An Extension of Gurson Model to Ductile Nanoporous Media

L. Dormieux and D. Kondo

Abstract We extend the classical Gurson model for ductile porous media by incorporating the surface/interface stresses effect which characterizes pores at nanoscale. For interface stresses obeying a von Mises criterion, we derive closed-form expressions of the parametric equations defining the yield surface. The magnitude of the interface effect is proved to be controlled by a non dimensional parameter depending on the voids characteristic size. It is observed that nanoporous materials can be made more strengthened than non-porous counterparts.

Keywords Ductile nanoporous materials · Micromechanics · Surface stress · Interfaces · Nanovoids · Yield function · Gurson model

1 Introduction

Investigation of size-dependent effects in nanomaterials including materials containing nano-voids has focused the attention of many researchers during the last decade. Early works have tried to model the transition zone between the nano-inclusion and the surrounding matrix as a thin but still three-dimensional layer [1, 2]. An alternative approach consists in adopting an interface description which is two-dimensional in nature. Concerning inclusion size effects on the effective elastic properties, some progresses have been gained in their understanding. Classical homogenization schemes as well as first order bounds in the theory of elastic heterogeneous media have been extended in order to incorporate interface and interface stresses (see e.g. [3–5]). Recent studies by [6] and [7] extended Hashin-Shtrikman bound to the above class of materials.

In contrast, it seems that few attention has been paid so far to the question of the effective strength of nanomaterials with account for interface effects. Mention can

L. Dormieux (✉)
Laboratoire Navier, UMR 8205 CNRS, ENPC, Champs-sur-Marne, France
e-mail: dormieux@enpc.fr

D. Kondo
Institut D'Alembert, UMR 7190 CNRS, UPMC, Paris, France
e-mail: djimedo.kondo@upmc.fr

G. Saxcé et al. (eds.), *Limit State of Materials and Structures*,
DOI 10.1007/978-94-007-5425-6_11,

be made of recent works by [8] who used the modified secant moduli approach. In the context of the ductile failure of porous materials, the Gurson model [9] is well known to provide an efficient approach of the strength reduction due to the porosity. The purpose of the present paper is to extend this model in order to capture the influence of interface stresses.

To begin with, in view of subsequent extensions, the basic features of the classical Gurson approach are recalled. Then, the mechanical model of interface stress is introduced. Finally, the case of interface stresses obeying a von Mises failure criterion is considered.

2 Ductile Failure of Porous Media and Gurson Model

Let us consider a *r.e.v.* Ω of a porous material with porosity f. The solid domain is $\Omega^s \subset \Omega$. The average on Ω (resp. Ω^s) of a field $a(\underline{z})$ is denoted by $\overline{a}$ (resp. $\overline{a}^s$):

$$\overline{a} = \frac{1}{|\Omega|} \int_{\Omega} a(\underline{z})\, dV; \qquad \overline{a}^s = \frac{1}{|\Omega^s|} \int_{\Omega^s} a(\underline{z})\, dV \tag{1}$$

Let $\boldsymbol{\Sigma}$ and $\mathbf{D}$ respectively denote the macroscopic stress and strain rate tensors. $\mathscr{V}(\mathbf{D})$ is the set of microscopic velocity fields, $\underline{v}(\underline{z})$ being kinematically admissible with $\mathbf{D}$. The latter are defined by uniform strain boundary conditions:

$$\mathscr{V}(\mathbf{D}) = \left\{\underline{v},\ (\forall \underline{z} \in \partial\Omega)\ \underline{v}(\underline{z}) = \mathbf{D} \cdot \underline{z}\right\} \tag{2}$$

Let us consider a microscopic stress field $\boldsymbol{\sigma}(\underline{z})$ in equilibrium with $\boldsymbol{\Sigma}$ in the sense of the average rule $\boldsymbol{\Sigma} = \overline{\boldsymbol{\sigma}}$. Hill's lemma states that:

$$\boldsymbol{\Sigma} : \mathbf{D} = \frac{1}{|\Omega|} \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d}\, dV \tag{3}$$

The strength of the solid phase is characterized by the convex set G^s of admissible stress states, which in turn is defined by a convex strength criterion $f^s(\sigma)$:

$$G^s = \left\{\sigma,\ f^s(\sigma) \le 0\right\} \tag{4}$$

The dual definition of the strength criterion consists in introducing the support function $\pi^s(\mathbf{d})$ of G^s, which is defined on the set of symmetric second order tensors $\mathbf{d}$ and is convex w.r.t. $\mathbf{d}$:

$$\pi^s(\mathbf{d}) = \sup\left(\boldsymbol{\sigma} : \mathbf{d}, \boldsymbol{\sigma} \in G^s\right) \tag{5}$$

