

Chapter 10

Dubislav and Bolzano

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10.1 Brief Introduction

Walter Dubislav (1895–1937) was an active member of the Berlin Group of logical empiricism in the early 1930s. A philosopher, mathematician and logician, he shared the thematic focus of the Berlin Group on the natural sciences, mathematics and logic. He shared the methodological demand of the Berlin Group that philosophical method of inquiry should follow the rigor and precision of formal sciences in exposition and logical reasoning (Rescher 2006, 283). A rigorous methodology for philosophy was also required by Bernard Bolzano (1781–1848), the Prague mathematician, logician and philosopher. Was it Bolzano's efforts to separate logic from psychology in the *Theory of Science* (Bolzano 1837) or his reconstruction of mathematics in the *Contributions to a Better Founded Exposition of Mathematics* (1810) which attracted Walter Dubislav's attention?

Dubislav was not interested in Bolzano's early attempts to develop a mathematical method for expounding objective dependence relations which hold between judgments as grounds and consequences (Bolzano 1810, II, § 2). His research is focused on the later Bolzano (1837). In a series of papers published between 1929 and 1931, he deals with Bolzano's Kant-criticism and Bolzano's contribution to modern logic. More specifically, he examines what he calls Bolzano's propositional functions (*Aussage- oder Satzfunktion*), his notion of analyticity and analytic statements, as well as his notions of probability (*Wahrscheinlichkeit*) and derivability (*Ableitbarkeit*).

My thanks go to Nikolay Milkov for his helpful advice.

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10.2 From Kant to Bolzano: Dubislav on Bolzano's Kant-Criticism

Dubislav was drawn to Bolzano's Kant-criticism and his meticulous efforts to secure a pre-Kantian (or rather, a pre-transcendental idealist) position in philosophy, as is testified by his 1929 article "Ueber Bolzano als Kritiker Kants" and his planned edition of František Příhonský's *Neuer Anti-Kant* (1850) in collaboration with Heinrich Scholz. Příhonský was a member of Bolzano's school and in the *Neuer Anti-Kant oder Prüfung der Kritik der reinen Vernunft*, Příhonský systematizes Bolzano's criticisms of Kant which are scattered across many papers between 1818 and 1837. Dubislav and Scholz (1931c) prepared a critical commentary on Bolzano-Příhonský's dispute with Kant. The edition never appeared but their commentaries were published in Edgar Morscher's (2003) edition of the *New Anti-Kant*. Kantian philosophy was still extraordinarily influential in Dubislav's time and hence he approved the Prague philosophers' examination of the explanations and proofs Kant puts forward for the claims set down in his philosophical system. According to Dubislav, if one accepted the view that Kant's philosophical system could be appropriately evaluated by scientific means, a critical appreciation from a scientific perspective would ascertain which of Kant's doctrines are true and which are false (Dubislav 1931c, 203).

Dubislav (1929, 358) explains that the scientific value of Bolzano's Kant-criticism lies in the exposition of unsolved problems in systematic philosophy by rebutting Kantian metaphysics and logic. Thus he argues that Bolzano did not deal with Kant because he was a famous philosopher but because he believed that a critical examination of Kant's doctrines would provide a convenient access to a series of important philosophical problems. On this view, criticizing Kant is an instrument for clarifying philosophical problems that go beyond Kant—such as the question whether epistemology underwrites metaphysics and logic or the Kantian division between mathematics and philosophy supplemented by the latter's claim that exact definitions and strict demonstrations cannot occur in philosophical investigations (Kant 1789, B754–5, B759). Such claims were a thorn in the side of the Berlin Group as much as they were in Bolzano's. Hence Dubislav in turn reconstructs Bolzano's critical reconstruction of Kant because a proper display of Kant's doctrines would clarify not only Bolzano's views, but also Dubislav's own. For instance, Bolzano (1837, § 65) elaborates Kant's definition of analyticity and his distinction between analytic and synthetic judgments.¹ Dubislav, for his part, pays tribute to Bolzano's own notion of analyticity and analytic propositions. He thus conceives Kant-criticism as an exercise in defining and specifying the viewpoint

¹Bolzano writes: "Kant penetrated this distinction the deepest and it is to him that the author of this book owes his correct view on this issue. [...] It suffices to grasp this distinction appropriately, in order to understand that there are attributes (*Beschaffenheiten*) which belong to an object and necessarily belong to it according to the concept we form of that object, without being presented as components of this concept." (1837, § 65.8, cf. also § 148, my translation—A. K.).

of those who criticize him (Dubislav 1929, 358). In this way, Bolzano, Přihonský and Dubislav avoid the paradox of criticizing a *Critique*. By reconstructing Kant's doctrines clearly and concisely, they present their own philosophical position. For example, Kant's account of how cognitions are produced is appropriately expressed by Bolzano's statement that "the possibility of cognizing (*Erkennbarkeit*) an object is the possibility of pronouncing a true judgment on it." (Bolzano 1837, § 26.4, my translation—A. K).

Dubislav (1929, 363–365) expounds Bolzano's critical examination of Kant's distinctions between analytic and synthetic judgments and Kant's notion of analyticity. I analyse Dubislav's reconstruction of Bolzano's Kant-criticism and Bolzano's own view, since Dubislav (1929, 1930a, b, 1931a, b, c, d) pays special attention to Bolzano's notion of analyticity. According to Kant, a judgment is analytic if its predicate-concept is (covertly) contained in or by its subject-concept. His well-known example of an analytic judgment is "all bodies are extended" where the predicate-concept does not add anything to the subject-concept that is not already contained in it (Kant 1789, B10–11, JL, § 36). On this view, analytic judgments are affirmative judgments and have the subject-copula-predicate form of categorical judgments: "All *A* are *B*". In addition, analytic judgments are epistemologically warranted by the principle of contradiction: the truth of an analytic judgment must be cognizable in accordance with the principle of contradiction (Kant 1789, B190).

Dubislav (1929, 364) remarks with Bolzano that this definition excludes hypothetical judgments which, according to Kant's table of judgment-forms, do not have the subject-copula-predicate form (Kant 1789, B95) and that Kant's analytic judgments are trivial and affirmative judgments only. When advancing the principle of contradiction as an epistemological warrant for analytic judgments, however, Kant also considers negative analytic judgments such as: "no unlearned person is learned" (Kant 1789, B192). If Bolzano (and Dubislav) omit Kant's negative analytic judgments, it is probably due to the fact that these latter do not modify the Kantian notion of analytic judgments. The predicate-concept of negative analytic judgments covertly or implicitly includes a partial negation of the subject-concept: "is learned" partially negates "no unlearned person". But this negation does not advance our knowledge of the subject-concept 'no unlearned person', since the negation is contained by the subject-concept.² Dubislav discusses Bolzano-Přihonský's objections that Kant's definition of analytic judgments is (i) too wide because it also applies to judgments such as: "the father of Alexander, King of Macedonia, was King of Macedonia" and "a triangle similar to an isosceles triangle is itself isosceles" which have the form "an *A* which has *B*

²Pace Y. Bar-Hillel (1950, 97), who notes Dubislav's (1926) "return to Bolzano's proposal" to accept analytically false as well as analytically true statements. Bar Hillel writes "[i]t is well known that a term corresponding to Bolzano's 'analytically false' lacked in Kant's terminology, that therefore Kant's classification of propositions into analytic and synthetic ones was by no means exhaustive." While Kant's classification may not be exhaustive, this is not because he did not accept analytically false statements but rather because he lacked Bolzano's innovative notion of statements with a variable component (Bar-Hillel 1950, 97).

is *A*". The predicate-concept merely repeats the subject-concept without clarifying the meaning of 'father of Alexander King of Macedonia' or 'triangle similar to isosceles triangle'. On the other hand, however, if the predicate-concept is contained as an essential mark in the subject-component, the definition of analytic judgments is (ii) too narrow because it excludes judgments such as "every object is either *B* or non-*B*" (1837 § 148; NAK, 34–35).³

