The Mathematics of Change and Variation from a Millennial Perspective: New Content, New Context

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This chapter raises three broad questions for the present day:

1. Will the movement of mathematics from static-inert to dynamic-computational media lead to a widening of mathematical genres and forms of mathematical reasoning?
2. Will mathematical activity within computational media lead to a democratization of access to (potentially new forms of) mathematical reasoning?
3. Can these changes transform our notions of a core mathematics curriculum for all learners?

But before going further, by way of starting points, we should like to give a broad view of what we take mathematics to be. We regard mathematics as a culturally shared study of patterns and languages that is applied and extended through systematic forms of reasoning and argument. It is both an object of understanding and a means of understanding. These patterns and languages are an essential way of understanding the worlds we experience—physical, social, and even mathematical. While our universe of experience can be apprehended and organized in many ways—through the arts, the humanities, the physical and social sciences—important aspects of our experience can be approached through systematic study of patterns. In addition, mathematics embodies languages for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing—all extending the limited powers of the human mind. Finally, mathematics embodies systematic forms of

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reasoning and argument to help establish the certainty, generality, and reliability of our mathematical assertions. We take as a starting point that all of these aspects of mathematics change over time, and that they are especially sensitive to the media and representation systems in which they are instantiated.

1 A Condensed National History of Representation

While the evolutionary history of representational competence goes back to the beginnings of human evolution (Donald, 1991, 1993; Mithen, 1996), and can be linked to the evolution of the physiology of the brain (Bradshaw and Rogers, 1993; Calvin, 1990; Lieberman, 1991; Wills, 1993), with three exceptions this history is beyond our scope. The first is simply to recognize that representational competence, reflected in spoken and then written languages, both pictographic and phonetic, in visual representations of every sort, is a defining feature of our humanity. It is reflected in our physiology, our cultures, and our technologies, physical and cognitive.

The second exception involves the two-step evolution of writing systems from the need to create quantified records (Schmandt-Besserat, 1988, 1992). As convincingly described by Schmandt-Besserat (1980, 1981, 1985), clay tokens were first used in clay envelopes to record quantities of grain and other materials in storage and commercial and tax transactions, i.e., a given number of grain-tokens represented a certain number of bushels. Before being put inside the soft clay envelopes, these tokens were pressed into the exterior, leaving an image of the envelope's contents. Over many generations, the envelope markings replaced the tokens. The envelopes evolved into tablets, and the representations led to pictographic writing. The second step for western civilization was the invention of phonetic writing. Arbitrary characters were used to encode arbitrary sounds (phonemes), giving rise to abstract expression (Logan, 1986, 1995). This supported new written structures such as codified law, for example Hammurabic code and Moses' commandments, and, when the idea reached Greece, it enabled the expression of science, mathematics, logic, and rational philosophy (McLuhan and Logan, 1977). We draw two broad, albeit unsurprising, inferences from this history. One is that quantification—mathematics—and the functioning of human society have been inextricably linked, beginning as early as the invention of writing. The second is that representational changes in the constraints and affordances of concrete media play a critical role in how we organize our worlds (Goodman, 1978).

The third major historical event to which we direct attention is the invention of the printing press. Of special interest are three consequences. First, there was a standardization of dialects and vernaculars used in spoken language in Europe, first in England with English—as opposed to the existing standard *formal* languages, namely Latin and Greek. Second, and closely related, was the democratization of literacy (Innis, 1951; McLuhan, 1962). Up until that time, there was a small collection of written works and a tiny elite who read and commented on them. Indeed, you could fit almost all of the available written classics on to a decent sized bookshelf. Along with the democratization of literacy came a critically important third event, the dramatic widening of literary forms and the rapid proliferation of original literary material based in everyday life (as opposed to narrowly academic commentary on the classics). For example, the novel was invented. Importantly, these events occurred largely outside of, and independently of, either the universities or the monasteries. A fifteenth-century monk would not today recognize the "language arts" curriculum as being about the "literacy," which was practiced and taught before the invention of the printing press. We need to remind ourselves that Shakespeare, and virtually all fiction of the sixteenth and seventeenth centuries, were regarded as "vulgar" literature, not admitted as the subject of academic study.

