# Chapter 2
# The Dirac Electron: Spin, *Zitterbewegung*, the Compton Wavelength, and the Kinetic Foundation of Rest Mass

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**Abstract** The Dirac equation, which was derived by combining, in a consistent manner, the relativistic invariance condition with the quantum superposition principle, has shown its fecundity by explaining the electron spin, predicting antimatter, and enabling Schrödinger's trembling motion (*Zitterbewegung*). It has also yielded as expectation value for the electron speed the velocity of light. But the question has hardly been raised as to the effect of this intrinsic motion on the electron mass. In this chapter, we conjecture that the internal structure of the electron should consist of a massless charge describing, at light velocity, a vibrating motion in a domain defined by the Compton wavelength, the measured rest mass being generated by this very internal motion.

> Around 1950, I had the rare opportunity of meeting Albert Einstein .... The professor addressed my colleague: '*Vot* are you studying?' 'I'm doing a thesis on quantum theory'. '*Ach*!' said Einstein, 'a *vaste* of time!'
>
> He turned to me: 'And *vot* are you doing?' I was more confident: 'I'm studying experimentally the properties of pions'. 'Pions, pions! *Ach*, *vee* don't understand *de* electron! *Vy* bother *mit* pions?' ...
>
> Leon Lederman: *Life in Physics and the Crucial Sense of Wonder*, CERN Courier, 10 September 2009

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K. Nishikawa et al. (eds.), *Quantum Systems in Chemistry and Physics*, Progress in Theoretical Chemistry and Physics 26, DOI 10.1007/978-94-007-5297-9_2,

## 2.1 Introduction

The atomic theory of matter, which was conjectured on qualitative empirical grounds as early as the sixth century BC, was shown to be consistent with increasing experimental and theoretical developments since the seventeenth century AD, and definitely proven by the quantitative explanation of the Brownian motion by Einstein and Perrin early in the twentieth century [1]. It then took no more than a century between the first measurements of the electron properties in 1896 and of the proton properties in 1919 and the explosion of the number of so-called elementary particles – and their antiparticles – observed in modern accelerators to several hundred (most of which are very short lived and some, not even isolated). Today, the ‘standard model’ assumes all particles to be built from three groups of four basic fermions – some endowed with exotic characteristics – interacting through four basic forces mediated by bosons – usually with zero charge and mass and with integer spin [2].

In this zoo of particles, only the *electron*, which was discovered even before the atomic theory was proven and the atomic structure was known, is really unsecable, stable, and isolatable. The *proton* also is stable and isolatable, but it is made up of two quarks *up* (with charge +2/3) and one quark *down* (with charge −1/3). As for the *quarks*, while expected to be stable, they have not been isolated. The other particle constitutive of the atomic nucleus, the *neutron*, is also made up of three quarks, one *up* and two *down*, but it is not stable when isolated, decaying into a proton, an electron, and an antineutrino (with a 15-min lifetime). The fermions in each of the higher two classes of the *electron* family (*muon* and *tau*) and of the two *quark* families (*strange/charmed* and *bottom/top*) are unstable (and not isolatable for the quarks). Only the elusive *neutrinos* in the three classes, which were postulated to ensure conservation laws in weak interaction processes, are also considered as being unsecable, stable, and isolatable.

Although quantum chromodynamics has endeavoured to rationalize the world of quarks, gluons, the strong interaction, and composite particles [2], it is not as in a developed stage as quantum electrodynamics, where electrons, photons, the electromagnetic interaction, and the whole domain of chemical physics are unified in a refined manner [3, 4]. This latter theory is but an extension of the Dirac theory [5, 6], which treated the electron in a *consistent* quantum-relativistic manner while its interaction with the electromagnetic field was considered semi-classically, to a full quantum-relativistic treatment of charged particles interacting with each other and with a *quantized* electromagnetic field by exchanging virtual photons.

Traditional attributes of matter are opacity (to light), resistance (to penetration), inertia (to motion), and weight. A transparent glass has no opacity (to visible light), but it requires a very hard material (a diamond cutter) to be penetrated. Pure air also shows transparency, but it shows resistance to penetration only at very high speeds (blasts, storms, planes, parachutes). These two attributes are well understood today as quantum effects due to the interactions of molecules with electromagnetic fields and with other molecules.

The attribute of inertia was identified by Galileo as being a resistance to acceleration/deceleration (rather than to uniform linear motion), while the attribute of weight (also investigated by Galileo) was related by Newton to the attraction by a massive body (as expressed in Kepler's rules). These two attributes were later correlated in general relativity theory by Einstein. But the quantum theory has not been directly involved in either inertia or weight until Dirac's attempt to bring together quantum and relativistic conditions in a matrix linear equation for the electron, using the total energy $mc^2$ rather than the kinetic energy $p^2/2m_0$ in his Hamiltonian operator.

In this chapter, we shall reassess some of the physical implications of the Dirac equation [5, 6], which were somehow overlooked in the sophisticated formal developments of quantum electrodynamics. We will conjecture that the *internal structure* of the electron should consist of a *massless charge* describing at *light velocity* an oscillatory motion (*Zitterbewegung*) in a small domain defined by the *Compton wavelength*, the observed *spin momentum* and *rest mass* being jointly generated by this very internal motion.

## 2.2 Compton Wavelength and de Broglie Wavelength

Although the corpuscular aspect of electromagnetic radiation, which was surmised by Newton in the seventeenth century, was used by Planck in 1900 to explain Wien's black body radiation law and by Einstein in 1905 to explain Lenard's photoelectric effect, its most spectacular demonstration was Compton's explanation in 1923 of the anomalous scattering of X-rays by bound electrons.

If an incident photon ($\boldsymbol{p}_1, E_1 = p_1 c$) hits an electron considered as nearly at rest ($0, m_0 c^2$), producing an electron recoil ($\boldsymbol{p}_0, E_0$), the direction of the scattered photon ($\boldsymbol{p}_2, E_2 = p_2 c$) makes an angle $\theta$ with that of the incident photon. Applying the laws of conservation of energy and momentum to the scattering process:

$$\boldsymbol{p}_1 = \boldsymbol{p}_2 + \boldsymbol{p}_0, \quad p_1 c + m_0 c^2 = p_2 c + \left(m_0^2 c^4 + p_0^2 c^2\right)^{\frac{1}{2}}, \tag{2.1}$$

one derives

$$m_0 c(p_1 - p_2) = p_1 p_2 (1 - \cos\theta). \tag{2.2}$$

Using the incident and scattered photon wavelengths, $\lambda_1 = h/p_1$, $\lambda_2 = h/p_2$, and introducing the electron *Compton wavelength*, $\lambda_C = h/m_0 c$, one obtains

$$\lambda_2 - \lambda_1 = \lambda_C (1 - \cos\theta). \tag{2.3}$$

This expression is rigorous with the relativistic treatment we have used. But the occurrence of the Compton wavelength $\lambda_C$ is not a relativistic effect since Eq. (2.2) also holds (to first order, except around $\theta = 0°$) if one uses the classical formula, $E_0 = p_0^2/2m_0$, for the kinetic energy of the ejected electron. In fact, the occurrence of this electron wavelength stems from the assumption that light is made of particles endowed with kinetic momentum, $p = h/\lambda$, as well as with energy, $E = p\,c$.

The question remains as to how the electron interacts, at the subquantum level, to scatter the photon. One could speculate on the fact that for $\theta = \pi/2$ (orthogonal scattering) the Compton wavelength adds to the photon wavelength while the electron recoils along $\phi \sim -\pi/4$ (as would a tiny mirror inclined at $\pi/4$), while for $\theta = 0$ (no scattering) the photon wavelength remains unchanged and the electron unmoved. Adding the electron *Compton wavelength* to the *orthogonally scattered* photon wavelength reduces the photon energy by the amount used for the electron ejection.

The *Compton wavelength*, $\lambda_C = h/m_0c$, is different from the *de Broglie wavelength*, $\lambda_B = h/m_0v$, in that it is unrelated to the particle velocity but solely depends on its rest mass (and light velocity). The larger the rest mass, the smaller the wavelength or, one could say, *the larger the Compton wavelength, the smaller the particle rest mass*.

## 2.3 The Dirac Equation

It will be useful to recall the Lorentz transformation equations of the space and time coordinates of a free particle between two inertial frames $S$ and $S'$:

$$x' = \gamma(x - \beta ct) \tag{2.4a}$$

$$ct' = \gamma(-\beta x + ct) \tag{2.4b}$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$, $v$ being the velocity of frame $S'$ relative to frame $S$ and $c$, the velocity of light. In similar transformation equations for the electromagnetic field (ruled by Maxwell's equations), the electric field components play the role of space coordinates and the magnetic field's that of a time coordinate.