$\pi^s(\mathbf{d})$ represents the maximum "plastic" dissipation capacity the material can afford. In the absence of interface effect, the macroscopic counterpart of $\pi^s(\mathbf{d})$ is defined as:

$$\Pi^{hom}(\mathbf{D}) = (1-f) \inf_{\underline{v} \in \mathscr{V}(\mathbf{D})} \overline{\pi^s(\mathbf{d})}^s \quad \text{with} \quad \mathbf{d} = \frac{1}{2}\left(\text{grad}\, \underline{v} + {}^t\, \text{grad}\, \underline{v}\right) \tag{6}$$

Using Eq. (3) together with the definition equation (6), it can be shown that Π^{hom} is the support function of the domain G^{hom} of macroscopic admissible stresses:

$$\Pi^{hom}(\mathbf{D}) = \sup\big(\boldsymbol{\Sigma} : \mathbf{D}, \boldsymbol{\Sigma} \in G^{hom}\big) \tag{7}$$

The limit stress states at the macroscopic scale are shown to be of the form $\boldsymbol{\Sigma} = \partial \Pi^{hom}/\partial \mathbf{D}$.

Starting from this general framework, the classical Gurson approach devoted to porous media deals with the case of a von Mises solid phase:

$$f^s(\boldsymbol{\sigma}) = \frac{3}{2}\boldsymbol{\sigma}_d : \boldsymbol{\sigma}_d - \sigma_o^2 \tag{8}$$

where $\boldsymbol{\sigma}_d$ is the deviatoric part of $\boldsymbol{\sigma}$. The support function $\pi^s(\mathbf{d})$ accordingly reads:

$$\begin{aligned} &\operatorname{tr}\mathbf{d} = 0 : \pi^s(\mathbf{d}) = \sigma_o d_{eq} \quad \text{with } d_{eq} = \sqrt{\frac{2}{3}\mathbf{d} : \mathbf{d}} \\ &\operatorname{tr}\mathbf{d} \neq 0 : \pi^s(\mathbf{d}) = +\infty \end{aligned} \tag{9}$$

The Gurson model introduces two simplifications. It first consists in representing the morphology of the porous material by a hollow sphere instead of the *r.e.v.* Let R_e (resp. R_i) denote the external (resp. cavity) radius. The volume fraction of the cavity in the sphere is equal to the porosity $f = (R_i/R_e)^3$. Then, instead of seeking the infimum in Eq. (6), $\Pi^{hom}(\mathbf{D})$ is estimated by a particular microscopic velocity field $\underline{v}(\underline{z})$. In the solid, the latter is defined as the sum of a linear part involving a second order tensor $\mathbf{A}$ and of the solution to an isotropic expansion in an incompressible medium. In spherical coordinates, it thus reads:

$$\underline{v}^G(\underline{z}) = \mathbf{A} \cdot \underline{z} + \alpha \frac{R_i^3}{r^2} \underline{e}_r \tag{10}$$

In the pore, the strain rate is defined from the velocity at the cavity wall:

$$\mathbf{d}^I = \mathbf{A} + \alpha \mathbf{1} \tag{11}$$

The local condition $\operatorname{tr}\mathbf{d} = 0$ has to be satisfied in the case of a von Mises material (see Eq. (9)). It follows then that $\mathbf{A}$ is a deviatoric tensor: $\operatorname{tr}\mathbf{A} = 0$. Furthermore, the boundary condition equation (2) at $r = R_e$ yields:

$$\mathbf{D} = \mathbf{A} + \alpha f \mathbf{1} \tag{12}$$

which reveals that $\mathbf{A}$ is the deviatoric part $\mathbf{D}_d$ of $\mathbf{D}$, while α is related to its spherical part:

$$\mathbf{A} = \mathbf{D}_d; \quad \alpha = \frac{1}{3f}\operatorname{tr}\mathbf{D} \tag{13}$$

The combination of Eq. (11) and Eq. (13) also yields:

$$\mathbf{d}^I = \mathbf{D}_d + \frac{\mathrm{tr}\,\mathbf{D}}{3f}\mathbf{1} \tag{14}$$