More recently, we have seen the invention of dynamic visual media, film, and especially television. These have again led to a democratization of visual culture and a widening of dynamic visual forms (McLuhan, 1967). Almost from the beginning, films, for example, were not sequential representations of visual events. Film generated new art forms, much in the way that new literary art forms flourished after the development of the printing press (Arnheim, 1957). There has been a democratization of visually mediated culture (Salomon, 1979). Most people enjoy film and can understand its idioms. Most people can follow the extraordinary visual and auditory feats of contemporary television, despite the rapid sequences of images and semiotic complexity (Fiske and Hartley, 1978; Williams, 1974). This democratization of visual culture occurred without formal instruction or education, outside the academic realm. Indeed, the former masters of the visual arts had rather little ability to guide the new genres that arose in Hollywood and Madison Avenue. These genres built upon naturally occurring visual and language-interpretation capabilities widely distributed across the population, and now continue with the ready ability of ways for everyday people to author and distribute videos

The invention of manipulable formalisms, numeric and algebraic, occurred around the same time that the printing press was invented. The first of these, the Hindu-Arabic place-holder system for numbers, was intimately involved in the commercial economy of the time (Swetz, 1987). And perhaps even more important for the longer term was the rapid development of an algebraic symbol system with syntax for manipulation. This was tied to an explosion of mathematics and science development that is continuing today. Importantly, this mathematics and science, and the notation systems in which it was encoded were developed by and for an intellectual élite—far less than one percent of the population. Until very recent times, only a very tiny minority of the population who were expected to learn these symbol systems and use them productively—and thus no effort went into designing the symbol systems to be readily learnable by the majority of the population. Whereas a more accessible vernacular style of writing emerged in literature and journalism (for example), mathematics has yet to develop an equally accessible style of reading and writing algebra for everyday use by the general public. Indeed, each of these successive inventions, writing and the printing technology that democratized it, dynamic visual forms, and now interactive digital notations, are much more deeply embedded in ordinary life than is "classic" school mathematics.

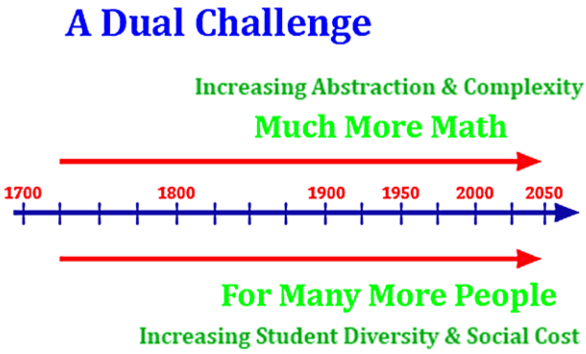

Fig. 1 A long-term trend: much more math for many more people

2 Dual Challenges: Much More Mathematics for Many More People

At the end of the twentieth century, we face a dual challenge in mathematics education at all levels, from kindergarten to adult education: we need to teach much more mathematics to many more people. The radical increase in the numbers of people who are expected to know and use mathematics is leading to a corresponding increase in student diversity and increases in the social cost of mathematics education—to near the limits for which societies are willing to pay. We need to achieve dramatic new efficiencies across the entire K-12 mathematics curriculum. These trends, as indicated in Fig. 1, have been under way for centuries and are expected to continue.

To illustrate the change in content, we recall a story from Tobias Dantzig (1954):

> It appears that a [German] merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which, in his opinion, was the only country where such advanced instruction could be obtained.

This story has been well corroborated by historians of mathematics, for example, Swetz (1987). While it concerns commercial mathematics, it makes the point that there was a time when even this mathematics was the province of a very élite group of specialists. Over time, the fact that widespread acceptance of the newly available notation system had large impact on the larger population's access to what we now term "shopkeeper arithmetic."

