

# Chapter 3

## Introduction to Data Envelopment Analysis

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**Abstract** This chapter introduces the basics of data envelopment analysis techniques, with a short historical introduction and examples of the constant returns to scale model (CRS) and the variable returns to scale (VRS) model. The ratio models are linearized and for both orientations primal and dual models are presented.

**Keywords** Data envelopment analysis • Efficiency evaluation

### 3.1 Introduction

*Data envelopment analysis* (DEA) is a mathematical programming-based technique to evaluate the relative performance of organisations. While the main applications have been in the evaluation of not-for-profit organisations, the technique can be successfully applied to other situations competing with other techniques as cost benefit analysis and multi criteria decision making as can be seen, for instance, in a recent study about the best choice for traffic planning, namely, the design and location of a highway in Memphis (Bougnol et al. 2005).

DEA is suited for this type of evaluation because it enables results to be compared making allowances for factors (Thanassoulis and Dunstan 1994). DEA makes it possible to identify efficient and inefficient units in a framework where results are

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considered in their particular context. In addition, DEA also provides information that enables the comparison of each inefficient unit with its “peer group”, that is, a group of efficient units that are identical with the units under analysis. These role-model units can then be studied in order to identify the success factors which other comparable units can attempt to follow. Thanassoulis (1993) argue that DEA is preferable to other methods, such as regression analysis, which also make it possible to contextualise results.

This chapter is structured as follows. The next section describes the development and fields of application of the technique, while Sect. 3.3 introduces the DEA models followed by a numerical and graphical example. Section 3.4 presents the mathematical formulation and the last one ends up with the main conclusions.

## 3.2 History and Applications of DEA

DEA is a mathematical programming technique presented in 1978 by Charnes et al. (1978), although its roots may be found as early as 1957 in Farrell’s seminal work (Farrell 1957) or even to Debreu’s, which introduced in the early fifties the “coefficient of resource utilisation” (Debreu 1951). It deserves special attention and also the work of the Dutch Nobel-prized Tjalling Koopmans and his “activity analysis concepts” (Koopmans 1951).

The DEA technique is usually introduced as a non-parametric one, but in fact it rests on the assumption of linearity (Chang and Guh 1991) and for the original constant returns to scale (CRS) models even in the more stringent assumption of proportionality.

Its application has been focused mainly on the efficiency assessment of not-for-profit organisations, since these cannot be evaluated on the basis of traditional economic and financial indicators used for commercial companies.

The first application of DEA was in the agriculture field; as a matter of fact, Farrell applied it to 1950 data of 48 states in the United States of America, considering 4 inputs and 2 outputs. At that time, the DEA term was not yet created, so in fact the first time the term DEA and that technique was applied was in the area of education, specifically in the analysis of Program Follow Through, conducted in the USA, in the late seventies (Rhodes 1978). Since then it has been used to assess efficiency in areas such as health (Wilson et al. 2012), county goals (Seiford and Zhu 2002), courts (Schneider 2005), universities (Bougnol et al. 2010) and many other not-for-profit sectors. Nowadays DEA can be seen to have spread to other fields such as transit (Chiu et al. 2011), mining, (Chen et al. 2010), air transportation (Pestana e Dieke 2007) and even banking (Emrouznejad and Anouze 2010).

In data envelopment analysis, the organisational units to be assessed should be relatively homogeneous and were originally termed decision-making units (DMUs). As the whole technique is based on comparison of each DMU with all the remaining ones, a considerable large set of units is necessary for the assessment to be meaningful. We will assume that each DMU produces  $N$  outputs by means of  $M$  inputs.

### 3.3 The Meaning of DEA Efficiency

We will introduce some simple examples based on the following data set with 4 variables: 2 inputs –  $X_1$  and  $X_2$  – and 2 outputs –  $Y_1$  and  $Y_2$ . Since this data will be used for pedagogical purposes, it is a small data set with just 12 decision-making units (DMU) (Table 3.1).

