^{(0)} \left[\cos kM - \cos kM_0 \right] \right\} \tag{5.134}$$

$$\tau^{(1)}(t) = \tau^{(0)} + a_0^{(0)}(t - t_0) + \sum_{k=1}^{\infty} \frac{1}{kn} \left\{ a_k^{(0)} \left[\sin kM - \sin kM_0 \right] + b_k^{(0)} \left[\cos kM - \cos kM_0 \right] \right\}. \tag{5.135}$$

Thus, when the function does not depend explicitly on time t, solutions (5.134) and (5.135) of Eq. (5.110) have three analytically different parts. The first is a constant term, depending on the initial values of the arbitrary constants. It is usually called the constant term of perturbation of the first order. The second part is a function monotonically increasing in time. It is called the secular term of the perturbation of the first order. The third part consists of an infinite set of trigonometric terms. All of them are periodic functions of and consequently of time t. This is called periodic perturbation.

Similarly, we can obtain solutions in the second, third, and so on, orders. Here we limit our consideration only within the first order of perturbation theory. In practice, few terms of the periodic perturbation can be taken into account, and the solution obtained becomes effective only for a short period of time.

When the perturbation function is a periodic function of some argument $'$,

$$M' = n'\left(t - \tau'\right),$$

the right-hand side of the system of Eqs. (5.128) are periodic functions of the two independent arguments and $'$. Therefore, they can be expanded into a double Fourier series in terms of sine and cosine of the linear combination of arguments and $'$. Then, in the first approximation of perturbation theory, we obtain the following system of equations:

$$\frac{\mathrm{d}C^{(1)}}{\mathrm{d}t} = A_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[A_{k,k'}^{(0)} \cos\left(kM + k'M'\right) + B_{k,k'}^{(0)} \sin\left(kM + k'M'\right) \right], \tag{5.136}$$

$$\frac{\mathrm{d}\tau^{(1)}}{\mathrm{d}t} = a_{00}^{(0)} + \sum_{k',k=-\infty}^{\infty} \left[a_{k,k'}^{(0)} \cos\left(kM + k'M'\right) + b_{k,k'}^{(0)} \sin\left(kM + k'M'\right)\right]. \tag{5.137}$$

Integrating equations (5.136) and (5.137) with respect to time, we obtain a solution of the system:

$$\begin{aligned} C^{(1)}(t) =& C^{(0)} + A_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \left\{A_{k,k'}^{(0)} \left[\cos\left(kM + k'M'\right)\right.\right. \\ & - \left.\cos\left(kM_0 + k'M'_0\right)\right] \\ & + \left.B_{k,k'}^{(0)} \left[\sin\left(kM + k'M'\right) - \sin\left(kM_0 + k'M'_0\right)\right]\right\} \end{aligned} \tag{5.138}$$

$$\begin{aligned} \tau^{(1)}(t) =& \tau^{(0)} + a_{00}^{(0)}(t - t_0) + \sum_{k',k=-\infty}^{\infty} \frac{1}{kn + k'n'} \left\{a_{k,k'}^{(0)} \left[\cos\left(kM + k'M'\right)\right.\right. \\ & - \left.\cos\left(kM_0 + k'M'_0\right)\right] \\ & + \left.b_{k,k'}^{(0)} \left[\sin\left(kM + k'M'\right) - \sin\left(kM_0 + k'M'_0\right)\right]\right\}. \end{aligned} \tag{5.139}$$

Equations (5.138) and (5.139) have the same analytical structure as (5.134) and (5.135). At the same time, in this case, the periodic part of the perturbation can be divided into two groups, depending on the value of the divisor $kn + k'n'$. If the values of k and k' are such that the divisor is sufficiently large, then period $T_{k,k'} = 2\pi/(kn + k'n')$ of the corresponding inequality will be rather small. Such inequalities are called short periodic. Their amplitudes are also rather small, and they can play a role only within short period of time.

If the values of k and k' are such that the divisor $kn + k'n'$ is sufficiently small but unequal to zero, then the period of the corresponding inequality will become large. The amplitude of such terms could also be large and play a role within large periods of time. Such terms form series of long-periodic inequalities. In the case of such k and k', when $kn + k'n' = 0$, the corresponding terms are independent of t and change the value of the secular term in the solutions (5.138) and (5.139).

5.8 Solution of the Virial Equation for a Dissipative System

In Chap. 4, we derived Jacobi's virial equation for a nonconservative system in the form

$$\ddot{\Phi} = 2E_0 [1 + q(t)] - U - k\dot{\Phi}. \tag{5.140}$$

At $k \ll 1$, $t \gg t_0$, $|U|\sqrt{\Phi} =\quad = \text{const.}$, $2E_0 = -A_0$, and when the magnitude of the term $k\dot{\Phi}$ is sufficiently small, Eq. (5.140) can be rewritten in a parametric form

$$\ddot{\Phi} = -A_0\left[1 + q(t)\right] + \frac{B}{\sqrt{\Phi}}, \tag{5.141}$$

where $q(t)$ is a monotonically increasing function of time due to dissipation of energy during "smooth" evolution of a system within a time interval $t \in [0, \tau]$.

Using the theorem of continuous solution depending on the parameter, we write the solution of Eq. (5.141) as follows:

$$- \arccos W + \arccos W_0 - \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}}\sqrt{1 - W^2} + \sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \sqrt{\frac{\left(2A_0\left[1 + q(t)\right]\right)^{3/2}}{4B}}\,(t - t_0)\,, \tag{5.142}$$

$$\arccos W - \arccos W_0 + \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}}\sqrt{1 - W^2} - \sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \sqrt{\frac{\left(2A_0\left[1 + q(t)\right]\right)^{3/2}}{4B}}\,(t - t_0)\,, \tag{5.143}$$

where

$$W = \frac{\frac{A_0[1+q(t)]}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{A_0[1+q(t)]C}{2B^2}}}, \quad W_0 = \frac{\frac{A_0}{B}\sqrt{\Phi} - 1}{\sqrt{1 - \frac{A_0 C}{2B^2}}},$$

$$A_0\left[1 + q(t)\right] > 0, \quad C < \frac{2B^2}{A_0\left[1 + q(t)\right]},$$

$$\left|-A_0\left[1 + q(t)\right]\sqrt{\Phi} + B\right| < B\sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}},$$

$$C = -2A_0\Phi_0 + 4B\sqrt{\Phi_0} - \dot{\Phi}_0^2.$$

Equations of discriminant curves that bound oscillations of the Jacobi function Φ by analogy with the case of the conservative system can be written as

$$\sqrt{\Phi_1} = \frac{B}{A_0\left[1 + q(t)\right]}\left[1 + \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}}\right] \quad t \in [0, \tau]\,, \tag{5.144}$$

$$\sqrt{\Phi_2} = \frac{B}{A_0\left[1 + q(t)\right]}\left[1 - \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}}\right], \quad t \in [0, \tau]\,. \tag{5.145}$$

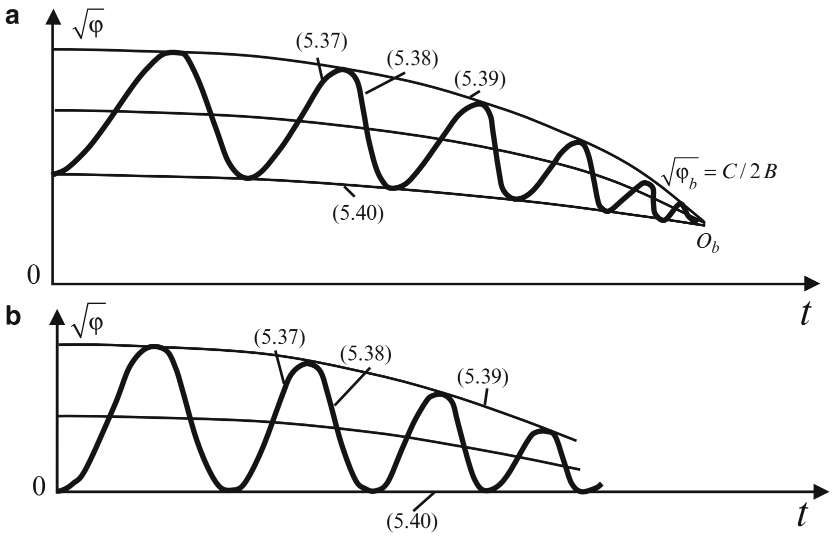


Fig. 5.5 Virial oscillations of Jacobi function in time for nonconservative system ( ) and for general (Wintner's) case (**b**)

It is obvious that the solution of Jacobi's virial equation for a nonconservative system is quasiperiodic with period

$$T_v(q) = \frac{8\pi B}{\left(2A_0\,[1+q(t)]\right)^{3/2}} \tag{5.146}$$

and an amplitude of Jacobi function oscillations

$$\Delta\sqrt{\Phi} = \frac{B}{A_0\,[1+q(t)]}\left(1 - \frac{A_0\,[1+q(t)]\,C}{2B^2}\right)^{1/2}. \tag{5.147}$$

As $q(t)$ is monotonically and continuously increasing the parameter confined in time, the period and the amplitude of the oscillations will gradually decrease and tend to zero in the time limit.

In Fig. 5.5 , the integral curves (5.142) and (5.143) and the discriminant curves (5.144) and (5.145) are shown in a general case when $0 < C < 2B^2/A_0$. At the point O_b, the integral and discriminant curves tend to coincide, and the value of the amplitude of the Jacobi function (polar moment of inertia) oscillations of the system goes to zero.

When $= 0$ (Fig. 5.5), the discriminant line (5.144) coincides with the axis of abscissa, $\Phi_2 = 0$. In the accepted case of constancy of the system mass, the point O_b, where the integral and discriminant curves coincide, will be reached in the time limit $t \to \infty$.

When $2\ {}^2/A_0 \to C$ and < 0, the solutions (5.142), (5.143), (5.144), and (5.145) could be complex so the processes considered are not physical.

We note that by analogy with the case for conservative system, considered in Chap. 4, we can show here that the asymptotic relations (5.136) and (5.137) for the solutions (5.142) and (5.143) of Jacobi's equation (5.141) in the points of contact of the discriminant line $\Phi_2 = 0$ are justified. In the points of contact for the integral curves (5.142) and (5.143) and the discriminant curves (5.144) and (5.145) for which Φ_1 and Φ_2 are not equal to zero, the following asymptotic relations are also justified:

$$\left(\sqrt{\Phi_1} - \sqrt{\Phi}\right) \propto \left(t' - t\right)^2, \tag{5.148}$$

$$\left(\sqrt{\Phi} - \sqrt{\Phi_2}\right) \propto \left(t - t'\right)^2, \tag{5.149}$$

where t' is time of a tangency point for the corresponding integral curve of the discriminant lines $\Phi_{1,2}$ when $\Phi_{1,2} \neq 0$.

5.9 Solution of the Virial Equation for a System with Friction

Let us consider the solution of Jacobi's virial equation for conservative systems, but let the relationship between its potential energy and the Jacobi function be as follows:

$$U\sqrt{\Phi} = B + k\dot{\Phi}. \tag{5.150}$$

In this case, the equation of virial oscillations (5.107) can be written as

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - k\frac{\dot{\Phi}}{\Phi}. \tag{5.151}$$

The term $-k\dot{\Phi}\Big/\sqrt{\Phi}$ in (5.151) plays the role of perturbation function, reflecting the effect of internal friction of the matter while the system is oscillating.

In principle, Eq. (5.151) can be solved using the above perturbation theory methods. However, we can show that a particular solution exists for the system of two differential equations of the second order, which satisfies Eq. (5.151). These differential equations are as follows:

$$\left(\sqrt{\Phi}\right)'' + \sqrt{\frac{2}{A}}k\left(\sqrt{\Phi}\right)' + \sqrt{\Phi} = \frac{B}{A}, \tag{5.152}$$

$$t'' + \sqrt{\frac{2}{A}}kt' + t = \frac{4B}{(2A)^{3/2}}\lambda. \tag{5.153}$$

In Eqs. (5.152) and (5.153), we introduced a new variable λ, so the primes at Φ and t mean differentiation with respect to λ. Note also that time t here is not an independent variable. This allowed us to transfer the nonlinear equation into two linear equations. The partial solution of Eqs. (5.152) and (5.153) containing two integration constants is

$$\sqrt{\Phi} = \frac{B}{A}\left[1 - \varepsilon e^{-\gamma/2\sqrt{2/A}\lambda}\cos\left(\sqrt{\frac{4A-2k^2}{4A}}\lambda + \psi + \tau\right)\right], \tag{5.154}$$

$$t = \frac{4B}{(2A)}\left[\lambda - \varepsilon e^{-\gamma/2\sqrt{2/A}\lambda}\sin\left(\sqrt{\frac{4A-2k^2}{4A}}\lambda + \psi\right)\right] - \frac{4B}{(2A^{3/2})}\sqrt{\frac{2}{A}}k, \tag{5.155}$$

where ε and ψ are arbitrary constants

$$\tau = \mathrm{arc}tg\sqrt{\frac{2}{A}}k\left(\frac{4A-2k^2}{4A}\right)^{-1/2}.$$

To show that Eqs. (5.154) and (5.155) of the two linear differential equations (5.152) and (5.153) are also general solutions of (5.46), let us do as follows.

Differentiating (5.151) with respect to λ, we obtain

$$t' = \sqrt{\frac{2}{A}}\sqrt{\Phi}. \tag{5.156}$$

We write the derivative from function $\sqrt{\Phi}$ with respect to λ using Eq. (5.156) in the form

$$\left(\sqrt{\Phi}\right)' = \frac{\dot{\Phi}}{\sqrt{2A}}. \tag{5.157}$$

We write the derivative from function $\sqrt{\Phi}$ with respect to λ using Eq. (5.156) in the form

$$\left(\sqrt{\Phi}\right)'' = \frac{\ddot{\Phi}}{\sqrt{2A}}t' = \frac{\ddot{\Phi}\sqrt{\Phi}}{A}. \tag{5.158}$$

Substituting Eqs. (5.157) and (5.158) for $\left(\sqrt{\Phi}\right)'$ and $\left(\sqrt{\Phi}\right)''$ into Eq. (5.152), we obtain

$$\frac{\ddot{\Phi}\sqrt{\Phi}}{A} + \sqrt{\frac{2}{A}}k\frac{\dot{\Phi}}{\sqrt{\Phi}} + \sqrt{\Phi} = \frac{B}{A}. \tag{5.159}$$

Dividing Eq. (5.159) by $\sqrt{\Phi}/A$, we have

$$\ddot{\Phi} + k\frac{\dot{\Phi}}{\sqrt{\Phi}} + A = \frac{B}{\sqrt{\Phi}},$$

which is in fact our Eq. (5.151). This means that Eqs. (5.154) and (5.155) are the general solutions of Eq. (5.151).

Note that Eq. (5.155) differs in general from Kepler's equation, both by the exponential factor before the sine function and by the constant term in the right-hand side of Eq. (5.155). In addition, it follows from Eq. (5.154) that the period of virial oscillations of the Jacobi function depends on the parameter k. Therefore, when λ changes its value by $2\pi \Big/ \Big\lfloor \sqrt{(4A - 2k^2)/4A} \Big\rfloor$, the value of $\sqrt{\Phi}$ remains unchanged (we neglect the changes of the amplitude of virial oscillations due to existence of the exponential factor) assuming that

$$\frac{k}{2}\sqrt{\frac{2}{A}}\frac{2\pi}{\sqrt{\frac{4A-2k^2}{4A}}} \gg 1.$$

It follows from Eq. (5.155) that time t changes by the relationship of $T = 8\pi B \big/ (2A)^{3/2} \sqrt{(4A - 2k^2)/4A}$ defining the period of the damping virial oscillations. Therefore, from solutions (5.154) and (5.155) of Eq. (5.151), it follows that if during the evolution of the system the value $U\sqrt{\Phi}$ varies only slightly around the constant, this leads to damping of the virial oscillations of the integral characteristics of the system around their averaged virial theorem value.

In conclusion, we have to note that derivation of the equation of dynamical equilibrium and its solution for conservative and dissipative systems shows that dynamics of celestial bodies in their own force field puts forward wide class of geophysical, astrophysical, and geodetic problems that can be solved by the methods of celestial mechanics and with new physical concepts we considered.

References

Duboshin GN (1975) Celestial mechanics: the main problems and the methods. Nauka, Moscow
Duboshin GN (1978) Celestial mechanics: analytical and qualitative methods. Nauka, Moscow
Ferronsky VI, Denisik SA, Ferronsky SV (1984) Virial approach to solution of the problem of global oscillations of the Earth atmosphere. Phys Atmos Oceans 20:802–809
Ferronsky VI, Denisik SA, Ferronsky SV (1987) Jacobi dynamics. Reidel, Dordrecht
Ferronsky VI, Denisik SA, Ferronsky SV (2011) Jacobi dynamics, 2nd edn. Springer, Dordrecht/Heidelberg
Zeldovich YB, Novikov ID (1967) Relativistic astrophysics. Nauka, Moscow

Chapter 6
The Nature of the Solar System Bodies Creation

Abstract The irresistible difficulty in cosmogony is the observed fact that planets of the solar system having only $\sim$0.015% of the system mass possess 98% of the orbital angular momentum. At the same time, $\sim$99.85% of the Sun's mass produce no more than 2% of the angular momentum, which is accepted to be the conservative parameter. This fact was found in the framework of the hydrostatic equilibrium of the bodies. It is shown in this chapter that in the framework of the Jacobi dynamics, creation of a new body occurs within the parental cloud as a result of its separation in the density on shells, where the outer shell reaches the state of weightlessness. Here the orbital moment of the momentum of a created secondary body represents the total kinetic moment of the parental body's cloud owing to the energy conservation law. It means that the orbital moment of momentum of each planet represents the kinetic momentum of the protosun at the time of planet separation and orbiting. The planet's orbital moment of momentum is formed by the total potential energy of the protosun. But the planet's angular moment of the axial rotation is formed by the tangential component of the own planet's potential energy. So the above fact appears to be a misunderstanding.

Appearance of weightlessness of the upper shell during the body's matter differentiation, conditions of a body separation and orbiting, the structure of the potential and kinetic energies of a nonuniform body, conditions of dynamical equilibrium of oscillation and rotation of a body, equations of oscillation and rotation of a body and their solution, the nature and mechanism of body shell differentiation, and physical meaning of the Archimedes and Coriolis forces are considered as particular tasks, for which mathematical solutions are found here. The problem of initial values of mean density and radius of a created body has also its solution.

The discussed physics and kinematics of creation and separation of the solar system bodies prove the Huygens law of motion on semicubical parabola of his watch pendulum, which synchronously follows the Earth's motion. The relationship between the evolute and evolvent represents the relationship between function and its derivative or between function and its integral. In the case of the Huygens

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_6,

oscillating pendulum, the suspension filament starts unrolling in a fixed point. In the case of a celestial body, creation of a satellite starts in a fixed point of its parental body where the initial conditions are transferred by Kepler's third law, which is the consequence of a body creation.

It is known that all the solar system bodies (the planets, their satellites, comets, and meteoric bodies) are identical in their substantial and chemical content, and in this respect, they are of common origin. But the search of a unified mechanism of the body creation has encountered an irresistible difficulty in their dynamics. The point is that planets having only ~0.015% of the system mass possess 98% of the orbital angular momentum. At the same time, ~99.85% of the Sun's mass produce no more than 2% of the angular momentum, which is accepted to be a conservative parameter. Also, the specific (for unit of the mass) angular momentum of planets is increased with the distance from the Sun. As discussed in the previous chapters, the above results follow calculation model based on hydrostatic equilibrium state of the system, where the body motion results from the outer forces. It was shown that the hydrostatic equilibrium of celestial body dynamics appeared to be not the correct physical conception.

We analyze the evolutionary problem of the solar system based on fundamentals of the Jacobi dynamics, where the body motion initiates by the inner forces' action. Here, the energy loss in the form of radiation is accepted as the physical basis of the body evolution, and the effect of elementary particles' collision and scattering appears to be the mechanism of the energy generation. It is clear from observation that all celestial bodies are self-gravitating systems.

It is shown next that the creation of a new body occurs within the parental cloud because of its separation in density on shells by the Archimedes law, when the outer shell reaches the state of weightlessness. Here, the orbital moment of momentum of a created secondary body represents the total kinetic moment of the parental body's cloud owing to the energy conservation law, as was shown in (3.17) and (3.18):

$$Q = \sum_i p_i r_i = \sum_i m_i v_i r_i = \sum_i m_i \dot{r}_i r_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i \frac{m_i r_i^2}{2}\right) = \frac{1}{2}\dot{I}_\mathrm{p},$$
$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{2}\ddot{I}_\mathrm{p} = \sum_i \dot{r}_i \cdot p_i + \sum_i \dot{p}_i \cdot r_i, \qquad (6.1)$$

where Q is the moment of momentum of the parental cloud, p is moment of a particle, r is the radius, and I_p is the polar moment of inertia of the cloud.

It means that the orbital moment of momentum of each planet represents the kinetic momentum of the protosun at the time of planet separation and orbiting. The kinetic moment of a body is equal to the sum of the rotational and oscillating moments, and the kinetic energy is equal to the sum of the rotational and oscillating energies, which follows from the energy conservation law. At the same time, the planet's orbital moment of momentum is formed by the total potential energy of the protosun. But the planet's angular moment of the axial rotation is formed by the tangential component of the own planet's potential energy (see Eqs. 6.8 and 6.9). Here, the energy of axial rotation compiles small portion of the oscillating energy. As noted in Sect. 3.3, the kinetic energy of Earth, Mars, Jupiter, Saturn, Uranus, and Neptune compiles 10^{-3}–10^{-2}, and that of Mercury, Venus, the Moon, and the Sun is about 10^{-4} from the total kinetic energy. For bodies with a uniform mass density distribution, the kinetic energy of rotation is equal to zero.

The interaction (collision and scattering) of mass particles is accompanied by continuous redistribution of the body's mass density. According to the Roche's tidal dynamics, redistribution of the mass density leads to the shell separation. It will be shown later that when the density of the upper shell reaches less than two third with respect to the underlying shells, then the upper shell becomes weightless (i.e., it loses weight). From the physical point of view, it means that the own force field of the upper shell is in dynamical equilibrium with the parental force field. In this case, if the density of the upper shell has nonuniform density distribution, then by the difference in the potentials of the force field, the shell is converted into the secondary body. If the upper shell has uniform mass density distribution, then the shell forms a ring around equatorial plane of the parental body. In general case, the upper weightless shell is decayed into fragments with different amounts of mass. The comets were formed from the solar shell; the satellites and meteorites were created from the planet's shells. During the evolution of a nonuniform gaseous body, it undergoes the axial and equatorial oblateness by an outer force field of the central parental body. This can be observed by inclination of the planet's and satellite's orbital plane slope relative to the parental equatorial plane. The polar outer force field pressure appears to be higher than equatorial. As a result, the outer polar force field values appeared to be higher than equatorial. Because of this, the polar matter of the upper shell is continuously removed from the equatorial plane. This is why the created bodies are formed mainly in the equatorial plane and form equatorial disk.

So the orbital motion of separated secondary body is defined by the outer force field at the surface of the parental body. The value of this field at the body's surface is a fundamental parameter, which is determined by the body's law of the energy conservation. That is why the orbital velocity of a newly created body is equal to its parental body's first cosmic velocity. The direction of the orbital motion is determined by the Lenz law (see Fig. 3.2). In this connection, it is worth to note that from the point of view of the solar system creation problem, the attempts to find explanation of the observed distribution of the moment of momentum between the axial rotation of the Sun and the planets' orbital motion are not rightful. This is because the planets' orbital velocity demonstrates parental relationship between the planets and the Sun by proving its identity with the first cosmic velocity and the law of the energy conservation.

Thus, it follows from the above scenario that the induced by the matter interaction outer force field of the Sun is responsible for the orbital motion of the planets.

Analogously planets' force field is orbiting their satellites. Doing so, each body with high accuracy records the value of the parent's potential energy at the moment of orbiting. As to the shell's axial rotation, its potential energy is determined by the value of its tangential component. The normal and tangential components of the body's potential energy comprise the total potential energy, which is a conservative parameter.

Justifications of the above dynamical effects, which take part in the creation of the solar system bodies in the framework of Jacobi's dynamics, are presented below. The main dynamical effect, related to the nature of the solar system body creation, is proved by observational data seen in Tables 1.1 and 1.2.

6.1 The Conditions of a Body Separation and Orbiting

It was shown earlier in Sect. 5.1 that in the framework of the Jacobi dynamics, the solution of Kepler's problem is given by the following equations:

$$\sqrt{\Phi} = \frac{B}{A}\left[1 - \varepsilon \cos(\lambda - \psi)\right], \tag{6.2}$$

$$t = \frac{4B}{(2A)^{3/2}}\left[\lambda - \varepsilon \sin(\lambda - \psi)\right] \tag{6.3}$$

$$\omega = \frac{2\pi}{T} = \frac{(2A)^{3/2}}{4B} = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{4}{3}\pi G\rho_0}, \tag{6.4}$$

where ε and ψ are the integration constants depending on the initial values of Jacobi's function Φ and its first derivative $\dot{\Phi}$ at the time moment t_0 (the time here is an independent variable); is the period of virial oscillations; ω is the oscillation; λ is the auxiliary independent variable; $= A_0 = 1/2E > 0$, $B = B_0 = U\sqrt{\Phi_0}$ for radial oscillations; and $= A_r = 1/2E > 0$, $B = B_r = U\sqrt{\Phi_r}$ for rotation of the body.

The product of the oscillation frequency ω of the outer force field and R of the body gives the value of the first cosmic velocity of an artificial satellite, that is, the velocity with which the satellite undertakes gravity attraction (the pressure induced by the outer force field). In order to undertake the attraction, satellite uses its own inner energy of the reactive engine. In this way, the satellite reaches the first cosmic velocity and becomes weightless, that is, its own outer force field reaches equilibrium with the planet's outer force field. After that, the engine is switched off and its motion continues by the energy of the outer force field. In order to be separated from the parental body, its outer shell must reach the state of weightlessness, that is, its own force field has to be in dynamical equilibrium with the parental force field. The secondary body, created from the outer shell, being completely in non-weighty state and in dynamical equilibrium with the parental outer force field, moves farther by that force field with the first cosmic velocity.

The data of Tables 1.1 and 1.2 show that the existing discrepancies in the moment of momentum distribution between the Sun and the planets and also the problems of capture or separation of the planets' cloud are taking off. The secondary body at its creation conserves the parental potential energy through the first cosmic velocity. As to the direction of orbital motion, the Lenz law works here, which evidences about the common nature of the gravity and electromagnetic fields. The specific value (per mass unit) of the planets' and satellites' orbital moment of momentum, which increases with the distance from a central body, has found explanation by the same reasons.

Now we come to the problem of weightlessness for the body's outer shell at the evolution by radiation of energy. First, the structure of the potential and kinetic energies of a celestial body is discussed.

6.2 The Structure of Potential and Kinetic Energies of a Nonuniform Body

In fact, all the celestial bodies of the solar system, including the Sun, are nonuniform creatures. They have a shell structure and the shells themselves are also nonuniform components of the body. It was shown in Sect. 2.2 that according to the artificial satellite data, all the measured gravitational moments of the Earth, including tesseral ones, have significant values. In geophysics, this fact is interpreted as a deviation of the Earth from the hydrostatic equilibrium and attendance of the tangential forces that are continuously developed inside the body. From the viewpoint of the planet's dynamical equilibrium, the fact of the measured zonal and tesseral gravitational moments is a direct evidence of the permanent development of the normal and tangential volumetric forces that are the components of the inner gravitational force field. In order to identify the above effects, the inner force field of the body should be accordingly separated.

The expressions (3.36), (3.37), (3.38), and (3.39) in Chap. 3 indicate that the force function and the polar moment of a nonuniform self-gravitating sphere can be expanded with respect to their components related to the uniform mean density mass and its nonuniformities. In accordance with the superposition principle, these components are responsible for the normal and tangential dynamical effects of a nonuniform body. Such a separation of the potential energy and polar moment of inertia through their dimensionless form factors α and β was done by Garcia et al. (1985) with our interpretation (Ferronsky et al. 1996). Taking into account that the observed satellite irregularities are caused by the nonuniform distribution of the mass density, an auxiliary function relative to the radial density distribution was introduced for separation:

$$\Psi(s) = \int_0^s \frac{(\rho_\mathrm{r} - \rho_0)}{\rho_0} x^2 \,\mathrm{d}x, \tag{6.5}$$

where $s = r/R$ is the ratio of the running radius to the radius of the sphere R, ρ_0 is the mean density of the sphere of radius r, ρ_r is the radial density, x is the running coordinate, and the value (ρ_r–ρ_0) satisfies $\int_0^R (\rho_r - \rho_0) r^2\,dr = 0$ and the function $\Psi(1) = 0$.

The function $\Psi(s)$ expresses a radial change in the mass density of the nonuniform sphere relative to its mean value at the distance r/R. Now we can write expressions for the force function and the moment of inertia by using the structural form factors α and β that were introduced in Sect. 3.6:

$$U = \alpha \frac{GM^2}{R} = 4\pi G \int_0^R r\rho(r)m(r)\,dr, \tag{6.6}$$

$$I = \beta^2 MR^2 = 4\pi \int_0^R r^4 \rho(r)\,dr. \tag{6.7}$$

By (6.5), we can do the corresponding changes of variables. As a result, the expressions for the potential energy U and polar moment of inertia I are found in the form of their components composed of their uniform and nonuniform constituents (Garcia et al. 1985; Ferronsky et al. 1996):

$$\begin{aligned} U &= 4\pi G \int_0^R r\rho(r)m(r)\,dr = \alpha \frac{GM^2}{R} \\ &= \left[\frac{3}{5} + 3\int_0^1 \psi x\,dx + \frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 dx\right] \frac{GM^2}{R}, \end{aligned} \tag{6.8}$$

$$I = \beta^2 MR^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x\,dx\right] MR^2. \tag{6.9}$$

It is known that the moment of inertia multiplied by the square of the frequency ω of the oscillatory–rotational motion of the mass is the kinetic energy of the body. Then, Eq. (6.9) can be rewritten as

$$K = I\omega^2 = \beta^2 MR^2\omega^2 = \left[\frac{3}{5} - 6\int_0^1 \psi x\,dx\right] MR^2\omega^2. \tag{6.10}$$

Let us clarify the physical meaning of the terms in expressions (6.8) and (6.10) of the potential and kinetic energies.

As it follows from (3.36) and Table 3.1, the first terms in (6.8) and (6.10), numerically equal to 3/5, represent α_0 and β_0^2, which are the structural coefficients of the uniform sphere with radius r, the density of which is equal to its mean value. The ratio of the potential and kinetic energies of such a sphere corresponds to the condition of the body's dynamical equilibrium when its kinetic energy is realized in the form of oscillations.

The second terms of the expressions can be rewritten in the form

$$3\int_0^1 \psi x \mathrm{d}x \equiv 3\int_0^1 \left(\frac{\psi}{x}\right) x^2 \,\mathrm{d}x, \tag{6.11}$$

$$-6\int_0^1 \psi x \mathrm{d}x \equiv -6\int_0^1 \left(\frac{\psi}{x}\right) x^2 \,\mathrm{d}x. \tag{6.12}$$

One can see here that the additive parts of the potential and kinetic energies of the interacting masses of the nonuniformities of each sphere shell with the uniform sphere having a radius r of the sphere shell are written. Note that the structural coefficient β of the kinetic energy is twice as high as the potential energy and has the minus sign. It is known from physics that interaction of mass particles, uniform and nonuniform with respect to density, is accompanied by their elastic and inelastic scattering of energy and appearance of a tangential component in their trajectories of motion. In this particular case, the second terms in Eqs. (6.8) and (6.10) express the tangential (torque) component of the potential and kinetic energies of the body. Moreover, the rotational component of the kinetic energy is twice as much as the potential one.

The third term of Eq. (6.8) can be rewritten as

$$\frac{9}{2}\int_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2 \,\mathrm{d}x. \tag{6.13}$$

Here, there is another additive part of the potential energy of the interacting nonuniformities. It is the nonequilibrated part of the potential energy that does not have an appropriate part of the reactive kinetic energy and represents a dissipative component. Dissipative energy represents the electromagnetic energy that is emitted by the body, and it determines the body's evolutionary effects. This energy forms the electromagnetic field of the body (see Chap. 8).

Nonuniformity of the density in this case and later is determined as difference between the density of the given spherical layer and mean value of the density of the sphere with radius of the spherical layer.

Thus, by expansion of the expression of the potential energy and the polar moment of inertia, we obtained the components of both forms of energy that are responsible for oscillation and rotation of the nonuniform body. Applying the above results, we can write separate conditions of the dynamical equilibrium for each form of the motion and separate virial equations of the dynamical equilibrium of their motion.

6.3 Conditions of Dynamical Equilibrium of Oscillation and Rotation of a Body

Equations (6.8) and (6.10) can be written in the form

$$U = (\alpha_0 + \alpha_{\mathrm{t}} + \alpha_\gamma)\frac{GM^2}{R}, \tag{6.14}$$

$$K = \left(\beta_0^2 - 2\beta_{\mathrm{t}}^2\right) MR^2\omega^2, \tag{6.15}$$

where $\alpha_0 = \beta_0^2$ and $\alpha_{\mathrm{t}} = -2\ \beta_{\mathrm{t}}^2$ and the subscripts 0, t, and γ define the radial, tangential, and dissipative components of the considered values.

Because the potential and kinetic energies of the uniform body are equal ($\alpha_0 = \beta_0^2 = 3/5$), from (6.8) and (6.10), one has

$$U_0 = K_0, \tag{6.16}$$

$$E_0 = U_0 + K_0 = 2U_0. \tag{6.17}$$

In order to express dynamical equilibrium between the potential and kinetic energies of the nonuniform interacting masses, we can write, from (6.8) and (6.10),

$$U_{\mathrm{t}} = 2K_{\mathrm{t}}, \tag{6.18}$$

$$E_{\mathrm{t}} = U_{\mathrm{t}} + K_{\mathrm{t}} = 3U_{\mathrm{t}}, \tag{6.19}$$

where E_{t},U_0, K_0, U_{t}, and K_{t} are the total, potential, and kinetic energies of oscillation and rotation, respectively. Note that the energy is always a positive value.

Equations (6.16), (6.17), (6.18), and (6.19) present expressions for uniform and nonuniform components of an oscillating system that serves as the conditions of their dynamical equilibrium. Evidently, the potential energy U_γ of interaction between the nonuniformities, being irradiated from the body's outer shell, is irretrievably lost and provides a mechanism of body's evolution.

In accordance with the classical mechanics, for the above-considered nonuniform gravitating body, being a dissipative system, the torque N is not equal to zero, the angular momentum L of the sphere is not a conservative parameter, and its energy is continuously spent during the motion, that is,

$$N = \frac{\mathrm{d}L}{\mathrm{d}t} \neq 0, \quad L \neq \text{const.}, \quad E \neq \text{const.} > 0.$$

A system physically cannot be conservative if friction or other dissipation forces are present, because $F{\cdot}\mathrm{d}s$ due to friction is always positive and an integral cannot vanish (Goldstein 1980),

$$\oint F \cdot \mathrm{d}s > 0.$$

6.4 Equations of Oscillation and Rotation of a Body and Their Solution

After we have found that the resultant of the body's gravitational field is not equal to zero and the system's dynamical equilibrium is maintained by the virial relationship between the potential and kinetic energies, the equations of a self-gravitating body motion can be written.

Earlier, we (Ferronsky et al. 1987) obtained virial equation for describing and studying the motion of both uniform and nonuniform self-gravitating spheres. Jacobi (1884) derived it from Newton's equations of motion of n mass points and reduced the n-body problem to the particular case of the one-body task with two independent variables, namely, the force function U and the polar moment of inertia $\dot{\Phi}$, in the form

$$\ddot{\Phi} = 2E - U. \tag{6.20}$$

Equation (6.20) represents the energy conservation law and describes the system in scalar U and Φ volumetric characteristics. In Chap. 4, it was shown that Eq. (6.20) is also derived from Euler's equations for a continuous medium and from the equations of Hamilton, Einstein, and quantum mechanics. Its time-averaged form gives the Clausius virial theorem for a system with outer source of forces. It was earlier mentioned that Clausius was deducing the theorem for the application in thermodynamics and, in particular, applied to the assessment and designing of Carnot's machines. As the machines operate in the Earth's outer force field, Clausius introduced the coefficient 1/2 to the term of "living force" or kinetic energy, that is,

$$K = \frac{1}{2}\sum_i m_i v_i^2.$$

As Jacobi has noted, the meaning of the introduced coefficient was to take into account only the kinetic energy generated by the machine, but not by the Earth's gravitational force. That was demonstrated, for instance, by the work of a steam hammer for driving piles. The machine raises the hammer, but it falls down under the action of the force of the Earth's gravity. That is why the coefficient 1/2 of the kinetic energy of a uniform self-gravitating body in Eqs. (6.8), (6.9) and (6.10) has disappeared. In its own force field, the body moves due to the release of its own energy.

Earlier, by means of the relation $U\sqrt{\Phi} \approx$ const., an approximate solution of Eq. (6.20) for a nonuniform body was obtained (Ferronsky et al. 1987, 2011). Now, after expansion of the force function and polar moment of inertia, at $U_\gamma = 0$ and taking into account the conditions of the dynamical equilibrium (6.17) and (6.19), Eq. (6.20) can be written separately for the radial and tangential components in the form

$$\ddot{\Phi}_0 = \frac{1}{2} E_0 - U_0, \tag{6.21}$$

$$\ddot{\Phi}_\mathrm{t} = \frac{1}{3} E_\mathrm{t} - U_\mathrm{t}. \tag{6.22}$$

Taking into account the functional relationship between the potential energy and the polar moment of inertia

$$|U|\sqrt{\Phi} = B = \text{const.}$$

and also taking into account that the structural coefficients $\alpha_0 = \beta_0^2$ and $2\alpha_0 = \beta_\mathrm{t}^2$, both Eqs. (6.21) and (6.22) are reduced to an equation with one variable and have a rigorous solution:

$$\Phi_\mathrm{n} = -A + \frac{B_\mathrm{n}}{\sqrt{\Phi_\mathrm{n}}}, \tag{6.23}$$

where A_n and B_n are the constant values and subscript "n" defines the nonuniform body.

The general solution of Eq. (6.23) is (5.14) and (5.15):

$$\sqrt{\Phi_\mathrm{n}} = \frac{B_\mathrm{n}}{A_\mathrm{n}}\left[1 - \varepsilon \cos(\xi - \varphi)\right], \tag{6.24}$$

$$\omega = \frac{2\pi}{T_\mathrm{v}} = \frac{(2A_\mathrm{n})^{3/2}}{4B_\mathrm{n}}, \tag{6.25}$$

where ε and φ are, as previously, the integration constants depending on the initial values of Jacobi's function Φ_n and its first derivative $\dot{\Phi}_\mathrm{n}$ at the time moment t_0 (the time here is an independent variable); T_v is the period of virial oscillations; ω is the oscillation frequency; ξ is the auxiliary independent variable; $A_\mathrm{n} = A_0 - 1/2E_0 > 0$; $B_\mathrm{n} = B_0 = U_0\sqrt{\Phi_0}$ for radial oscillations; $A_\mathrm{n} = A_\mathrm{t} = -1/3E_\mathrm{t}, > 0$; and $B_\mathrm{n} = B_\mathrm{t} = U_\mathrm{t}\sqrt{\Phi_\mathrm{t}}$ for rotation of the body.

The expressions for the Jacobi function and its first derivative in an explicit form can be obtained after transforming them into the Lagrange series:

$$\sqrt{\Phi_n} = \frac{B}{A}\left[1 + \frac{\varepsilon^2}{2} + \left(-\varepsilon + \frac{3}{8}\varepsilon^3\right)\cos\, M_c - \frac{\varepsilon^2}{2}\cos\, 2M_c - \frac{3}{8}\varepsilon^3\cos\, 3M_c + \cdots\right],$$
$$\Phi_n = \frac{B^2}{A^2}\left[1 + \frac{3}{2}\varepsilon^2 + \left(-2\varepsilon + \frac{\varepsilon^3}{4}\right)\cos\, M_c - \frac{\varepsilon^2}{2}\cos\, 2M_c - \frac{\varepsilon^3}{4}\cos\, 3M_c + \cdots\right],$$
$$\dot{\Phi}_n = \sqrt{\frac{2}{A}}\varepsilon B\left[\sin\, M_c + \frac{1}{2}\varepsilon\sin\, 2M_c + \frac{\varepsilon^2}{2}\sin\, M_c\left(2\cos^2 M_c - \sin^2 M_c\right) + \cdots\right]. \tag{6.26}$$

Radial frequency of oscillation ω_{0r} and angular velocity of rotation ω_{tr} of the shells of radius r can be rewritten from (6.25) as

$$\omega_{0r} = \frac{(2A_0)^{3/2}}{4B_0} = \sqrt{\frac{U_{0r}}{J_{0r}}} = \sqrt{\frac{\alpha_{0r}^2 G m_r}{\beta_{0r}^2 r^3}} = \sqrt{\frac{4}{3}\pi G\rho_{0r}}, \tag{6.27}$$

$$\omega_{tr} = \frac{(2A_t)^{3/2}}{4B_t} = \sqrt{\frac{2U_{tr}}{J_{tr}}} = \sqrt{\frac{2\alpha_{tr}^2 G m_r}{\beta_{tr}^2 r^3}} = \sqrt{\frac{4}{3}\pi G\rho_{0r}k_{er}}, \tag{6.28}$$

where U_{0r} and U_{tr} are the radial and tangential components of the force function (potential energy), J_{0r} and $J_{tr} = 2/3J_{0r}$ are the polar and axial moments of inertia, $\rho_{0r} = \frac{1}{V_r}\int_{V_r}\rho(r)\mathrm{d}V_r$, $\rho(r)$ is the law of radial density distribution, ρ_{0r} is the mean density value of the sphere with a radius r, V_r is the sphere volume with a radius r, $2\alpha_{tr} = \beta_{tr}^2$, and k_{er} is the dimensionless coefficient of the energy dissipation or tidal friction of the shells equal to the shell's oblateness.

The relations (6.24) and (6.25) represent Kepler's laws of body rotation in dynamical equilibrium. In the case of uniform mass density distribution, the frequency of oscillation of the sphere's shells with radius r is $\omega_{0r} = \omega_0 = \text{const}$. It means that here all the shells are oscillating with the same frequency. Thus, it appears that only nonuniform bodies are rotating systems.