Another useful orienting statistic is derived from U.S. Department of Education (1996). In the United States, 3.5 % of the 17- to 18-year-old population cohort took high school Advanced Placement Calculus (successful completion of the associated test allows them to substitute this course for a corresponding one in the university). *This is almost exactly the percentage of students graduating from high school in the U.S. a century earlier.* Perhaps 2 % of the US population was expected to learn algebra a century ago; today in the US almost all students are expected to take an

Algebra course even before they attend high school. Similar dramatic changes in expectations regarding who can or should learn mathematics have occurred internationally, throughout both the developed and developing world.

Indeed, reflecting on these longer-term trends, the question we asked about whether a democratization of mathematical reasoning would occur, now shifts from the interrogative to the imperative: we *must* teach much more mathematics to many more people. But how can we speak of much more mathematics when the curriculum is already overflowing? And for many more people when most of those people presently end up disliking mathematics intensely?

Before replying, we ask how many people can travel 50 miles per hour? Or can fly? Or can speak and be heard a thousand miles away? Answer: most of us. Rendering much more mathematics learnable by many more people will require at least the levels of co-ordinated innovation standing behind the automobile, airplane or telephone. Let's step back a bit and examine these other innovations.

First note, the automobile involved considerably more than the invention of the internal combustion engine. Automobiles are embedded in a sophisticated system of interrelated innovations and practices that cover a wide range of systems, mechanical, hydraulic, electronic, as well as roadways, laws, and maps. Then there is the matter of educating and organizing the people to build, operate, and market them. Of course, jet airplanes, airports, navigation systems, worldwide communication systems, airline reservation systems, radar-based flight controllers, are at least as great a miracle. With a very occasional exception, all these staples of the late twentieth century operate with extraordinary efficiency in the service of quite ordinary people—*and are expected to!*

We see parallel developments now becoming possible in educational technology. Much attention has been drawn to graphical and dynamic media, with its attendant possibilities for engaging children in constructing, reasoning, and communicating across multiple representational forms. Likewise, the ubiquitous availability of social networking suggests a broad change to communication is now afoot, with potentially as far-reaching consequences. Yet new media and networking are incomplete without a third development: the possibility of mass-producing personalizable educational content. Just as transportation required Henry Ford's assembly-line-produced Model T, and communications required the dial tone, educational technology needs a wave of modularization, substitutability, and combinatoric composition. This is now becoming possible under the rubric of "component software architectures" (Cox, 1996) which allow for the mix-and-match interoperability, integration, and customization of modular functionalities: notebooks, graphs, calculators, simulations, algebraic formulae, annotation tools, etc. Component software architectures bring the possibility of constructing large complex systems through a highly distributed effort among developers, researchers, activity authors, curriculum experts, publishers, teachers, and students, among others. As we argue elsewhere (Roschelle and Kaput, 1996; Roschelle et al., 1998), the integration of media, networks, and component architecture can begin to allow us to approach educational problems on a scale that was formerly inconceivable.

With our confidence stiffened by clear success in transportation and communication, and with an understanding that the infrastructure for similar advances in educational technology is now emerging, let us now turn to our particular interest in present day reform—democratizing access to powerful mathematics.

3 Access to Powerful Mathematics Through New Representational Forms

Educational innovators have long experimented with the construction of alternative notational systems to enable learning of mathematics and science. One well-established method is to embed mathematics in computer languages (Ayers et al., 1988; diSessa et al., 1995; Hatfield and Kieren, 1972; Noss and Hoyles, 1996; Papert, 1980; Sfard and Leron, 1996). Familiar examples could include Turtle Geometry (Abelson and diSessa, 1980), mathematical programming in ISETL (Dubinsky, 1991), and spreadsheets (Neuwirth, 1995). Another method is to embed the content in activities such as computer games (Kraus, 1982; Shaffer et al., 2005). Here we argue for a representational alternative: embedding mathematics in direct manipulation of dynamic spatial forms and conversation over those forms (Kaput, 1992). Dynamic geometry is one example of alternative notational form based upon direct manipulation of spatial forms (Jackiw, 1991–2009; Goldenberg, 1997; Laborde, 1984–2009). Direct manipulation of two-dimensional vectors is another (Roschelle, 1991). For the mathematics of change and variation, our SimCalc project has chosen to focus on directly manipulable Cartesian graphs that control the action of animations.