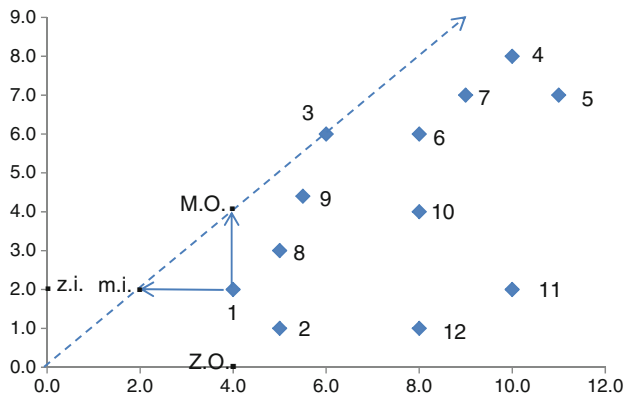
First, to introduce some basic concepts, we will suppose that only  $X_1$  and  $Y_1$  would be important for our analysis.

In this case, we can graph the 12 observations on a scatter plot, and it would be obvious that the most efficient one will be the DMU 3 since a straight line originating at the (0;0) point towards DMU 3 has the higher slope than any of the remaining (Fig. 3.1).

The straight line originating at the (0;0) point towards DMU 3 is called the efficiency frontier and together with the  $X$  axis it defines a cone with its vertex

**Table 3.1** Illustrative data set

DMU	$X_1$	$X_2$	$Y_1$	$Y_2$
1	4,0	140,0	2,0	28,0
2	5,0	90,0	1,0	22,5
3	6,0	36,0	6,0	12,0
4	10,0	300,0	8,0	60,0
5	11,0	66,0	7,0	16,5
6	8,0	36,0	6,0	12,0
7	9,0	12,0	7,0	6,0
8	5,0	210,0	3,0	30,0
9	5,5	33,0	4,4	5,5
10	8,0	288,0	4,0	72,0
11	10,0	80,0	2,0	20,0
12	8,0	8,0	1,0	4,0



**Fig. 3.1** Example with 12 hypothetical farms consuming  $X_1$  and producing  $Y_1$

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	DMU	Score	$X_1\{I\}\{V\}$	$Y_1\{O\}\{V\}$	Benchmarks	$\{S\}X_1\{I\}$	$\{S\}Y_1\{O\}$
1	Farm 1	50,00%	1,00000	1,00000	3 (0,33)	0,00000	0,00000
2	Farm 2	20,00%	1,00000	1,00000	3 (0,17)	0,00000	0,00000
3	Farm 3	100,00%	1,00000	1,00000	11		
4	Farm 4	80,00%	1,00000	1,00000	3 (1,33)	0,00000	0,00000
5	Farm 5	63,64%	1,00000	1,00000	3 (1,17)	0,00000	0,00000
6	Farm 6	75,00%	1,00000	1,00000	3 (1,00)	0,00000	0,00000
7	Farm 7	77,78%	1,00000	1,00000	3 (1,17)	0,00000	0,00000
8	Farm 8	60,00%	1,00000	1,00000	3 (0,50)	0,00000	0,00000
9	Farm 9	80,00%	1,00000	1,00000	3 (0,73)	0,00000	0,00000
10	Farm 10	50,00%	1,00000	1,00000	3 (0,67)	0,00000	0,00000
11	Farm 11	20,00%	1,00000	1,00000	3 (0,33)	0,00000	0,00000
12	Farm 12	12,50%	1,00000	1,00000	3 (0,17)	0,00000	0,00000

**Fig. 3.2** Example of EMS results under constant returns to scale minimisation of inputs

at the origin. This cone is called the production possibility set, since it contains all real data, and according to DEA axioms, only points inside this cone correspond to possible working conditions based on best-achieved performance.

We will analyse in greater detail unit 1. There are two ways for DMU 1 to reach efficiency: increasing output, till it reaches M.O. (maximisation of output) or reducing the input till m.i. (minimisation of input).

The actual value for efficiency is defined as the ratio between the distances:  $d(m.i. - z.i.) / d(1 - z.i.) = (2 - 0) / (4 - 0) = 50\%$ . On a similar way, the actual value for inefficiency is defined as the ratio between the distances:  $d(M.O. - Z.O.) / d(1 - Z.O.) = (4 - 0) / (2 - 0) = 100\%$ . In these calculations, z.i. means the zero input point and Z.O. the zero output point for the DMU 1.