10.2.1 Dubislav on Bolzano's Notion of Analyticity

Having examined Bolzano's criticism of Kant's analytic judgments, Dubislav (1926, 1929, 1930a, b, 1931a, b, c, d) reconstructs Bolzano's own notion of analyticity which is based on the method of variation of presentations (*Vorstellungen*) in a proposition (*Satz*) and the notion of validity (1837, §§ 147, 148.1). According to Bolzano, a declarative statement is analytic if and only if it contains at least one presentation which can be arbitrarily varied without disturbing its truth or falsity. In addition, a declarative statement is analytic if and only if all the statements which could be obtained by the arbitrary variation of this presentation, are either all true or all false, provided only their subject-presentation is objectual (*gegenständlich*) (1837, § 147). In other words, analytic statements produced by the process of variation must (i) be either all true or all false and (ii) have the same truth value as the original proposition. In addition (iii) the process of variation must produce an objectual (*gegenständliche*) statement, that is, a declarative statement which has an actual or possible referent. Hence the subject-presentation, as a component of the statement, has to have a referential relation to its object, regardless of whether that object actually exists. (1837, § 137). Dubislav (1931c, 224) accepts Bolzano's claim that the objectuality constraint holds for general assertions because, on his view, an assertion such as "all triangles have three angles" is applicable if and only if the respective presentation is objectual. He adds that although in mathematical logic this constraint is obsolete, Bolzano's claim is equivalent to the logicist interpretation: "it is the case for all *x*: '*x* is a triangle' implies '*x* has three angles'", and the statement in quotation marks is true if the sentential function '*x* is a triangle' is "always false".

Bolzano's explication of analytic statements (as well as his notion of probability) is also based on the notion of validity (*Gültigkeit*) (1837, § 148.1). Although

³Quine, in "Two Dogmas of Empiricism", has similar objections against the Kantian notion of analytic judgments (Quine 1951, 21). Cf. Edgar Morscher (2003).

N.B. Kant could rebut (ii) because, in addition, he accepts judgments as analytic if they rest on the principle of contradiction. He gives the following examples of analytic geometrical principles: " $a = a$ ", "the whole is equal to itself", " $(a + b) > a$, i.e., the whole is greater than its part" (Kant 1789, B 16). But discussing the relevance of Bolzano- Přihonský's Kant-criticism goes beyond the scope of this chapter. Let it suffice to say with Dubislav that via Kant, Bolzano worked his way into crucial problems of philosophy and discovered solutions which anticipate views in modern logic and philosophy.

Dubislav discusses validity in relation to probability and derivability, he does not mention it with regard to analytic statements.⁴ Bolzano says that a declarative statement P is analytic if and only if it contains a replaceable presentation R which can be arbitrarily varied without disturbing the truth or falsity of P . The resulting variants of P are either all true or all false, so that P is universally valid or universally contra-valid in regard to R . In § 147, Bolzano introduces the notion of universal validity of declarative statements as such as follows: a proposition as such is universally valid in regard to the collection of true statements T if all T -variants of P are true. P has a degree of validity (*Grad der Gültigkeit*) which is defined as the ratio of the number of true variants to the total of variants. If all the variants are true, P is universally valid and its validity is 1. If all the variants are false, P is universally contra-valid and its validity is 0. If some variants are true and some are false, P 's degree of validity is a fraction between 0 and 1. Bolzano's validity is a relative notion, since the degree of validity of a given statement is always relative to a given variable component, so that one and the same statement can have different validities.

In addition, Bolzano distinguishes between analytic and logically analytic statements by relating the notion of universal validity to his notion of analyticity when he defines logically analytic statements as logically and universally valid or logically and universally contra-valid. (1837, § 148.3). The difference between analytic and logically analytic statements is that the invariable presentations of the latter are logical concepts such as the copula 'is' or 'has', the concept of negation or the concept 'something'. Bolzano uses an epistemological criterion for distinguishing between analytical and logically analytical statements: he says that for assessing the analytical nature of this sub-species of analytical statements, only logical knowledge is necessary, whereas for assessing the truth or falsity of analytical statements in the wider sense, a completely different kind of knowledge is required, since extra-logical concepts are brought in. Moreover, Bolzano admits that the distinction between analytic and logically analytic statements is rather unstable, for the domain of concepts belonging to logic is not so sharply delimited that disputes could never arise.⁵

According to some commentators, Bolzano's analytic statements are propositional forms: if a sequence of presentations in a declarative statement is replaced by another sequence, by means of such a uniform variation of presentations in

⁴See on this Jan Berg (Berg 1999, 122–124).

⁵"[Logically analytic statements] differ from [analytic statements in the wider sense] in that for an assessment of the analytic nature of the former, only logical knowledge is necessary because the concepts which form the invariable part of those statements all belong to logic. The assessment of the truth and falsity of propositions of the former, however, require a wholly different kind of knowledge, since concepts alien to logic intrude. This distinction is admittedly unstable, for the domain of concepts belonging to logic is not that sharply delimited so that some controversy is inevitable." (1837, § 148.3, my translation—A. K.). Bolzano's distinction between analytic statements and logically analytic statements is famously discussed by Bar-Hillel (1950), Berg (1999), and Künne (2008).

a statement, a propositional form or invariant is produced which is analytic if and only if all its component presentations are either true or false and if and only if it remains true or false, despite the changes produced in some of those component presentations. (1837, § 148, NAK, p.36, Bar-Hillel 1952, 67; Morscher 2003, XLV).⁶ Bolzano uses the notion “propositional form”, or “sentential form”, depending on our interpretation of “*Satzform*”—according to him, all propositions have the uniform subject-copula-predicate structure “A has *b*” (1837, § 127). Following Russellian usage, *Satz* is usually translated as *proposition* and, following Morscher, this translation is applied to Bolzano, based on Bolzano’s claim that “*Sätze an sich*” are not linguistic expressions but the sense of those expressions (1837, § 19) which is independent of the mental or linguistic acts in which it is expressed. *Sätze* on the other hand, are declarative statements and that is the translation used in this chapter. Furthermore, in recent years (Textor 1997; Berg 1999; Sebestik 2007), Bolzano’s “*Satzform*” has been considered as sentential, rather than propositional because it is an expression which becomes a sentence and it is obtained by considering some parts of a sentence variable. Propositions are either true or false and do not have variable parts because they are the sense expressed by a sentence or sentential form. The latter is indeterminate.

Bolzano does not have Frege’s two-place casting mould of ‘function-argument’, in which expressions fit together as ‘complete’ and ‘incomplete’ and hence he also lacks the Fregean notion of a function is an incomplete expression which takes a number of names as arguments and produces one proposition as the value. Nonetheless, Bolzano did not explicate statements as connections of presentations related by the copula but assigned primacy to statements over presentations by suggesting that we do not use the concept of presentation for defining the concept of a statement, since presentations are merely those parts of a statement which are not themselves statements (1837, § 128). In this sense, Bolzano anticipated Frege.

There is no consensus amongst commentators about whether Bolzano’s standard form of statements “A has *b*” is a propositional form or a sentential form (*Satzform*), let alone an abstraction of propositional forms or propositional function $A(b)$ as Dubislav claims. Some commentators (Berg 1999; Sebestik 2007; Textor 1997) hold (*contra* Dubislav 1929, 1930a, b, 1931b, c; Morscher 1999b, 2003; Siebel 1996, 1999) that the variable presentations are parts of declarative statements expressing a *Satzform* (sentential form). Thus “Caius is a man” and “Caius has wisdom” express the sentential form “A has *b*”. On a semantic level, sentences express propositions (*Sätze an sich*), that is, linguistic senses. Bolzanian statements as such (*Sätze an sich*) are roughly equivalent to Fregean thoughts: they are linguistic senses or

⁶“But suppose a statement contains just a single presentation which could be arbitrarily varied without disturbing the truth or falsity of the statement; i.e. if all statements obtainable from it by arbitrarily substituting this presentation by others, are either all be true or all false, provided only they have objectuality (*Gegenständlichkeit*). This property of the statement is already sufficiently remarkable to differentiate it from all those statements for which this is not the case. Hence I allow myself to call statements of this kind *analytic*, borrowing an expression from Kant.” (1837, § 148.1, my translation—A. K.).

possible contents of a sentence, expressible or thinkable, which are either true or false.⁷ The standard form of sentences (*Satzform*) [*A* has *b*] is also either true or false (1837, §§ 19, 28, 126, 127) but, unlike Bolzanian statements as such, it is a linguistic expression obtained by considering parts of a sentence variable (Sebestik 2007; Textor 1997).