Recalling Eq. (6), the use of $\underline{v}^G$ (giving strain rate $\mathbf{d}^G$) provides an upper bound of Π^{hom}:

$$\Pi^{hom}(\mathbf{D}) \leq (1-f)\overline{\pi^s\left(\mathbf{d}^G\right)}^s \tag{15}$$

Using Eq. (9), the derivation of the right hand side in Eq. (15) requires to determine the average of d_{eq} over Ω^s. In order to obtain an analytical expression, it is convenient to apply the following inequality to $\mathscr{G} = \mathbf{d} : \mathbf{d} = 3d_{eq}^2/2$ [9]:

$$\int_{\Omega^s} \sqrt{\mathscr{G}(r,\theta,\varphi)}dV \leq 4\pi \int_{R_i}^{R_e} r^2\left(\langle\mathscr{G}\rangle_{\mathscr{S}(r)}\right)^{1/2} dr \tag{16}$$

where $\mathscr{S}(r)$ is the sphere of radius r and $\langle\mathscr{G}\rangle_{\mathscr{S}(r)}$ is the average of $\mathscr{G}(r,\theta,\varphi)$ over all the orientations:

$$\langle\mathscr{G}\rangle_{\mathscr{S}(r)} = \frac{1}{4\pi r^2}\int_{\mathscr{S}(r)} \mathscr{G}(r,\theta,\varphi)\, dS \tag{17}$$

This eventually yields the following upper bound of $\Pi^{hom}(\mathbf{D})$:

$$\Pi_G^{hom}(\mathbf{D}) = \sigma_o f D_{eq}\left(\xi\left(\operatorname{arcsinh}(\xi) - \operatorname{arcsinh}(f\xi)\right) + \frac{\sqrt{1+f^2\xi^2}}{f} - \sqrt{1+\xi^2}\right) \tag{18}$$

with $D_{eq} = \sqrt{2\mathbf{D}_d : \mathbf{D}_d/3}$ and $\xi = 2\alpha/D_{eq}$. In the standard case (no interface effect), it is emphasized that the pore size R_i does not matter by itself since only the ratio $R_i/R_e = f^{1/3}$ intervenes in the expression Eq. (18).

The last step is the derivation of the limit states $\boldsymbol{\Sigma} = \partial\Pi_G^{hom}/\partial\mathbf{D}$. It is first observed that $\Pi_G^{hom}(\mathbf{D})$ is in fact a function of $\mathbf{D}$ through α and D_{eq}:

$$\boldsymbol{\Sigma} = \frac{\partial\Pi_G^{hom}}{\partial\alpha}\frac{\partial\alpha}{\partial\mathbf{D}} + \frac{\partial\Pi_G^{hom}}{\partial D_{eq}}\frac{\partial D_{eq}}{\partial\mathbf{D}} \tag{19}$$

where

$$\frac{\partial\alpha}{\partial\mathbf{D}} = \frac{1}{3f}\mathbf{1}; \qquad \frac{\partial D_{eq}}{\partial\mathbf{D}} = \frac{2}{3D_{eq}}\mathbf{D}_d \tag{20}$$

The combination of Eq. (19) and Eq. (20) also yields:

$$\mathrm{tr}\,\boldsymbol{\Sigma} = \frac{1}{f}\frac{\partial\Pi_G^{hom}}{\partial\alpha}; \qquad \Sigma_{eq} = \sqrt{3\boldsymbol{\Sigma}_d : \boldsymbol{\Sigma}_d/2} = \frac{\partial\Pi_G^{hom}}{\partial D_{eq}} \tag{21}$$

In turn, Eq. (18) leads to:

$$\begin{aligned} \mathrm{tr}\,\boldsymbol{\Sigma} &= 2\sigma_o\big(\mathrm{arcsinh}(\xi) - \mathrm{arcsinh}(f\xi)\big) \\ \Sigma_{eq} &= \sigma_o\left(\sqrt{1+f^2\xi^2} - f\sqrt{1+\xi^2}\right) \end{aligned} \tag{22}$$

Eliminating ξ between the spherical and deviatoric parts of $\boldsymbol{\Sigma}$ eventually leads to the well known Gurson strength criterion:

$$\frac{\Sigma_{eq}^2}{\sigma_o^2} + 2f\cosh\left(\frac{\mathrm{tr}\,\boldsymbol{\Sigma}}{2\sigma_o}\right) - 1 - f^2 = 0 \tag{23}$$

This equation characterizes the boundary of the domain G_G^{hom} which support function is Π_G^{hom}. This domain is in fact an upper bound of the exact domain G^{hom} of macroscopic admissible stresses, that is, $G^{hom} \subset G_G^{hom}$.