We conclude that in DEA, there are two different means of optimization related with radial measures of score: input minimisation and output maximisation. The artificial points M.O. and m.i. are termed targets or composite points for DMU 1. Point 3 is the only efficient one, and it is the peer for all remaining points.

Under constant returns to scale, efficiency is the reciprocal of inefficiency; the peer set is also the same, regardless of the orientation, although their targets are different as we conclude from the exposition above.

Typical DEA software like efficiency measurement system (EMS), developed by Holger Scheel (Scheel 2000) at Dortmund University (using Csaba Mészáros' BPMPD interior-point solver) would give us results as depicted in Fig. 3.2.

EMS highlights the efficient units and provides us all the necessary values:

- Score means efficiency.
- $X_1\{I\}\{V\}$  presents the virtual input of variable  $X_1$ , in a similar way  $Y_1\{O\}\{V\}$  presents the virtual output of variable  $Y_1$ .

	DMU	Score	$X_1\{I\}\{V\}$	$Y_1\{O\}\{V\}$	Benchmarks	$\{S\}X_1\{I\}$	$\{S\}Y_1\{O\}$
1	Farm 1	200.00%	1.00000	1.00000	3 (0.67)	0.00000	0.00000
2	Farm 2	500.00%	1.00000	1.00000	3 (0.83)	0.00000	0.00000
3	Farm 3	100.00%	1.00000	1.00000	11		
4	Farm 4	125.00%	1.00000	1.00000	3 (1.67)	0.00000	0.00000
5	Farm 5	157.14%	1.00000	1.00000	3 (1.83)	0.00000	0.00000
6	Farm 6	133.33%	1.00000	1.00000	3 (1.33)	0.00000	0.00000
7	Farm 7	128.57%	1.00000	1.00000	3 (1.50)	0.00000	0.00000
8	Farm 8	166.67%	1.00000	1.00000	3 (0.83)	0.00000	0.00000
9	Farm 9	125.00%	1.00000	1.00000	3 (0.92)	0.00000	0.00000
10	Farm 10	200.00%	1.00000	1.00000	3 (1.33)	0.00000	0.00000
11	Farm 11	500.00%	1.00000	1.00000	3 (1.67)	0.00000	0.00000
12	Farm 12	800.00%	1.00000	1.00000	3 (1.33)	0.00000	0.00000

Fig. 3.3 Example of EMS results under constant returns to scale maximisation of outputs

- *Benchmarks* means that all farms, except for the third, are benchmarked against farm 3 and so is used 11 times for comparison. The coefficients in this column mean that if we multiply farm 3 by those values, it will result in the composite-projected target point.
- $\{S\}X_1\{I\}$  presents the slacks for input  $X_1$ , in a similar way  $\{S\}Y_1\{O\}$  presents the slacks for output  $Y_1$ .

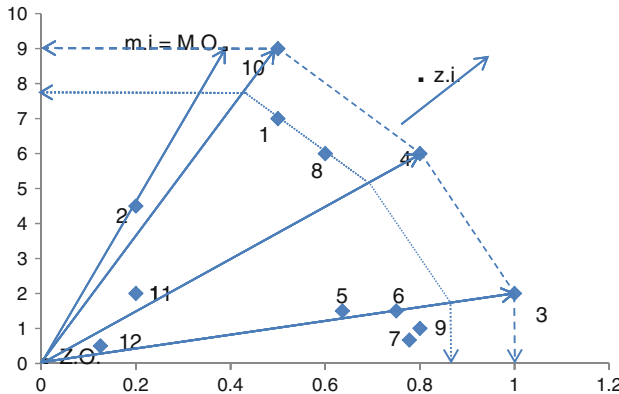
Thinking again around unit 1, it means that its target under input minimisation is  $0,33 \times (6;6) = (2;2)$ . These are exactly the coordinates of point m.i. in Fig. 3.1. The concepts of virtual multipliers and slacks will be clarified after the presentation of the mathematical formulation.

So far it is very important to emphasise that there are two different radial orientations: input minimisation and output maximisation. There are target points and a set of efficient units that benchmark the non-efficient ones.

The results for output maximisation are depicted on Fig. 3.3.