It can be seen that, while the space and time coordinates depend on the reference frame, the combination

$$x_0^2 \equiv (ct)^2 - \underline{r}^2 \equiv x_4^2 - x_1^2 - x_2^2 - x_3^2 \tag{2.5a}$$

is relativistically invariant under any change of frame (its square root is Minkowski's *proper interval*). This formula can alternatively be written as

$$x_4^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{2.5b}$$

The dependence of the measured time on the inertial frame (the $-\beta x$ term in Eq. 2.4b), which entails $\gamma \neq 1$, stems from the invariance of $c$ with respect to the frame. Einstein's equivalence relation $E = mc^2$ arises from the resulting intrication of space and time. One of the clues that led de Broglie to the idea of matter waves (and to the explanation of quantization rules in atomic spectra by assuming standing waves in electron orbits) was a comparison of this relation with that expressing the quantization of light, $E = h\,c/\lambda$, which yields $m = h/\lambda\, c$ for photons and, by analogy, $\lambda = h/mv$ for particles with non-zero rest mass.

The Dirac equation was derived in several steps [5, 6], starting with the time-dependent wave equation for a free particle in the Schrödinger representation:

$$i\hbar\,\frac{\partial\Psi}{\partial t} = H\;\Psi, \quad \text{or} \quad i\hbar\,\frac{\partial\Psi}{\partial(ct)} = mc\;\Psi, \tag{2.6}$$

where the Hamiltonian operator was given the relativistic form: $H = mc^2$. The term expressing the external motion is embedded in the relativistic formula for the mass: $m = m_0\gamma$. In order to unveil this term, $H$ is transformed to the form

$$H = mc^2 = \left[\frac{m_0^2c^6}{(c^2 - v^2)}\right]^{1/2} = \left[m_0^2c^4 + \frac{m_0^2c^4v^2}{(c^2 - v^2)}\right]^{1/2} =$$
$$= \left(m_0^2c^4 + p^2c^2\right)^{1/2} = \left(m_0^2c^2 + p^2\right)^{1/2}c,$$
$$\text{or} \quad mc = \left(m_0^2c^2 + p^2\right)^{1/2}, \tag{2.7a}$$

with $p = m_0\gamma v = mv = p_0\gamma$. When $v \ll c$, $H$ reduces to the usual form: $H_0 = (m_0c^2 +)\; p_0^2/2m_0\; (+\ldots)$.

In Eq. (2.7a), $p^2 = p_1^2 + p_2^2 + p_3^2$ with $p_i = mv_i$ along $x_i$, and from Eqs. (2.5) and (2.6) one can define an *additional 'momentum'* $\boldsymbol{p_4 \equiv mc}$, corresponding to the *time 'coordinate'* $\boldsymbol{x_4 \equiv ct}$, and an *invariant 'momentum'* $\boldsymbol{p_0 \equiv m_0c}$, for a *particle at rest*. Equation (2.7a) can then be written as

$$p_4^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2. \tag{2.7b}$$

Comparing Eqs. (2.7b) and (2.5b) shows that the *relativistically invariant 'momentum'* $p_0$ corresponds to the *relativistically invariant 'coordinate'* $x_0$. To the 'Pythagorean relation' between the *generalized coordinates*, $x_4^2 = x_0^2 + \underline{r}^2$, corresponds a similar relation between the *generalized momenta*, $p_4^2 = p_0^2 + \underline{p}^2$.

By analogy with the non-relativistic case, one can write

$$p_1 \to -i\hbar\,\frac{\partial}{\partial x},\; p_2 \to -i\hbar\,\frac{\partial}{\partial y},\; p_3 \to -i\hbar\,\frac{\partial}{\partial z},\; p_4 \to i\hbar\,\frac{\partial}{\partial(ct)}, \tag{2.8}$$

the last expression being introduced to bring time on the same footing as the space coordinates. At this stage, the operator associated with $p_0$ is just $p_0$. Equation (2.6) can then be written as

$$\left[p_4 - \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{1/2}\right]\Psi = 0, \tag{2.9}$$

which is linear in $p_4$ but not in the other $p_i$'s and, therefore, not fully satisfactory from the relativistic point of view.

The *second step* was thus to multiply this equation on the left side by $\left[p_4 + \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{1/2}\right]$, yielding the more symmetric form

$$\left[p_4^2 - \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)\right]\Psi = 0, \tag{2.10}$$

where only those solutions belonging to positive values of $p_4$ are also solutions of Eq. (2.9). This is the so-called Klein-Gordon equation, which reduces to the wave equation for $m_0 = 0$ and is suitable for the description of zero-spin free particles.

Although Eq. (2.10) fulfils the relativistic condition of space-time equivalence, it does not fulfil the quantum requirement of linearity so that the superposition principle, probability density formula and uncertainty principle could apply [5, 6].

The *third step* was to look for an analogous equation *linear* in all $p_\mu$'s, that is,

$$\left[p_4 - (\boldsymbol{\alpha}_0 p_0 + \boldsymbol{\alpha}_1 p_1 + \boldsymbol{\alpha}_2 p_2 + \boldsymbol{\alpha}_3 p_3)\right]\Psi = 0, \tag{2.11}$$

where the $\boldsymbol{\alpha}_\mu$'s must be matrices independent of the $p_\mu$'s and of the $x_\mu$'s in free space. Multiplying to the left side by $[p_4 + (\boldsymbol{\alpha}_0 p_0 + \boldsymbol{\alpha}_1 p_1 + \boldsymbol{\alpha}_2 p_2 + \boldsymbol{\alpha}_3 p_3)]$ yields

$$\left[p_4^2 - (\boldsymbol{\alpha}_0 p_0 + \boldsymbol{\alpha}_1 p_1 + \boldsymbol{\alpha}_2 p_2 + \boldsymbol{\alpha}_3 p_3)^2\right]\Psi = 0. \tag{2.12}$$

This coincides with Eq. (2.10) only if one has, for $\mu, \nu = 0, 1, 2, 3$:

$$\boldsymbol{\alpha}_\mu^2 = 1, \quad \boldsymbol{\alpha}_\mu \boldsymbol{\alpha}_\nu + \boldsymbol{\alpha}_\nu \boldsymbol{\alpha}_\mu = 0. \tag{2.13}$$

In addition to being normalized and anticommutative, these matrices, of course, must be Hermitian. These conditions are similar to those for the three components $\sigma_x, \sigma_y, \sigma_z$ of the spin operator $\boldsymbol{\sigma}$ and of their Pauli representations as 2D matrices:

$$\sigma_x \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y \sim \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad \sigma_z \sim \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.14}$$

But now we have four components for the four-vector $(p_1, p_2, p_3, p_0)$, and the four $\boldsymbol{\alpha}_\mu$ matrices fulfil the above requirements only if they possess at least four dimensions; e.g. [5, 6], using the 2D Pauli matrices as off-diagonal elements of the 4D Dirac matrices relative to the $p_\mu$'s:

$$\boldsymbol{\alpha}_1 \equiv \begin{pmatrix} 0 & \boldsymbol{\sigma}_x \\ \boldsymbol{\sigma}_x & 0 \end{pmatrix} \quad \boldsymbol{\alpha}_2 \equiv \begin{pmatrix} 0 & \boldsymbol{\sigma}_y \\ \boldsymbol{\sigma}_y & 0 \end{pmatrix} \quad \boldsymbol{\alpha}_3 \equiv \begin{pmatrix} 0 & \boldsymbol{\sigma}_z \\ \boldsymbol{\sigma}_z & 0 \end{pmatrix} \qquad \boldsymbol{\alpha}_0 \equiv \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

A result is that for a vector to be representative of the wave function $\Psi$ it must have four components or, alternatively, that $\Psi$ must contain a variable taking on four values. Dirac has explained why the electron has spin, which was known as requiring the wave function $\Psi$ to have two components, and that this number must be doubled because the quasi-linear Eq. (2.11), which is equivalent to the quadratic Eq. (2.10) under the conditions (2.13), has additional, negative-energy solutions, which he assigned to an antielectron having opposite charge [5].

As expected, Eq. (2.11) is invariant under Lorentz transformations [5, 6]. It was noticed by de Broglie [6] that the process leading from Eq. (2.10) to (2.11) is similar to that leading from the second-order equations for the electric and magnetic fields $\underline{E}$ and $\underline{B}$ of electromagnetic radiation to the four coupled, first-order, Lorentz-invariant Maxwell equations.