Rotation of each body's shell depends on the effect of the potential energy scattering at the interaction of masses of different density. As a result, a tangential component of energy appears, which is defined by the coefficient k_{er}. In geodynamics, the coefficient is known as the geodynamical parameter. Its value is equal to the ratio of the radial oscillation frequency and the angular velocity of a shell and can be obtained from Eqs. (6.27) to (6.28), that is,

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{{\omega_t}^2}{\frac{4}{3}\pi G\rho_0}. \tag{6.29}$$

It was found that in the general case of a three-axial (a, b, c) ellipsoid with the ellipsoidal law of density distribution, the dimensionless coefficient $k_e \in [0,1]$ is equal to (Ferronsky et al. 1987, 2011)

$$k_r = \frac{F(\varphi, f)}{\sin\,\varphi} \bigg/ \frac{a^2 + b^2 + c^2}{3a^2},$$

where $\varphi = \arcsin\sqrt{\frac{(a^2 - c^2)}{a^2}}$, $f = \sqrt{\frac{(a^2 - b^2)}{(a^2 - c^2)}}$, and $F(\varphi, f)$ is an incomplete elliptic integral of the first degree in the normal Legendre form.

Thus, in addition to the solution of radial oscillations obtained earlier (Ferronsky et al. 1987, 2011), now we have a solution of its rotation. It is seen from expression (6.27) that the shell oscillation does not depend on the phase state of the body's mass and is determined by its density.

It follows from Eqs. (6.27) to (6.28) that in order to obtain the frequency of oscillation and angular velocity of rotation of a nonuniform body, the law of radial density distribution should be revealed. This problem will be considered later on. But before that, the problem of the nature of a body shell separation with respect to its density needs to be solved.

6.5 The Nature and Mechanism of Body Shell Differentiation

It is well known that celestial bodies have a quasi-spherical shell structure. This phenomenon has been confirmed by recording and interpretation of seismic longitudinal and transversal wave propagation during earthquakes. In order to understand the physics and mechanism of a body mass differentiation with respect to its density, we apply Roche's tidal dynamics.

Newton's theorem of gravitational interaction between a material point and a spherical layer states that the layer does not affect a point located inside the layer. On the contrary, the outside-located material point is affected by the spherical layer. Roche's tidal dynamics is based on the above theorem. His approach is as follows (Ferronsky et al. 1996).

There are two bodies of masses M and m interacting in accordance with Newton's law (Fig. 6.1a).

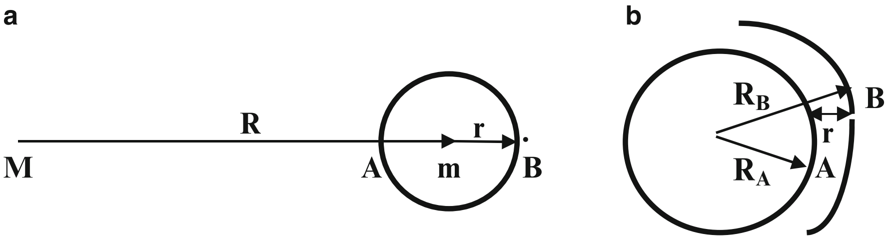


Fig. 6.1 The tidal gravitational stability of a sphere (**a**) and the sphere layer (**b**)

Let $M \gg m$ and $R \gg r$, where r is the radius of the body m and R is the distance between the bodies M and m. Assuming that the mass of the body M is uniformly distributed within a sphere of radius R, we can write the accelerations of the points A and B of the body m as

$$q_A = \frac{GM}{(R-r)^2} - \frac{Gm}{r^2}, \quad q_B = \frac{GM}{(R+r)^2} + \frac{Gm}{r^2}.$$

The relative tidal acceleration of the points A and B is

$$\begin{aligned} q_{AB} &= G\left[\frac{M}{(R-r)^2} - \frac{M}{(R+r)^2} - \frac{2m}{r^2}\right] \\ &= \frac{4\pi}{3} G\left[\rho_M R^3 \frac{Rr}{(R^2-r^2)^2} - 2\rho_m r\right] \\ &\approx \frac{8\pi}{3} Gr\left(2\rho_M - \rho_m\right). \end{aligned} \tag{6.30}$$

Here $\rho_M = M/\frac{4}{3}\pi R^3$ and $\rho_m = m/\frac{4}{3}\pi r^3$ are the mean density distributions for the spheres of radius R and r. Roche's criterion states that the body with mass m is stable against the tidal force disruption of the body M if the mean density of the body m is at least twice as high as that of the body M in the sphere with radius R. Roche considered the problem of the interaction between two spherical bodies without any interest in their creation history and in how the forces appeared. From the point of view of the origin of celestial bodies and of the interpretation of dynamical effects, we are interested in the tidal stability of separate envelopes of the same body. For this purpose, we can apply Roche's tidal dynamics to study the stability of a nonuniform spherical envelope.

Let us assess the tidal stability of a spherical layer of radius R and thickness $r = R_B - R_A$ (Fig. 6.1b). The layer of mass m and mean density $\rho_m = m/4\pi\, {R_A}^2 r$ is affected at point A by the tidal force of the sphere of radius R_A. The mass of the sphere is M and mean density $\rho_M = M/(\frac{4}{3})\pi R_A^3$. The tidal force in point B is generated by the sphere of radius $R + r$ and mass $M + m$. Then, the accelerations of the points A and B are

$$q_A = \frac{GM}{R_A^2}, \quad q_B = \frac{G\,(M+m)}{(R_A+r)^2}.$$

The relative tidal acceleration of the points and is

$$\begin{aligned} q_{AB} &= GM\left[\frac{1}{R_A^2} - \frac{1}{(R_A+r)^2}\right] - \frac{Gm}{(R_A+r)^2} \\ &= \left(\frac{8}{3}\pi G\rho_M - 4\pi G\rho_m\right) r = 4\pi Gr\left(\frac{2}{3}\rho_M - \rho_m\right), \quad (R \gg r). \end{aligned} \tag{6.31}$$

Equations (6.30) and (6.31) give the possibility to understand the nature of a body shell separation including some other dynamical effects.

6.6 Self-Similarity Principle and Radial Component of Nonuniform Sphere

It follows from Eq. (6.31) that in the case of the uniform density distribution ($\rho_m = \rho$), all spherical layers of the gravitating sphere move to the center with accelerations and velocities that are proportional to the distance from the center. It means that such a sphere contracts without loss of its uniformity. This property of self-similarity of a dynamical system without any discrete scale is unique for a uniform body (Ferronsky et al. 1996).

A continuous system with a uniform density distribution is also ideal from the point of view of Roche's criterion of stability with respect to the tidal effect. That is why there is a deep physical meaning in the separation of the first term of potential energy in expression (6.8). A uniform sphere is always similar in its structure in spite of the fact that it is a continuously contracting system. Here, we do not consider the Coulomb force effect. In this case, we have considered the specific proton and electron branches of the evolution of the body (see Chap. 8).

Note that in Newton's interpretation, the potential energy has a nonadditive category. It cannot be localized even in the simplest case of the interaction between two mass points. In our case of a gravitating sphere as a continuous body, for the interpretation of the additive component of the potential energy, we can apply Hooke's concept. According to Hooke, there is a linear relationship between the force and the caused displacement. Therefore, the displacement is in square dependence on the potential energy. Hooke's energy belongs to the additive parameters. In the considered case of a gravitating sphere, the Newton force acting on each spherical layer is proportional to its distance from the center. Thus, from a physical point of view, the interpretations of Newton and Hooke are identical.

At the same time, in the two approaches, there is a principal difference even in the case of uniform distribution of the body density. According to Hooke, the cause of displacement, relative to the system, is the action of the outer force. And if the total energy is equal to the potential energy, then equilibrium of the body is achieved. The potential energy plays here the role of elastic energy. The same uniform sphere with Newton's forces will be contracted. The body's all the elementary shells will move without a change in uniformity in the density distribution. But the first terms of Eqs. (6.8), (6.9), and (6.10) show that the tidal effects of a uniform body restrict the motion of the interacting shells toward the center. In accordance with Newton's third law and the d'Alembert principle, the attraction forces, under the action of which the shells move, should have equal and opposite direct forces of Hooke's elastic counteraction. In the framework of the elastic gravitational interaction of shells, the dynamical equilibrium of a uniform sphere is achieved in the form of its elastic oscillations with equality between the potential and kinetic energies. The uniform sphere is dynamically stable relative to the tidal forces in all of its shells during the time of the system contraction. Because the potential and kinetic energies of a sphere are equal, its total energy in the framework of the averaged virial theorem within one period of oscillation is accepted formally as equal to zero. Equality of the potential and kinetic energies of each shell means the equality of the centripetal (gravitational) and centrifugal (elastic constraint) accelerations. This guarantees the system remaining in dynamical equilibrium. On the contrary, all the spherical shells will be contracted toward the gravity center, which, in the case of the sphere, coincides with the inertia center but does not coincide with the geometric center of the masses. Because the gravitational forces are acting continuously, the elastic constraint forces of the body's shells are also reacting continuously. The physical meaning of the self-gravitation of a continuous body consists in the permanent work that applies the energy of the interacting shell masses on one side and the energy of the elastic reaction of the same masses in the form of oscillating motion on the other side. At dynamical equilibrium, the body's equality of potential and kinetic energies means that the shell motion should be restricted by the elastic oscillation amplitude of the system. Such an oscillation is similar to the standing wave that appears without the transfer of energy into outer space. In this case, the radial forces of the shell's elastic interactions along the outer boundary sphere should have a dynamical equilibrium with the forces of the outer gravitational field. This is the condition of the system to be held in the outer force field of the mother's body. Because of this, while studying the dynamics of a conservative system, its rejected outer force field should be replaced with the corresponding equilibrated forces as they do, for instance, in Hooke's theory of elasticity.

Thus, from the point of view of dynamical equilibrium, the first terms in Eqs. (6.8) and (6.10) represent the energy that provides the field of the radial forces in a nonuniform sphere. Here, the potential energy of the uniform component plays the role of the active force function, and the kinetic energy is the function of the elastic constraint forces.

6.7 Charge-Like Motion of Nonuniformities and Tangential Component of the Force Function

Let us now discuss the tidal motion of nonuniformities due to their interactions with the uniform body. The potential and kinetic energies of these interactions are given by the second terms in Eqs. (6.8) and (6.10). In accordance with (6.31), the nonuniformity motion looks like the motion of electrical charges interacting on the background of a uniform sphere contraction. Spherical layers with densities exceeding those of the uniform body (positive anomalies) come together and move to the center in elliptic trajectories. The layers with a deficit of the density (negative anomalies) come together but move from the center on the parabolic path. Similar anomalies come together, but those with the opposite sign are dispersed with forces proportional to the layer radius. In general, the system tends to reach a uniform and equilibrium state by means of redistribution of its density up to the uniform limit. Both motions happen not relative to the empty space, but relative to the oscillating motion of the uniform sphere with a mean density. Separate consideration of motion of the uniform and nonuniform components of a heterogeneous sphere is justified by the superposition principle of the action of forces that we keep here in mind. The considered motion of the nonuniformities looks like the motion of the positive and negative charges interacting on the background of the field of the uniformly dense sphere (Ferronsky et al. 1996). One can see here that in the case of gravitational interaction of mass particles of a continuous body, their motion is the consequence not only of mutual attraction but also of mutual repulsion by the same law $1/r^2$. In fact, in the case of a real natural nonuniform body, it appears that the Newton and Coulomb laws are identical in details. Later on, while considering a body's by-density differentiated masses, the same picture of motion of the positive and negative anomalies will be seen.

If the sphere shells, in turn, include density nonuniformities, then by means of Roche's dynamics, it is possible to show that the picture of the nonuniformity motion does not differ from that considered above.

In physics, the process of interaction of particles with different masses without redistribution of their moments is called elastic scattering. The interaction process resulting in the redistribution of their moments and change in the inner state or structure is called inelastic scattering. In classical mechanics, while solving the problems of motion of the uniform conservative systems (like motion of the material point in the central field or motion of the rigid body), the effects of the energy scattering do not appear. In the problem of dynamics of the self-gravitating body, where interaction of the shells with different masses and densities is considered, the elastic and inelastic scattering of the energy becomes an evident fact followed from the consideration of the physical meaning of the expansion of the energy expressions in the form of (6.8) and (6.10). In particular, their second terms represent the potential and kinetic energies of gravitational interaction of masses having a nonuniform density with the uniform mass and express the effect of elastic scattering of density-different shells. Both terms differ only in the numeric coefficient and sign. The difference in the numerical coefficient evidences that the potential energy here is equal to half of the kinetic one ($U_t = 1/2K_t$). This part of the active and reactive force function characterizes the degree of the noncoincidence of the volumetric center of inertia and that of the gravity center of the system expressed by Eqs. (3.38) and (3.39). This effect is realized in the form of the angular momentum relative to the inertia center.

Thus, we find that inelastic interaction of the nonuniformities with the uniform component of the system generates the tangential force field that is responsible for the system rotation. In other words, in the scalar force field of the by-density uniform body, the vector component appears. In such a case, we can say that, by analogy with an electromagnetic field, in the gravitational scalar potential field of the nonuniform sphere $U(R, t)$, the vector potential $A(R, t)$ appears for which $U = \text{rot } A$, and the field $U(R, t)$ will be solenoidal. In this field, the conditions for vortex motion of the masses are born, where $\text{div } A = 0$. This vector field, which in electrodynamics is called solenoidal, can be represented by the sum of the potential and vector fields. The fields, in addition to the energy, acquire moments and have a discrete-wave structure. In our case, the source of the wave effects appears to be the interaction between the elementary shells of the masses by means of which we can construct a continuous body with a high symmetry of forms and properties. The source of the discrete effects can be represented by the interacting structural components of the shells, namely, atoms, molecules, and their aggregates. We shall continue the discussion about the nature of the gravitational and electromagnetic energies in Chap. 8.

6.8 Physical Meaning of the Archimedes and Coriolis Forces

The Archimedes principle states that *The apparent loss in weight of a body totally or partially immersed in a liquid is equal to the weight of the liquid displaced.* We saw in Sect. 6.5 that the principle is described by Eq. (6.31) and the forces that sink down or push out the body or the shell are of a gravitational nature. In fact, in the case of $\rho_m = \rho_M$, the body immersed in a liquid (or in any other medium) is kept in place due to equilibrium between the forces of the body's weight and the forces of the liquid reaction. In the case of $\rho_m > \rho_M$ or $\rho_m < \rho_M$, the body is sinking or floating up depending on the resultant of the above forces. Thus, the Archimedes forces seem to have a gravity nature and are the radial component of the body's inner force field.

It is assumed that the Coriolis forces appeared as an effect of the body motion in the rotational system of coordinates relative to the inertial reference system. In this case, rotation of the body is accepted as the inertial motion, and the Coriolis forces appear to be the inertial ones. It follows from the solution of Eq. (6.22) that the Coriolis forces appear to be the tangential component of body's inner force field, and the body rotation is caused by the moment of those forces that are relative to the three-dimensional center of inertia, which also does not coincide with the three-dimensional gravity center.

In accordance with Eq. (6.31) of the tidal acceleration of an outer nonuniform spherical layer at $\rho \neq \rho_m$, the mechanism of the gravitational density differentiation of masses is revealed. If $\rho < \rho_m$, then the shell immerses (is attracted) up to the level where $\rho = \rho_m$. At $\rho > \rho_m$, the shell floats up to the level where $\rho = \rho_m$, and at $\rho > 2/3\rho_m$, the shell becomes a self-gravitating one. Thus, in the case when the density increases toward the sphere's center, which is the Earth's case, each overlying stratum appears to be in a suspended state due to repulsion by the Archimedes forces, which, in fact, are a radial component of the gravitational interaction forces.

The effect of the gravitational differentiation of masses explains the nature of creation of shell-structured celestial bodies and corresponding processes (for instance, the Earth's crust and its oceans, geotectonic, orogenic, and seismic processes, including earthquakes). All these phenomena appear to be a consequence of the continuous gravitational differentiation in the density of the planet's masses. We assume that creation of the Earth and the solar system as a whole was resulted by this effect. For instance, the mean value of the Moon's density is less than two third of the Earth's, that is, $\rho < 2/3\rho_m$. If one assumes that this relation was maintained during the Moon's formation, then, in accordance with Eq. (6.31), this body was separated at the earliest stage of the Earth's mass differentiation. Creation of the body from the separated shell should occur by means of the cyclonic eddy mechanism, which was proposed in due time by Descartes and which was unjustly rejected. If we take into account the existence of the tangential forces in the nonuniform mass, then the above mechanism seems to be realistic.

Thus, we learned the nature and mechanism of initially heavy outer shell of a self-gravitating body into a weightless state. Such a weightless shell by its own tangential component of the potential energy is transferred into vortex cloud and, after reaching the dynamical equilibrium (self-gravitating state), becomes a planet, satellite, or any other body. In the case of uniform density of the weightless shell, it transfers into a nebula, equatorial ring, or diffuse matter. The orbital motion of a newly created planet, satellite of other body, is determined by the first cosmic velocity of the parental body. And the axial rotation depends on the value of nonuniformity in density.

6.9 Initial Values of Mean Density and Radius of a Body

Thus, it follows from Eq. (6.31) that the outer shell of a gaseous body after reaching its density equal to two third from the mean value of the total body becomes weightless. If the own shell's density is nonuniform, then by its tangential component of the energy, the shell is transferred into a secondary body in the form of vortex creature. As it is seen from observation, new bodies are formed in different regions of a protoparent body's surface. The large-in-mass bodies like stars, planets, and satellites are firmed mainly in the equatorial zone due to difference in value for the polar and equatorial outer force field. Because of this, a new body inherits the polar and equatorial obliquity, the value of which reflects the degree of the nonuniformity of its density. The comets, asteroids, and smaller bodies are formed in the other regions of the parental bodies. The high eccentric orbits of such bodies prove this fact. The inclination of the new body's orbital plane relative to the parental equatorial plane can be up to close to 180°.

The following initial values of density ρ_i and radius R_i of the protosun and protoplanets can be obtained on the basis of their dynamic equilibrium state.

The protosolar gaseous cloud has separated from the protogalaxy body when its outer shell in the equatorial domain has reached the state of weightlessness. In fact, the gaseous cloud should represent chemically nonhomogeneous rotating body. As it follows from Roche's dynamics (Eq. 6.31), the mean density of the gaseous protogalaxy outer shell should be $\rho_s = 2/3\rho_g$. The condition $\rho_s = 2/3\rho_g$ is the starting point of separation and creation of the protosun from the outer protogalaxy shell. Accepting the above-described mechanism of formation of the secondary body, we can find the mean density of the protogalaxy at the moment of the protosun separation as

$$\rho_g = \frac{m_\Gamma}{\frac{4}{3}\pi R^3} = \frac{2.5 \cdot 10^{41}}{\frac{4}{3} \cdot 3.14 \cdot (2.5 \cdot 10^{20})^3} = 1.67 \cdot 10^{-21}\,\text{kg/m}^3 = 1.67 \cdot 10^{-24}\,\text{g/cm}^3.$$

Here, the protogalaxy radius is equal to the semimajor orbital axis of the protosun, that is, $R_u = 2.5 \cdot 10^{20}$ m.

The mean density of the separated protogalaxy shell is

$$\rho_c = 2/3\rho_g = 2/3 \cdot 1.67 \cdot 10^{-24} = 1.11 \cdot 10^{-24}\,\text{g/cm}^3.$$

In accordance with Eq. (6.30), the mean density and radius of the initially created protosun body should be

$$\rho_s = 2\rho_g = 2 \cdot 1.67 \cdot 10^{-24} = 3.34 \cdot 10^{-24}\,\text{g/cm}^3;$$

$$R_c = \sqrt[3]{\frac{2 \cdot 10^{33}}{\frac{4}{3} 3.34 \cdot 10^{-24}}} = 7.5 \cdot 10^{18}\,\text{cm} = 7.5 \cdot 10^{16}\,\text{m}.$$

The mean density and the radius of the initially created proto-Jupiter, proto-Earth, and proto-Moon are as follows:

$$\text{The Proto-Jupiter}: \ \rho_j = 2 \cdot 10^{-9}\,\text{g/cm}^3, \quad R_j = 6.2 \cdot 10^{13}\,\text{cm} = 6.2 \cdot 10^{11}\,\text{m};$$

$$\text{The proto-Earth}: \ \rho_e = 2.85 \cdot 10^{-7}\,\text{g/cm}^3, \ R_e = 1.9 \cdot 10^{11}\,\text{cm} = 1.9 \cdot 10^{9}\,\text{m};$$

$$\text{The Proto-Moon}: \rho_m = 5 \cdot 10^{-4}\,\text{g/cm}^3, \quad R_m = 1.1 \cdot 10^{9}\,\text{cm} = 1.1 \cdot 10^{7}\,\text{m};$$

Analogous unified process was repeated for all the planets and their satellites.

From the analysis of the above observational and calculated data, the following conclusions are made:

1. The planets of the solar system were created from a common nonuniform in density self-gravitating protosolar cloud, which has separated during evolution on shells with different densities. In accordance with the Roche's tidal dynamics, after the outer shell reaching density equal to two third from the cloud's mean value (the condition of the weightlessness relative to the total body), by the inner force field and the tangential component of the potential energy, the protoplanets after becoming self-gravitating bodies were formed and separated. Analogous processes have taken place at creation of the satellites from the planets. In addition, accumulation of the "light" matter in the outer shells took place gradually and accompanied by separation of small portions in the form of comets and other bodies and dust matter being weightless relative to the surrounding weighted bodies.

2. The orbital velocities of the planets and satellites, which correspond to the first cosmic velocity of the parental bodies, appear to be an effect of the outer force field, which is realized at the moment when the shell reaches its weightlessness state. The orbital motion of the planets, satellites, and other bodies in the outer force field results from the laws of electrodynamics.
3. The small planets of the asteroid belt have created from the protosolar cloud by the common law. Appraising by the orbital velocities, there are no features of their creation because of a body destruction.
4. The axial rotation of the Sun, planets, and satellites has taken and takes place by tangential component of the inner force field. The axial rotation has never been inertial like rigid body. The body's angular moment depends on the friction (weight) of the rotating masses, and to the contrary of the orbital moment of momentum, it has not remained at a conservative value. The orbital angular momentum is the fundamental and conservative parameter because it expresses the law of the body's energy conservation law. The angular momentum of the Sun itself expresses only the tangential component of its potential energy, which is a small part of the total potential energy of the body (see minus sign in Eq. 6.10). The direction of revolution and rotation of all the planets and satellites is governed by the force field of the parental body and determined, as in electrodynamics, by the Lenz law.

The discussed physics and kinematics of creation and separation of the solar system bodies prove the Huygens law of motion on semicubical parabola of his watch pendulum, which synchronously follows the Earth's motion. Relationship between the evolute and the evolvent represents the relationship between function and its derivative or between function and its integral. For the Huygens oscillating pendulum, the suspension filament starts unrolling in a fixed point. In the case of a celestial body, creation of a satellite starts in a fixed point of its parental body where the initial conditions are transferred by Kepler's third law, which is the consequence of a body creation.

References

Ferronsky VI, Denisik SA, Ferronsky SV (1987) Jacobi dynamics. Reidel, Dordrecht

Ferronsky VI, Denisik SA, Ferronsky SV (1996) Virial oscillations of celestial bodies: V. The structure of the potential and kinetic energies of a celestial body as a record of its creation history. Celest Mech Dyn Astron 64:167–183

Ferronsky VI, Denisik SA, Ferronsky SV (2011) Jacobi dynamics, 2nd edn. Springer, Dordrecht/Heidelberg

Garcia LD, Mosconi MB, Sersic JL (1985) A global model for violent relaxation. Astrophys Space Sci 113:89–98

Goldstein H (1980) Classical mechanics, 2nd edn. Addison-Wesley, Reading

Jacobi CGJ (1884) Vorlesungen über Dynamik. Klebsch, Berlin

Chapter 7
The Body's Evolutionary Processes as Effects of Energy Emission

Abstract Several problems of the gravitational evolution of a gaseous sphere, based on Jacobi's virial equation and the relationship between the potential energy and the moment of inertia of the sphere in the form $-U\sqrt{\Phi} = \alpha\beta G m^{5/2}$, are considered. The solving problems are as follows:

– Equilibrium boundary conditions for a self-gravitating gaseous sphere
– Velocity of gravitational differentiation of a gaseous sphere
– The luminosity–mass relationship
– Bifurcation of a dissipative system
– Cosmochemical effects
– Radial distribution of mass density and the body's inner force field
– Oscillation frequency and angular velocity of shell rotation
– The nature of precession, nutation, and body's equatorial plane obliquity
– The nature of Chandler's effect of the Earth pole wobbling

All the above tasks have physical formulation and mathematical solution.

We consider here several problems of the gravitational evolution of a gaseous sphere based on Jacobi's virial equation and the relationship between the potential energy and the moment of inertia of the sphere in the form

$$-U\sqrt{\Phi} = \alpha \frac{Gm^2}{R} \sqrt{m(\beta R)^2} = \alpha\beta G m^{5/2}, \tag{7.1}$$

where U is the potential energy of the sphere, I is the polar moment of inertia, G is the gravitational constant, m is the body mass, R is the sphere radius, and α and β are dimensionless structural parameters depending on the radial mass density distribution of the spherical body.

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_7,

From (7.1), and taking into account Eqs. (3.33), (3.35), (6.14), and (6.15), we have the following relations between the structural form factors:

$$\alpha = \frac{r_g}{R} \quad \text{and} \quad \beta = \frac{r_m}{R}, \tag{7.2}$$

$$\alpha\beta = \text{const.}, \tag{7.3}$$

where $\alpha = (\alpha_0 + \alpha_t + \alpha_\gamma)$; $\beta = (\beta_0 - \beta_t)$; $\alpha_0 = \beta_0^2 = 0.6$; $\alpha_t = 2\beta_t^2$; $\alpha_0\beta_0 = a_0 =$ const; r_g and r_m are the reduced gravity radius and radius of inertia; and α_0, β_0, α_t, α_γ, and β_t are form factors of the normal, tangential, and dissipative components of the energy for nonuniform mass density distribution of a system.

In Chap. 6, we found that the constancy of the form factor product (7.3) is independent of the body mass, radius, and radial mass density distribution for spherical and elliptic bodies. Equation (7.1) is therefore a key expression in our further consideration.

7.1 Equilibrium Boundary Conditions for a Self-Gravitating Gaseous Sphere

It is well known that polytropic models require the boundary mass density of a gravitating body to be rigorously equal to zero. Hence, this condition gives us no opportunity to consider any physical processes during evolution.

If Eq. (7.1) for the spherical and elliptical gravitating system is valid, it allows us to consider convenient boundary conditions that can be used in the study of evolutionary problem.

In deriving the physical boundary conditions for a self-gravitating and rotating gaseous sphere, we consider its rotation as an effect of the tangential component of energy generated by the interacted nonuniform particles. As shown in Chap. 6, the ellipticity of the body is formed not as a result of its rotation but because of its self-gravitation. The key relationship (7.1) used here as the basis of our consideration prevents any possible errors. When we have to introduce the moment of inertia, the rotating sphere boundary at the equator will be defined by Kepler's law.

The fact that gaseous sphere boundary equilibrium conditions differ from those of the interior explains the difference between a free molecular boundary particle movement and an internal chaotic one. It is a consequence of the discrete matter structure dominant at the boundary (Jeans 1919).

Let us now consider the thermodynamic boundary conditions. Surely, we can define the boundary temperature only in the case of its real existence, which, in turn, depends on the existence of the thermodynamic equilibrium between matter and radiation. Otherwise, it cannot be considered as black body radiation, and the Stefan–Boltzmann equation is inapplicable.

Thermodynamic equilibrium at the boundary can be reached only when the energy and the momentum carried away by the radiation flow is greater than that carried away by the flow of particles from the sphere surface per unit time. Such a surface cannot increase further without disturbing the thermodynamic equilibrium.

We shall consider the evolutionary process of the gaseous sphere to be a successive series of hydrodynamic states in equilibrium. We shall also assume that the radiation energy loss causes the sphere to contract during the time periods between the equilibrium states.

Taking these ideas into account, we can express the hydrodynamic equilibrium at the boundary either by an expression representing particle flow "locking" by the gravitational force or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface, which can be written in the form

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{v}^2}{R}, \tag{7.4}$$

where μ is the mass of the particle and $\bar{v}$ is the velocity of the particle heat movement at the sphere boundary of the pole (more precisely, it is the velocity of a particle running from the gravitational field).

For gravitational contraction between any two equilibrium states, Eq. (7.4) must be written as

$$\frac{Gm\mu}{R^2} > \frac{\mu\bar{v}^2}{R}. \tag{7.5}$$

Let us prove that expression (7.4) for the gaseous spherical body boundary satisfies the virial relations.

First, we consider one particle at the sphere boundary surface with mass μ and moving in the volumetric central field of the body with mass m and radius R. Then, it is easy to see that

$$\left(\frac{\mu\ddot{R}^2}{2}\right) = \mu\left[\ddot{\bar{R}}\bar{R} + \left(\dot{\bar{R}}\right)^2\right], \tag{7.6}$$

where the kinetic energy K_p of the particle is

$$\mu\left(\dot{\bar{R}}\right)^2 = \frac{2\mu v^2}{2} = 2K_\mathrm{p} \tag{7.7}$$

From Newton's law, we have

$$\ddot{\bar{R}} = -\frac{Gm}{R^3}R. \tag{7.8}$$

The potential energy U_p of the particle in the gravitational field of the body is

$$\mu \ddot{\bar{R}} \bar{R} = -\frac{Gm\mu}{R^3} \left(\bar{R} \bar{R} \right) = -\frac{Gm\mu}{R} = U_p. \tag{7.9}$$

Therefore,

$$\frac{d^2}{dt^2} \left(\frac{\mu R^2}{2} \right) = U_p + 2K_p. \tag{7.10}$$

Summing over all particles at the boundary layer and neglecting their interaction energy, we obtain

$$\frac{d^2}{dt^2} \left(\frac{m_s R^2}{2} \right) = U_s + 2K_s, \tag{7.11}$$

where m_s is the mass of the boundary spherical layer.

Or finally,

$$\frac{3}{4} \ddot{I}_s = U_s + 2K_s,$$

$$\ddot{\Phi}_s = U_s + 2K_s, \tag{7.12}$$

which represents Jacobi's virial equation for a spherical gaseous layer.

The exchange of particles between the gaseous body and its boundary layer takes place at the same radius R and lasts for a short time, while the total mass of the layer remains constant. So Eq. (7.12) is rigorous.

The solution of Eq. (7.12) will be exactly the same as that obtained in Chap. 5 for a gravitating sphere, except that the corresponding parameters of the sphere must be replaced by those of the boundary layer.

If one time averages over time intervals that are longer than the period of boundary-layer oscillations, then the left-hand side of Eq. (7.12) tends to zero (i.e., the layer enters into the outer force field) and a quasi-equilibrium boundary state is obtained, determined by the generalized classical virial relation between the potential and kinetic energies:

$$\dot{\Phi} = U_s + 2K_s. \tag{7.13}$$

Thus, we have proved that Eq. (7.4) written for the gaseous sphere boundary is a virial relation. We shall use this expression further in solving the problem of contraction velocity for gravitating gaseous sphere.

7.2 Velocity of Gravitational Differentiation of a Gaseous Sphere

In considering the evolution of a gaseous sphere, one does not usually take into account its rotation because the total kinetic energy exceeds the rotational energy. Other authors who accepted the rotation of the gaseous sphere could not manage with the angular momentum accepted as conservative value during contraction (Zeldovich and Novikov 1967; Spitzer 1968; Alfven and Arrhenius 1970).

It was shown in Chaps. 3 and 6 that the main part of kinetic energy of a celestial body is represented by the oscillatory energy of the interacting elementary particles. The rotational part is much smaller than oscillatory energy and appears to be an indication of degree of the body matter non-homogeneity. Slow rotating bodies like the Sun, Mercury, Venus, and Moon have more homogeneous density distribution. Their part of rotational energy from the total kinetic one is $\sim 1/10^4$. For the other planets of the solar system, this figure is $\sim 1/300$. It follows from (6.10) of Chap. 6 that the value of oscillatory energy for a body as a whole is a conservative parameter. The value of rotary energy is a changeable parameter.

The solution of the virial equation obtained earlier enables us to propose the following mechanism for gravitational contraction of a gaseous sphere. During each period of the sphere's oscillation, a certain amount of energy is lost through radiation. Hence, the contraction amplitude is larger than the expansion amplitude. The difference between the two amplitudes is the value of the gaseous sphere contraction averaged over one period of oscillation. Taking into account the adiabatic invariant relation (Landau and Lifshitz 1973), we shall consider the problem of the gravitational contraction of a gaseous sphere using the virial relations and the key relationships (7.1) and (7.3). Note that we consider here the process of evolution without loss of body equilibrium.

Since we consider the evolution process of a gaseous sphere as a successive moment from one equilibrium state to another, it is natural that the minimum time interval for averaging varying parameters should be a little larger than that required for establishing the hydrodynamic equilibrium. So it is not difficult to control the variations of parameters during evolution that are not in contradiction with the equilibrium. (Later, we shall consider these restrictions to be nonexistent.)

It is convenient for our purpose to write the generalized virial theorem in the form

$$-U = -2(E - E_\gamma) \equiv 2(E_\gamma - E), \tag{7.14}$$

where $E = U + K$ is the total energy of the gaseous sphere, which is a constant over time, E_γ is the electromagnetic energy radiated up to the considered moment of time, K is kinetic energy, which includes the energy of rotation and oscillation of the interacted mass particles, and E and U are negative parameters.

The time derivative of E_γ is the gaseous sphere luminosity L, which is a function of the sphere radius R and the boundary surface temperature T_0:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma\right) = L = 4\pi\sigma R^2 T_0^4, \tag{7.15}$$

where σ is the Stefan–Boltzmann constant.

From Eq. (7.14), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma\right) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(E_\gamma - E\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(-U\right).$$

The potential energy is in turn a function of the radius R:

$$-U = \alpha\frac{Gm^2}{R}.$$

The time derivative of $(-U)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-U\right) = v_\mathrm{c}\frac{\mathrm{d}}{\mathrm{d}R}\left(-U\right),$$

where $v_\mathrm{c} = \mathrm{d}R/\mathrm{d}t$ is the gaseous sphere contraction velocity. To find this velocity, we write

$$\frac{1}{2}v_\mathrm{c}\frac{\mathrm{d}}{\mathrm{d}R}\left(\alpha\frac{Gm^2}{R}\right) = \frac{\mathrm{d}E_\gamma}{\mathrm{d}t} = L$$

and finally, with the help of Eq. (7.3), we obtain

$$v_\mathrm{c} = \frac{8\pi\sigma}{Gm^2}\frac{R^2T_0^4}{(\mathrm{d}/\mathrm{d}R)(\alpha/R)} \tag{7.16}$$

From Eq. (7.16), it is easy to see that v_c contains two unknown functions: $\alpha = \alpha(R)$ and $T_0 = T_0(R)$.

As was found in Chap. 3, the structural form factor α, as well as β, is the function of radial mass density distribution of the sphere. In Chap. 6, we considered this function presented by (7.2) and (7.3). It was found that the contraction velocity of the gaseous sphere depends on the mass density redistribution, which determines the kinetic energy of the body and its shells. So the function $\beta = \beta(R)$ can be found from the condition of kinetic energy conservation of the body's upper shell after its separation.

It follows from (7.3) that during the gravitational contraction of the gaseous sphere, its radius $R \to R_1$ and $\beta \to 1$ (where R_1 is the orbital radius of separation). If $R \to R_1$, then velocity of rotation $v \to v_1$ (v_1 is the first cosmic velocity).

The kinetic energy of the body's upper shell before K_b and after K_a shell is written as

$$K_b = I\omega^2 = \beta^2 m\omega^2 R^2, \tag{7.17}$$

$$K_a = mv_1^2 = m_s\,\omega^2 R_1^2, \tag{7.18}$$

where I is the polar moment of inertia of the body, ω is the frequency of the radial oscillations, m and m_s are the body and its upper shell mass, and $R - R_1$ is the thickness of the upper shell or the contraction value.

From Eqs. (7.17) and (7.18), we can write

$$\beta^2 = \frac{m_s\,\omega^2 R_1^2}{m\omega^2 R^2} = \kappa\frac{R_1^2}{R^2},$$
$$\beta = \sqrt{\kappa}\frac{R_1}{R},$$
$$\alpha = \frac{a}{\beta} = \frac{a}{\sqrt{\kappa}}\frac{R}{R_1}, \tag{7.19}$$

where   is the ratio of the protosun's mass to the mass of a separated body.

Thus, we obtained an expression for α as a function of R, which is valid when the kinetic energy of the upper body's shell conserves in the orbital motion of the separated creature.

Let us now try to obtain the relationship between the gaseous sphere boundary temperature T_0 and the radius R. We introduced the virial equilibrium boundary conditions by Eq. (7.4). This equilibrium was defined as particle flow "locking" by the gravitational force or, equivalently, by an equation showing the absence of particle dissipation from the boundary surface. Let us now rewrite it thus:

$$\frac{Gm\mu}{R^2} = \frac{\mu\bar{\nu}^2}{R}. \tag{7.20}$$

The heat velocity $\bar{\nu}^2$ depends on the boundary temperature T_0 as

$$\mu\bar{\text{v}}^2 = 3kT_0, \tag{7.21}$$

where k is the Boltzmann constant.

Therefore, we can rewrite the condition for particle flow "locking" (7.20) with the help of Eq. (7.21) as

$$\frac{Gm\mu}{3k} = T_0 R. \tag{7.22}$$

From the law of equal energy distribution over the degrees of freedom for the case of a gas particle mixture in equilibrium, it follows that

$$\mu_1 \bar{v}_1^2 = \mu_2 \bar{v}_2^2. \tag{7.23}$$

It is easy to see from (7.22) that the equilibrium radius of a gaseous sphere depends on the chemical composition of the gas. This conclusion follows from Eqs. (7.22) and (6.71) of Chap. 6, where the mechanical equilibrium condition of a body's upper shell is considered. Those results explain the effect of the particle flow "locking" on the pole by the gravitational force, which is based on the concept of mass and radiation equilibrium. Care must therefore be taken when the gas mixture is analyzed; that is, if there are a small number of particles with light masses, the mixture will dissipate easily and the particle flow "locking" will take place in the case of the heavier particles of the gas mixture. The results explain also the observing orbital motion of planets and satellites mainly in the equatorial plane of the parental body. The conditions here for body separation from the viewpoint of dynamical equilibrium appear to be preferential.

When the quantities of the various mass particles are approximately equal, the particle flow "locking" condition can be found only by a numerical solution. The gaseous sphere radius can be determined only after the equilibrium equation is solved, and to solve it, we must consider all the given types and concentrations of particles in the flow. Formally, we can apply the effective particle mass μ, which depends on a value averaged over all the particle masses. The problem can also be solved by numerical methods for a gas mixture consisting of many particles and especially when the processes of ionization and recombination and chemical reactions occur.

Another interesting phenomenon, which we shall discuss, arises from the fact that electromagnetic forces are much stronger than gravitational forces. When some electrons escape the gravitating body, it becomes positive by charges that create huge forces, which tend to stop the process of electron dissipation. That is why it is necessary to use the proton mass μ_p when the gaseous cloud consists of neutral hydrogen partly ionized at the gaseous sphere boundary surface (the position of the boundary shell is specified by the radius R). The flow of electrons will be "locked" by the extra forces appearing as a result of their primary dissipation. In addition, this uncompensated positive charge should have a drift at the boundary surface, and small flow of cold plasma should be observed.

In the course of contraction of the gaseous sphere and the increase in its average temperature, the process of gas ionization should also increase. When the flow of electrons is large enough, and the limiting equilibrium between the gravitational forces and the charged protons is achieved, the protons should also start to run off the body's gravitational field. In this case, the increasing electron flux has to be "locked" by electrostatic forces. The boundary equilibrium change from the proton "locking" to electron "locking" should start at this moment.

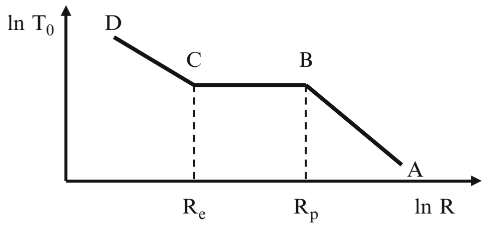


Fig. 7.1 Proton *AB* and electron *CD* equilibrium phases of the boundary shell of contracting gaseous sphere

Thus, we come to the conclusion that at least two phases of gaseous sphere evolutions should exist: that of the proton and that of the electron, with a transitional domain between them that can be calculated by numerical methods in each specific case.

Figure 7.1 illustrates all that we have said. The process of gravitational contraction of the gaseous sphere is represented by the curve *ABCD*. Within the *AB* range, the body equilibrium is kept by the gravitational field "locking" of the proton flow (the proton phase). Within the same range of sphere contraction, the radius R decreases, while the temperature T_0 increases. Point B is the critical one; here, the transformation of equilibrium boundary conditions from proton "locking" to electron "locking" begins. The process spreads up to point *C*. While the sphere radius decreases in the range *BC*, the boundary temperature remains constant.

In the electron equilibrium phase in the range *CD*, we can see that during the contraction process, the boundary temperature increases again.

Let us check the derived expression (7.22) and the conclusion concerning the existence of two boundary equilibrium phases on the observed Sun data.

First, we calculate the numerical value of T_0R in the CGS system with the help of Eq. (7.22). Assuming numerical values for proton and electron masses, we obtain

$$T_{\mathrm{p}}R_{\mathrm{p}} = A_{\mathrm{p}} = \frac{Gm\mu_{\mathrm{p}}}{3k} = \frac{6.67\cdot10^{-8}\cdot2\cdot10^{33}\cdot1.67\cdot10^{-24}}{3\cdot1.38\cdot10^{-16}} = 5\cdot10^{17}\,\mathrm{cm\cdot K},$$

$$T_{\mathrm{e}}R_{\mathrm{e}} = A_{\mathrm{e}} = \frac{Gm\mu_{\mathrm{e}}}{3k} = \frac{6.67\cdot10^{-8}\cdot2\cdot10^{33}\cdot9.1\cdot10^{-28}}{3\cdot1.38\cdot10^{-16}} = 2.73\cdot10^{14}\,\mathrm{cm\cdot K}.$$

For the contemporary Sun, we know that $R = 7\cdot10^{10}$ cm and $T_0 = 5{,}000$ K so that $T_0R = 3.5\cdot10^{14}$ cm K.