We can confirm that the conclusions are very similar to those from Fig. 3.2; more specifically the score now is the inefficiency, the reciprocal of the scores under input minimisation. Slacks are identically null as in the minimisation of inputs.

We can proceed with our analysis including output  $Y_2$  in our analysis. In this case we will have 3 variables, and we cannot represent them on the  $XY$  plane. Anyway, since we assume that constant returns to scale prevail, we can normalise the outputs by the single input  $X_1$ . That way, we get the Fig. 3.4, where we can easily spot three efficient farms: Farm 3 remains efficient (whenever we add new variables, efficiencies never decrease), and farms 4 and 10 turn out efficient, joining the new efficiency frontier.



**Fig. 3.4** Cut of the PPS at  $X_1 = 1$ , under CRS, two outputs:  $Y_1$  and  $Y_2$

We point out in Fig. 3.4 the zero output point (Z.O.) at the origin and the zero input area that lies outside the production possibility space (PPS). The PPS is now a polyhedral cone (an inverted pyramid) with 3 specific edges: the rays that depart from the origin to each of the 3 efficient units (3;4;10). There are other 3 edges that are not specific, since they exist always regardless of the specific data set; one belongs to the plane defined by  $X = 0$ , the other belongs to the plane with  $Y = 0$  and finally the third is the Z axis with both  $X = 0$  and  $Y = 0$  (Fig. 3.4).

The graph in Fig. 3.4 is a cut of the PPS at  $X_1 = 1$ , since we normalised the outputs. The dashed line is the intersection of the horizontal plane  $X_1 = 1$  with the efficiency frontier (the polyhedral cone). The dotted line is an isoefficiency contour.

It is clear that DEA clusters DMU according to their specificities; farms 1 and 8 are related to farms 4 and 10, their peer group. The same way farm 6 will be similar to farm 3, and so on. A different situation arises with farm 2, which cannot be expressed as a finite linear positive combination of the efficient ones; in that case, it is necessary to have a slack in  $Y_1$ . The same situation happens with farms 7 and 9, they “need” slacks in  $Y_2$ , and this means that after their projection on the vertical part of the dashed line, they still have to increase its production of  $Y_2$ . This can be confirmed by the EMS results for this case. Typically this kind of figure is associated with input minimisation, since we have only one input to reduce. It can be associated also to output maximisation, where an equiproportional maximisation of both outputs (keeping the output mix constant) is required.

Finally, we examine the case for two inputs and a single output  $Y_2$ . This time we cannot anticipate the results, since we did not study efficiencies for pairs of these three variables. Again, since we assume that constant returns to scale prevail, we can normalise the inputs by the single output  $Y_2$ . That way we get Fig. 3.5 where we plotted  $X_2/Y_2$  versus  $X_1/Y_2$  where we can easily spot three efficient farms:

The PPS is again a polyhedral cone (an inverted unbounded pyramid) with 3 specific edges: the rays that depart from the origin to each of the 3 efficient units (3;7;10). There are no other edges since the inefficient units are located in an unbounded region that tends to infinity as outputs approach zero.

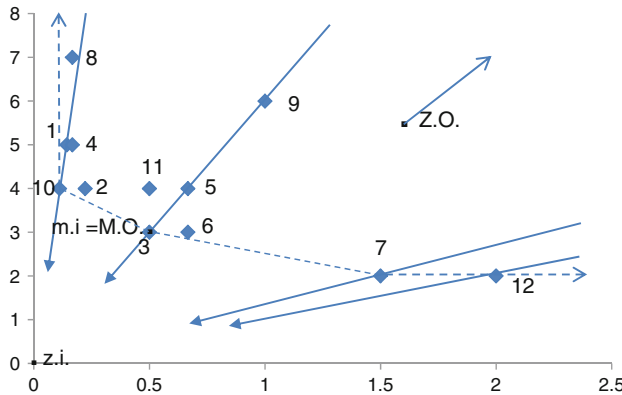
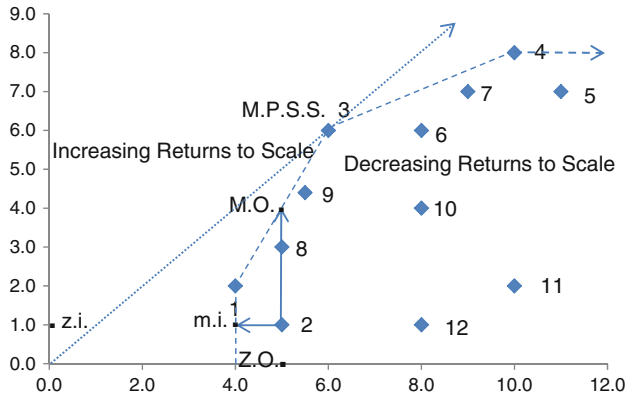