By contrast, Dubislav's colleague Heinrich Scholz (1931c, 208–210) suggests the notion of *perfect assertive-form* (*perfekte Aussageform*), or a form in which all components we explicate as variable are replaced by the appropriate signs. On Scholz's reading of Bolzano's § 148, a statement is analytic, if and only if it can be obtained through replacement (*Einsetzung*) from a perfect statement-form which always turns out either true or false. Scholz's distinction between a sentential form and a perfect sentential form corresponds to Bolzano's distinction between analytic and logically analytic statements.

10.3 Dubislav on Bolzano as a Precursor of Modern Formal Logic

Dubislav argues that Bolzano anticipated the thinkers of his time and that is why there was no fruitful interaction between his theories and theirs, because he was misunderstood or ignored by his contemporaries.⁸ Dubislav attempts to bring Bolzano into the contemporary discussion by reformulating his discoveries in contemporary terms, so as to show the Prague philosopher's relevance to contemporary views, at the risk of misrepresenting the latter's claims. Thus Yehoshua Bar-Hillel (1952, 337–338) acknowledges Dubislav's evaluation of Bolzano's contributions to logic whilst rejecting Dubislav's (1931b) claim that Bolzano anticipated modern *mathematical* logic.⁹ Bar Hillel also rejects the claim

⁷Bolzano holds that if a proposition is true, it expresses the sense of a certain combination of words. Omnipotence can be predicated of God if and only if the subject "God" actually has this property, otherwise the proposition is false and has no sense (1837, § 28).

⁸If Bolzano was little known at his time, the main reason was political rather than scientific: a Roman Catholic priest and professor of theology, Bolzano was removed from his post at the German University of Prague in 1820, after nearly being excommunicated for criticizing the official theological manual. The mathematical discoveries of the young Bolzano, such as the 1917 theorem that, given any bounded sequence (a_n) of real numbers, there exists a convergent sub-sequence (a_{n_j}) which was later called the "Bolzano–Weierstraß theorem", remained unnoticed until it was independently re-discovered by Weierstraß 50 years later. Dubislav (1931d, 344, 1931e) briefly mentions Bolzano's contributions to mathematics. Bolzano's logical and philosophical teachings were, however, propagated in the Danube Monarchy by his students R. Zimmermann and F. Přihonský and influenced philosophers such as Husserl and Meinong.

⁹"The expression 'mathematical logic' is not free of ambiguities, but if its component 'mathematical' is not to be devoid of any literal value, then we cannot assent to Dubislav when he calls Bolzano "a forerunner of *mathematical* logic". There seem to be among German logicians a certainly understandable tendency to praise Bolzano beyond his certainly great merits. Even if

of Scholz–Dubislav (Dubislav 1931c) that Bolzano anticipated modern semantic logic. Borrowing Carnap’s terminology he argues that Bolzanian concepts such as statement as such or proposition (*Satz an sich*), presentation (*Vorstellung*) and variable presentation (*veränderliche Vorstellung*) “belong to the *non-semiotical* part of the meta-language of the object language dealt with, which was colloquial German for Bolzano and is ordinary English with us. They do not belong to the *semantical* part of the metalanguage, and in their definition no mention is made of any semantical concepts such as ‘designate’, ‘express’, etc.” (1952, 324). *Pace* Bar-Hillel, Bolzano does not distinguish between meta-language and object-language as Carnap and Tarski did, though he did make semantic innovations, for example by introducing the distinctions between subjective and objective presentations and subjective and objective statements—the latter being the significations of mental or linguistic expressions. In addition, Bolzano was attentive to the role of signs and their signification which are key semantic and semiotic notions. He also introduced the notion of explication (*Verständigung*), a statement communicating the meaning or signification an interlocutor relates to a certain sign (1837, §§ 285, 668).¹⁰ Nonetheless, Bar-Hillel has a point in saying that these innovations do not belong to semantic *logic*—or to formal semantics.

Although Bolzano did not formalize his theory, modern and contemporary logicians who are his commentators, did.¹¹ Bolzano’s contribution to modern formal logic is his method of variation, his notion of analytic statements and what Dubislav (1929, 1930a, b, 1931b, d) and some other commentators (Bar Hillel 1952) call propositional functions. Whether or not this name is appropriate for Bolzano’s analytic statements is discussed below. Another Bolzanian innovation is his notion of derivability (*Ableitbarkeit*) which is a precursor of the modern notion of logical consequence—or, as Dubislav (1930a, b, 1931b, d) holds, the notion of implication—as well as the notion of grounding (*Abfolge*) which contributes to the logic of explanation and to methods of deductive knowledge.

10.3.1 *Dubislav on Bolzanian Propositional Functions*

Bolzano’s notion of analytic statements is arguably a predecessor of propositional functions in the sense that their analyticity depends on their containing at least one presentation which may be arbitrarily varied to produce either true or false variants of the original proposition. But, *pace* Dubislav, Bolzano’s equivalent to propositional functions is not so much the notion of propositional forms but rather

he did not anticipate either semantics or mathematics, he did investigate topics far beyond his own time and created foundations for many disciplines of actual value.” (1952, 337–338).

¹⁰See on this Kasabova (2006).

¹¹Bar Hillel (1950, 1952), Corcoran (1993), Etchemendy (1999), Künne (2006, 2008), Dubislav (1931c), Siebel (1996, 1999, 2002), and Tatzel (2002).

the variation of presentations, even though the “*veränderliche Vorstellung*” is not, strictly speaking, a variable quantity which may assume any one of a set of values. “Variable” translates *veränderlich*, though neither Bolzano nor Přihonský used the word “variable” in its contemporary mathematical sense. In their usage, a *variable* is not a letter but refers to a constant which can be replaced and produce new statements.

Dubislav repeatedly praises Bolzano’s “uncovery of those judgment-forms containing variables” (“*Aufdeckung von derartigen Variable enthaltenden Urteilsformen*”) and the “so-called assertive or propositional functions” (“*sogenannte Aussage- oder Satzfunktionen*”) (1929, 365, 1930a, 408, 1930b, 265, 1931b, 449–450, 1931c, 206, 1931d, 341, 1932e).

“By uncovering those judgment-forms containing variables, Bolzano made one of the deepest discoveries in the domain of elementary logic. These formations (*Gebilde*), which Bolzano designated as statements with variable presentations, are called propositional functions. These formations are such that, if the variables contained in them are replaced by their values according to a rule of substitution (*Substitutionsvorschrift*), one obtains statements in the usual sense of the word. So we can designate those assertive or propositional functions as casting moulds for sentences.”¹² (1929, 365, my translation—A. K.)

By way of criticizing Kant’s philosophical claims and reconstructing his views using a rigorous method of inquiry, Bolzano discovered important logical and philosophical principles. According to Dubislav, Bolzano’s discovery of propositional forms is one of the reasons why (1929, 1931b) Bolzano is not only ‘Kant’s critic’ but also ‘a precursor of mathematical logic’. In words that Dubislav borrowed from the French logician and mathematician, Louis Couturat (1905), sentential or, as Russell says, propositional forms, are casting moulds of linguistic expressions.¹³ Dubislav (1931b, 450–451, 1931c, 341) considers Bolzano’s use of the method of variation of the component-parts of propositions a “classical discovery” of the “so-called assertive or propositional function”: “He characterizes an assertive function as follows, which we render in the terminology used today: an assertive function is

¹²“Mit der Aufdeckung von derartigen Variable enthaltenden Urteilsformen hat nun Bolzano eine der tiefsten Entdeckungen auf dem Gebiete der elementaren Logik gemacht. Man nennt diese Gebilde, die Bolzano selbst als Sätze mit veränderlichen Vorstellungen bezeichnet hat, Satzfunktionen. Es sind also Gebilde so beschaffen, daß, wenn man die in ihnen enthaltenden Variablen nach einer Substitutionsvorschrift durch Werte derselben ersetzt, Sätze im üblichen Sinne des Wortes resultieren. Man kann also anschaulich derartige Satzfunktionen mit L. Couturat als Gießformen für Sätze bezeichnen.”