As at $T_0 = 5{,}000$ K, where gas ionization must be fairly complete, we have a very good coincidence of the calculated and the observed data for the products T_0R and $T_{\mathrm{e}}R_{\mathrm{e}}$. For this temperature, the proton radius of the Sun R_{p} is equal to $10^{14}T_0R$ cm, which corresponds to the orbit radius of Jupiter.

Thus, we have found α, β, and T_0 as functions of the radius R. We can now obtain the gaseous sphere contraction velocity. We rewrite Eq. (7.16):

$$v_{\mathrm{c}} = \frac{8\pi\sigma}{Gm^2}\frac{(RT_0)^4}{R^2\,(\mathrm{d}/\mathrm{d}R)\,(\alpha/R)}. \tag{7.24}$$

and, using (7.19), we can evaluate the denominator as

$$R^2\left|\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{\alpha}{R}\right)\right| = R^2\left|\frac{\mathrm{d}}{\mathrm{d}R}\left[\frac{a}{\sqrt{\kappa}}\frac{1}{R}\frac{R}{R_1}\right]\right| = \frac{a}{2\sqrt{\kappa}}\frac{R}{R_1}$$

Finally, we write contraction velocity v_c as

$$v_c = \frac{16\pi\sigma}{Gm^2}\frac{A^4}{\mathrm{a}}\sqrt{\kappa}\frac{R_1}{R}, \tag{7.25}$$

where $A = A_e = R_e T_e$ and $A = A_p = R_p T_p$ are for the electron and the proton phases of the gaseous sphere evolution, respectively.

Let us use Eq. (7.25) to obtain the contraction velocity and the time of contraction of the protosun during the proton and the electron phases of the gaseous sphere evolution using its corresponding parameters.

If we take for the proton phase of the protosun, after its separation from the protogalaxy, $A_p = 5{\cdot}10^{17}$ cm·K, initial radius $R = 7.5{\cdot}10^{18}$ cm, final radius of the proton phase evolution (at the asteroid belt, after separation of the proto-Jupiter), $R_1 = 4.2{\cdot}10^{13}$ cm, $a = 0.46$, and $\sigma = 5.76{\cdot}10^{-5}$ erg·cm^{-2} s·(K)^4 as initial, we obtain

$$\bar{v}_{\mathrm{orb}} = \frac{16{\cdot}3.14{\cdot}5.76{\cdot}10^{-5}\left(5{\cdot}10^{17}\right)^4}{6.67{\cdot}10^{-8}(2{\cdot}10^{33})^2{\cdot}0.46}\cdot\sqrt{\frac{2{\cdot}10^{33}}{2.65{\cdot}10^{30}}}\cdot\frac{4.2{\cdot}10^{13}}{7.5{\cdot}10^{18}} = 2.23{\cdot}10^5 \mathrm{cm{\cdot}s^{-1}},$$

$$t_{bI3} = \frac{7.5{\cdot}10^{18}}{2.23{\cdot}10^5} = 3.36{\cdot}10^{13}\ \mathrm{s} = 1.06{\cdot}10^6\ \text{years}$$

We can now find the contraction velocity and the time of contraction of the protosun during the electron phase of the gaseous sphere evolution. We take now for the electron phase $A_e = 2.73{\cdot}10^{14}$ cm·K, initial radius of the protosun, after separation of the proto-Jupiter, $R = 4.2{\cdot}10^{13}$ m, final radius of the electron phase let the present-day value be $R_1 = 7{\cdot}10^{10}$ m, $a = 0.46$, and $\sigma = 5.76{\cdot}10^{-5}$ erg·cm^{-2}·s·K^4. Then, we obtain

$$v_{\mathrm{cse}} = \frac{16{\cdot}3.14{\cdot}5.76{\cdot}10^{-5}\left(2.73{\cdot}10^{14}\right)^4}{6.67{\cdot}10^{-8}(2{\cdot}10^{33})^2{\cdot}0.46}\cdot\sqrt{\frac{2{\cdot}10^{33}}{1.18{\cdot}10^{28}}}\cdot\frac{7{\cdot}10^{10}}{4.2{\cdot}10^{13}} = 8.96{\cdot}10^{-5}\ \mathrm{cm{\cdot}s^{-1}},$$

$$t_{\mathrm{se}} = \frac{4.2{\cdot}10^{13}}{8.96{\cdot}10^{-5}} = 4.7{\cdot}10^{17}\mathrm{s} = 14.9{\cdot}10^9\ \text{years}.$$

The found values show that the contemporary solar system has formed during the proton phase (Jupiter's group of planets) within 1 million years and during the electron phase (the Earth's group of planets) within the next 15 billion years. Here, we have not taken into account the effects of chemistry of the gaseous sphere on the equilibrium boundary conditions of the evolutionary process. But the obtained figures of evolution time show that our calculations give good approximation to the reality.

7.3 The Luminosity–Mass Relationship

To obtain the luminosity–mass relationship, we again consider the gaseous sphere evolution plot given in Fig. 7.1. It follows from (7.15) that in proton (*AB*) and electron (*CD*) evolutionary phases, the gaseous sphere luminosity is proportional to $1/R^2$. The boundary surface temperature T_0 remains practically constant during the transition period (*BC*), when the equilibrium transformation from the proton to the electron phase takes place. But the gaseous sphere luminosity will decrease sharply. One can see that the luminosity decrease here is proportional to

$$L \propto \frac{\mu_{\mathrm{p}}^2}{\mu_{\mathrm{e}}^2}, \tag{7.26}$$

that is, it is proportional to the ratio of the proton and electron mass squared as the gaseous sphere surface decreases proportionally to R^2. Thus, while going from point *B* to point *C* of the plot, the luminosity of the contracting body decreases by six orders of magnitude. We can suppose that the observed variations of variable star brightness are related to their virial energy pulsations, when stars at the stage of evolution are being considered.

As shown in the previous section, the most continuous period of proton or electron phase evolution is on the right end of the plot intercept (*AB*) and (*CD*). For these principal evolution time intervals, we can write

$$L = 4\pi\sigma R^2 T_0^4 = \frac{(RT_0)^4}{R^2} \propto m^4. \tag{7.27}$$

This expression, derived from our theoretical considerations, is in good agreement with the well-known luminosity–mass relation, which follows from observations. That is why Eq. (7.22) can be considered as an additional relation between the luminosity, the radius, and the boundary surface temperature.

Let us take one more example. In Campbell's work (1962), 13 elliptical galaxies from the Virgo Cluster are considered, and an analysis of the mass–radius relation for the observed data is given. To interpret these data, Jeans's relation (Jeans 1919) is used:

$$Gm\mu = \frac{3}{2}kT_0R \quad \text{or} \quad \frac{m}{R} = \frac{3kT_0}{2G\mu}, \tag{7.28}$$

where μ is the proton mass.

On the plot presented in this work reflecting the mass–radius dependence, all the points are found to lie on a straight line with slope corresponding to $T_0 \approx 1.5{\cdot}10^7$ K. Campbell concludes from this that the Jeans condition of self-gravitational instability is valid.

We note that Jeans's formula was derived on the assumption of low gas temperature and that all the kinetic energy of the gas is used for particle heat movement. The radiation energy was not taken into account.

Because of the absence of direct temperature measurements, the theoretically found high-temperature values at very steep line slopes need other explanations. We must stress that in the observational data presented, the distance to the objects (in relative units) has been found with high degree of precision so that the experimentally derived constancy of the line slope should be trusted.

We interpret Campbell's data on the basis of our expression (7.22), where we consider the mass–radius relation to be dependent on electron temperature. That is why, contrary to Jeans, we write

$$\frac{m}{R_e} = \frac{3kT_e}{G\mu_e}.$$

Now, the value of the boundary surface temperature of Campbell's galaxies is $T_0 \approx 4{,}000$ K. This value corresponds to the usual boundary temperatures of celestial bodies whose evolution goes according to the electron phase of the equilibrium.

Hence, the experimental data presented by Campbell in his paper confirm once more the validity of Eq. (7.22) and the assumption of the existence of two evolutionary phases for celestial bodies.

In connection with the interpretation of Campbell's data, it is possible to use Eq. (7.22) to obtain the limiting temperature that should be reached by a gaseous sphere in its evolution. We write (7.22) as

$$\frac{Gm}{c^2}\frac{1}{R} = \frac{3kT_e}{\mu c^2} \quad \text{or} \quad \frac{R_g}{R} = \frac{3kT_0}{\mu c^2}. \tag{7.29}$$

Hence, during the evolution of a gaseous sphere through the electron phase of equilibrium, when $R \to R_g < T_0 \to \mu_e c^2/3k$ or, equally,

$$3kT_0 \to \mu_e c^2 \approx 0.5\,\text{MeV},$$

$$T \approx 5{\cdot}10^9\,\text{K}.$$

This means that the temperature of the bodies approaches the electron temperature.

7.4 Bifurcation of a Dissipative System

In Chap. 5, we considered the dynamics of a dissipative system, assuming that its evolution is a consequence of the loss of energy due to its radiation. Let us consider the problem in some details.

Jacobi's virial equation for a system was written as

$$\ddot{\Phi} = -A_0\left[1 + q(t)\right] + \frac{B}{\sqrt{\Phi}}, \tag{7.30}$$

where the function $A_0[1 + q(t)] = E - E_\gamma$ increases monotonically, reflecting the change of the total energy of a system as a function of time, and E_γ is the energy radiated up to time $t[E_\gamma > 0]$.

The solution of Eq. (7.30) was found to be

$$\begin{aligned} &-\arccos\, W + \arccos\, W_0 - \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}} = \sqrt{1 - W^2} \\ &+ \sqrt{1 - \frac{A_0 C}{2B^2}}\sqrt{1 - W_0^2} = \pm\frac{\left[2A_9\left(1 + q(t)\right)\right]^{3/2}}{4B}\left(t - t_0\right). \end{aligned} \tag{7.31}$$

Equations of the discriminant curves that bound oscillations of the moment of inertia (Jacobi function) (see Fig. 5.5) are

$$\sqrt{I_{1,2}} = \frac{2B}{A_0\left[1 + q(t)\right]}\left\{1 \pm \sqrt{1 - \frac{A_0\left[1 + q(t)\right]C}{2B^2}}\right\}. \tag{7.32}$$

From the analysis of the solution of Eq. (7.30), it follows that the dissipative system during its evolution must inevitably reach the state when its stability breaks; that moment (see Fig. 5.5) can be defined by the point O_b, which is the physical bifurcation point. The position of the point can be defined by Eq. (7.32) as

$$\frac{2B^2}{A_0\left[1 + q\left(t_b\right)\right]} = C \tag{7.33}$$

where $q\left(t_b\right)$ is the parameter of the bifurcation point that can be found from condition (7.33):

$$q\left(t_b\right) = \frac{2B^2}{A_0 C} - 1 \tag{7.34}$$

The moment of inertia (Jacobi function) of the system corresponding to the bifurcation point, where the discriminant lines coincide, is

$$I_b = \frac{B}{A_0\left(1 + \dfrac{2B^2}{A_0 C} - 1\right)} = \frac{C^2}{4B^2} \tag{7.35}$$

To find the moment of time of t_b where the system reaches its bifurcation point, one must know the law of energy radiation of the body $q(t)$ or $E_\gamma(t)$, entering Eq. (7.30).

We give below our model solution for $E_\gamma(t)$.

The solution for the energy $E_\gamma(t)$ radiation up to t is based on the assumed existence of the proton and the electron phases of evolution for celestial bodies proposed in this chapter. On this basis, we have found a relationship between the body luminosity L and its radius R. During "smooth" intervals of the body evolution, when $E_\gamma(t)$ is a continuous and monotonic function of time, the following relation holds:

$$\frac{Gm\mu_\mathrm{p}}{3k} = RT_0 \tag{7.36}$$

where μ_p is the mass of the particle (proton or electron) that provides the boundary heat equilibrium of the body, k is the Boltzmann constant, and T_0 is the gaseous sphere boundary temperature.

Let us write down the expression for the body luminosity L in relation to the time derivative of E_γ:

$$\frac{\mathrm{d}E_\gamma}{\mathrm{d}t} = L = 4\pi\sigma R^2 T_0^4 \tag{7.37}$$

where σ is the Stefan–Boltzmann constant.

Now, we shall find an explicit expression for $E_\gamma(t)$ with the initial condition $E_\gamma\,(t_0)\,|_{t_0=0}\; = 0$

Equation (7.37) between the limits 0 and t can be integrated with the help of (7.36):

$$E_\gamma(t)=\int_0^t 4\pi\sigma R^2 T_0^4\,\mathrm{d}t=\int_0^t \frac{4\pi\sigma R^4 T_0^4}{R^2}\,\mathrm{d}t=\int_0^t \frac{4\pi\sigma\left(Gm\mu_\mathrm{p}\right)^4}{(3k)^4}\frac{1}{R^2}\,\mathrm{d}t=\int_0^t \frac{K}{R^2}\,\mathrm{d}t \tag{7.38}$$

where $K = 4\pi\sigma\left(Gm\mu_\mathrm{p}\right)^4(3k)^4$.

Now, let us make use of expression (7.25) for the velocity of the gravitational contraction of the gaseous sphere v_c, which we had found earlier in this chapter:

$$v_\mathrm{c} = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_\mathrm{p}}{3k}\right)^4\frac{\sqrt{\kappa}}{a}\sqrt[4]{\frac{R_1}{R}} \tag{7.39}$$

Integrating this equation,

$$\int_0^R R^{1/4}\mathrm{d}R = -\frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_\mathrm{p}}{3k}\right)^4\frac{1}{a}\sqrt[4]{\kappa^2 R_1}\int_0^t \mathrm{d}t$$

we obtain

$$\frac{4}{5}R^{5/4} - \frac{4}{5}R_0^{5/4} = -\frac{32}{3}\frac{\pi\sigma}{Gm^2}\left(\frac{Gm\mu_\mathrm{p}}{3k}\right)^4\frac{1}{a}\left(\sqrt[4]{\kappa^2 R_1}\right)t \tag{7.40}$$

Then

$$R = \left(-Dt + R_0^{5/4}\right)^{4/5}$$

where

$$D = \frac{40\pi\sigma}{3Gm^2}\left(\frac{Gm\mu_p}{3k}\right)^4 \frac{1}{a}\sqrt[4]{\kappa^2 R_1}$$

Finally, substituting the found expression for (7.40) into (7.38), we have

$$E_\gamma(t) = \int_0^t \frac{K\,\mathrm{d}t}{\left(-Dt + R_0^{5/4}\right)^{8/5}} = \frac{5K}{3D}\left[\left(R_0^{5/4} - Dt\right)^{-3/5} - R_0^{3/4}\right]$$
$$= \frac{5}{3}\frac{K}{D}\left[\frac{1}{\left(R_0^{5/4} - Dt\right)} - \frac{1}{R_0^{3/4}}\right] \tag{7.41}$$

Thus, we have obtained an expression in explicit form that can be used to calculate the energy loss by radiation during the time intervals of "smooth" evolution of celestial bodies and, hence, to find the parameters of the bifurcation point of a dissipative system.

7.5 Cosmochemical Effects

From the analysis of the solution of Eq. (7.30) for a dissipative system, we found that, because of energy loss, a celestial body reaches a bifurcation point, characterized by separation of its outer shell in which angular frequency coincides with the frequency of virial oscillations. According to our theory of bifurcational creation of secondary bodies (in Alfven's definition), some portion of the mass of the rotating primordial cloud reaches equilibrium relative to the inner force field of the whole cloud at the bifurcation point and moves further in a Kepler's orbit. As a result, during the subsequent dissipation of energy, the primary body continues its contraction by means of redistribution of the mass density without a separated secondary body. This secondary body conserves the corresponding angular moment $M_1 = mv_1R_1 = mv_1^2/\omega$, which in fact is the kinetic energy divided by frequency of the interacted mass particles. In accordance with (7.3), the value of this tangential component of the kinetic energy is equal to half of the potential energy ($2\beta_t = \alpha_t$) at the moment of a secondary body separation.

It is commonly known that when both the gravitational and electromagnetic interactions are taken into account, the condition to attain an equilibrium state by some portion of the mass (secondary body) can be written in the form suggested by Chandrasekhar and Fermi (1953):

$$\int\limits_{(V)} \left[\rho \bar{v}^2 + 3p + \frac{H^2 + E^2}{8\pi} - \frac{(\nabla U)^2}{8\pi G} \right] \mathrm{d}V = 0, \tag{7.42}$$

where ρ is the density of the substance of the secondary body, v the mean velocity, p the internal pressure, H and E the components of the electromagnetic field, G the gravitational constant, V the volume of the system, and ∇U the gradient of the gravitational field.

Since the bifurcational point of a system is characterized by the zero amplitude of the virial oscillations, the kinetic terms in Eq. (7.42) are small compared to the mass terms. In this case, Eq. (7.42) can be rewritten as (Ferronsky et al. 1981a, b, 1996)

$$\int\limits_{(V)} \left[3p - \frac{(\nabla U)^2}{8\pi G} \right] \mathrm{d}V \approx 0$$

or

$$\int\limits_{(V)} 3p \, \mathrm{d}V \approx 0.1 \frac{Gm^2}{R}, \tag{7.43}$$

where the coefficient 0.1 represents the electromagnetic component in expansion of the potential energy (7.43) found by astronomical observation of the equilibrium nebulae (Ferronsky et al. 1996).

The left-hand side of (7.43) is proportional to the energy of the Coulomb interactions of the charged particles (electrons, protons, ionized atoms, and molecules). The right-hand side of this expression is proportional to the energy of the gravitational interaction of the particles.

Thus, assuming the separated secondary body to have mass m and radius R and the average mass of its constituent particles to be μ, expression (7.43) can be rewritten in the form of an equality of the energies of the gravitational and Coulomb interactions or Madelung's energy (Kittel 1968):

$$0.1 \frac{Gm^2}{R} \propto \frac{m}{\mu} \frac{e^2}{R \sqrt[3]{\mu/m}}, \tag{7.44}$$

where $e = 4.8 \cdot 10^{-10}$ e.s.u. is the electron charge.

Expression (7.44) is the equivalent of

$$m\mu^2 \propto \frac{e^3}{G^{3/2}}. \tag{7.45}$$

Table 7.1 Critical and averaged masses of the constituent particles for the planets

Planets	m_c(g)	μ_a(g)	μ_a(aum)
Mercury	$0.33{\cdot}10^{27}$	$0.78{\cdot}10^{-21}$	469
Earth	$5.97{\cdot}10^{27}$	$0.18{\cdot}10^{-21}$	114
Jupiter	$2{\cdot}10^{30}$	$0.00{\cdot}10^{-23}$	6.02
Saturn	$0.57{\cdot}10^{30}$	$1.87{\cdot}10^{-23}$	11.3
Uranus	$0.087{\cdot}10^{30}$	$4.79{\cdot}10^{-23}$	28.8

The last expression relates the critical mass m_c of the separated secondary body to the averaged mass μ_a of its constituent particles (electron, proton, molecules), responsible for the hydrodynamic equilibrium of the body, as

$$m_c\mu_a^2 \propto \left(\frac{e^2}{G}\right)^{3/2} = \text{const.} = 2{\cdot}10^{-16}\ \text{g}^3. \tag{7.46}$$

To illustrate this relationship, we determined the average values for the masses of the individual particles constituting the planets, stars, and galaxies.

7.5.1 Planets

Table 7.1 shows critical masses of the constituent particles for the planets of the solar system.

Thus, assuming that the bifurcation theory describes the formation of the solar system correctly, the particles determining the hydrodynamic gas pressure in the case of the considered planet at the moment of their separation from the protosolar cloud could have been composed of such elements as H, He, O, Si, Mn, and Fe in atomic or molecular form. The average masses of the particles obtained can be used as a criterion in the development of cosmochemical models of planets with a complicated chemical composition at the moment of their separation from the protosolar cloud and also for the construction of their chemical evolution models.

7.5.2 Stars

From (7.46), the boundary values for all stellar critical masses can be found, corresponding to the masses of the proton and the electron—particles that can be responsible for the hydrodynamic pressure inside the stellar cloud at the moment of separation at the bifurcation point of the protogalactic cloud.

For the mass of the proton $\mu_p = 1.6{\cdot}10^{-24}$ g, $m_c = 10^{32}$ g
For the mass of the electron $\mu_e = 0.9{\cdot}10^{-27}$ g, $m_c = 2{\cdot}10^{38}$ g
In the case of $\mu_a = \sqrt{\mu_p\mu_a} = 0.4{\cdot}10^{-25}$ g, $m_c = 10^{35}$ g

Therefore, considering a typical stellar mass to be $\sim 10^{33}$ g, we obtain that the hydrodynamic equilibrium of the gas at the moment of separation of the protostellar cloud is supported both by electron and by proton in the framework of the bifurcation theory of formation of celestial bodies.

7.5.3 Galaxies

The presence of the factor $(e^2/G)^{3/2}$ in the right-hand side of (7.46) allows us to carry out the following transformations:

$$m_{\mathrm{c}}\mu_{\mathrm{a}}^2 = \left(\frac{e^2}{\hbar c}\right)^{3/2}\left(\frac{\hbar c}{G}\right) = \left(\frac{1}{137}\right)^{3/2} m_{\mathrm{p}}^3, \tag{7.47}$$

where $\hbar$ is Planck's constant, c the velocity of light, and m_{p} Planck's mass.

Thus, in the right-hand side of (7.47), there are two fundamental constants: Planck's mass m_{p} ($2.2{\cdot}10^{-5}$ g) and the fine-structure constant $\alpha = 1/137$. The presence of the constant α in the right-hand side of (7.47), being the universal constant of the weak and electromagnetic interactions, shows that this relation is applicable not only to electromagnetic but also to weak interactions. Then, putting the experimentally found values for the neutrino mass $\mu_{\nu} = 10^{-30}$ g (Shirkov 1980) into (7.44), we obtain

$$m_{\mathrm{c}} = \frac{2{\cdot}10^{-16}}{(10^{-30})^2} = 2{\cdot}10^{44}\mathrm{g}. \tag{7.48}$$

This mass, following from (7.48), is a typical mass of galaxies. Therefore, in the framework of the bifurcation theory of formation of celestial bodies, the hydrodynamic equilibrium (7.41) of the substances of galaxies at the moment of their formation can be provided by the pressure of neutrinos.

7.5.4 Universe

In the framework of the virial oscillation theory, the evolution of the universe can be described by a pulsating model (for $c =$ constant) of the system of material elementary particles. Such a system exists for indefinitely long time. The mass of the particle responsible for hydrodynamic equilibrium of the universe at the moment of its maximal compression (singularity stage) can be obtained from the same expression (7.46). Assuming $m_{\mathrm{c}} \approx 10^{56}$ g, we obtain

$$\mu_{\mathrm{a}} \approx 10^{-36}\mathrm{g}. \tag{7.49}$$

In the bifurcation theory, the maximal average mass of particles in cosmic space can be determined from the condition $\mu_a = m_c$. Then,

$$\mu_{max} = 6{\cdot}10^{-6}\text{g}.$$

This value is close to Planck's mass.

7.6 Radial Distribution of Mass Density and the Body's Inner Force Field

At present, only the Earth has experimental data that allow to interpret them with respect to radial distribution of the body's mass density. Taking into account our consideration of dynamics of celestial bodies as self-gravitating systems, we assume that formation of the Earth's mass density distribution is typical at least for all the planets and satellites.

The existent idea about the radial mass density distribution of the Earth is based on interpretation of transmission velocity of the longitudinal and transverse seismic waves. Figure 7.2 presents the classic curve of transmission velocities of the longitudinal and transverse seismic waves in the Earth plotted after generalization of numerous experimental data (Jeffreys 1970; Melchior 1972; Zharkov 1978). The curves of the radial density and hydrostatic pressure distribution based on interpretation of the velocities of the longitudinal and transverse seismic waves are also shown.

The picture of the transmission velocities of the seismic waves was obtained by observations and therefore is realistic and correct. But interpretation of the obtained data was based on the idea of hydrostatic equilibrium of the Earth. It leads to incredibly high pressures in the core and high values of the mass density.

In accordance with Bullen's approach for interpretation of the seismic data, the density distribution is characterized by the following values (Bullen 1974; Melchior 1972; Zharkov 1978). The density of the crust rocks is 2.7–2.8 g/cm^3 and increases toward the center by a certain curve up to $\sim$13.0 g/ m^3 with jumps at the Mohorovičić–Gutenberg discontinuity, between the upper and lower mantles, and on the border of the outer core. Within the core, the values of the transverse seismic waves are equal to zero. Despite the jump of the longitudinal seismic wave velocity at the outer core border dropping down, Bullen accepted that the density increases toward the center. It was done after his unsuccessful attempt to approximate the seismic data of the parabolic curve that gives a decrease of density in the core. Such a tendency is not consistent with the idea of iron core content. Bullen certainly had no idea that the radius of inertia and radius of gravity of the body do not coincide with its geometric center of mass, and, therefore, the maximum value of density is not located there. In accordance with our concept of the equilibrium condition of the planet and its dynamical parameters, the approach to interpretation of the seismic data related to the radial density and radial pressure distribution should be done on a new basis.

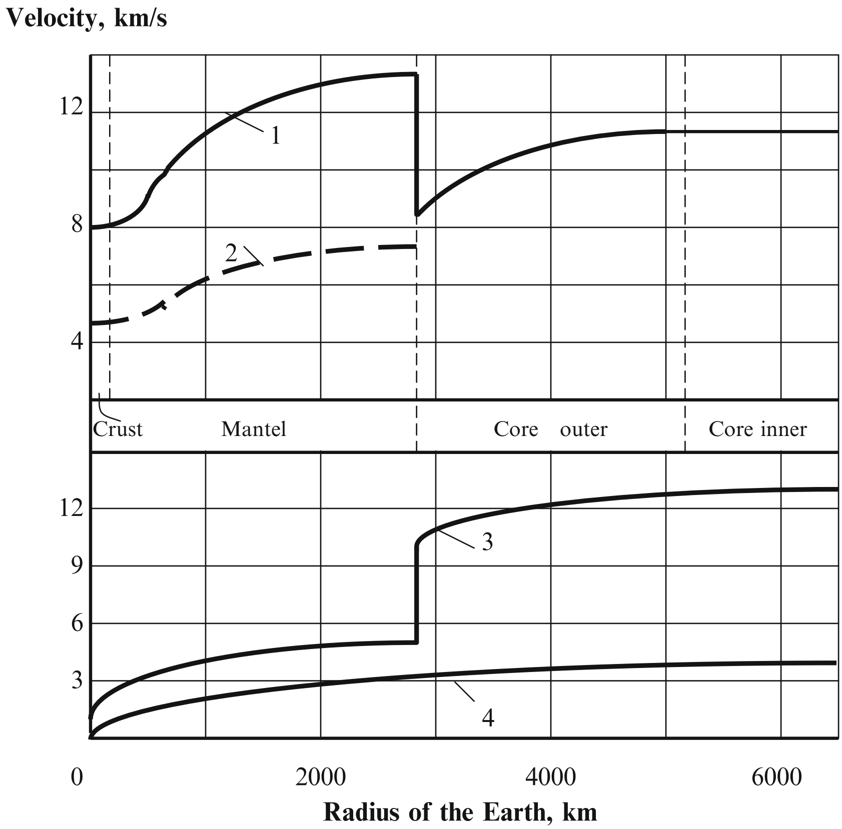


Fig. 7.2 Present-day interpretation of the curves of transmission velocities of longitudinal (*1*) and transverse (*2*) seismic waves, density (*3*), and hydrostatic pressure (*4*) in the Earth

Now, when we accept the concept of dynamical equilibrium of the Earth and refuse its hydrostatic version, the basic idea to search for a solution of the problem seems to be the found relationship between the polar moment of inertia and the potential (kinetic) energy. The value of the structural form factor of the Earth's mean axial moment of inertia $\beta_\perp^2 = J_\perp / \quad R^2 = 0.3315$ found by artificial satellites (Zharkov 1978) should be taken as a starting point. The mean polar moment of inertia of the assumed spherical nonuniform planet is equal to $\beta^2 = (3/2)\beta_\perp^2 = 0.49725$. We accept this value for the development of the methodology.

Let us take the found mechanism of the shell separation as a basis with respect to the mass density that was presented in Sects. 6.5, 6.6, 6.7, and 6.8. The conditions and mechanism of the shell separation into radial and tangential components of the inner force field (by the Archimedes and Coriolis forces) represent continually acting effects and create physics for the Earth's structure formation. These effects explain the jumps between the shells observed by seismic data density. We take also into account the effect, according to which the velocity of the sound recorded by the transmission velocity of the longitudinal and transverse seismic waves quantitatively characterizes the energy of the elastic deformation of the media and velocity of its transmission there (Ferronsky and Ferronsky 2010).

Applying the conception of Sect. 6.7, we accept that the nonuniformities of the spherical shells come together and, after their density becomes lower than that of the mean density of the inner sphere, move from the center by the parabolic law because they interact according to the law $1/r^2$. So we can find a probable law of the radial density distribution in the form

$$\rho(r) = \rho_0(ax^2 + bx + c), \tag{7.50}$$

where $= r/R$ is the ratio of the running and the final radius of the planet; ρ_0 is the body's mean density; and a, b, and are the numerical coefficients.

The numerical coefficients were selected for different densities for the upper shell in such a way that the planet's total mass M would be constant, that is,

$$M = 4\pi \int_0^R r^2 \rho(r)\,\mathrm{d}r = 4\pi \int_0^R r^2 \rho_0 \left(-a\frac{r^2}{R^2} + b\frac{r}{R} + c \right) \mathrm{d}r$$
$$= \frac{4}{3}\pi R^3 \rho_0 \left(-\frac{3}{5}a + \frac{3}{4}b + c \right).$$

Here, the term $(3/5)a + (3/4)b + c = 1$ in the right-hand side of the expression allows us to calculate and plot the distribution density curves in a dimensionless form.

We accepted three most typical parabolas (7.51), which satisfy the condition of equality of their moment of inertia, found by artificial satellite data, namely, the axial moment of inertia $J_{\perp} = \beta_{\perp}^2 mR^2 = 0.3315\ mR^2$ or the polar moment of inertia $J = \beta^2 mR^2 = 0.4973\,mR^2$. In addition, the first relation in (7.51) represents the straight line for which the surface mass density and that in the center correspond to the present-day version and to the form factor $\beta_{\perp}^2$. The fifth straight line represents the uniform spherical planet. The curve equations with selected numerical coefficients a, b, and are as follows:

$$1.\quad \rho(r) = \rho_0 \left(-2\frac{r}{R} + 2.495 \right), \quad a = 0, \quad \rho_s = 2.73\ \mathrm{g/cm^3};$$

$$2.\quad \rho(r) = \rho_0 \left(-1.51\frac{r^2}{R^2} + 0.016\frac{r}{R} + 1.894 \right), \quad \rho_s = 2.08\ \mathrm{g/cm^3};$$

$$3.\quad \rho(r) = \rho_0 \left(-3.26\frac{r^2}{R^2} + 2.146\frac{r}{R} + 1.3465 \right), \quad \rho_s = 1.28\,\mathrm{g/cm^3};$$

$$4.\quad \rho(r) = \rho_0 \left(-5.24\frac{r^2}{R^2} + 5.132\frac{r}{R} + 0.295 \right), \quad \rho_s = 1.03224\,\mathrm{g/cm^3}.$$

$$5.\quad \rho(r) = \rho_0 = \mathrm{const.} \tag{7.51}$$

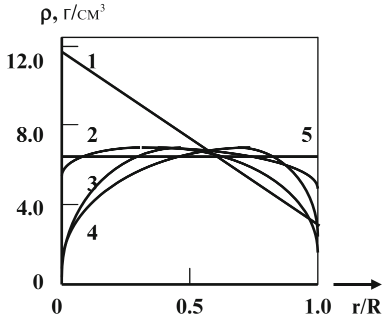


Fig. 7.3 Parabolic curves of radial density distribution calculated by Eq. (7.51)

Figure 7.3 shows all the curves of (7.51). They intersect the straight line 5 of the mean density in the common point that corresponds to the value $r/R = 0.61475$.

Using Eq. (7.51) and the found (by observations) form factor $\beta_\perp^2 = 0.3315$, the main dynamical parameters were calculated for all four curves. The calculations were done by the known formulae of the theory of interaction (Duboshin 1975) and taking into account the relations of (6.8) and (6.9) obtained in Sect. 6.2. These calculations are presented below for equation (4), as an example.

The potential energy of the nonuniform sphere with the density distribution law $\rho(r)$ is found from the equation

$$U = 4\pi G \int_0^R r\rho(r)m(r)\,\mathrm{d}r, \tag{7.52}$$

where

$$\rho(r) = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right), \quad a = -5.24; \quad b = 5,132; \quad c = 0.295;$$

$$m(r) = 4\pi\int_0^r r^2\rho(r)\mathrm{d}r = 4\pi\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r = \frac{4}{3}\pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right).$$

Then

$$\begin{aligned} U(r) &= 4\pi G\int_0^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\frac{4}{3}\pi r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)\mathrm{d}r \\ &= \left(\frac{4}{3}\pi\rho_0\right)^2 GR^5\frac{R}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{9}{28}b^2 + \frac{24}{35}ac + \frac{7}{8}bc + \frac{3}{5}c^2\right) \\ &= 0.0660143\frac{GM^2}{R}. \end{aligned} \tag{7.53}$$

The form factor of the potential energy is $\alpha = r_g/R = 0.660143$, and the reduced radius of gravity is $r_g = \sqrt{0.660143\, R^2} = 0.8124918\, R$.

In accordance with (6.8), the potential energy of the nonuniform sphere is expanded into the components

$$U = U_0 + U_t + U_\gamma. \tag{7.54}$$

The potential energy of the uniform sphere is equal to

$$U_0 = \frac{3}{5}\frac{GM^2}{R}, \tag{7.55}$$

where form factors of potential and kinetic energies are equal to $\alpha_0 = 0.6$ and $\beta_0^2 = 0.6$.

In accordance with the second term of the right-hand side of Eq. (6.8), the tangential component of the nonuniform sphere is written as

$$U_t = -\frac{1}{2}4\pi G\int_0^R r\rho_t(r)m_0(r)\,\mathrm{d}r, \tag{7.56}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right),$$

$$m_0(r) = 4\pi\int_0^r r^2\rho_0\,\mathrm{d}r = \frac{4}{3}\pi\rho_0 r^3.$$

The coefficient ½ in (7.56) is taken as the ratio of the second term of the right-hand side of Eqs. (6.8) and (6.9) as, in this particular case, the tangential component of the potential energy is determined through the tangential component of the kinetic energy and is equal to half its value. Then

$$U_t = -\frac{1}{2}4\frac{4}{3}(\pi\rho_0)^2 G\int_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\mathrm{d}r$$
$$= -\frac{1}{2}\frac{GM^2}{R}\left(\frac{3}{7}a + \frac{1}{2}b + \frac{3}{5}c - \frac{3}{5}\right) = 0.0513571\frac{GM^2}{R}.$$

The form factors of the tangential components of the potential and kinetic energies are equal to $\alpha_t = 0.051357$ and $\beta_t^2 = 2{\cdot}0.051357 = 0.102714$.

In accordance with the third term in the right-hand side of Eq. (6.8), the dissipative component of the potential energy of the nonuniform sphere is

$$U_\gamma = 4\pi G \int_0^R r\rho_t(r) m_t(r)\,\mathrm{d}r, \tag{7.57}$$

where

$$\rho_t(r) = \rho(r) - \rho_0 = \rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right),$$

$$m_t(r) = 4\pi \int_0^r r^2\rho_t(r)\,\mathrm{d}r = 4\pi \int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\mathrm{d}r$$
$$= \frac{4}{3}\pi\rho_0 r^3\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right).$$

then

$$U_\gamma = 4\frac{4}{3}(\pi\rho_0)^2 G \int_0^R r^4\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c - 1\right)\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c - 1\right)\mathrm{d}r$$
$$= \frac{GM^2}{R}\left(\frac{1}{5}a^2 + \frac{81}{160}ab + \frac{24}{35}ac - \frac{24}{35}a + \frac{9}{28}b^2 + \frac{7}{8}bc - \frac{7}{8}b + \frac{3}{5}c^2 - \frac{5}{6}c + \frac{3}{5}\right)$$
$$= 0.008786\,\frac{GM^2}{R}. \tag{7.58}$$

So the value of the form factor of the dissipative component is $\alpha_\gamma = 0.008786$.

The radial distribution of the potential energy for interaction of a test mass point with the nonuniform sphere is

$$U(r) = \frac{4\pi G}{r}\int_0^r r^2\rho(r)\,\mathrm{d}r + 4\pi G\int_r^R r\rho(r)\,\mathrm{d}r = \frac{4\pi G}{r}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r$$
$$+ 4\pi G\int_r^R r\rho_0\left(a\frac{r^2}{R^2} + b\frac{r}{R} + c\right)\mathrm{d}r$$
$$= \frac{GMm_1}{R}\left(-\frac{3}{20}a\frac{r^4}{R^4} - \frac{1}{4}b\frac{r^3}{R^3} - \frac{1}{2}c\frac{r^2}{R^2} + \frac{3}{4a} + b + \frac{3}{2}c\right)$$
$$= \frac{GMm_1}{R}\left(0.786\frac{r^4}{R^4} - 1.283\frac{r^3}{R^3} - 0.1475\frac{r^2}{R^2} + 1.6445\right). \tag{7.59}$$

At $r/R = 0$, $\alpha_v(r) = 1.6445$, and at $r/R = 1$, $\alpha_v(r) = 1$.

Table 7.2 Physical and dynamical parameters of the Earth for the density distribution presented by Eq. (7.51)

Equation N	1	2	3	4
ρ_s, g/ m^3	2.76	2.08	1.65	1.03224
ρ , g/ m^3	13.8	10.455	6.315	1.6284
ρ_{max}, g/ m^3/km	13.8/0	10.455/0	8.26/2096	8.57/3122
$\beta_\perp^2$	0.3315	0.3315	0.3315	0.3315238
β^2	0.49725	0.49725	0.49725	0.49725858
β_t^2	0.10275	0.10275	0.102752	0.102714
α	0.660737	0.660737	0.660737	0.660143
α_t	0.051371	0.051371	0.0513714	0.0513571
α_γ	0.009366	0.009366	0.009366	0.0087859
r_g, m	5178.6	5178.7	5178.6	5176.4
r_m, m	4492.6	4492.6	4492.6	4492.7

Here, ρ_s, ρ , and ρ_{max} are the density on the sphere's surface, in the center, and maximal accordingly; $\beta_\perp^2$, β^2, and β_t^2 are the form factors of the axial, polar, and tangential components of the radius of inertia accordingly; α, α_t, and α_γ are the form factors of the radial, tangential, and dissipative components of the force function accordingly; r_g and r_m are the radiuses of the gravity and inertia

The radial distribution of the interaction force of the test mass point with the nonuniform sphere is

$$q(r) = -\frac{4\pi G}{r^2}\int_0^r r^2\rho(r)\,\mathrm{d}r = -\frac{4\pi G}{r^2}\int_0^r r^2\rho_0\left(a\frac{r^2}{R^2}+b\frac{r}{R}+c\right)\mathrm{d}r$$
$$= -\frac{GMm_1}{R^2}\left(\frac{3}{5}a\frac{r^3}{R^3}+\frac{3}{4}b\frac{r^2}{R^2}+c\frac{r}{R}\right)$$
$$= -\frac{GMm_1}{R^2}\left(-3.144\frac{r^3}{R^3}+3.849\frac{r^2}{R^2}+0.295\frac{r}{R}\right). \quad (7.60)$$

At $r/R = 0$, $\alpha_\gamma(r) = 0$, and at $r/R = 1$, $\alpha_\gamma(r) = 1$.

Table 7.2 demonstrates the results of the calculated dynamical parameters for all the density curves (7.51), and Fig. 7.4 shows the curves of radial distribution of the potential energy and gravity force for the test mass point.

We wish to evaluate all four curves of mass density distribution in order to recognize which one is closer to the real Earth. In this case, we keep in mind that the observed density jumps can be obtained for any curve by approximation of its continuous section with the mean value for each shell.

Figure 7.4 shows that the radial density values are substantially different for each curve. It refers, first of all, to the surface and center of the body. At the same time, Table 7.2 demonstrates the complete identity of the dynamical parameters of all the nonuniform spheres. It means that a fixed value of the polar moment of inertia

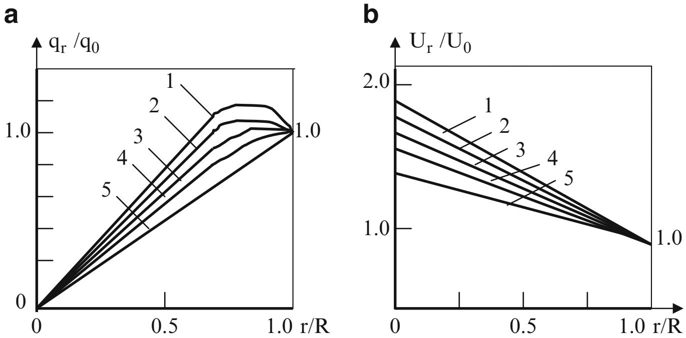
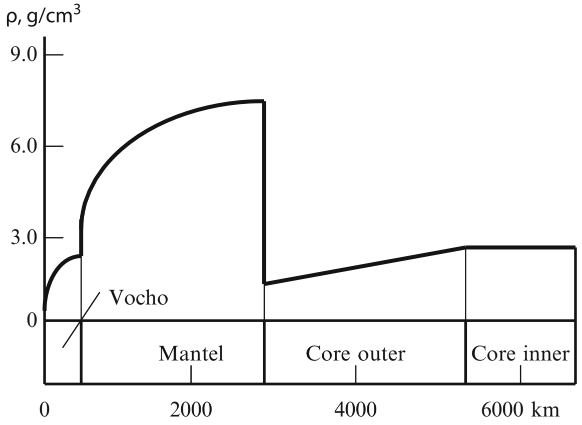


Fig. 7.4 The curves of the radial distribution of the potential energy (**a**) and gravity force (**b**) for the mass point test done by Eqs. (7.53) and (7.60)

permits us to have a multiplicity of curves of the radial density distribution with identical dynamical parameters of the body. The found property of the nonuniform self-gravitating sphere proves the rigor of the discovered functional relationship between the potential (kinetic) energy and the polar moment of inertia of the sphere. This property, in turn, is explained by the energy conservation law of a body during its motion and evolution in the form of the dynamical equilibrium equation or generalized virial theorem.