Fig. 3.5 Cut of the PPS at  $Y_2 = 1$ , under CRS, two inputs:  $X_1$  and  $X_2$

	DMU	Score	$X_1\{I\}\{V\}$	$X_2\{I\}\{V\}$	$Y_2\{O\}\{V\}$	Benchmarks	$\{S\}X_1\{I\}$	$\{S\}X_2\{I\}$	$\{S\}Y_2\{O\}$
1	Farm 1	125.24%	0.06844	0.93156	1.00000	3 (0.02) 10 (0.48)	0.00000	0.00000	0.00000
2	Farm 2	106.67%	0.12500	0.87500	1.00000	3 (0.50) 10 (0.25)	0.00000	0.00000	0.00000
3	Farm 3	100.00%	0.23077	0.76923	1.00000	7			
4	Farm 4	126.67%	0.07895	0.92105	1.00000	3 (0.33) 10 (1.00)	0.00000	0.00000	0.00000
5	Farm 5	133.33%	0.21751	0.78249	1.00000	3 (1.83)	0.00000	0.00000	0.00000
6	Farm 6	104.76%	0.18182	0.81818	1.00000	3 (0.90) 7 (0.29)	0.00000	0.00000	0.00000
7	Farm 7	100.00%	0.33333	0.66667	1.00000	2			
8	Farm 8	150.00%	1.00000	0.00000	1.00000	10 (0.62)	0.00000	30.00000	0.00000
9	Farm 9	200.00%	0.22861	0.77139	1.00000	3 (0.92)	0.00000	0.00000	0.00000
10	Farm 10	100.00%	1.00000	0.00000	1.00000	5			
11	Farm 11	123.33%	0.24324	0.75676	1.00000	3 (1.56) 10 (0.08)	0.00000	0.00000	0.00000
12	Farm 12	100.00%	0.00000	1.00000	1.00000	7 (0.67)	2.00000	0.00000	0.00000

Fig. 3.6 EMS results for the two inputs,  $X_1$  and  $X_2$ , and a single output,  $Y_2$

Farm 12 is not strongly efficient, since farm 7 has the same ratio on  $X_2/Y_2$ , but a smaller one on  $X_1/Y_2$ . Indeed there is a slack on  $X_1$ . We can see in the graph that the two points differ 0.5 on  $X_1/Y_2$ . This happens at the cutting plane  $Y_2 = 1$ , as the true scale of operation of farm 12 is  $Y_2$ . The slack is  $0.5 \times 4 = 2$  as can be seen in Fig. 3.6 where we present the results for this case. We can notice that the only farms that EMS classifies as efficient are farms 3, 7 and 10, those that are on shadowed lines of the table. Farm 12 presents a score of 100%, although it is not efficient since it has 2-unit slack in  $X_1$ ; its efficiency according to the Charnes Cooper and Rhodes model (CCR model) is  $100\% - 2\varepsilon$  where  $\varepsilon$  is a small non-Archimedean entity. These units that are on the “horizontal or vertical” parts of the efficiency frontier are termed “weakly efficient DMUs”; we will come back to this matter when we present the models in the remaining of this section.



**Fig. 3.7** Comparison of efficiency frontiers of the CCR and BCC models

So far we have assumed we are operating in a constant returns to scale situation. We will now present only one graphical example for the variable returns to scale (VRS) model, introduced by Banker, Charnes and Cooper in 1984 (Banker et al. 1984).

We will study again the one input ( $X_1$ ) producing one single output ( $Y_1$ ) case, but according to Banker, Charnes and Cooper the efficiency frontier is not anymore a polyhedral cone, but instead the intersection of all the polyhedral cones, one for each DMU; this result is based on the convex hull of the all DMUs. In the CCR model, we had flexibility on the choice of the weights; now in the BCC model, we are free to choose also the origin of our data (the vertex of the polyhedral cone).