Although spin was first introduced phenomenologically (see Sect. 2.4) and shown to require only 2D matrices for its representation (Eq. 2.14), the theoretical proof for its existence required a four-component wave vector, yielding additional negative-energy states. This hints that *spin*, as well as *Zitterbewegung* (see Sect. 2.4), must be related to these states. This appears in the entanglement of the four components of $\Psi$ when Eq. (2.11) is written explicitly in the form of four coupled equations [6].

One may notice that the matrices $\boldsymbol{\alpha}_i$ multiplying the components $p_i$ of the momentum that describe the *external trajectory* of the particle are off-diagonal, whereas the matrix $\boldsymbol{\alpha}_0$ multiplying the momentum $p_0$ related to the rest mass energy $m_0c^2$ is diagonal. This suggests there is some *internal motion* orthogonal to the external trajectory, as hinted in Eq. (2.7b) where the generalized momentum $\boldsymbol{mc}$ appears as a *Pythagorean sum* of the two orthogonal momenta $\boldsymbol{m}_0\boldsymbol{c}$ and $\boldsymbol{p}$.

Indeed, three internal motions (which have been shown to be related) have been discussed by Dirac from his equation. One involves the well-established *spin* angular momentum, which gives rise to the *measured* magnetic moment; another is the *Zitterbewegung* (proper oscillatory motion) derived by Schrödinger from Dirac's equation; and finally there is an internal motion adding to that defining the external trajectory of the particle to give it the *computed* velocity $c$. We shall comment on these three motions.

## 2.4 The Electron Internal Motion: Spin, *Zitterbewegung*, and Light Velocity

The electron spin entered quantum mechanics in two different ways. The first was the explanation, by Goudsmit and Uhlenbeck (1925), of the Zeeman splitting of the spectral lines of atoms by a magnetic field (1896) and of the Stern and Gerlach deflection of the trajectory of atoms by an inhomogeneous field (1922). The electron was endowed with an *intrinsic* magnetic moment and, since it has electric charge, with a *rotational* internal motion adding to its quantized motion around a nucleus. This electron property was later shown to be responsible for most of materials' magnetism, known for long: ferro (and anti) and ferri (and anti), as well as para (but not dia). Electron paramagnetic resonance (EPR) spectroscopy and related techniques [7] are based on this property, and on a similar property proposed by Pauli for nuclei [1924], which is at the basis of nuclear magnetic resonance (NMR).

Various models have been designed to account for the magnetic properties of the electron [6]. In the simple model of a loop with radius $r$ described by a point charge $-e$, the *measured* magnitude of the induced magnetic moment $\underline{\mu}$ orthogonal to the loop can be used to derive the *rotational* velocity $v$:

$$\begin{aligned}\mu = I.S = \left(\frac{-e.v}{2\pi r}\right).\pi r^2 &= -\frac{e.v\ r}{2}\\ &= -\left(\frac{1}{2}\right)\frac{e\ \hbar}{2m_0} \rightarrow v = \frac{\hbar}{2m_0 r}.\end{aligned} \tag{2.16}$$

If one identifies $r$ with the *measured* Compton radius, $r_C = \hbar/2\,m_0 c$ (Sect. 2.2 and Eq. 2.34), this formula yields: $v = c$!

The second intrusion of the electron spin came through a non-energetic, symmetry requirement, the so-called Fermi-Dirac statistics for systems of identical, half-integer spin particles, which results in total antisymmetry of the Schrödinger wave function in a combined space and spin coordinate domain. This entails the Pauli exclusion principle (1925) in the framework of the independent-particle, Slater-determinantal model. The expression of atomic and molecular wave functions as linear combinations of Slater determinants has been the basis of most of the subsequent methodologies of quantum chemistry, thermodynamics, and spectroscopy.

These two aspects of the electron spin, that of an internal dynamical variable introduced to satisfy a symmetry requirement and that related to an intrinsic magnetic moment interacting with an external field, were elucidated by Dirac from his quantum-relativistic equation. But it also yielded an electron moving at the speed of light!

To have the electron magnetic moment show up, it is necessary to make it interact with an external magnetic field; and to have its spin momentum appear, it has to be combined with an orbital momentum. Equation (2.11) was thus extended to include interactions with an electromagnetic field. Let us call $A_4$ and $\underline{A}$ the scalar and vector potentials in *MKSA* units (in earlier formulations of the Dirac equation [5, 6], $\underline{A}$ was divided by $c$ due to the use of *cgs* units). We can write

$$\left[\left(p_4 + \frac{e\ A_4}{c}\right) - \boldsymbol{\alpha}_0 p_0 - \underline{\boldsymbol{\alpha}}.\left(\underline{p} + e\ \underline{A}\right)\right]\Psi = 0. \tag{2.17}$$

It can be noticed that the *internal* momentum $p_0$ remains *unchanged* in the presence of a field. In the Heisenberg picture, which is more suitable to make comparisons between classical and quantum mechanics, the equations of motion are determined by the Hamiltonian

$$H = c\ p_4 = -e\ A_4 + c\boldsymbol{\alpha}_0 p_0 + c\underline{\boldsymbol{\alpha}}.(\underline{p} + e\ \underline{A})]. \tag{2.18}$$

This gives, using the forms and properties of the $\boldsymbol{\alpha}_\mu$ matrices (Eqs. 2.13, 2.14, and 2.15), especially the fact that $\boldsymbol{\alpha}_0$ is normalized and anticommutes with $\boldsymbol{\alpha}_i$ ($i = 1, 2, 3$) while commuting with $(\underline{p} + e\underline{A})$:

$$\left(p_4 + \frac{e\ A_4}{c}\right)^2 = \left[\boldsymbol{\alpha}_0 p_0 + \underline{\boldsymbol{\alpha}}.(\underline{p} + e\ \underline{A})\right]^2 = p_0^2 + \left[\underline{\boldsymbol{\sigma}}.(\underline{p} + e\ \underline{A})\right]^2. \tag{2.19}$$

If one uses the general relation for any two 3D vectors $\underline{C}$ and $\underline{D}$ commuting with the $\boldsymbol{\sigma}_i$'s, which results from the properties of the Pauli matrices (Eqs. 2.14),

$$(\underline{\boldsymbol{\sigma}}.\underline{C}).(\underline{\boldsymbol{\sigma}}.\underline{D}) - \underline{C}.\underline{D} = i\ \underline{\boldsymbol{\sigma}}.\underline{C}\times\underline{D},$$

one obtains for $\underline{C} = \underline{D} = (\underline{p} + e\underline{A})$, substituting $\underline{p} = -i\,\hbar\underline{\nabla}$ then $\underline{B}\,(\underline{r}, t) = \underline{\nabla}\times\underline{A}\,(\underline{r}, t)$,

$$\left[\underline{\boldsymbol{\sigma}}.(\underline{p} + e\ \underline{A})\right]^2 - (\underline{p} + e\ \underline{A})^2 = i\ e\,\underline{\boldsymbol{\sigma}}.\ (\underline{p}\times\underline{A} + \underline{A}\times\underline{p}) =$$
$$= \hbar\,e\,\underline{\boldsymbol{\sigma}}.\ \underline{\nabla}\times\underline{A} = \hbar\ e\ \underline{\boldsymbol{\sigma}}.\underline{B}.$$

Equation (2.19) then becomes

$$\left(p_4 + \frac{e\ A_4}{c}\right)^2 = p_0^2 + (\underline{p} + e\ \underline{A})^2 + e\ \hbar\ \underline{\boldsymbol{\sigma}}.\underline{B}. \tag{2.20}$$

In order to compare this expression with the non-relativistic one, $H$ is written in the perturbative form: $H = m_0c^2 + H'$. To first order, this yields

$$H' = -e\ A_4 + \frac{(\underline{p} + e\ \underline{A})^2}{2m_0} + \left(\frac{e\ \hbar}{2m_0}\right)\underline{\boldsymbol{\sigma}}.\underline{B}. \tag{2.21}$$

In addition to the potential and kinetic energy terms of the classical Hamiltonian for a slow electron, there appears an extra term, which can be seen as expressing the interaction of the electron with a magnetic field $\underline{B}$ through an *intrinsic magnetic moment*, $\underline{\mu} = -(e\ \hbar/2m_0)\ \underline{\boldsymbol{\sigma}}$, in agreement with Eq. (2.16). This extra term arises naturally from the factor $\underline{\boldsymbol{\sigma}}$ embedded in Eq. (2.19).