3 Interfaces and Interface Stresses

The recent literature devoted to nanocomposites has extensively presented the concepts of interface and interface stresses [4, 10–13]. In fact, these concepts are already present in the modeling of capillary forces [14]. The interface itself is a mathematical model for a thin layer between two phases across which the traction vector undergoes a discontinuity. In contrast, the displacement and the tangential strain components are continuous (see [3]). Introducing the local unit normal vector $\mathbf{n}$ to the interface S, the stress discontinuity $[\boldsymbol{\sigma}]\cdot\mathbf{n}$ is related to the interface stresses $\boldsymbol{\tau}$ by the generalized Laplace equations which physically represent the condition for the mechanical equilibrium of the interface [15]:

$$\begin{aligned} \mathbf{n}\cdot[\boldsymbol{\sigma}]\cdot\mathbf{n} &= -\boldsymbol{\tau}:\boldsymbol{\kappa} \\ \mathbf{P}\cdot\mathbf{n} &= -\nabla_S\cdot\boldsymbol{\tau} \end{aligned} \tag{24}$$

where $\nabla_S\cdot$ denotes the divergence operator defined on the interface S; tensor $\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n}$ and $\boldsymbol{\kappa}$ is the curvature tensor. The stress state $\boldsymbol{\tau}$ locally meets the plane stress conditions w.r.t. the tangent plane to the interface. We herein consider that the pore/solid boundary is such an interface.

The interface stresses also manifest themselves by a specific contribution to the energy $\mathscr{W}$ developed by the internal forces in the strain rate field $\mathbf{d}$:

$$\mathscr{W} = \int_\Omega \boldsymbol{\sigma}:\mathbf{d}\,dV = \int_{\Omega^s} \boldsymbol{\sigma}:\mathbf{d}\,dV + \int_S \boldsymbol{\tau}:\mathbf{d}\,dS \tag{25}$$

From a mathematical point of view, Eq. (25) amounts to saying that the internal forces can be represented by the sum of a standard Cauchy stress field $\boldsymbol{\sigma}$ in the solid and by a Dirac distribution $\boldsymbol{\tau}$ of stresses of support S. Hence, the integral in the left-hand side of Eq. (25) must be understood in the sense of the distribution theory.

Since the interface stress state is a plane stress one, the work it develops in the strain rate $\mathbf{d}$ only depends on the projection $\mathbf{d}^{int}$ of $\mathbf{d}$ on the local tangent plane, which is defined as [5]:

$$\mathbf{d}^{int} = \mathbb{T} : \mathbf{d} \quad \text{with } \mathbb{T} = \mathbf{P}\underline{\overline{\otimes}}\mathbf{P} \tag{26}$$

with $\mathbf{A}\underline{\overline{\otimes}}\mathbf{B}_{ijkl} = (A_{ik}B_{jl} + A_{il}B_{jk})/2$.

The surface integral in the expression of $\mathscr{W}$ has a counterpart in the homogenized support function $\Pi^{hom}(\mathbf{D})$ which now reads:

$$\Pi^{hom}_{int}(\mathbf{D}) = \inf_{\underline{v}\in\mathscr{V}(\mathbf{D})}\left((1-f)\overline{\pi^s(\mathbf{d})}^s + \frac{1}{|\Omega|}\int_S \pi^{int}(\mathbb{T} : \mathbf{d})\, dS\right) \tag{27}$$

π^{int} denotes the support function of the domain G^{int} of admissible surface stresses (see also Eq. (5)):

$$\pi^{int}(\mathbb{T} : \mathbf{d}) = \sup\left(\boldsymbol{\tau} : \mathbb{T} : \mathbf{d}, \boldsymbol{\tau} \in G^{int}\right) \tag{28}$$

It is emphasized that the latter meet the local plane stress conditions.