In Fig. 3.7, we have a vertical facet that ends at DMU 1; this facet means that we assume a fixed amount of input ( $X_1 = 4$ ) necessary just to start the business. DMU 2 under VRS will have a much higher efficiency than under CRS, especially in the minimisation of inputs case. DMU 2 is spending 5 units of input, and it was expected to spend just one in the CRS case; this leads to a CRS efficiency of  $1/5 = 20\%$ ; in the more benevolent BCC model, it was expected to spend 4 units, so its efficiency rises to  $4/5 = 80\%$ . The ratio between CRS efficiency and VRS efficiency is called scale efficiency and increases from the origin ( $X=0; Y=0$ ) till DMU 3 and then starts decreasing. The first region is the increasing returns to scale region, and the second is the decreasing returns to scale region. In higher dimensions it seems more difficult, but it is much easier than it seems at first look. The most productive scale size (MPSS) is simply the intersection between the VRS and the CRS efficiency frontiers. From the origin till the MPSS, we are in the increasing returns to scale region; from the MPSS till infinity, we are on the decreasing returns to scale region.

In CRS, we had a simple relation between the scores of input minimisation and output maximisation (they were reciprocals: efficiency =  $1/\text{inefficiency}$ ). Under VRS it does not happen anymore. In fact there is no clear relation between those two scores since the peer set most of the times differs from one orientation to the other. For instance, DMU 10 under input minimisation has DMU 1 and 3 as peers, but if we intend to maximise its output, the reference set is 3 and 4.



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	DMU	Score	X1{O}{V}	Y1{O}{V}	Benchmarks	{S}X1{O}	{S}Y1{O}
1	Farm 1	100.00%	1.00000	1.00000	6		
2	Farm 2	80.00%	1.00000	1.00000	1 (1.00)	0,00000	1,00000
3	Farm 3	100.00%	1.00000	1.00000	6		
4	Farm 4	100.00%	1.00000	1.00000	2		
5	Farm 5	72.73%	1.00000	1.00000	3 (0.50) 4 (0.50)	0,00000	0,00000
6	Farm 6	75.00%	1.00000	1.00000	3 (1.00)	0,00000	0,00000
7	Farm 7	88.89%	1.00000	1.00000	3 (0.50) 4 (0.50)	0,00000	0,00000
8	Farm 8	90.00%	1.00000	1.00000	1 (0.75) 3 (0.25)	0,00000	0,00000
9	Farm 9	94.55%	1.00000	1.00000	1 (0.40) 3 (0.60)	0,00000	0,00000
10	Farm 10	62.50%	1.00000	1.00000	1 (0.50) 3 (0.50)	0,00000	0,00000
11	Farm 11	40.00%	1.00000	1.00000	1 (1.00)	0,00000	0,00000
12	Farm 12	50.00%	1.00000	1.00000	1 (1.00)	0,00000	1,00000

Fig. 3.8 EMS results for the one input,  $X_1$ , and a single output,  $Y_1$ , under VRS

In Fig. 3.8, we present the results for the one input,  $X_1$ , and a single output,  $Y_1$ , under VRS and minimisation of inputs. We cannot represent graphically any 3 variables case under variable returns to scale, so now we have to introduce the mathematical formulation that relies on linear programming and duality.

### 3.4 Mathematical Formulation for DEA

In DEA, efficiency ( $Ef_a$ ) of a specific decision-making unit ( $DMU_a$ ) under analysis is defined as the ratio between a weighted sum of its  $s$  outputs  $Y_{ra}$  and a weighted sum of its  $m$  inputs  $X_{ia}$ , a natural extension of the concept of efficiency used in the fields of physics and engineering (Charnes et al. 1978).

$$Ef_a = \frac{\sum_{r=1}^s \mu_{ra} Y_{ra}}{\sum_{i=1}^m v_{ia} X_{ia}} \tag{3.1}$$

When assessing a set of  $J$  organisations, where  $X_{ik}$  stands for the  $i$ th input of the  $k$ th DMU, with a similar meaning for  $Y_{rk}$ , the weights  $\mu_{rk}$  and  $v_{ik}$ , in Eq. (3.1), are chosen for each  $DMU_j$ , under evaluation as those that maximise its efficiency as defined by  $Ef_a$ . Several constraints have to be added to the maximisation problem:

- The strict positivity (Charnes et al. 1978) of the weights  $\mu_{rk}$  and  $v_{ik}$ , (also known as virtual multipliers).