The properties of graphs suggest interesting answers to the three major questions we posed earlier:

1. Widening of forms? Graphs already support a range of forms that is considerably wider than can be expressed in closed-form symbolic algebra (Kaput, 1994), and more specific to particular reasoning techniques. For example, as we shall describe below, graphs can easily support manipulation of piecewise defined functions, a form that is extremely cumbersome in traditional algebra.
2. Democratization of access? Graphs are already a more democratic form, appearing frequently in newspapers, television, business presentations, and even U.S. presidential campaign speeches—at least in terms of reading and interpretation, as opposed to writing and manipulating graphs. These are all places where equations are seldom found, and indeed usually taboo. As was the case with the explosion of literary forms, graphs appear to draw upon cognitive capabilities, which are more widespread or accessible than formal mathematical symbols, although not without challenges (Leinhardt et al., 1990; McDermott et al., 1987). We shall deal with the matter of writing and manipulating graphs shortly.
3. New core curriculum? Most of the basic characteristics of mathematical thinking outlined at the beginning of this chapter and highlighted in modern curriculum standards can be carried over to graphical representational forms, allowing students to begin grappling with powerful concepts earlier and more successfully. In the next millennium, graphical mathematics will need to be part of the basic

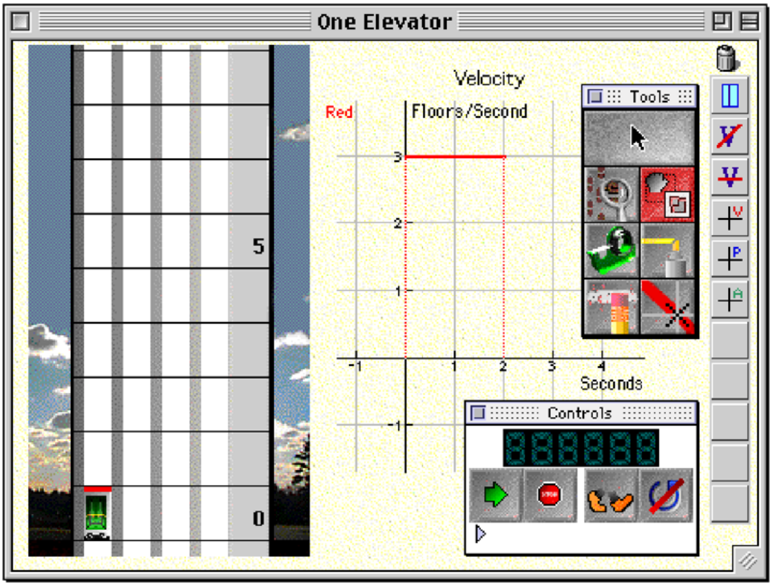

Fig. 2 An introductory elevators activity in SimCalc MathWorlds® software

mainstream experience for all students. But a major step in this direction will require a move from static graphs that are merely read and interpreted to dynamically manipulable graphs that can be linked to phenomena and simulations of various kinds. And this change must occur in concert with substantial changes in how the content is organized and experienced.

Our SimCalc Project is building upon the unique potential of manipulable graphs in our *MathWorlds* software, which supports learning about rates of change of objects in motion, along with many related concepts. Figure 2 shows a screen from an introductory activity, with a moving elevator controlled by a piecewise-defined step function for velocity. In this activity (designed by our colleague, Walter Stroup), middle-school students make velocity functions that occupy six grid squares of area. The students are asked to make as many different (positive) functions as they can, and compare similarities and differences. As mathematicians well know, all such functions will cause the elevator to move upwards 6 floors, but will vary in times and speeds. Pictured is a very simple one-piece velocity function. As we discuss elsewhere (Roschelle et al., 2000), velocity step functions also draw upon students' prior knowledge and skills: students can compute the integral by multiplying the sides of the rectangles or simply counting squares. They can readily distinguish duration (width) from speed (height), and distance travelled (area). Questions about the meaning of negative areas (below the axis) arise naturally and their resolution can be grounded in the motion of the simulated elevator.