If we accept the conditions of the mass density separation presented in Sect. 6.5, 6.6, 6.7, and 6.8, then the range of curves of the density distribution gives a principal picture of its evolutionary redistribution and can be applied for reconstruction of the Earth's history. It follows from Eq. (6.31) that the density value of each overlying shell of the created Earth should be higher than the mean density of the inner mass. Otherwise, such a shell cannot be retained and should be dispersed by the tidal forces. It follows from this that the planet's formation process should be strictly operated by the dynamical laws of motion in the form of the virial oscillations and accompanied by differentiation of the nonuniform shells. The model of a cyclonic vortex that was proposed by Descartes is the most acceptable from the point of view of the considered ideas of planets' and satellites' creation from a common nebula. This problem needs a separate consideration. We only note here that from the presented curves of radial density distribution, the parabola (4) more closely reflects the present-day planet's evolution as fixed by observations. In this case, location of the Earth's reduced inertia radius falls on the lower mantle and the reduced gravity radius on the upper mantle. The density maximum falls also on the lower mantle. Its value is found by ordinary means, namely, by taking the derivative from the density distribution law as equated to zero. From here, $\rho_{\max} = 8.57$ g/ m^3 is found to be at a distance of $r = 3{,}122$ km. It means that the density maximum comes close to the border of the outer core where, as seismic observations show, the main density jump occurs. Curve (4) corrects the values of the radial density distribution in the mantle and changes its earlier interpretation in the outer and inner core. Because of zero values of the transverse velocities, the matter of the inner core

Fig. 7.5 Radial density distribution of the Earth by the authors' interpretation



has a uniform density structure and, from the point of view of the equilibrium state, seems to be in a gaseous state at a pressure of 1–2 atm. Taking into account the location of the maximum density value, there is a reason to assume that the outer core matter stays in the liquid or supercritical gaseous stage. In any case, the density and pressure of the inner and outer cores are much lower and should have values corresponding to the seismic wave velocities. On the basis of the equation of mass density differentiation (6.31), we interpret the density jumps observed (by seismic data) nearby the Mohorovičić–Gutenberg and at the outer core borders as the borders of the shell's dynamical equilibrium. A shell that is found over that border appears in a suspended state due to the action of the radial component of the gravitational pressure developed by the denser underlying shell. While the thickness of the suspended shell is growing, it acquires its own equilibrium pressure (iceberg effect). The extremely high pressures in the Earth's interior, which follow from the hydrostatic equilibrium conditions, are impossible in its own force field.

The concept discussed above in relation to the Earth's density distribution is illustrated in Fig. 7.5.

The polar moment of inertia here is $r_{\mathrm{m}} = 3/2 r_{\mathrm{m}}^{\perp} = \sqrt{1.5 \cdot 0.3315\,\mathrm{R}^2} = 0.70516\,\mathrm{R} = 4493 \cdot 10^3$ m and the radius of gravity is $r_{\mathrm{g}} = 0.8164\,\mathrm{R} = 5{,}201 \cdot 10^3$ m.

7.7 Oscillation Frequency and Angular Velocity of Shell Rotation

Let us continue the discussion about the nature of the Earth's dynamical parameters as an example. In order to determine numerical values of frequency of the virial oscillations and the angular velocities, which are the main dynamical parameters of the Earth's shells, we accept equation (4) of the density distribution (7.51) as the first approximation. All further relevant calculations can be made by applying this equation.

We know the mean values of the planet's density $\rho_0 = 5.519$ g/ m^3 and angular velocity of the upper shell $\omega_t = 7.29{\cdot}10^{-5}$ s^{-1}. Applying these values, the frequency and the period of the virial oscillations, and the coefficient k_e of the tangential component of the inner forces, can be found. In accordance with Eq. (6.27), the frequency of the upper shell is equal to

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G\rho_0(r)} = \sqrt{\frac{4}{3}3.14{\cdot}6.67{\cdot}10^{-8}{\cdot}5.519} = 1.24{\cdot}10^{-3}\ \text{s}^{-1}.$$

The period of oscillation is found from the expression

$$T_\omega = \frac{2\pi}{\omega_0(r)} = \frac{6.28}{1.24{\cdot}10^{-3}} = 5060.4 \quad c = 1.405\ \text{h}$$

The product of the found frequency and the Earth's radius gives the value of the planet's first cosmic velocity, the mean value of which is

$$v = \omega(r)r_c = \left(1.24{\cdot}10^{-3}\right){\cdot}6370 = 7.9\ \text{km/s}.$$

Unlike the usual expression for the first cosmic velocity in the form of $v_1 = \sqrt{GM/r}$, we used here the physical condition of the dynamical equilibrium at the Earth's surface between the inner gravitational pressure of interacting masses and the outer background pressure including atmospheric pressure.

Given below, our own observation data on the near-surface atmospheric pressure and temperature oscillations at the near-surface layer and the results of the spectral analysis prove the above theoretical calculations of the planet's frequency of virial oscillations (Ferronsky and Ferronsky 2010).

Now, applying the known mean value of the Earth's angular velocity $\omega_t = 7.29{\cdot}10^{-5}$ s^{-1} and the known value of the frequency of virial oscillations for the upper shell $\omega_0 = 1.24{\cdot}10^{-3}$ s^{-1} by Eq. (6.29), the coefficient k_e can be found:

$$k_e = \frac{\omega_t^2}{\omega_0^2} = \frac{\left(7.29{\cdot}10^{-5}\right)^2}{(1.24{\cdot}10^{-3})^2} = \frac{1}{289.33} = 0.003456.$$

The coefficient k_e is known in geodynamics as a parameter that shows the ratio between the centrifugal force at the Earth's equator and the acceleration of the gravity force there, which is equal to $k_e = 1/289.37$ (Melchior 1972). The parameter is used to study the Earth's figure based on the Clairaut hydrostatic theory.

7.7.1 Thickness of the Upper Earth's Rotating Shell

It is known that the value of the mean linear velocity of the upper planet's shell is $v = 0.465$ km/c. We can find the thickness h at which the velocity v corresponds to the found frequency of radial oscillations of the shell $\omega_0 = 1.24{\cdot}10^{-3}$ s^{-1}:

$$h_e = \frac{v}{\omega_0(r)} = \frac{0.465}{1.24 \cdot 10^{-3}} = 375\,\text{km}. \tag{7.61}$$

Such is the thickness of the upper shell of the Earth, which is rotating by forces in its own force field. It is assumed that the shell is found in the solid state. In reality, it is known that the rigid shell has a thickness less than 50 km. The remaining more-than-300-km-thick part of the shell has a viscous–plastic consistency, the density of which increases with depth. The border of the shell has a decreased density because of the melted substance due to high friction and saturation by a gaseous component. The border plays a role of some sort of spherical hinge. Because the density of the Earth's crust is lower than that of the underlying matter, it occurs in the suspended state. During the oscillating motion, the crust shells are affected by the alternating-sign acceleration and the inertial hydrostatic equilibrium.

7.7.2 Oscillation of the Earth's Shells

Let us obtain the expression of virial oscillations for the Earth's other shells by applying expression (4) of (7.51) for the radial density distribution. Write Eq. (6.27)

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G \rho_0(r)},$$

where

$$\rho_0(r) = \frac{m_0(r)}{\frac{4}{3}\pi r^3} = \frac{4\pi \int\limits_0^r r^2 \rho(r)\,dr}{\frac{4}{3}\pi r^3} = \frac{\frac{4}{3}\pi r^3 \rho_0 \left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)}{\frac{4}{3}\pi r^3}$$

$$= \rho_0 \left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right).$$

Then

$$\omega_0(r) = \sqrt{\frac{4}{3}\pi G \rho_0(r)} = \sqrt{\frac{4}{3}\pi G \rho_0 \left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)}$$

$$= 1.24 \cdot 10^{-3}\sqrt{\left(-3.144\frac{r^2}{R^2} + 3.849\frac{r}{R} + 0.295\right)}. \tag{7.62}$$

At $r/R = 0$, $\omega_0(r) = 0.6743 \cdot 10^{-3}\ \text{s}^{-1}$; at $r/R = 1$, $\omega_0(r) = 1.24 \cdot 10^{-3}\ \text{s}^{-1}$; and at $\rho_{max} = 8.57$ g/ m^3, $\omega_0(r) = 1.486 \cdot 10^{-3}\ \text{s}^{-1}$, where $r/R = 0.49$.

Figure 7.6 shows changes in the virial oscillation frequencies of the Earth's shells.

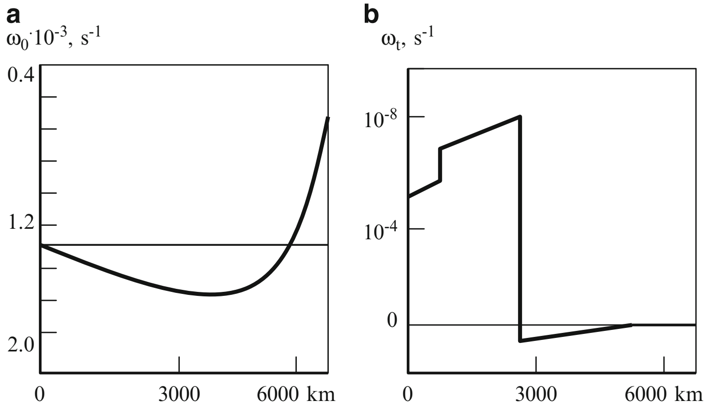


Fig. 7.6 Radial change in virial oscillation frequencies (**a**) and angular velocity of rotation (**b**) according to Eqs. (7.62) and (7.63)

7.7.3 Angular Velocity of Shell Rotation

Angular velocity of the Earth's shell rotations is determined from Eq. (6.28):

$$\omega_t(r) = \sqrt{\frac{4}{3}\pi G \rho_t(r)} = \sqrt{\frac{4}{3}\pi G \rho_0(r)\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)k_e(r)}$$
$$= \omega_0(r)\sqrt{\left(\frac{3}{5}a\frac{r^2}{R^2} + \frac{3}{4}b\frac{r}{R} + c\right)k_e(r)}$$
$$= \omega_0(r)\sqrt{\left(-3.144\frac{r^2}{R^2} + 3.8475\frac{r}{R} + 0.295\right)k_e(r)}, \tag{7.63}$$

where $\omega_t(r)$ is the angular velocity of the shell rotation and $\omega_0(r)$ is the shell oscillation frequency that is determined by Eq. (7.62).

The geodynamic parameter $k_e(r)$, which expresses the ratio of the tangential component of the force field and the gravity force acceleration for the upper shell, is approximated as

$$k_e(r) = \frac{\omega_t^2(r)}{\omega_0^2(r)}.$$

At $r/R = 1$, $k_e(r) = 0.003456$; at $r/R = 0$, $k_e(r) = 1$, and $\omega_t(0) = \omega_0(0)$; that is, the virial oscillation frequency corresponds to the gravity pressure of the uniform density masses. In this particular case, we are interested in changes of the angular velocity of rotation of the upper (1,000 km) and lower (up to the core border) mantle

(2,900 km) shells. Figure 7.6b shows the radial change of the angular velocity of rotation calculated by Eq. (7.63). It is seen that the angular velocity at the lower mantle–outer core is close to zero but changes its direction.

We emphasize once more that Eqs. (7.62) and (7.63) express the third Kepler's law, which determines radial distribution of both the virial oscillation frequencies and the angular velocities of rotation. Numerical values of these parameters are determined by the radial density distribution law. It also determines the density jumps that mark the effect of the shell's hydrostatic equilibrium.

7.8 The Nature of Precession, Nutation, and Body's Equatorial Plane Obliquity

The most noteworthy effects of dynamics of the Earth and other bodies are the interrelated phenomena of the precession and nutation of the axis of rotation, the tidal effects of the oceans and atmosphere, the axial obliquity and declination of the plumb line, and the gravity change at each point of the planet's outer force field. The present-day ideas about the nature of these phenomena were formed on the basis of the Earth's hydrostatic equilibrium and, since old times, were considered as effects of perturbation from the Sun, the Moon, and other planets. All the above phenomena represent periodic processes, and many observational and analytical works were done for their understanding and description. The present-day studies of these processes are still continuing to be specified and corrected. This is because such topical problems as correct time, ocean dynamics, short- and long-term weather and climate changes, and other environmental changes are important for everyday human life.

Now, after it was found that the conditions of the hydrostatic equilibrium are not acceptable for the study of the Earth's dynamics, we reconsider the nature of the phenomena by applying the concept of the planet's dynamical equilibrium and developing a novel approach to solving the problem.

7.8.1 Phenomenon of Precession

The first discovered phenomenon was the precession of equinoxes. It was observed already in the second century BP by the Greek astronomer and mathematician Hipparchus. His discovery was based on the comparison of longitudes of the far stars with the longitudes of the same stars determined 150 years ago by the other astronomers.

Inertial rotation of a symmetrical rigid body with a fixed point gives the classical explanation of precession. Such a motion of the body, presented in Fig. 7.7, includes its rotation with angular velocity Ω relative to the axis z, fixed in the body, and

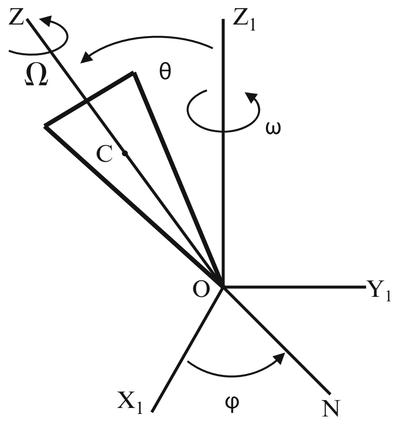


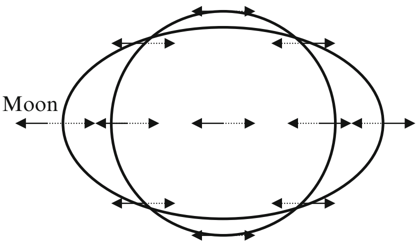
Fig. 7.7 Classical explanation of precession motion

from rotation with angular velocity ω around the axis z_1. Here, the axes $_1$, y_1, and z_1 are accepted to be immobile because motion of the body is considered just relative to them. The straight line ON perpendicular to the plane $z_1O\ z$ is called the line of nodes, and angle $\psi = x_1ON$ is the precession angle. Together with precession, the body performs the nutation motions (axis wobbling) that cause changes in the nutation angle $\Theta = z_1Oz$.

Perturbation of the Earth's inertial rotation is considered as a result of the applied solar–moon force couple, the axis of which is at right angles to the rotation axis; the body turns around the third mutually perpendicular axis. The Earth is accepted as a rigid body oblate along the rotary axis. Newton's idea was that the spherical body has an equatorial bulge that appeared as the result of the planet's oblateness. In this case, the Sun attracts stronger, the body's equatorial bulge, and it tends to decrease the inclination of the Earth's equatorial plane to the ecliptic. The Moon affects analogously but two times as powerfully due to close distance. The common effect of the Sun and Moon on the equatorial excess of the rotating Earth mass leads to the rotary axis precession. Because the induced precession forces are continuously varying due to changes in the Sun and Moon position relative to the Earth, then additional nutations (wobble) of the axis are observed during translational motion of the planet. In addition to the moon–solar precession, the effect of the other planets of about few tenth of an arc second is observed. The combined Earth precession rate is estimated to be equal to $\sim$50.3$''$ per year or one complete rotation in $\sim$26,000 years.

The theory of the precession and nutation of the Earth's axis of rotation based on the hydrostatics was developing in the works of D'Alembert, Laplace, and Euler. The precession values were calculated by Bessel and Struve and under verifying till up to now. Physical basis of the modern studies remains unchanged. The main accent in the studies is made on consideration of the elastic and rheological properties of the planet and the effects of dynamics of the atmosphere and the oceans and dynamics of the liquid core, the probability of which is assumed (Jeffreys 1970; Munk and MacDonald 1960; Melchior 1972–1973; Sabadini and Vermeersen 2004; Molodensky and Kramer 1961; Magnitsky 1965).

Fig. 7.8 Scheme of mass interaction between the Moon and the Earth for explanation of tidal effects (by Pariysky 1975)



7.8.2 Tidal Effects

The theory of the ocean tides was also presented first by Newton in his *Principia,* Proposition XXIV, Theorem XIX. He stated that the tides are caused by action of the Moon and the Sun. It follows from the Corollaries I and (Proposition LXVI, Book I) that the sea should rise and subside twice per every lunar and twice per every solar day, and the highest tide in the free and deep seas should appear less than 6 h after the tide body has passed the place meridian. And it happens like that along all the east Atlantic and Pacific shores. The effects of both tide bodies are summed up. At joining and opposing positions of the bodies, their effects are summed up and provide the highest or lowest tide. Observation shows that the tide effect of the Moon is stronger than the Sun.

Modern studies in the theory of precession and nutation remain on the physical basis described by Newton. Besides, all the above phenomena are considered in close relationship, and their amplitudes and periods are described by common equations that follow from the attraction theory (Melchior (1972–1973)).

The modern physical picture for explanation of the tidal interaction is presented as follows (Pariysky 1975). The tidal force is equal to a difference between any Moon-attracted placed on the Earth (including the atmosphere, the oceans, and the solid body) and the same particle replaced by the center of the planet (Fig. 7.8).

The normal tidal forces are proportional to the mass of the Moon m and the distance to the center of the Earth r, and to inverse cubic distance between the Moon and the Earth R, and zenith distance of the Moon z. The vertical component of the tidal force per the mass unit F_v is changing the gravity force into the value

$$F_v = 3G\frac{mr}{R^3}\left(\cos^2 z - \frac{1}{3}\right), \tag{7.64}$$

where G is the gravity constant.

The gravity force decreases by 0.1 mgal or by 10^{-7} of its value on the Earth's surface when the Moon stays in zenith or nadir, and twice increases when the Moon rises or sets.

The horizontal component of the tidal force is equal to zero when the Moon stays zenith, nadir, and on the horizon. Its maximum value reaches 0.08 mgal at zenith distance of the Moon equal to 45°:

$$F_{\mathrm{h}} = 3G\frac{mr}{R^3}\sin^2 z. \tag{7.65}$$

The tidal force of the Sun is formed analogously. But because of distance, its value is by 2.16 times less than of the lunar one. Due to rotational and orbital motion of the Earth, the Moon, and the Sun, the tidal force of each point in the atmosphere, the oceans, and the planet's surface continuously changes in time. The tables of integral values of the tidal forces in the form of the sums of periodic components ($\sim$500 terms or more) calculated by the theory of motion of the Moon round the Earth and the Earth round the Sun were compiled.

By estimation of many authors, the total tidal-slowing down Earth rotation amounts to 3.5 ms in 100 years. By astronomic observation, the Earth's rotation is accelerated by 1.5 ms per 100 years.

Note that in the framework of the hydrostatic approach, the problems of the nature of the obliquity of axis of the Earth's rotation to the ecliptic and the nature of the obliquity of axes of the Moon and the Sun to their orbit planes and their obliquity to the ecliptic are not discussed. These problems have no even formulation.

7.8.3 The Nature of Perturbations Based on Dynamic Equilibrium

In the beginning, let us consider physical meaning of the gravitational perturbation for interacted volumetric (but not point) body masses. To the contrary of the hydrostatics, where the measure of perturbation in the precession–nutation and the tidal phenomena is the perturbing force, in the dynamic approach, that measure of perturbation is power's pressure. In Chap. 2, we came to a conclusion that the mass points and the vector forces as a physical and mathematical instrument in the problem solution of dynamics of the Earth in their own force field are inapplicable. This is because the outer vector central force field of the interacted volumetric masses expresses incorrectly dynamical effects of their interaction. As a result, the kinetic effect of interaction of the mass particles, namely, the kinetic energy of their oscillation, is lost. And also, the geometric center of a body is accepted as the gravity center and center of the inertia (reaction). In dynamics, it leads to wrong results and conclusions. In this connection, we found that in the dynamics of a self-gravitating body, the effect of gravitational interaction of mass particles should be considered as the power's pressure. In addition, in this case, we are free in the choice of a reference system. Our conclusion does not contradict to Newton's physical ideas that are presented in Book I of his *Principia* where he says:

> I approach to state a theory about the motion of bodies tending to each other with centripetal forces, although to express that physically it should be called more correct as pressure. But we are dealing now with mathematics and in order to be understandable for mathematicians let us leave aside physical discussion and apply the force as its usual name.

Accepting the power pressure as an effect of gravitational interaction, we come to an understanding that in the considered problem of the mutual perturbations among the Earth, the Moon, and the Sun, the interaction results not between the body centers or shells along straight lines but between the outer force fields of the bodies and between their inner force fields of the shells. Satellite observations show that the outer force field, induced by the Earth's mass, has 4π-outward direction of propagation and acquires the wave nature. We consider this outer wave force field as a physical media by which the bodies transmit their energy. Thus, the Earth and other planets are held and move on the orbits by the power of the outer force field of the Sun. This statement is proved by the discovered solar system bodies' origin. This energy represents the integral effect of the Sun's mass interaction, and this energy conserves by orbiting motion of separating body (see Tables 1.1 and 1.2 of Chap. 1). The frequency of the gravity interaction determines the border equality of energies for interacting bodies. In this case, equality of the frequencies is the condition of their dynamical equilibrium.

Now, we can write the condition of the interacted force fields and find the physical border of such equilibrium between the Sun and the Earth in the form

$$\omega_s(R_s) = \omega_e(R_e), \tag{7.66}$$

where R_s and R_e are the radius of the Sun's and radius of the Earth's outer force field where the energies caring by the frequencies are equal to one another.

Analogously, equilibrium of the field energy for the Earth and the Moon is written as

$$(\omega_e)R_e = \omega_m(R_m), \tag{7.67}$$

where R are R_m are the radius of the Earth's and radius of the Moon's outer force field.

The mean value of the radius in (7.66) can be found from the equality of the frequencies of two bodies:

$$\sqrt{\frac{GM_s}{R_s^3}} = \sqrt{\frac{GM_e}{R_e^3}}, \tag{7.68}$$

and

$$R_s + R_e = R_{se}. \tag{7.69}$$

After transforming the equations and substituting numerical values of the solar mass $M_s = 1.99{\cdot}10^{30}$ kg, mass of the Earth $M = 5.976{\cdot}10^{24}$ kg, and the mean distance between the two bodies $R_s = 1.496{\cdot}10^{11}$ m, we obtain the cubic equation

$$R_e^3 - 1.35{\cdot}10^8 R_e^2 + 2.02{\cdot}10^{21} R_e - 10^{34} = 0. \tag{7.70}$$

Compiling and solving analogous equations for the Earth's position in perihelion ($R_{pe} = 1.471{\cdot}10^{11}$ m) and in aphelion ($R_{ae} = 1.521{\cdot}10^{11}$ m), we can find the corresponding mean radius in the knots, perihelion, and aphelion of the Earth's field equilibrium state:

$$R_{ek} \approx 2.131{\cdot}10^9\,\text{m}; \quad R_{ep} \approx 2.1277{\cdot}10^9\,\text{m}; \quad R_{ea} \approx 2.1335{\cdot}10^9\,\text{m}. \tag{7.71}$$

The corresponding energy caring by the frequency of dynamical equilibrium of the Earth's field in the found points, using Eq. (6.43), is the following:

$$\omega_{ek} \approx 4.1183{\cdot}10^{-7}\text{s}^{-1}; \quad \omega_{ep} \approx 4.1374{\cdot}10^{-7}\text{s}^{-1}; \quad \omega_{ea} \approx 4.1038{\cdot}10^{-7}\text{s}^{-1}. \tag{7.72}$$

It follows from the equilibrium condition (7.66) that the found frequency values (7.72) for the Earth should coincide with the frequencies of the Sun's oscillations of the force field in the corresponding points of Earth's orbit.

By the same method, the corresponding values of radiuses of the outer force field and the frequencies of the Moon locating in the Earth's force field can be found. According to (7.68) and (7.69) for the Moon at its mass $M_m = 7.35{\cdot}10^{22}$ kg and at the mean value of distance between two bodies $R_{km} = 3.844{\cdot}10^8$ m, in the perigee $R\ _m = 3.644{\cdot}10^8$ m and apogee $R\ _m = 4.068{\cdot}10^8$ m, for the radius, Eq. (7.70) is written in the form

$$R_m^3 - 0.14{\cdot}10^8 R_m^2 + 0.54{\cdot}10^{16} R_m - 0.69{\cdot}10^{24} = 0. \tag{7.73}$$

The values of radiuses in the knots, perigee, and apogee will be written as

$$R_{mk} \approx 0.72{\cdot}10^8\text{m}; \quad R_{mp} \approx 0.724{\cdot}10^8\text{m}; \quad R_{ma} \approx 0.716{\cdot}10^8\text{ m}. \tag{7.74}$$

The corresponding energy caring by frequencies in the above points of dynamical equilibrium of the Earth's field in the found points, using Eq. (6.43), is the following:

$$\omega_{mk} = 3.6242{\cdot}10^{-6}\text{s}^{-1}; \quad \omega_{mp} = 3.5942{\cdot}10^{-6}\text{s}^{-1}; \quad \omega_{ma} = 3.6546{\cdot}10^{-6}\text{s}^{-1}. \tag{7.75}$$

The found frequency values (7.75) for the Earth should coincide with the frequencies of the Moon's oscillations of the force field in the each point of the Moon's orbit.

The above results mean that the Earth's dynamical equilibrium at its orbital motion around the Sun and the Moon's dynamic equilibrium at its motion around the Earth are determined by the frequency equality of the outer force fields of the two bodies in every point of their orbits. Because the frequency of oscillation in a given point of the force field is a function of the body's mass density distribution, the inclination of the Earth's and the Moon's orbital plane to the equatorial plane of the Sun and the Earth is determined by asymmetry in mass density distribution of the two bodies. The observed inclination of the orbital planes is determined by asymmetry in mass density distribution of the Sun and the Earth. The observed parameters of the orbits and their inclination relative to the plane diameters of the Sun, the Earth, and the Moon give a general view of the asymmetric distribution of the body's masses. In particular, the northern hemisphere of the Earth is more massive than the southern one. In the perihelion, the northern hemisphere is turned to the less massive hemisphere of the Sun so that the polar oblateness of each body controls the location of its pericenter and apocenter, and the equatorial oblateness of each body responds to the location of its nodes. Thus, the body motion in the outer force field of its parent occurs under strict conditions of dynamic equilibrium, which is also the main condition of its separation. It follows from the condition of dynamic equilibrium that the orbital motion of the Earth and the Moon reflects asymmetry in mass density distribution of the Sun, the Earth, and the Moon and asymmetry in the potential of the outer wave field distribution. Only the structure of the Sun's outer wave field controls the Earth's trajectory at the orbital motion, and the Earth's force field manages the orbital motion of the Moon, but not vice versa or somehow else.

7.8.4 Rotation of the Outer Force Field and the Nature of Precession and Nutation

At the right time of motion of the bodies with the outer wave fields, their mutual perturbations are transferred not directly from each body to the other one or from their shells but through the outer fields by means of the corresponding active and reactive wave pressures of the interacting fields. There is an important dynamic effect of all the perturbations. This is the continuous change in the outer wave field of each body that proceeds from its nonuniform radial distribution of the mass density. As it was earlier shown, the nonuniform radial distribution of mass density initiates the differential rotation of the body shells. And, in accordance with Eqs. (6.27) and (6.28) expressing Kepler's third law, the reduced body shells' perturbing effects are transferred to the other body by means of the outer wave field. The Sun, for instance, transfers all the perturbations resulting during rotation of the interacting masses of the shells to the Earth continuously through its outer wave field. The Earth, in the framework of the energy conservation law, demonstrates all the perturbations by changes in its orbit turns around the Sun (see Fig. 7.9).

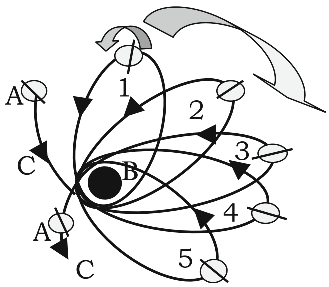


Fig. 7.9 Real picture of motion of body *A* in the force field of body *B*. Digits identify succession of turns of body *A* moving around body *B* along the open orbit *C*

Earlier, it was shown that in the case of nonuniform distribution of mass density, the body's potential and kinetic energies have radial and tangential components that induce oscillation and rotation of the shells. It was defined by Eq. (7.61) that the observed daily rotation of the Earth concerns only the upper shell with a thickness of ~375 km and reaches the nearby Mohorovičić discontinuity. By the same reasoning, it is not difficult to find the thickness of the upper shells for the Sun and the Moon correspondingly equal to

$$h_{\mathrm{s}} = \frac{v_{\mathrm{s}}}{\omega_{0\mathrm{s}}\,(R_{\mathrm{s}})} \approx \frac{2}{6.28{\cdot}10^{-4}} \approx 3,180\ \mathrm{km}, \tag{7.76}$$

$$h_{\mathrm{m}} = \frac{v_{\mathrm{m}}}{\omega_{0\mathrm{m}}\,(R_{\mathrm{m}})} \approx \frac{4.56{\cdot}10^{-3}}{9.66{\cdot}10^{-4}} \approx 4.72\ \mathrm{km}. \tag{7.77}$$

We do not know real values and angular velocities for the inner shells of the three bodies. These velocities have a direct interrelation with the observed changes in parameters of the orbital motion of the Earth and the Moon including the retrograde motion of the orbital nodes and the apsidal line. In this connection, let us try to understand first of all the nature of precession and nutation of the bodies from the viewpoint of the dynamic approach.

It was noted above that, in accordance with the hydrostatic approach, precession of the equinoxes of the Earth is an effect of the net torque of the Moon and the Sun on the equatorial "bulge" aroused from gravitational attraction. The torque aspires to diminish inclination of the equatorial belt with surplus mass relative to the ecliptic and induce the retrograde motion of the nodal line. In addition, because the ratio of distance between the interacted bodies is changed, the relationship between the forces is also changed. In this connection, the precession is accompanied by nutation (wobbling) motion of the axes of rotation.

Analysis of orbits of the artificial satellite motion around the Earth shows that, in spite of absence of the equatorial "bulge" of mass, the apparatus demonstrates the precession effect. Its orbital plane has a clockwise rotation with retrograde motion of the nodal line. But a new explanation of the phenomenon is given. It appears that the retrograde motion of the nodal line associates with the Earth's equatorial and polar oblateness. The amplitude of the nodal line shift depends on the satellite orbit inclination to the Earth's equatorial plane. In the case of the poles' orbital plane, the nodal line shift is completely absent. This is because the pole motion excludes both the polar and the equatorial oblatenesses of the Earth. The direction of motion of the apsidal line depends on the satellite's orbit inclination and is determined by the Lenz law.

It is also known that for the other free-of-satellite planets, the retrograde motion of the nodal line is also a characteristic phenomenon called the "secular perihelion shift." It was found from the observation of Mercury, Venus, Earth, and Mars that their secular perihelion shifts are decreased from $\sim 40''$ through $\sim 8.5''$, $\sim 5''$ to $\sim 1.5''$ accordingly (Chebotarev 1974).

All these facts imply that the explanation given for the satellites' precession depending on their orbital inclination to the ecliptic is correct. But the nature of this unique phenomenon, characteristic for all celestial bodies, is inconsistent with the hydrostatic approach and should be reconsidered, taking also into account the satellite observations.

The precession of the Earth, the Moon, and the artificial satellites in the form of motion of an orbital plane toward the backward direction of the body's motion should be considered as a virtual explanation of the phenomenon. In fact, the orbit's plane is a geometric shape traced by the body. And there is no reason to consider its movement without the body itself. There is no difficulty to present the real body motion in space in two opposite directions synchronously. In particular, the actual picture of the Earth, the Moon, and the satellite motion in counterclockwise direction and retrograde movement of the nodal line is shown in Fig. 7.9.

Here, the satellite is moving in the counterclockwise direction along unlocked elliptic orbit 1 in the continuously changing (perturbed by oblatenesses) force field of the planet. Because of the counterclockwise rotation of the Earth's mass, the satellite in perigee started to move on orbit 2 and makes a shift in retrograde direction in the ascending and descending nodes. At the same time, the eccentricity of orbit 2 changes by a proper value. Analogously, the body passes on orbits 3, 4, 5, and so on. The theory of dynamic equilibrium of the Earth explains the physics of the observed phenomenon as follows.

The dynamic equilibrium theory assumes that the Earth is a self-gravitating body, the interacting mass particles of which induce the inner and outer force fields. Separation of the planet's asymmetric shells results by the inner force field and depends on the law of the radial mass density distribution. The normal component of the body's power pressure provides oscillation, and the tangential component induces rotation of the shells having a different angular velocity. At the same time, the mantle shells A and the outer shell of the core B may have the same (Fig. 7.10a) or opposite direction (Fig. 7.10b) of rotation, depending on the radial mass density distribution.

The seismic data show that the inner core C has a uniform density distribution. Because of this, it does not rotate and its potential energy is realized in the form of oscillation of the interacting particles. The potential of the outer force field is controlled by integral effect of the interacted masses of all the shells and presented by the reduced shell D having continuously changing power.

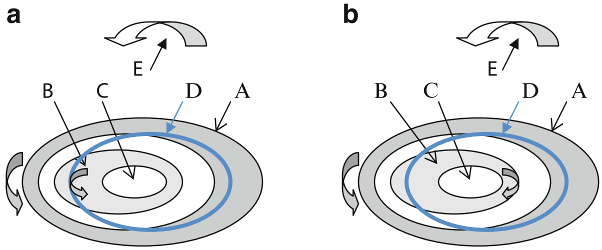


Fig. 7.10 Sketch of rotation of the Earth's shells by action of the inner force field: is the mantle shells; *B* is the outer core; *C* is the inner core; *E* is the outer force field; and *D* is the reduced shell of the inner force field of the planet for direct (**a**) and for opposite direction (**b**) of the shell rotation

The energy of the Earth's outer force field is changed from the body surface in accordance with the $1/r$ law and, at every r, is continuously varied because of differences in the angular velocity of rotation of the shell's masses. This force field controls the direction and the angular velocity of orbital motion of a satellite. Taking into account the nonuniform and asymmetric distribution of the masses of rotating shells, the change in the trajectory of the body motion is accompanied by a corresponding change in eccentricity of the orbit both at each and subsequent turns. Its maximum value is reached when the nonuniformities of the rotating masses coincide and the minimal value appears at the opposite position.

It is worth noting that the effect of retrograde motion of the nodal line of the Earth, the Moon, and artificial satellites appears to be a common phenomenon because the induced by the outer force fields of the Sun and the Earth are changing with a finite velocity. The conclusion follows from here that the Sun has the same effects in its shell structure and motion. It is obvious that the other planets with their satellites have the same character of structure and motion.

If one takes into account the effect of a planet's orbital plane inclination to the equatorial plane of the Sun, then the above changes are found to follow the law of $1/r$. This observable fact proves our conclusion that the changes in the outer force field of a body are controlled by rotation of its reduced inner force shell (see the force shell *D* in Fig. 7.10). It explains why Mercury has maximal value of the "secular perihelion shift" between the other planets.

Thus, the Earth's orbital motion and retrograde movement of its nodal line are controlled by the Sun's dynamics of the masses through the outer force field. The Earth plays the same role for the Moon and the artificial satellites. As to the nutation motion, its nature is related to the same peculiarities in the structure and motion of the bodies, but the effects of their perturbations are fixed by the axis wobbling.

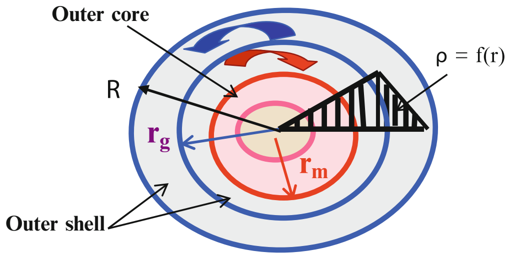


Fig. 7.11 Dependence of the parabolic law of radial density distribution on the shell rotation for the Earth. Here r_m and r_g are the reduced radiuses of inertia and gravitation

7.8.5 The Nature of Possible Clockwise Rotation of the Outer Core of the Earth

The question arises why the outer planet's core may have a clockwise rotation. It was shown in Sect. 3.6 that the law of radial density distribution determines the direction of a body's shell rotation.

It was found that in the case of uniform mass density distribution, all energy of the mass interaction is realized in the form of oscillation of the interacting particles (Fig. 3.2a). If the density increases from the body's surface to the center, then there are oscillations and counterclockwise rotation of shells (Fig. 3.2b). Increase of mass density from center to surface leads to oscillation and clockwise rotation with different angular velocities of the body shells (Fig. 3.2c). Finally, the parabolic law of radial density distribution (Fig. 7.11), where the density increases from the surface and then it decreases, leads to oscillation and reverse directions of rotation. Namely, the upper shells have a counterclockwise and the central shells have a clockwise rotation. The case demonstrated on Fig. 7.11, obviously, is characteristic for a self-gravitating body.

Note that direction of the body rotation depends on radial density distribution and corresponds with the Lenz right-hand or right-screw rule, well known in electrodynamics. Taking into account the observed effect of the retrograde motion of the satellite nodal line, the gravitational induction of the inner and outer force fields of the Earth has a common nature with electromagnetic induction noted earlier. Just Fig. 7.11 may explain the nature of the retrograde motion of the nodal line of a satellite orbit related to the finite velocity in the potential changes of the outer Earth's force field induced by the interacted mass particles. The continuous and opposite-directed movement of the asymmetric mass density distribution of the mantle and the outer core (Fig. 7.11) seems to be the physical cause of precession, nutation, and variation of the inner and outer force fields observed by satellites. This idea is proved by the satellite data about the retrograde motion of the nodal line depending on the inclination of its orbital plane with respect to the planet's equatorial plane.

It is worth recalling, from the literature, that the idea of dynamical effects of the probably liquid core of the Earth has been discussed among geophysicists for a long time (Melchior 1972).

7.8.6 The Nature of the Earth's Orbit Plane Obliquity to the Sun's Equatorial Plane

Celestial mechanics does not discuss the problem of obliquity of the planet's and satellite's orbit planes and accepts it as an observable fact. From the viewpoint of dynamical equilibrium, creating and orbiting of the planets and satellites originated from the parental upper weightlessness shell with the first cosmic velocity. The separation could have happened at any point of the body's surface, depending on the stage of evolution and the radial mass density distribution. It is known from the observation that in most cases, it occurred in the parental equatorial zone. But there are observational data, from which some planets and satellites separated under higher angle of inclination. It is known from the experience of the artificial satellite launching that the angle of inclination to the Earth's equatorial plane is determined by the parameters of satellite motion and dynamical effects of the Earth's force field.

Unfortunately, up to now, we fix inclination of orbital planes of all planets and even the Sun relative to the Earth's orbital plane accepted as ecliptic. This is a residual of the Ptolemaeus heliocentric system of the world, which was preserved in order to use the observational data. This is the cause of changes in virtual direction of the apsidal line at orbital inclination about 63° and the orbital plane rotation of the planets and satellites.

Despite this, we conclude that the Sun has a shell structure and its outer force field is rotating with angular velocity equal to the velocity of the Earth retrograde motion of the knots. Thus, we find that the Earth's and planet's precession of the axes of rotation is the effect of difference in the Sun's velocity of the shells and correspondingly outer force field rotation. Taking into account observational data, the Earth's annual value in our time is equal to $\sim 50''$. But in longer time scale, this value is changing. This is because of changes in the period of rotation of the reduced shell of the inner force field of the Sun (see reduced shell *D* in Fig. 7.10). By the Earth's climatic changes, the period of rotation of the Sun's inner force field reduced shell changes, which is between $\sim$50,000 and $\sim$120,000 years (see Fig. 7.12).

The problem of a body motion on non-closed rotating trajectory, shown in Fig. 7.9, in classical mechanics, is known as a problem of the finite motion in the central force field in the domain restricted by the radiuses $r_{\max}$ and $r_{\min}$ from the aphelion to the perihelion (Landau and Lifshitz 1969). The trajectory can be closed after n turns at the condition of radius vector r turn on the angle $\Delta\varphi$, which is equal to the rational part from 2π, that is, at $\Delta\varphi = 2\,\pi n_1/n_2$, but n_1 and n_2 should be equal to an integer.

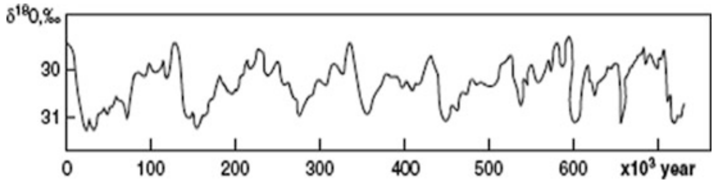


Fig. 7.12 Isotopic composition of oxygen in shells of mollusk *Globigerinoides sacculifer* within a time period of 0–730,000 years (Emiliani 1978)

In our case, by observation, the annual precession of the Earth's axes of rotation is equal to $\Delta\varphi \approx 50''$. In addition, the upper more-light-in-density shell of the Sun with a thickness of 3,200 km, which has the observed daily angular velocity of 14.4°, and also the Earth's rotating force field generate extra perturbation. So strictly speaking, the Earth's trajectory remains non-closed. Thus, the above figures in first approximation can be used in practical geophysics for characterization of the integral rotation of masses of the Sun's shells.

The rotating Sun's upper shell appears to be an additional source of the Earth's perturbations, which developed in nutations of its upper shell. Speculative perception of the nutations is understood as a wobbling of the planet's axes of rotation in different time scales from daily to annual.