- For scaling purposes, all  $n$  DMUs under analysis must have efficiencies not exceeding an agreed value, typically one or 100%, as is usual in engineering definitions of efficiency.
- A third kind of restriction has to be included since otherwise this linear fractional program would yield an infinite number of solutions. In fact, if a set of weights  $\mu_{rk}$  and  $v_{ik}$  returns the optimal solution, so would  $k\mu_{rk}$  and  $kv_{ik}$ . Making the denominator in Eq. (3.1) equal to one or 100% circumvents this situation.

So, we have to solve the following linear programming maximisation problem for each one of the  $J$  DMUs under analysis:

$$\max \text{Ef}_a = \frac{\sum_{r=1}^s \mu_{ra} y_{ra}}{\sum_{i=1}^m v_{ia} x_{ia}} \quad (3.2)$$

$$\text{s.t. } \mu_{ra} \geq \varepsilon > 0 \quad r = 1 \dots s \quad (3.3)$$

$$v_{ia} \geq \varepsilon > 0 \quad i = 1 \dots m \quad (3.4)$$

$$\text{Ef}_k = \frac{\sum_{r=1}^s \mu_{rk} y_{rk}}{\sum_{i=1}^m v_{ik} x_{ik}} \leq 1 \quad k = 1 \dots n \quad (3.5)$$

$$\sum_{i=1}^m v_{ia} x_{ia} = 1 \quad (3.6)$$

This fractional linear program can be solved by the Charnes and Cooper transformation (Charnes and Cooper 1962) which yield the following linear program:

$$\text{Max Ef}_a = \sum_{r=1}^s \mu_{ra} y_{ra} \quad (3.7)$$

s.t.

$$\sum_{i=1}^m v_{ia} x_{ia} = 1 \quad (3.8)$$

$$\sum_{i=1}^m v_{ia} x_{ik} - \sum_{r=1}^s \mu_{ra} y_{rk} \leq 0 \quad j = 1 \dots J \quad (3.9)$$

$$\mu_{ra} \geq \varepsilon > 0 \quad r = 1 \dots s \quad (3.10)$$

$$v_{ia} \geq \varepsilon > 0 \quad i = 1 \dots m \quad (3.11)$$

The problem above is known as the multiplier problem, since its unknowns are the weights, which are usually lower bounded by a small quantity non-Archimedean

entity,  $\varepsilon$  (Eqs. 3.10 and 3.11), so that all inputs and outputs are considered in the evaluation (Charnes et al. 1979) even if with a minor weight  $\varepsilon$ , set typically equal to  $10^{-6}$ .

The dual of this problem, which we shall call the envelopment problem, provides important information about economies that could be achieved in all the inputs; it also indicates which efficient units the inefficient unit being assessed should emulate. Those efficient units are usually referred to as the reference set or peer group of the unit under evaluation.

Any linear problem has a dual, which we will call the envelopment problem that can be written as follows:

$$\min : Z_a - \varepsilon \left\{ \sum_{r=1}^s S_{ra}^+ + \sum_{i=1}^m S_{ia}^- \right\} \quad (3.12)$$

$$\text{s.t. } \sum_{k=1}^n \lambda_k x_{ik} + S_{ia}^- = Z_a x_{ia} \quad i = 1 \dots m \quad (3.13)$$

$$\sum_{k=1}^n \lambda_k y_{rk} - S_{ra}^+ = y_{ra} \quad r = 1 \dots s \quad (3.14)$$

$$\lambda_k \geq 0 \quad k = 1 \dots n \quad (3.15)$$

$$S_{ra}^+ \geq 0 \quad r = 1 \dots s \quad (3.16)$$

$$S_{ia}^- \geq 0 \quad i = 1 \dots m \quad (3.17)$$

This formulation can be interpreted as follows: Given  $DMU_a$ , find the composite unit which has no smaller outputs than this one and whose inputs are smaller than those of  $DMU_a$  scaled down by a factor  $Z_a$  as small as possible. This is why this formulation is known as input minimisation; since we are minimising  $Z_a$ , we are seeking the minimal inputs that, based on best-achieved performance, could still produce the same amount of outputs as  $DMU_a$  is currently producing. These composite units are finite linear combinations of efficient units more specifically finite positive combinations which lie on the efficiency frontier. These efficient DMUs are known as the *peer group*, the role-model units that inefficient  $DMU_a$  should try to emulate.