The extension of the Gurson model to interface effects simply consists in estimating the support function $\Pi^{hom}(\mathbf{D})$ by the upper bound obtained for the velocity field $\underline{v}^G$ introduced in Eq. (10):

$$\Pi^{hom}_{G,int}(\mathbf{D}) = \Pi^{hom}_G(\mathbf{D}) + \frac{1}{|\Omega|}\int_S \pi^{int}\left(\mathbb{T} : \mathbf{d}^G\right) dS \tag{29}$$

Clearly, we are left with the determination of the interface correcting term, which has to be added to the standard expression (18).

4 Extension of the Gurson Model: The von Mises Interface

We now assume that the strength of the interface can be described by a von Mises criterion

$$\frac{3}{2}\boldsymbol{\tau}_d : \boldsymbol{\tau}_d - k^2_{int} \le 0 \tag{30}$$

in *plane stress* condition, where $\boldsymbol{\tau}_d$ denotes the deviatoric part of the interface stress $\boldsymbol{\tau}$. The strength of the interface is then similar in nature to that of the matrix, up to the fact that it has a bidimensional character. In the local tangent plane which unit normal vector is $\underline{n} = \underline{e}_r$, the support function of the domain G^{int} then reads (see [16]):

$$\pi(\mathbb{T} : \mathbf{d}) = 2k^{int}\sqrt{\frac{1}{3}\left(d^2_{\theta\theta} + d^2_{\varphi\varphi} + d^2_{\varphi\theta} + d_{\theta\theta}d_{\varphi\varphi}\right)} \tag{31}$$

where k^{int} has the physical dimension of a membrane stress, that is, a force per unit length. The tensor $\mathbf{d}$ whose components appear in Eq. (31) is the pore strain rate $\mathbf{d}^I$ given in Eq. (14), which is then projected on the tangent plane by the operator $\mathbb{T}$. The projection operator $\mathbb{T}(\theta, \varphi)$ depends on the location on the spherical cavity wall (see Eq. (26)):

$$\mathbb{T} = \mathbf{P} \underline{\overline{\otimes}} \mathbf{P} \quad \text{with } \mathbf{P} = \mathbf{1} - \underline{e}_r \otimes \underline{e}_r \tag{32}$$

The components of the strain rate tensor appearing in Eq. (31) are then given by

$$d_{\alpha\beta} = \underline{e}_\alpha \overset{s}{\otimes} \underline{e}_\beta : \mathbb{T} : \mathbf{d}^I \tag{33}$$

with $\alpha,\ \beta = \theta$ or φ, that is:

$$d_{\alpha\beta} = \mathbf{T}^{\alpha\beta} : \mathbf{d}^I \tag{34}$$

with $\mathbf{T}^{\alpha\beta} = \underline{e}_\alpha \overset{s}{\otimes} \underline{e}_\beta : \mathbb{T}$. It is therefore convenient to introduce the fourth-order tensor $\mathbb{M}$:

$$\mathbb{M} = \mathbf{T}^{\varphi\varphi} \otimes \mathbf{T}^{\varphi\varphi} + \mathbf{T}^{\theta\theta} \otimes \mathbf{T}^{\theta\theta} + \mathbf{T}^{\varphi\theta} \otimes \mathbf{T}^{\varphi\theta} + \mathbf{T}^{\varphi\varphi} \otimes \mathbf{T}^{\theta\theta} \tag{35}$$

such that

$$\pi^{int}(\mathbb{T} : \mathbf{d}) = 2k_{int}\sqrt{\frac{1}{3}\mathbf{d}^I : \mathbb{M} : \mathbf{d}^I} \tag{36}$$

In order to determine the contribution Π^{int} of the interface to $\Pi^{hom}(\mathbf{D})$ (see Eq. (27)), we are left with the integration over the spherical interface:

$$\Pi^{int} = \frac{2k_{int}}{|\Omega|} \int_S \sqrt{\frac{1}{3}\mathbf{d}^I : \mathbb{M} : \mathbf{d}^I}\, dS \tag{37}$$

As in the classical derivation of the Gurson criterion, we have to replace Π^{int} by an upper bound in order to obtain an analytical expression:

$$\Pi^{int} \le \frac{2k_{int} R_i^2}{|\Omega|} \sqrt{\frac{4\pi}{3} \int_{S_o} \mathbf{d}^I : \mathbb{M} : \mathbf{d}^I\, dS} \tag{38}$$

where S_o is the (boundary of the) unit sphere. Since $\mathbf{d}^I$ is a constant, the right hand side in Eq. (38) can be put in the form:

$$\Pi^{int} \le \frac{2k_{int} R_i^2}{|\Omega|} \sqrt{\frac{4\pi}{3}\, \mathbf{d}^I : \left(\int_{S_o} \mathbb{M}(\theta, \phi)\, dS \right) : \mathbf{d}^I} \tag{39}$$