¹³Unfortunately Dubislav gives no reference for Louis Couturat. In 1905 Couturat published *Les Principes des Mathématiques: avec un appendice sur la philosophie des mathématiques de Kant, L’Algèbre de la logique, Les définitions mathématiques* and *Définitions et démonstrations mathématiques*. The last two works are cited in the bibliographie of *Die Definition* (1931a), as well as the German translation of *Les principes des mathématiques* (1908). It is likely that Dubislav refers to Couturat (1905) when citing the expression “Gießformen”.

a formation containing one or several place-holders such that a proposition (*Satz*) results, if a place-holder is filled in according to a rule of replacement.” (Dubislav 1931b, 450–451, my translation—A. K.).¹⁴

In order to emphasize Bolzano’s anticipation of modern and mathematical logic, Dubislav’s reconstruction sometimes departs from the former’s original account. Thus Bolzano does not use concepts such as “*Leerstelle*”, (“place-holder”) or “*Einsatzvorschrift*” (“*rule of substitution*”) but “variable presentations” which are replaceable by other presentations (1837, 147). In footnotes to (1929, 449, 1931c, 340), Dubislav remarks that Bolzano designates what he himself calls propositional functions as (declarative) statements with variable presentations (*Sätze mit veränderlichen Vorstellungen*), a hint that commentators inevitably construct their own views by reconstructing the theory of an earlier author. Sure enough, Dubislav’s (1931a, 116) notion of determination of concepts is very close to the ‘propositional function’ he imputes to Bolzano. This is interesting for two reasons:

1. Dubislav’s notion of concept corresponds to his characterization of Bolzano’s variable presentations: “Concepts in a logical sense are merely signs of a particular kind, namely signs in the shape of sentential- or (as they are also called) propositional functions of a variable. Such a propositional function [. . .] is taken as [. . .] a casting mould for statements [. . .]. A propositional function of a variable is produced if, in a statement, a sign is substituted by a variable [. . .].” (1931a, 116, my translation—A. K.).¹⁵ The mathematical logician Kurt Grelling (1932, 197), an active member of the Berlin Group until 1937, objects against Dubislav that, on the latter’s view, propositional functions are signs but Dubislav does not explain what they designate, i.e. what they mean or signify (*bedeuten*). Could Dubislav’s propositional functions be the meanings or significations (*Bedeutungen*) of concepts? Apparently not, because Dubislav (*ibid*) also claims that a propositional function represents or stands for the concept “prime number”. Hence propositional functions are signs and concepts are meanings or significations of signs.
2. Dubislav’s (1931a, 116–117) view that concepts are meanings of signs is very close to Bolzano’s account of *Vorstellungen an sich* (which may or may not be arbitrarily replaceable in a declarative statement).¹⁶ Bolzano distinguishes between subjective presentations which are mental or linguistic acts and

¹⁴“Eine Aussagefunktion charakterisiert er folgendermaßen, wobei wir die heute übliche Terminologie benutzen: eine Aussagefunktion ist ein Gebilde, welches ein oder mehrere Leerstellen dergestalt enthält, daß, wenn man die Leerstelle nach Maßgabe einer Einsatzvorschrift ausfüllt, eine Aussage resultiert”.

¹⁵“Begriffe im Sinne der Logik sind lediglich Zeichen besonderer Art. Und zwar Zeichen in Gestalt von Aussage- oder, wie man sie auch genannt hat, Satzfunktionen einer Variablen. Unter einer derartigen Aussagefunktion [. . .] versteht man [. . .] eine Gießform für Aussagen [. . .] Eine Aussagefunktion einer Variablen resultiert, wenn man sich innerhalb einer Aussage ein Zeichen durch eine Variable [. . .] ersetzt denkt”.

¹⁶“Vorstellung [. . .] welche sich willkürlich abändern läßt” (1837, § 148.1)

objective presentations or presentations as such which are constituent parts of declarative statements as such or propositions (1837, § 50). He characterizes objective presentations (*Vorstellungen an sich*) as significations or meanings (*Bedeutungen*) of signs which are designated by signs (*ibid.*, § 285). Dubislav (1931a, 117) comments: “If one were to strip Bolzano’s explications of their mystical character, that which he wanted to be understood as a presentation as such can be determined as a propositional function of a variable which has precisely those attributes which Bolzano ascribed to his presentations as such.” (my translation—A. K.).¹⁷

Hence Dubislav’s view that concepts are significations or meanings of signs is Bolzanian but is it formalist? In *Die Definition* (1931a, 113) Dubislav classifies Bolzano’s account of determination of concepts (*Begriffsbestimmung*) as idealist, distinguishing Bolzano’s notion from his own which he calls formalist. Is Dubislav’s view formalist? Grelling (1932, 198) says ‘no’. On Dubislav’s view, it seems that for a formalist, concepts are signs without meaning or signification and thus Grelling (1932) objects that Dubislav has not explained what a concept designates. If Dubislav holds that it designates a class, as he seems to (1931a, 116), what are the criteria for designating an object as belonging to one class rather than another? Grelling concludes his objection against Dubislav’s ‘formalist’ definition of concepts with the deadly question which level of existence he would ascribe to a class?

“Physical reality can hardly be ascribed to [a class]—it is not something one can meet in the woods [. . .]. Then again one can hardly express it as a presentation so, if at all, one would have to ascribe ideal existence to it, which [. . .] amounts to having jumped out of the frying pan into the fire, or [. . .] having cast out the devil with Beelzebub.” (1932, 198, my translation—A. K.).¹⁸

10.3.2 *Dubislav on Bolzano’s Notions of Derivability and Probability*

Bolzano’s notion of derivability (*Ableitbarkeit*) is characterized by a compatibility constraint (*Verträglichkeit*) and a substitutional criterion. Bolzano uses the method of variation—the idea that components of declarative statements can be varied or

¹⁷“Wenn man die Erläuterungen, die Bolzano für das, was er unter einer Vorstellung an sich verstanden wissen wollte, ihres mystischen Charakters entkleidet, dann ist festzustellen, daß eine Aussagefunktion einer Variablen im obigen Sinne gerade diejenigen Beschaffenheiten besitzt, die Bolzano seinen Vorstellungen an sich zuschrieb.”

¹⁸“Physische Wirklichkeit kann man ihr nicht gut zuschreiben, man kann ihr nicht im Walde begegnen [. . .]. Da man sie auch nicht gut als Vorstellung aussprechen kann, so muß man ihr, wenn überhaupt, eine ideale Existenz zuschreiben, womit man also vom Regen in die Traufe gekommen ist, oder [. . .] den Teufel mit Beelzebub ausgetrieben hat.”

substituted—in his account of derivability (*Ableitbarkeit*). Some statements P_1 to P_n are derivable from other statements Q_1 to Q_n with respect to common components i_1 to i_n if and only if all substitutions of presentations i_1 to i_n which produce only true statements in P_1 to P_n also produce only true statements in Q_1 to Q_n and the propositions are *compatible* relative to their variable components i_1 to i_n if at least one substitution for i_1 to i_n produces only true statements in P_1 to P_n (WLII, §§ 154, 155.2).