In the envelopment problem, it is easy to understand the role of the small non-Archimedean entities  $\varepsilon$ ; they simply multiply by the sum of slacks so that slacks are not ignored in the overall score.

As a whole, the interpretation of the DEA technique is straightforward and can be put in the following terms:

### 3.4.1 Multiplier Problem

Evaluate each DMU with the set of weights which maximises its efficiency, provided that all other DMUs, rated with that set of weights, have efficiency not greater than unity.

### 3.4.2 Envelopment Problem

Find the smallest proportion  $Z_a$  of inputs that would bring the current  $DMU_a$  to the enveloping surface of all DMUs.

The model presented above, named CCR after the acronym of the authors Charnes et al. (1978), assumes constant returns to scale. This means that when the input of an efficient unit is multiplied by a given factor, its output level is also multiplied by the same factor. In this case, the production possibilities set will be a closed convex polyhedral cone in  $\mathcal{R}^{n+m}$  in the positive orthant.

In other situations this is not the case. The scale of operations may have an impact on the outputs, creating “economies” or “diseconomies” of scale. The BCC model, developed by Banker et al. (1984), can deal with variable returns to scale.

The BCC envelopment model can be obtained from the CCR envelopment model by adding the convexity constraint to the envelopment problem:

$$\sum_{k=1}^n \lambda_k = 1 \quad (3.18)$$

The multiplier problem is the dual of the envelopment formulation, and the extra restriction originates an extra free variable. It is important to note that if we relax the convexity constraint to  $\sum_{k=1}^n \lambda_k \geq 1$  than its dual, variable will become non-negative and it becomes clear that this model (the non-decreasing returns to scale model) is nothing else but a CCR model with an extra output identically unitary.

The same analysis could be conducted based on the maximisation of outputs, leading to similar formulations.

For instance, DMU 10 will be rated under VRS as 5/8 efficient and  $4/8 = 50\%$  under CRS. It is straightforward to conclude that VRS efficiency is never smaller than the CRS efficiency. This became clear if we note that the envelopment problem has an extra constraint, and it is a minimising program. The ratio between those two efficiencies is called scale efficiency, and it is important in the efficiency analysis. All those definitions can be made also for the output maximisation orientation.

In VRS we can define several regions, the increasing returns to scale and the decreasing returns to scale that expands until the infinity. Those regions are separated by the most productive scale size, where the frontier of the pointed cone

intersects the convex hull of the DMUs. In Fig. 3.7 it corresponds to point 3, the intersection between the dotted line (a linear hyperplane of dimension 1) and the dashed polygon.

### 3.5 Concluding Remarks

Using all the typical graphical representations of data envelopment analysis, the authors introduced based on a visual and spatial approach the basic ideas and principles of DEA. The two models for CRS and VRS, known as the CCR and BCC models, respectively, were presented, as well as both input minimisation and output maximisation orientations; this makes a total of 4 different models, and for each of those 4, there is a dual (weights = multipliers versus envelopment problems).

Many new developments of DEA could be presented here, most specifically weights restrictions, non-controllable inputs and outputs and what may be the most important one, the superefficiency model, that was introduced by Andersen and Petersen from Odense University by 1993 (Andersen and Petersen 1993). This technique is rather appealing because they allow efficiency to be greater than one, discriminating between efficient DMUs that otherwise are all ranked equal; it avoids also ambiguity on weights of those units (multiple solutions on the weights = multipliers problem, degeneracy in its dual).

The equations presented allow the implementation of DEA software in many programs as Excel, SAS, Mosel, Stata, OpenOffice and on any other program with linear programming solving capabilities.

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