The spin angular momentum itself does not give rise to any *potential* energy. To show its existence, Dirac computed the angular momentum integrals for an electron moving in a central electric field, that is, from Eq. (2.18):

$$H = -e\ \underline{A}_4(r) + c\ \boldsymbol{\alpha}_0 p_0 + c\ \underline{\boldsymbol{\alpha}}.\underline{p}. \tag{2.22}$$

In the Heisenberg picture, one obtains, for the $l_1$ component, say, of the orbital angular momentum $\underline{l} = -i\hbar \underline{r} \times \underline{\nabla}$,

$$i\hbar \frac{\partial l_1}{\partial t} = [l_1, H] = c\,[l_1.(\underline{\boldsymbol{\alpha}}\,\underline{p}) - (\underline{\boldsymbol{\alpha}}\,\underline{p}).l_1] = \\ = c\,\underline{\boldsymbol{\alpha}}(l_1.\underline{p} - \underline{p}.\,l_1) = -i\hbar\, c(\boldsymbol{\alpha}_3.p_2 - \boldsymbol{\alpha}_2.p_3) \neq 0; \tag{2.23}$$

similarly, for the corresponding component of the Pauli matrix operator,

$$i\hbar \frac{\partial \sigma_1}{\partial t} = [\sigma_1, H] = c\,[\sigma_1.(\underline{\boldsymbol{\alpha}}\,\underline{p}) - (\underline{\boldsymbol{\alpha}}\,\underline{p}).\sigma_1] = \\ = c\,(\sigma_1\underline{\boldsymbol{\alpha}} - \underline{\boldsymbol{\alpha}}\sigma_1).\underline{p} = 2\,i\,c\,(\boldsymbol{\alpha}_3.p_2 - \boldsymbol{\alpha}_2.p_3) \neq 0. \tag{2.24}$$

From Eq. (2.23) it is seen that $l_1$ is *not* a constant of the motion, but from Eq. (2.24) it is seen that

$$\frac{\partial l_1}{\partial t} + \left(\frac{\hbar}{2}\right)\frac{\partial \sigma_1}{\partial t} = 0. \tag{2.25}$$

Dirac interpreted this as the electron having a *spin* angular momentum, $\underline{s} = (\hbar/2)$ $\underline{\boldsymbol{\sigma}}$, that has to be *added* to the *orbital* angular momentum $\underline{l}$ to get a constant of the motion. It is the same matrix/operator vector $\underline{\boldsymbol{\sigma}}$ that fixes the direction of $\underline{s}$ and that of the magnetic moment $\underline{\boldsymbol{\mu}}$ derived from Eq. (2.21), and this justifies the simple model leading to Eq. (2.16).

Following considerations developed by Bohr, Darwin, and Pauli, de Broglie [6] showed that it is not possible to separate the electron spin momentum from its orbital momentum because, in any direct measurement, the uncertainties on the components of the orbital momentum would be larger than the spin momentum. This is due to the electron having a finite size, defined by the Compton radius.

Equations (2.25) and (2.21) do not tell us *at which velocity* the electron 'rotates' to acquire kinetic and magnetic spin momenta. This is provided by another computation by Dirac [5]. He used a Heisenberg picture with a field-free Hamiltonian (but the conclusion would also hold with a field present):

$$H = c\,(\boldsymbol{\alpha}_0 p_0 + \boldsymbol{\alpha}_1 p_1 + \boldsymbol{\alpha}_2 p_2 + \boldsymbol{\alpha}_3 p_3). \tag{2.26}$$

The *linear momentum* $\underline{p}$ obviously commutes with $H$ and thus is a *constant of the motion*. Making use of the properties of the $\boldsymbol{\alpha}_k$'s (Eqs. 2.13), one can further write, for an arbitrary component $v_k$ ($k = 1, 2, 3$) of the electron velocity,

$$i\hbar \frac{\partial x_k}{\partial t} = [x_k, H] = c\,(x_k\,\underline{\boldsymbol{\alpha}}.\underline{p} - \underline{\boldsymbol{\alpha}}.\underline{p}\,x_k) = c\,\boldsymbol{\alpha}_k\,(x_k p_k - p_k x_k) \\ = i\hbar\,c\,\boldsymbol{\alpha}_k \rightarrow v_k = \left|\frac{\partial x_k}{\partial t}\right| = \pm c, \tag{2.27}$$

showing the electron moves at light velocity! If we used the classical expression for the energy of a free particle, $H = p^2/2\,m_0$, in Eq. (2.26), we would recover, through Eq. (2.27), the classical relation between velocity and momentum, $v_k = p_k/m_0$, which we expect also to hold in the relativistic case.

The paradox was elucidated through the 'trembling motion' (*Zitterbewegung*) discovered by Schrödinger [8] while investigating the velocity operators $\boldsymbol{\alpha}_k$ introduced by Dirac to linearize his equation. The equation of motion of a velocity component, $v_k = c\boldsymbol{\alpha}_k$, can be written as

$$i\hbar\,\frac{\partial \alpha_k}{\partial t} = \alpha_k H - H\alpha_k.$$

Since $c\,\boldsymbol{\alpha}_k$ anticommutes with all the terms in Eq. (2.26) except $c\,\boldsymbol{\alpha}_k p_k$, one also has

$$\boldsymbol{\alpha}_k H + H\boldsymbol{\alpha}_k = \boldsymbol{\alpha}_k(c\;\boldsymbol{\alpha}_k\,p_k) + (c\;\boldsymbol{\alpha}_k\,p_k)\;\boldsymbol{\alpha}_k = 2c\,p_k.$$

These two equations together yield

$$i\hbar\,\frac{\partial \boldsymbol{\alpha}_k}{\partial t} = 2\boldsymbol{\alpha}_k H - 2c\,p_k.$$

Since $H$ and $p_k$ are time independent, this entails

$$i\hbar\,\frac{\partial^2 \boldsymbol{\alpha}_k}{\partial t^2} = 2\left(\frac{\partial \boldsymbol{\alpha}_k}{\partial t}\right) H.$$

This differential equation in $\partial\boldsymbol{\alpha}_k/\partial t$ can be integrated twice, yielding the explicit time dependence of the velocity, then position, operators. One first obtains

$$v_k = c\;\boldsymbol{\alpha}_k = c^2 p_k H^{-1} + \left(\frac{i\hbar c}{2}\right)\gamma_k^0 e^{-i\omega t} H^{-1}, \tag{2.28}$$

where $\omega = 2H/\hbar$ and $\gamma_k^0 = \partial\boldsymbol{\alpha}_k/\partial t$ at $t = 0$. As $H = mc^2$, the first term is a constant of the order of $p_k/m$, the classical relation between momentum and velocity. But there is an extra term, here also, oscillating at the frequency:

$$\nu' = \frac{2mc^2}{h}, \tag{2.29}$$

which stems mainly from the rest mass energy $m_0c^2$ in the power expansion of $H$ following Eq. (2.7a).

Only the *constant part* is observed in a practical measurement, which gives the *average velocity* through a time interval much larger than $\nu^{-1}$; whereas the *oscillatory part* explains why the *instantaneous velocity* has eigenvalues $\pm c$ [5, 6]. Further integration yields the time dependence of the electron coordinate $x_k$, and it is seen that the *amplitude* of the oscillatory motion is of the order of $\hbar/2m_0c$, the Compton radius of the relativistic electron (Sect. 2.2 and Eq. 2.34).

*Zitterbewegung* vanishes when one takes expectation values over wave packets made up solely of positive (or negative) energy states [8], which are not full solutions of the wave equation because of the coupling of the four components of $\Psi$ in Eq. (2.11). This motion was interpreted as being due to a *wave beat* between the states with energies $\pm mc^2$, the beat frequency being the difference of the two wave frequencies: $\pm mc^2/h$ [6]. It was also shown (e.g. [9]) that transitions between positive and negative energy states are possible whenever the electron potential energy undergoes variations of at least $m_0c^2$ over distances of at most $h/m_0c$. This is another clue that the Compton wavelength, internal motion, and negative energy states are deeply related. Recently [10] it has been shown that *Zitterbewegung* can affect harmonic generation by strong laser pulse and that stimulated *Zitterbewegung* can be generated by laser-induced transitions between positive and negative energy states.

Comparing the preceding results with those expressed in Eqs. (2.16) and (2.21) makes it clear that the *internal motion* giving rise to the kinetic and magnetic *spin momenta* is nothing but *Zitterbewegung*. A classical relativistic model was proposed [11] in which *spin* appears as the *orbital* angular momentum of *Zitterbewegung*. Moreover, the quantum-relativistic relation of the *Zitterbewegung frequency* to the *inertial mass* together with the general-relativistic equivalence of this latter to the *gravitational mass* establish a link between *spin* and *gravitation*. In a stochastic electrodynamics (SED) model [12], *Zitterbewegung* arises from the *electromagnetic interaction* of a semi-classical particle with the *vacuum zero-point field*, and the *van der Waals* force generated by this oscillatory motion is identified with the *Newtonian* gravitational field. More generally, there have been various attempts to involve general relativity into quantum mechanics (e.g. [13, 14]) or to derive one from the other (e.g. [15, 16]).