Analogous phenomenon is observed for the Moon's motion around the Earth. Looking at the period of the main nutation, the integral period of rotation of all the shells of the planet should be equal to 18.6 years. The angular velocity of rotation of reduced shell of the Earth's inner force field should be equal to 19.35° per year, 1.61° per month, and 3.18′ per day. The period of the Moon's revolution around the Earth is 27.3 days. The Moon's daily angular orbital velocity is equal to 13.19° and is the same as the value around its own axes of rotation. During one period of its turn, the Moon delaying in the motion in arc distance $\Delta\varphi = 42.54'$, which we accept as a retrograde motion of the knots. The main period of the Earth's nutation seems to be the period of the Moon's precession of the axes. Because the daily and monthly time scales of motion of the Earth and the Moon do not coincide, the arc values of their retrograde knots motion are continuously changing with a period of 18.6 years. The values of the daily, monthly, and yearly nutations of the Earth's upper shell are correspondingly changing because of the Moon's perturbations.

7.8.7 The Nature of Chandler's Effect of the Earth Pole Wobbling

As it was noticed, changes in the planet's inner force field are observed in the form of nutation or wobbling of the axis of rotation. The axis itself reflects the dynamics of the upper planet's shell, the thickness of which, by our estimate, is about 375 km. The Moon is rotating around the Sun in the force field of the Earth, which is perturbed by its natural satellite. Its maximum yearly perturbation should be the Chandler effect. The Moon's yearly cycle seems to be the ratio of the Earth's to the Moon's month (in days). Then, this cycle is 365 (30.5/27) ≈ 410 days (see Fig. 7.13).

7.8.8 *The Nature of Obliquity of the Earth's Equatorial Plane to the Ecliptic*

It is obvious that the obliquity of the planet's equatorial plane is related to the polar and equatorial oblateness of the Earth's masses. It follows from Eq. (6.28) that the obliquity, in turn, is determined by the tangential component of the inner force pressure generated by the nonuniform radial mass density distribution. This tangential component of the inner force field induces the inner field of the rotary moments, the energy of which was discussed in Sect. 7.8.6 and presented in Table 7.2. The obliquity value can be obtained from the ratio of the potential energy of the uniform U_0 and nonuniform U_{t} body of the same mass. Accepting this physical idea and the data of Table 7.2, we can write and obtain

$$\cos\,\theta = \frac{U_0}{U_1} = \frac{\alpha_0}{\alpha_1} = \frac{0.6}{0.66} = 0.909, \quad \Theta = 24.5°, \tag{7.78}$$

where α_0^2 and α_{t}^2 are the structural form factors taken from Table 7.2.

The error obtained in the calculation of obliquity by formula (7.78) equal to about 1° or $\Delta\alpha_{\mathrm{t}}^2 = 0.006$. can be explained by the accepted law of the continuous radial distribution of the planet's mass density.

Equation (7.78) expresses the integral effect of the obliquity of the planet's equatorial plane, which is observed on the surface of the upper rotating shell. It was shown earlier that the observed obliquity is really an integral dynamical effect of the Earth's mass including the upper part of the Gutenberg shell. But being in a suspended state, relative to the other parts of the body, the upper shell is able to wobble as if on a hinge joint by perturbation from the Sun and the Moon. This effect of the upper shell wobbling gives an impression of the axial wobbling.

By the same cause, the obliquity of the ecliptic with respect to the solar equator is determined by the Sun's polar and equatorial oblateness. The trajectory of the Earth's orbital motion at each point is controlled by the outer asymmetric solar force field in accordance with the dynamic equilibrium conditions. And only in the nodes, which are common points for equatorial oblateness of the Sun and the Earth, is Huygens' effect of the innate initial conditions fixed by the third Kepler's law.

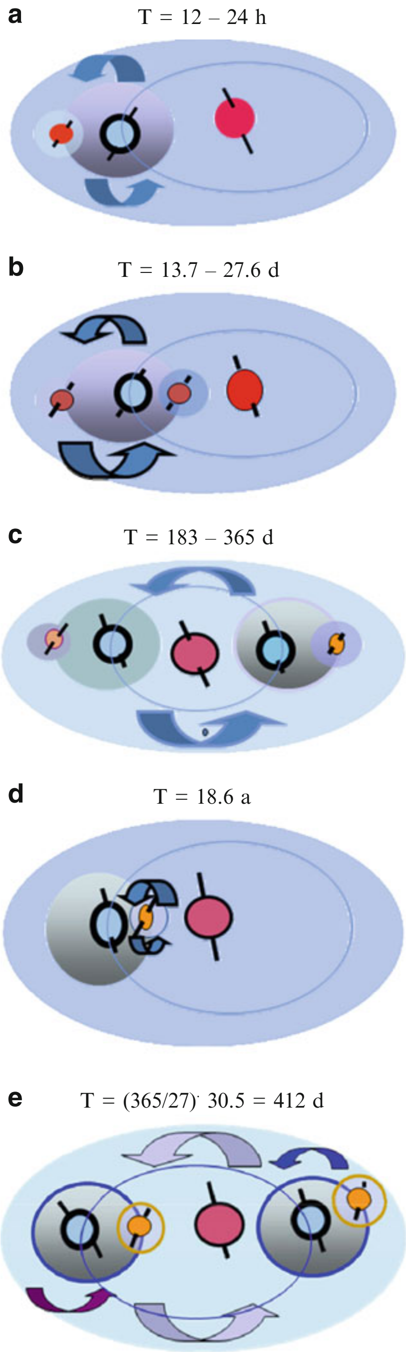


Fig. 7.13 Effects of inner and outer force fields of the three bodies (the Sun, the Earth, and the Moon) on the Earth's axes nutation: (**a**) diurnal and semidiurnal caused by the Earth's upper shell rotation, (**b**) monthly and semimonthly initiated by the Moon at its elliptic orbit revolution, (**c**) annual and semiannual caused by the nonuniform outer solar force field, (**d**) 18.6-year periodic nutation caused by the Moon's precession (the outer force field rotation), (**e**) Chandler's effect caused by the Moon's yearly cycle

7.8.9 Tidal Interaction of Two Bodies

Let us consider the mechanism and effects of interaction of the outer force pressure of two bodies being in dynamic equilibrium. Come back to the mechanism and conditions of separation of a body mass with respect to its density when a shell with light density is extruded to the surface. Rewrite Eq. (6.31) for acceleration of the gravity force in points and of the two body shells (Fig. 6.1b) and their densities ρ and ρ_m.

$$q_{AB} = 4\pi G r \left(\frac{2}{3} \rho_M - \rho_m \right), \tag{7.79}$$

After the shell with density ρ_m appears on the outer surface of the body, the condition of its separation by Eq. (7.79) will be

$$\rho_M > 2/3 \rho_m. \tag{7.80}$$

The gravitational pressure will replace the shell up to the radius $+\delta$, where the condition of its equilibrium reaches $\rho = \rho_m$. This condition is kept on the new borderline between the body and its upper shell. Taking into account that the shell in any case has a thickness, then, by the Archimedes law, the body will be subject to its hydrostatic pressure. If the separated shell is nonuniform with respect to density, then a component of the tangential force pressure appears in it, and the secondary self-gravitating body (satellite) is formed. The new body will be kept on the orbit by the normal and equal tangential components of the outer force pressure. In this case, the reaction of the normal gravitational pressure will be local and nonuniform. If the upper shell is uniform with respect to density, then the reaction of the normal gravitational pressure along the whole surface of the body and the shell remains uniform. In this case, the separated shell remains in the form of a uniform ring.

The above schematic description of the physical picture of the separation and creation of a secondary body can be used for the construction of a mechanism of the tidal phenomena in the oceans, the atmosphere, and the upper solid shell at interaction between the Earth and the Moon. The outer gravitational pressure of the Moon, due to which it maintains itself in equilibrium on the orbit, at the same time renders hydrostatic pressure on the Earth's atmosphere, oceans, and upper solid shell through its outer force field. This effect determines the tidal wave in the oceans and takes active part in the formation and motion of cyclonic and anticyclonic vortexes. In accordance with the Pascal law, the reaction of the Moon's hydrostatic pressure is propagated within the total mass of the ocean water and forms two tidal bulges. Because the upper shell of the Earth is moving faster relative to the motion of the Moon, the front tidal bulge appears ahead of the moving planet. Our perception of the ocean tides as an effect of attraction of the Moon appears to be speculative.

7.8.10 *Change in Climate as an Effect of Changes of the Earth's Orbit*

The above analysis of dynamical effects of the Earth's shells is based first of all on the data of satellite's orbit changes and measurements of the planet's force field. Unfortunately, a specific feature of an artificial satellite orbital motion is its artificial velocity, which is ~16 times higher than the angular velocity of the upper Earth's shell. In this connection, all its parameters of satellite motion are unnatural. So we cannot directly divide the natural component of its nodal retrograde shift in order to get the total picture of perturbations that propagate the Earth's inner shells. This is an experimental problem.

But there are also long-term astronomical observations of the Earth's dynamics relative to the far stars, the results of which correspond to the presented ones. In addition, periodicity in rotation of asymmetric inner shells of the Sun can be fixed by climatic changes on the Earth over a long period of time. Such changes were being studied, for instance, by data of the oxygen isotopic composition in mollusk shells over a number of years. Figure 7.12 demonstrates the results of Emiliani (1978), who studied the core obtained during deep sea drilling in the Caribbean basin.

The author obtained the picture of climate change in the Pleistocene era over 700,000 years. It is seen that the periods of climate change vary from 50,000 to 120,000 years. It means that the pure period of rotation of the asymmetric mass shells of the Sun is absent, and the orbital trajectory has not been locked into place during the studied time.

References

Alfven H, Arrhenius G (1970) Origin and evolution of the solar system. Astrophys Space Sci 9:3–33

Bullen RE (1974) Introduction to the theory of seismology, Cambridge University Press, London

Campbell PM (1962) Proc Natl Acad Sci 48(12):1993

Chandrasekhar S, Fermi E (1953) Problems of gravitational stability in the presence of a magnetic field. Astrophys J 118:113

Chebotarev GA (1974) Celestial mechanics. In: The Great Soviet encyclopedia, 3rd edn. Moscow, pp 386–388

Duboshin GN (1975) Celestial mechanics: the main problems and the methods. Nauka, Moscow

Emiliani C (1978) The cause of the ice ages. Earth Planet Sci Lett 37:349–352

Ferronsky VI, Ferronsky SV (2010) Dynamics of the earth. Springer, Dordrecht/Heidelberg

Ferronsky VI, Denisik SA, Ferronsky SV (1981a) On the relationship between the total mass of a celestial body and the averaged mass of its constituent particles. Phys Lett 84A:223–225

Ferronsky VI, Denisik SA, Ferronsky SV (1981b) On the relationship between the total mass of a celestial body and the averaged mass of its constituent particles. Phys Lett 84A:223–225

Ferronsky VI, Denisik SA, Ferronsky SV (1996) Virial oscillations of celestial bodies: V. The structure of the potential and kinetic energies of a celestial body as a record of its creation history. Celest Mech Dyn Astron 64:167–183

Jeans JH (1919) Problems of cosmogony and stellar dynamics. Cambridge University Press, Cambridge

Jeffreys H (1970) The earth: its origin, history and physical constitution, 5th edn. Cambridge University Press, Cambridge

Kittel C (1968) Introduction to solid state physics, 3rd edn. Wiley, New York

Landau LD, Lifshitz EM (1969) Mechanics, elektrodynamics. Nauka, Moscow

Landau LD, Lifshitz EM (1973) Mechanics. Nauka, Moscow

Magnitsky VA (1965) Inner structure and physics of the earth. Nauka, Moscow

Melchior P (1972) Physique et dynamique planetaires. Vander-Editeur, Bruxelles

Molodensky MS, Kramer MV (1961) The earth's tidals and nutations of the planet. Nauka, Moscow

Munk W, MacDonald G (1960) Rotation of the Earth (transl. from English). Mir, Moscow

Pariysky NN (1975) Tides. In: Great Soviet Encyklopedy, 3rd edn. Moscow, 580–582

Sabadini R, Vermeersen B (2004) Global dynamics of the earth. Kluwer, Dordrecht

Shirkov DV (ed) (1980) Microcosmos physics, small encyclopedy. Nauka, Moscow

Spitzer L Jr (1968) Diffuse matter in space. Interscience, New York

Zeldovich YB, Novikov ID (1967) Relativistic astrophysics. Nauka, Moscow

Zharkov VN (1978) Inner structure of the earth and planets. Nauka, Moscow

Chapter 8
The Nature of Electromagnetic Field of a Celestial Body and Mechanism of Its Energy Generation

Abstract In order to find a solution of the problem, a novel idea based on the innate capacity of body's energy for performing motion is discussed. The energy is the measure of the motion and interaction of particles of any kind of body's matter. The various forms of energy are interconvertible, and its sum for a system remains constant. The above unique properties of the energy, with its oscillating mode of the motion in our dynamics, make it possible to consider the nature of the electromagnetic and gravitational effects of celestial bodies as interconnected events.

Applying the dynamical approach and the results obtained, it is shown that the nature of creation of the electromagnetic field and the mechanism of its energy generation appear to be the effect of the volumetric gravitational oscillation of the body's mass. This effect is also characteristic for any celestial body. A number of tasks were considered in this chapter, namely, electromagnetic component of the interacted masses, potential energy of the Coulomb interaction of mass particles, emission of electromagnetic energy by a celestial body as an electric dipole, quantum effects of generated electromagnetic energy, and the nature of the star-emitted radiation spectrum.

The relationship between the gravitational field (potential energy) and the polar moment of inertia of the Earth, discovered by the artificial satellites, leads to understand the nature and the mechanism of a celestial body's energy generation as the force function of all the dynamical processes releases in the form of oscillation and rotation of the matter. Through the energy nature, we understand the unity of forms of the gravitational and electromagnetic interactions, which, in fact, are the two sides of the same natural effect.

The hydromagnetic dynamo, the action of which is provided by the planet's liquid metal core or the solar gas plasma, is the most popular idea for the explanation of a body electromagnetic field generation. Its essence is in the motion of the conducting liquid core where self-excitation of the electric and magnetic poloidal (meridional)

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_8,

and toroidal (parallel) fields happened. During the rotation of the inner planet's shells with different angular velocities, in the case of asymmetric thermal convection of the shell mass, the intensity of the fields is increased. This condition, for example, for the Earth is achieved because the rotation and magnetic axes are not coincided and the thermal convection supposedly takes place. But physically justified theory of the observed planets and solar phenomenon of electromagnetic field is absent. There is no explanation for the mechanism of generation of the energy of this field, except for general physical principle of the mass and charge interaction. Also the ideas or hypotheses about the source of refilling of the planets' energy that is spent for the gravitational and thermal irradiation are absent. The only source of the solar- and star-irradiated energy is accepted to be the interior nuclear fusion. In this chapter, we discuss this problem from the position of the Jacobi dynamics effects.

In order to find a solution to this problem, in this chapter, we discuss a novel idea based on the innate capacity of body's energy for performing motion. As shown in Chaps. 3 and 4, the energy is the measure of the motion and interaction of particles of any kind of body's matter. The various forms of energy are interconvertible, and its sum for a system remains constant. The above unique properties of the energy, with its oscillating mode of the motion in our dynamics, make it possible to consider the nature of the electromagnetic and gravitational effects of celestial bodies as interconnected events.

It was shown in Chap. 7 that the body's gravitational (potential) energy results in the body's matter volumetric pulsations, having oscillating regime, frequencies of which depend on the mass density. In our consideration, the planets and stars are accepted as self-gravitating bodies. Their dynamics is based on their own internal force field, and the potential and kinetic energies are controlled by the energy of oscillation of the polar moment of inertia, that is, by interaction of the body's elementary particles.

Applying the dynamical approach and the results obtained, we show below that the nature of creation of the electromagnetic field and the mechanism of its energy generation appear to be the effect of the volumetric gravitational oscillation of the body's masses. This effect is also characteristic for any celestial body.

8.1 Electromagnetic Component of the Interacted Masses

It was shown in Sect. 6.2 that the electromagnetic energy is a component of the expanded analytical expression of the potential energy. The expansion was done by means of the auxiliary function of the density variation relative to its mean value. The expression of the body's potential gravitational energy in the expanded form (6.8) was found as

$$U = \alpha \frac{GM^2}{R} = \left[\frac{3}{5} + 3 \int_0^1 \psi \, x \, \mathrm{d}x + \frac{9}{2} \int_0^1 \left(\frac{\psi}{x} \right)^2 \mathrm{d}x \right] \frac{GM^2}{R}, \qquad (8.1)$$

where U is the potential energy of the gravitational interaction, α is the form factor of the force function, G is the gravity constant, M is the body mass, R is its radius, and $\psi(s)$ is the auxiliary function of radial density distribution ρ_r relative to its mean value ρ_0:

$$\psi(s) = \int\limits_0^s \frac{(\rho_\mathrm{r} - \rho_0)}{\rho_0} x^2\,\mathrm{d}x. \tag{8.2}$$

We have considered and applied the first two right-hand side terms of Eq. (8.1). The third term in dimensionless form represents an additive part of the potential energy of the interaction of the nonuniformities between themselves, which was written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int\limits_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int\limits_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2\,\mathrm{d}x. \tag{8.3}$$

where

$$\lambda = \int\limits_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \geq 0.$$

The nonuniformities are determined as the difference between the given density of a spherical layer and the mean density of the body within the radius of the considered layer. For interpretation of the third term, we apply the analogy of electrodynamics (Ferronsky et al. 1996). For each particle, there generates an external field, which determines its energy. The energies of some other interacted particles and their own charges are determined by this field. As far as the potential of the field is expressed by means of the Poisson's equation through the density of charge in the same point, the total energy can be presented in additive form through the application of the squared field potential. If the body mass is considered as a moving system, then Maxwell's radiation field applies.

In our solution, the dimensionless third term of the field energy is written as

$$\frac{9}{2}\lambda = \frac{9}{2}\int\limits_0^1 \left(\frac{\psi}{x}\right)^2 \mathrm{d}x \equiv \frac{9}{2}\int\limits_0^1 \left(\frac{\psi}{x^2}\right)^2 x^2\,\mathrm{d}x \equiv \frac{9}{2}\int\limits_0^1 E^2\,\mathrm{d}V, \tag{8.4}$$

where $= \psi/x^2$ is the dimensionless form of the electromagnetic field potential that is a part of the gravitational potential, ψ plays the role of the charge, and $\mathrm{d}V = x^2\,\mathrm{d}x$ is the volume element in dimensionless form.

Table 8.1 Observational parameters of equilibrium nebulae

Parameters	Visible dark nebulae			
	Small globula	Large globula	Intermediate cloud	Large cloud
m/m_{Sun}	>0.1	3	$8{\cdot}10^2$	$1.8{\cdot}10^4$
$R(pc)$	0.03	0.25	100	20
$(\ / \ \text{m}^3)$	$>4{\cdot}10^4$	$1.6{\cdot}10^3$	100	20
$m/\pi R^2$ (g/ m^2)	$>10^{-2}$	$3{\cdot}10^{-3}$	$3{\cdot}10^{-3}$	$3{\cdot}10^{-3}$

In order to determine the numerical value of λ, the calculations for a sphere with different laws of radial density distribution including the polytropic model were done (Ferronsky et al. 1996). These models were used in our earlier numerical calculations of the form factors α and β. The results show that for the density distributions that have physical meaning (Dirac's envelopes, and Gaussian and exponential distributions) and also for the polytropes with index 1.5, the parameter λ has the same constant value. We interpret this fact for a steady-state dynamical system as an evidence of the existence of equilibrium radiation between a celestial body and the external flow. The numerical value of the parameter λ is equal to 0.022. There is also an observational confirmation of this conclusion. Spitzer (1968) demonstrates observational results of nebulae of different mass and size in Table 8.1.

Thus, in this case, the energy of the equilibrium electromagnetic field of radiation is equal to

$$U_\gamma = \left(\frac{9}{2} \int_0^1 E^2 \, \mathrm{d}V \right) \frac{GM^2}{R} = 0.1 \frac{GM^2}{R}. \tag{8.5}$$

Thus, the virial approach to the problem solution of the Earth's global dynamics gives a novel idea about the nature of the planet's electromagnetic field. The energy of this field appears to be the component of the potential energy of the interacted masses. The question arises about the mechanism of the body's energy generation, which provides radiation in a wide range of the wave spectrum from radio through thermal and optical to x and γ rays.

8.2 Potential Energy of the Coulomb Interaction of Mass Particles

With the help of model solution, we can show that for the Coulomb interactions of the charged particles, constituting a celestial body, the relationship between the potential energy of a self-gravitating system and its Jacobi function

$$U\sqrt{\Phi} = \text{const.} \tag{8.6}$$

remains (Ferronsky et al. 1981).

Derivation of the expression for the potential energy of the Coulomb interactions of a celestial body is based on the concept of an atom following, for example, from the Thomas–Fermi model (Flügge 1971). In our problem, this approach does not result in limited conclusions since the expression for the potential energy, which we write, will be correct within a constant factor.

Let us consider a one-component, ionized, quasi-neutral, and gravitating gaseous cloud with a spherical symmetrical mass distribution and radius of the sphere R. We shall not consider here the problem of its stability, assuming that the potential energy of interaction of charged particles is represented by the Coulomb energy. Therefore, in order to prove relationship (8.6), it is necessary to obtain the energy of the Coulomb interactions of positively charged ions with their electron clouds.

Assume that each ion of the gaseous cloud has the mass number $_{\mathrm{i}}$ and the order number Z, and the function $\rho(r)$ expresses the law of mass distribution inside the gaseous cloud. The mass of the ion will be ${}_{\mathrm{i}}m_{\mathrm{p}}$ (where $m_{\mathrm{p}} = 4.8 \cdot 10^{-24}$ g is mass of the proton), and its total charge will be $+Ze$ (where $e = 4.8 \cdot 10^{-10}$ GCSE is an elementary charge). Then, let the total charge of the electron cloud, which is equal to $-Ze$, be distributed around the ion in the spherically symmetrical volume of radius r_{i} with charge density $q_{\mathrm{e}}(r_{\mathrm{e}})$, $r_{\mathrm{e}} \in [0, r_{\mathrm{i}}]$. Radius r_{i} of the effective volume of the ion may be expressed through the mass density distribution $\rho(r)$ by the relation

$$\frac{4}{3}\pi r_{\mathrm{i}}^{3} = \frac{A_{\mathrm{i}} m_{\mathrm{p}}}{\rho(r)}. \tag{8.7}$$

Then,

$$r_{\mathrm{i}} = \sqrt[3]{\frac{3A_{\mathrm{i}} m_{\mathrm{p}}}{4\pi\rho(r)}}. \tag{8.8}$$

Let us calculate the Coulomb energy U'_{c} per ion, using relation (8.8). Assuming that the charge distribution law in the effective volume of radius r_{i} is given, we may write U'_{c} in the form

$$U'_{\mathrm{c}} = U_{\mathrm{c}}^{(+)} + U_{\mathrm{c}}^{(-)}, \tag{8.9}$$

where $U_{\mathrm{c}}^{(-)}$ is the potential energy of the Coulomb repulsion of electrons inside the effective volume radius r_{i} and $U_{\mathrm{c}}^{(+)}$ is the potential energy of attraction of the electron cloud to the positive ion.

Let the charge distribution law $q_{\mathrm{e}}(r_{\mathrm{e}}) = q_0(r_{\mathrm{e}})$ inside the electron cloud be given. Then, normalization of the electron charge of the cloud, surrounding the ion, may be written in the form

$$-Ze = \int_0^{r_1} 4\pi q_{\mathrm{e}}(r_{\mathrm{e}})\, r_{\mathrm{e}}^2\, \mathrm{d}r_{\mathrm{e}}. \tag{8.10}$$

From expression (8.10), we may obtain the normalization constant q_0, which will depend on the given law of charge distribution, as

$$q_0 = -\frac{Ze}{4\pi \int_0^{r_1} r_e^2 f(r_e)\,dr_e}. \tag{8.11}$$

Now it is easy to obtain expressions for $U_c^{(-)}$ and $U_c^{(+)}$ in the form

$$U_c^{(-)} = (4\pi)^2 q_0^2 \int_0^{r_1} r_e f(r_e)\,dr_e \int_0^{r_1} \left(r'_e\right)^2 f\left(r'_e\right)\,dr'_e, \tag{8.12}$$

$$U_c^{(+)} = 4\pi Zeq_0 \int_0^{r_1} r_e f(r_e)\,dr_e. \tag{8.13}$$

Finally, expression (8.8) for the potential energy U'_c corresponding to one ion may be rewritten using (8.11), (8.12), and (8.13) in the form

$$U'_c = -e^2Z^2 \left[\frac{\int_0^{r_1} r_e f(r_e)\,dr_e}{\int_0^{r_1} r_e^2 f(r_e)\,dr_e} - \frac{\int_0^{r_e} r_e f(r_e)\,dr_e \int_0^{r_e} (r'_e) f(r'_e)\,dr'_e}{\left(\int_0^{r_1} r_e^2 f(r_e)\,dr_e\right)^2}\right]. \tag{8.14}$$

It is easy to see that the right-hand side of Eq. (8.14), the expression enclosed in brackets, determines the inverse value of some effective diameter of the electron cloud, which may be expressed through the form factor α_i^2 of the ion radius r_i, that is,

$$\frac{\int_0^{r_1} r_e f(r_e)\,dr_e}{\int_0^{r_1} r_e^2 f(r_e)\,dr_e} - \frac{\int_0^{r_e} r_e f(r_e)\,dr_e \int_0^{r_e} (r'_e) f(r'_e)\,dr'_e}{\left(\int_0^{r_1} r_e^2 f(r_e)\,dr_e\right)^2} = -\frac{\alpha_i}{r_i^2}. \tag{8.15}$$

Thus, expression (8.14), using (8.15), yields

$$-U'_c = \alpha \frac{e^2 Z^2}{r_i^2}. \tag{8.16}$$

The numerical values of the form factor α_i depending on the charge distribution $q_e(r_e)$ inside the electron cloud are given in Table 8.2, and their calculations were given in our work (Ferronsky et al. 1981).

Table 8.2 Numerical values of the form factors α_i for different radial charge distribution of the electron cloud around the ion

The law of charge distribution[a]	α_i^2
$q_e(r_e) = q_0 = \text{const}$	0.9
$q_e(r_e) = q_0(1 - r_e/r_i)$	1.257
$q_e(r_e) = q_0(1 - r_e/r_i)^n$	$\frac{(n+3)\left(11n^2+41n+36\right)}{8(2n+3)(2n+5)}$
$q_e(r_e) = q_0(r_e/r_i)^n$	$\frac{(n+3)^2}{(n+2)(2n+5)}$
The same for $\to\infty$	$\alpha_i^2 \to 1/2$

[a]Here q_0 is the charge value in the center of the sphere; r_e is the parameter of radius, $r_e \in [0, r_i]$; $n = 0, 1, 2, \ldots$ is an arbitrary number

Using expression (8.16), the total energy of the Coulomb interaction of particles may be written as

$$-U_c = 4\pi \int_0^R \frac{\rho(r)}{A_i m_p} U'_c r^2 \, dr = \frac{3\alpha_i e^2 Z^2}{R} \int_0^R R r^2 \left(\frac{4\pi\rho(r)}{3A_i m_p}\right)^{4/3} dr. \tag{8.17}$$

Introducing the form factor of the Coulomb energy α_i in expression (8.17), depending on the mass distribution in the gaseous cloud and on the charge distribution inside the effective volume of the ion, we obtain

$$-U_c = \alpha_c \frac{e^2 Z^2}{r_i^2} \left(\frac{m}{A_i m_p}\right)^{4/3}, \tag{8.18}$$

where

$$\alpha_c = \frac{3\alpha_i \int_0^R [(4\pi/3)\rho(r)]^{4/3} R r^2 \, dr}{m^{4/3}},$$

$$m = \sum_{i=1}^N m_i = 4\pi \int_0^R r^2 \rho(r) \, dr.$$

Since the total number of ions N in the gaseous cloud is equal to

$$N = \frac{m}{A_i m_p},$$

and the relation between the radius of the cloud and the radius of the ion may be obtained from the relationship of the corresponding volumes

$$\frac{4}{3}\pi R^3 = N \frac{4}{3}\pi r_i^3,$$

then the expression (8.18) may be rewritten in the following form:

$$-U_{\mathrm{c}} = \alpha_{\mathrm{c}} \frac{N^{4/3} e^2 Z^2}{R^2} = \alpha_{\mathrm{c}}^2 N \frac{e^2 Z^2}{r_{\mathrm{i}}^2}. \tag{8.19}$$

Hence, the form factor entering the expression for the potential energy of the Coulomb interaction acquires the same physical meaning, what it has in the expression for the potential energy of the gravitational interaction of the masses considered in Sect. 2.6. It represents the effective shell to which the charges in the sphere are reduced, that is,

$$\alpha_{\mathrm{c}} = \frac{r_{\mathrm{i}}}{r_{\mathrm{ei}}}. \tag{8.20}$$

Taking into account that the moment of inertia of the body is $I = \beta^2 m R^2$, then the relation (8.6) can be written in the form

$$-U_{\mathrm{c}} \sqrt{I} = \alpha_{\mathrm{c}} N^{4/3} \frac{e Z^2}{R} \sqrt{\beta^2 m R^2} = \alpha_{\mathrm{c}} \beta^2 N^{4/3} m^{1/2} e^2 Z^2 = \text{const.} \tag{8.21}$$

Since we have assumed that the mass of the system and its ion composition are constants, examination of Eq. (8.6) will be equivalent to the analysis of the product of the form factors α_{c} and β. Equation (8.6) holds if

$$\alpha\beta = \frac{r_{\mathrm{i}}}{r_{\mathrm{ei}}} \approx \text{const.}$$

The results of the numerical calculations of the form factors α_{c} and β for different mass distribution in the cloud are shown in Table 8.3, and calculations were carried out in our work (Ferronsky et al. 1981). The values of the form factor α_{i} of the ion, the numerical value of which depends on the choice of charge distribution $q_{\mathrm{e}}(r_{\mathrm{e}})$, are shown in Table 8.2.

In Table 8.3, the numerical values of the form factor α_{c} and the product of the form factors $\alpha_{\mathrm{c}}\beta$ are given for the case of homogeneous distribution of the electron charge around ion, that is, when $q_{\mathrm{e}}(r_{\mathrm{e}}) = \text{constant}$. From Table 8.3, it follows that for different laws of mass distribution, when the mass increases to the center, the product of form factors α_{c} and β remains constant, and therefore Eq. (8.6) holds, with the same comments as were made previously.

From Eq. (8.20), it follows, however, that the form factor of the Coulomb energy α_{c} becomes infinite, when the volume occupied by the ions tends to zero. Correspondingly, the Coulomb energy in this case will also tend to infinity. In Table 8.3, there are two laws of mass distribution for which the last condition holds. They are $\rho(r) = \rho_0(1 - (r/R)^n$ for $n \to \infty$. When the particles of the system are gathering at the shell of the finite radius, the energy of the Coulomb interaction tends to infinity, whereas the energy of gravitational interaction has a finite value.

Table 8.3 Numerical values of the form factors α_c and β product for different laws of radial mass distribution

The law of mass distribution[a]	α_c/α_i	α_c at $\alpha_i = 0.9$[b]	β	$\alpha_c\beta$ at $\alpha_i = 0.9$[b]
$\rho(r) = \text{const}$	1	0.9	0.6324	0.5692
$\rho(r) = \rho_0(1 - r/R)$	1.1303	1.0173	0.5163	0.5253
$\rho(r) = \rho_0(1 - r/R)^2$	1.3331	1.1998	0.4364	0.5236
$\rho(r) = \rho_0(1 - r/R)^3$	1.5510	1.3959	0.3779	0.5276
$\rho(r) = \rho_0(1 - r/R)^n$	$\frac{27}{\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\frac{243}{10\sqrt[3]{6}}\frac{[(n+1)(n+2)(n+3)]^{4/3}}{(4n+3)(4n+6)(4n+9)}$	$\sqrt{\frac{8}{(n+4)(n+5)}}$ for $n \to \infty$	0.5909
$\rho(r) = \rho_0(r/R)$	1.0159	0.9143	2/3	0.6095
$\rho(r) = \rho_0(r/R)^2$	1.0461	0.9415	0.6900	0.6497
$\rho(r) = \rho_0(r/R)^n$	$\frac{27}{10\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\frac{81}{100\sqrt[3]{3}}\frac{(n+3)^{4/3}}{(4n+9)}$	$\sqrt{\frac{2n+3}{3n+5}}$ for $n \to \infty$,	$\to \infty$
$\rho(r) = \rho_0 e^{-k(r/R)}$	$0.2321k$	$0.2089k$	$2\sqrt{2}/k$	0.5909
$\rho(r) = \rho_0 e^{-k(r/R)^2}$	$0.5907k^{1/2}$	$0.5316k^{1/2}$	$(1/k)^{1/2}$	0.5316

[a] Here, ρ_0 is the normalization constant; r is the parameter of the radius $r \in [0, R]$; n and k are arbitrary numbers, $n = 0, 1, 2, \ldots$

[b] The value α_i corresponds to the homogeneous charge distribution in the electron cloud, surrounding the ion $q_e(r_e) = \text{constant}$ (see Table 8.2)

When the mass distribution is $\rho(r) = \rho_0(1-(r/R)^n$, the form factors of gravitational and Coulomb energies are both finite. But the form factors of the Jacobi function of the system in this case tend to zero, a circumstance that provides the constancy of the product of the form factors α_c and β. This difference might play a decisive role in the evolution of the system.

In conclusion, we note that the results of the study on the relationship between the Jacobi function and the potential energy allow us to consider that the transfer from Jacobi's equation into the equations of virial oscillations is from the point of view of physics justified. This justification has been achieved in the framework of Newton and Coulomb interactions of the particles of the system.

8.3 Emission of Electromagnetic Energy by a Celestial Body as an Electric Dipole

In Chap. 5, we considered the solution of the virial equation of dynamical equilibrium for dissipative systems written in the form

$$\ddot{\Phi} = -A_0\,[1 - q(t)] + \frac{B}{\sqrt{\Phi}}. \tag{8.22}$$

Here, the function of the energy emission $[1 - q(t)]$ was accepted on the basis of the Stefan–Boltzmann law without an explanation of the nature of the radiation process. Now, after the analysis of the relationship between the potential energy and the polar moment of inertia, considered in the previous section, and taking into account the observed relationship by artificial satellites, we try to obtain the same relation for the celestial body as an oscillating electric dipole (Ferronsky et al. 1987).

Equation (8.22) for a celestial body as a dissipative system can be rewritten as

$$X\,(t - t_0) = E_\gamma\,(t - t_0)\,.$$

The electromagnetic field formed by the body is described by Maxwell's equations, which can be derived from Einstein's equations written for the energy–momentum tensor of electromagnetic energy. In this case, only the general property of the curvature tensor in the form of Bianchi's contracted identity is used. We recall briefly this derivation sketch (Misner et al. 1975).

Let us write Einstein's equation in geometric form

$$G = 8\pi T, \tag{8.23}$$

where G is an Einstein tensor and is an energy–momentum tensor.

In the absence of mass, the energy–momentum tensor of the electromagnetic field can be written in arbitrary coordinates in the

$$4\pi T^{\mu\nu} = F^{\mu\alpha} F^{\nu\beta} g_{\alpha\beta} - \frac{1}{4} g^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}, \tag{8.24}$$

where $g_{\alpha\beta}$ is the metric tensor in coordinates and $F^{\mu\nu}$ is the tensor of the electromagnetic field.

From Bianchi's identity,

$$\nabla G \equiv 0, \tag{8.25}$$

where ∇ is a covariant 4-delta and follows the equation expressing the energy–momentum conservation law:

$$\nabla T \equiv 0. \tag{8.26}$$

In the component form, the equation is

$$F^{\mu\alpha}_{;\sigma} g_{\alpha\tau} F^{\sigma\tau} + F^{\mu\alpha}_{;\tau} g_{\alpha\sigma} F^{\tau\alpha} = g^{\mu\nu} \left(F_{\nu\tau;\sigma} + F_{\sigma\nu;\tau} \right) F^{\sigma\tau}. \tag{8.27}$$

After a series of simple transformations, we finally have

$$F^{\beta\nu}_{;\nu} = 0. \tag{8.28}$$

Here and above, the symbol "**;**" defines covariant differentiation.

To obtain the total power of the electromagnetic energy emitted by the body, Maxwell's equations should be solved, taking into account the motion of the charges constituting the body. In the general case, the expressions for the scalar and vector potentials are

$$4\pi\varphi = \int\limits_{(V)} \frac{[\rho]\,\mathrm{d}V}{R}, \tag{8.29}$$

$$4\pi \bar{A} = \int\limits_{(V)} \frac{[j]\,\mathrm{d}V}{R}, \tag{8.30}$$

where ρ and j are densities of charge and current, $[j]$ denotes the retarding effect (i.e., the value of function j at the time moment $t - R/c$), R is the distance between the point of integration and that of observation, and the velocity of light.

In this case, however, it seems more convenient to use the Hertz vector Z of the retarded dipole $(t - R/c)$ (Tamm 1976). The Hertz vector is defined as

$$4\pi Z = \frac{1}{R} \rho \left(t - \frac{R}{c} \right). \tag{8.31}$$

Electromagnetic field potentials of the Hertz dipole can be determined from the expressions

$$\varphi = -\mathrm{div} Z, \tag{8.32}$$

$$\bar{A} = \frac{1}{c}\frac{\mathrm{d}Z}{\mathrm{d}t}. \tag{8.33}$$

Moreover, the Hertz vector satisfies the equation

$$\Box Z \equiv \left(\nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial^2}{\partial t^2}\right) Z = 0, \tag{8.34}$$

where $\Box$ is the d'Alembertian operator.

The intensities of the electric and magnetic fields and are expressed in terms $\bar{Z}$ by means of the equations

$$\bar{H} = \mathrm{rot}\,\dot{\bar{Z}}, \tag{8.35}$$

$$\bar{E} = \mathrm{grad} \quad \mathrm{div}\,\bar{Z} - \frac{1}{c}\ddot{\bar{Z}}. \tag{8.36}$$

The radiation of the system can be described with the help of the Hertz vector of the dipole $\bar{p} = q\bar{r}$, where q is the charge and r is the distance of the vector from the charge $(+q)$ to $(-q)$.

From the sense of the retardation of the dipole $(t - R/c)$, we can write the following relations:

$$\frac{\mathrm{d}\bar{p}}{\partial R} = -\frac{1}{c}\dot{\bar{p}}, \quad \frac{\mathrm{d}^2\bar{p}}{\mathrm{d}R^2} = \frac{1}{c}\ddot{\bar{p}}.$$

Then, the components of the fields $\bar{E}$ and $\bar{H}$ of the dipole are as follows:

$$H_\varphi = \frac{\sin\,\theta}{c^2 R}\ddot{\bar{p}}\left(t - \frac{R}{c}\right), \tag{8.37}$$

$$E_\theta = \frac{\sin\,\theta}{c^2 R}\ddot{\bar{p}}\left(t - \frac{R}{c}\right), \tag{8.38}$$

where θ is the angle between and $\bar{R}$; $H_\varphi \perp E_\theta$ and $\perp R$; the other components of E and H in the wave zone are tending to zero quicker than $1/R$ in the limit $R \to \infty$.

The flux of energy (per unit area) is equal to

$$S = \frac{c}{4\pi}E_\theta H_\varphi = \frac{1}{4\pi c^2}\frac{\sin\,\theta}{R^2}\left(\ddot{\bar{p}}\right)^2. \tag{8.39}$$

The total energy radiated per unit time is given by

$$\oiint S \, \mathrm{d}\sigma = \frac{2}{3c^3} \left(\ddot{\bar{p}}\right)^2. \tag{8.40}$$

Thus, transforming the dissipative system to an electric dipole by means of the Hertz vector, we have reduced the task of a celestial body model construction to the determination of the dipole charges $+Q$ and $-Q$ through the effective parameters of the body.

This problem can be solved by equating expression (8.40) for the total radiation of a celestial body as an oscillating electric dipole. In addition, the relation for the black body radiation expressing through effective parameters was presented below in Sect. 8.5.

The expression (8.40) for the total rate of the electromagnetic radiation J of the electric dipole can be written in the form (Landau and Lifshitz 1973)

$$J = \frac{2}{3} \frac{Q^2}{c^3} \left(\ddot{\bar{r}}\right), \tag{8.41}$$

where Q is the absolute value of each of the dipole charges and r is the vector distance between the polar charges of the dipole. Its length in our case is equal to the effective radius of the body.

In our elliptic motion model of the two equal masses, the vector $\bar{r}$ satisfies the equation

$$\ddot{\bar{r}} = -Gm \frac{\bar{r}}{r^3}. \tag{8.42}$$

Thus, the total rate of the electromagnetic radiation of the dipole is

$$J = \frac{2}{3} \frac{Q^2}{c^3} \frac{(Gm)^2}{r^4}. \tag{8.43}$$

In order to obtain the average flux of electromagnetic energy radiation, the value of the factor $1/r^4$ should be calculated and averaged during the time period of one oscillation. Using the angular momentum conservation law, we can replace the time averaging by angular averaging, taking into consideration that

$$\mathrm{d}t = \frac{mr^2}{2M} \mathrm{d}\varphi, \tag{8.44}$$

where is angular momentum and φ is the polar angle.

The equation of the elliptical motion is

$$\frac{1}{r} = \frac{1}{a\left(1 - \varepsilon^2\right)} \left(1 + \varepsilon \cos\varphi\right), \tag{8.45}$$

where is the semimajor axis and ε is the eccentricity of the elliptical orbit.

The value of $1/r^4$ can be found by integration. In our case of small eccentricities, we neglect the value of ε^2 and write

$$\overline{\left(\frac{1}{r^4}\right)} = \frac{1}{a^4}. \tag{8.46}$$

Finally, we obtain

$$\bar{J} = \frac{2}{3}\frac{Q^2}{c^3}\frac{Gm^2}{a^4}. \tag{8.47}$$

Earlier, it was shown (Ferronsky et al. 1987) that

$$\bar{J} = 4\pi\sigma\frac{1}{a^2}A_{\mathrm{c}}^4, \tag{8.48}$$

where σ is the Stefan–Boltzmann constant, $= Gm\mu\ /3k$ is the electron branch constant, μ is the electron mass, and k is the Boltzmann constant.

Equating relations (8.47) and (8.48), we find the expression for the effective charge Q as follows:

$$Q = \sqrt{6\pi\sigma}\frac{A_{\mathrm{e}}^2}{cr_{\mathrm{g}}}, \tag{8.49}$$

where $r_{\mathrm{g}} = Gm/c^2$ is the gravitation radius of the body.