Over many weeks with MathWorlds, students can study the properties of velocity graphs in relation to motion, then position graphs, then relations between the two, and finally (for older students) acceleration graphs. Along the way, students also work with manipulable graphs that are piecewise linear (instead of step functions), continuous instead of discontinuous, and varying arbitrarily (not just linearly). The various representations are each dynamically linked, so that students can directly observe the effects of changing a velocity graph upon position, or vice versa.

4 Relationships Among Representations Move to the Center

As mentioned above, the printing press led to increasing diversity of literary forms including, for example, the novel. We argued that computational representations are doing the same for mathematics, and that new forms of graphs are likely to become common tools for mathematical reasoning. Here we push our millennial comparison one step further. Among the new literary forms that emerged, the novel stands out as creating a more participatory experience for the reader; readers of novels are swept into a fully articulated world that at times seems as real as the familiar world. Indeed, the great achievement of successful authors is to relate experience in the reader's personal world to the new imaginary worlds. Moreover, novelists were now free to treat topics that were neither religious, nor mythical, nor heroic—contemporary life became the subject of literary experience. Of course, these new forms did not arrive without precedent; oral story-telling traditions paved the way; and contemporary novels are certainly no more constrained to common experience than film-makers are bound to reproducing common events.

This trend has its parallel in technology that brings motion experiences into the mathematics classroom, and thus ties the mathematics of change to its historical and familiar roots in experienced motion. Motion can be represented cybernetically (as an animation or simulation), as we described above in MathWorlds. And motion can also be represented physically, in experiences of students' own body movement, or objects that they move. When desired, these physical motions can be digitized and imported as data into the computer, attached to actors, repeated, edited, and so on. Below we discuss three ways in which SimCalc is using the relationship between physical and cybernetic experience to give students new opportunities to make sense of traditionally difficult concepts such as mean value, limits, and continuity.

When using MathWorlds in classrooms, especially with young children, we often begin with physical motion, unconnected to the computer at all. For example, students might be asked to walk along a line, with speeds qualitatively described as "fast," "medium," or "slow." The class can then measure the time to cover a fixed distance, beginning the slow process of building and differentiating the quantities of distance, rate, and time and their relationships. Later students move to the computer and use an activity that displays a "walking world" with animated characters whose velocities are constrained to three fixed heights, corresponding to fast, medium and slow. With the greater precision and control supported by the computer representations, students can now begin to make quantitative comparisons. Here we use the