Dubislav (1930a, b, 1931d), points out that derivability is a special kind of compatibility (*Art der Verträglichkeit* 1931b, 451) and corresponds to the notion of formal implication in mathematical logic. Having asserted that Bolzanian derivability is a precursor of formal implication, Dubislav remarks that in the latter accounts the compatibility constraint is left out (Dubislav 1931b, 452). Berg (1999) and Siebel (1996, 1999) concur that Bolzano's account of derivability is not easily reconciled with modern logic. The reasons are twofold: first compatibility is a three-place relation between statements as such on one hand, and presentations, on the other, since two statements as such (or two classes of statements as such) are compatible with regard to their variable components (the presentations). (1837, § 154) Second, derivability is not a relation between linguistic signs but the semantic content or sense of linguistic signs.

Derivability, as Dubislav (1931b) explains, is a relation of implication which relates to the validity (*Gültigkeit*) of formal or logical implication as well as material or relative implication, for Bolzano examines the process of deduction leading from premises to conclusions and the validity of implicational statements or inferences. Thus Q is deducible from P in a step-by-step deduction showing that Q is true if P is true. Dubislav (1931b) also notes that Bolzano uses two kinds of derivability (formal and material). Bolzano's distinction between derivability in a broad and narrow sense is similar to his distinction between analytical statements and logically analytic statements. As Dubislav puts it, Bolzano “realized that there are two kinds of derivability relations: first, those which mere logical knowledge suffices to determine. Second, there are those relations which can only be determined by means of extra-logical knowledge”.¹⁹ (1931b, 452–453, my translation—A. K.)

Bolzano's derivability in a broad sense (material implication) holds for conditionals related by the ‘*if . . . then*’ conjunct where ‘implies’ relates parts of a sentence to make a more complex sentence. Bolzano considers such implications as conditionals “in the broad sense of derivability” which require knowledge outside the domain of logic (1837, § 223). He gives the following example: Caius is a man

¹⁹“Bolzano erkannte nämlich, daß es zwei Arten von Ableitsbeziehungen gibt. Erstens solche, zu deren Feststellung man lediglich logischer Kenntnisse bedarf, und zweitens solche, zu deren Feststellung außerlogische Kenntnisse herangezogen werden müssen.” Apparently Bar-Hillel (1952) did not read this part of Dubislav's reconstruction of Bolzano because he claims that Bolzano “does not distinguish, strangely enough, between material and formal derivability, but he does so, for instance, with respect to a closely related concept, that of consequence (*Abfolge*).” (Bar-Hillel 1952, 86). *Pace* Bar-Hillel, Bolzano's logical derivability is close to the modern notion of *consequence*, whereas *Abfolge* is *grounding* (or ground-consequence).

implies Caius has an immortal soul, where ‘implies’ is relative to Caius. To accept or understand (*einsehen*) this, “we must know that all human souls are immortal”. But in order to know that the implication is correct (*richtig*), it suffices to recognize it as an instance of the inference scheme ‘for every x , if x is a man, then x has an immortal soul’. Cases where ‘implies’ denotes a material implication, that is, if the ‘ A implies B ’ means that A is false or B is true are problematic because of counter-intuitive results—the so-called paradoxes of material implication: either the whole conditional is true whenever the antecedent is false or the whole conditional is true whenever the consequent is true.

Later commentators (Corcoran 1993; Etchemendy 1999) hold that Bolzano’s notion of derivability is a (primitive) precursor of Tarski’s (1936) notion of logical consequence. Bolzano’s notion of logical derivability (derivability in a narrow sense) is close to Tarski’s logical consequence, since Bolzano says that in cases of derivability such as ‘ A implies B or A implies not- B ’ all except the logical presentations have to be varied. Logical derivability is a relation using the *if . . . then* construction denoted by the verb ‘*implies*’ which can relate either sentence-schemas, as in ‘ B is a bachelor, implies B is unmarried’ or parts of a sentence, as in conditional clauses.

But the analogy between Bolzano and Tarski is limited (Siebel 1996, 1999). Tarski’s notion of logical consequence does not have a compatibility constraint, nor does it hold between the contents of statements. In addition, it concerns a meta-linguistic framework and an interpreted language and turns on truth-conditions or satisfaction conditions of propositional functions. A propositional function Fd is satisfied if and only if all properties of F are satisfied by a domain or set of individuals d which is defined by the properties of F . For Tarski, “[t]he sentence X follows logically from the sentences of the class K if and only if every model of the class K is also a model of the sentence X ” (1936, 417). For Tarski, the replaceable elements are the objects falling under the variables of the non-logical constants, whereas for Bolzano, the replaceable elements are non-logical presentations which are parts of statements as such.

Dubislav does not mention Bolzano’s notion of grounding (*Abfolge*): a statement that p is true because q . The grounding relation provides the semantic conditions for a deduction (*Ableitung*, *Herleitung*). These conditions are designated by the conjunct ‘*because*’ which denotes the grounding relation, a very peculiar relation, by virtue of which some terms act as grounds to others (Bolzano 1837, § 162).²⁰ The explanatory force of the grounding relation lies in “drawing out the elements of an implicit deduction”, by means of which we “obtain the key to new truths which were not clear to common sense”.²¹ Unlike derivability, the grounding relation holds

²⁰“Ein sehr merkwürdiges Verhältnis, vermöge dessen sich einige derselben zu andern als Gründe zu ihren Folgen verhalten.” Bolzano 1837, § 162; § 221.note: “der Begriff einer solchen Anordnung unter den Wahrheiten, vermöge deren sich aus der geringsten Anzahl einfacher Vordersätze die möglich größte Anzahl der übrigen Wahrheiten als bloßer Schlußsätze ableiten lasse”.

²¹1810, *Beyträge* II, § 2; 1837, § 401.

only between true sentences of the form: ‘ p because q ’ which are compatible as ground and consequence and its terms are either single sentences or collections of sentences.

Instead, Dubislav (1930a, 409, 1930b, 264–265, 1931d, 343) relates Bolzano’s notion of derivability to the latter’s account of probability (*Wahrscheinlichkeit*). He considers Bolzano’s probability as containing (*enthalten*) a derivability relation.²² Dubislav’s reconstruction is based on the premise that Bolzano’s notions of derivability and probability are characterized by the compatibility constraint. A derivability relation is a probability relation with the numerical value $P = 1$ and the numerical values of this relation lie in the interval $0,1$ (1930b, 264). More recently, Jan Sebestik (2007, 38–39) takes up Dubislav’s point, adding that it is Bolzano’s compatibility constraint which enables “the extension of deductive logic to inductive logic via probability.” Sebestik praises what he calls Bolzano’s extraordinary achievement in providing “the first logical definition of probability. For the first time deductive logic and inductive logic are united in a global theory and the former appears as a limit case of the latter.” (ibid., 38–39).

Bolzano was hardly the first mathematician and logician to deal with probability—he Bolzano agrees with Laplace that probability is a relation in which the number of propitious (*günstige*) cases stands to the number of possible cases (1837, § 161, note 2). His innovations are (1) that he provided a systematized account of probability as a property (*Beschaffenheit*) of statements and (2) that he introduced the distinction between objective probability or the ratio of the number of true variants to the number of (collection of) statements and subjective probability or degrees of confidence and credibility. A statement M has objective probability $P = 1$ or certainty if M is derivable from a collection of statements C relative to variable presentations i . If M is not derivable from C relative to variable presentations i , it has the probability $P = 0$, that is, M and C are incompatible. Thus certainty and incompatibility are the limits of probability with the values 1 and 0. In addition, Bolzano introduces conditional probability: a statement M is conditionally probable if its probability is $0 < P < 1$ (Bolzano 1837, § 161.1). Dubislav leaves out Bolzano’s account of subjective probability: the degree of confidence with which we judge that p (or take p to be true) is Bolzano’s tool for determining the limits between a cognition (*Erkenntnis*) and an error (*Irrtum*) (Bolzano 1837, § 317).