We have thus demonstrated that it is possible to construct a simple model of the radiation emitted by a celestial body, using only the effective radius and the charge of the body. Moreover, a practical method of determining the effective charge using the body temperature from observed data is shown.

The logical question raised is thus: what is the mechanism of the energy generation of the bodies that they emitted in the wide range of oscillating frequency spectrum? Let us consider this important question at least in first approximation.

8.4 Quantum Effects of Generated Electromagnetic Energy

The problem of the energy generation technology for human practical use has been solved far before. In the beginning, it was understood how to transfer the wind and fire energy into the energy of translational and rotary motion. Later on, people have learned about production of the electric and atomic energies. Technology of the thermonuclear fusion energy generation is the next step. It is assumed that the Sun replenishes its emitted energy by the thermonuclear fusion of hydrogen, helium, and carbon. The Earth's thermal energy loss is considered to be filled up by convection

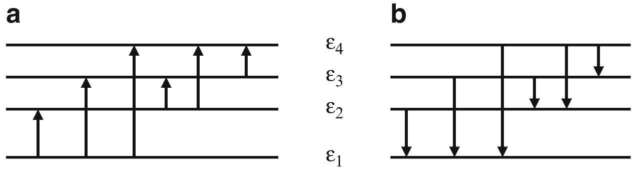


Fig. 8.1 Quantum transition of energy levels at contraction phase of the body mass (**a**) and inversion at the phase of its expansion (**b**); ε_1, ε_2, ε_3, and ε_4 are levels of energy

of the masses and thermal conductivity. But the source of energy for convection of the masses is not known.

The obtained solution of the problem of volumetric pulsations for a self-gravitating body based on their dynamical equilibrium creates real physical basis to formulate and solve the problem. In fact, if a body performs gravitational pulsations (mechanical motions of masses) with strict parameters of contraction and expansion of any as much as desired small volume of the mass, then such a body, like a quantum generator, should generate electromagnetic energy by means of its transformation from mechanical form through the forced energy level transitions and their inversion on both the atomic and nuclear levels. In short, the considered process represents transfer of mechanical energy of the mass pulsation to the energy of electromagnetic field (Fig. 8.1).

An interpretation of the process can be presented as follows. While pulsating and acting in the regime of the quantum generator, the body should generate and emit coherent electromagnetic radiation. Its intensity and wave spectrum should depend on the body mass and its radial density distribution and chemical (atomic) content. As shown in Sect. 6.4, the body with uniform density and atomic content provides pulsations of uniform frequency within the entire volume. In this case, the energy generated during the contraction phase will be completely absorbed at the expansion phase. The radiation that appeared at the body's boundary surface must be in equilibrium with the outer flux of radiation. The phenomenon like this seems to be characteristic for the equilibrated galaxy nebulae and for the Earth's water vapor in anticyclonic atmosphere.

The pulsation frequencies of the shell-structured bodies are different but steady for each shell density. In the case of density increase toward the body center, the radiation generating at the contraction phase will be partially absorbed by an overlying stratum at the expansion phase. The other part of radiation will be summed up and transferred to the body surface. That radiation forms an outer electromagnetic field and is equilibrated by interaction with the outer radiation flux. The rest of the nonequilibrated and more energetic radiation in the spectrum moves to the space. The coherent radiation that reaches the boundary surface has a pertinent potential and wave spectrum depending on the mass and atomic content of the interacted shells in accordance with Moseley law. The Earth emits infrared thermal radiation in an optical shortwave range of spectrum. The Sun and other stars cover the spectrum of electromagnetic radiation from radio through optical, *x,* and gamma rays of wave ranges. The observed spectra of star radiation show that total mass of a body takes part in the generation and formation of surface radiation. According to the accepted parabolic law of density distribution of the Earth, it has maximum density value near the lower mantle boundary. The value of the outer core density has jump-like fall, and the inner core density seems to be uniform up to the body center. The discussed mechanism of the energy generation is justified by the observed seismic data of density distribution. It is assumed that the excess of generated electromagnetic energy from the outer core comes to the inner core and keeps the pressure of dynamical equilibrium at the body pulsation there during the entire time of the evolution. The parabolic distribution of density seems to be characteristic for most of the celestial bodies.

In connection with the discussed problem, it is worth to consider the equilibrium conditions between radiation and matter on the body boundary surface.

8.5 The Nature of the Star-Emitted Radiation Spectrum

We assume that the novas and supernovas after explosion and collapse pass into neutron stars, white dwarfs, quasars, black holes, and other exotic creatures that emit electromagnetic radiation in different ranges of the wave spectrum. The effects discussed in this book based on dynamical equilibrium evolution of self-gravitating celestial bodies allow the exotic stars to be interpreted from a new position. We consider the observed explosions of stars as a natural logical step of evolution related to their mass differentiation with respect to the density. The process is completed by separation of the upper "light" shell. At the same time, the wave parameters of the generated energy of the star after shell separation are changed because of changes in density and atomic contents. As a result, the frequency intensity and spectrum of the coherent electromagnetic radiation on the boundary surface are changing. For example, instead of radiation in optical range, the coherent emission in x or gamma ray range takes place. But the body's dynamical equilibrium should remain during all the time of evolution. The loss of the upper body shell leads to decrease in the angular velocity and increase in the oscillation frequency. The idea of the star gravitational collapse seems to be an effect of the hydrostatic equilibrium theory (Ferronsky 2005).

As to the high temperature on the body surface, the order of which from Rayleigh–Jeans' equation is 10^7 K and more, in our interpretation as applying Eq. (7.22) for evolution of a star of solar mass at the electron phase (Fig. 8.1), the limiting temperate $_0 \to \mu\ ^2/3k$ or (Ferronsky et al. 1996)

$$3kT_0 \to \mu_e c^2 \approx 0.5\,\text{MeV},$$

$$T \approx 5 \cdot 10^9\,\text{K}.$$

This means that on the body surface, the gas approaches to the electron temperature because the velocity of its oscillating motion runs to c.

The energy is a quantitative measure of interaction and motion of all the forms of the matter. In accordance with the law of conservation, the energy does not disappear and does not appear itself. It only passes from one form to another. For a self-gravitating body, the energy of mechanical oscillations, induced by the gravitational interactions, passes to the electromagnetic energy of the radiation emission and vice versa. The process results by the induced quantum transition of the energy levels and their inversion. Here, transition of the gravitational energy into electromagnetic and vice versa results in the self-oscillating regime. In the outer space of the body's border, the emitted radiation energy forms the equilibrium electromagnetic field. The nonequilibrium part of the energy in the corresponding wave range of the spectrum is irradiated to the outer space. The irreversible loss of the emitted energy is compensated by means of the binding energy (mass defect) at the fission and fusion of molecules, atoms, and nuclei. The body works in the regime of a quantum generator. Those are conclusions followed from the theory based on the body dynamical equilibrium.

In conclusion, we wish to stress that the relationship between the gravitational field (potential energy) and the polar moment of inertia of the Earth, discovered by the artificial satellites, leads to understand the nature and the mechanism of the planet's energy generation as the force function of all the dynamical processes release in the form of oscillation and rotation of the matter. Through the energy nature, we understand the unity of forms of the gravitational and electromagnetic interactions, which, in fact, are the two sides of the same natural effect.

References

Ferronsky VI (2005) Virial approach to solve the problem of global dynamics of the Earth. Investigated in Russia. http://zhurnal.ape.relarn/articles/2005/120.pdf, pp 1207–1228

Ferronsky VI, Denisik SA, Ferronsky SV (1981) Virial oscillations of celestial bodies: I. The effect of electromagnetic interactions. Celest Mech 23:243–267

Ferronsky VI, Denisik SA, Ferronsky SV (1987) Jacobi dynamics. Reidel, Dordrecht

Ferronsky VI, Denisik SA, Ferronsky SV (1996) Virial oscillations of celestial bodies: V. The structure of the potential and kinetic energies of a celestial body as a record of its creation history. Celest Mech Dyn Astron 64:167–183

Flügge S (1971) Practical quantum mechanics. Springer, Berlin

Landau LD, Lifshitz EM (1973) Field theory. Nauka, Moscow

Misner CW, Thorne KS, Wheeler JA (1975) Gravitation. Freeman, San Francisco

Spitzer L Jr (1968) Diffuse matter in space. Interscience, New York

Tamm IE (1976) Fundamentals on theory of electricity. Nauka, Moscow

Chapter 9
Creation and Decay of a Hierarchic Body System at Expansion and Attraction of the Force Field

Abstract All small and large celestial bodies appear to present some clots of energy in the form of condensed discrete infinitesimal particles. In this case, at some stage of the universe evolution, there was a common or a number of smaller clots of the matter energy. Once created, they started to decay. After decay, they were created again. This phenomenon looks like the water cycle in the nature, during which the initial "dark" energy is converted into condensate. The main part of that "dark" energy remains in the form of the background or of the force field. During the universe expansion, which we observe, the initially condensed energy is also expanding by the inner pressure and emits energy in the form of discrete infinitesimal weightless particles.

On the basis of the above-considered analysis of dynamical effects related to the origin of the solar system bodies, one may note that the basic point of the process of the initial condensate decay is the interaction of elementary particles and the energy loss in the form of radiation emission. The interaction of particles results by their collision and crushing, which looks like acceleration and collision of the protons in a collider. The radiation is a flux of weightlessness with respect to given body particles. These particles have mass, but it is a defect, that is, weightlessness with respect to the body matter. The radiated energy is the basis and content of a celestial body evolution. The loss of energy finally leads to differentiation of matter in density and to the shell separation. The outer shell appears to be most light in density, and at some stages of evolution, its inner force field overtakes the weightless threshold with respect to the parental body. This is the way of the outer shell separation and creation of secondary body. Creation of the hierarchic subsystem of bodies like galaxy, star, planet, and satellite in the scale of the whole universe, in fact, is the process of body's weighted matter decay. The universe expansion is the observable fact and the evidence of creation and decay of the weightless and weightlessness matter of the same energy by means of oscillating motion.

In the last years, the scenario of Big Bang in connection with the universe origin was widely discussed. From the viewpoint of the Jacobi dynamics, the idea of Big Bang corresponds to the stage of expansion in the framework of Jacobi's pulsating

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_9,

model of the universe. The experimental research is developed by the collider in CERN in search of the Higgs boson, which is an elementary particle in the quantum field, named after the English physicist Peter Higgs. There is information about the evidence of existence of such a scalar particle with a mass equivalent to $\sim$125 GeV of energy. It is assumed that this is a fundamental particle of the universe creation according to Big Bang theory.

If one accepts the idea of the existence of the universe origin, then in the framework of the Jacobi dynamics, its expansion should have physical limit in time. This limit should be reached when all the hierarchic subsystems of bodies decay up to the level of elementary scalar particles. After that, the stage of attraction (fall down) of the particles will start. The attraction of mass particles (electrons and nuclei of known and unknown elements) should continue up to their turn to expansion. The attraction process will be finished when the pressure in the universe's inner and outer fields come to the equilibrium. After that, because of mass particle energy radiation, the process of mass particle decay and the expansion and creation of the hierarchic subsystem bodies will start again.

Note that in the stage of universe attraction, the interacted elementary particles at their synthesis into mass particles (electrons, nuclei, molecules) absorb energy in the form of mass defect, which is used for the binding of the nuclei components.

The process of the decay up to the level of elementary particles and attraction up to the stage of galaxies composed of atoms and molecules can continue for infinitely long time.

In light of the possible scenario of decay and creation of the universe, the phenomenon of creation of weighted mass particles (electrons and nuclei of atoms) by the synthesis of elementary particles is of interest. In the framework of the Jacobi dynamics, this problem based on the effect of simultaneous collision of *n* particles has mathematical solution and is presented in this chapter.

Thus, we discovered an interesting natural phenomenon. All small and large celestial bodies appear to be some clots of energy in the form of condensed discrete infinitesimal particles. It is possible to assume that at some initial stage, there was a common or a number of smaller clots of the matter energy. Once created, they decayed. After decay, they were created again. This phenomenon looks like the water cycle in the nature, during which, as the water moisture, not all the initial "dark" energy is converted into condensate. The main part of that "dark" energy remains in the form of the background or of the force field. During the universe expansion, which we observe, the initially condensed energy is also expanding by the inner pressure and emits energy in the form of discrete infinitesimal particles.

On the basis of the above-considered analysis of dynamical effects related to the origin of the solar system bodies, one may note that the basic point of the process of the initial condensate decay is the interaction of elementary particles and the energy loss in the form of radiation emission. The interaction of particles results by their collision and crushing, which looks like acceleration and collision of the protons in the collider. The radiation is a flux of weightlessness with respect to given body particles. These particles have mass, but it is a defect, that is, weightlessness with respect to the body matter. The radiated energy is the basis and content of a celestial body evolution. The loss of energy finally leads to differentiation of matter in density and to the shell separation. The outer shell appears to be most light in density, and at some stages of evolution, its inner force field overtakes the weightless threshold with respect to the parental body. This is the way of the outer shell separation and creation of secondary body. Creation of the hierarchic subsystem of bodies like galaxy, star, planet, and satellite in the scale of the whole universe, in fact, is the process of body's weighted matter decay. The universe expansion is the observable fact and the evidence of creation and decay of the weightless and weightlessness matter of the same energy by means of oscillating motion.

The question is rising: how long the universe expansion will continue? There are two options in the answer. The process will either continue for infinitely long time or there is a time and physical limit for it. Infinite expansion needs infinite energy. There should be infinite number of universes in the case of limiting expansion of ours. Or there is some new model of the space geometry. We do not have any data for discussing a problem like this.

In the last years, the scenario of Big Bang in connection with the universe origin is widely discussed. From the viewpoint of the Jacobi dynamics, the idea of Big Bang corresponds to the stage of expansion in Jacobi's pulsating model of the universe. The fundamental experimental research is developed by the collider in CERN in search of the Higgs boson, which is an elementary particle in the quantum field, named after the English physicist Peter Higgs. There is preliminary information about the evidence of existence of such a scalar particle with a mass equivalent to $\sim$125 GeV of energy. It is assumed that this is a fundamental particle of the universe creation according to Big Bang theory.

If one accepts the idea of the existence of an oscillating universe, then in framework of the Jacobi dynamics, its expansion should have physical limit in time. This limit should be reached when all the hierarchic subsystems of bodies decay up to the level of elementary scalar particles. After that, the stage of attraction (fall down) of the particles will start. The attraction of mass particles, electrons and nuclei of known and unknown elements, should continue up to their turn to expansion. The attraction process will be finished when the pressure in the universe's inner and outer fields come to the equilibrium. After that, because of mass particle energy radiation, the process of mass particle decay and the expansion and creation of the hierarchic subsystem bodies will start again.

Note that in the stage of universe attraction, the interacted elementary particles at their synthesis into mass particles (electrons, nuclei, molecules) absorb energy in the form of mass defect for the binding of the nuclei components.

In light of the possible scenario of decay and creation of the universe, the phenomenon of creation of weighted mass particles (electrons and nuclei of atoms) by the synthesis of elementary particles is of interest. The known regularities in the periodical table, the achievements of experimental physics in synthesis of the super-heavy elements and also in the study of fusion production, attest about the reality of such a scenario. In the framework of the Jacobi dynamics, this problem, based on the effect of simultaneous collision of n particles for obtaining stable mass particles by that mechanism, has mathematical solution and is presented below.

9.1 Relationship of the Jacobi Function and Potential Energy at Simultaneous Collision of *n* Particles

In the previous chapters, we have considered the general approaches to the formulation and solution of the Jacobi dynamics problems connected with the evolutionary processes of celestial bodies. For this purpose, we have transformed Jacobi's virial equations for conservative and nonconservative systems

$$\ddot{\Phi} = 2E - U, \tag{9.1}$$

$$\ddot{\Phi} = 2E - U + X(t, \Phi, \dot{\Phi}) \tag{9.2}$$

into equations of virial oscillations in the following form:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{9.3}$$

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} + X(t, \Phi, \dot{\Phi}). \tag{9.4}$$

The transfer from Eqs. (9.1) and (9.2) to Eqs. (9.3) and (9.4) has been made by using the following relationship between the Jacobi function and potential energy:

$$U\sqrt{\Phi} = B = \text{const.} \tag{9.5}$$

As shown in Chap. 5, the validity of the relationship (9.5) for explicitly solved cases of the many-body problem in mechanics and physics is an obvious fact. Consequently, for example, in the case of two-body problem, which represents the conservative system, the solutions of Eq. (9.3) will be analogous to Keplerian equations of conic sections according to which the Jacobi function (or potential energy) changes with time. In the same manner, the solution of the generalized equation of virial oscillations (9.4) in celestial mechanics will correspond to the solution for the periodic motion in the two-body problem obtained by perturbation theory methods.

The validity of Eq. (9.5) for a many-body system, including the problem of the synthesis of the mass points at simultaneous collision of n elementary particles, in a general case is not obvious despite the fact that both volumetric integral characteristics considered are functions of the distribution of mass density of a system.

In this chapter, we consider in detail the main physical aspect of the relationship between the Jacobi function and the potential energy of a system.

9.2 Asymptotic Limit of Simultaneous Collision of Mass Points for Conservative System

We take the advantage of the results presented by Wintner (1941) in order to study the many-body problem. From such a study, it follows that for a conservative system of n mass points of arbitrary configuration interacting according to Newton's law, the following statement is valid.

If the motion of the material points of a system of arbitrary configuration has the consequence that all of them tend to simultaneously collide, then the relationship $U\sqrt{\Phi}$ approaches a constant value. This result obtained by Wintner supplements the general properties of conservative systems of material points interacting according to Newton's law when their number remains constant all the time. The condition of constancy of the number of mass points of a system is equivalent to that of the distance $\Delta_{ij} = |r_i - r_j|$ between any pair of points at any moment of time and should be $\Delta_{ij} > 0$, where r_i and r_j denote the three vectors of the coordinates of mass points in the barycentric coordinate system.

For such a system, from the analysis of Jacobi's virial equation (9.1) and the expression for the Jacobi function Φ,

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i\, m_j\, \Delta_{ij}^2, \tag{9.6}$$

for kinetic energy T

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i\, m_j\, (\dot{r}_i - \dot{r}_j)^2, \tag{9.7}$$

and for potential energy U

$$U = -G \sum_{1 \le i < j \le n} \frac{m_i\, m_j}{\Delta_{ij}}, \tag{9.8}$$

Three inequalities were obtained that produce restrictions on the Jacobi function (or potential energy) and its derivatives. These inequalities can be written in the following form:

$$\left|\dddot{\Phi}\right| \le \eta\left(\left|\ddot{\Phi}\right| + 2\,|E|\right)^{5/2}, \tag{9.9}$$

$$\left(\ddot{\Phi} - 2E\right) \Phi^{1/2} \geq \mu > 0, \tag{9.10}$$

$$\ddot{\Phi} - E - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi} \geq \frac{M^2}{4\Phi}, \tag{9.11}$$

where constants

$$\eta = \frac{\sqrt{2m}}{G} \sum_{1 \leq i < j \leq n} (m_i\, m_j)^{-3/2} > 0,$$

$$\mu = \frac{G}{\sqrt{2m}} \sum_{1 \leq i < j \leq n} (m_i\, m_j)^{3/2} > 0,$$

$$M^2 = C_1^2 + C_2^2 + C_3^2,$$

$$m = \sum_{i=1} m_i,$$

m_i is the mass of the ith point, $E = T + U$ is the total energy, and C_1, C_2, and C_3 are projections of the angular momentum M on the axes.

The third inequality (9.11) is more complicated than the others as it contains the value M of the constant angular momentum besides the constant E, which is the total energy of the system.

It has been shown by Wintner (1941) that if the motion of material points of an arbitrary configuration system provides their simultaneous collision, then the system possesses zero angular momentum and a simultaneous collision will occur in the finite interval of time. In addition, the behavior of the Jacobi function in the vicinity of the time moment t_0 of simultaneous collision is defined by the following asymptotics:

$$\Phi \propto (t - t_0)^{4/3}, \tag{9.12}$$

$$\Phi \propto (t - t_0)^{1/3}, \tag{9.13}$$

$$\Phi \propto (t - t_0)^{-2/3}. \tag{9.14}$$

Following Wintner (1941), we introduce the definition of a central configuration, which is needed for further consideration of the problem. If the positions of the material points in the system are such that the following relation is satisfied:

$$U_n = \sigma m_i r_i, \tag{9.15}$$

then the configuration of the system is called central.

Here, in Eq. (9.15),

$$\sigma = -\frac{U}{2\Phi}.$$

The definition (9.15) of the central configuration can be rewritten in the following equivalent form:

$$(U^2\Phi)_n = 0. \tag{9.16}$$

As proved by Wintner (1941), the important relation follows from asymptotics (9.12), (9.13), and (9.14) at $t \to t_0$:

$$\left(U^2\Phi\right)_n \to 0 \tag{9.17}$$

which, together with the definition of the central configuration, leads to the following theorem:

Any arbitrary configuration of material points in the asymptotic time limit of simultaneous collisions of all the mass points tends to the central configuration.

It follows from this that

$$\lim_{t \to t_0} |U|\sqrt{\Phi} = \text{const.} \tag{9.18}$$

This theorem (9.18) justifies the transformation of Jacobi's virial equations (9.1) and (9.2) into equations of virial oscillations (9.3) and (9.4) within the framework of Newton's law of interaction of material points of a conservative system.

9.3 Asymptotic Limit of Simultaneous Collision of Mass Points for Nonconservative System

The model of a conservative system permits a limited number of problems to be solved. In reality, all natural systems are nonconservative. Study of the dynamics of such systems is the main object of the problem of evolution.

It is well known from the observations described in the general course of physics by Kittel et al. (1965) that the gravitating systems in nature are contracting while losing part of their total energy through friction and electromagnetic radiation. From the kinematic point of view, this gravitational contraction is equivalent to the simultaneous collision of all n mass points of the system. We consider below the validity of the theorem expressed by Eq. (9.18) for nonconservative systems.

Let the motion of a system of n mass points occur by means of the gravitational interaction and Newtonian friction of the mass points. Then, Jacobi's virial equation can be written as

$$\ddot{\Phi} = 2E(t) - U(t) - k\dot{\Phi}, \tag{9.19}$$

where $E(t)$ is the value of the total energy of the system at the moment of time t.

From the analysis of the equations of motion resulting in (4.23), it follows that

$$E(t) = E_0 - 2k \int_{t_0}^{t} T(t)\, \mathrm{d}t = E_0 \left[1 + q(t)\right],$$

where E_0 is the value of the total energy of the system at the initial moment of time t_0 and $q(t)$ is a monotonically increasing function of time.

We also accept the condition of the constancy of the number of mass particles in the system, from which it follows that the distance between any pairs of points $\Delta_{ij} > 0$ and the following relation is correct:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\Delta_{ij}\right| \le \left|\dot{r}_i - \dot{r}_j\right|.$$

In the framework of this essentially important condition that forbids paired, threefold, and higher-fold collisions, we obtain three inequalities analogous to (9.9), (9.10), and (9.11). The inequalities are valid at any stage of the system's evolution and place restrictions on the Jacobi function and its derivatives.

From expression (9.8) for the potential energy of the system, the following inequalities can be written:

$$\left|\dot{U}\right| = \left|G \sum_{1\le i<j\le n} \frac{m_i m_j}{\Delta_{ij}^2} \dot{\Delta}_{ij}\right| \le G \sum_{1\le i<j\le n} \frac{m_i m_j}{\Delta_{ij}^2} \left|\dot{r}_i - \dot{r}_j\right| \tag{9.20}$$

and

$$\frac{Gm_i m_j}{\Delta_{ij}} < -U,$$

where r_i and r_j are three vectors of coordinates of mass points in the barycentric coordinate system.

Substituting the last inequality into (9.20), we obtain

$$\left|\dot{U}\right| \le \frac{U^2}{G} \sum_{1\le i<j\le n} \frac{\left|\dot{r}_i - \dot{r}_j\right|}{m_i m_j}.$$

Since

$$m_i m_j (\dot{r}_i - r_j) \le 2mT,$$

and assuming

$$\eta = \frac{1}{G} \sum_{1 \le i < j \le n} \frac{m^{1/2}}{(m_i m_j)^{3/2}},$$

we obtain

$$\left|\dot{U}\right| \le U^2 \eta (2T)^{1/2}. \tag{9.21}$$

Then, using Eq. (9.19) in the form

$$U = 2E_0 \left[1 + q(t)\right] - \ddot{\Phi} - k\dot{\Phi} \tag{9.22}$$

and the law of conservation of energy for a dissipative system

$$U + T = E_0 \left[1 + q(t)\right], \tag{9.23}$$

we rewrite the inequality (9.21) in the form

$$\begin{aligned} \left|\dot{U}\right| &\le \left\{2\left|E_0\right|\left[1+q(t)\right]+\left|\ddot{\Phi}\right\}+k\ \left|\dot{\Phi}\right|\right\}^2 \eta\sqrt{2}\left\{2\left|E_0\right|\left[1+q(t)\right]+\left|\ddot{\Phi}\right\}+k\ \left|\dot{\Phi}\right|\right\}^{1/2} \\ &= \sqrt{2}\eta\left\{2\left|E_0\right|\left[1+q(t)\right]+\left|\ddot{\Phi}\right\}+k\left|\ \dot{\Phi}\right\}^{5/2}. \end{aligned} \tag{9.24}$$

Differentiating (9.22) with respect to time and substituting this into (9.24), we finally obtain the first inequality:

$$\left|\dddot{\Phi} + k\ddot{\Phi} - 2E_0\dot{q}(t)\right| \le \sqrt{2}\eta\left\{2\left|E_0\right|\left[1+q(t)\right]+\left|\ddot{\Phi}\right|+k\left|\dot{\Phi}\right|\right\}^{5/2}. \tag{9.25}$$

In the same way, it follows from (9.6) that

$$\Phi^{1/2} \ge \frac{1}{(2m)^{1/2}} (m_i\, m_j)^{1/2} \Delta_{ij}.$$

Then,

$$\frac{\Phi^{1/2} m_i m_j}{\Delta_{ij}} \ge (2m)^{1/2} (m_i\, m_j)^{1/2}.$$

By virtue of (9.4) and (9.8),

$$\ddot{\Phi} + k\dot{\Phi} - 2E\left[1 - q(t)\right] = G \sum_{1 \le i < j \le n} \frac{m_i\, m_j}{\Delta_{ij}}.$$

The second inequality has the form

$$\ddot{\Phi} + k\dot{\Phi} - 2E\,[1 + q(t)]\,\Phi^{1/2} \geq \mu > 0, \tag{9.26}$$

where

$$\mu = \frac{G}{(2m)^{1/2}} \sum_{1 \leq i < j \leq n} (m_i\, m_j)^{3/2}.$$

Now let us derive the third inequality followed from the Cauchy–Bunyakovsky inequality, which is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Since

$$r_i^2 = |r_i|^2 \quad \text{and} \quad |(r_i \cdot \dot{r}_i)| = (|r_i| \cdot |\dot{r}_i|),$$

and from the definition of the Jacobi function, one obtains

$$\Phi = \sum_{i=1}^{n} m_i\,(|r_i| \cdot |\dot{r}|)\,.$$

Applying the Cauchy–Bunyakovsky inequality to this expression at

$$a_i = m_i^{1/2}|r| \quad \text{and} \quad b_i = m_i^{1/2}|r|,$$

we can write

$$\dot{\Phi}^2 \leq 2\Phi \sum_{i=1}^{n} m_i\,|\dot{r}_i|^2 = 2\Phi \sum_{i=1}^{n} \frac{m_i\,(r_i \cdot \dot{r}_i)^2}{r_i^2}.$$

Assuming

$$a_i = m_i^{1/2}\,|r_i|\,, \quad A_i = \frac{m_i^{1/2}\,[r_i X \dot{r}_i]}{|r_i|},$$

the vector of the angular momentum M is

$$M = \sum_{i=1}^{n} a_i\,A_i\,.$$

Then, in a similar way, we write

$$M^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} A_i^2\right) \equiv 2\Phi \sum_{i=1}^{n} \frac{m_i\,[r_i X \dot{r}_i]}{r_i^2}.$$

The addition of the last two inequalities yields

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} \frac{m_i \left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}_i]^2\right\}}{r_i^2}.$$

But since

$$\left\{(r_i \cdot \dot{r}_i)^2 + [r_i X \dot{r}_i]^2\right\} = r_i^2 \cdot \dot{r}_i^2,$$

we have

$$\dot{\Phi}^2 + M^2 \le 2\Phi \sum_{i=1}^{n} m_i \dot{r}_i.$$

As Jacobi's equation can be written in the form

$$\ddot{\Phi} + k\dot{\Phi} - E_0\,[1 + q(t)] = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2,$$

after substitution of this into the right-hand side of the last inequality, we obtain

$$\dot{\Phi}^2 + M^2 \le 4\Phi\left\{\ddot{\Phi} + k\dot{\Phi} - E_0\,[1 + q(t)]\right\}.$$

Hence, the third inequality can be written as

$$\ddot{\Phi} + k\dot{\Phi} - E_0\,[1 + q(t)] - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}. \tag{9.27}$$

Let us now analyze the behavior of the Jacobi function Φ and its derivatives. For this purpose, we introduce the auxiliary function $Q = \underline{Q}(t)$ equal to

$$Q = k\dot{\Phi}\Phi^{1/2} - E_0\,[1 + q(t)]\,\Phi^{1/2} + \frac{1/4\dot{\Phi}^2 + 1/4M}{\Phi^{1/2}}, \tag{9.28}$$

where $\Phi^{1/2} > 0$.

Then differentiating (9.28) and using

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^{1/2}\right) = \frac{\dot{\Phi}}{2\Phi^{1/2}},$$

we obtain

$$\dot{Q} = \frac{1}{2}\frac{\dot{\Phi}}{\Phi^{1/2}}\left\{\ddot{\Phi} + k\dot{\Phi} - E_0\left[1 + q(t)\right] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{4\Phi}\right\} + \Phi^{1/2}\left[k\ddot{\Phi} - E_0\dot{q}(t)\right],$$

where $\Phi^{1/2} > 0$, $\dot{q}(t) > 0$, and in agreement with (9.27),

$$\left\{\ddot{\Phi} + k\dot{\Phi} - E_0\left[1 + q(t)\right] - \frac{1}{4}\frac{M^2}{\Phi} - \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi}\right\} \geq 0.$$

Let t_0 be the time of simultaneous collision of all the particles of the system. Then for $t \to t_0$ ($t \to t_0$) $\Phi \to 0$. Let us show that the necessary condition for the existence of such t_0 for which $\Phi \to 0$ (if $t \to t_0$) is that the constant angular momentum M must be zero.

Note that if, for $t \to t_0$, $\Phi \to 0$, then all mutual $\triangle_{ij} = |r_i - r_j|$ also tend to zero, and the potential energy $U \to -\infty$.

Since

$$\ddot{\Phi} = 2E_0\left[1 + q(t)\right] - U - k\dot{\Phi},$$

where $E_0 = \text{const.}$, $|\dot{\Phi}| \to \infty$, $|q(t)|$, $|\dot{q}(t)| < \infty$, then, for $t \to t_0$, $\ddot{\Phi} \to \infty$. Thus, for t sufficiently close to t_0, we have $\ddot{\Phi} > 0$ and therefore the derivative $\dot{\Phi}$ increases and does not change its sign. Since $\Phi > 0$ and $\Phi \to 0$, Φ is a monotonically decreasing function. It therefore follows from the expression for $\dot{Q}$ that the function Q in (9.28) for t sufficiently close to t_0 must decrease and its time limit for $t \to t_0$ might be $-\infty$, but cannot be $+\infty$. Moreover, it follows from the above statement that for $t \to t_0$, the limit of function (9.28) is

$$\lim_{t \to t_0} Q = \lim_{t \to t_0} \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}, \tag{9.29}$$

but since $\Phi^{1/2} > 0$, the time limit (6.29) must be finite and nonnegative. Hence, for $t \to t_0$ and $\Phi \to 0$, the value $M^2/\Phi^{1/2}$ must remain limited. Therefore, since $M^2 = \text{const.}$, then $M \equiv 0$ and proof is completed.

The above analysis shows that at $t \to t_0$, $\ddot{\Phi} \to \infty$, and it therefore follows from (9.25) that

$$\left|\dddot{\Phi} = 2E_0\dot{q}(t) + k\ddot{\Phi}\right| \leq \text{const.}\left(\left|\ddot{\Phi}\right| + k\left|\dot{\Phi}\right|\right)^{5/2}. \tag{9.30}$$

Using the second inequality (9.26), it can be shown that if t_0 is the time moment of simultaneous collision of all the particles of the system, then as $\Phi^{1/2} > 0$ at $t \to t_0$, the ratio $\dot{\Phi}/\Phi^{1/2}$ tends to a finite and positive limit.

In fact, as has been shown above, the limit (9.29) of the function (9.28) for $t \to t_0$ has a finite value. Since $M = 0$,

$$\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

will also be finite and nonnegative. Let us show that this limit cannot be equal to zero.

Since for $t \to t_{0,} M = 0$, $\Phi^{1/2} \to 0$, the function (9.28) and its limit (9.29) may be written in the form

$$Q = k\dot{\Phi}\Phi^{1/2} - E_0\,[1 - q(t)]\,\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{9.31}$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}, \tag{9.32}$$

where

$$\mu_0 = \lim_{t-t_0} Q.$$

From (9.31), we find that

$$2Q\,\Phi^{1/2} = k\dot{\Phi}\Phi - 2E_0\,[1 - q(t)]\,\Phi + \frac{1}{2}\dot{\Phi}^2.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(2Q\,\Phi^{1/2}\right) = \ddot{\Phi}\dot{\Phi} + k\ddot{\Phi}\Phi + 2k\dot{\Phi}^2 - 2E_0\,[1 - q(t)]\,\dot{\Phi} - 2E_0\Phi\dot{q}.$$

Let us carry out the integration between the limit t_0 and $\bar{t}$ of the last relation where t_0 has a fixed value and $\bar{t} \to t_0$. We take into account that

$$\lim_{t-t_0} \Phi^{1/2} = 0,$$

$$\mu_0 = \frac{1}{4}\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} < \infty.$$

Then, we write

$$2Q\,\Phi^{1/2} = \int_{t_0}^{\bar{t}} \left\{\left[\ddot{\Phi} - 2E_0\,(1 - q(t)) + 2k\dot{\Phi}\right]\dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q}\right]\Phi\right\}\mathrm{d}t.$$

As shown above, the derivative Φ retains its sign in the sufficiently small neighborhood of point t_0. Since $\Phi \geq 0$ and $q > 0$, the positive constant μ in the inequality (9.26) will be such that in the sufficiently small neighborhood of t_0, we have

$$2|Q|\Phi^{1/2} \geq \int_{t_0}^{\bar{t}} \left\{ \frac{\mu}{\Phi^{1/2}}\dot{\Phi} + \left[2k\ddot{\Phi} - 2E_0\dot{q}\right]\Phi \right\} \mathrm{d}t.$$

The first integral to the right of this inequality being equal to $2\mu\Phi^{1/2}$, and $\Phi^{1/2} \to 0$ with $t \to t_0$, then, in the sufficient small neighborhood of t_0, we have

$$2\,|Q|\,\Phi^{1/2} \geq 2\mu\Phi^{1/2} \quad \text{or} \quad |Q| \geq \mu.$$

Since $\mu > 0$ and taking into account the existence of the time limit (9.32), we have finished the proof of correctness of the inequality

$$\lim_{t\to t_0} \left(\frac{\dot{\Phi}}{\Phi^{1/2}} \right) > 0.$$

The above analysis allows us to obtain the following asymptotic relations for the Jacobi function when $t \to t_0$.

Since the limit

$$\mu_0 = \frac{1}{4} \lim_{t\to t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}}$$

has a nonzero value, the function $\Phi = \Phi(t) > 0$ tends to zero as $t \to t_0$ in such a way that, in neighborhood of t_0, it is proportional to $(t - t_0)^{4/3}$ with a coefficient of proportionality of $((9/4)\ \mu_0)^{2/3}$, and one can differentiate this asymptotic relation with respect to t. Hence, the following asymptotic relations are satisfied:

$$\Phi \propto \left(3/2\mu_0^{1/2}\right)^{4/3} (t - t_0)^{4/3}, \tag{9.33}$$

$$\Phi \propto \left(12\mu_0^2\right)^{1/3} (t - t_0)^{1/3}. \tag{9.34}$$

In fact, (9.34) follows from (9.33) not only from groundless differentiation but actually from (9.33), if (9.32) is taken into account. The asymptotic relation (9.33) itself follows from (9.32), if we write the last relation in the form

$$\pm \frac{\mathrm{d}t}{\mathrm{d}\Phi} \propto \frac{1}{2}\mu_0^{-1/2}\Phi^{-1/4}$$

and then integrate it between the limits $\Phi = 0$ and $\Phi > 0$ but sufficiently close to $\Phi = 0$. Integration (but not differentiation) of such an asymptotic relation is always an allowed procedure, and hence, the asymptotic relations (9.33) and (9.34) are satisfied.

Let us show that besides (9.32), (9.33), and (9.34), the following asymptotic relations are also available:

$$\mu_0 = \pm \lim_{t \to t_0} \Phi^{1/2} \ddot{\Phi}, \tag{9.35}$$

$$\Phi \propto \left(2/3\mu_0^{1/2}\right)^{2/3} (t - t_0)^{-2/3}. \tag{9.36}$$

To prove relation (9.35), we multiply (9.27) by $\Phi^{1/2}$. Assuming for $t \to t_0$ and $\Phi^{1/2} \to 0$, $|E_0|(1 + q(t)) < \infty$, $M \equiv 0$, and using (9.32), we find that the lower limit $\underline{\lim}\Phi^{1/2}\ddot{\Phi} \geq \mu_0$. Since (9.35) is equivalent to (9.36), this asymptotic relation will be proved, if the upper limit $li\bar{m}\Phi^{1/2}\ddot{\Phi} \leq \mu_0$.

For the proof, we assume that $F = \left(\dot{\Phi}\right)^3$, so that

$$\ddot{F} = 6\dot{\Phi}\ddot{\Phi}^2 + 3\dot{\Phi}^2\dddot{\Phi}.$$

Then, with the aid of (9.30)

$$\left|\dddot{\Phi} - 2E_0\dot{q} + k\ddot{\Phi}\right| \leq \text{const.}\left(|\ddot{\Phi}| + k|\dot{\Phi}|\right)^{5/2},$$

and expressing $\dot{\Phi}$ and $\ddot{\Phi}$ through the function $\dot{F} = \Phi^3$ and $\dot{F} = 3\dot{\Phi}^2\ddot{\Phi}$, we find

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const.}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{|F|}.$$

On the right-hand side of this inequality, we find from (9.34) where $\dot{\Phi} = F^{1/3}$ that for $t \to t_0$

$$\left|\ddot{F} + 6\dot{q}(t)F^{2/3}\right| < \text{const.}\frac{\dot{F}^2 + \left(|\dot{F}|\right)^{5/2}}{t - t_0}. \tag{9.37}$$

Finally, if v_0 is a positive constant equal to $m\ (12\ \mu_0)^2$, then for $t \to t_0$

$$F \propto v_0(t \to t_0), \tag{9.38}$$

$$\underline{\lim}\dot{F} \geq v_0. \tag{9.39}$$

In fact, if $F = \dot{\Phi}^3$, then (9.38) is equivalent to (9.34). At the same time, by virtue of the relation $v_0 = (12\ \mu_0)^2$, $F = \dot{\Phi}^3$, $\dot{F} = 3\dot{\Phi}^2\ddot{\Phi}$, and (9.32), the inequality (9.39) is another form of the inequality $\underline{\lim}\Phi^{1/2}\ddot{\Phi} \geq \mu_0$, which we have already proved. Therefore, we are bound to prove the inequality that can be written in the form $li\bar{m}\dot{\Phi} \leq \mu_0$ by analogy with (9.39). Hence, we must prove that the asymptotic relations (9.38) and (9.39) with the aid of the “Tauberian condition” (9.37) yield the inequality lim $\dot{F} \le \nu_0$, which denotes that $F \to \nu_0$. From this inequality and from (9.39), the existence of the succession of time intervals follows:

$$t_1^{\mathrm{I}} < t < t_1^{\mathrm{II}}, \ldots, t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$$

which tends to t_0 as $k \to \infty$ in such a way that whenever $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$,

$$0 < \upsilon_0 < p = \dot{F}\left(t_k^{\mathrm{I}}\right) < \dot{F}(t) < \dot{F}\left(t_k^{\mathrm{II}}\right) < q \tag{9.40}$$

where p and q are some fixed numbers that are chosen between the limits $\underline{\lim}\dot{F}$ and $li\bar{m}\dot{F}$ ($\le\infty$) of the conditions function lim $\dot{F}(t)$. It is obvious that we can assume that $t_0 = 0$. If we accept const. = const. $(p^2 + p^{5/2})$, then for any t in any of the time intervals $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$, by virtue of (9.37) and (9.40), we find that the following inequality holds:

$$\left|\ddot{F}(t) + 6\dot{q}(t)F^{2/3}(t)\right| < \frac{\text{const.}}{|t|}.$$

Since t tends to $t_0 = 0$, increasing or decreasing, all t_k^{I} and t_k^{II} lie on the same side of $t_0 = 0$. Integration of the inequality (6.40) between the limits t_k^{I} and t_k^{II} yields

$$\left|\dot{F}\left(t_k^{\mathrm{II}}\right) - \dot{F}\left(t_k^{\mathrm{I}}\right) + \int\limits_{t_k^{\mathrm{I}}}^{t_k^{\mathrm{II}}} 6\dot{q}(t)F^{2/3}(t)\,\mathrm{d}t\right| < \text{const.} \quad \log\left|\frac{t_k^{\mathrm{II}}}{t_k^{\mathrm{I}}}\right|.$$

By virtue of (9.40), the difference $\dot{F}\left(t_k^{\mathrm{II}}\right) - \dot{F}\left(t_k^{\mathrm{I}}\right)$ is equal to a positive constant $p - q$ and

$$\int\limits_{t_k^{\mathrm{I}}}^{t_k^{\mathrm{II}}} 6\dot{q}(t)F^{2/3}(t)\,\mathrm{d}t > 0.$$

Hence, the limit log $|t_k^{\mathrm{II}}/t_k^{\mathrm{I}}|$, as $k \to \infty$, is greater than a certain positive number. For this reason, when $k \to \infty$, there exists a certain positive number λ that satisfies the relation

$$\frac{t_k^{\mathrm{II}}}{t_k^{\mathrm{I}}} > \lambda > 0. \tag{9.41}$$

Then with the aid of (9.38), it follows that

$$\frac{\left|F\left(t_k^{\mathrm{I}}\right)\right|}{\left|t_k^{\mathrm{I}}\right|} \to \nu_0 \quad \text{and} \quad \frac{\left|F\left(t_k^{\mathrm{II}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \to \nu_0,$$

since

$$t_k^{\mathrm{I}} \to t_0, \quad t_k^{\mathrm{II}} \to t_0, \quad t_0 = 0, \quad \nu = 0.$$

On the other hand, if k is sufficiently large, the following inequality is valid:

$$\frac{\left|F\left(t_k^{\mathrm{II}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \cdot \left|\frac{\left|t_k^{\mathrm{II}}\right|}{t_k^{\mathrm{I}}}\right| - \frac{\left|F\left(t_k^{\mathrm{I}}\right)\right|}{\left|t_k^{\mathrm{II}}\right|} \cdot \left|\frac{\left|t_k^{\mathrm{II}}\right|}{F\left|t_k^{\mathrm{II}}\right|}\right| > p\left|\frac{\left|t_k^{\mathrm{II}}\right|}{\left|t_k^{\mathrm{I}}\right|} - 1\right|. \tag{9.42}$$

In fact, all t_k^{I} and t_k^{II} lie on the same side of t_0 and then

$$\left|\left|t_k^{\mathrm{II}}\right| - \left|t_k^{\mathrm{I}}\right|\right| = t_k^{\mathrm{II}} - t_k^{\mathrm{I}}.$$

Since $t_k^{\mathrm{I}} \to t_0$ and $t_k^{\mathrm{II}} \to t_0$, then for sufficiently large k, all $F\left(t_k^{\mathrm{I}}\right)$ and $F\left(t_k^{\mathrm{II}}\right)$ have the same sign. Hence, (9.42) can be written in the form

$$\left|\left|F\left(t_k^{\mathrm{II}}\right)\right| - \left|F\left(t_k^{\mathrm{I}}\right)\right|\right| > p\left|\left|t_k^{\mathrm{II}}\right| - \left|t_k^{\mathrm{I}}\right|\right|$$

and is equivalent to the inequality

$$\left|F\left(t_k^{\mathrm{II}}\right) - F\left(t_k^{\mathrm{I}}\right)\right| > p\left|t_k^{\mathrm{II}} - t_k^{\mathrm{I}}\right|.$$

The validity of the last inequality is obvious, since by virtue of (9.40) for $t_k^{\mathrm{I}} < t <_k^{\mathrm{II}}$, we have $\dot{F}(t) > p > o$. Therefore, inequality (9.42) also holds.