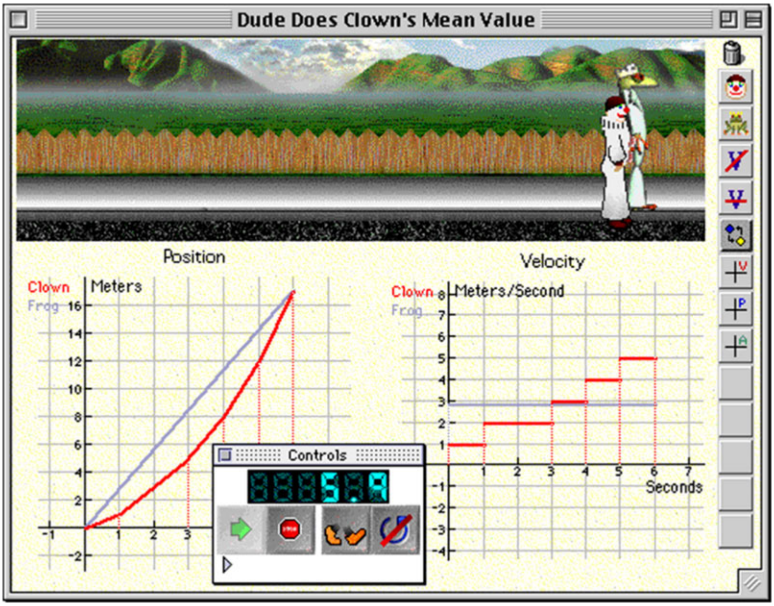

Fig. 3 Dude does Clown's mean value

kinesthetically rich experiences of the physical world to present difficult quantification challenges for students who have only the vaguest idea of what one might measure and why (Piaget, 1970; Thompson and Thompson, 1995). Animated clowns, on the other hand, are less grounded in real experience (indeed their gaits are cartoonish at best), but easier to control, measure, examine, and repeat. They provide pedagogically powerful intermediate idealizations of motion phenomena.

In later activities, physical and cybernetic experience can be connected directly through data. For example, in Roschelle et al. (1998), we describe an activity sequence in which students explore the concept of mean value—a mathematically central concept upon which much theoretical structure depends (Fleming and Kaput, 1979). Here a student's body motion is captured with a motion sensor, and input into MathWorlds, where it becomes replayable as the motion of the walking "Clown" character. The motion also appears in a MathWorlds graph as a continuously varying velocity function (assuming the walking student varied speed). Students may now construct a second animated character ("Dude") whose motion is controlled by a single constant velocity function (see Fig. 3). The challenge is to find the correct velocity to arrive at the same final location at exactly the same time—thus finding the mean value.

It is exciting for students to find the mean value of their own variable-velocity body motion. While a large body of research exists regarding student mathematization of motion, it focuses mainly on high school and college age students, and generally deals with "regular" motion that is describable algebraically (McDermott et al., 1987; Thornton, 1992). In contrast, we allow students to work with highly irregular motion, generated by their own bodies, and leading towards the important technique of approximating continuous variation with piecewise linear functions.

Once students are comfortable with the concept of mean value, they can use piecewise segments to try to approximate a varying motion. In the example given above, the two motions will only intersect at one place, the final location. But in MathWorlds students can use two piecewise linear velocity segments (each comprising half the given interval) and thus intersect at two places, the final location and the mid-duration location. This process can be repeated with more and more (and smaller and smaller) segments, making the two characters meet 2, 4, 8, 16, 32, or more times. Of course, as the number of segments increases, the approximation between the motions becomes greater, giving students a concrete sense of taking a limit. Here students see how an idealized abstraction (a linear velocity segment) can model a continuously varying real world variable (their varying body position) with as much precision as required (see Roschelle et al., 1998, for more detail).

Although the activity sequence described above was important in early teaching experiments with MathWorlds and shows how dynamic representations can fruitfully reinterpret ideas in a Calculus course, over time our work has come to emphasize position graphs before velocity graphs. When we initiated a sequence of projects in Texas to test SimCalc curriculum at scale, our entry point was 7th grade mathematics. At this level, SimCalc relates best not to mean value (a more advanced concept) but rather to the core construct of proportionality.

In traditional seventh grade teaching, proportionality is addressed through the equality of two ratios: $a/b = c/d$. Problems rely heavily on the useful but inscrutable cross-multiplication procedure, which transforms this equality to $a * d = c * b$, and the process of identifying three given quantities and calculating a fourth. While useful, this conceptualization of proportionality is narrowly useful and is not particularly fertile for students' further mathematical development. We found that SimCalc could bring the "democratizing access to the mathematics of change" theme to life at this grade level by refocusing proportionality on its relationship to rate, as in the speed of a moving object. Further, graphs could be used to visualize how speed connects changes in time to changes in position, e.g., through the graph of a line. This could then be developed into the first important mathematical function that students encounter, $y = kx$ (which is further developed into $y = mx + b$ in our eighth grade curriculum unit).