Noting from Eq. (35) that:

$$\int_{S_o} \mathbb{M}\, d\sigma = \pi \left(\frac{6}{5}\mathbb{K} + 4\mathbb{J} \right) \tag{40}$$

the contribution of the interface to $\Pi^{hom}(\mathbf{D})$ can be estimated by the following upper bound:

$$\begin{aligned}\Pi^{int} &\leq 6f\frac{k_{int}}{R_i}\sqrt{\mathbf{d}^I:\left(\frac{1}{10}\mathbb{K}+\frac{1}{3}\mathbb{J}\right):\mathbf{d}^I} \\ &= 3f\frac{k_{int}}{R_i}D_{eq}\sqrt{\xi^2+\frac{3}{5}} \quad \text{with } \xi = 2\alpha/D_{eq}\end{aligned} \tag{41}$$

in which it is recalled that R_i represents the radius of the pores. The term provided by (41) is to be added to Eq. (18) in view of the derivation of the strength criterion. The comparison of the respective contributions of the solid equation (18) and of the interface equation (41) is controlled by the nondimensional parameter

$$\Gamma = k_{int}/(R_i\sigma_o) \tag{42}$$

which is pore size-dependent. The smaller the pores the greater the influence of the interface effects on the strength.

We note that Eq. (19) and Eq. (21) are still valid provided that Π_G^{hom} is replaced by $\Pi_{G,int}^{hom} = \Pi_G^{hom} + \Pi^{int}$. This leads to the parametric equations

$$\begin{aligned}\text{tr}\,\boldsymbol{\Sigma} &= \sigma_o\left(2\big(\text{arcsinh}(\xi) - \text{arcsinh}(f\xi)\big) + \Gamma\frac{6\xi}{\sqrt{\xi^2+3/5}}\right) \\ \Sigma_{eq} &= \sigma_o\left(\sqrt{1+f^2\xi^2} - f\sqrt{1+\xi^2} + \Gamma\frac{9f}{5\sqrt{\xi^2+3/5}}\right)\end{aligned} \tag{43}$$

Note that this boundary is symmetric w.r.t. the $\text{tr}\,\boldsymbol{\Sigma} = 0$ axis. Let us emphasize that, by the presence of term $\Gamma = k_{int}/(R_i\sigma_o)$, (43) explicitly shows that the yield strength depends on the voids size. In order to get a closer insight into the influence of the interface on the effective strength, it is useful to provide an analytical approximation of the boundary of the domain defined by Eq. (43) in the form $\mathscr{F}(\Sigma_{eq}, \text{tr}\,\boldsymbol{\Sigma}) = 0$. This can be done by means of expansions of Eq. (43) in the vicinity of $\xi = 0$ and $\xi = \infty$. First, in the vicinity of the maximum deviatoric strength ($\xi = 0$, low stress triaxiality), the boundary can be approximated by a parabola in the ($\text{tr}\,\boldsymbol{\Sigma}$, Σ_{eq}) plane:

$$\frac{\Sigma_{eq}}{\sigma_o} = 1 - f + \Gamma\frac{9f}{\sqrt{15}} - \frac{f}{8(1-f+\Gamma\sqrt{15})}\left(\frac{\text{tr}\,\boldsymbol{\Sigma}}{\sigma_o^2}\right)^2 \tag{44}$$

In turn, in the vicinity of the pure isotropic tensile/compression loading ($\xi = \pm\infty$), the boundary can be approximated by another parabola:

$$\frac{\Sigma_{eq}^2}{\sigma_o^2} = \frac{3}{2}\left(1 - f^2 + \frac{18}{5}\Gamma f^2\right)\left(-\frac{2}{3}\log f + 2\Gamma \pm \frac{\text{tr}\,\boldsymbol{\Sigma}}{3\sigma_o}\right) \tag{45}$$