From (9.42) in the limit $k \to \infty$ and with the aid of (9.41) where $\nu_0 > 0$, we obtain the following inequality:

$$\nu_0\left|\lambda - \nu_0\nu_0^{-1}\right| = p\left|\lambda - 1\right|.$$

Finally, by virtue of (9.41),

$$\left|\lambda - \nu_0\nu_0^{-1}\right| = \left|\lambda - 1\right| > 0,$$

and hence $\nu_0 \geq p$.

On the other hand, by virtue of (9.40), $p \geq \nu_0$. The observed contradiction is that the supposition we made at the beginning ($\overline{lim}\,\dot{F} \gg \nu_0$) is false. Thus, we have proved the validity of the increase inequality $\overline{lim}\,\dot{F} \leq \nu_0$, and this completes the proof of the relations (9.35) and (9.36).

Let us now show that if the motion of n points with masses m_i in the time limit $t \to t_0$ produces their simultaneous collision, then the configuration of these n particles tends to central configuration (9.15) as $t \to t_0$. In the proof, we shall use the asymptotic relations (9.33), (9.34), and (9.36), and the Tauberian lemma, which states that if the function $g(u)$ has continuous derivatives $\dot{g}(u)$ and $\ddot{g}(u)$ for $u \to \infty$ and tends, as $u \to \infty$, to a finite limit and $\ddot{g}(u) <$ const., then $\dot{g}(u) \to 0$.

There is no loss of generality in assuming that $t \to t_0 \to 0$, so that $t \to t_0 > 0$. Then, the asymptotic relations (9.33), (9.34), and (9.36) are simply equivalent to:

$$t^{-4/3}\Phi \to \mu_1 > 0, \tag{9.43}$$

$$t\left(\dot{t^{-4/3}\Phi}\right) \to 0, \tag{9.44}$$

$$t\left(\ddot{t^{-4/3}\Phi}\right) \to 0 \tag{9.45}$$

where

$$\mu_1 = \left(\frac{3}{2}\mu_0^{1/2}\right)^{4/3} \quad \text{and} \quad t \to 0$$

Since

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2,$$

it follows from (9.43) that when the time limit $t \to 0$, all n mass particles collide at the origin of the barycentric coordinate system OXYZ in such a way that, for sufficiently small t, the linear dimensions of the configuration will be proportional to $t^{2/3}$. For this reason, we eliminate this factor $t^{2/3}$ simply by multiplying the unit of length by the factor $t^{-2/3}$. Then, we consider instead of the values

$$r_i, \quad \Delta_{ij} = \left|r_i - r_j\right|,$$

$$\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i r_i^2, \tag{9.46}$$

$$U = -G\sum_{1 \le i < j \le n} \frac{m_i m_j}{\Delta_{ij}}, \tag{9.47}$$

the corresponding values

$$\bar{r} = t^{-2/3} r_i, \quad \bar{\Delta}_{ij} = \left|\bar{r}_i - \bar{r}_j\right| = t^{-2/3}\Delta_{ij},$$

$$\bar{\Phi} = t^{-4/3}\Phi = \frac{1}{2}\sum_{i=1}^{n} m_i\, \bar{r}_i^2,$$

$$U = t^{2/3} U = -G\sum_{1 \le i < j \le n} \frac{m_i\, m_j}{\Delta_{ij}}.$$

The procedure is permissible since the definition of the central configuration is an invariant relative scale transformation of all the coordinates $r_i \to \delta r_i$, where δ is an arbitrary nonzero factor. Then, the relation (9.15) is invalid for the fixed $t \neq 0$, but

$$\left(\bar{\Phi}\bar{U}^2\right)_{\bar{r}_i} = 0 \tag{9.48}$$

where $I = 1, 2, \ldots, n$ in the same limit $t \to 0$.

The proof of this theorem, the mathematically precise formulation of which is expressed by (9.48), has several stages.

First, we show that in the time limit $t \to 0$,

$$\frac{4}{9}\bar{\Phi} + U \to 0, \tag{9.49}$$

and

$$\bar{\Delta}_{ij} > \text{const.} > 0. \tag{9.50}$$

Let us introduce a time transformation, changing t to $\bar{t} = -\ln t$ in such a way to have $\bar{t} \to \infty$ for $t \to 0$. Let this transformation be

$$t = e^{-\bar{t}}. \tag{9.51}$$

Then, if the arbitrary function f depends on time t, we have

$$t\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{\mathrm{d}f}{\mathrm{d}\bar{t}}, \tag{9.52}$$

$$t^2\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}\bar{t}^2} + \frac{\mathrm{d}f}{\mathrm{d}\bar{t}}. \tag{9.53}$$

With the aid of (9.51), (9.52), and (9.53), we rewrite the equation of motion

$$m_i\ddot{r}_i = -U_{\eta}$$

in the form

$$m_i\left(\ddot{\bar{r}}_i - \frac{1}{3}\dot{\bar{r}}_i - \frac{2}{9}\bar{r}_i\right) = -\bar{U}_{\bar{r}_i} - k\dot{\bar{r}}_i, \tag{9.54}$$

where derivatives are written with respect to $\bar{t}$ and $\bar{U}_{\bar{r}_i} = t^{4/3}U_{\bar{r}_i}$.

Similarly, let us rewrite the energy conservation law and Jacobi's virial equation in the form

$$\frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{\bar{r}}_i - \frac{2}{3}\bar{r}_i\right)^2 + \bar{U} = E_0\left[1 + q(t)\right]e^{-2/3\bar{t}}, \tag{9.55}$$

$$\ddot{\bar{\Phi}} - \frac{5}{3}\dot{\bar{\Phi}} - \frac{4}{9}\bar{\Phi} = -U + 2E_0\left[1 + q(t)\right]e^{-2/3\bar{t}}. \tag{9.56}$$

Assuming $f = \Phi$ in (9.52) and (9.53), we obtain relations that are valid in the time limit $\bar{t} \to \infty$ and similar to (9.43), (9.44), and (9.45) as $t \to 0$:

$$\bar{\Phi} \to \mu_1 > 0, \tag{9.57}$$

$$\dot{\bar{\Phi}} \to 0, \tag{9.58}$$

$$\ddot{\bar{\Phi}} \to 0. \tag{9.59}$$

In the limit $\bar{t} \to \infty$ from (9.56), where $E_0(1 + q(t))$ is finite, with the aid of (9.57), (9.58), and (9.59), it follows that (9.49) is valid. Moreover, it is obvious from (9.49) and (9.57) that the potential energy U tends to a finite value and hence (9.50) follows from (9.47).

Second, let us show that the time limit $\bar{t} \to +\infty$ $(t \to 0)$:

$$\dot{\bar{r}} \to 0, \tag{9.60}$$

$$\ddot{\bar{r}} < \text{const.}, \tag{9.61}$$

$$\dddot{\bar{r}} < \text{const.}, \tag{9.62}$$

Note that (9.46) yields

$$\dot{\bar{\Phi}} = \sum_{i=1}^{n} m_i\,\dot{\bar{r}}_i\bar{r}_i. \tag{9.63}$$

Then, in the time limit $\bar{t} \to \infty$ and with the aid of (9.49) and (9.63), we obtain

$$\sum_{i=1}^{n} m_i\,\dot{\bar{r}}_i^2 \to 0,$$

which gives (9.60). Furthermore,

$$\bar{r} < \text{const.}, \tag{9.64}$$

$$\left|\bar{U}_{r_i}\right| < \text{const.} \tag{9.65}$$

In fact, Eq. (9.64) follows from (9.57) by virtue of Eq. (9.46). At the same time, Eq. (9.65) follows from (9.47) and (9.50). Equation (9.56) follows from (9.54), (9.60), (9.64), and (9.65). Finally, by differentiating (9.56) with respect to $\bar{t}$ and then using (9.60) and (9.61), it is easy to see that for the proof of (9.62), it is sufficient to show the boundedness of the second derivatives of the functions $\bar{U}\ (\bar{r}_1, r_2, \ldots, r_n)$ in the time limit $t \to \infty$. But the boundedness of these derivatives follows obviously from (9.47), (9.50), and (9.64).

Finally, in accordance with (9.60) and (9.62), the Tauberian lemma is valid if we consider the function $g(u) = \dot{\bar{r}}_i$, where $u = \bar{t}$. Hence, not only $\dot{\bar{r}}_i \to 0$ but $\ddot{\bar{r}}_i \to 0$.

It follows therefore from (9.54) that

$$\frac{2}{9} m_i\, \bar{r}_i - U_{\bar{r}_i} \to 0.$$

Then, by virtue of (9.46),

$$\frac{2}{9} \bar{\Phi}_{\bar{\eta}} - U_{\bar{r}_i} \to 0.$$

From the last expression, with the aid of (9.49) and (9.57), it follows that

$$\left(\bar{\Phi}\bar{U}^2\right)_{\bar{r}_i} = \bar{\Phi}_{\bar{r}_i} \bar{U}^2_{\bar{r}_i} + 2\bar{\Phi}\bar{U}\,\bar{U}_{\bar{r}_i} \to 0$$

and therefore

$$\left(\Phi U^2\right)_{\eta} \to 0$$

at $t \to 0$.

The last expression completes the proof of the theorem that an arbitrary nonconservative system tends to have the central configuration in the asymptotic limit of simultaneous collision of all its particles.

This theorem is the physical and mathematical bases for the explanation of mechanism of the electrons and atomic nuclei creation, including the transuranium elements, during attraction of the universe and their decay at expansion.

9.4 Asymptotic Limit of Simultaneous Collision of Charged Particles of a System

The following analysis is given for a system consisting of a large number of charged material particles. The particles considered are positively charged nuclei of atoms and electrons.

The objective is to prove the statement that the arbitrary configuration of a system of charged particles interacting according to an inverse law (i.e., gravitational or

Coulomb) in the asymptotic time limit of simultaneous collision of all the particles (for $t \to t_0$) tends to a central configuration.

Using the definition of central configuration (9.15) (Wintner 1941) and assuming its uniqueness, the statement to be proved can be written in the form

$$\lim_{t \to t_0} \left(|U_\Sigma| \sqrt{\Phi} \right) = \text{const.} \tag{9.66}$$

where $U_\Sigma = U + U_c$ is the potential energy of the system, which is equal to the sum of the gravitational potential energy of Coulomb interactions.

Using Wintner's method (Wintner 1941), we have previously studied the asymptotic time limit of (9.66) for conservative and nonconservative systems whose particles are interacting according to the law of gravitation. Since the relationship (9.66) is linear as a function of potential energy, we have to prove it only for Coulomb interactions of system particles. The proof given below for a nonconservative system is also based on Wintner's method, modified for the case of charged particles.

So for a nonconservative system of n particles interacting according to the Coulomb law, let us write down in an inertial barycentric coordinate system the Jacobi function, functions of the potential and kinetic energies, as well as the energy conservation law and Jacobi's virial equation as follows:

$$\Phi = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i \, m_j \, \Delta_{ij}^2, \tag{9.67}$$

$$T = \frac{1}{2m} \sum_{1 \le i < j \le n} m_i \, m_j \left(\dot{r}_i - \dot{r}_j \right)^2, \tag{9.68}$$

$$U = -G \sum_{1 \le i < j \le n} \frac{q_i \, q_j}{\Delta_{ij}}, \tag{9.69}$$

$$E = E(t) = E_0 - E_\gamma = T + U_c, \tag{9.70}$$

$$\ddot{\Phi} = 2E(t) - E_c, \tag{9.71}$$

where $q_i = eZ_i$ is the charge of ith particle with mass m_i; $Z_i = -1, 1, +2, \ldots, N \le n$; m is the total mass of the system; $E_\gamma < \infty$; and $\dot{E}\gamma < \infty$; that is, the total energy and the luminosity of the system at any time t are functions monotonically bounded from above.

The proof of the relationship (9.66) can easily be obtained from the asymptotic expressions for the Jacobi function and its first and second derivatives as

$$\Phi \propto (t - t_0)^{4/3}, \tag{9.72}$$

$$\dot{\Phi} \propto (t - t_0)^{1/3}, \tag{9.73}$$

$$\ddot{\Phi} \propto (t - t_0)^{-2/3}, \tag{9.74}$$

where $t \to t_0$ and t_0 is the moment of simultaneous collision of the charged particles of the system.

From the expressions (9.72), (9.73), and (9.74), the limit (9.66), which we are proving, follows from exact repetition of Wintner's arguments (1941). However, Eq. (9.72), (9.73), and (9.74) follows from the existence of the limits

$$\lim_{t-t_0} \frac{\dot{\Phi}^2}{\Phi^{1/2}} = \mu_0 = \text{const.} > 0, \tag{9.75}$$

$$\lim_{t \to t_0} \ddot{\Phi} \Phi^{1/2} = \eta_0 = \text{const.} > 0. \tag{9.76}$$

The limits (9.75) and (9.76) may be obtained in the future from analysis of the Jacobi function in the neighborhood of t_0, using the auxiliary function

$$Q = -(E - E\gamma)\,\Phi^{1/2} + \frac{1}{4}\frac{\dot{\Phi}^2 + M^2}{\Phi^{1/2}}$$

and the three inequalities, correct in the most general case, that is, not especially in the close neighborhood of the point of simultaneous collision of particles. These inequalities are

$$\left|\dddot{\Phi} + 2E_\gamma\right| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right)^{5/2} \eta_0, \tag{9.77}$$

$$\left[\ddot{\Phi} - 2\left(E - E_\gamma\right)\Phi^{1/2}\right] \ge \eta_0 > 0, \tag{9.78}$$

$$\ddot{\Phi} - E + E_\gamma - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}, \tag{9.79}$$

where M is the angular moment of the system.

Let us prove inequalities (9.77), (9.78), and (9.79) for a system of particles interacting according to Coulomb law.

To prove the inequality (9.77), it is essential that the absolute value of the total potential energy of the system of particles is less than the absolute value of the energy of mutual interactions of any pair of charged particles, that is,

$$\frac{q_i\, q_j}{\Delta_{ij}} \le |U_c|\,. \tag{9.80}$$

Since

$$\left|\dot{U}_{\mathrm{c}}\right| = \left|\sum_{1\le i<j\le n} \frac{q_i\, q_j}{\Delta_{ij}^2} \frac{\mathrm{d}}{\mathrm{d}t} \Delta_{ij}\right| \le \sum_{1\le i<j\le n} \frac{\left|q_i\, q_j\right|}{\Delta_{ij}^2} \left|\dot{r}_i - \dot{r}_j\right|$$

and

$$\frac{1}{\Delta_{ij}} \le \frac{|U_{\mathrm{c}}|}{\left|q_i\, q_j\right|^2},$$

$$\left|\dot{U}_{\mathrm{c}}\right| \le |U_{\mathrm{c}}|^2 \sum_{1\le i<j\le n} \frac{\left|\dot{r}_i - \dot{r}_j\right|}{\left|q_i\, q_j\right|}.$$

Analogously, since

$$m_i\, m_j \left|\dot{r}_i - \dot{r}_j\right|^2 \le 2mT$$

and

$$m_i \ge \frac{|q_i|}{e} \mu_{\mathrm{e}},$$

$$2mT \ge \frac{\left|q_i q_j\right|}{e^2} \mu_{\mathrm{e}}^2 \left|\dot{r}_i - \dot{r}_j\right|^2,$$

and, therefore,

$$\left|\dot{U}_{\mathrm{c}}\right| \le \left|\dot{U}_{\mathrm{c}}\right|^2 T^{1/2} \frac{(2m)^{1/2}}{\mu_{\mathrm{e}}} \sum_{1\le i<j\le n} \frac{1}{\left|q_i\, q_j\right|^{3/2}},$$

where μ_{e} is the electron mass.

From Jacobi's equation and the law of conservation of energy, it follows that

$$\left|\dot{U}_{\mathrm{c}}\right| = \left|\dddot{\Phi} + 2\dot{E}_\gamma\right|,$$

$$|U_{\mathrm{c}}| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right),$$

$$|T| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right),$$

and, finally, we obtain the first inequality:

$$\left|\dddot{\Phi} + 2\dot{E}_\gamma\right| \le \left(\left|\ddot{\Phi}\right| + 2\left|E - E_\gamma\right|\right)^{5/2} \eta_0,$$

$$\eta_0 = \frac{(2m)^{1/2}}{\mu_c} e \sum_{1 \le i < j \le n} \frac{1}{\left(q_i\, q_j\right)^{3/2}} > 0.$$

The second inequality (9.78) may be derived from Jacobi's equation:

$$\ddot{\Phi} - 2\,(E - E\gamma) = -U_c = -\sum_{1 \le i < j \le n} \frac{q_i\, q_j}{\Delta_{ij}} = |U_c| \ge \frac{\left|q_i\, q_j\right|}{\Delta_{ij}}$$

and the inequality followed from the definition of the Jacobi function:

$$2m\Phi \ge m_i\, m_j\, \Delta_{ij},$$

$$\frac{1}{\Delta_{ij}} \ge \frac{\left(m_i\, m_j\right)^{1/2}}{(2\mathrm{m})^{1/2}\Phi^{1/2}}.$$

Thus, finally, we have

$$\left[\ddot{\Phi} - 2\left(E - E_\gamma\right)\right]\Phi^{1/2} \ge \frac{\left|q_i\, q_j\right|\left(m_i\, m_j\right)^{1/2}}{(2m)^{1/2}} = \mu_0 > 0.$$

The derivation of the third inequality (9.79) is based on the Cauchy–Bunyakovsky inequality:

$$\left(\sum_{1 \le i \le n}^{n} a_i\, b_i\right)^2 \le \left(\sum_{1 \le i \le n}^{n} a_i^2\right)\left(\sum_{1 \le i \le n}^{n} b_i^2\right).$$

Substituting into it

$$a_i = m_i^{1/2}\,|r_i|\,, \quad b_i = m_i^{1/2}\frac{\mathrm{d}}{\mathrm{d}t}\,|r_i|\,,$$

we have

$$\dot{\Phi} = \sum_{1 \le i \le n} m_i\,|r_i|\,\frac{\mathrm{d}}{\mathrm{d}t}\,|r_i|\,,$$

$$\left(\dot{\Phi}\right)^2 \le 2\Phi \sum_{1 \le i \le n} m_i \frac{\left(r_i \frac{\mathrm{d}}{\mathrm{d}t} r_i\right)^2}{|r_i|^2}.$$

Substituting as before

$$a_i = m_i^{1/2}\,|r_i|\,, \quad b_i = m_i^{1/2}\frac{[r_i\, \dot{r}_i]}{|r_i|},$$

we obtain

$$M^2 \le 2\Phi \sum_{1 \le i \le n} \frac{m_i |r_i \, \dot{r}_i|^2}{|r_i|^2},$$

where M is the angular momentum of the system equal to

$$M = \sum_{1 \le i \le n} m_i \, [r_i \, \dot{r}_i] \, .$$

Summing up the two inequalities just obtained, we have

$$\left(\dot{\Phi}\right)^2 + M^2 \le 2\Phi \sum_{1 \le i \le n} \frac{m_i}{|r_i|^2} \left\{ (r_i \, \dot{r}_i)^2 + [r_i \, \dot{r}_i]^2 \right\}$$

$$= 2\Phi \sum_{1 \le i \le n} m_i \, (\dot{r}_i)^2 = 4T\Phi = 4\Phi \left[\ddot{\Phi} - \left(E - E_\gamma \right) \right].$$

We finally obtain an expression for the third inequality (9.79):

$$\ddot{\Phi} - E + E_\gamma - \frac{\dot{\Phi}^2}{4\Phi} \ge \frac{M^2}{4\Phi}.$$

This ends the proof of the expression (9.66) for the Coulomb interactions of charged particles of the system in the asymptotic time limit of their simultaneous collision.

9.5 Relationship Between the Jacobi Function and Potential Energy for a System with High Symmetry

It was shown in Chaps. 3 and 7 that the relationship of Jacobi's function and the potential energy

$$|U| \sqrt{\Phi} = B \tag{9.81}$$

does not change for different mass density distribution laws and configurations of the system. In this case, Jacobi's virial equation

$$\ddot{\Phi} = 2E - U, \tag{9.82}$$

by means of (9.81), we transfer into the equation of virial oscillations:

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}. \tag{9.83}$$

However, even for a system with spherical symmetry and fixed mass, the value of (9.81) changes for different laws of distribution of the mass density $\rho(r)$ (where r is the radius of the shell with density $\rho(r)$); $r \in [0,R]$. In this connection, transformation of Eq. (9.82) into (9.83) is possible only after special study, which is described below.

We pay special attention to the systems with high symmetry, namely, spherical and elliptical. This is because most of the natural systems from galaxies, stars, planets, and their satellites and also liquids and DT targets for carrying out the nuclear synthesis works to atoms possess such a symmetry. The systems with charged particles are also included here. We consider below the conditions that allow us to transform Eq. (9.82) into (9.83) for systems with spherical and elliptical symmetry.

9.5.1 Systems with Spherical Symmetry

Let us begin by considering the value of Eq. (9.81) for a spherical system. It is convenient to start such a study after rewriting the expressions for the Jacobi function and the potential energy in the form

$$\Phi = \frac{1}{2}\beta^2 m R^2, \tag{9.84}$$

$$U = -\alpha \frac{Gm^2}{R}, \tag{9.85}$$

where α^2 and β^2 are dimensionless form factors independent of radius R and mass m of the spherical system (see Sect. 3.6).

We now rewrite (9.81), using (9.84) and (9.85), as

$$B = \alpha\beta G m^{5/2}. \tag{9.86}$$

Use of form factors α and β allows us to show that the parameter B in (9.81) does not depend on radius of the spherical system. The product of α and β depends on mass density distribution law $\rho(r)$ and does not depend on the total mass of the system. Hence, the problem of the study of the changes of parameter B in (9.81) for an arbitrary spherical system is reduced to consideration of the dependence of the product of the $\alpha\beta$ form factors on the mass density distribution law for the sphere with radius unity and mass unity. Let us consider such a sphere and calculate the value

$$a = \alpha\beta. \tag{9.87}$$

For the arbitrarily given law of density distribution $\rho(k)$, $k \in [0, 1]$, satisfying the condition

$$\int\limits_{(V)} \rho(k)\,\mathrm{d}V(k) = 1,$$

the volume of the sphere with radius unity is

$$V = \iiint\limits_{(V)} \mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z = \int\limits_0^1 k^2\,\mathrm{d}k \int\limits_0^\pi \sin\theta\,\mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = \frac{4}{3}\pi.$$

The volume of the sphere with radius k is

$$V(k) = \int\limits_0^k k'^2\,\mathrm{d}k' \int\limits_0^\pi \sin\theta\,\mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = \frac{4}{3}\pi k^3. \tag{9.88}$$

The volume of the spherical shell with radius k and thickness dk is

$$\mathrm{d}V(k) = k^2\,\mathrm{d}k \int\limits_0^\pi \sin\,\theta\,\mathrm{d}\theta \int\limits_0^{2\pi} \mathrm{d}\varphi = 4\pi k^2\,\mathrm{d}k. \tag{9.89}$$

The mass of the spherical shell with radius k and thickness dk is

$$dm(k) = 4\pi\rho(k)k^2\,\mathrm{d}k.$$

The mass of the sphere with radius k is

$$m(k) = 4\pi \int\limits_0^k \rho\left(k'\right)\left(k'\right)^2 \mathrm{d}k'. \tag{9.90}$$

The mass of the sphere as a whole is

$$m = 4\pi \int\limits_0^1 \rho(k)k^2\,\mathrm{d}k = 1. \tag{9.91}$$

The polar moment of inertia of the shell with radius k and thickness dk is

$$\mathrm{d}I(k) = k^2\,\mathrm{d}m(k) = 4\pi\rho(k)k^4\,\mathrm{d}k,$$

and the Jacobi function of the sphere is

$$\Phi = \frac{4\pi}{2} \int_0^1 \rho(k) k^4 \, \mathrm{d}k. \tag{9.92}$$

We can write the expression for the form factor β from (9.84) using (9.91) and (9.92):

$$\beta = \sqrt{\frac{\Phi}{\frac{1}{2}m}} = \frac{\sqrt{\int_0^1 \rho(k) k^4 \, \mathrm{d}k}}{\int_0^1 \rho(k) k^2 \, \mathrm{d}k}. \tag{9.93}$$

The potential energy of the shell with radius k and thickness dk in the gravitational field of the sphere of radius k is

$$\mathrm{d}U(k) = -G \frac{m(k) \, \mathrm{d}m(k)}{k} = -G \frac{16\pi^2 \rho(k) k^2 \, \mathrm{d}k \int_0^k \rho(k') (k')^2 \mathrm{d}k'}{k}.$$

The potential energy of the sphere as a whole is

$$U = -16\pi^2 G \int_0^1 \rho(k) k \, \mathrm{d}k \int_0^k \rho(k') (k')^2 \mathrm{d}k'. \tag{9.94}$$

We can write the expression for the form factor α using (9.85), (9.91), and (9.94) as

$$\alpha = -\frac{U}{Gm^2} \frac{\int_0^1 \rho(k) k \, \mathrm{d}k \int_0^k \rho(k') (k')^2 \mathrm{d}k'}{\left(\int_0^1 \rho(k) k^2 \, \mathrm{d}k \right)^2}. \tag{9.95}$$

Finally, the product of form factors α and β represents the functional of the function of mass density distribution $\underline{\rho}(k)$:

$$a = \alpha\beta = \frac{\int_0^1 k\rho(k) \, \mathrm{d}k \int_0^k \rho(k') (k')^2 \mathrm{d}k' \sqrt{\int_0^1 \rho(k) k^4 \, \mathrm{d}k}}{\left(\int_0^1 \rho(k) k^2 \, \mathrm{d}k \right)^{5/2}}. \tag{9.96}$$

Table 9.1 Numerical values of form factors α and β and their product $\alpha\beta$ for various formal laws of radial mass density distribution of the spherical system

Law of mass density distribution $\rho(k)$, $k \in [0, 1]$	α	$\beta_{\perp}^2$	β^2	$\alpha\beta$
$\rho(r) = \rho_0$	0.6	0.4	0.6	0.46
$\rho(r) = \rho_0(1-k)$	0.728	0.27	0.4	0.47
$\rho(r) = \rho_0(1-k^2)$,	0.7142	0.29	0.42	0.46
$\rho(r) = \rho_0(1-k)^n$	$\frac{(5m+8)(m+3)^2}{8(2n+3)(2n+5)}$	$\frac{8}{(n+4)(n+5)}$	$\frac{12}{(n+4)(n+5)}$, at $n \to \infty$	0.54
$\rho(r) = \rho_0 k^n$	$\frac{n+3}{2n+5}$	$\frac{2n+9}{6n+15}$	$\frac{n+3}{2n+5}$, at $n \to \infty$	0.5
$\rho(r) = \rho_0\delta(1-k)$	0.5	0.67	1.0	0.5

The values of the form factors α and β and of their product $\alpha\beta$ for different formal laws of mass density distribution are given in Table 9.1. The numerical calculations of this table can be found in our paper (Ferronsky et al. 1978).

It can be seen from Table 9.1 that the form factor β changes from 0 to 1: $\beta \in [0, 1]$. It reaches the value of unity in the case when the entire mass of the sphere is distributed within its outer shell (at $k = 1$). The minimal value of the form factor β must be when the entire mass concentrates in the center of the sphere (at $k = 0$). But if we do not place any strong restrictions on the function $\rho(k)$, that is, in the general case, nothing can be said about the changing interval of the value $a = \alpha\beta$ (9.85). It is only possible to note that $a = \alpha\beta$ always has a positive value. From Table 9.1, it can also be assumed that the value of a is more than $(3/5)^{3/2} \approx 0.46$, which corresponds to the homogeneous distribution of the mass density within the sphere. It is known also from Chap. 6 that the homogeneous sphere, while contracting under gravitational forces, conserves its homogeneity up to the moment of simultaneous collision of all its particles. Thus, according to Wintner's terminology, a uniform body appears to be the central configuration.

The sphere expands and then (the time is reversible in classical physics) becomes homogeneous again. So in accordance with the definitions given in the previous section, the homogeneous sphere appears to be the central configuration. Applying the main idea of the central configuration theorem discussed above in the general case, we assume the following qualitative picture of the evolution of a heterogeneous spherical system. During the contraction of the system, the $\alpha\beta$ decreases and tends to the quantity $(3/5)^{3/2}$, reaching this value at the moment of simultaneous collision of all the particles. If the expansion starts before the moment of simultaneous collision of the matter (at the neighborhood of singularity), then the value of $\alpha\beta$ again increases. Thus, there is a case of perturbed virial oscillations of the system. This case is known in the literature as "stormy relaxation" of a gaseous sphere and is described quantitatively by the following equation of change of value of $|U|\sqrt{\Phi}$ (Ferronsky et al. 1984):

$$U\sqrt{\Phi} = B - k\Phi$$

where B is a constant and k is also a constant.

This law of change of value of $|U|\sqrt{\Phi}$ will be considered in detail in Chap. 8, which is devoted to astrophysics applications. Here we note that the only mechanism that drives the matter of a system toward simultaneous collision is the loss of energy through radiation. So, for conservative systems, the equation of virial oscillations has the form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}} - \frac{k\dot{\Phi}}{\sqrt{\Phi}}.$$

The term $k\dot{\Phi}/\sqrt{\Phi}$ is part of the perturbation function. It does not lead to the loss of total energy of the system, and we can call it internal friction.

9.5.2 Polytropic Gas Sphere Model

The laws of mass density distribution in the previous section were considered formally, neglecting the requirement of hydrodynamic stability of the system. However, it is well known that for the many really existing celestial gas bodies, a polytropic model in the central domain is a good one.

Let us study the value of the form factors α and β and their product $\alpha\beta$ for the polytropic gas sphere model at various quantities of polytropic index. The equation of state for a gas sphere is

$$\frac{\mathrm{d}p(k)}{\mathrm{d}k} = -G\frac{m(k)\rho(k)}{k^2}, \tag{9.97}$$

where $p(k)$ is the gas pressure, $\rho(k)$ is the mass density of the gas, and G is the gravitational constant.

Using Eq. (9.97), we can rewrite it for the sphere with radius k and mass m in the form

$$\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\left|\frac{k^2}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}\right| = -4\pi G\rho(k). \tag{9.98}$$

This is one of the basic equations in the theory of the internal structure of the stars used up to now.

It is assumed that for polytropic models, the two independent characteristics in Eq. (9.98), namely, pressure $p(k)$ and mass density $\rho(k)$, are linked by the relationship:

$$p(k) = C\rho^b(k), \tag{9.99}$$

where C and b are constants.

From (9.99), it follows that

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k} = C\frac{b}{b-1}\frac{\mathrm{d}\rho^{b-1}(k)}{\mathrm{d}k}. \tag{9.100}$$

Substituting (9.100) into (9.98) and introducing specification

$$\rho^{b-1}(k) = u(k) \quad n = \frac{1}{b-1}, \tag{9.101}$$

we obtain

$$C\,(1+n)\,\frac{1}{k^2}\frac{\mathrm{d}}{\mathrm{d}k}\left|k^2\frac{\mathrm{d}u(k)}{\mathrm{d}k}\right| = 4\pi G u^n(k). \tag{9.102}$$

Equation (9.102) can be simplified if dimensionless variables $\Theta(x) = u(x)/u_0$ and $x = \lambda k$ are introduced. Here u_0 is the value $u(k)$ in the center of the sphere, that is, at $k = 0$. The coefficient λ is selected with the condition that, after substitution of the function $\Theta(x)$ into (9.102), all the constants should be canceled. Then, the following relationship for λ can be obtained:

$$C(1+n)\lambda^2 = 4\pi G u_0^{n-1}, \tag{9.103}$$

and Eq. (9.102), known as the Emden equation, takes the form

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left|x^2\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right| = -\Theta^n(x). \tag{9.104}$$

It is obvious that for $x = 0$, the function $\Theta(x)$, known as the Emden function, should satisfy two conditions:

$$\Theta(x)\,|_{x=0} = 1, \qquad \frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\,|_{x=0} = 0. \tag{9.105}$$

We now obtain the expression for the form factor α for a sphere with polytropic index n. For this purpose, we write the expression of potential energy in the form

$$U = -G\int\frac{m(k)\,\mathrm{d}m(k)}{k}.$$

Using Eq. (9.97) for the gas sphere and the expression for $dm(k)$, we rewrite (9.105) as follows:

$$U = \int\frac{k}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}\,\mathrm{d}m(k) = 4\pi\int k^3\,\mathrm{d}p(k). \tag{9.106}$$

After integration by parts of the right-hand side of (9.105), we obtain

$$U = -12\pi \int_0^1 k^2 p(k)\,\mathrm{d}k. \tag{9.107}$$

On the other hand, (9.105) can be rewritten in the form

$$U = -\frac{G}{2}\int \frac{\mathrm{d}m^2(k)}{k}.$$

Integrating the right-hand side of the last relationship by parts, we obtain

$$U = -\frac{G}{2}\frac{m^2(k)}{k}\Big|_{k=0}^{k=1} - \frac{G}{2}\int \frac{m^2(k)\,\mathrm{d}k}{k^2}. \tag{9.108}$$

The integral in the right-hand side of (9.108) is transformed with the help of (9.97) as follows:

$$-\frac{G}{2}\int \frac{m^2(k)\,\mathrm{d}k}{k^2} = \frac{1}{2}\int \frac{m(k)}{\rho(k)}\frac{\mathrm{d}p\,(k)}{\mathrm{d}k}\mathrm{d}k.$$

Thus, using (9.100), we obtain

$$-\frac{G}{2}\int \frac{m^2(k)\,\mathrm{d}k}{k^2} = \frac{1}{2}\int m(k)C\frac{b}{b-1}\,\mathrm{d}\rho^{b-1}(k),$$

and integrating by parts, we have

$$\begin{aligned}-\frac{G}{2}\int \frac{m^2(k)\,\mathrm{d}k}{k^2} &= \frac{1}{2}C\frac{b}{b-1}\rho^{b-1}(k)m(k)\Big|_{k=0}^{k=1} - \frac{1}{2}\int C\frac{b}{b-1}\rho^{b-1}(k)4\pi k^2\rho(k)\,\mathrm{d}k\\ &= -\frac{1}{2}\int (n+1)4\pi k^2 p(k)\,\mathrm{d}k.\end{aligned} \tag{9.109}$$

Substituting (9.109) into (9.108), we obtain the second expression for the potential energy:

$$U = -\frac{G}{2} - \frac{4\pi\,(n+1)}{2}\int_0^1 k^2\rho(k)\,\mathrm{d}k, \tag{9.110}$$

where the condition $m(1) = 1$ has been taken into account.

Solving the system of equation (9.110) and (9.107) with respect to U, we find that

$$U = -G\frac{3}{5-n},$$

and hence

$$\alpha = \frac{3}{5-n}. \tag{9.111}$$

Now we derive the expression for the form factor β. For this purpose, we write the Jacobi function expression for a polytropic sphere:

$$\Phi = \frac{4\pi}{2}\int_0^1 k^4\rho(k)\,\mathrm{d}k = \frac{4\pi}{2}\int_0^{x_1}\frac{\Theta^n(x)x^4\,\mathrm{d}x}{\lambda^5}, \tag{9.112}$$

where x_1 is the first root of the equation $\Theta(x) = 0$.

Let us specify

$$\nu = \int_0^{x_1}\Theta^n(x)x^4\,\mathrm{d}x.$$

And taking into account (9.103), we write

$$C(1+n)\lambda^2 = 4\pi G u_0^{n-1}.$$

Then

$$\Phi = \frac{4\pi\nu}{2}\frac{u_0^n}{\lambda^5} = \frac{4\pi\nu}{2}\frac{[C(1+n)]\,n/n-1}{(4\pi G)n/n-1}\lambda^{(5-3n)/n-1}. \tag{9.113}$$

Now we obtain the second expression for the Jacobi function using the condition of equation (9.99) at the border surface of the sphere, that is, at $k = 1$. Then,

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}\Big|_{k=1} = -\frac{Gm(k)}{k^2}\Big|_{k=1} \tag{9.114}$$

and

$$m(k)k^2\Big|_{k=1} = -\frac{k^4}{G}\frac{1}{\rho(k)}\frac{\mathrm{d}p\,(k)}{\mathrm{d}k}\Big|_{k=1}.$$

Table 9.2 Numerical values of form factors α and β and their product $\alpha\beta$ for different values of polytropic index n

Index n	α^2	x_1	$-x^2 \frac{B\Theta(x)}{dx}$	ν	β	$\alpha\beta$
0	0.6	2.45	4.9	17.63	0.77	0.46
1	0.75	3.14	3.14	12.15	0.62	0.465
1.5	0.87	3.63	2.71	11.12	0.55	0.475
2	1.0	4.35	2.41	10.61	0.48	0.482
3	1.5	6.89	2.01	10.85	0.34	0.502
3.5	2.0	9.53	1.89	11.74	0.26	0.52

The left-hand side of Eq. (9.114), taking into account (9.100) and (9.101), is

$$\frac{1}{\rho(k)}\frac{\mathrm{d}p(k)}{\mathrm{d}k}\,|_{k=1} = C\frac{b}{b-1}\frac{\mathrm{d}\rho^{b-1}(k)}{\mathrm{d}k} = C(n-1)\frac{\mathrm{d}u\,(k)}{\mathrm{d}k}. \tag{9.115}$$

Finally, we obtain

$$\begin{aligned}\Phi &= \frac{1}{2}\beta^2 m(k)k^2\,|_{k=1} = -\frac{1}{2}\beta^2\frac{C(n+1)}{G}k^4\frac{\mathrm{d}u\,(k)}{\mathrm{d}k}\,|_{k=1}\\ &= -\frac{1}{2}\beta^2\frac{C(n+1)}{G}u_0\frac{x^4}{\lambda^3}\frac{\mathrm{d}\Theta\,(x)}{\mathrm{d}k}\,|_{x=x_1}\,.\end{aligned}$$

Or when using (9.103),

$$\Phi = \frac{1}{2}\pi\beta^2\frac{C(1+n)^{n/n-1}}{(4\pi G)^{n/n-1}}\lambda^{(5-3n)/n-1}\left|x^4\frac{\mathrm{d}\Theta(x)}{\mathrm{d}k}\right|\,|_{x=x_1}\,. \tag{9.116}$$

Dividing (9.116) by (9.113), we obtain

$$\beta = \sqrt{\frac{\nu}{\left[-x^4\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x}\right]|_{x=x_1}}}. \tag{9.117}$$

We calculated the values of α and β and their product $\alpha\beta$ using the data for ν, x_1, and

$$\frac{-x^2\mathrm{d}\Theta(x)}{\mathrm{d}x}\,|_{x=x_1}$$

at different polytropic index values, taken from Chandrasekhar (1939, 1942). The calculated data are shown in Table 9.2. It is interesting to note that in the framework of the really existing physical laws of mass density distribution $\rho(k)$, the quantity $\alpha\beta$ changes within the narrow limits despite the fact that each of the form factors α and β varies almost three times the variation of the polytropic index from 0 to 3.5.

9.5.3 System with Elliptical Symmetry

We have shown in the previous section that the property of the central configurations consisting in the constancy of the product $\alpha\beta$ holds for system with spherical symmetry.

Now we prove that this property holds for elliptical symmetry with an ellipsoidal mass distribution. Moreover, we show that among all the configurations, only ellipsoidal mass distribution possesses this property of central configurations.

Let us write the equation of the general ellipsoid with semi-axes a, b, c:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c_2} = 1, \tag{9.118}$$

where x, y, and z are the Cartesian coordinates of the surface of this ellipsoid.