²²Twice Dubislav (1930a, 409, 1931d, 343), Dubislav cites the following passage in the *Wissenschaftslehre* (1837, § 161.1): “Let us consider certain presentations $i, j \dots$ in a single proposition A or in several propositions A, B, C, D, \dots as variable, and in the latter case suppose that propositions A, B, C, D are in a relation of compatibility in regard to these presentations. Then it will often be particularly important to know the relation of the collection of cases in which propositions $A, B, C, D \dots$ all become true, stands to the collection of those cases in which an additional proposition M becomes true, and whether we should also take M to be true or not. For if the latter collection comes to half of the former, we can hold M to be true merely on account of the truth of propositions $A, B, C, D \dots$ and if this is not the case, then we cannot. So I permit myself to call this relation between said collections the *relative validity* of proposition M in regard to propositions A, B, C, D , or the *probability* proposition M attains from the *presuppositions* A, B, C, D .” (my translation—A. K.).

10.4 Dubislav and Bolzano on Definition

Dubislav (1931a) distinguishes between the following accounts of definition (1) determination of essence or *definitio rei* (*Wesensbestimmung*, *Sacherklärung*); (2) determination of concepts or *definitio nominis* (*Begriffsbestimmung*); (3) setting conditions for the meaning or usage of a sign or *definitio lexicalis* and (4) stipulating the meaning of a new sign or the new usage of a familiar sign or *definitio stipulationis*. In addition, he expounds the concept of definition, determines the relation between *definiens* and *definiendum* and examines rules for use of concepts in definitions. He is particularly interested in (2) or the definability of concepts (*Begriffsbestimmung*).

As I mentioned in Sect. 10.2.1, Dubislav (1931a, 116) considers concepts as meanings of signs and signs as sentential forms of variables (*Aussagefunktionen von Variablen*).²³ However, Dubislav (1931a, 1932) does not rebut Grelling's (1932, 194) objection that he fails to distinguish between two kinds of dependence relations: (i) the relation between assumptions in a system of assumptions and (ii) the relation between the sense of statements and the definition of the signs occurring in them. For Dubislav (1931a, 117) claims that the *sense* of statements depends on the propositional functions which depend on a system of assumptions (*System der Voraussetzungen*)—that is, he aims at (i) whilst dealing with (ii). According to Grelling (1932, 193–194), the *sense* of a statement depends on the definition of the signs occurring in it. Thus the sense of the statement “the events *A* and *B* occurring at places *a* and *b* are simultaneous” can be quite different, depending on how simultaneity is defined, but this sense does not depend on a whole system of assumptions which mutually support each other.

Given Dubislav's investigations of Bolzano's contributions to modern logical theories, it is somewhat surprising that he did not heed the latter's views on definition in (Dubislav 1931a). Despite Dubislav's (1931a, 114–17) abridged reading of Bolzano as a Platonist idealist (which is wrong, because Bolzano did not postulate a ‘third realm’ of mind-independent entities), he does, as he puts it, extract the notion of propositional function from the latter's presentations as such (ibid., 117). Dubislav's (1931a) take on Bolzano appears slightly confusing: first, he classifies the latter's account of definition in (2) as an idealist determination of concepts (ibid., 117) and then he considers it as a case of (1) or determination of essences in the Aristotelian tradition (ibid., 133–134) which he subsequently rejects. In order to clarify Dubislav's confusion and to evaluate his objection, I reconstruct Bolzano's account of definition.

²³In a reply to Grelling (1932) and Dubislav (1932, 203), concedes having tacitly accepted Pascal's characterization of definition that, he now admits, is too narrow because it does not allow for inductive definitions in which newly introduced signs are not eliminable.

10.4.1 Bolzano on Definition

Dubislav's account of Bolzano is actually correct: Bolzano subscribes to (1) but, in different parts of the *Wissenschaftslehre* he also elaborates (2) and (4).²⁴ On Bolzano's 'Aristotelian' or 'Porphyrian' view (1), a definition says what something is, based on (essential) predication (1837, §§ 127, 128, 136, 137). A thing is defined by predicates ascribing to it its essential properties. Thus a definition answers the question "what is it?" Bolzano agrees with Dubislav that definition as predication presupposes a (linguistic) system in which a presentation is segmented into a subject- and predicate-signification or, as Bolzano says, a subject-presentation (*Subjectvorstellung*) and predicate-presentation (*Prädicativvorstellung*) which he sometimes calls grammatical subject (*Unterlage*) and grammatical predicate (*Aussageteil*). Bolzano claims that declarative statements are reducible to or transformable into a canonical form $[A \text{ has } b]$ or $[A \text{ has non-}b]$ —a uniform structure which holds for all declarative statements of natural language.²⁵ In other words, "is F " is predicated of a grammatical subject x and the predicate either says or does not say what x is. In addition, Bolzano stipulates a truth condition for the canonical form: $[A \text{ has } b]$ is true if and only the statement that A has b is objectual (*gegenständlich*), that is, if it asserts of its object (*Gegenstand*) that which actually belongs (*wirklich zukommt*) to it (1837, § 28, 124). Bolzano's canonical form of declarative statements is thus a *definitio rei*, as Dubislav (1931a, 133–134) points out. But Dubislav criticizes Bolzano's version of (1) or the characterization of a thing's essential properties:

We should ask how Bolzano can ground his claim that the given attribute-presentation b belongs to the objects in question by virtue of the mere concept under which we usually grasp them? He is obliged, for this purpose, to refer to his theory of truths and presentations as such [...] [and] to ascertain that such statements are valid not only relative to a system of basic presuppositions assumed as true, but *per se*. As a result, in our view his attempt at grounding becomes untenable. (Dubislav 1931a, 234, my translation—A. K.).²⁶

²⁴Bolzano gives an account of stipulative definition in part 4 of the *Theory of Science* and it is reconstructed in Kasabova (2006).

²⁵Bolzano famously claims that all statements in natural language are expressible by a uniform structure: "that the following holds of all propositions in general. The concept of *having* [...] the concept signified by the word *has* occurs in all propositions. Besides this one component two others occur [...] in all propositions connected with each other by a *has* as indicated in the expression $A \text{ has } b$. One of these components, namely the one indicated by A , stands as if it were to present the object dealt with in the proposition and the other, b , as if it were to present the *attribute* (*Beschaffenheit*) the proposition ascribes to that object. Therefore I permit myself to call [...] A the *supporting* or subject-presentation; [...] and b the *assertive part* (*Aussageteil*) or *predicate presentation*." (1837, § 127, my translation—A. K.). Cf. on this Textor (1997).

²⁶"Wie kann aber Bolzano, dass ist zu fragen, seine These begründen, daß die genannte Beschaffenheitsvorstellung b den fraglichen Gegenständen vermöge des bloßen Begriffes zukommt, unter dem wir sie aufzufassen pflegen? Er ist genötigt, sich zu diesem Zwecke auf seine Lehre von den Wahrheiten an sich und Vorstellungen an sich zu beziehen [...], zu ermitteln, daß derartige

Pace Dubislav, Bolzano does not characterize a thing's essential properties by virtue of truths and representations as such, but by means of a definition with a grammatical subject-predicate structure—that is, by virtue of a linguistic system which determines the linguistic structure of definitions. This structure is syntactic: the order [subject-presentation—copula-presentation—predicate-presentation] and semantic: the sense of the statement [A has *b*] is what is expressible by [A has *b*] or what is meant. In other words, the components of a declarative statement are determined by the order of that statement. Dubislav may have been misled by Bolzano's truth condition: "a statement is true if it ascribes to its object something that belongs to it" (1837, § 124). But Bolzano does not merely provide a truth condition for the *definitio rei*; rather, his analysis concerns the structure of *Sätze an sich* and a semantic account of truth as a property of propositions (Textor 1997; Künne 2006).