These curricula units developed a rich network of connections among representations along two dimensions, a familiar-formal dimension and a graphical-linguistic representation. For instance, in initial exercises, an animated motion (familiar, graphical) is related to a story of a race (familiar, linguistic) and a graph of the motion of the actors in the race (formal, graphical). As the curriculum progresses, algebraic expressions (formal, linguistic) are also introduced in relationship to graphs, using tables as a stepping stone. Overall, we have argued that optimal curricular sequences using new representational media should focus on connections between types of representations and should develop more formal representations as their more familiar counterparts are better understood.

As reported elsewhere in this volume, we have had considerable success in demonstrating the effectiveness of these curricular units for seventh and eighth grade mathematics through large-scale randomized experiments. We attribute some of this success to the idea that "new technology without new curriculum isn't worth the silicon it's written in." To realize the potential of dynamic representations, new curricular pathways must be envisioned and made concrete for teachers in curriculum workbooks and teacher professional development.

To summarize, we see new technologies creating a possibility to reconnect mathematical representations and concepts to directly perceived phenomena, as well as to strengthen students' understanding of connections among different forms of mathematical representation. By starting from more familiar antecedents, such as graphs and motion, both in kinesthetic and cybernetic form, and developing towards more compact and formal mathematical representations, we see an opportunity to create a new path of access to mathematics that has too often remained the province of a narrow elite.

5 Discussion: Mathematics Education at the Beginning of a New Millennium

By momentarily rising from the trenches of mathematics education reform to a larger time scale, we identified a major long-term trend: *Computational media are reshaping mathematics, both in the hands of mathematicians and in the hands of students as they explore new, more intimate connections to everyday life.* As we mentioned earlier, we can already be fairly certain this will lead to widening of mathematical forms, just as the printing press increased the range of acceptable literary forms. Already, we are seeing new forms such as spreadsheets and data visualization becoming prominent in everyday life. Computational tools are leading to new epistemological methods as professional mathematicians explore the extraordinary graphical phenomenology of dynamical systems (Stewart, 1990). Educators are rapidly inventing new, additional forms—such as mathematical programming languages and construction kits for dynamic geometry.

Based on our experiences and experimental investigations with SimCalc, we are more and more optimistic about the second question we raised: democratic access. Some representational forms, like directly editable graphs, can make difficult concepts such as mean value, limit, and continuity in calculus newly available to ordinary middle-school (10- to 12-year-old) students and can lead to fertile reinterpretations of existing concepts, such as rate and proportionality. The technology's capability to provide better links between graphical representations and phenomena (physical and cybernetic) also appear essential, as this linkage grounds concepts in familiar semantic referents.

Yet as Sherin (1996) points out, new forms also lead to changes in the meaning of the concepts; in Sherin's studies, students who learned physics by programming in a computer language learned a set of physics concepts subtly different from those learned by students using traditional algebraic symbols. Indeed, in our work with SimCalc, we are currently working out the curricular bridge from editable piecewise functions back to traditional algebra. It is by no means easy. And this is just the beginning! We want to lead students towards understandings of the larger mathematics of change and variation that includes dynamical systems because this relatively new mathematical form, with roots in Poincaré's work at the end of the previous century, is revolutionizing many sciences simultaneously as we approach the next century (Hall, 1992). However, the major long-term educational experiment with systems concepts, based in the use of *Stella*, is far from a clear success (Doerr, 1996). Other innovative approaches to systems concepts, *StarLogo* (Resnick, 1994) and *AgentSheets* (Repenning, 1994) look promising but present difficulties in linking back to commonplace notation. Nonetheless, we believe that with time and effort, innovations in computational representations will make democratic access to systems dynamics possible.

To harness this potential fully, however, reformers will need to rise to the challenge of our third question: Can these new possibilities transform our notion of a core mathematics curriculum for all learners? The technological revolutions in transportation and communications would be meaningless or impossible if core societal institutions and infrastructures remained unchanged in their wake. Today's overnight shipments and telecommuting workers would be a shock to our forebears 100 years ago, but our curriculum would be recognized as quite familiar. If we are to overcome this stasis, we must seize the opportunities implicit in new dynamic notations to reorganize the curriculum to enable extraordinary achievement from ordinary learners.

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