In his detailed analysis of Dirac's theory [6], de Broglie pointed out that, in spite of his equation being Lorentz invariant and its four-component wave function providing tensorial forms for all physical properties in space-time, it does not have space and time playing full symmetrical roles, in part because the condition of hermiticity for quantum operators is defined in the space domain while time appears only as a parameter. In addition, space-time relativistic symmetry requires that Heisenberg's uncertainty relations,

$$\Delta p_i.\Delta x_i \sim \hbar \quad (i = 1, 2, 3), \tag{2.30}$$

be completed by a similar relation for the energy, the 'time component' of the four-vector momentum whose space components are the $p_i$'s. This did not seem to be consistent with the energy corresponding to the Hamiltonian $H$ rather than to the operator $i\hbar\ \partial/\partial t$. However, consistency can be recovered by writing

$$\Delta H.\Delta t = \Delta(mc^2).\Delta t = \Delta(mc).\Delta(ct) = \Delta(p_4).\Delta(x_4) \sim \hbar, \tag{2.31}$$

assigning the *full* momentum $p_4 = mc$ to the *time* component $x_4 = ct$, the corresponding operator being $i\hbar\ \partial/\partial(ct)$, in accordance with Eq. (2.8).

If, in Eq. (2.31), $mc$ is replaced by $m_0\gamma c$ (with $\gamma$ defined in Eqs. 2.4), it comes

$$\Delta(m_0\gamma c).\Delta(ct) = \Delta(m_0c).\Delta(ct\gamma) = \Delta(m_0c).\Delta(c\tau_0) \sim \hbar, \tag{2.32}$$

where $\tau_0$ is the *proper time* of the electron, which defines its *internal clock*. To the *internal* time coordinate $c\tau_0 = x_0$ is associated the *rest* mass momentum $m_0c = p_0$. If one removes the $\Delta$'s, one obtains

$$m_0c.c\tau_0 \sim \hbar \to \tau_0 \sim \frac{\hbar}{m_0c^2} = \frac{1}{2\pi\nu_0}, \tag{2.33}$$

where $\nu_0$ is half the *Zitterbewegung* frequency for the electron *at rest*. For this latter, $p_i = 0$ ($i = 1, 2, 3$) and, using the expression for $\boldsymbol{\alpha}_0$ in Eq. (2.15) and the vector form for $\Psi$, Eq. (2.11) reduces to

$$i\hbar\,\frac{\partial\Psi_j}{\partial t} = +\,m_0c^2\Psi_j \to \Psi_j = \Psi_{j0}\exp\left(-2\pi i\,\nu_0 t\right) = \Psi_{j0}\exp\left(-\frac{i\ t}{\tau_0}\right),$$
$$i\hbar\,\frac{\partial\Psi_k}{\partial t} = -\,m_0c^2\Psi_k \to \Psi_k = \Psi_{k0}\exp\left(+2\pi i\,\nu_0 t\right) = \Psi_{k0}\exp\left(+\frac{i\ t}{\tau_0}\right),$$

where $j = 1, 2$; $k = 3, 4$; and $\nu_0 = m_0c^2/h$. The difference (beat) frequency $\nu_0' = 2\nu_0$ of the positive and negative energy states is the *Zitterbewegung* frequency for the electron *at rest*. In the complex exponential argument, $\tau_0 \sim 1.29 \times 10^{-21}$ s defines the *time scale* of the electron internal motion.

## 2.5 The Electron Radii

The spin angular momentum and associated magnetic moment of the electron emerged naturally from Dirac's quantum-relativistic treatment. What also came out from the Dirac equation is that the oscillatory motion (*Zitterbewegung*) giving rise to these momenta involves negative energy states and takes place at light velocity. As the rest masses of both electron and positron are non-zero, one may wonder why they do not go to infinity at that velocity. A first clue is that, since the electron and positron 'rest masses' are opposite and since the 'trembling motion' involves both positive and negative energy states, the 'vibrating entity' has zero average mass, departures from this value being allowed by Heisenberg's uncertainty principle.

There have been a number of speculations on the foundations of inertia, gravitation, and mass (e.g. [15–17]). In the following, we present a novel conjecture based on the previous discussion.

Let us consider again the simple classical picture of a particle endowed with charge $e$ and mass $m_0$ moving at velocity $c$ around a loop of radius $r_C$. In this picture, the intrinsic angular momentum would be $s = m_0c.r_C = r_C.2\pi\hbar/\lambda_C$, from the definition of $\lambda_C$ in Eq. (2.3). As in the Bohr model for the orbital motion of an electron around a nucleus, the spin $s/\hbar$ of the electron takes a (half) integer value if the loop circumference $2\pi r_C$ involves a (half) integer number of wavelengths $\lambda_C$ (the 'half' stemming from the loop being actually a sphere in space-time). This 'loop' could then be considered as some kind of 'intrinsic orbit' with radius $r_C = \lambda_C/4\pi$. From the definition of the Compton wavelength (Eq. 2.3), one may express the rest mass as a function of the inverse of this 'orbit radius':

$$m_0 = \frac{\hbar}{2c\ r_C}, \quad r_C = \frac{\lambda_C}{4\pi}. \tag{2.34}$$

One may then say that this ***intrinsic orbit*** (which defines the 'internal structure' of the particle) is described at velocity $c$ (as results from the Dirac equation), while the ***external orbit*** (in an atom for instance) is described at velocity $v$. However, this makes it necessary to consider that the charged entity describing the intrinsic orbit has ***zero rest mass***. This suggests that the ***rest mass observed*** with respect to an external body (such as an atomic nucleus) arises from the very intrinsic motion of the charged entity at velocity $c$.

The above picture should, of course, be amended to account for the contraction of the loop radius with this fast motion. In fact, if a charged entity describes a spherical motion at light velocity it should look as punctual to an external observer (or a nucleus). But this would violate Heisenberg's uncertainty principle. The quantization condition of the 'intrinsic orbit' can actually be recovered from the relation: $\Delta p.\Delta r \sim \hbar/2$ (the quotient 2 being due to the half-integer value of the spin). If one replaces $\Delta r$ by $r_C$ and $\Delta p$ by $m_0 c$ then $r_C$ can be written as $r_C \sim \hbar/2\, m_0 c$, yielding $4\pi r_C \sim h/m_0 c = \lambda_C$, the *Compton wavelength*. This derivation is similar to that of the *Bohr radius* $a_0$ (which expresses the non-collapse of the electron onto the nucleus) by substituting $\Delta r$ by $a_0$ and $\Delta p$ by $p$ in the quantum condition, $\Delta p.\Delta r \sim \hbar$, and using the balance condition: $p^2/m\ a_0 = e^2/4\pi\varepsilon_0 a_0^2$.

It should be noted, however, that, while we know what holds the electron in a confined region around the *Bohr radius*, the *attraction* by the nucleus, we do not know what holds the conjectured, massless charged entity in a confined region around the *Compton radius*. One may think of a *pressure* generated by interactions with virtual particles of the Dirac sea, yielding a kind of Brownian motion at the subquantum level, the *Zitterbewegung*. However, contrary to the Brownian motion, the electron internal motion is not random, since it gives rise to observable spin momentum and magnetic moment.

Another property of the electron is the so-called *classical radius* $r_0$, which is the size that the electron would need to have its rest mass $m_0$ entirely due to its electric potential energy $E_0$. According to classical electrostatics, the energy required to assemble a sphere of radius $r_0$ and charge $e$ is given by $E_0 = k\ e^2/4\pi\varepsilon_0 r_0$, where $k = ½$ if the charge is evenly distributed on the surface and grows larger for a density increasing towards the centre. Assuming all the rest mass energy $m_0 c^2$ is of electrostatic origin yields, for $k = 1$, $r_0 = e^2/4\pi\varepsilon_0 m_0 c^2$ (Table 2.1). This is the length scale at which renormalization becomes important in quantum electrodynamics.