At first blush it seems that for Bolzano, the essence of a thing is the collection (*Inbegriff*) of all properties derivable or inferrable from the concept (*Begriff*) of that thing (1837, §§ 111, 502) which corresponds to Dubislav's suggestion: Bolzano's *definitio rei* names a presentation such that we may infer from it all essential properties of the correlative object. Bolzano (*ibid.*, § 502), however, revises the explanation given in § 111 and proposes to narrow it down by distinguishing between essential and derived properties of things. The first are necessary and the second are accidental. He would reply to Dubislav that an essential property of things must also be their necessary property and vice versa. For example an essential and necessary property of a triangle is that it is a system of three points. On the other hand, the property that the sum of all angles is equal to two right angles is a derived (*abgeleitete*) property of a triangle which objectively follows from (*abfolgt*) its essential property.²⁷ This latter property is not an essential property in the narrow sense. Thus the essence (*Grundwesen*) of a thing is the collection (*Inbegriff*) of only those properties yielded by its concept which are not inferrable (*herleiten*) from any other concept of that thing as consequences from a ground.²⁸ In addition, Bolzano might have asked Dubislav in regard to (1) whether a determination of essence (*Wesensbestimmung*) is equivalent to a *Sacherklärung* and whether a *definitio rei* determines a thing or a presentation of that thing. In the latter case, is the definition a *definitio rei* or a *definitio nominis*?

Aussagen nicht nur relativ zu einem als wahr unterstellten System von Grundvoraussetzungen gelten, sondern schlechthin. Damit wird aber sein Begründungsversuch für uns hinfällig."

²⁷"It does not lie as a constituent in the concept of a triangle, but is only a consequence ensuing from this concept (*nur eine aus diesem Begriffe sich ergebende Folgerung*), that a triangle could be equilateral." (1837 § 55.10c, my translation—A. K.).

²⁸"In this narrower meaning (*Bedeutung*) one takes the essence (*Wesen*) of a thing, also called the grounding essence (*Grundwesen*) to discern it better, as the collection of only those attributes ensuing from its mere concept, which cannot be objectively derived (*herleiten*) from any other concept of it (i.e. as consequences from their ground, § 198)." (1837, § 502, my translation—A. K.).

10.4.1.1 Bolzano on Nominal Definition

In addition to the traditional *definitio rei*, Bolzano works out an account of *definitio nominis*. He applies the distinction between constitutive (*constitutiven*) and derived (*abgeleiteten*) distinctive features (*Merkmale*) not only to things but also to presentations (1837, § 65.10, 120). He also introduces a crucial distinction between the components (*Bestandteile*) of a presentation (namely a property concept) and the attributes (*Beschaffenheiten, Merkmale*) of an object. Let us reconsider Dubislav's question: "We should ask how Bolzano can ground his claim that the given attribute-presentation *b* belongs to the objects in question by virtue of the mere concept under which we usually grasp them?" (Dubislav 1931a, 134). Dubislav is asking how Bolzano can justify his claim that a property concept belongs to the objects in question by virtue of the concept by which we grasp them.

Bolzano has a ready reply: a property concept is not composed of the features of its object but the properties of an object can be derived or inferred from the concept of that object (the property concept) without being thought as constitutive parts of that concept. "Wie es aber möglich sey, daß ein Gegenstand Theile habe, deren Vorhandenseyn aus unserer Vorstellung gefolgert werden kann, ohne daß ihrer darin gedacht wird; daß läßt sich freilich nicht eher wohl begreifen, bis man den Unterschied, der zwischen *Bestandtheilen* und *Merkmalen* obwaltet, deutlich eingesehen hat." (1837, § 65.8).

Bolzano expounds a general account of definition where non-essential properties and non-essential property-presentations are inferrable from the essential properties of objects. Only the latter are also constituents of presentations. Otherwise he would have to accept the erroneous claim (which he rebuts) that if an object has an infinite number of properties, the concept of that object would have to have an infinite number of constituents.²⁹ In order to protect this account against a conflation between properties of objects and constituents of presentations, Bolzano therefore has to reject structural isomorphism between objects and presentations.³⁰ Consequently, the first of Bolzano's conditions for a *definitio nominis* is that (i) there is no structural isomorphism between a presentation and an object: the components of a presentation are not to be confused with the attributes of its object (*ibid.*, §§ 63, 64, 65).

The second condition, also related to the rejection of structural isomorphism, is (ii) the distinction between components or constituents (*Bestandtheile*) of

²⁹"In my view it is by no means necessary that a concept ensuing that the object corresponding to it is composed of so and so many parts, should be composed of just as many parts (such as the presentations of those particular parts)" (1837, § 65.7, my translation—A. K.).

³⁰A further reason for rejecting structural isomorphism are, as Bolzano points out, cases of complex objectless presentations such as [a regular 10-chiliagon (*Zehntausendeck*)], [round square], [blue yellow] or [golden mountain] which have no corresponding object, as well as objectual presentations comprising relative clauses, such as [a land without mountains] or [a book without copper] in which the attributive concept does not correspond to any property of objects falling under that concept but to properties the object is lacking (1837, §§ 63, 66, 70).

presentations and distinctive features (*Merkmale*) or attributes (*Beschaffenheiten*). An attribute of an object is not a component of the concept under which that object falls. Nor is the collection of properties belonging to and determining an object structurally isomorphic with the collection of the presentations (*Inbegriff*) of these attributes (*ibid.*, § 64). In addition, attributes that necessarily belong to the presentation of an object are not presented as constituents of that concept. For instance, an equilateral triangle necessarily falls under the concept [equiangularity], yet [equiangularity] is not a necessary constituent of the concept [equilaterality]. Equiangularity is a necessary attribute of an equilateral triangle without necessarily being thought or presented in the attribute-concept [equilaterality] defining that object—it is inferrable from the concept [equilaterality] (*ibid.*, § 64). The third condition (iii) for nominal definitions is thus the distinction between constituents or components of presentations on one hand and the presentations of attributes or features of objects, on the other (*ibid.*, § 65.9). As Bolzano explains in the *Paradoxes of the Infinite*, in order to think a collection (*Inbegriff*) it is not necessary to think all the objects composing it (1831, § 14). For instance, I can think of an orchestra without thinking of all its players. In fact, I wouldn't need to think set-theoretically of all its members or mereologically of all its parts, because a collection is defined by what it does—thus the bass viol and the violin are essential and necessary constituents of the collection [orchestra] which I might think of, whilst the other constituents are inferrable.³¹ Likewise, equiangularity and equilaterality are attributes of triangles which are inferrable from the concept [triangle] but they are not components of that concept (1837, § 65.10).

Bolzano's *definitio nominis* therefore allows for the inferrability of attributes from the concept of a given object without conflating concepts with their objects. The inferrable or derivable attributes are non-essential properties of the object and therefore they are not constituents of the concept of that object. Essential attributes such as “triangularity”, however, are also constituents of the concept [triangle] since they belong to the nominal definition of a triangle. Thus we can interpret Bolzano's claim that “whatever one must necessarily think in order to have really thought a given presentation is also a constituent of the latter” (*ibid.*, § 64.2, my translation—A. K.). Bolzano's account of nominal definition involves a clarification of the notion of intension or content of a presentation and of the relation between intension and extension. Having distinguished between the distinctive features or attributes of an object and the components of attribute-concepts, Bolzano rejects the structural isomorphism of objects and concepts which implies (i) that the content of a presentation is composed of the attributes of an object and (ii) that the content of a presentation is composed of subordinate presentations which stand under it. His clarification of the notion of intension involves criticizing Kant's notion of inclusion which is the latter's criterion for determining analytic judgements, discussed in Sect. 10.2 of this chapter. On Kant's view (shared by the young Bolzano, 1810,

³¹ Pace Kneale and Kneale (1962, 364), “Bolzano seems to be in danger of confusing a whole of parts with a set of members.”

§ 17), analytic judgments are those in which the predicate-concept (denoting the genus) is covertly contained in the subject-concept (denoting the species). On this view, analytic judgments function as nominal definitions because they relate *genus proximum* and *differentiam specificam*: the genus “extended” can be extracted from the species “bodies”—on the assumption that the content of a concept is composed of the sum total of partial concepts which are also attributes of the objects falling under that concept.