The equation of a set of similar ellipsoidal shells of this ellipsoid with the ellipsoidal mass distribution $\rho(x)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c_2} = k^2, \tag{9.119}$$

where $k\in$ [0, 1] is a parameter of the homogeneous ellipsoidal shell.

The gravitational potential inside this ellipsoidal shell is equal to a constant at an arbitrary point (x, y, z)

$$F(x, y, z) = -\frac{Gm_s}{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}, \tag{9.120}$$

where m_s is the mass of the shell and u is a parameter of integration.

We write down the form factor α_e of the potential energy U of this ellipsoid as

$$\alpha_c = -\frac{aU}{Gm^2}, \tag{9.121}$$

where a is semimajor axis in the equatorial plane and m is the total mass.

The volume of an ellipsoid bounded by the surface (9.119) with the parameter k is

$$V(k) = \frac{4}{3}\pi abck^3. \tag{9.122}$$

The volume of the thin shell bounded by ellipsoidal surfaces with the parameters k and $k + \mathrm{d}k$ is

$$\mathrm{d}V(k) = 4\pi abck^2\,\mathrm{d}k. \tag{9.123}$$

The mass of this shell is expressed as

$$\mathrm{d}m_{\mathrm{s}}(k) = 4\pi abck^2\rho(k)\,\mathrm{d}k. \tag{9.124}$$

Then, the total mass of the ellipsoid is

$$m = 4\pi abc\int_0^1 k^2\rho(k)\mathrm{d}k. \tag{9.125}$$

The mass of an ellipsoid bounded by the surface with the parameter k is

$$m(k) = 4\pi abc\int_0^k \left(k'\right)^2\rho\left(k'\right)\,\mathrm{d}k'. \tag{9.126}$$

Using the reciprocation theorem (Duboshin 1975), we write the potential energy of the ellipsoid in the form

$$U = -\int_0^1 m(k)\,\mathrm{d}F\,(k). \tag{9.127}$$

The gravitational potential inside the thin shell bounded by elliptical surface with parameters k and $k + \mathrm{d}k$ (9.120) is

$$\mathrm{d}F(k) = 2\pi Gabck\,\rho(k)\,\mathrm{d}k\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}. \tag{9.128}$$

Now we write the expression for the form factor α_{e} using the corresponding values of U and m as

$$\alpha_{\mathrm{e}} = -\frac{aU}{Gm^2} = \frac{a}{2}\frac{\int_0^1 k\,\rho(k)\,\mathrm{d}k\int_0^k (k')^2\rho\left(k'\right)\,\mathrm{d}k'}{\left[\int_0^1 k^2\rho(k)\,\mathrm{d}k\right]^2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}$$

$$= \alpha\frac{a}{2}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}, \tag{9.129}$$

where α is the potential energy form factor corresponding to the radial mass distribution law $\rho(k)$.

It is easy to see from Eq. (9.129) that when $a = b$, we obtain the value of the form factor α_e for the ellipsoid of rotation

$$\alpha_e = \alpha \frac{\arcsin e}{e}. \tag{9.130}$$

Since

$$e = \sqrt{\frac{a^2 - c^2}{a^2}} \in |0, 1|,$$

$$\alpha_e \in \left[, \frac{\pi}{2}\alpha\right].$$

When $a > b > c$, Eq. (9.129) be (Janke et al. 1960)

$$\alpha_e = \alpha \int_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$
$$= \alpha \frac{a}{\sqrt{a^2 - c^2}} F\left(\arcsin\sqrt{\frac{a^2 - c^2}{a^2}}, \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\right).$$

Denoting

$$\arcsin\sqrt{\frac{a^2 - c^2}{a^2}} = \arcsin e_1 = \varphi \quad \text{and} \quad \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} = \frac{e_2}{e_1} = \sin\alpha = f,$$

we obtain

$$\alpha_e = \alpha \frac{F(\varphi, f)}{\sin\varphi}, \tag{9.131}$$

where $F(\varphi, f)$ is an incomplete elliptical integral of the first degree in the normal Legendre form. If $e_1 < 0.999$ and $0 < e_2 < e_1$, the function $F(\varphi, f)\sin^{-1}\varphi \in [1.000; 3.999]$ (Janke et al. 1960). When the arguments φ and f increase, the function $F(\varphi, f)\sin^{-1}\varphi$ also increases continuously.

Let us now consider the form factor β, which may be written as

$$\beta = \left[\frac{\Phi}{ma^2}\right]^{1/2}. \tag{9.132}$$

Obviously, β can be obtained by corresponding integration over the parameter $k \in [0, 1]$, if one writes the Jacobi function for the homogeneous thin shell bounded by the surfaces within the parameters k and $k + \mathrm{d}k$ and with mass distribution $\rho(k)$ in the integrand.

Since the Jacobi function for a homogeneous ellipsoid with mass density ρ_0 is

$$\Phi = \frac{2}{15}\pi abc\rho_0\left(a^2 + b^2 + c^2\right), \tag{9.133}$$

the Jacobi function for a thin ellipsoid shell may be written as

$$\mathrm{d}\Phi\,(\mathrm{k}) = \frac{2}{3}\pi abc\,\rho(k)\,k^4\,\mathrm{d}k\,\left(a^2 + b^2 + c^2\right). \tag{9.134}$$

Consequently, the Jacobi function Φ of the ellipsoid is equal to

$$\Phi = \frac{2}{3}\pi abc\left(a^2 + b^2 + c^2\right)\int\limits_0^1 \rho(k)k^4\,\mathrm{d}k. \tag{9.135}$$

Finally, using (9.135) and (9.125), Eq. (9.132) for the form factor β will be

$$\beta_\mathrm{e} = \left[\frac{a^2 + b^2 + c^2}{3a^2}\frac{\int_0^1 \rho(k)k^4\,\mathrm{d}k}{\int_0^1 \rho(k)k^2\,\mathrm{d}k}\right]^{1/2} = \beta\left[\frac{a^2 + b^2 + c^2}{3a^2}\right]^{1/2}, \tag{9.136}$$

where β is a form factor of the Jacobi function of the system with radial mass distribution $\rho(k)$ and the expression

$$\left[\frac{a^2 + b^2 + c^2}{3a^2}\right]^{1/2} \in \left[\frac{1}{\sqrt{3}}, 1\right].$$

So the value a_e is equal to

$$a_\mathrm{e} = \alpha_\mathrm{e}\beta_\mathrm{e} = a\frac{F\,(\varphi, f)}{\sin\,\varphi}\left[\frac{a^2 + b^2 + c^2}{3a^2}\right]^{1/2}. \tag{9.137}$$

Now it can be shown that the property (9.137) of the product $\alpha\beta$ constancy is possessed only by systems with elliptical symmetry and ellipsoidal mass density distribution. This means that for such systems, the form factors α and β may be expressed as a product of corresponding form factors of the sphere and terms depending on the form of the boundary surface.

For this proof, we consider an arbitrary system with a similar law of mass distribution $\rho(k)$, $k \in |0, 1|$, and the boundary surface S. Then, since we consider only one-dimensional $\rho(k)$, mass density will be constant on any surface with a fixed parameter k and similar to S. The area of this surface is

$$S'(k) = Sk^2. \tag{9.138}$$

If the volume of the body is equal to V, then the volume of the part of the body bounded by the surface $S'(k)$ is

$$V'(k) = Vk^3, \tag{9.139}$$

and its mass is

$$m(k) = V\int_0^1 k^2\rho(k)\,\mathrm{d}k. \tag{9.140}$$

Let us introduce the Cartesian coordinate system OXYZ with an origin coinciding with the center of similarity. Let us denote by h in the equatorial plane OXY the longest distance from the center of similarity to the boundary and assume thatf the form factor $\alpha^2{}_\mathrm{e}$ of the body can be expressed as a product of the form factors of the potential energy α for the radial mass density distribution law and some term $\triangle(S)$ depending on the form of the boundary surface:

$$\alpha_\mathrm{e} = -\frac{Uh}{Gm^2} = \alpha\Delta(S) = \frac{\int_0^1 k\rho(k)\,\mathrm{d}k \int_0^k (k')^2\rho\left(k'\right)\,\mathrm{d}k'}{\left(\int_0^1 k^2\rho(k)\,\mathrm{d}k\right)^2}\Delta(S). \tag{9.141}$$

From Eq. (9.141) we can obtain the potential energy in the form

$$U = -\frac{Gm^2}{h}\alpha\Delta(S) = -\frac{GV}{h}\int_0^1 k\rho(k)\,\mathrm{d}k\;\Delta(S). \tag{9.142}$$

Since the terms G, V, H, and $\triangle(S)$ do not depend on the parameter k, let us put them into the integrand and denote

$$\frac{GV}{h}k\rho(k)\,\Delta(S)\,\mathrm{d}k = F(k).$$

Then, Eq. (9.142) may be written as

$$U = -\int_0^1 m(k)\,\mathrm{d}F(k). \tag{9.143}$$

Comparing Eqs. (9.143) and (9.127), one can see that Eq. (9.143) is an equation for the reciprocation theorem, whose validity is based on the constancy of the gravitational potential $\mathrm{d}F(k)$ inside the thin shell bounded by the similar and similarly situated surfaces with parameters k and $k + \mathrm{d}k$. But as shown in the work of Dive (1931), where one can find rigorous proof of the reverse Newton theorem, only ellipsoidal shells possess such a property. Therefore, the body with the one-dimensional mass distribution law $\rho(k)$ for which the form factor α_{e} is equal to the product of the form factor of the sphere and some term depending on the form of the boundary surface $\triangle(S)$ must satisfy the equation of the ellipsoid (9.118).

9.5.4 System with Charged Particles

In Sect. 8.2, it was shown by modeling solution that for the Coulomb interactions of charged particles, constituting a system, Eq. (9.5) holds under the same conditions as the previous models discussed above.

Considering a one-component ionized quasi-neutral and self-gravitating gaseous cloud with spherically symmetric mass density distribution, we found that the form factors in the expression for the potential energy of the Coulomb interaction have the same physical meaning, which has the gravity mass interaction. It represents the shell to which the sphere of charges is reduced.

The task about the Coulomb potential energy of the interacting charged particles proves legality of solution of the Jacobi virial equation for the study of the celestial body's electromagnetic effects.

But it follows from Eq. (8.3) that the form factor α_{c} of the Coulomb energy becomes an infinite value when ion's volume tends to zero; in this case, the Coulomb energy tends also to infinity. In Table 8.3, there are two laws of the mass density distribution for which the last condition holds. These laws are $\rho(r) = \rho_0\ (r/R)^n$ and $\rho(r) = \rho_0(1 - r/R)^n$ at $n \to \infty$. When particles come together in the shell of infinite radius, the Coulomb interaction energy becomes infinitely large. When the mass density distribution law is $\rho(r) = \rho_0(1{-}r/R)^n$, then the form factors of the gravity and Coulomb energy have a finite value. In this case, the form factors of Jacobi's function of a system make the constant $a = \alpha\ \cdot\beta$ equal to zero. This effect can play a decisive role in the evolution of the system.

We note in conclusion that the analysis of relationship between the Jacobi function and potential energy from physical viewpoint justifies transfer from Jacobi's equations (9.1) and (9.2) to equations of virial oscillation (9.3) and (9.4). At the same time, it is possible to meet deviation of in Eq. (9.5) from the constant value because of small effects of perturbations, which can take part at the evolution of heterogenic systems.

9.6 Direct Derivation of the Equation of Virial Oscillation from Einstein's Equations

Weinberg (1972) reduced Einstein's equation for homogeneous isotropic space, with the help of the Robertson–Walker metric, to the following scalar form:

$$3\ddot{R} = -4G(\rho + 3p)R, \tag{9.144}$$

$$\ddot{R}R + 2(\dot{R})^2 + 2k = 4\pi G(\rho - p)R^2, \tag{9.145}$$

where R is the radius of the universe, p radiation pressure (mass defect), and ρ the density of matter without mass defect.

Multiplying Eq. (9.144) by $R/3$ and summing it with (9.145), we obtain

$$\left(\ddot{R}^2\right) + 2k = 8\pi G R^2 \left(\frac{1}{3}\rho - p\right). \tag{9.146}$$

When $\rho \ll p$ and ρR^3 is a constant (dust cloud), and taking into account that for curved space (Landau and Lifshitz 1973)

$$\rho R^3 = \frac{m}{2\pi^2}, \tag{9.147}$$

where m is the total mass of the particles constituting the cloud, expression (9.147) is transformed into

$$\left(\ddot{R}^2\right) + 2k = \frac{8\pi}{3} G \frac{m}{2\pi^2} \frac{1}{R}. \tag{9.148}$$

Since from the Jacobi function we have $\Phi = mR^2/2$, Eq. (9.148) can be rewritten as

$$\ddot{\Phi} + km = \frac{2}{3\pi} G m^2 \sqrt{\frac{m}{2}} \frac{1}{\sqrt{\Phi}} \tag{9.149}$$

or

$$\ddot{\Phi} + km = \frac{\sqrt{2}}{3\pi} \sqrt{G^2 m^5} \frac{1}{\sqrt{\Phi}}. \tag{9.150}$$

Finally, the equation of virial oscillations can be easily obtained in the known form

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}, \tag{9.151}$$

where $A = km = E$ is the total energy and B is a constant equal to $Gm^{5/2}$ multiplied by a factor that depends on the realization of the mass defect and on the period of the $\alpha\beta$ form factors (equal to $1/\sqrt{2}$).

When $p = \rho/3$, the equation of virial oscillations for radiation can be obtained from Eq. (9.146):

$$\ddot{\Phi} = -A.$$

Equations (9.144) and (9.145) are valid for all natural systems that exhibit a central symmetry of mass distribution. For celestial bodies, Eq. (9.146) is written as

$$\rho R^3 = \frac{3m}{4\pi}.$$

Then, from (9.146), it follows that

$$\left(\ddot{R}\right)^2 + 2k = \frac{8\pi}{3} G R^2 \left(1 - \frac{3p}{\rho}\right) = \frac{2Gm}{R}\left(1 - \frac{3p}{\rho}\right).$$

Now, Eq. (9.151) becomes

$$\ddot{\Phi} = -A + \frac{B}{\sqrt{\Phi}}\left(1 - \frac{3p}{\rho}\right). \tag{9.152}$$

As Weinberg (1972) pointed out, the inequality $0 < 3p \leq \rho$ holds for celestial bodies, and in the most general case, we can write

$$p = (\gamma - 1)(\rho - n\mu),$$

where n is the density of particles and μ is the mass of a particle.

Therefore, $(\rho - n\mu)$ is the mass defect and γ is the polytropic index, which for stable system ranges from 0 to 5/3 for nonrelativistic objects and $\gamma \geq 4/3$ for ultra-relativistic objects. For $\gamma > 5/3$, the body expands indefinitely, and at $\gamma \leq 4/3$, collapse of the body occurs.

For actually existing celestial bodies, where the absence of heat equilibrium is taken into account (in the case of a discrete system), pressure is defined as (Weinberg 1972)

$$p = \frac{1}{3}\left[\rho + f(\rho, n)\right],$$

where $f(\rho, n) = T_\alpha^0$ is a function of the energy density ρ and the density n (number of particles per unit volume). This function is equal to zero in the ultra-relativistic limit, and in the nonrelativistic limit, it is equal to

$$[-n\mu + (\rho - n\mu)] = -2n\mu + \rho.$$

In both limiting cases, pressure p is

$$p = \frac{1}{3}\rho \quad \text{and} \quad p = \frac{2}{3}(\rho - n\mu).$$

Hence, in Eq. (9.152), the undetermined factor in B is equal to zero and $[(2\mu n/\rho) - 1)$ or $(1 - (2\Delta/\rho)]$, where $\Delta = \mu n - \rho$ is the mass defect.

Finally, taking into account the mass defect in Eq. (9.152) shows that the constant $B = B_0 D$, where B_0 is of Newtonian nature ($aGm^{5/2}$) and D, a relativistic correction, is smaller than 1.

Now let us estimate this correction D in the case of the white dwarf and the neutron star models according to Weinberg.

The equation determining the density of particles when Fermi–Dirac statistics hold can be written as

$$n = \frac{k_{\mathrm{F}}^3}{3\pi\hbar^2},$$

where n is the number of particles in the volume, k_{F} the radius of the Fermi sphere, and $\hbar$ is Planck's constant.

The density of matter of a star is written as

$$\rho = n\mu_{\mathrm{p}}n_{\mathrm{p}},$$

where μ_{p} is the mass of a proton and n_{p} the average number of protons in a nuclei.

The critical density of matter in a star is

$$\rho_{\mathrm{cr}} = \frac{\mu_{\mathrm{p}}n_{\mathrm{p}}\mu_{\mathrm{e}}^3}{3\pi\hbar^3},$$

where μ_{e} is the electron mass.

Introducing the new variables $Z_1 = \rho/\rho_{\mathrm{cr}}$ and $Z_2 = \rho/\rho_{\mathrm{cr}}$, the equation of state for white dwarfs can be rewritten as follows:

$$Z_1 = \frac{3\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}} F_1(Z_1),$$

$$Z_2 = \frac{3\mu_{\mathrm{e}}}{\mu_{\mathrm{p}}} F_2(Z_2),$$

where F_1 and F_2 are some transcendental functions.

For neutron stars, the critical density is

$$\rho_{cr} = \frac{\mu_{\mathrm{p}}^4}{3\pi\hbar^3},$$

and the equations of state are written as

$$Z_1 = 3F_1(Z_1),$$

$$Z_2 = 3F_2(Z_2).$$

Solving the equations of state for the two limiting cases when $\rho \ll \rho_{cr}$ (i.e., when the polytropic indexes are 5/3 and 4/3, respectively), we obtain for white dwarfs, respectively

$$\rho_e = \frac{3}{2}p \quad \text{and} \quad \rho_e = 3p.$$

For neutron stars, in the limiting cases ($\rho \ll \rho_{cr}$) and ($\rho \gg \rho_{cr}$), we have the same form of relations:

$$\rho = \frac{3}{2}p \quad \text{and} \quad \rho = 3p,$$

where ρ is the total density of matter.

In the ultra-relativistic limit, the relativistic correction will have very large values ($D = 0$), which means that the total collapse of the star (Oppenheimer–Volkoff limit) is leading to the formation of a black hole.

Thus, we have obtained the equation of virial oscillations (9.152) directly in the most general case and without having to assume the constancy of the form factor product $\alpha\beta$. Since the same equation follows from Jacobi's equation with the use of the hypothesis, we conclude that the relation $\alpha\beta =$ constant was proven.

We should also note that modern astrophysical studies of the oscillation of celestial bodies in the nonrelativistic approximation are based on the supposition that these movements have a homologous structure (Misner et al. 1973; Weinberg 1972; Frank-Kamenetsky 1959; Zeldovich and Novikov 1967). It can easily be verified that the supposition of homology is a sufficient condition to prove the constancy of the form factor product $\alpha\beta$, which is the main point in the derivation of the equation of virial oscillations from Jacobi's equation.

The mathematical formulation of the homologous motion of matter in the course of oscillation of a celestial body is written as follows:

$$r(t) = t(0) \cdot f(t),$$

where $r(t)$ is the radius of a given layer shell of the body and $f(t)$ is an arbitrary function of time.

Let us introduce the Lagrange coordinates, where m is the mass inside the sphere of radius r and dm is the mass of shell of radius r and thickness dr. According to the property of Lagrange coordinates, they are independent of time.

Then, the Jacobi function and the potential energy are written as

$$\Phi = \frac{1}{2}\int_0^m r^2\, \mathrm{d}m,$$

$$|U| = G\int_0^m \frac{m\, \mathrm{d}m}{r}.$$

Using the assumption that the motion is homologous, these expressions can be rewritten as

$$\Phi = \frac{1}{2} f^2(t)\int_0^m r^2(0)\, \mathrm{d}m,$$

$$U = \frac{G}{f(t)} G\int_0^m \frac{m\, \mathrm{d}m}{r(0)}.$$

Integrals on the right-hand side of these expressions do not depend on time and are therefore constants. Thus, the product $U^2\Phi$ does not depend on time and is also a constant.

Note that in the works of the authors mentioned above, the formula for the pulsation frequency of celestial bodies has been obtained assuming small amplitudes and the validity of the harmonic law of pulsations. Our approach allows the same frequency of pulsations to be obtained without the above restricting assumptions. Moreover, by comparing the two expressions that give equivalent results, it is possible to obtain the polytropic index that enters into the astrophysical formula for the frequency of pulsations.

References

Chandrasekhar S (1939) An introduction to the study of stellar structure. Chicago University Press, Chicago

Chandrasekhar S (1942) Principles of stellar dynamics. Chicago University Press, Chicago

Dive P (1931) Bull Soc Math France 59:128

Duboshin GN (1975) Celestial mechanics: the main problems and the methods. Nauka, Moscow

Ferronsky VI, Denisik SA, Ferronsky SV (1978) The solution of Jacobi's virial equation for celestial bodies. Celest Mech 18:113–140

Ferronsky VI, Denisik SA, Ferronsky SV (1984) Virial approach to solution of the problem of global oscillations of the earth atmosphere. Phys Atmos Ocean 20:802–809

Frank-Kamenetsky LA (1959) Physical processes in the interior of stars. Fizmatgiz, Moscow

Janke E, Emde F, Lösch F (1960) Tafeln höherer funktionen, 6 Aufgabe, Stuttgart

Kittel C, Knight WD, Ruderman MW (1965) Mechanics, Berkeley physics course, vol 1. McGraw Hill, New York

Landau LD, Lifshitz EM (1973) Field theory. Nauka, Moscow

Misner CW, Thorne KS, Wheller JA (1975) Gravitation. Freeman, San Francisco

Weinberg S (1972) Gravitation and cosmology. Wiley, New York

Wintner A (1941) The analytical foundations of celestial mechanics. Princeton University Press, Princeton

Zeldovich YB, Novikov ID (1967) Relativistic astrophysics. Nauka, Moscow

Chapter 10
Conclusions

Let us summarize the obtained results of the work.

The discovered nature of the planets' and satellites' orbital motion by the first cosmic velocity of their protoparents proves the correctness of initial fundamentals for the dynamics of the natural systems based on dynamical equilibrium. The conditions of hydrostatic equilibrium used earlier for such a problem appear to be incorrect. This might be the reason why the nature of Newton's gravity force has not yet been disclosed. The solution of this problem is ordinary. The force of gravitation is the first derivative in time from the inner energy of the interacted elementary matter particles of the system. The energy itself, being the measure of this interaction, in general case, is the second derivative in time from the moment of inertia of the system. In the case of uniform system, the energy is equal to the first derivative in time from the moment of inertia.

The only change of the force as the measure of interaction by the energy brings together the nature of the gravity and electromagnetic interactions. If one takes into account the inner and outer force fields, then both interactions become equivalent in their capacity.

It was found that the process of the planets' and satellites' creation by separation from the parental bodies is coupled with conditions of the universe expansion. The hierarchic subsystems appear only in these conditions. The inner energy of the parental bodies is only released in the expansion conditions. In reality, the process of a subsystem creation is only a part of a more general process of decay of a system by elementary particles accompanied by the release of the bonded energy. The main points of dynamical effects of this process are self-gravitation (weightness) and weightlessness. The process of the universe attraction appears to be the next stage of its evolution. If the law of energy conservation is held, then this stage must come. The process starts with the creation of mass particles including electrons, nuclei of atoms, and their isotopes by synthesis of the elementary weightlessness scalar particles. The main point of the dynamic effect of this process should be the simultaneous collision of n elementary particles with creation of a mass particle. The synthesis of the scalar elementary particles will be accompanied by the absorption

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4_10,

of energy from the force field for binding the particles. The hydrostatic pressure also takes part in the process of attraction of the oscillating system in the framework of the Archimedes law.

It can be noted that the problem of creation of the solar system bodies considered here, despite its apparent complexity, is a very simple and natural process. The difficulty lies in the perception of the new physical conception, based on the effect of the inner energy irradiation, which is in fact the observable phenomenon.

It might be surprising that the integral approach to the description of dynamics of natural systems, which has a number of obvious advantages, has been underdeveloped compared with the differential hydrostatic approach. However, if we consider the development of the apparatus of mathematical physics from this viewpoint, the picture changes completely.

In fact, as soon as the concept of the field was formulated—even though initially this concept was a purely mathematical one (e.g., of the electrostatic and magnetic fields)—Gauss's theorem relating to the flux of a field vector through a closed surface was put forward. This integral characteristic of a field enclosed within a surface is an invariant of the field. In the case of electrostatics, it is the charge that gives rise to the field.

The concept of vector flux through a closed surface has been generalized and developed. For example, such a generalization is Stokes's theorem relating to the circulation of a vector around a closed circuit, which can be used to identify vortex sources in vector fields. These theorems, which by their very nature are distinctly integral ones, have served as the basis for the whole mathematical theory of continuum mechanics, the electromagnetic theory of Maxwell, and Poisson's theory of Newtonian gravitation.

Thus, the development of the mathematical apparatus of physics has taken the course of the integral approach to the description of natural phenomena. The concepts of divergence and the rotor introduced in this connection have served as instruments for finding the sources and sinks of a field and its vortices.

However, the idea of the continuity of a field, which gave rise to these concepts, itself placed a limit on them because the size of the region in which the charge was enclosed by a surface had to tend to zero. The Gaussian surface integral was thus replaced by divergence as a differential operation.

Circulation was similarly replaced by the rotor as a differential operation. It is these operations that are used in the Maxwellian field theory. This is because of the erroneous idea that the electric charges giving rise to the field are themselves continuous quantities distributed over the volume and also over the surface of dielectrics and conductors. The theorems of Gauss and Stokes are therefore limited to volumes shrinking to nil, and the theory became a purely differential one. This situation was later improved by Lorentz, who introduced into the field discrete charge points of finite magnitude scattered in empty space. According to his theory, Maxwell's equations remain applicable in the empty space between the small regions enclosing point singularities. On the closed surfaces surrounding these regions containing field singularities, the solutions to the field equations satisfy the integral conditions. The flax of the field vector through these surfaces is equal to the sum of discrete charges enclosed by the total surface.

With the solution averaged over space, Lorentz's theory led to Maxwell's theory, which was in fact his objective. This is how the integral approach to the description of natural phenomena came into being.

The same approach was used by Einstein in the interpretation of his general theory of relativity and for deriving the equations of motion of matter in accordance with Newton's theory from his own equations.

It is, of course, well known that Einstein constructed his general theory of relativity as a relativistic theory of gravitation. For this, he first wrote Newton's equations in the form of field equations using Poisson's equation and then gave the latter a relativistic, generalized character.

Einstein went further and abandoned inertial counting system, which had been accorded a position of privilege. Thus, the invariance was no longer assumed to be Lorentzian but universal in relation to any improper continuous transformation. Here, use was also made of Lorentz's idea, which we have mentioned earlier, of the discrete nature of the distribution of matter. Matter is concentrated in point singularities of a field, and between them, there is empty space for which Einstein's field equations hold true. The equations are not satisfied at singular points, which must be surrounded by closed surfaces. For the latter, the integral relations of Gauss in turn hold true; that is, the flux of the field through these surfaces is equal to the charges found inside them. It should be emphasized once again that the actual fields inside these regions need not satisfy the conditions of Einstein's equations.

Einstein's theory is, therefore, by its very nature and because of the basis on which it is constructed, an integral one. This fact is not usually realized, which is why we draw attention to it. It is by this condition, which in mathematical terms amounts to the requirement that the divergence of the original tensor should become exactly nil, that the nature of Einstein's tensor is uniquely defined. Such a tensor is one, the divergence of which is twice the contracted Bianchi identity for the Riemann curvature tensor.

If all the singularities of a field are surrounded by small spheres, in the space between them, the field will everywhere be regular and its equations can be expanded in descending series in terms of the reciprocals of the velocities of light. Upon equating the coefficients in terms of the same powers, we obtain a series of equations. Every such system contains new quantities not found in the previous systems and is easily solved.

The motion of singularities (i.e., of particles) is determined by virtue of the fact that the left-hand sides of the systems of equations being solved satisfy four identities. The right-hand side of these equations must therefore also satisfy these identities or, with the singularities taken into account, the integral conditions. In the absence of singularities, these conditions are automatically satisfied and provide nothing new. But if they are present, they determine the equations of motion. Einstein followed all calculations through and obtained Newton's equations. This method can also be used when gravitational and magnetic fields exist simultaneously, and the result of the calculation is positive. In this way, Einstein showed that even the classical interaction of mass points is caused by the nonlinearity of the field equations. This fact is usually emphasized, but the role of integral conditions tends not to be mentioned.

Einstein's equations therefore contain Newton's equations and thus also their solutions and combinations.

Jacobi's virial equation is derived from Newton's equations and, consequently, must itself be contained in Einstein's equations. However, it is not immediately apparent whether Newton's or Jacobi's equation is the more fundamental. Newton's equations were obtained by Einstein from his second-order equations by approximation. Jacobi's equation was obtained from Einstein's by the method of oscillation moments, also in second-order but by an exact method. This makes Jacobi's equation the more fundamental one; moreover, unlike Newton's equation, it remains integral and dynamical in nature.

As we have mentioned, the way in which the whole problem is formulated gives Jacobi's moment equation an exact, closed form which in fact solves the problem itself. In the case of the universe, the problem is also one of its non-steady-state natures. A clever solution to this problem was found earlier by Friedmann. His solution is a solution to Jacobi's equation or to the smoothed Einstein's equation. This is an analogue of Maxwell's equation in the form of a smoothed Lorentzian equation for charge points.

For the empty space between point singularities, an anisotropic solution to Einstein's equation has been found (also by indirect means). This solution is Kasner's metric. Analysis of this metric shows the empty space being considered pulsates. It is compressed on two axes, expands on one, and vice versa. Since this solution has been obtained for the case of space without matter, that is, without its interaction, so that the law of interaction is without significance, the oscillatory nature of processes in nature is universal. The solution, however, is a formal one and its physical significance needs to be elucidated.

In fact, in Newton's well-known law of gravitation for two masses, it is assumed that these are mass points. Otherwise, the inverse square law ceases to apply to their interaction. This in turn contravenes the law of remote screening mentioned in Chap. 1, which makes it impossible for approximately isolated (conservative) systems to exist.

The law of gravitation thus permits the existence of infinitely small radii of curvature and thereby of an infinitely large curvature of space–time, that is, of singularities. There are other examples of motion toward or away from a singularity, such as the formation of stars and planets and the expansion of the universe. Newton's law of gravitation therefore non-explicitly reflects the conditions for the existence of singularities, and the generalization of his theory by Einstein retains and, on the basis of the principle of equivalence, clearly demonstrates these singularities.

Singularities are therefore an empirical fact. So what are they?

In accordance with Einstein's theory, curvature is produced by mass. Consequently, empty space–time is not abstract emptiness but a physical vacuum with its own structure and also an analogue of mass, which in fact reflects Kasner's solution to Einstein's equation. This view is now widely held. In most models, a vacuum is considered to be a quantum mechanical system of virtual particles and to behave in a way similar to an elastic medium. Belinsky et al. (1970) studied the behavior of Einstein's equation for nonempty space–time but near a singularity. They showed that with increasing proximity to (distance from) the singularity, a moment is approached at which the vacuum curvature exceeds the curvature from matter, and the solution to Einstein's equation again becomes Kasner's solution.

Its solution, however, is a case of uniform—although anisotropic—space–time. Belinsky, Lifshitz, and Khalatnikov also examined the case of inhomogeneous space–time and came to a conclusion that the nature of the solution was the same but that the Kasner parameters were dependent on the coordinates and time.

In the case of further evolution of Kasner's solution with the expansion of space away from the singularity, the original anisotropic space gradually converted into isotropic space, that is, into the Friedmann model, which is a solution to the second-order virial equation.

The oscillatory law of the dynamics of natural processes is thus a universal law of nature. It should, however, be noted that into all the approaches mentioned above, the concept of finite time and of a beginning of time counting has been introduced. In some models, there is also the concept of the end of the world. Only in one of them (in which the average density of matter for the space being considered is strictly determined) do the periodically alternating processes of expansion and contraction infinitely. It is this mode that is determined by the solution to Jacobi's virial equation.

A special feature of Kasner's solution for the general anisotropic case of space–time is the appearance of dependence of metric coefficients of time in it in accordance with the $|t|^{2/3}$ law, where t is a time interval. This law was found for the most general case in which there is no external symmetry, that is, no symmetry that is not only associated with the internal arrangement of singularities.

The sources of the important relation $|t|^{2/3}$ go back to Kepler, who found the law experimentally in accordance with which the squares of the periods of rotation of bodies of the solar system are the cubes of the semi-axes of the ellipses in which they undergo motion.

It was pointed out in Chap. 6 that in Newton's theory about the attraction of mass points, such a law is also found to be asymptotic for the case in which n-bodies collide simultaneously. It was also shown there that within this asymptotic limit, the simultaneous collision of n-bodies leads to a homologous configuration. And, in turn, the condition of the applicability of Jacobi's general virial equation with two functions holds true. Thus, using a solution of the Kasner type, the applicability of Jacobi's virial equation within the asymptotic limit of simultaneous collision between n-bodies that was found earlier for Newton's theory is extended to the case of the solution of Einstein's general equation. This indicates the universal nature of Jacobi's virial equation in dynamics.

Let us note a further important aspect of the solutions under consideration, which relates to the change of the Kasner epochs. Their number is infinitely independent of whether the world has a beginning or an end. This occurs as a result of a decrease in the duration of an individual epoch as a singularity is approached.

Let us now consider yet another aspect of the fundamental nature of Jacobi's virial equation. As we have already pointed out, Newton's law of gravitation permits the existence of a curvature in space–time, which is derived from Einstein's theory. However, there is one fundamental difference between the two theories. According to Newton, the gravitational interaction is a long-range one, corresponding to an infinite velocity of propagation of the interaction. Einstein assumes a short-range interaction. It is propagated at finite velocity (at the velocity of light). Consequently, Newton's theory is formulated in terms of Euclidian geometry. Nevertheless, with both theories, space–time is distorted.

Newton's theory is constructed on the basis of a simple empirical law of Kepler and does not make use of another empirical law, namely, the principle of equivalence derived from the experiments of Eötvös.

So what common ground is there between the theories?

The fact is that Newton's theory is constructed as Newtonian mechanics plus his own law of gravitation. In Newtonian mechanics, there are three axioms, but the type of interaction is not determined; this is done experimentally. In generalized Newton's theory, it is the mechanics that should have been generalized and not the type of interaction.

With Einstein, the type of interaction is replaced by the principle of equivalence. The mechanics, on the other hand, is generalized in accordance with the principle of the invariance of equations. Long-range interaction is thus not involved here, and the type of interaction makes no difference.

Jacobi's virial equation, which was obtained from Newton's equations, also does not so much generalize the type of interaction law, in the way that this was done in his (Jacobi's) conclusions, as taken into account the mass defect (potential energy). It is therefore linked with the principle of equivalence. The mass defect, in turn, is determined by a system that has already been formed and, consequently, does not depend on the type of interaction during the process of formation (long range or short range).

As was thought by Wintner, Jacobi's virial equation therefore reflects the type of interaction law only integrally over the whole period of time in which the mass defect is formed. Also, if there is no delay, as in the case of Newton's long-range law of gravitation, it will be simultaneously a specific and instantaneous type of interaction, as pointed out by Wintner.

If a delay does take place, for example, in accordance with Einstein's short-range interaction law, instead of a specific, instantaneous type of interaction, the equation will include an expression that has been strongly averaged over time, and the dependence on the type of interaction will cease to be of significance. It will be replaced by an assertion about the dependence on instantaneous mass or on the mass defect that has built up over a long time.

This is the answer to the question posed. At the same time, the strength of Jacobi's equation is evident. Since in the general theory of relativity the usual problems in the framework of a short time interval—and even the classical two-body problem—are not solved, the enormous practical significance of solving Jacobi's virial equation becomes obvious. The fact that there are oscillations even in empty space–time indicates the exclusively fundamental nature of this equation.

Moreover, it has now become obvious that Jacobi's virial equation, which was obtained from Newton's equations, is a particular case of more general virial equation derived from Einstein's equations. This equation will thus be studied from the most general global points of view, namely, that of empty space–time, which will not be called a vacuum, and that of models of an evolving universe. It should be noted here that the models that have so far been developed from Jacobi's equation of an open, a closed, and a pulsating universe have been obtained automatically as its natural solutions as a function of the source data—the quantities of total moment and mass defect. In this case, all possible types of solution are encompassed, and the question of the completeness of the set of possible models of the universe is thereby solved.

Let us now consider an example that demonstrates the use of the integral approach for constructing a complete closed theory based on Hooke's law. The concerned theory is the theory of elasticity.

In this theory, for any volume of a continuum, only quantities and parameters that are integral from the point of view of an external observer are considered, namely, deformation, stress, and modulus of elasticity. The elements of the volume interact through their surface. A quantitative measure of their interaction is provided by strains and a quantitative measure of the results of interaction by relative changes in the external dimensions of elements, in other words, their deformation. The internal structure of the material is demonstrated quantitatively by means of integral parameters, namely, the mass density, the modulus of elasticity, and Poisson's coefficient.

The interaction between the element of interest of a body and the external world takes place through external surface and volume forces. The external surface forces act only on the surfaces of the whole body and not on that of any of its elements. External volume forces amount to the application of surface forces to the surfaces of any element and, thereby, to tensions. Here, external surface forces do not come into the equilibrium equations but into the boundary conditions of the problem and are thus excluded as forces.

It is important to stress this point. It was mentioned earlier by Hertz, who set himself the problem of constructing a system of mechanics without forces. The fact that he was relatively unsuccessful is because in his days, Minkowski's idea about the unity of space–time was as yet unknown. The link between static and dynamics was not as clear as it would be after Minkowski.

Hooke's theory is a strictly linear one. The two states of object it considers are the initial and the final states before and after the application of the forces. One of them is generally the equilibrium state. If these two states of one and the same system occur at different times, displacement deformations are replaced by velocity deformations. In this case, the approach followed takes the form of the theory of viscous or liquid media of gases.

For a fluid, Hooke's law is written in the form of Pascal's law. In this way of writing, it expresses the condition of equilibrium of the medium, where the stresses on the main axes are equal to the pressure of the fluid. Another condition of equilibrium for a fluid is the law of the conservation of matter.

If, in the context of Hooke's law, we move to the point of view of Minkowskian unified space–time and effect a Lorentzian transformation from a stationary system of coordinates to a moving one, the equilibrium conditions in accordance with Pascal's law or in the form of any other Hookean tension law can be expressed as Euler's equations and as an equation of the continuity of the medium. Here, it is important to note that, when deriving Euler's equations of motion, it is not obligatory to use Newton's second law of mechanics and that a Hookean system equilibrium equation can be used.

Nor are any dynamic laws used to justify the Minkowski approach, which is based directly on experimental values and is considered to be valid.

It should be noted that, in the context of Hooke's law, a rigorous solution can be found to Jacobi's virial equation for conservative system. In this case, Hooke's law determines the constancy of the product of potential energy and of the Jacobi function; this constancy is written in the form $|U|\,\sqrt{\Phi} = aGm^{5/2}$.

In this relation, the coefficient $a = \alpha\beta$ (which stands for the product of form factors included in the expressions for the potential energy and the Jacobi function) acts as a modulus of the dynamic elasticity of the system. It remains a constant and reflects the constancy of the law of mass density distribution of the system within the limits of its elastic deformations with virial oscillations. The deformation of the system is characterized by its integral parameter, the Jacobi function, and the stresses are determined by the term $Gm^{5/2}/U$. As a result, the virial pulses of the system will be strictly periodic, and the deformations will be found to be elastic and therefore reversible.

On this basis, it was shown in Chap. 7 that the parameters of the virial oscillations of the Earth, which are detected, can be used as if the Earth were an elastic body for determining its potential energy. This option remains open for when natural systems are being examined in the framework of other models of continuous and discrete media.

We have mentioned a number of aspects of the universality of Jacobi dynamics in the examination of natural systems. We shall now consider the prospects for solving a number of practical problems in the context of the dynamic approach.

One of these problems is that of the dynamics of the solar system, of its evolution, and of its origin. In Chaps. 1, 6, and 7, we made a first step and obtained the basic common solution on the creation of celestial bodies and their systems. It appears that any rank of new celestial body (from galaxy to meteorite and even to molecule and atom) is born by self-gravitating parent in consequence of loss of its energy by radiation. It means that the stage of self-gravitation and separation must be changed by the stage of gravitation and joining of the matter. Thus, the present-day stage of the expansion of the universe after total separation of the matter should come to the stage of its contraction and gathering. Generally saying, our universe is a closed pulsating and perpetual system. New more detailed solutions in this direction are desirable.

The problem of the dynamics of bodies in the solar system is a traditional practical problem of classical celestial mechanics. The key to it is solution of the two-body problem, one of the bodies being the Sun, while the other is one of the planets of the system. The influence of other planets is taken into account by using the methods of perturbation theory.

In the context of the dynamic approach, a new problem of the dynamics of the self-gravitating Earth and its interaction with the Sun and the Moon were considered in Chap. 7. The found normal and tangential components of the potential and kinetic energies of a self-gravitating body made it possible to understand the mechanism of separation of the body's shells, their oscillation, and rotation by the inner force field. It was understood that the induced outer force field, which has all the properties of the electromagnetic field, acquires the property to conserve the irradiated energy and potential in the orbital motion of its secondary body. But because of the limited velocity of propagation of the changing potential, the orbital trajectory is found to be open. This fact is proved both by the artificial satellites and by the observed precession of all the planets and the moons. The found important effect makes it possible to interpret inner structure of the Sun, the Earth, the Moon, and other celestial bodies. And also, it raises the problem of improving Kepler's approximation of the Earth's and other body's orbits, which are found to be too rough (Severny 1988).

In our opinion, the dynamics of the microcosm is a very interesting field for the application of Jacobi dynamics. This book takes only the first step in this direction. It is shown that Jacobi dynamics is also applicable for the solution of this type of problem. An attractive idea is to use the dynamic approach for studying the physics of molecules, atoms, and nuclei as dissipative systems, which might lead to the discovery of many interesting effects.

References

Belinsky VA, Lifshitz EM, Khalatnikov IM (1970) Usp Fiz Nauk 102:463

Severny AB (1988) Some problems on physics of the sun. Nauka, Moscow

Index

V.I. Ferronsky and S.V. Ferronsky, *Formation of the Solar System: A New Theory of the Creation and Decay of the Celestial Bodies*, DOI 10.1007/978-94-007-5908-4,