**Table 2.1** Some universal constants and electron and proton properties

| Name | Symbol | Formula | Dimension | Value | Unit |
|---|---|---|---|---|---|
| *Gravitational constant* | $G$ | $F_{grav} = G\, m\, m' / d^2$ | $M^{-1}L^3T^{-2}$ | $6.672 \times 10^{-11}$ | $N.m^2kg^{-2}$ |
| *Free space permittivity* | $\varepsilon_0$ | $F_{elec} = (4\pi\varepsilon_0)^{-1}\, e\, e'/d^2$ | $M^{-1}L^{-3}T^4I^2$ | $8.85419 \times 10^{-12}$ | $F.m^{-1}$ |
| *Light velocity* | $c$ | *Constant in all frames* | $L.T^{-1}$ | $2.99792 \times 10^8$ | $m.s^{-1}$ |
| *Planck's constant* | $h$ | $\Delta E = h\nu$ $\hbar = h/2\pi$ | $M.L^2T^{-1}$ | $6.62618 \times 10^{-34}$ | J.s |
| *Elementary charge* | $e$ | *Negative or positive* | I.T | $1.60219 \times 10^{-19}$ | C |
| *Fine-structure constant* | $\alpha$ | $e^2/4\pi\varepsilon_0\hbar c$ | Dimensionless | 1/137.036 | Pure number |
| *Electron rest mass* | $m_e$ | *Negative for positrons* | M | $9.10953 \times 10^{-31}$ | kg |
| *Gravitational invariant* | $\delta$ | $Gm_e{}^2/\hbar c$ | Dimensionless | $1.75122 \times 10^{-45}$ | Pure number |
| *Classical electron radius* | $r_0$ | $e^2/4\pi\varepsilon_0 m_e c^2$ | L | $2.81794 \times 10^{-15}$ | m |
| *Compton electron radius* | $r_C$ | $\hbar/2m_e c$ | L | $1.93080 \times 10^{-13}$ | m |
| *Hydrogen Bohr radius* | $a_0$ | $4\pi\varepsilon_0\hbar^2/m_e e^2$ | L | $5.29177 \times 10^{-11}$ | m |
| *Gravitational electron radius* | $r_G$ | $(G/c^2)\, m_e$ | L | $6.763 \times 10^{-58}$ | m |
| *Electron mass density* | $\rho_e$ | $m_e/(4\pi/3)r_C{}^3$ | $M.L^{-3}$ | $30.2131 \times 10^6$ | $kg.m^{-3}$ |
| *Proton rest mass* | $m_P$ | *Negative for antiprotons* | M | $1.67265 \times 10^{-27}$ | kg |
| *Proton mass density* | $\rho_P$ | $m_P/(4\pi/3)r_P{}^3$ | $M.L^{-3}$ | $34.3425 \times 10^{19}$ | $kg.m^{-3}$ |
| *Hydrogen non-rel. I.P.* | $I_H\,(1s)$ | $e^2/8\pi\varepsilon_0 a_0$ | $M.L^2T^{-2}$ | 13.6058 | eV |

The classical radius $r_0$ is related to the Compton radius $r_C$ by the relation: $r_0/2r_C = e^2/4\pi\varepsilon_0\hbar c = \alpha$, $\alpha$ being the fine-structure constant ($\alpha = c^{-1}$ in atomic units). The electron classical radius $r_0$ is also related to the hydrogen Bohr radius $a_0$ (Table 2.1) by the relation: $r_0.a_0 = \hbar^2/m_0^2c^2$, or $2\pi r_0.2\pi a_0 = \lambda_C^2 = (4\pi r_C)^2$. This shows that the *Compton radius* $r_C$ is a kind of *geometric average* of the *classical radius* $r_0$ and the *Bohr radius* $a_0$; hence, the *harmonic relation*

$$\frac{2r_C}{a_0} = \frac{r_0}{2r_C} = \alpha, \quad \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \sim 0.7297 \times 10^{-2}. \tag{2.35}$$

If then the *electron* is considered as the 'lowest (stable) state' of some kind of 'hidden structure' similar to the Bohr atom, the related *muon* and *tau* particles could be seen as 'excited (unstable) states' of this internal system. In hydrogenoid atoms, the *smaller* the 'Bohr' (average) radius $<r>_n$ of a given (spherically symmetric) *ns* orbital, the *larger* the ionization energy $I_n$ from this state, according to the formula: $I_n <r>_n \sim (Z/4\pi\varepsilon_0)\ e^2$. Analogically, in the electron family, the *smaller* the 'Compton' radius $r_C$ of a particle, the *larger* its rest mass energy $m_0c^2$: according to Eq. (2.34), $m_0c^2 r_C \sim \hbar c/2$. However, $I_n$ (governed by the electromagnetic interaction) increases practically as the square of the radial quantum number $n$, while $m_0$ (governed by an undetermined interaction) increases hyper-exponentially with the rank of the particle ($n = 1, 2, 3$ for the *electron*, *muon*, and *tau* particles).

Other radii that could be considered are those related to the *space-time curvature* in general relativity theory. If the electron is viewed as a *micro-universe* with a rest mass $m_0$ uniformly distributed within a 3D sphere of radius $r_G$, then the space-time '*inside*' the electron would be endowed with a Gaussian 2D curvature increasing with the mass-energy density $\rho_G$, according to the formula [18, 19]

$$\frac{6}{r_G^2} = 8\pi \left(\frac{G}{c^4}\right) \rho_G. \tag{2.36}$$

Using $\rho_G = m_0\ c^2/(4\pi\ /3)\ r_G^3$ yields

$$r_G = \left(\frac{G}{c^2}\right) m_0, \tag{2.37}$$

which is about $6.763 \times 10^{-58}$ m with the values listed in Table 2.1. This electron 'gravitational radius' $r_G$ is over $10^{42}$ times smaller than the 'classical radius' $r_0$ because the gravitational interaction is that smaller than the electromagnetic interaction, the two radii being in the same ratio as the two forces (Table 2.1):

$$\begin{aligned} \frac{r_0}{r_G} = \left(\frac{e^2}{4\pi\varepsilon_0 m_0 c^2}\right) \times \left(\frac{c^2}{G\ m_0}\right) &= \frac{(4\pi\varepsilon_0)^{-1} e^2}{G\ m_0^2} \\ &= \frac{F_{\text{elec}}}{F_{\text{grav}}} = 4.167 \times 10^{42}. \end{aligned} \tag{2.38}$$

Another point of view is to consider the space-time curvature induced by the rest mass $m_0$ of the electron '*outside'* a volume of radius $r_Q$. According to general relativity theory, the curvature radius $R_G$ around the electron would be given by [18, 19]:

$$R_G = \left(\frac{c^2}{G}\right) \frac{r_Q^2}{m_0} = \frac{r_Q^2}{r_G}. \tag{2.39}$$

This yields a *harmonic relation* similar to Eq. (2.35) which, *when* $r_Q$ *is the Compton radius* $r_C$, relates to *a gravitational invariant* $\delta$ similar to the *fine-structure constant* $\alpha$ (Table 2.1):

$$\frac{r_G}{2r_C} = \frac{2r_C}{4R_G} = \delta, \quad \delta = \frac{Gm_0^2}{\hbar c} \sim 0.175 \times 10^{-44}. \tag{2.40}$$

If $r_Q$ is the Compton radius $r_C$, then $R_G = 5.512 \times 10^{31}$ m $\sim 0.58 \times 10^{16}$ light years. The outside curvature $R_G$ equals the frontier radius $r_Q$ only if $r_Q = r_G$, which is far below the Compton radius.

For the proton, due to Eq. (2.34), Eq. (2.39) gives a value $(1{,}836.15)^3$ times smaller: $R_G = 0.8904 \times 10^{22}$ m $\sim 0.94 \times 10^6$ light years. For a quasi-fermion with the mass and size of the Earth ($M = 5.974 \times 10^{24}$ kg, $R = 6.371 \times 10^6$ m), it gives $R_G = 9.118 \times 10^{15}$ m $\sim 0.97$ light year; and with the mass and size of the Sun ($M = 1.989 \times 10^{30}$ kg, $R = 6.970 \times 10^8$ m), $R_G = 0.329 \times 10^{15}$ m $\sim 12.78$ light days. The contribution of the electron rest mass to the space-time curvature is absolutely negligible - even in its vicinity - relative to that of the other masses in the universe (which result in an overall radius $r_G \sim 13.7 \times 10^9$ light years).

From Eqs. (2.37) and (2.39), it is clear that the confinement of a charged entity oscillating at light velocity within a Compton radius defined by Eq. (2.34) cannot be related directly to the gravitational space-time curving.

To summarize the above discussion, the Compton radius $r_C$ appears as playing a privileged role in the description of the electron. If one considers the *electromagnetic force*, $r_C$ is the geometric average of the *classical electron radius* $r_0$ and the *Bohr hydrogen radius* $a_0$, yielding a harmonic relation with $\alpha$ as ratio. If one considers the *gravitational force*, $r_C$ is the geometric average of the *curvature* $r_G$ *within the particle* and the *curvature* $R_G$ *at distance* $r_C$ *from the core*, also yielding a harmonic relation with a ratio $\delta$ related to $\alpha$ by the ratio of the two forces.