10.4.1.2 Bolzano and Dubislav on the Canon of Reciprocity

The later Bolzano (1837, § 120) explains that he was able to avoid the mistake of conflating the properties of an object with the components of its presentation and overloading (*überfüllen*) the intension of a concept by considering the content as a composition of its parts, due to Kant’s distinction between analytic and synthetic judgments which compelled him to clarify the relation between the intension and extension of a concept. Bolzano (*ibid.*, §§ 65, 120) and Dubislav (1931a, 12) note that Kant supported the so-called canon of reciprocity, namely that the intension and extension of a concept stand in an inverse relation: if the intension of a concept is conceived as a conjunction (*Knüpfoperation*) of attributes and its extension is conceived as the collection of objects comprised by this concept, then the more attributes or properties of objects are contained in its concept, the fewer are comprised (*umfassen*) by it or fall under it.

Dubislav (*ibid.*, 12–13) pays tribute to Bolzano’s critique of the erroneous canon of reciprocity,

according to which extension and intension of a concept stand in a reciprocal relation. Furthermore, this theory of concepts is connected with the claim [...] that the so-called partial presentations of a concept are always also features of the objects falling under that concept [...], a claim Bolzano also proved as incorrect. Hence the confusion of the two states of affairs ‘comprised by a concept’ and ‘falling under a concept’, produced the bewildering terminology in which so-called partial presentations of a concept are called features of that concept since, according to the above-mentioned claim, partial components of a concept comprise those features under certain conditions.³² (Dubislav, *ibid.*, 12–13, my translation—A. K.)

Dubislav comments that Bolzano clarifies the confusion between intensive and extensive relations with concept-concept and concept-object relations: subordination or comprehension (*Umfassung*) are relations of inclusion between concepts,

³²“wonach Umfang und Inhalt eines Begriffes sich zueinander reziprok verhalten sollen. Ferner wird mit dieser Begriffslehre die [...] ebenfalls von Bolzano als unrichtig erwiesene Behauptung verbunden, daß die sogenannten Teilvorstellungen eines Begriffes immer zugleich auch Merkmale der unter den Begriff fallenden Gegenstände [...] sein sollen. Daraus hat sich dann bei Verwechslung der beiden Sachverhalte “Von einem Begriffe umfaßt worden” und “Unter einen Begriff fallen” die verwirrende Terminologie entwickelt, die sogenannten Teilvorstellungen eines Begriffes Merkmale desselben zu nennen, weil unter der erwähnten Annahme die Teilvorstellungen eines Begriffes u. U. umfassen würden.”

whereas subsumption, ‘falling under’, ‘contained under’ or ‘contained by’ is a relation of extension between concepts and objects. He does not mention that Bolzano’s clarification is important for the notion of nominal definition.

Bolzano (1837, § 120) rejects both parts of the canon of reciprocity, as follows: (1) ‘the intension of a presentation may be increased without increasing its extension.’ (i) Consider redundant concepts such as [triangle which has the attribute equilaterality] in which the attribute of equilaterality is an added constituent of the concept [triangle] without increasing its extension. (ii) Consider auxiliary or adjunctive concepts which increase the content of the nominal concept without increasing its extension: the concept [round ball] has a larger content than the concept [ball] but their extension is the same. Bolzano’s example is, however, problematic, as well as his rejection of the second part of the canon. (iii) By adding a new constituent to a concept, it is possible to increase its extension by increasing its intension. Bolzano also uses this condition for rejecting (2) ‘the extension of a presentation may be increased without increasing its intension.’ He gives the following example: the concept [a man who understands all European languages] is increased in extension by adding [living] to its intension. Unfortunately for Bolzano, his example shows the validity of the canon he rejects: the concept [a man who understands all living European languages] has an increased intension but a decreased extension, for [all European languages] are thus limited to the living ones, excluding the dead languages which are included in the former.

Bolzano offers a better argument for (2): (iv) a subordinate concept may be built (*bilden*), increasing the extension of the main concept without adding something to its content, since it is not necessary for a subordinate concept to be partly composed of the concept comprising it. The concept [actual] is not a component of the concept [possible] although [actual] is subordinate to and inferrable from [possible] (*ibid.*, § 65.10).³³ As Dubislav (1931a, 12) says, “subordinated to” does not imply “a part of”. Bolzano would add that, precisely for this reason, analyticity is not correctly defined as an inclusion of the predicate-concept in the content of the subject-concept, nor is a concept appropriately defined by decomposing it. Instead, analyticity is based on the method of variation of presentations and a concept is adequately defined (essentially as well as nominally), if we distinguish between its intension and extension.

Bolzano’s contribution to the development of formal semantics is that his distinction between the content of a concept (*Bestandteile*) and its range of applicability over the particular objects it denotes (*Merkmale*) prefigures the distinction between intension and extension, the origin of which is officially attributed to Frege’s famous distinction between *Sinn* and *Bedeutung* (Frege 1892).³⁴ In my view, however, (*pace*

³³Cf. on this Künne (2008, 212–215).

³⁴Roman Jakobson (1980) notes Bolzano’s distinction between the meaning (*Bedeutung*) of a sign as such and the sense (*Sinn*) that this sign acquires in the context of the present circumstance. Unlike Frege, Bolzano uses *Bedeutung* to denote the presentation of a sign, which is why ‘meaning’ is the appropriate translation. Cf. Kasabova (2006).

Dubislav), Bolzano does not really refute the canon of reciprocity – the inverse relation of a concept’s intension and extension is still valid for nominal definitions, for the explanation of what a word or concept means and how it is used does not rely on investigating or enumerating the attributes of the thing(s) denoted by this word or concept. As Bolzano’s own example of adjunctive concepts such as [round ball] shows, on pain of circularity, the extension is not larger than the intension.

Unfortunately, Dubislav omits Bolzano’s important contribution to the notion of stipulative definition, expounded in part 4 of the *Theory of Science*, in a chapter called: *Theory of Signs or Semiotics* (see also 1837, § 637).³⁵ Bolzano (ibid., § 668.9) prefigures Carnap by advancing the notion of explication (*Verständigung*) as definition. An explication improves the existing notion in a particular context by creating a new usage (ibid., § 284). Bolzano uses stipulative definition as a kind of explication for presenting the key notions of the *Theory of Science* (ibid., § 668.9). He introduces the notion of presentations and propositions as such by specifying the new usage of a familiar concept.

Bolzano’s stipulative definition is based on the grounding relation (*Abfolge*): the property *isosceles* is an essential property of triangles because being triangular is inferrable from [isosceles], hence for Bolzano this kind of definition is inferential (ibid., §§ 111, 162, 198, 221.note).³⁶ In addition, inferential definition is important for determining infinite collections: a collection can comprise infinitely many items because it is determined by a generic concept and a classificatory principle: ‘belongs to *A* or does not belong to *A*’ (1831, § 14). Accordingly, Bolzano defines the concept [actual] as inferrable from [possible] (1837, § 65.10).

10.5 Conclusion

In this chapter I reconstruct Dubislav’s perspective on Bolzano, relating it to more recent discussions amongst Bolzanians. At times the discussion is underpinned by Kantian notions the critique of which has long since become a philosophical commonplace. Dubislav’s views on Bolzano—and Bolzano’s views on notions such as analyticity, validity, variation, derivability, probability and definition—are of interest for historians of logic and philosophy.

³⁵Jakobson (1980) points out Bolzano’s contribution to semiotics, although logicians and philosophers usually neglect this fact. Bolzano considered the theory of signs as belonging to methodology or the theory of science proper. Logic taken in a wide sense is a theory of science and the theory of science proper is the *organon* which regulates our acquisition of knowledge and includes a didactic theory of signs because Bolzano subscribes to the view that the correct understanding and use of words are based on a correct understanding of signs. See on this Kasabova (2006).

³⁶Jan Sebestik (1992, 139) notes that in Bolzano’s notion of explication paraphrastic elucidations or contextual definitions appear for the first time in the history of logic. Cf. Kasabova (2006, 13).

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