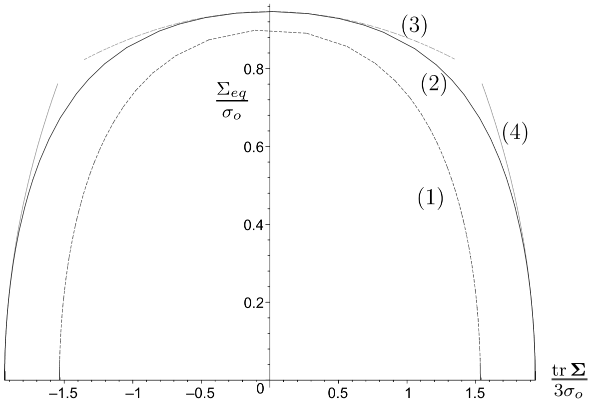

Fig. 1 (*1*): classical Gurson model; (*2*): extended Gurson model with $f = 0.1$ and $\Gamma = 0.2$; (*3*): parabola of (44); (*4*): parabola of (45)

Illustrations of the results are provided on Fig. 1. Continuous lines correspond to the parametric formulation of the macroscopic yield function (see (43)) while discontinuous lines are associated to the two above expansions derived in the form of parabola. First, the results clearly show a significant effect of the void size. Note that k_{int} being fixed, a decrease of R_i is represented by an increase of Γ. Moreover, the two proposed expansions appear accurate for a large range of triaxiality.

5 Isotropic Tensile/Compressive Strength

In the framework of the geometrical model of hollow sphere, the classical Gurson model (no interface stress) is known to provide an exact result as regards the isotropic tensile/compressive strength.

With $\Sigma_{eq} = 0$, the solutions to Eq. (23) are the isotropic stress tensors $\pm\Sigma^+\mathbf{1}$ with $\Sigma^+ = -2\sigma_o \log f/3$. As a matter of fact, the Gurson approach shows that an admissible isotropic macroscopic stress state $\boldsymbol{\Sigma} = \Sigma\mathbf{1}$ is subjected to the condition $|\Sigma| \leq \Sigma^+$. Conversely, let us consider the microscopic stress state defined in the solid in spherical coordinates by:

$$\boldsymbol{\sigma} = \varepsilon \frac{3\Sigma^+}{2\log f}\left(2\log\frac{R_i}{r}\mathbf{1} - \mathbf{P}\right) \quad \text{with } \varepsilon = \pm 1 \tag{46}$$

It is readily seen that the latter is in equilibrium with the macroscopic stress state $\varepsilon\Sigma^+\mathbf{1}$ since it satisfies the momentum balance condition $\mathrm{div}\,\boldsymbol{\sigma} = 0$ and the boundary conditions $\boldsymbol{\sigma}\cdot\underline{e}_r = 0$ at $r = R_i$ and $\boldsymbol{\sigma}\cdot\underline{e}_r = \varepsilon\Sigma^+\underline{e}_r$ at $r = R_e$. Furthermore, it meets the von Mises criterion equation (8). This proves that such a macroscopic stress state is admissible and furthermore, that Σ^+ is indeed the isotropic tensile/compressive strength.

Let us now examine the effect of interface stresses on the isotropic tensile/compressive strength. Consider the case of the von Mises interface. According to the extended Gurson model equation (45), the necessary condition for an isotropic macroscopic stress state $\boldsymbol{\Sigma} = \Sigma \mathbf{1}$ to be admissible reads $|\Sigma| \leq \Sigma^+ + 2\Gamma\sigma_o$. Conversely, let us consider the microscopic stress state defined in the solid in spherical coordinates by:

$$\boldsymbol{\sigma} = \varepsilon\left(\frac{3\Sigma^+}{2\log f}\left(2\log\frac{R_i}{r}\mathbf{1} - \mathbf{P}\right) + 2\Gamma\sigma_o\mathbf{1}\right) \quad \text{with } \varepsilon = \pm 1 \tag{47}$$

and on the interface S by $\boldsymbol{\tau} = \varepsilon k_{int}\mathbf{P}$ (recall that $k_{int} = \Gamma\sigma_o R_i$). It satisfies the momentum balance equation $\operatorname{div}\boldsymbol{\sigma} = 0$ and the boundary condition $\boldsymbol{\sigma} \cdot \underline{e}_r = \varepsilon(\Sigma^+ + 2\Gamma\sigma_o)\underline{e}_r$ at $r = R_e$. It also satisfies the generalized Laplace equations (24). Furthermore, it meets the von Mises interface criterion equation (30). This establishes that $\Sigma^+ + 2\Gamma\sigma_o$ is the isotropic tensile/compressive strength.

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