Of the various definitions of electron radii, only that emerging from the description of the Compton scattering has direct experimental evidence. This radius also defines the amplitude of the ‘trembling’ (oscillatory) motion, which is responsible for the spin momentum and magnetic moment of the electron.

It should be noted that in this model, where the electron appears as a quasi-Bohr subsystem with radius $r_C$, there is *no Coulomb singularity*, according to Gauss’ theorem, and *no cusp condition* is required if the wave equation is reformulated to account for the electron size.

## 2.6 The Rest Mass as Related to the Spin Motion

The essential idea in this chapter is that the *rest mass* of the electron stems from the *rotational motion at light velocity*, in a confined region defined by the *Compton radius*, of a *massless charged entity*. That a mass may stem from motion is nothing new since an inertial mass $m_0$ gains extra value with increasing speed $v$, according to the relativistic formula: $m_v = m_0/(1 - v^2/c^2)^{1/2}$ ($\rightarrow \infty$ when $v \rightarrow c$). That a massless entity travelling at light velocity may display mass properties is nothing new either since a photon has a kinetic momentum (e.g. in the Compton effect) defined by $p = h/\lambda$ and a gravitational mass (e.g. in the Mössbauer shift) defined by $m = p/c$.

Relativity theory tells that length, interval, *and mass* vary with velocity, *not charge*. If the electron mass essentially results from the rotational motion, at light velocity, of a massless charge on a sphere of radius $r_C$, then the contribution of the electrostatic potential due to the charge distribution over this sphere is

$$E_0 \sim \frac{e^2}{8\pi\varepsilon_0 r_C} = \alpha.m_0c^2, \tag{2.41}$$

that is, less than 1% of the rest mass energy (electromagnetic and gravitational contributions are even smaller). But this contribution is still $2/\alpha$ times larger than the potential (ionization) energy of the electron in a hydrogen 1s orbital (Table 2.1):

$$I_H(1\mathrm{s}) = \frac{e^2}{8\pi\varepsilon_0 a_0} = \left(\frac{1}{2}\right)\alpha^2 m_0 c^2. \tag{2.42}$$

How does the *hidden confined motion* of the massless charge at velocity $c$ relate to the *visible free motion* of the resulting particle at velocity $v$? If one uses again the semi-classical picture of an electron ball, the radius $r_v$ parallel to the direction of the motion decreases as $r_v = r_C\,(1 - v^2/c^2)^{1/2}$ ($\rightarrow 0$ when $v \rightarrow c$), yielding the expected mass increase:

$$m_v = \frac{\hbar}{2r_v c} = \frac{m_C}{(1 - v^2/c^2)^{1/2}}. \tag{2.43}$$

The contraction of the radius of the *visible particle* along the direction of the *outer motion* when its *velocity* increases entails a decrease in the *amplitude* of the *inner motion* of the *hidden entity.* Resistance to acceleration (inertia) can then be seen as a resistance to the resulting 'motion distortion'. If indeed the spin motion occurs at light velocity and if the rest mass stems from this very motion, this may be the deep reason why $c$ is a limiting speed for all motions and why inertial frames play a specific role in relativity theory.

These are only qualitative considerations. The problem of combining a spherical motion approaching light velocity [20–22] with a linear motion of increasing speed is very complex indeed and requires the mathematical formalism of general relativity theory. This will be the subject of further work.

## 2.7 Other Elementary Particles

The number of so-called elementary particles – and their antiparticles – observed in modern accelerators has reached several hundred (most of them being very short lived and some, not even isolated). Ultimately, they disintegrate into nucleons (made

**Table 2.2** For all known particles of the electron family and for a few other common particles, measured rest mass (in MeV) and computed 'Compton radius' (in nm)

| Particle | Rest mass/MeV | Charge/e | Spin/$\hbar$ | 'Compton radius'/nm | Lifetime/s | Discovery |
|---|---|---|---|---|---|---|
| *Electron* | **0.5110** | −1 | 1/2 | $1.931\times10^{-2}$ | Stable | 1896 Cambridge |
| *Muon* | **105** | −1 | 1/2 | $0.940\times10^{-4}$ | $\sim2\times10^{-6}$ | 1936 Caltech |
| *Tau* | **1,700** | −1 | 1/2 | $0.580\times10^{-5}$ | $\sim3\times10^{-13}$ | 1975 Stanford |
| *Neutrino(s)* | $<\mathbf{10^{-6}}$ | 0 | 1/2 | $>0.987\times10^{4}$ $\sim 10\ \mu$ | Oscillating | 1956–1962 –2001 |
| *Proton* | **938.272** | +1 | 1/2 | $10.508\times10^{-6}$ *$0.842\times10^{-6}$* | $\sim10^{34}$ years | 1919 |
| *Neutron* | **939.565** | 0 | 1/2 | $10.501\times10^{-6}$ *$0.341\times10^{-6}$* | ~15 min | 1932 |
| *Photon* | $<0.76\times10^{-37}$ | 0 | 1 | $>\mathbf{1.30\times10^{35}}$ | Exchanging | 1905 |
| *Big Bang singularity* | $\sim10^{22}$ | ? | ? | $\sim10^{-26}$ | $\sim10^{-43}$ | ~1930's |

The charge, spin, and measured lifetime of these particles and values (in *italics*) of the proton and neutron *charge radii* measured by *electron scattering* are also given. The correspondence between units used in Tables 2.1 and 2.2 is: 1 MeV = $1.60219\times10^{-13}$ J = $1.78268\times10^{-30}$ kg; 1 light year = $0.94605\times10^{25}$ nm

up of quarks), electrons, and neutrinos. In addition to the electron, the only stable and isolatable particles are the proton and the neutrino. One may add the neutron, which decomposes into a proton, an electron, and an antineutrino when isolated. As the Dirac equation in free space does not refer to the charge (or the stability) of the electron, the only conditions for other particles to obey this equation are to have a rest mass and spin ½. All that was said for the electron should then hold for these three particles, as well as for the others in the electron and neutrino families.

In Table 2.2, we have gathered the *measured* rest mass and *computed* 'Compton radius' for these particles. The *electron*, *muon*, and *tau* form a homogeneous family, which shows decreasing lifetime with increasing mass. The *proton* and the *neutron,* being sensitive to the strong interaction, belong to a different family. Although they are not sensitive to the electromagnetic field, particles of the *neutrino* family, which are endowed with spin ½, should follow the Dirac equation, if they have non-zero rest mass. The charge does not enter when one uses Heisenberg's uncertainty relation to estimate the Compton radius of a particle. However, for neutrinos (not composite as the neutron), there is no magnetic moment associated with the spin.

The *proton* and the *neutron* being *composite* particles, their *measured* radius $r_N$ ($N$ standing for nucleon) strongly differs from their 'Compton radius' $r_C$ and their magnetic moment $\mu_N$ from the nuclear magneton $\mu_P = e\,c\,r_P$ ($r_P$ being the Compton radius for the proton) by factors 2.79285 and −1.91315, respectively (for electrons, the corresponding factor is 1.00116, the decimals stemming from *qed* effects).

Although the *photon* has zero rest mass and spin 1 and thus does not follow the Dirac equation [5], Table 2.2 also gives a computed 'rest mass' for a photon travelling freely across the universe, assuming for the latter a radius of $\sim 13.7\times 10^9$ light years. If the universe were flat and infinite, the photon 'rest mass' would be zero. The value given here is purely formal, not only because it is very small but also because it could be detected only by an observer 'external' to our universe!

In the lower row of Table 2.2, there are also given the so-called Planck's energy $E_P$, Planck's length $r_P$, and Planck's time $\tau_P$, which define the *Big Bang singularity* and are similarly, in accordance with Heisenberg's uncertainty principle, related through Compton's formula: $r_P \sim \hbar/2m_P c \sim \hbar\, c/2E_P$ and $\tau_P \sim \hbar/E_P \sim 2\, r_P/c$.

It may be interesting to assess what would be the equivalent of the *Bohr radius* for a *neutrino* orbiting around a *neutron* under the sole influence of *gravitation*, the two particles being deprived of charge. They are also sensitive to the weak interaction but this latter, though much larger than gravitation, is very short ranged and negligible at these distances.

Assuming Heisenberg's relations can still be used for the gravitational field, one can write that at the equilibrium, 'Bohr-like' distance $a_\nu$, if $n$ and $\nu$ are the neutron and neutrino masses, respectively, the 'inertial force', ${p_\nu}^2/\,\nu\, a_\nu$, is balanced by the 'gravitational force', $G\, n\, \nu/{a_\nu}^2$, yielding

$$\Delta p^2.\Delta r^2 \sim p_\nu^2.a_\nu^2 \sim G\; n\nu^2 a_\nu \sim \hbar^2 \to a_\nu = \frac{\hbar^2}{G\; n\nu^2}. \tag{2.44}$$

This is about $3.28\times 10^{24}$ light years with the numerical values given in Tables 2.1 and 2.2. Comparing the above formula with that for the neutrino 'Compton radius', $r_C = \hbar\,/2\nu\, c$, yields the ratio

$$\frac{2r_C}{a_\nu} = \frac{G\; n\nu}{\hbar c} = \frac{\delta\; n\nu}{m_0^2} = 2.11 \times 10^{15}\; n\nu. \tag{2.45}$$

This equation is similar to Eq. (2.35), with $\alpha$ replaced by $\delta$ defined in Eq. (2.40).

## 2.8 The Photon as a Composite of Two Conjugate Fermions

In one of his conjectures [23], de Broglie described the photon as resulting from the 'fusion' of two spin-½ particles, an electron and a positron (whose spins would add and charges cancel) or a neutrino and its antineutrino. Although de Broglie managed to derive Maxwell's equations from this model, his idea was not retained in further developments of quantum electrodynamics. But it was somehow revived in the 'standard model' of quantum chromodynamics, where it is assumed that the strong interaction between quarks constitutive of nucleons is mediated by massless vector gauge gluons, each gluon carrying a 'colour charge' (blue, green, or red) and an 'anticolour charge' (antiblue, antigreen, or antired), while mesons result from the 'fusion' of two quarks of a given colour and the corresponding anticolour.

The 'fusion' model can be simply pictured as follows. An electron approaching light velocity would appear to an external observer as a flattened ellipsoid orthogonal to the direction of the motion, our 'massless charged entity' rotating around the linear motion axis, say $z$. A positron could then be seen as a similar entity rotating in the *opposite* sense. The composition of the two motions yields 0 along an axis orthogonal to $z$, say $x$, and, along the third axis, $y$, it yields $y = 2r_e \cos 2\pi \nu_e t$, with $2r_e = \lambda_C/2\pi$ (Eq. 2.34) and $\nu_e = c/2\pi r_e$ (the two entities rotating at light velocity around $z$). The maximum (positive) value of $y$ for the particle-antiparticle pair is reached when $2\pi \nu_e \tau_e = 2k\pi$ $(k = 0, \pm 1, \ldots)$, at time intervals given by $\tau_e = 1/\nu_e$.

During this rotating period, the pair has travelled, at light velocity, over the linear distance $c.\tau_e = c/\nu_e = 2\pi r_e = \lambda_C/2$. This is the distance on the linear path of two maxima along the circular path and thus has the meaning of a wavelength. If one identifies the pair with the photon then one can write: $E = h\,\nu_e = 2h\,c/\lambda_C = 2m_0c^2$, the sum of the two particle energies, or the energy required for a $\gamma$ photon to yield an electron-positron pair.

Also according to this model, the metastable hydrogenoid species *positronium* ($\tau \sim 0.1$ ns) may be seen as a couple of oppositely charged vortices (with Compton radius $r_C = a_0.\alpha/2$ and velocity $c$) rotating around a barycentre at distance $a_e = 4a_0$ with velocity $v = c.\alpha/4$ ($a_0$ being the Bohr radius and $\alpha$ the fine-structure constant, Table 2.1). The spins of the two vortices may be opposite ($S = 0$) or aligned ($S = 1$). As in the 'Fujiwara effect' in fluid dynamics [24], the two vortices would attract each other when they spin in the same direction and eventually merge into a single vortex, which would be our 'compound' photon (the positive charge vortex being equivalent to a reversed negative charge vortex).

In the above description of a photon as a 'fusion' of an electron and a positron, an electron charge would oscillate along the $y$ axis, say, generating an electromagnetic field with the oscillating electric component parallel to the motion of the charge and the in-phase magnetic component orthogonal to $y$ and $z$.

However, this description holds only for photons with energies $E = 2m_0c^2$. But electromagnetic radiation ranges from radio waves to cosmic rays. One could then conjecture that, whereas there is a discrete spectrum of rest masses (and other properties) for particles that can be isolated, photons are made up of 'virtual' particles that exist only in combination. A similar assumption is made in quantum chromodynamics, where quarks exist only in combinations in gluons, mesons, or baryons. A photon of arbitrary energy $E' = h\nu'$ could then be seen as a 'virtual' particle-antiparticle pair with 'Compton wavelength' $2c/\nu'$ and 'rest mass' $h\nu'/2c^2$. When a photon transfers part of its energy to an electron, as in the Compton effect (Sect. 2.2), it trades with the 'Dirac sea' its constitutive 'virtual pair' of Compton wavelength $\lambda_1$ against a lower-energy 'virtual pair' with $\lambda_2$ given by Eq. (2.3).

## 2.9 Conclusions

In this chapter, we have revisited the Dirac equation in its original form and investigated its implications regarding the electron structure and rest mass. On the basis of this discussion, the following conclusions have been drawn:

1. The *spin angular momentum* and intrinsic magnetic moment of the electron (or positron) stem from its 'trembling motion' (*Zitterbewegung*). This latter is due to a *wave beat* of coupled positive and negative energy states with energies corresponding to the *electron and positron rest masses*. The *value ½ of the spin* results from the factor 2 in the difference of the interfering frequencies: $\pm m_0c^2/h$. Therefore, *every particle endowed with spin ½*, including neutrinos, *should have rest mass*, however small it may be.
2. Alternatively, the electron (or positron) *rest mass* can be seen as arising from the *spinning motion* of a *massless charge* at *light speed*. The rest mass involved in external motions (or interactions) would then be due mainly to this internal motion. The ratio between the electrostatic (classical) and kinetic (rotational) contributions to the rest mass in this model is equal to the *fine-structure constant*: $\alpha \sim 1/137$.
3. The magnitudes of the *spinning radius* $r_C$ and of the *rest mass* $m_0$ are related by the *Compton formula:* $r_C.m_0c = \hbar/2$, which expresses the uncertainty principle for 'position' $r_C$ and 'momentum' $m_0c$. *Rest mass* and *spin motion* thus appear as *essentially related quantum properties*, a kind of zero-point vibration energy for a charged entity with respect to some inertial frame.
4. The *Compton diameter* $2r_C$ is the *geometric average* of the classical electron radius $r_0$ and the Bohr hydrogen radius $a_0$, the ratio of this harmonic relation being the *fine-structure constant:* $\alpha \sim 1/137$. It is also the *geometric average* of the gravitational curvature radii 'inside' and 'outside' the electron, $r_G$ and $4R_G$ respectively, the ratio of this harmonic relation being a *gravitational invariant:* $\delta \sim 1.75\times10^{-45}$.
5. Due to the connection between *spin motion* and *inertial mass* revealed by the Dirac equation, and to the equivalence between *inertial* and *gravitational* masses implied by general relativity theory, there is a deep (though not yet very clear) connection between *spin* and *gravitation*.
6. By relating the rest mass to the internal motion, quantum theory brings an insight into the bearing of such relativistic concepts as Lorentz-invariant, Minkowski's *proper interval* $x_0$. As the property $m_0c$ is the 'residual momentum' when the *linear* part $\underline{p}^2$ is subtracted from the *total* entity $m^2c^2$ (Eq. 2.7b), the property $x_0$ is the 'residual interval' when the *space* coordinate $\underline{r}^2$ is subtracted from the *time* coordinate $c^2t^2$ (Eq. 2.5b).
7. The reason why *time* plays a specific role in physics may then be that it relates to the inner clock, $\tau_0 = x_0/c$; spin momentum, $s = p_0r_C$; and rest mass, $m_0 = p_0/c$, of the matter particles. This may also be why *inertial frames*, which involve *time* through *spin* and *mass*, play a privileged role in physics. Time would not exist

in a universe made solely of light, where there would be no inertial frames to measure velocities. The emergence of time seems to be intimately related to the 'splitting' of 'genderless' photon (or boson) particles to yield matter (and anti-matter) particles.

8. If the *electron* is seen as the 'ground state' of a *subsystem* analogous to the Bohr atom, then the parent *mu* and *tau* leptons could be seen as its 'excited states', with a Compton radius decreasing as the rest mass increases.
9. This picture is consistent with de Broglie's theory of photons resulting from the 'fusion' of particle pairs.

Formal developments on the combination of circular and linear motions at relativistic speeds are in progress.

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